On Markov Processes characterised via Martingale Problems

Abhay G. Bhatt

Thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirement for the award of the degree of **Doctor of Philosophy.**

October 1992

Stat-Math Unit
Indian Statistical Institute
New Delhi

TO MY PARENTS

Latika G. Bhatt and Gopal A. Bhatt

Acknowledgements

I am greatly indebted to Prof. Rajeeva L. Karandikar, without whose guidance, help and encouragement this work would not have been possible. Conversations with him have always had a stimulating effect and I have enjoyed them thoroughly. He has painstakingkly gone through the many drafts of the thesis, weeded out many errors and suggested changes that have made the text clearer and more precise. It will be my endeavour to try and justify the faith that he has in my ability.

I thank Prof. G. kallianpur and Prof. Karandikar for allowing me to include parts of joint work in the thesis.

I also thank all the members of the Stat-Math faculty of the Indian Statistical Institute, Delhi center, for introducing me to the mathematics which has been useful in this work. They have not only kept my interest in the subject alive but have also added to it. It was due to Mr. Rajesh David that I came to ISI after my graduation. My decision to take up research was influenced by Prof. R. B. Bapat. I thank both of them.

All the members of my family have been a constant source of love for me. They have always encouraged me in my academic efforts and have stood by my decisions even if I have sometimes gone against their wishes. It is my family which has been my strength in difficult times.

I am also indebted to many friends for their help and support during my stay at ISI. I would have missed a lot in life but for them.

I thank the National Board For Higher Mathematics and the Indian Statistical Institute for providing financial support in the form of research awards.

Contents

Introduction

Martingale approach to the study of finite dimensional diffusions was initiated by Stroock - Varadhan, who coined the term martingale problem. Their success led to a similar approach being used to study Markov processes occuring in other areas such as infinite particle systems, branching processes, genetic models, density dependent population processes, random evolutions *etc.*

Suppose X is a Markov process corresponding to a semigroup $(T_t)_{t\geq 0}$ with generator L. Then all the information about X is contained in L. We also have that

$$M^f(t) := f(X(t)) - \int_0^t Lf(X(s))ds$$

is a martingale for every $f \in \mathcal{D}(L)$. *i.e.* X is a solution to the martingale problem for L. Now instead of the generator L, if we start with an operator A, such that there exists a unique solution to the martingale problem for A, then under some further conditions the solution is a Markov process corresponding to a semigroup which is given by a transition probability function. Hence the operator A determines the semigroup $(T_t)_{t\geq 0}$. A is then a restriction of the generator of $(T_t)_{t\geq 0}$ to $\mathcal{D}(A)$. It is natural to expect that in this case the operator A contains all the information about the Markov process X. For example, it is well known that $\int Lf d\mu = 0$ for all $f \in \mathcal{D}(L)$ implies that μ is an invariant measure for the Markov process and one may expect that $\int Af d\mu = 0$ for all $f \in \mathcal{D}(A)$ would also imply that μ is an invariant measure.

In this thesis, we address this question as well as prove uniqueness of a measure valued evolution equation corresponding to this Markov process when the test functions are taken from $\mathcal{D}(A)$. Weak convergence of a sequence of processes when the limiting process arises as a unique solution to a martingale problem is also studied. All the results are for the case when the state space of the underlying process is a complete separable metric space and thus the results can also be applied to processes taking values in infinite dimensional spaces.

In the first two sections in chapter 1 preliminary (known) results on martingale problems and their connections to Markov Processes are collected together.

The first section starts with the definition of the martingale problem corresponding to an operator A on $C_b(E)$, where E is a complete separable metric space. Since progressively measurable solutions play an important role in the later chapters, a distinction is made between well posedness in the class of progressively measurable solutions and well posedness in the class of solutions which have right continuous paths with left limits, *i.e.* r.c.l.l. solutions. The following ***separability*** condition is assumed throughout the thesis.

(I) There exists a countable subset $\{f_n\} \subset \mathcal{D}(A)$ such that

$$bp-closure(\{(f_n, Af_n) : n \geq 1\}) \supset \{(f, Af) : f \in \mathcal{D}(A)\}.$$

It is shown that this condition and well - posedness of the martingale problem for degenerate initials implies that the solution is a Markov process admitting a one parameter semigroup given by a transition probability function. Results are proved which show that for the question of uniqueness it suffices to look at the one dimensional distributions of the solutions. A result on r.c.l.l. modification of solutions when the state space E is compact is proved. It is shown that there exists a solution to the martingale problem for a perturbation of the operator A if the martingale problem for A itself admits a solution. In the end, the time inhomogeneous martingale problem is defined. The presentation here follows that in [7]. Some proofs are taken from [15].

In the second section, a weak convergence result from [7] is stated and a consequence is obtained which is applicable in the martingale problem context.

The third section is on examples. Some results of Stroock - Varadhan on finite dimensional diffusions ([15]) and those of Yor on Hilbert space valued diffusions ([16]) are stated. In both the cases it is proved that the martingale problem is well posed in the class of progressively measurable solutions.

The last section is on Hilbert space valued stochastic evolution equation and the corresponding martingale problem. The results are new (at least in the generality considered in the thesis) and hence proofs of existence and uniqueness of solutions to the stochastic evolution equation and that of their equivalence with solutions to the corresponding martingale problem are included. These are taken from [1].

Chapter 2 deals with invariant measures for Markov Processes characterised via martingale problems. The first section includes some auxillary results. A generalisation of the Riesz representation theorem for the space $E \times E$ when each of the two marginals on E are countably additive is proved.

In section 2.2. the main result of this thesis, a criterion for invariant measures of a Markov process is proved. In [5], Echeverria proved that when E is a compact metric space or a locally compact separable metric space and A is an operator for which the martingale problem is well - posed in the class of r.c.l.l. solutions with domain being an algebra, then μ is an invariant measure for A if it satisfies

$$\int_E Af d\mu = 0 \quad \forall f \in \mathcal{D}(A). \tag{0.0.1}$$

Here we show that the same result holds when E is a complete, separable metric space. The additional condition required is that the martingale problem for A is well - posed in the class of progressively measurable solutions. It may be noted that this condition is satisfied for the generator L of a Markov process.

In the course of the proof we *imbed* the martingale problem for A into a compact sapce $\hat{E}$. The imbedding is such that the two martingale problems, on E and $\hat{E}$, are equivalent in a certain sense, *viz.* there is a one to one correspondence between solutions X and Z to the martingale problems on E and $\hat{E}$ respectively. It is to be noted that even if Z is an r.c.l.l. process, X, which is got from Z by a certain transformation need not be r.c.l.l. and hence the condition about the well - posedness in the class of progressively measurable solutions. This imbedding and the above mentioned generalisation of the Riesz representation theorem are the key tools in the proof of the main theorem.

In chapter 3 the measure valued evolution equation

$$\int_E f d\nu_t = \int_E f d\nu_0 - \int_0^t \left(\int_E (Af - \lambda(\cdot)f) d\nu_s\right) ds \quad f \in \mathcal{D}(A) \tag{0.0.2}$$

is considered, where A is an operator on $C_b(E)$ and $\lambda \in C_b(E)$. It is shown that when A satisfies the conditions of the main result in chapter 2 (Theorem

2.2.3), there exists a unique solution to (0.0.2) in the class of positive measures. The criterion for invariant measures is used crucially in proving this result.

In section 2, uniqueness of solutions to the evolution equation (time inhomogeneous case) is deduced using results in the previous section. Applications to filtering theory are discussed in the last section. The results presented in chapters 2 and 3 are from [2].

Chapter 4 includes results from [3] . The imbedding of the martingale problem for A into a compact space $\hat{E}$ is made use of to prove results on convergence of a sequence of processes X_n to a given Markov process characterised via the martingale problem for A. Two results on weak convergence of the sequence of processes and two results on the convergence of finite dimensional distributions are proved. We are able to get rid of the *compact containment condition* on X_n. This condition occurs in many results on convergence of processes. (See *e.g.* [7] and [6]). But when the state space is infinite dimensional this is not an easy condition to verify. Similar results using the martingale approach when the state space is locally compact can also be found in [7].

In section 2, these are applied to yield convergence results for Hilbert space valued processes. The first example is illustrative, and it shows the power of the method by deducing Donsker's invariance principle for Hilbert space valued random variables from the Central limit theorem via some simple computations. The other two applications are about Hilbert space valued diffusions and we show that these diffusions depend continuously on the coefficients.

Chapter 1

Preliminaries

1.1 The Martingale Problem

Let (E, d) be a complete separable metric space. $\mathcal{E}$ will denote the Borel σ- field on E, $B(E)$ will denote the space of bounded measurable real valued functions on E, $C(E)$ will denote the space of real valued continuous functions on E, $C_b(E)$ will denote the space of bounded continuous real valued functions on E, $\mathcal{P}(E)$ will denote the space of probability measures on $(E, \mathcal{E})$. Note that $B(E)$ as well as $C_b(E)$ are Banach spaces with the supremum norm $\|f\| := \sup_{x\in E}|f(x)|$.

We consider operators A on $C_b(E)$ with domain denoted by $\mathcal{D}(A)$. Let $\mu \in \mathcal{P}(E)$.

Definition 1. A (E- valued) measurable process X defined on some probability space $(\Omega, \mathcal{F}, P)$ is said to be a solution to the martingale problem for (A, μ) with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$ if

(i) $P \circ X(0)^{-1} = \mu$.

(ii) For all $f \in \mathcal{D}(A), f(X(t)) - \int_0^t Af(X(s))ds$ is a $(\mathcal{G}_t)_{t\geq 0}$ - martingale.

When $\mathcal{G}_t = {}^*\mathcal{F}_t^X := \sigma\{X(s), \int_0^s h(X(u))du : s \leq t, h \in C_b(E)\}$, the σ-fields are dropped from the statement. Note that when X is a process which has paths which are right continuous and have left limits, ${}^*\mathcal{F}_t^X$ is same as $\mathcal{F}_t^X := \sigma\{X(s) : s \leq t\}$. It is clear that if X is a solution to the martingale problem for (A, μ) with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$, then it is also a solution with respect to the filtration $({}^*\mathcal{F}_t^X)_{t\geq 0}$.

Lemma 1.1.1 *A process X is a solution to the martingale problem for A if and only if*

$$\mathbb{E}[(f(X(t_{n+1})) - f(X(t_n)) - \int_{t_n}^{t_{n+1}} Af(X(s))ds) \prod_{k=1}^{n} h_k(X(t_k))] = 0 \quad (1.1.1)$$

for all $f \in \mathcal{D}(A), , 0 \le t_1 < t_2 < \ldots < t_{n+1}, h_1, h_2, \ldots, h_n \in B(E)$, and $n \ge 1$.

Proof. Clearly if X is a solution to the martingale problem for A, (1.1.1) holds since $\prod_{k=1}^{n} h_k(X(t_k))$ is ${}^*\mathcal{F}_{t_n}^X$ measurable. Also,

$$M^f(t) := f(X(t)) - \int_0^t Af(X(s))ds$$

is ${}^*\mathcal{F}_t^X$ measurable. Hence to prove the converse assertion it suffices to prove that for $t_n < t_{n+1}$

$$\mathbb{E}[(M^f(t_{n+1}) - M^f(t_n))I_B] = 0 \quad \forall B \in {}^*\mathcal{F}_{t_n}^X.$$

Consider the functions of the form $[\prod_{k=1}^{n} h_k(X(t_k))][\prod_{l=1}^{m} \int_0^{s_m} f_l(X(s))ds]$, where $n, m \ge 1$, $0 \le t_1 < t_2 < \ldots \le t_n$, $0 \le s_1, s_2, \ldots s_m \le t_n$, and $h_1, h_2, \ldots, h_n, f_1, f_2, \ldots f_m \in B(E)$. These functions generate $({}^*\mathcal{F}_{t_n}^X)$. Hence the result follows if we can show that

$$\mathbb{E}[(M^f(t_{n+1}) - M^f(t_n))(\prod_{k=1}^{n} h_k(X(t_k)))(\prod_{l=1}^{m} \int_0^{s_m} f_l(X(s))ds)] = 0. \quad (1.1.2)$$

Using (1.1.1) , one has

$$\mathbb{E}[(M^f(t_{n+1}) - M^f(t_n))(\prod_{k=1}^{n} h_k(X(t_k)))(\prod_{l=1}^{m} f_l(X(s))ds)] = 0.$$

Now (1.1.2) follows from this by Fubini's theorem. ∎

When the process X has paths which are right continuous and have left limits it is useful to consider the law of the process X as the solution to the martingale problem. Frequently we are interested in solutions which have such paths. We will denote by $D([0, \infty), E)$ the space of all E valued r.c.l.l. functions on $[0, \infty)$ equipped with the Skorokhod topology, and by $\mathcal{S}_E$, the Borel σ-field on $D([0, \infty), E)$.

Definition 2. A probability measure $P \in \mathcal{P}(D([0,\infty),E))$ is said to be a solution to the martingale problem for (A,μ) if there exists a $D([0,\infty),E)$-valued process X (defined on some $(\Gamma,\mathcal{G},Q)$) with $\mathcal{L}(X) = P$ and such that X is a solution to the martingale problem for (A,μ) in the sense of Definition 1.

Or, equivalently, $P \in \mathcal{P}(D([0,\infty),E))$ is a solution if the co-cordinate process $\theta(t,\omega) := \omega(t)$ defined on $(D([0,\infty),E),\mathcal{S}_E,P)$ is a solution in the sense of Definition 1.

Note that if X satisfies (1.1.1) and Y has the same finite dimensional distributions as that of X then Y also satisfies (1.1.1) . Hence from Lemma 1.1.1 it follows that if X is a solution to the martingale problem for an operator A and Y is a measurable modification of X then Y is also a solution. This prompts the following definition. Let $\mathcal{C}$ be a class of processes.

Definition 3. The martingale problem for (A,μ) is said to be well - posed in the class $\mathcal{C}$ if there exists a solution $X \in \mathcal{C}$ to the martingale problem for (A,μ) and if $Y \in \mathcal{C}$ is also a solution to the martingale problem for (A,μ), then X and Y have the same finite dimensional distributions.

When $\mathcal{C}$ is the class of all measurable processes then we just say that the martingale problem is well - posed.

Definition 4. The $D([0,\infty),E)$ - martingale problem for (A,μ) is said to be well - posed if there exists a solution $P \in \mathcal{P}(D([0,\infty),E))$ to the $D([0,\infty),E)$ - martingale problem for (A,μ) and if Q is any solution to the $D([0,\infty),E)$ - martingale problem for (A,μ) then $P = Q$.

Since finite dimensional distributions characterise the probability measures on $D([0,\infty),E)$, well - posedness in the class of r.c.l.l. solutions is same as well - posedness of the $D([0,\infty),E)$ - martingale problem.

Well - posedness requires uniqueness in the class of all measurable solutions, whereas well - posedness of the $D([0,\infty),E)$ - martingale problem requires uniqueness only among r.c.l.l. solutions. Of course, well - posedness of the $D([0,\infty),E)$ - martingale problem requires existence of r.c.l.l. solutions. Also well - posedness and existence of r.c.l.l. solutions together imply well - posedness of the $D([0,\infty),E)$ - martingale problem. However, there exist operators A for which the $D([0,\infty),E)$ - martingale problem is well - posed but the martingale problem for A is not well - posed. (See [7, p.265]).

Definition 5. The ($D([0,\infty),E)$-) martingale problem for A is well - posed if the ($D([0,\infty),E)$-) martingale problem for (A,μ) is well - posed for every $\mu \in \mathcal{P}(E)$.

Definition 6. For $f_k, f \in B(E)$, we say that $f_k \stackrel{bp}{\to} f$ (where bp stands for bounded pointwise) if $\|f_k\| \leq M$ and $f_k(x) \to f(x)$ for all $x \in E$. A class of functions $\mathcal{U} \subset B(E)$ is said to be bp-closed if $f_k \in \mathcal{U}, f_k \stackrel{bp}{\to} f$ implies $f \in \mathcal{U}$. bp-closure($\mathcal{U}$) is defined to be the smallest class of functions in $B(E)$ which contains $\mathcal{U}$ and is bp-closed.

It should be noted that bp-closure is not closure in a topological sense. For example, if $\mathcal{E}_0$ is a field that generates $\mathcal{E}$ and $\mathcal{H}$ is the class of $\mathcal{E}_0$ - simple functions, then bp-closure $\mathcal{H}$ is the class of all $\mathcal{E}$ - measurable bounded functions.

The following *separability* condition on A which will be assumed throughout this thesis plays a very important role in what follows.

(I) There exists a countable subset $\{f_n\} \subset \mathcal{D}(A)$ such that

$$bp - closure(\{(f_n, Af_n) : n \geq 1\}) \supset \{(f, Af) : f \in \mathcal{D}(A)\}.$$

Suppose A satisfies condition (I). From definitions 1 and 5 and a simple application of the Dominated Convergence Theorem it follows easily that a process X is a solution to the martingale problem for A if and only if it is a solution to the martingale problem for the restriction of A to the countable subset $\{f_n\}$. Recall that $\theta(t)$ denotes the co-ordinate process on $D([0,\infty),E)$.

Theorem 1.1.2 *Suppose the $D([0,\infty),E)$ - martingale problem for (A,δ_x) is well - posed for each $x \in E$. Let P_x denote the solution. Suppose further that A satisfies the separability condition (I). Then*

(i) *$x \mapsto P_x(C)$ is measurable for all $C \in S_E$.*

(ii) *For all $\mu \in \mathcal{P}(E)$, the $D([0,\infty),E)$ - martingale problem for (A,μ) is well - posed, with the solution P_μ given by*

$$P_\mu(C) = \int_E P_x(C)\mu(dx). \tag{1.1.3}$$

(iii) *Under P_μ, $\theta(t)$ is a Markov process with transition function P given by*

$$P(s,x,F) = P_x(\theta(s) \in F) \tag{1.1.4}$$

Proof. We will first show that the set $\mathcal{M} := \{P_x : x \in E\}$ is a Borel set in $\mathcal{P}(D([0,\infty),E))$. To this end first choose a countable subset $M \subset C_b(E)$ such that $B(E) \subset$ bp-closure(M). (That such a countable set exists can be seen, e.g. from Proposition 4.2, Chapter III, [7]). Let H be the collection of functions of the form

$$\eta(\theta) = [(f_n(\theta(t_{m+1})) - f_n(\theta(t_m)) - \int_{t_m}^{t_{m+1}} Af_n(\theta(s))ds) \prod_{k=1}^{m} h_k(\theta(t_k))] \tag{1.1.5}$$

where $h_1, h_2, \ldots, h_m \in M, 0 \le t_1 < t_2 \ldots < t_{m+1} \subset Q$ and $\{f_n\} \subset \mathcal{D}(A)$ is as in condition (I). Then H is countable. It follows from Lemma 1.1.1 that the set $\mathcal{M}_1 = \cap_{\eta \in H}\{P : \int \eta dP = 0\}$ is the set of solutions of the martingale problem for A. $\mathcal{M}_1$ is a Borel set since the map $P \mapsto \int \eta dP$ is a continuous map. Set $G : \mathcal{P}(D([0,\infty),E)) \to \mathcal{P}(E)$ by $G(P) = P \circ \theta(0)^{-1}$. Then G is continuous and note that $\mathcal{M} = \mathcal{M}_1 \cap G^{-1}(\{\delta_x : x \in E\})$. Hence $\mathcal{M}$ is a Borel set. Also, the well - posedness hypothesis implies that G restricted to $\mathcal{M}$ is one-to-one mapping onto $\{\delta_x : x \in E\}$. Thus the inverse map G^{-1} from $\{\delta_x : x \in E\}$ to $\mathcal{M}$ is Borel measurable. (See *e.g.* [14, p.22]). Note that $G(P_x) = \delta_x$ and hence the map $\delta_x \mapsto P_x$ is measurable which in turn implies (i).

Now let P_μ be defined as in (1.1.3) .Then for any set $F \in \mathcal{E}$

$$P_\mu \circ \theta(0)^{-1}(F) = \int_E P_x \circ \theta(0)^{-1}(F)\mu(dx) = \int_E \delta_x(F)\mu(dx) = \mu(F).$$

And for $\eta \in H$,

$$\int_{D([0,\infty),E)} \eta dP_\mu = \int_E \int_{D([0,\infty),E)} \eta dP_x \mu(dx) = 0.$$

Hence P_μ is a solution to the martingale problem for (A,μ).

The rest of the proof is on the lines of the proof of Theorem 6.2.1 in [15]. Let Q be a solution to the $D([0,\infty),E)$- martingale problem for (A,μ) and Q_ω be the regular conditional probability of Q given $\theta(0)$. Let η be given by (1.1.5) and $h \in C_b(E)$. Let $\eta'(\theta) = \eta(\theta)h(\theta(0))$. Then $\eta' \in H$ and hence

$$\mathbb{E}^Q[\eta(\theta)h(\theta(0))] = \mathbb{E}^Q[\eta'] = 0. \tag{1.1.6}$$

Since (1.1.6) holds for all $h \in C_b(E)$, it follows that

$$\mathbb{E}^{Q_\omega}[\eta] = \mathbb{E}^{Q}[\eta|\theta(0)] = 0 \text{ a.s. - } Q.$$

Since H is countable, we can get a Q-null set N_0 such that for all $\omega \notin N_0$,

$$\mathbb{E}^{Q_\omega}[\eta] = 0 \quad \forall \ \eta \in H.$$

This shows that for $\omega \notin N_0$, Q_ω is a solution to the martingale problem for A with initial distribution $\delta_{(\theta(0,\omega))}$. Now well - posedness implies that for $\omega \notin N_0$, $Q_\omega = P_{\theta(0,\omega)}$ which in turn implies that $Q = P_\mu$. This proves (ii).

Fix s and let $\theta'(t) = \theta(t+s)$. Let Q'_ω be the regular conditional probability distribution of θ' (under P_x) given $\mathcal{F}_s$. (See [4]). Then arguing as in part (ii) it follows that Q'_ω is a solution to the martingale problem for $(A, \delta_{(\theta(s,\omega))})$. Hence by well - posedness it follows that $Q'_\omega(\theta'(t) \in F) = P(t, \theta(s,\omega), F)$ where $P(s,x,\cdot)$ is defined by (1.1.4) . Hence for $f \in B(E)$,

$$\begin{aligned}
\mathbb{E}^{P_x} f(\theta(t+s)) &= \mathbb{E}^{P_x}[\mathbb{E}^{P_x}[f(\theta(t+s))|\mathcal{F}_s]] \\
&= \mathbb{E}^{P_x}[\int_E f(y) P(t, \theta(s,\cdot), dy)] \\
&= \int_E \int_E f(y_2) P(t, y_1, dy_2) P(s, x, dy_1).
\end{aligned}$$

In particular

$$P_x(\theta(t+s) \in F) = \int_E P(t,y,F) P(s,x,dy) \quad \forall \ F \in \mathcal{E}.$$

i.e.

$$P(s+t,x,F) = \int_E P(t,y,F) P(s,x,dy) \quad \forall \ F \in \mathcal{E}. \qquad (1.1.7)$$

Hence the transition probability function $P(\cdot,\cdot,\cdot)$ satisfies the Kolmogorov Chapman equation. (iii) now follows from the above discussion. ■

According to Definition 3, well - posedness of the martingale problem for A holds if any two solutions to the martingale problem have the same finite dimensional distrtibutions. In fact, one needs to check only that the one dimensional distributions of any two solutions are the same, as is demonstrated in the next result.

Theorem 1.1.3 *Suppose that for each $\mu \in \mathcal{P}(E)$, any two solutions X and Y (defined respectively on $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$) of the martingale problem for (A, μ) have the same one-dimensional distributions, then X and Y have the same finite dimensional distributions, i.e. the martingale problem is well-posed.*

Proof. We want to show that

$$\mathbb{E}^{P_1}[\prod_{k=1}^{m} f_k(X(t_k))] = \mathbb{E}^{P_2}[\prod_{k=1}^{m} f_k(Y(t_k))] \tag{1.1.8}$$

for all $0 \leq t_1 < t_2 < \ldots t_m, f_1, f_2, \ldots, f_m \in B(E)$ and $m \geq 1$. Note that (1.1.8) holds for $m = 1$ by hypothesis of the theorem. We will prove (1.1.8) by induction on m. Suppose that (1.1.8) is true for $m = n$. Fix $0 \leq t_1 < t_2 < \ldots t_n, f_1, f_2, \ldots, f_n \in B(E), f_k > 0$. Define

$$\begin{aligned} Q_1(F_1) &= \frac{\mathbb{E}^{P_1}[I_{F_1} \prod_{k=1}^{n} f_k(X(t_k))]}{\mathbb{E}^{P_1}[\prod_{k=1}^{n} f_k(X(t_k))]} \quad \forall\, F_1 \in \mathcal{F}_1 \\ Q_2(F_2) &= \frac{\mathbb{E}^{P_2}[I_{F_2} \prod_{k=1}^{n} f_k(Y(t_k))]}{\mathbb{E}^{P_2}[\prod_{k=1}^{n} f_k(Y(t_k))]} \quad \forall\, F_2 \in \mathcal{F}_2 \end{aligned}$$

Let $\tilde{X}(t) = X(t_n + t)$, $\tilde{Y}(t) = Y(t_n + t)$. Let $\eta(\cdot)$ be defined by (1.1.5) for $0 \leq s_1 < s_2 < \ldots, s_{m+1} = t, h_1, h_2, \ldots, h_m \in B(E)$ and $f \in \mathcal{D}(A)$. Note that

$$\begin{aligned} \mathbb{E}^{P_1}[\eta(X(t_n + \cdot)) \prod_{k=1}^{n} f_k(X(t_k))] &= \mathbb{E}^{P_1}[(f(X(s_{m+1} + t_n)) - f(X(s_m + t_n)) \\ &- \int_{t_n+s_m}^{t_n+s_{m+1}} Af(X(u))du) \prod_{j=1}^{m} h_j(X(t_n + s_j))][\prod_{k=1}^{n} f_k(X(t_k))] = 0. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}^{Q_1}[\eta(\tilde{X})] &= \frac{\mathbb{E}^{P_1}[\eta(X(t_n + \cdot)) \prod_{k=1}^{n} f_k(X(t_k))]}{\mathbb{E}^{P_1}[\prod_{k=1}^{n} f_k(X(t_k))]} \\ &= 0. \end{aligned}$$

Similarly

$$\mathbb{E}^{Q_2}[\eta(\tilde{Y})] = 0.$$

Hence $\tilde{X}$ defined on $(\Omega_1, \mathcal{F}_1, Q_1)$ and $\tilde{Y}$ defined on $(\Omega_2, \mathcal{F}_2, Q_2)$ are solutions to the martingale problem for A. Also by (1.1.8) for $m = n$,

$$\begin{aligned}\mathbb{E}^{Q_1}[f(\tilde{X}(0))] &= \frac{\mathbb{E}^{P_1}[f(X(t_n))\prod_{k=1}^{n} f_k(X(t_k))]}{\mathbb{E}^{P_1}[\prod_{k=1}^{n} f_k(X(t_k))]} \\ &= \frac{\mathbb{E}^{P_2}[f(Y(t_n))\prod_{k=1}^{n} f_k(Y(t_k))]}{\mathbb{E}^{P_2}[\prod_{k=1}^{n} f_k(Y(t_k))]} \\ &= \mathbb{E}^{Q_2}[f(\tilde{Y}(0))] \quad \forall\, f \in B(E).\end{aligned}$$

Hence $\tilde{X}$ and $\tilde{Y}$ have the same initial distribution. The hypothesis of the theorem implies

$$\mathbb{E}^{Q_1}[f(\tilde{X}(t)] = \mathbb{E}^{Q_2}[f(\tilde{Y}(t)] \quad \forall\, t \geq, f \in B(E).$$

i.e.

$$\mathbb{E}^{P_1}[f(X(t_n + t)) \prod_{k=1}^{n} f_k(X(t_k))] = \mathbb{E}^{P_2}[f(X(t_n + t)) \prod_{k=1}^{n} f_k(X(t_k))],$$

and we get (1.1.8) for $m = n + 1$ by setting $t_{n+1} = t_n + t$.

∎

When A satisfies the conditions of Theorem 1.1.2, we associate the following Markov semigroup $(T_t)_{t \geq 0}$ to A.

$$T_t f(x) = \int_E f(y) P(t, x, dy) \tag{1.1.9}$$

for $f \in B(E)$, where $P(\cdot, \cdot, \cdot)$ is given by (1.1.4) .

Theorem 1.1.4 *Suppose that the $D([0,\infty), E)$- martingale problem for A is well - posed. Suppose further that the semigroup T_t is associated to A (by (1.1.4) and (1.1.9)).*

Let X, defined on $(\Omega, \mathcal{F}, P)$, be a solution of the martingale problem for A (with respect to $(\mathcal{G}_t)_{t \geq 0}$). Let τ be a finite stop time. Then for $f \in B(E), t \geq 0$,

$$\mathbb{E}[f(X(\tau + t)) | \mathcal{G}_\tau] = T_t f(X(\tau)). \tag{1.1.10}$$

In particular

$$P((X(\tau + t) \in \Gamma) | \mathcal{G}_\tau) = P(t, X(\tau), \Gamma) \; \forall\, \Gamma \in \mathcal{E}. \tag{1.1.11}$$

Proof. Let P_1, P_2 be defined on $(\Omega, \mathcal{F})$ by

$$\begin{aligned} P_1(B) &= \frac{\mathbb{E}[I_F P_{X(\tau)}(B)]}{P(F)} \\ P_2(B) &= \frac{\mathbb{E}[I_F \mathbb{E}[I_B(X(\tau+\cdot))|\mathcal{G}_\tau]}{P(F)} \end{aligned}$$

where $F \in \mathcal{G}_\tau$ with $P(F) > 0$. Then P_1, P_2 define solutions to the martingale problem for A with the same initial distribution and proceeding as in the proof of Theorem 1.1.3 we get (1.1.10) . ∎

It can be shown that when L is the generator of a Markov process then the martingale problem for L is well - posed. (See [7] , Theorem IV.4.1). The semigroup $(T_t)_{t\geq 0}$ is then the semigroup generated by L. On the other hand, if A satisfies the conditions of Theorem 1.1.2, the weak generator L of the asociated semigroup is an extension of A.

At this point, we state a result which says that any solution to the martingale problem admits a r.c.l.l. modification when the state space is compact.

Theorem 1.1.5 *Let E be a compact metric space. Let A be an operator on $C(E)$ such that $\mathcal{D}(A)$ is measure determining and contains a countable subset that separates points in E. Let X, defined on $(\Omega, \mathcal{F}, P)$, be a solution to the martingale problem for A. Then X has a modification with sample paths in $D([0,\infty), E)$.*

Proof. Let $\{g_k : k \geq 1\} \subset \mathcal{D}(A)$ separate points in E. Now

$$M_k(t) := g_k(X(t)) - \int_0^t Ag_k(X(s))ds$$

is a martingale for all k. Then it is well known that

$$\lim_{\substack{s\uparrow t\\ s\in Q}} M_k(s) \ , \ \lim_{\substack{s\downarrow t\\ s\in Q}} M_k(s)$$

exist a.s. Here Q denotes the set of all rationals in $\mathbb{R}$. It follows that

$$\lim_{\substack{s\uparrow t\\ s\in Q}} g_k(X(s)) \ , \ \lim_{\substack{s\downarrow t\\ s\in Q}} g_k(X(s)) \tag{1.1.12}$$

exist a.s.

Let $\Omega' \subset \Omega$ be such that $P(\Omega') = 1$ and (1.1.12) holds for all $t \geq 0, k \geq 1$, and $\omega \in \Omega'$.

Since E is compact, for each $t \geq 0$, and every sequence $\{s_n\} \subseteq Q$ with $s_n > t, \lim_{n\to\infty} s_n = t$ and for $\omega \in \Omega'$ there exists a subsequence $\{s_{n_i}\}$ such that $\lim_{i\to\infty} X(s_{n_i}, \omega)$ exists. Clearly

$$g_k(\lim_{i\to\infty} X(s_{n_i}, \omega)) = \lim_{\substack{s \downarrow t \\ s \in Q}} g_k(X(s, \omega)).$$

Since $\{g_k : k \geq 1\}$ separate points in E, it follows that $\lim_{\substack{s \downarrow t \\ s \in Q}} X(s, \omega)$ exists. Similarly $\lim_{\substack{s \uparrow t \\ s \in Q}} X(s, \omega)$ exists for all $t \geq 0$ and all $\omega \in \Omega'$. Define

$$Y(t, \omega) := \lim_{\substack{s \downarrow t \\ s \in Q}} X(s, \omega)$$

Then it follows that for $w \in \Omega'$, $Y(t, \omega)$ is r.c.l.l. and

$$Y^-(t, \omega) = \lim_{\substack{s \uparrow t \\ s \in Q}} X(s, \omega).$$

Defining Y suitably for $\omega \notin \Omega'$ it follows that Y has sample paths in $D([0, \infty), E)$.

Since X is a solution to the martingale problem for A, we get for $f \in \mathcal{D}(A)$

$$\begin{aligned} I\!E[f(Y(t))|\mathcal{F}_t^X] &= \lim_{\substack{s \downarrow t \\ s \in Q}} I\!E[f(X(s))|\mathcal{F}_t^X] \\ &= f(X(t)). \end{aligned}$$

Since $\mathcal{D}(A)$ is measure determining the above result follows from the following lemma.

Lemma 1.1.6 *Suppose that X, Y, defined on $(\Omega, \mathcal{F}, P)$ satisfy*

$$I\!E[f(Y)|\mathcal{G}] = f(X) \tag{1.1.13}$$

for all $f \in M \subset C_b(E)$ where $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$, with respect to which X is measurable, and M is a measure determining set. Then $X = Y$ a.s.

Proof. We have for $B \in \mathcal{G}$ and $f \in M$,

$$\int_B f(Y)dP = \int_B f(X)dP. \tag{1.1.14}$$

Fix $B \in \mathcal{G}$. Define P_1 and P_2 on $(E, \mathcal{E})$ by

$$\begin{aligned} P_1(F) &= P(B \cap \{Y \in F\}) \\ P_2(F) &= P(B \cap \{X \in F\}) \end{aligned}$$

for all $F \in \mathcal{E}$. Now (1.1.14) implies

$$\int_E f dP_1 = \int_E f dP_2 \quad \forall\, f \in M$$

which in turn implies $P_1 = P_2$. *i.e.*

$$P(B \cap \{Y \in F\}) = P(B \cap \{X \in F\}) \quad \forall\, B \in \mathcal{G}, F \in \mathcal{E}.$$

Substituting $B = \Omega$, we get

$$P(\{Y \in F\}) = P(\{X \in F\}) \quad \forall\, F \in \mathcal{E}. \tag{1.1.15}$$

And for $B = \{X \in F\}$, we get

$$P(\{Y \in F\} \cap \{X \in F\}) = P(\{X \in F\}). \tag{1.1.16}$$

Combiming (1.1.15) and (1.1.16) , we get

$$P(\{Y \in F\} \cap \{X \notin F\}) + P(\{Y \notin F\} \cap \{X \in F\}) = 0.$$

This holds for all $F \in \mathcal{E}$. Hence we get $P(X = Y) = 1$. ∎

Lemma 1.1.7 *Suppose that there exists a solution X_x to the martingale problem for (A, δ_x) for each $x \in E$. Then A is dissipative i.e., for every $f \in \mathcal{D}(A)$ and every $\lambda > 0$, we have*

$$\|(\lambda - A)f\| \geq \lambda \|f\|. \tag{1.1.17}$$

Proof. We know that

$$f(X_x(t)) - \int_0^t Af(X_x(s))ds \tag{1.1.18}$$

is a martingale. An 'integration by parts' argument (see [15, p.26], [7, p.65 and p.92]) implies that for every$\lambda > 0$,

$$e^{-\lambda t} f(X_x(t)) + \int_0^t e^{-\lambda s}((\lambda - A)f(X_x(s)))ds \tag{1.1.19}$$

is a martingale. Hence for every $x \in E$,

$$f(x) = \mathbb{E}[\int_0^\infty e^{-\lambda s}((\lambda - A)f(X_x(s)))ds].$$

Therefore

$$|f(x)| \leq \int_0^\infty e^{-\lambda s}\|(\lambda - A)f\|ds \leq \lambda^{-1}\|(\lambda - A)f\|$$

and

$$\lambda\|f\| \leq \|(\lambda - A)f\|.$$

This proves the lemma. ∎

We now prove existence of solutions for perturbations of the operator A.

Theorem 1.1.8 *Let A be an operator with $\mathcal{D}(A) \subset C_b(E)$. Let $\lambda \in C_b(E)$ be non-negative and let $\eta(x, \Gamma)$ be a transition function on $E \times \mathcal{E}$. Let*

$$Bf(x) = \lambda(x) \int_E (f(y) - f(x))\eta(x, dy) \;\; f \in B(E).$$

Suppose that for every $\mu \in \mathcal{P}(E)$, there exists a solution to the $D([0,\infty), E)$ martingale problem for (A, μ). Then for every $\mu \in \mathcal{P}(E)$ there exists a solution to the martingale problem for $(A + B, \mu)$.

Proof. To simplify matters let us assume that $\lambda > 0$ is a constant. The general case can be deduced from this by considering $\lambda_1 = \sup_{x\in E} \lambda(x)$ and

$$\eta'(x, \Gamma) = (1 - \frac{\lambda(x)}{\lambda_1})\delta_x(\Gamma) + \frac{\lambda(x)}{\lambda_1}\eta(x, \Gamma)$$

and noting that $Bf(x)$ can also be represented as

$$Bf(x) = \lambda_1 \int_E (f(y) - f(x))\eta'(x, dy) \quad \forall\ f \in B(E).$$

Let $\Omega = \prod_{k=1}^{\infty}(D([0,\infty),E) \times [0,\infty))$. Denote by Ω_k and Ω_k^0 the k^{th} copy of $D([0,\infty),E)$ and $[0,\infty)$ respectively. Let θ_k and ξ_k denote the co-ordinate random variables and $\mathcal{F}_k, \mathcal{F}_k^0$ be the Borel σ-fields on Ω_k and Ω_k^0 respectively. Let $\mathcal{F}$ be the product σ-field on Ω.

Intutively, a solution to the martingale problem for $A+B$, evolves in E as a solution to the martingale problem for A till an exponentially distributed time with parameter λ and which is independent of the past. At this time if the process is at x, it jumps to y with probability $\eta(x,dy)$ and then continues evolving as a solution to the martingale problem for (A,δ_y). To put this in a mathematical framework, we consider that between the k^{th} and the $(k+1)^{th}$ jump (dictated by B), the process lies in Ω_k. The k^{th} copy of the exponential time is a random variable in Ω_k^0. Now to see that such a process does exist, we proceed as follows.

Let $\mathcal{G}_k$ be a sub σ-algebra of $\mathcal{F}$ that is generated by cylinder sets $C_1 \times \prod_{i=k+1}^{\infty}(\Omega_i \times \Omega_i^0)$, where $C_1 \in \mathcal{F}_1 \otimes \mathcal{F}_1^0 \otimes \ldots \otimes \mathcal{F}_k \otimes \mathcal{F}_k^0$. Similarly, let $\mathcal{G}^k \subset \mathcal{F}$ be the σ-algebra generated by sets of the form $\prod_{i=1}^{k}(\Omega_i \times \Omega_i^0) \times C_2$, where $C_2 \in \mathcal{F}_k \otimes \mathcal{F}_k^0 \otimes \ldots$.

Let P_x, P_μ denote solutions to the martingale problems for (A,δ_x) and (A,μ) respectively. Let γ be the exponential distribution with parameter λ. Fix $\mu \in \mathcal{P}(E)$. Define, for $\Gamma_1 \in \mathcal{F}_1, \ldots, \Gamma_k \in \mathcal{F}_k$, $F_1 \in \mathcal{F}_1^0, \ldots, F_k \in \mathcal{F}_k^0$,

$$\begin{aligned}
P_1(\Gamma_1) &= P_\mu(\Gamma_1) \\
P_1^0(\theta_1, F_1) &= \gamma(F_1) \\
\vdots \quad & \quad \vdots \\
P_k(\theta_1,\xi_1,\ldots,\theta_{k-1},\xi_{k-1},\Gamma_k) &= \int_E P_x(\Gamma_k)\eta(\theta_{k-1}(\xi_{k-1}),dx) \\
P_k^0(\theta_1,\ldots,\xi_{k-1},\theta_k,F_k) &= \gamma(F_k)
\end{aligned} \tag{1.1.20}$$

and so on. Note that $P_1 \in \mathcal{P}(\Omega_1)$ and $P_1^0, P_2, P_2^0, \ldots$ are transition probability functions. Then Tulcea's theorem applies and we get a probability P on $(\Omega,\mathcal{F})$ satisfying for $C_1 \in \mathcal{F}_1 \otimes \mathcal{F}_1^0 \otimes \ldots \otimes \mathcal{F}_k \otimes \mathcal{F}_k^0$,

$$\begin{aligned}
&P(C_1 \times \Omega_{k+1} \times \Omega_{k+1}^0 \times \Omega_{k+2} \times \Omega_{k+2}^0 \times \ldots) \\
&\quad = \int\int \ldots \int I_{C_1}(\theta_1,\xi_1,\ldots\theta_k,\xi_k)P_k^0(\theta_1,\ldots\theta_k,d\xi_k)\ldots P_1^0(\theta_1,d\xi_1)P_1(d\theta_1).
\end{aligned}$$

For $C \in \mathcal{G}_k$ and $C' \in \mathcal{G}^{k+1}$

$$P(C \cap C') = \mathbb{E}[\int_E P(C'|\theta_{k+1}(0) = x)\eta(\theta_k(\xi_k), dx) I_C]. \tag{1.1.21}$$

To see that (1.1.21) holds, note that

$$P(C \cap C') = \mathbb{E}[I_C I_{C'}] = \mathbb{E}[I_C \mathbb{E}[I_{C'}|\mathcal{G}_k]].$$

When C' is a cylinder set in $\mathcal{G}^{k+1}$, using (1.1.20) , it is easy to see that

$$\mathbb{E}[I_{C'}|\mathcal{G}_k] = \int_E P(C'|\theta_{k+1}(0) = x)\eta(\theta_k(\xi_k), dx).$$

Now (1.1.21) follows from this and usual measure theoretic arguments.

Define $\tau_0 = 0, \tau_k = \sum_{i=1}^k \xi_i$, and $N(t) = k$ for $\tau_k \le t < \tau_{k+1}$. Note that N is a Poisson process with parameter λ. Define

$$X(t) = \theta_{k+1}(t - \tau_k), \;\; \tau_k \le t < \tau_{k+1}, \tag{1.1.22}$$

and $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^N$. We claim that X is a solution of the martingale problem for $(A + B, \mu)$ with respect to $(\mathcal{F}_t)_{t\ge 0}$.

It is easy to see that for $f \in \mathcal{D}(A)$

$$f(\theta_{k+1}((t \vee \tau_k) \wedge \tau_{k+1} - \tau_k)) - f(\theta_{k+1}(0)) - \int_{\tau_k}^{(t\vee\tau_k)\wedge\tau_{k+1}} Af(\theta_{k+1}(s - \tau_k))ds$$

is an $(\mathcal{F}_t)_{t\ge 0}$ martingale. Summing over k we get

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \sum_{k=1}^{N(t)}(f(\theta_{k+1}(0)) - f(\theta_k(\xi_k))) \tag{1.1.23}$$

is an $(\mathcal{F}_t)_{t\ge 0}$ martingale. Also note that

$$\sum_{k=1}^{N(t)}(f(\theta_{k+1}(0)) - \int_E f(y)(\eta(\theta_k(\xi_k), dy)) \tag{1.1.24}$$

and

$$\int_0^t \int_E (f(y) - f(X(s-)))\eta(X(s-), dy)d(N(s) - \lambda s) \tag{1.1.25}$$

are $(\mathcal{F}_t)_{t\ge 0}$ martingales. Adding (1.1.24) and (1.1.25) to (1.1.23) we get that

$$f(X(t)) - f(X(0)) - \int_0^t (Af(X(s)) + Bf(X(s)))ds \tag{1.1.26}$$

is an $(\mathcal{F}_t)_{t\geq 0}$ martingale. ∎

Till now we have considered only the time homogeneous martingale problem. We now define the time inhomogeneous martingale problem.

For $t \geq 0$, let $(A_t)_{t\geq 0}$ be linear operators on $C_b(E)$ with a common domain $\mathcal{D} \subset C_b(E)$.

Definition 7. A measurable process X defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ is said to be a solution to the martingale problem for $(A_t)_{t\geq 0}$ with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$ if for any $f \in \mathcal{D}$

$$f(X(t)) - \int_0^t A_s f(X(s))ds \tag{1.1.27}$$

is a $(\mathcal{G}_t)$ - martingale.

Definition 8. Let $\mu \in \mathcal{P}(E)$. The martingale problem for $(A_t)_{t\geq 0}$ is said to be well - posed if there exists a solution to the martingale problem and if any two solutions X and Y, defined respectively on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $(\tilde{\Gamma}, \tilde{\mathcal{G}}, \tilde{Q})$ with respect to some filtrations and satisfying $\tilde{P} \circ X(0)^{-1} = \tilde{Q} \circ Y(0)^{-1} = \mu$ have the same finite dimensional distributions.

We say that the martingale problem for $(A_t)_{t\geq 0}$ is well - posed if the martingale problem for $((A_t)_{t\geq 0}, \mu)$ is well - posed for every $\mu \in \mathcal{P}(E)$. Well - posedness of the $D([0,\infty), E)$ - martingale problem is defined similarly.

Let $E^0 = [0,\infty) \times E$. Most of the results on martingale problems can be extended to the time dependent case by considering the space-time process

$$X^0(t) = (t, X(t)) \tag{1.1.28}$$

and the corresponding martingale operator A^0 with domain $\mathcal{D}' \subset C_b(E^0)$. The following theorem is the main tool in doing so.

Let $C_0^1([0,\infty))$ denote the space of continuously differentiable functions on $[0,\infty)$ with compact support.

Theorem 1.1.9 *Let $\mathcal{D}' \subset C_b(E^0)$ consist of functions of the form*

$$g(t,x) = \sum_{i=1}^k h_i(t) f_i(x) \quad h_i \in C_0^1([0,\infty)), f_i \in \mathcal{D} \tag{1.1.29}$$

Define an operator A^0 on $C_b(E^0)$ with domain $\mathcal{D}'$ by

$$A^0 g(t,x) = \sum_{i=1}^{k} [f_i(x) \frac{\partial}{\partial t} h_i(t) + h_i(t) A_t f_i(x)] \tag{1.1.30}$$

Then X is a solution to the martingale problem for $(A_t)_{t\geq 0}$ if and only if X^0 is a solution to the martingale problem for A^0 .

Proof. Let X be a solution (with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$) to the martingale problem for $(A_t)_{t\geq 0}$. Clearly it suffices to check that

$$fh(X^0(t)) - \int_0^t A^0 fh(X^0(s))ds \tag{1.1.31}$$

is a $(\mathcal{G}_t)_{t\geq 0}$-martingale for $f \in \mathcal{D}, h \in C_0^1([0,\infty))$.

For $0 < s < t$, let $g(t) = \mathbb{E}[f(X(t))|\mathcal{G}_s]$. Then

$$g(t) - g(s) = \int_s^t \mathbb{E}[A_u f(X(u))|\mathcal{G}_s] du \tag{1.1.32}$$

Then

$$\begin{aligned} g(t)h(t) - g(s)h(s) &= \int_s^t \frac{\partial}{\partial u}[g(u)h(u)]du \\ &= \int_s^t \{h(u)\mathbb{E}[A_u f(X(u))|\mathcal{G}_s] + g(u)\frac{\partial}{\partial u}h(u)\}du \\ &= \int_s^t \mathbb{E}[A^0 fh(X^0(u))|\mathcal{G}_s]du. \end{aligned}$$

The second equality above follows from (1.1.32) . Now, it follows that (1.1.31) is a martingale and hence X^0 is a solution to the martingale problem for A^0.

The converse follows by taking $h = 1$ on $[0,T], T > 0$. ∎

As before we impose the separability condition which is as follows. Here, $(f, A_\bullet f)$ stands for the function $(t,x) \to (f(x), [A_t f](x))$, which is the *graph* of (A_t).

$(I)'$ There exists a countable subset $\mathcal{D}_0$ of $\mathcal{D}$ such that $\{(f, A_\bullet f) : f \in \mathcal{D}\} \subset$ bp-closure $- \{(f, A_\bullet f) : f \in \mathcal{D}_0\}$.

Note that if $(A_t)_{t\geq 0}$ satisfy $(I)'$, then A^0, defined by (1.1.30) satisfies condition (I). This is so since $C_0^1([0,\infty))$ is dense in $C_b([0,\infty))$.

1.2 Weak Convergence

We will now state a compactness criterion for $D([0,\infty),E)$ valued processes. This is Theorem III.8.6 in [7]. This criterion is particularly useful in the context of martingale problems as is illustrated in Theorem 1.2.2 below.

Theorem 1.2.1 *Let Y_n be a sequence of $D([0,\infty),\mathbb{R})$-valued processes. Suppose that for every $\varepsilon > 0$ and $t \geq 0$, there exists a compact set $K_{\varepsilon,t} \subset E$ such that*

$$P\{Y_n(t) \in K_{\varepsilon,t}\} \geq 1-\varepsilon \quad \forall\, n. \tag{1.2.1}$$

Suppose that for each $T > 0$, there exists $\beta > 0$ and a family $\{\gamma_n(\delta) : 0 < \delta < 1, n \geq 1\}$ of nonnegative random variables satisfying

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbb{E}[\gamma_n(\delta)] = 0$$

and

$$\mathbb{E}[|Y_n(t+u)-Y_n(t)|^\beta|\mathcal{F}_t^n] \leq \mathbb{E}[\gamma_n(\delta)|\mathcal{F}_t^n]$$

for $0 \leq t \leq T, 0 \leq u \leq \delta$ where $\mathcal{F}_t^n = \mathcal{F}_t^{Y_n}$. Let $Q_n = P \circ Y_n^{-1}$. Then $\{Q_n : n \geq 1\}$ is a tight family of measures on $D([0,\infty),\mathbb{R})$.

Define for $p < \infty$, $\|h\|_{p,T} := [\int_0^T |h(t)|^p dt]^{\frac{1}{p}}$ and $\|h\|_{\infty,T} = \text{ess}\sup_{0\leq t\leq T}|h(t)|$.

Theorem 1.2.2 *Suppose that $\mathcal{D}(A) \subset C_b(E)$ is an algebra. Let A_n be operators on $B(E)$, $n = 1,2,\ldots$, and X_n be solutions to the $D([0,\infty),E)$- martingale problem for A_n with respect to filtrations $(\mathcal{G}_t^n)_{t\geq 0}$. Suppose that for every $f \in \mathcal{D}(A)$, there exists $f_n \in \mathcal{D}(A_n)$ satisfying*

$$\limsup_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} |f_n(X_n(t)) - f(X_n(t))|] = 0 \tag{1.2.2}$$

and

$$\limsup_{n\to\infty} \mathbb{E}[\|A_n f_n \circ X_n\|_{p,T} < \infty \text{ for some } p \in (1,\infty]. \tag{1.2.3}$$

Then $\{f \circ X_n\}_{n\geq 1}$ is relatively compact for each $f \in \mathcal{D}(A)$. More generally, $\{(f_1,f_2,\ldots,f_k)\circ X_n\}_{n\geq 1}$ is relatively compact in $D([0,\infty),\mathbb{R}^k)$ for all $f_1,f_2,\ldots,f_k \in \mathcal{D}(A), 1 \leq k \leq \infty$.

Proof. Note that for $f \in \mathcal{D}(A)$, f^2 also belongs to $\mathcal{D}(A)$. Choose f_n's and g_n's satisfying (1.2.2) and (1.2.3) for f and f^2 respectively for some p and p'. Let q and q' satisfy $\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{p'} + \frac{1}{q'}$.

Writing $(a-b)^2 = a^2 - b^2 - 2b(a-b)$ and and using the fact that X_n is a solution to the martingale problem for A_n, and with a view to applying Theorem 1.2.1 we get bounds on the *conditional modulus of continuity*. We get for $0 < u < \delta$,

$$\begin{aligned}
&\mathbb{E}[(f(X_n(t+u)) - f(X_n(t)))^2|\mathcal{G}_t^n] \\
&= \mathbb{E}[f^2(X_n(t+u)) - f^2(X_n(t))|\mathcal{G}_t^n] \\
&\quad -2f(X_n(t))\mathbb{E}[f(X_n(t+u)) - f(X_n(t))|\mathcal{G}_t^n] \\
&\le 2\mathbb{E}[\sup_{s\in[0,T+1]} |f^2(X_n(s)) - g_n(X_n(s))|\mathcal{G}_t^n] \\
&\quad +4\|f\|\mathbb{E}[\sup_{s\in[0,T+1]} |f(X_n(s)) - f_n(X_n(s))|\mathcal{G}_t^n] \\
&\quad +\mathbb{E}[\sup_{0\le r\le T} \int_r^{r+\delta} |A_n g_n(X_n(s))|ds|\mathcal{G}_t^n] \\
&\quad +2\|f\|\mathbb{E}[\sup_{0\le r\le T} \int_r^{r+\delta} |A_n f_n(X_n(s))|ds|\mathcal{G}_t^n] \\
&\le \mathbb{E}[\gamma_n(\delta)|\mathcal{G}_t^n]
\end{aligned} \tag{1.2.4}$$

where

$$\begin{aligned}
\gamma_n(\delta) = 2&\sup_{s\in[0,T+1]} |f^2(X_n(s)) - g_n(X_n(s))| \\
&+\ 4\|f\| \sup_{s\in[0,T+1]} |f(X_n(s)) - f_n(X_n(s))| \\
&+\ \delta^{\frac{1}{q'}}\|A_n g_n\|_{p',T+1} + 2\|f\|\delta^{\frac{1}{q}}\|A_n f_n\|_{p,T+1}.
\end{aligned} \tag{1.2.5}$$

The last inequality in (1.2.4) follows from the observation that $\int_r^{r+\delta} |h(s)|ds \le \delta^{\frac{1}{q}}\|h\|_{p,T+1}$ for $0 \le r \le T$. Now using (1.2.2) and (1.2.3) we get

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbb{E}[\gamma_n(\delta)] = 0 \tag{1.2.6}$$

and relative compactness of the sequence $f(X_n)$ follows from Theorem 1.2.1. To prove the result let $f_1, f_2, \ldots, f_k \in \mathcal{D}(A)$, and define $\gamma_n^j(\delta)$ by (1.2.5) for f_j.

Set $\gamma_n(\delta) = \sum_{j=1}^k \gamma_n^j(\delta)$. Then

$$\mathbb{E}[\sum_{j=1}^{k}(f_j(X_n(t+u)) - f_j(X_n(t)))^2|\mathcal{G}_t^n] \leq \mathbb{E}[\gamma_n(\delta)|\mathcal{G}_t^n]$$

for $0 \leq t \leq T$ and $0 \leq u \leq \delta$, and the $\gamma_n^j(\delta)$ can be so selected that (1.2.6) holds. Finally relative compactness for $k = \infty$ follows from relative compactness for all $k < \infty$. ∎

The following variation of the previous result is useful when the processes X_n do not arise as solutions to martingale problems and can be proved similarly.

Theorem 1.2.3 *Suppose that $\mathcal{D} \subset C_b(E)$ is an algebra and let X_n be $D([0,\infty), E)$ - valued processes. Suppose that for every $f \in \mathcal{D}$, there exist progressively measurable processes ξ_n and ϕ_n such that*

$$\xi_n(t) - \int_0^t \phi_n(s)ds \tag{1.2.7}$$

is a martingale with respect to filtration $(\mathcal{F}_t^{X_n})_{t\geq 0}$,

$$\limsup_{n\to\infty} \mathbb{E}[\sup_{t\in\mathbb{Q}\cap[0,T]} |\xi_n(t) - f(X_n(t))|] = 0 \tag{1.2.8}$$

and

$$\limsup_{n\to\infty} \mathbb{E}[\|\phi_n\|_{p,T} < \infty \text{ for some } p \in (1,\infty]. \tag{1.2.9}$$

Then $\{f \circ X_n\}_{n\geq 1}$ is relatively compact for each $f \in \mathcal{D}$. More generally, $\{(f_1, f_2, \ldots, f_k) \circ X_n\}_{n\geq 1}$ is relatively compact in $D([0,\infty), \mathbb{R}^k)$ for all $f_1, f_2, \ldots, f_k \in \mathcal{D}, 1 \leq k \leq \infty$.

1.3 Examples

1. Finite Dimensional Diffusion

Let $E = \mathbb{R}^m$ and let S_m^+ denote the space of $m \times m$ positive definite matrices. Let $a : \mathbb{R}^m \to S_m^+$ and $b : \mathbb{R}^m \to \mathbb{R}^m$ be bounded continuous functions. Define an operator A on $\mathcal{D}(A) := C_0^2(\mathbb{R}^m)$ by

$$Af(x) = \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(x)f_{ij}(x) + \sum_{i=1}^{m} b_i(x)f_i(x)$$

for $f \in C_0^2(\mathbb{R}^m)$ where $f_i = \frac{\partial}{\partial x_i} f$, $f_{ij} = \frac{\partial}{\partial x_j} f_i$ and $a(x) = ((a_{ij}(x)))_{1 \le i,j \le m}$ and $b(x) = (b_1(x), b_2(x), \dots, b_m(x))$.

It is clear that A satisfies the separability condition (I). Stroock - Varadhan ([15]) showed that the $D([0,\infty), E)$ - martingale problem for A is well - posed. The corresponding process is a diffusion on $\mathbb{R}^m$ with a and b as the diffusion and drift coefficients respectively. In fact the martingale problem for A is well - posed in the class of progressively measurable processes. This follows if we show that every progressively measurable solution to the martingale problem for A has a r.c.l.l. modification. We will show that this is indeed true.

Denote by $\hat{\mathbb{R}}^m$ the one point compactification of $\mathbb{R}^m$ and the point at infinity by Δ. Define an operator $\hat{A}$ on $C(\hat{\mathbb{R}}^m)$ with domain

$$\mathcal{D}(\hat{A}) = \{\hat{f} \in C(\hat{\mathbb{R}}^m) : \hat{f}|_{\mathbb{R}^m} \in \mathcal{D}(A), \hat{f}(\Delta) = 0\}$$

and for $\hat{f} \in \mathcal{D}(\hat{A})$,

$$\hat{A}\hat{f}(\Delta) = 0, \quad \hat{A}\hat{f}(x) = Af(x) \text{ for } x \in \mathbb{R}^m.$$

Let X be a progressively measurable solution to the martingale problem for (A, μ). Considered as a process taking values in $\hat{\mathbb{R}}^m$, X is a solution to the martingale problem for $(\hat{A}, \hat{\mu})$ where $\hat{\mu}(B) = \mu(B \cap \mathbb{R}^m)$ for all Borel sets $B \in \mathbb{R}^m$. Note $\hat{\mu}(\mathbb{R}^m) = 1$. Since $C_0^2(\mathbb{R}^m)$ is dense in $C_b(\mathbb{R}^m)$, $\mathcal{D}(\hat{A})$ satisfies the conditions of Theorem 1.1.5. It follows that X has a modification, Y, with sample paths in $D([0,\infty), \hat{\mathbb{R}}^m)$.

Let $B(0,n)$ denote the ball of radius n with centre 0. For $n \ge 1$, choose $f_n \in C_0^2(\mathbb{R}^m)$ such that $f_n(x) = 1$ for all $x \in B(0,n)$ and $f_n(x) = 0$ for all $x \in B(0, n+1)^c$ and such that $\sup_n \|f_n''\| < \infty$, where f_n'' denotes the second derivative of f_n. Note that $Af_n(x) = 0$ for all $x \notin \{n \le |x| \le n+1\}$. Also $(f_n, Af_n) \xrightarrow{bp} (1, 0)$. Extend f_n to $C(\hat{\mathbb{R}}^m)$ by defining $\hat{f}_n(\Delta) = 0$. Then $(\hat{f}_n, \hat{A}\hat{f}_n) \xrightarrow{bp} (I_{\mathbb{R}^m}, 0)$.

Let r denote the metric on $\hat{\mathbb{R}}^m$. For $m = 1, 2, \dots$, define the $(\mathcal{F}_{t+}^X)_{t \ge 0}$ stop time

$$\tau_m = \inf\{t : r(\Delta, Y(t)) < \frac{1}{m}\}. \tag{1.3.1}$$

Then $\tau_1 \le \tau_2 \le \dots$ and $\lim_{m\to\infty} Y(\tau_m \wedge t) = Z(t)$ exists. Note that $Z(t) = \Delta$ if and only if $\lim_{m\to\infty} \tau_m = \tau \le t$. Now using the fact that Y is a solution to the martingale problem for $\hat{A}$ and the optional sampling theorem we get for each $t \geq 0$,

$$\mathbb{E}[\hat{f}_n(Y(\tau_m \wedge t))] = \mathbb{E}[\hat{f}_n(Y(0))] + \mathbb{E}[\int_0^{\tau_m \wedge t} \hat{A}\hat{f}_n(Y(s))ds].$$

Letting $m \to \infty$, we have

$$\mathbb{E}[\hat{f}_n(Z(t))] = \mathbb{E}[\hat{f}_n(Y(0))] + \mathbb{E}[\int_0^{\tau \wedge t} \hat{A}\hat{f}_n(Y(s))ds],$$

and now an application of the Dominated convergence theorem gives

$$P\{\tau > t\} = P\{Y(t) \in \mathbb{R}^m\} = P\{Y(0) \in \mathbb{R}^m\} = 1.$$

Hence it follows that almost all sample paths of Y lie in $D([0,\infty), \mathbb{R}^m)$. Hence as remarked at the beginning this shows that the martingale problem for A is well - posed in the class of progressively measurable solutions.

2. Hilbert Space Valued Diffusion

Let $E = H$, a real, separable Hilbert space. The inner product on H will be denoted by $(\cdot,\cdot)$ and the norm by $\|\cdot\|$. $\mathcal{L}_2(H,H)$ will denote the space of Hilbert Schmidt operators on H and for $\Sigma \in \mathcal{L}_2(H,H)$, $\|\Sigma\|_{HS}$ will denote the Hilber Schmidt norm of Σ.

Let $\sigma : H \to \mathcal{L}_2(H,H)$ and $b : H \to H$ be measurable functions satisfying

$$\begin{aligned} \|\sigma(h)\|_{HS} &\leq K \\ \|b(h)\| &\leq K \\ \|\sigma(h_1) - \sigma(h_2)\|_{HS} &\leq K\|h_1 - h_2\| \\ \|b(h_1) - b(h_2)\| &\leq K\|h_1 - h_2\| \end{aligned}$$

for all $h, h_1, h_2 \in H$. Fix a CONS $\{\phi_i : i \geq 1\}$ in H and define $P_n : H \to \mathbb{R}^n$ by

$$P_n(h) = ((h,\phi_1),\ldots,(h,\phi_n)).$$

Define an operator A with domain $\mathcal{D}(A) = \{f \circ P_n : f \in C_0^2(\mathbb{R}^n), n \geq 1\}$ by

$$[A(f \circ P_n)](h) = \frac{1}{2}\sum_{i,j=1}^{n}(\sigma^*(h)\phi_i, \sigma^*(h)\phi_j)f_{ij} \circ P_n(h) + \sum_{i=1}^{n}(b(h),\phi_i)f_i \circ P_n(h)$$

where $f_i = \frac{\partial}{\partial x_i} f$ and $f_{ij} = \frac{\partial}{\partial x_j} f_i$.

Let $\mathcal{D}_m = \{f \circ P_m : f \in C_0^2(\mathbb{R}^m)\}$ and let A_m be the restriction of the operator A to $\mathcal{D}_m$. For every m, let $\{f^{m,n}\}_{n\geq 1} \subset C_0^2(\mathbb{R}^m)$ be such that

$$\text{bp-closure } - \{(f^{m,n} \circ P_m, A_m(f^{m,n} \circ P_m)) : n \geq 1\} \supset \{(g, A_m g) : g \in \mathcal{D}_m\}.$$

Let $\mathcal{D}_0 = \cup_{m=1}^{\infty} = \{f^{m,n} \circ P_m : n \geq 1\}$. Then it can be easily checked that this countable subset of $\mathcal{D}(A)$ satisfies the condition in (I), *i.e.* A satisfies the separabiltiy condition (I). Yor ([16]) showed that the $D([0,\infty), E)$ - martingale problem for A is well - posed. In fact the solution belongs to $C([0,\infty), E)$, the space of continuous functions on E. The solution is a Hilbert space valued diffusion.

If $(X(t))_{t\geq 0}$ is a progressively measurable process that is a solution to the martingale problem for (A, μ), then using arguments as in the finite dimensional case it can be shown that $(X^i(t))_{t\geq 0} := (X(t), \phi_i)_{t\geq 0}$ admits a continuous modification, say $(\tilde{X}^i(t))_{t\geq 0}$. Let

$$M^i(t) = \tilde{X}^i(t) - \tilde{X}^i(0) - \int_0^t (b(X(s)), \phi_i) ds$$

Then it follows that M^i is a continuous martingale and

$$\langle M^i, M^j \rangle_t = \int_0^t (\sigma^*(X(s))\phi_i, \sigma^*(X(s))\phi_j) ds$$

Now

$$\begin{aligned} \mathbb{E} \sup_{t\leq T} (M^j(t))^2 &\leq 4\mathbb{E}(M^j(T))^2 \\ &= 4\mathbb{E} \int_0^T (\sigma^*(X(s))\phi_j, \sigma^*(X(s))\phi_j) ds \qquad (1.3.2) \end{aligned}$$

Since

$$\begin{aligned} \mathbb{E} \sum_{j=1}^{\infty} \int_0^T (\sigma^*(X(s))\phi_j, \sigma^*(X(s))\phi_j) ds &= \mathbb{E} \int_0^T \|\sigma^*(X(s))\|_{HS}^2 ds \\ &\leq K^2 T, \qquad (1.3.3) \end{aligned}$$

it follows that $\mathbb{E} \sum_{j=1}^{\infty} \sup_{t\leq T} (M^j(t))^2 < \infty$. Hence

$$\sum_{j=1}^{\infty} \sup_{t\leq T} (M^j(t))^2 < \infty \quad \text{a.s.} \qquad (1.3.4)$$

From this it follows that $\sum_{j=1}^{n} M^j(t)\phi_j$ converges uniformly in [0,T] a.s. to say $M(t)$, and hence $(M(t))_{t\geq 0}$ is continuous. It is easy to see that

$$M(t) = X(t) - X(0) - \int_0^t b(X(s))ds \quad a.s.$$

Hence defining $\tilde{X}(t) := M(t) + X(0) + \int_0^t b(X(s))ds$, one gets that $\tilde{X}$ is a continuous modification of X. Well - posedness of the martingale problem for A in the class of progressively measurable solutions follows as in the last example.

1.4 Stochastic Evolutions

In this section we consider Hilbert space valued solutions of stochastic evolution equations and the corresponding martingale problem. Once again H will denote a real, separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. $\mathcal{L}(H,H)$ will denote the class of all continuous linear operators on H and $\mathcal{L}_2(H,H)$ the class of all Hilbert-Schmidt operators. For an operator F, the Hilbert-Schmidt norm will be denoted by $\|F\|_{HS}$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ assumed to satisfy the usual conditions. Let $(W(t))_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-cylindrical Brownian motion on H. Recall that for a progressively measurable H-valued process f such that $\int_0^t \|f(s)\|^2 ds < \infty$ a.s. for all t, the stochastic integral $\int_0^t \langle f(s), dW(s)\rangle$ is defined as follows. Let $\{\phi_k : k \geq 1\}$ be a CONS in H. Let $W^k(t) := W(t)(\phi_k)$. Then $(W^k(t))_{t\geq 0}$ is a sequence of independent real valued Brownian motions and

$$\int_0^t \langle f(s), dW(s)\rangle := \sum_{k=1}^{\infty} \int_0^t (f(s),\phi_k) dW^k(s).$$

It can be proved that the series appearing above converges unifromly in $t \in [0,\infty)$ for all ω outside a null set.

The indefinite integral is a continuous local martingale with quadratic variation process $\int_0^t \|f(s)\|^2 ds$. For a progressively measurable $\mathcal{L}_2(H,H)$ valued process F, with $\int_0^t \|F_s\|_{HS}^2 ds < \infty$ a.s. for all t, the stochastic integral $\int_0^t F_s dW(s)$ is defined and satisfies for $\phi \in H$,

$$\left(\int_0^t F_s dW(s), \phi\right) = \int_0^t \langle F_s^* \phi, dW(s)\rangle. \tag{1.4.1}$$

Here F_s^* denotes the adjoint of the operator F_s. Indeed, (1.4.1) can be taken as the definition of the stochastic integral

$$\int_0^t F_s dW(s).$$

We need the following estimate, which is Burkholder's inequality in this context. It is stated in a weaker form (without sup over t), suitable for use later.

Lemma 1.4.1 *For $2 \leq p < \infty$, there exist constants C_p depending only on p such that for a progressively measurable $\mathcal{L}_2(H, H)$ valued process F with*

$$\mathbb{E}[\{\int_0^t \|F_s\|_{HS}^2 ds\}^{p/2}] < \infty$$

one has

$$\mathbb{E}[\| \int_0^t F_s dW(s)\|^p] \leq C_p \mathbb{E}[\{\int_0^t \|F_s\|_{HS}^2 ds\}^{p/2}].$$

Outline of proof: Let $\{\phi_k : k \geq 1\}$ be a CONS in H. Using Burkholder's inequality (see e.g. [15], p117), we get

$$\mathbb{E}[\{\sum_{k=1}^d (\int_0^t \langle F_s^* \phi_k, dW(s)\rangle)^2\}^{p/2}] \leq C_p \mathbb{E}[\{\sum_{k=1}^d \int_0^t \|F_s^* \phi_k\|^2 ds\}^{p/2}].$$

The required inequality follows from this, using Fatou's lemma. ∎

We are going to consider the following Stochastic evolution equation.

$$dX(t) = -LX(t)dt + \sigma(t, X(t))dW(t) + b(t, X(t))dt \qquad (1.4.2)$$

where $X(0)$ is independent of $(W(t))_{t \geq 0}$. Here the operator L is assumed to satisfy the following conditions.

$$T_t := e^{-tL} \quad \textit{is a contraction semigroup on} \quad H, \qquad (1.4.3)$$

$$L^{-1} \quad \textit{is a bounded self adjoint operator with discrete spectrum.} \qquad (1.4.4)$$

Let $\{\phi_k : k \geq 1\}$ be the eigenfunctions of L^{-1}, which constitutes a CONS in H and let $\{\lambda_k^{-1} : k \geq 1\}$ be the corresponding eigenvalues. We assume also that

$\sigma : [0,T] \times H \to H$ and $b : [0,T] \times H \to \mathcal{L}(H,H)$ are continuous functions satisfying

$$\begin{aligned} |(b(t,h),\phi_k)| &\leq c_k(1+\|h\|^2)^{\frac{1}{2}} & (1.4.5)\\ \|\sigma^*(t,h)\phi_k\| &\leq d_k(1+\|h\|^2)^{\frac{1}{2}} & (1.4.6)\\ |((b(t,h_1)-b(t,h_2)),\phi_k)| &\leq c_k\|h_1-h_2\| & (1.4.7)\\ \|(\sigma^*(t,h_1)-\sigma^*(t,h_2))\phi_k\| &\leq d_k\|h_1-h_2\| & (1.4.8) \end{aligned}$$

for all $k \geq 1, t \in [0,T], h, h_1, h_2 \in H$, where σ^* is the adjoint of the operator σ and $\{c_k\}, \{d_k\}$ satisfy

$$\sum_{k=1}^{\infty} c_k^2 \lambda_k^{-1} := C_{2.1} < \infty \tag{1.4.9}$$

and

$$\sum_{k=1}^{\infty} d_k^2 \lambda_k^{-\theta} := C_{2.2} < \infty \tag{1.4.10}$$

for some θ, $0 < \theta < 1$. Note that (1.4.10) also implies

$$\sum_{k=1}^{\infty} d_k^2 \lambda_k^{-1} := C_{2.3} < \infty. \tag{1.4.11}$$

Under these conditions the stochastic integral $\int_0^t \sigma(s, X(s))dW(s)$ may not be defined. However, for any progressively measurable process $(X(s))$,

$$\begin{aligned} \int_0^t \|T_{t-s}\sigma(s,X(s))\|_{HS}^2 ds &\leq \int_0^t \sum_{k=1}^{\infty} e^{-2(t-s)\lambda_k} d_k^2(1+\|X(s)\|^2)ds \\ &= \int_0^t f_\sigma(t-s)(1+\|X(s)\|^2)ds \end{aligned} \tag{1.4.12}$$

where

$$f_\sigma(u) = \sum_{k=1}^{\infty} e^{-2u\lambda_k} d_k^2. \tag{1.4.13}$$

Since $\int_0^t f_\sigma(u)du \leq \sum_{k=1}^{\infty} d_k^2 \lambda_k^{-1} = C_{2.3}$ it follows that the stochastic integral referred to above exists if

$$\int_0^T \|X(s)\|^2 ds < \infty \text{ a.s.} \tag{1.4.14}$$

Similarly

$$\left[\int_0^t \|T_{t-s}b(s,X(s))\|ds\right]^2 \leq T\int_0^t \sum_{k=1}^{\infty} e^{-2(t-s)\lambda_k} c_k^2 (1+\|X(s)\|^2)ds$$
$$= \int_0^t f_b(t-s)(1+\|X(s)\|^2)ds \qquad (1.4.15)$$

where

$$f_b(u) = T\sum_{k=1}^{\infty} e^{-2u\lambda_k} c_k^2 \qquad (1.4.16)$$

and again we have that $\int_0^t f_b(u)du \leq TC_{2.1}$. Thus, for every ω such that (1.4.14) holds, we also have that the integral

$$\int_0^t T_{t-s}b(s,X(s))ds$$

is well defined.

Definition: A progressively measurable process $(X(t))_{t\geq 0}$ is said to be a ***solution*** to (1.4.2) if (1.4.14) holds and for every t, and

$$X(t) = T_tX(0) + \int_0^t T_{t-s}\sigma(s,X(s))dW(s) + \int_0^t T_{t-s}b(s,X(s))ds \quad \text{a.s.} \quad (1.4.17)$$

In the literature, this solution is also known as a mild solution or an evolution solution. Note that progressive measurabilty of $(X(t))$ implies that $X(0)$ is independent of $(W(t))$. It is easy to see that if $(X(t))_{t\geq 0}$ is a solution and $(X'(t))_{t\geq 0}$ is a progressively measurable modification of $(X(t))_{t\geq 0}$, i.e. $P(X(t) = X'(t)) = 1$ for all t, then $(X'(t))_{t\geq 0}$ is also a solution to (1.4.2) .

Remark 1.4.1 : Let us define a new probability measure $\tilde{P}$ on $\mathcal{F}$ as follows.

$$\tilde{P}(C) = \int_C \alpha \exp\{-\|X(0)\|\}dP, \qquad (1.4.18)$$

where the constant α is chosen such that $\tilde{P}(\Omega) = 1$. First note that $\tilde{P} << P$ and $P << \tilde{P}$, i.e. P and $\tilde{P}$ have same null sets. Here, $\frac{d\tilde{P}}{dP}$ is $\mathcal{F}_0$ measurable and hence $(W(t))_{t\geq 0}$, considered on the probability space $(\Omega, \mathcal{F}, \tilde{P})$, is again a cylindrical Brownian motion. And for F such that

$$\int_0^T \|F_s\|_{HS}^2 ds < \infty \quad \text{a.s.} \quad (P \quad \text{or} \quad \tilde{P}),$$

if $M(t) = \int_0^t F_s dW(s)$ on $(\Omega, \mathcal{F}, P)$ and $\tilde{M}(t) = \int_0^t F_s dW(s)$ on $(\Omega, \mathcal{F}, \tilde{P})$, then

$$P(M(t) = \tilde{M}(t) \text{ for all } t) = 1$$

$$\tilde{P}(M(t) = \tilde{M}(t) \text{ for all } t) = 1.$$

Thus $(X(t))_{t\geq 0}$ is a solution to (1.4.2) on $(\Omega, \mathcal{F}, P)$ if and only if $(X'(t))_{t\geq 0}$ is a solution to (1.4.2) on $(\Omega, \mathcal{F}, \tilde{P})$. And now we have for all $p < \infty$,

$$\mathbb{E}_{\tilde{P}} \|X(0)\|^p < \infty.$$

Here is a variant of Gronwall's lemma which is crucially used in proving existence and uniqueness results for the solution.

Lemma 1.4.2 *Let f, g and δ be positive integrable functions on $[0, T]$. Suppose for all $t \leq T$,*

$$g(t) \leq c + \int_0^t f(s)\{g(t-s) + \delta(t-s)\}ds. \tag{1.4.19}$$

Then there exists a finite measure μ on $[0, T]$ depending only on f such that

$$g(t) \leq c + \int_0^t [c + \delta(t-s)]\mu(ds)$$

Proof: Iterating the inequality (1.4.19) we get

$$\begin{aligned}
g(t) &\leq c + \int_0^t f(s_1)\{g(t-s_1) + \delta(t-s_1)\}ds_1 \\
&\leq c + \int_0^t f(s_1)\delta(t-s_1)ds_1 \\
&\quad + \int_0^t f(s_1)[c + \int_0^{t-s_1} f(s_2)\{g(t-s_1-s_2) + \delta(t-s_1-s_2)\}ds_2]ds_1. \\
&\vdots \\
&\leq c + \sum_{j=1}^k \int_0^t [c + \delta(t-u)]\mu_j(du) + \int_0^t g(t-u)\mu_k(du)
\end{aligned}$$

where

$$\mu_j([0,u]) = \int_0^T \cdots \int_0^T f(s_1)f(s_2)\dots f(s_j)1_{(\sum_{i=1}^j s_i \leq u)}ds_1 \dots ds_j.$$

We will first prove that $\sum_{j=1}^{\infty}\mu_j([0,T]) < \infty$. Choose $\alpha > 0$ such that $\int_0^T e^{-\alpha s} f(s)ds \le \frac{1}{2}$. This can always be done as $\int_0^T f(s)ds < \infty$.

$$\begin{aligned}\mu_j([0,T]) &\le e^{\alpha T}\int_0^T \dots \int_0^T f(s_1)f(s_2)\dots f(s_j)e^{-\alpha\sum_{k=1}^{j} s_k}ds_1\dots ds_j \\ &\le e^{\alpha T}(\frac{1}{2})^j.\end{aligned}$$

Hence $\sum_j \mu_j([0,T]) < \infty$. As a consequence $\mu(C) := \sum_{j=1}^{\infty}\mu_j(C)$, for C Borel in $[0,T]$, defines a finite measure. Since $\int_0^t g(t-s)\mu(ds) < \infty$, it follows that $\int_0^t g(t-s)\mu_k(ds)$ goes to zero as $k \to \infty$ and hence

$$g(t) \le c + \int_0^t [c + \delta(t-s)]\mu(ds).$$

■

We will now obtain an estimate on the second moment of a solution.

Theorem 1.4.3 *If $(X(t))_{t\ge0}$ is a solution to (1.4.2) satisfying $\mathbb{E}\|X(0)\|^2 < \infty$, then*

$$\sup_{t\le T}\mathbb{E}\|X(t)\|^2 \le C_{2.4}[1 + \mathbb{E}\|X(0)\|^2] \tag{1.4.20}$$

where $C_{2.4}$ is a constant depending only on the constants $C_{2.1}$ - $C_{2.3}$.

Proof. Let $(X(t))_{t\ge0}$ be a solution to (1.4.2) satisfying (1.4.14) . Then it follows that

$$\begin{aligned}(X(t),\phi_k) = e^{-t\lambda_k}(X(0),\phi_k) &+ \int_0^t \langle e^{-\lambda_k(t-s)}\sigma^*(s,X(s))\phi_k, dW(s)\rangle \\ &+ \int_0^t e^{-\lambda_k(t-s)}(b(s,X(s)),\phi_k)ds \end{aligned}\tag{1.4.21}$$

and hence that

$$d(X_t,\phi_k) = \langle\sigma^*(t,X(t))\phi_k, dW(t)\rangle + (b(t,X(t)) - \lambda_k X(t),\phi_k)dt. \tag{1.4.22}$$

Fix n and define a stop time τ_n by

$$\tau_n = \inf\{t \ge 0 : \int_0^t \|X(s)\|^2 ds \ge n\} \wedge T. \tag{1.4.23}$$

and let

$$\xi^k(t) := e^{\lambda_k(t\wedge\tau_n)}(X(t\wedge\tau_n),\phi_k).$$

Note that $\tau_n \to T$ since $(X(t))_{t\geq 0}$ is assumed to satisfy (1.4.14) . It is easy to see that

$$\begin{aligned} d\xi^k(t) = I_{(t<\tau_n)}\exp\{\lambda_k(t\wedge\tau_n)\}\langle\sigma^*(t,X(t))\phi_k, dW(t)\rangle \\ +I_{(t<\tau_n)}\exp\{\lambda_k(t\wedge\tau_n)\}(b(t,X(t)),\phi_k)dt \end{aligned}$$

or

$$\begin{aligned} \xi^k(t) = \xi^k(0) + \int_0^{t\wedge\tau_n} \exp\{\lambda_k s\}\langle\sigma^*(s,X(s))\phi_k, dW(s)\rangle \\ +\int_0^{t\wedge\tau_n} \exp\{\lambda_k s\}(b(s,X(s)),\phi_k)ds \end{aligned}$$

and hence that

$$\begin{aligned} \mathbb{E}|\xi^k(t)|^2 &\leq 3\mathbb{E}\Big[|\xi^k(0)|^2 + \int_0^{t\wedge\tau_n} e^{2\lambda_k s}\|\sigma^*(s,X(s))\phi_k\|^2 ds \\ &\quad +t\int_0^{t\wedge\tau_n} e^{2\lambda_k s}|(b(s,X(s)),\phi_k)|^2 ds\Big] \\ &\leq 3\mathbb{E}\Big[|\xi_0^k|^2 + \int_0^t e^{2\lambda_k s}(d_k^2 + Tc_k^2)(1+\|X(s)\|^2)1_{(s<\tau_n)}ds\Big] \end{aligned}$$

Using that $\mathbb{E}[\|X(t)\|^2 1_{(t<\tau_n)}] \leq \sum_k e^{-2\lambda_k t}\mathbb{E}|\xi^k(t)|^2$ we get

$$\begin{aligned} \mathbb{E}[\|X(t)\|^2 1_{(t<\tau_n)}] &\leq 3\Big[\mathbb{E}\|X(0)\|^2 + \int_0^t \sum_k 2e^{-2\lambda_k(t-s)}(d_k^2 + Tc_k^2) \\ &\qquad \mathbb{E}[1+\|X(s)\|^2 1_{(s<\tau_n)}]ds\Big] \\ &\leq 6\Big[\mathbb{E}\|X(0)\|^2 + \int_0^t f_0(t-s)\mathbb{E}[1+\|X(s)\|^2 1_{(s<\tau_n)}]ds\Big] \end{aligned}$$

where $f_0(u) = f_\sigma(u) + Tf_b(u)$ is an integrable function (see (1.4.13) ,(1.4.16)). Since

$$\int_0^T \mathbb{E}[\|X(s)\|^2 1_{(s<\tau_n)}]ds < \infty$$

by choice of τ_n, we can use Lemma 1.4.2 to conclude that

$$\mathbb{E}[\|X(t)\|^2 1_{(t<\tau_n)}] \leq C[1+\mathbb{E}\|X(0)\|^2]$$

where the constant C does not depend on n. Now the result follows from Fatou's lemma by letting $n \longrightarrow \infty$. ∎

The next step is to prove the existence and uniqueness of solution to (1.4.2) .

Theorem 1.4.4 *Suppose L, A, B satisfy (1.4.3) - (1.4.10) . Let $X(0)$ be an $\mathcal{F}_0$-measurable H- valued random variable and let $(W(t))_{t\geq 0}$ be an $(\mathcal{F}(t))_{t\geq 0}$-cylindrical Brownian motion. Then*
(i) there exists a solution $(\hat{X}(t))_{t\geq 0}$ of (1.4.2) satisfying (1.4.14) with $\hat{X}(0) = X(0)$.
(ii) Let $(X(t))_{t\geq 0}$ and $(U(t)_{t\geq 0}$ be solutions to (1.4.2) satisfying (1.4.14) , such that $X(0) = U(0)$. Then

$$P(X(t) = U(t)) = 1 \text{ for all } t. \tag{1.4.24}$$

Proof: (i) Let $\tilde{P}$ be defined by (1.4.18) . As noted in Remark 1.4.1, sufficies to construct a solution on $(\Omega, \mathcal{F}, \tilde{P})$. For $n \geq 1$ and $0 \leq i \leq n$, let $t_i^n = \frac{i}{n}T$. Define $(X^n(t), t_i^n < t \leq t_{i+1}^n)$ $i \geq 0$ inductively as follows. For $t_k^n < t \leq t_{k+1}^n$, let

$$\begin{aligned} X^n(t) = X_{t_k^n}^n + T_t X(0) - T_{t_k^n} X(0) \quad &+ \quad \int_{t_k^n}^t T_{t-u}\sigma(u, X^n(t_k^n))dW(u) \\ &+ \quad \int_{t_k^n}^t T_{t-u} b(u, X^n(t_k^n))du. \end{aligned} \tag{1.4.25}$$

Let $Y^n(t) = X_{t_k^n}^n$ for $t_k^n < t \leq t_{k+1}^n$. Then

$$X^n(t) = T_t X(0) + \int_0^t T_{t-u}\sigma(u, Y^n(u))dW(u) + \int_0^t T_{t-u} b(u, Y^n(u))du. \tag{1.4.26}$$

Proceeding as in (1.4.12) and (1.4.15) , it follows that

$$\tilde{E}\|X^n(t)\|^2 \leq 3[\tilde{E}\|X(0)\|^2 + \int_0^t f_0(t-s)(1 + \|Y^n(u)\|^2)du]$$

where $f_0 = f_\sigma + T f_b$. Using this, by induction on k, we can deduce that

$$\sup_{t_k^n < t \leq t_{k+1}^n} \tilde{E}\|X_t^n\|^2 < \infty.$$

Now writing $g_n(t) = \sup_{0\leq s\leq t} \tilde{E}\|X^n(t)\|^2$, it follows that g_n is an integrable (indeed bounded) function and that

$$g_n(t) \leq 3\big[\tilde{E}\|X(0)\|^2 + \int_0^t f_0(t-s)(1 + g_n(s))ds\big]$$

and hence Lemma 1.4.2 yields

$$\sup_{n\geq 1}\sup_{0\leq t\leq T}\tilde{E}\|X^n(t)\|^2 \leq C'[1+\tilde{E}\|X(0)\|^2] = C'' \tag{1.4.27}$$

Here, $\tilde{E}$ denotes integral w.r.t. $\tilde{P}$. Using (1.4.26) for n, m and using the Lipschitz conditions on σ and b, we get (calculations are similar to those in (1.4.12) ,(1.4.15))

$$\begin{aligned}\tilde{E}\|X^n(t)-X^m(t)\|^2 &\leq 2\tilde{E}\{\int_0^t \|T_{t-u}(\sigma(u,Y^n(u))-\sigma(u,Y^m(u)))\|_{HS}^2 du \\ &\quad + T\int_0^t \|T_{t-u}(b(u,Y^n(u))-b(u,Y^m(u))\|^2 du\} \\ &\leq 2\int_0^t f_0(t-u)\{\tilde{E}\|Y^n(u)-Y^m(u)\|^2\}du.\end{aligned}$$

Let $g_{n,m}(t) = \tilde{E}\|X^n(t)-X^m(t)\|^2$ and $\delta_{n,m}(t) = [\tilde{E}\|X^n(t)-Y^n(t)\|^2 + \tilde{E}\|X^m(t)-Y^m(t)\|^2]$. Then $g_{n,m}$ and $\delta_{n,m}$ are uniformly bounded (by (1.4.27)) and

$$g_{n,m}(t) \leq 6\int_0^t f_0(t-u)\{g_{n,m}(u)+\delta_{n,m}(u)\}du.$$

Using Lemma 1.4.2, we get that for a finite measure μ on $[0,T]$, we have

$$g_{n,m}(t) \leq \int_0^t \delta_{n,m}(t-u)\mu(du). \tag{1.4.28}$$

Now, for $t_i^n < s \leq t_{i+1}^n$

$$\begin{aligned}\tilde{E}\|X^n(s)-Y^n(s)\|^2 &= \tilde{E}\|X^n(s)-X_{t_i^n}^n\|^2 \\ &\leq 3[\tilde{E}\|T_{t_i^n}(T_{s-t_i^n}-I)X(0)\|^2 + \int_{t_i^n}^s \tilde{E}\|T_{s-u}\sigma(u,Y^n(u))\|_{HS}^2 du \\ &\quad +\tilde{E}(\int_{t_i^n}^s \|T_{s-u}b(u,Y^n(u))\|du)^2] \\ &\leq 3[\tilde{E}\|(T_{s-t_i^n}-I)X(0)\|^2 \\ &\quad +\int_{t_i^n}^s f_0(s-u)du[1+\tilde{E}\|X_{t_i^n}^n\|^2] \\ &\leq 3\sup_{u\leq[T/n]}[\tilde{E}\|(T_u-I)X(0)\|^2 + (1+C'')\int_0^{T/n} f_0(s)ds.\end{aligned}$$

Thus $\sup_{s\leq T}\tilde{E}\|X^n(s)-Y^n(s)\|^2 \to 0$ as $n\to\infty$. So $\sup_{t\leq T}\delta_{n,m}(t)\to 0$ as $n,m\to\infty$ and hence

$$\sup_{s\leq T}\tilde{E}\|X^n(s)-X^m(s)\|^2\to 0,\quad \sup_{s\leq T}\tilde{E}\|Y^n(s)-Y^m(s)\|^2\to 0 \qquad (1.4.29)$$

as $n,m\to\infty$. Note that since Y^n is a piecewise constant, adapted process it is progressively measurable. In view of (1.4.29) we can choose a subsequence $\{n_k\}$ such that $Z^k(s):=Y^{n_k}(s)$ satisfies

$$\sup_{s\leq T}\tilde{E}\|Z^k(s)-Z^{k+1}(s)\|^2\leq 2^{-k}.$$

Then it follows that $\sum_k\|Z^k(s)-Z^{k+1}(s)\|<\infty$ a.s. for all s. Thus, $Z^k(s)$ converges a.s. for each s. Define

$$\begin{aligned}\hat{X}(s,\omega) &= \lim Z^k(s,\omega) \quad \text{if it exists in } H\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Then $\hat{X}(s)$ is a progressively measurable process. Further, it follows that

$$\sup_{s\leq T}\tilde{E}\|Y^n(s)-\hat{X}(s)\|^2\to 0,\quad \sup_{s\leq T}\tilde{E}\|X^n(s)-\hat{X}(s)\|^2\to 0.$$

From this, it can be verified that $\hat{X}$ is a solution to (1.4.2) (on $(\Omega,\mathcal{F},\tilde{P})$) with $\hat{X}_0=X_0$ and that (1.4.14) holds. This completes the proof of (i).

For (ii), again, let $\tilde{P}$ be given by (1.4.18) . Then $(X(t))_{t\geq 0}$ and $(U(t))_{t\geq 0}$ are solutions to (1.4.2) on $(\Omega,\mathcal{F},\tilde{P})$ and in view of Theorem 1.4.3, $\int_0^T\tilde{E}\|X(t)-U(t)\|^2dt<\infty$. Using the Lipschitz conditions on σ and b, we can deduce

$$\tilde{E}\|X(t)-U(t)\|^2\leq 2\Big[\int_0^t f_0(t-s)\tilde{E}\|X(s)-U(s)\|^2ds\Big].$$

An application of Lemma 1.4.2, with $c=0$, $\delta\equiv 0$, yields

$$\tilde{E}\|X(t)-U(t)\|^2=0$$

for all t. Thus $\tilde{P}(X(t)=U(t))=1$ and hence (1.4.24) follows. ∎

We are now in a position to obtain an estimate on the growth of the p^{th} moment of the solution.

Theorem 1.4.5 *Let $(X(t))_{t\geq 0}$ be a solution to (1.4.2) satisfying (1.4.14). Then for $p \geq 2$, there exists a constant C'_p depending only on the constant C_p in Lemma 1.4.1 and on $C_{2.1}, C_{2.3}$ such that if $\mathbb{E}\|X(0)\|^p < \infty$, then*

$$\sup_{t\leq T} \mathbb{E}\|X(t)\|^p \leq C'_p[1 + \mathbb{E}\|X(0)\|^p]. \tag{1.4.30}$$

Proof. Let $X^n(t)$ be the approximation constructed in the proof of the previous theorem. Using Lemma 1.4.1, it follows from (1.4.26) that

$$\begin{aligned} \mathbb{E}\|X^n(t)\|^p \;\leq\; & C_1\Big[\mathbb{E}\|X(0)\|^p + C_p\mathbb{E}\Big(\int_0^t f_\sigma(t-s)(1+\|Y^n(s)\|)^2 ds\Big)^{\frac{p}{2}} \\ & + \mathbb{E}\Big(\int_0^t f_b(t-s)(1+\|Y^n(s)\|^2)ds\Big)^{\frac{p}{2}}\Big] \end{aligned} \tag{1.4.31}$$

Using Holder's inequality for the ds integrals, we get

$$\begin{aligned} \mathbb{E}\|X^n(t)\|^p \;\leq\; & C_1\Big[\mathbb{E}\|X(0)\|^p + C_p\Big(\int_0^t f_\sigma(t-s)ds\Big)^{\frac{p}{2}-1} \\ & \mathbb{E}\Big(\int_0^t f_\sigma(t-s)(1+\|Y^n(s)\|)^p ds\Big) \\ & + \Big(\int_0^t f_b(t-s)ds\Big)^{\frac{p}{2}-1}\mathbb{E}\Big(\int_0^t f_b(t-s)(1+\|Y^n(s)\|)^p ds\Big)\Big] \end{aligned} \tag{1.4.32}$$

From (1.4.32) it follows by induction on k that $\sup_{t_k^n \leq t \leq t_{k+1}^n} \mathbb{E}\|X^n(t)\|^p < \infty$. Writing $h_n(s) = \sup_{u\leq s} \mathbb{E}\|X^n(u)\|^p$, it follows that h_n is an integrable function. We can rewrite (1.4.32) as

$$h_n(t) \leq C'\Big[\mathbb{E}\|X(0)\|^p + \int_0^t f_0(t-s)(1+h_n(s))ds\Big].$$

From Lemma 1.4.2 it now follows that

$$h_n(t) \leq C''[1 + \mathbb{E}\|X(0)\|^p].$$

The constants C', C'' depend only on p and on $C_{2.1}, C_{2.3}$. As noted in the previous result, a subsequence of $X^n(s)$ converges to $\hat{X}(s)$, where $\hat{X}$ is a solution to (1.4.2). Hence using Fatou's lemma, it follows that the required moment estimate holds for $\hat{X}$. The result follows from this as $\hat{X}, X$ have the same finite dimensional distributions by the uniqueness part of the previous theorem. ∎

We now look at regularity of paths of the solution to (1.4.2).

Theorem 1.4.6 *Let $(X(t))_{t\geq 0}$ be a solution to (1.4.2). Then $(X(t))_{t\geq 0}$ admits a continuous modification, which is of course a solution to (1.4.2).*

Proof: Let $\tilde{P}$ be given by (1.4.18). In view of Remark 1.4.1, it suffices to prove that X has a continuous modification on $(\Omega, \mathcal{F}, \tilde{P})$. Let us write

$$X(t) = T_t X(0) + Y(t) + Z(t)$$

where $Y(t) = \int_0^t T_{t-u}\sigma(u, X(u))dW(u)$ and $Z(t) = \int_0^t T_{t-u}b(u, X(u))du$. Clearly, $T_tX(0,\omega)$ is continuous for all ω. For $0 \leq s \leq t \leq T$,

$$\begin{aligned}
\|Z(t) - Z(s)\|^2 &= \| \int_0^s (T_{t-u} - T_{s-u})b(u, X(u))du + \int_s^t T_{t-u}b(u, X(u))du\|^2 \\
&\leq 2\Big[\int_0^s \|(T_{t-u} - T_{s-u})b(u, X(u))\|du\Big]^2 \\
&\quad +2\Big[\int_s^t \|T_{t-u}b(u, X(u))\|du\Big]^2 \\
&\leq 2\Big[(\int_0^s \Big\{\sum_k (e^{-\lambda_k(t-u)} - e^{-\lambda_k(s-u)})^2 c_k^2(1 + \|X(u)\|^2)\Big\}^{\frac{1}{2}} du\Big]^2 \\
&\quad +2\Big[(\int_s^t \Big\{\sum_k e^{-2\lambda_k(t-u)} c_k^2(1 + \|X(u)\|^2)\Big\}^{\frac{1}{2}} du\Big]^2 \\
&\leq [\int_0^T (1 + \|X(u)\|^2)du]\alpha(s,t) \qquad (1.4.33)
\end{aligned}$$

where

$$\alpha(s,t) = \int_0^s \Big[\sum_k (e^{-\lambda_k(t-u)} - e^{-\lambda_k(s-u)})^2 c_k^2\Big] du + \int_s^t \Big[\sum_k e^{-2\lambda_k(t-u)} c_k^2\Big] ds.$$

The last step above follows from Holder's inequality. Note α can be computed and we can verify that $\alpha(s,t) \leq \beta(t-s)$ with

$$\beta(\delta) := \sum_{k=1}^{\infty} \frac{c_k^2}{2\lambda_k}[(1 - e^{-\delta\lambda_k})^2 + (1 - e^{-2\delta\lambda_k})]. \qquad (1.4.34)$$

Clearly (1.4.9) implies $\beta(\delta) \to 0$ as $\delta \to 0$. Using (1.4.14), it follows that

$$\lim_{\delta\to 0} \sup_{0 \leq t-s \leq \delta} \|Z(t) - Z(s)\|^2 = 0 \quad \text{a.s.}$$

Thus $(Z(t))_{t\geq 0}$ is continuous a.s.

It remains to show that $(Y(t))_{t\geq 0}$ admits a continuous modification. We shall achieve this via the Kolmogorov criterion. Choose p such that $(1-\theta)p > 2$, where θ is as in (1.4.10) . Recall that by the choice of $\tilde{P}$, $\tilde{E}\|X(0)\|^p < \infty$ and hence by Lemma 1.4.5, $\sup_{t\leq T} \tilde{E}\|X(t)\|^p < \infty$. As before, $\tilde{E}$ stands for integral with respect to $\tilde{P}$. For $s \leq t \leq T$, writing

$$Y(t) - Y(s) = \int_0^s (T_{t-u} - T_{s-u})\sigma(u, X(u))dW(u) + \int_s^t T_{t-u}\sigma(u, X(u))dW(u)$$

and using Lemma 1.4.1, we get

$$\begin{aligned}
\tilde{E}\|Y(t) - Y(s)\|^p &\leq 2^{p-1}C_p\tilde{E}\Big[\{\int_0^s \|(T_{t-u} - T_{s-u})\sigma^*(u, X(u))\|_{HS}^2 du\}^{p/2} \\
&\qquad + \{\int_s^t \|T_{t-u}\sigma^*(u, X(u))\|_{HS}^2 du\}^{p/2}\Big] \\
&\leq 2^{p-1}C_p\tilde{E}\Big[\{\int_0^s \sum_k (e^{-\lambda_k(t-u)} - e^{-\lambda_k(s-u)})^2 d_k^2(1 + \|X(u)\|)^2 du\}^{p/2} \\
&\qquad + \{\int_s^t \sum_k e^{-2\lambda_k(t-u)} d_k^2(1 + \|X(u)\|)^2 du\}^{p/2}\Big] \qquad (1.4.35)
\end{aligned}$$

Let us write $\psi_1(u) = \sum_k (e^{-\lambda_k(t-u)} - e^{-\lambda_k(s-u)})^2 d_k^2$ and $\psi_2(u) = \sum_k e^{-2\lambda_k(t-u)} d_k^2$. Now

$$\begin{aligned}
\tilde{E}[\int_0^s \psi_1(u)(1 + \|X(u)\|)^2 du]^{\frac{p}{2}} &\leq \tilde{E}\Big[\Big(\int_0^s \psi_1(u)du\Big)^{\frac{p}{2}-1} \\
&\qquad \Big(\int_0^s \psi_1(u)(1 + \|X(u)\|\Big)^p du\Big] \\
&\leq C_p'\tilde{E}(1 + \|X(0)\|)^p(\int_0^s \psi_1(u)du)^{\frac{p}{2}}
\end{aligned}$$

by Holder's inequality and (1.4.30) . Similarly estimating the second term in (1.4.35) , we get

$$\tilde{E}\|Y(t) - Y(s)\|^p \leq C_p'\tilde{E}(1 + \|X(0)\|))^p[(\int_0^s \psi_1(u)du)^{\frac{p}{2}} + (\int_0^s \psi_2(u)du)^{\frac{p}{2}}]. \qquad (1.4.36)$$

Evaluating the integrals, one obtains

$$\begin{aligned}
\tilde{E}\|Y(t) - Y(s)\|^p &\leq C_p'\tilde{E}(1 + \|X(0)\|)^p[\{\sum_k \frac{d_k^2}{2\lambda_k}(1 - e^{-\lambda_k(t-s)})^2\}^{p/2} \\
&\qquad + \{\sum_k \frac{d_k^2}{2\lambda_k}(1 - e^{-2\lambda_k(t-s)})\}^{p/2}]
\end{aligned}$$

Now using the obvious inequality $1 - e^{-x} \le x \wedge 1 \le x^{\delta}$ for $x > 0, 0 < \delta \le 1$, for $\delta = \frac{1-\theta}{2}$ and $\delta = 1 - \theta$ respectively, we get

$$\begin{aligned}\tilde{E}\|Y(t) - Y(s)\|^p &\le C'_p[\{\sum_k \frac{d_k^2}{2\lambda_k}(\lambda_k(t-s)))^{1-\theta}\}^{p/2} \\ &\quad + \{\sum_k \frac{d_k^2}{2\lambda_k}(2\lambda_k(t-s))^{1-\theta}\}^{p/2}] \\ &\le C'_p(\frac{1}{2} + \frac{1}{2^{\theta}})(\sum_k \frac{d_k^2}{\lambda_k^{\theta}})^{p/2}.(t-s)^{(1-\theta)p/2}.\end{aligned}$$

Recalling the assumption (1.4.10) and noting that by our choice, p satisfies $\frac{p}{2}(1-\theta) > 1$, we conclude that

$$\tilde{E}\|Y(t) - Y(s)\|^p \le C_{2.5}|t-s|^{1+\delta} \tag{1.4.37}$$

with $\delta = \frac{p}{2}(1-\theta) - 1$, where $C_{2.5}$ depends only on $p, C_{2.2}$. Thus $(Y(t))_{t\ge 0}$ has a continuous modification. ∎

Now the existence and uniqueness result, Theorem 1.4.4 can be recast as follows.

Theorem 1.4.7 *There exists a continuous solution X to the Stochastic evolution equation (1.4.2) . Further if X' is any other solution to (1.4.2) with continuous paths, then*

$$P(X(t) = X'(t) \text{ for all } t, 0 \le t \le T) = 1.$$

Our next step is to prove uniqueness in law of solutions to (1.4.2) .

Theorem 1.4.8 *Let $(X(t))_{t\ge 0}$ be a solution to (1.4.2) [on $(\Omega, \mathcal{F}, P)$] and let $(X'(t)_{t\ge 0}$ be a solution to (1.4.2) on $(\Omega', \mathcal{F}', P')$ w.r.t. some P' - cylindrical Brownian motion on H. Suppose that X, X' have continuous paths and suppose $P \circ (X(0))^{-1} = P' \circ (X'(0))^{-1}$. Then*

$$P \circ (X)^{-1} = P' \circ (X')^{-1}. \tag{1.4.38}$$

Proof: Let $(X^n(t))_{t\ge 0}$ be the approximation constructed in the previous theorem and let $(V^n(t))_{t\ge 0}$ be the approximation defined analogously on $(\Omega', \mathcal{F}', P')$

(with $X'(0)$ and $(W'(t))_{t\geq 0}$ in place of $X(0)$ and $(W(t))_{t\geq 0}$ in (1.4.25)). It is easy to see that the finite dimensional distributions of $(X^n(t))_{t\geq 0}$ and $(V^n(t))_{t\geq 0}$ are the same. Now $\tilde{E}\|X^n(t) - X(t)\|^2 \to 0$ implies that $P(\|X^n(t) - X(t)\| > \delta) \to 0$ for all $\delta > 0$. Similarly, $P'(\|V^n(t) - X'(t)\| > \delta) \to 0$. Thus the finite dimensional distributions of $(X(t))_{t\geq 0}$ and $(X'(t))_{t\geq 0}$ are the same. Since X, X' have continuous paths, this yields (1.4.38) . ∎

We will now consider the martingale problem corresponding to (1.4.2) . We impose the condition that the coefficients σ and b are bounded. *i.e.*

$$|(b(t,h),\phi_k)| \leq c_k \tag{1.4.39}$$

$$\|\sigma^*(t,h)\phi_k\| \leq d_k. \tag{1.4.40}$$

Condition (1.4.7) – (1.4.10) continue to hold.

For $f \in C_0^2(\mathbb{R}^n), n \geq 1$, let $U_n f : H \to \mathbb{R}$ be defined by

$$(U_n f)(h) = f((h,\phi_1),\ldots,(h,\phi_n)) \tag{1.4.41}$$

For $f \in C_0^2(\mathbb{R}^n)$, we will write $f_i = (\partial/\partial x_i)f$ and $f_{ij} = (\partial/\partial x_j)f_i$. Let

$$\mathcal{D} = \{U_n f : f \in C_0^2(\mathbb{R}^n), n \geq 1\} \tag{1.4.42}$$

Define A_t on $\mathcal{D}$ by

$$\begin{aligned} A_t(U_n f)(h) &= \frac{1}{2}\sum_{i,j=1}^n (\sigma^*(t,h)\phi_i, \sigma^*(t,h)\phi_j)(U_n f_{ij})(h) \\ &\quad + \sum_{i=1}^n (b(t,h) - \lambda_i h, \phi_i)(U_n f_i)(h) \end{aligned} \tag{1.4.43}$$

If $(X(t))_{t\geq 0}$ is a solution to (1.4.2) , then we have seen that (1.4.22) holds and hence it follows that for all $g \in \mathcal{D}$,

$$g(X(t)) - g(X(0)) - \int_0^t (A_s g)(X(s))ds \tag{1.4.44}$$

is also a martingale. In other words, if $(X(t))_{t\geq 0}$ is a solution to (1.4.2) then $(X(t))_{t\geq 0}$ is a solution to the martingale problem for $(A_t)_{t\geq 0}$. That the converse is also true is proved next.

Theorem 1.4.9 *Let $(X(t))_{t\geq 0}$ be a progressively measurable process satisfying (1.4.14) such that (1.4.44) is a martingale for all $g \in \mathcal{D}$. Then on the probability space $(\Omega', \mathcal{F}', P') = (\Omega, \mathcal{F}, P) \otimes ([0,1], \bar{\mathcal{B}}, \nu)$, there exists a cylindrical Brownian motion $(W(t))_{t\geq 0}$ on H with respect to a family $(\mathcal{G}_t)_{t\geq 0}$ such that*

(a) $(X(t))_{t\geq 0}$ is $(\mathcal{G}_t)_{t\geq 0}$-progressively measurable

and

(b) $(X(t))_{t\geq 0}$ is a solution to (1.4.2) . (Here ν is the Lebesgue measure on $[0,1]$ and $\bar{\mathcal{B}}$ is the ν-completion of the Borel σ field).

Proof: Using (1.4.44) for $g = U_n f, f \in C_0^2(\mathbb{R})$, we can first conclude as in example 1 of the previous section that $((X(t), \phi_i), 1 \leq i \leq n)$ has an r.c.l.l. modification and then further that it has a continuous modification. Let us denote the continuous version of $(X(t), \phi_i)$ by Y^i. Then we also deduce that

$$M^i(t) = Y^i(t) - Y^i(0) - \int_0^t \lambda_i Y^i(s)ds - \int_0^t (b(s, X(s)), \phi_i)ds$$

is a continuous local martingale and that

$$\langle M^i, M^j\rangle(t) = \int_0^t (\sigma^*(s, X(s))\phi_i, \sigma^*(s, X(s))\phi_j)ds.$$

As a consequence, using (1.4.40) we have

$$\mathbb{E}\sup_{s\leq t} |M^k(s)|^2 \leq 4\mathbb{E}\langle M^k, M^k\rangle(t) \leq d_k^2 t. \tag{1.4.45}$$

Let $N^k(t) := \lambda_k^{-1/2} M^k(t)$ Then using (1.4.11) and (1.4.45) we get

$$\mathbb{E}\sup_{s\leq t} \| \sum_{k=m}^{k=r} N^k(s)\phi_k\|^2 \to 0 \text{ as } m, r \to \infty.$$

Hence $N(t) := \sum_{k=1}^{\infty} N^k(t)\phi_k$ is an H-valued continuous local martingale. Here

$$\begin{aligned} \langle N^k, N^j\rangle(t) &= \int_0^t \lambda_k^{-1/2}\lambda_j^{-1/2}(\sigma^*(s, X(s))\phi_k, \sigma^*(s, X(s))\phi_j)ds \\ &= \int_0^t (G_s^*\phi_k, G_s^*\phi_j)ds \end{aligned}$$

where $G_s(\omega) = L^{-1/2}\sigma(s, X(s,\omega))$. Note that

$$\int_0^T \|G_s(\omega)\|_{HS}^2 ds < \infty$$

in view of the assumption (1.4.10) . Let $\{\beta^j(t), 1 \le j < \infty\}$ be a sequence of independent $(\mathcal{F}_t^1)_{t\ge0}$ -Brownian motions on $([0,1], \bar{\mathcal{B}}, \nu)$, where $(\mathcal{F}_t^1)_{t\ge0}$ satisfy the usual conditions. Let $\mathcal{G}_t^1 = \mathcal{F}_t \otimes \mathcal{F}_t^1, t \ge 0$. $\mathcal{G}_t^1$ is a σ-field on $\Omega' = \Omega \times [0,1]$. Let $(\mathcal{G}_t)_{t\ge0}$ be the smallest family of σ fields on Ω' satisfying the usual conditions such that $\mathcal{G}_t^1 \subseteq \mathcal{G}_t$. Using arguments as in the proof of Theorem IV 3.5. in [16], one can show that there exists a cylindrical Brownian motion $(W(t))_{t\ge0}$ on H w.r.t $(\mathcal{G}_t)_{t\ge0}$ such that

$$N(t) = \int_0^t G_s dW(s).$$

Then $N^k(t) = (N(t), \phi_k) = \int_0^t \langle \lambda_k^{-1/2} \sigma^*(s, X(s))\phi_k, dW(s)\rangle$ and hence

$$M^k(t) = \int_0^t \langle \sigma^*(s, X(s))\phi_k, dW(s)\rangle.$$

From here, it follows that $(X(t))_{t\ge0}$ satisfies (1.4.22) and hence that $(X(t))_{t\ge0}$ is a solution to (1.4.2) . ∎

In the light of Theorem 1.4.7, some of the results concerning the equation (1.4.2) proved earlier can be recast for the martingale problem for $(A_t)_{t\ge0}$ as follows.

Theorem 1.4.10 *(a). Let $(X(t))_{t\ge0}$ be a progressively measurable process satisfying (1.4.14) and suppose $(X(t))_{t\ge0}$ is a solution to the martingale problem for $(A_t)_{t\ge0}$. Then $(X(t))_{t\ge0}$ admits a continuous modification.*

(b). For all $\mu \in \mathcal{P}(H)$, there exists a continuous process $(X(t))_{t\ge0}$ such that (1.4.44) is a martingale for every $g \in \mathcal{D}$ and such that the law of $X(0)$ is μ. Further the law of the process X is uniquely determined.

(c). For $0 \le s \le T, x \in H$, there is a unique measure $P_{s,x}$ on $C([0,T], H)$ such that (writing the co-ordinate process on $C([0,T], H)$ as $\eta(t)$),

(i) $P_{s,x}(\eta(u) = x, 0 \le u \le s) = 1$.

(ii) $g(\eta(t)) - \int_s^t (A_u g)(\eta(u))du, \quad t \ge s$ is a $P_{s,x}$ martingale.

(d) Further, $(\eta(t))_{t\ge0}$ is a time inhomogeneous Markov process on the probability space $(\Omega', \mathcal{F}', P_{s,x})$ (where Ω' is $C([0,T], H)$ and $\mathcal{F}'$ is the Borel σ-field

on Ω') for each $(s,x) \in [0,T] \times H$. The (common) transition probability function $P(r,y,t,B)$ is given by

$$P(r,y,t,B) = P_{r,y}(\eta(t) \in B)$$

for $r \leq t \leq T, y \in H, B$ Borel in H.

Proof: (a),(b) follow from Theorems 1.4.4, 1.4.6 and 1.4.7. (c) is the same as (b) - with a change of origin from 0 to s in the time variable. For (d), let us note that if for each $n, \mathcal{C}_n$ is a countable dense subset of $C_0^2(\mathbb{R}^n)$ (in the norm, $\|f\|_0 = (\|f\| + \sum_i \|f_i\| + \sum_{ij} \|f_{ij}\|)$, $\|\cdot\|$, being sup norm) then

$$\mathcal{D}_0 = \{U_n f : f \in \mathcal{C}_n, n \geq 1\}$$

is a countable set and for every $g = U_n f \in \mathcal{D}$ we can get $g_k \in \mathcal{D}_0$ such that $g_k \to g$ and $A_t g_k \to A_t g$. Just take $g_k = U_n f_k$ where $f_k \in \mathcal{C}_n$ approximate f in $\|.\|_0$ norm. Hence the Markov property of $(\eta(t))_{t\geq 0}$ under $\{P_{s,x}\}$ and the expression for the transition function follow from well posedness. ∎

Remark 1.4.2 In the above setup if the condition (1.4.10) does not hold but (1.4.11) holds, one can still deduce from Theorem 1.4.4 that the martingale problem for $(A_t)_{t\geq 0}$ is well - posed in the class of progressively measurable solutions.

Chapter 2

A Criterion For Invariant Measures

2.1 Introduction

Suppose A satisfies the conditions of Theorem 1.1.2. Then we have seen that it determines a Markov process.

Definition 1. $\mu \in \mathcal{P}(E)$ is a ***stationary distribution*** or an ***invariant measure*** for the Markov process determined by A, if the solution X to the martingale problem for (A, μ) is a stationary process, *i.e.*, if $P\{X(t+s_1) \in \Gamma_1, \ldots, X(t+s_k) \in \Gamma_k\}$ is independent of $t \geq 0$ for all $0 \leq s_1 < s_2 < \ldots < s_k, \Gamma_1, \Gamma_2, \ldots, \Gamma_k \in \mathcal{E}$ and for all $k \geq 1$.

The transition probability P given by (1.1.4) then satisfies

$$\mu(\Gamma) = \int_E P(t, x, \Gamma)\mu(dx) \quad \forall\, t > 0 \tag{2.1.1}$$

and for all $\Gamma \in \mathcal{E}$.

For T_t as in (1.1.9) , it is easy to deduce that (2.1.1) implies

$$\int_E f d\mu = \int_E T_t f d\mu \quad \forall\, f \in B(E), t > 0. \tag{2.1.2}$$

Now suppose that L is the generator of a semigroup $(T_t)_{t\geq 0}$ corresponding to a Markov process X. Then one can show that (2.1.2) holds if and only if

$$\int_E (Lf) d\mu = 0, \qquad \forall\, f \in \mathcal{D}(L). \tag{2.1.3}$$

i.e. μ is an invariant measure if and only if (2.1.3) holds. This could be a useful criterion to test for invariant measures if we can describe $\mathcal{D}(L)$ completely. But in general $\mathcal{D}(L)$ can be very large and it may be difficult to describe it completely. Here we examine conditions under which the generator L in (2.1.3) can be replaced by an operator A for which the martingale problem is well -

posed. The advantage is that in some cases the domain of the operator A can be chosen to be much smaller than $\mathcal{D}(L)$.

Echeverria ([5]) showed that when the state space is a compact metric space or a locally compact separable metric space, and when the martingale problem for A is well - posed and $\mathcal{D}(A)$ is an algebra, then for μ to be an invariant measure it suffices to check $\int_E Af d\mu = 0$ for $f \in \mathcal{D}(A)$.

Here we look at this question when A is an operator on $C_b(E)$, where E is a complete, separable metric space. We are able to prove that (2.1.3) implies that μ is an invariant measure under some further conditions on the operator A.

We begin with the following result.

Lemma 2.1.1 *Let $\{g_k\} \subset C_b(E)$ be a countable subset that separates points in E and vanishes nowhere. Let U_n, U be E-valued random variables defined on $(\Omega_0, \mathcal{F}_0, P_0)$ such that $P_0 \circ U_n^{-1} = P_0 \circ U^{-1}$ for all n. Suppose $g_k(U_n) \to g_k(U)$ in probability as $n \to \infty$, $\forall\, k$. Then $U_n \to U$ in probability as $n \to \infty$.*

Proof. Note that $g_j(U_n)g_k(U) \to g_j(U)g_k(U)$ in probability $\forall j, k$. Let

$$\mathcal{D}_0 = \{h : E \times E \to \mathbb{R}; h(x_1, x_2) = g_j(x_1)g_k(x_2) \forall x_1, x_2, \text{ for some } j \geq 1, k \geq 1\}$$

and let $\mathcal{U}$ be the algebra generated by $\mathcal{D}_0$. Since $\{g_k\}$ separates points in E, $\mathcal{U}$ separates points in $E \times E$. Also for $h \in \mathcal{U}$

$$h(U_n, U) \to h(U, U) \quad \text{in probability.}$$

Also (U_n, U) is relatively compact by hypothesis. Let $\varepsilon > 0, \delta > 0$ be arbitrary but fixed. Choose a compact subset K_δ of $E \times E$ with

$$P_0\{(U_n, U) \in K_\delta\} \geq 1 - \delta \quad \forall\, n.$$

Now the metric d restricted to K_δ, which we continue to denote by d, is continuous and $\mathcal{U}' = \mathcal{U}|_{K_\delta}$ is an algebra that separates points and vanishes nowhere. Hence by the Stone-Weierstrass theorem $\mathcal{U}'$ is dense in $C(K_\delta)$ in the uniform topology. Choose $h_k \in \mathcal{U}'$ such that $\|h_k - d\| \to 0$ as $k \to \infty$. Then remembering that $d(U,U) = 0$, we have

$$\begin{aligned} P_0\{d(U_n,U) > \varepsilon\} &\le P_0\{d(U_n,U) > \varepsilon, (U_n,U) \in K_\delta\} + P_0\{(U_n,U) \notin K_\delta\} \\ &\le P_0\{|d(U_n,U) - h_k(U_n,U)| > \varepsilon/3, (U_n,U) \in K_\delta\} \\ &\quad + P_0\{|h_k(U_n,U) - h_k(U,U)| > \varepsilon/3, (U_n,U) \in K_\delta\} \\ &\quad + P_0\{|d(U,U) - h_k(U,U)| > \varepsilon/3, (U_n,U) \in K_\delta\} + \delta. \end{aligned}$$

Choosing k such that $\|h_k - d\| < \varepsilon/3$, we get

$$P_0\{d(U_n,U) > \varepsilon\} < P_0\{|h_k(U_n,U) - h_k(U,U)| > \varepsilon/3, (U_n,U) \in K_\delta\} + \delta.$$

Taking limsup over n, we conclude $\limsup P_0(d(U_n,U) > \varepsilon) \le \delta$. Hence $U_n \to U$ in probability as $n \to \infty$. ∎

The next result is a key step in the proof of our main result. This is a generalisation of the Riesz representaion theorem.

Theorem 2.1.2 *Let E be a complete separable metric space and let Λ be a positive linear functional on $C_b(E \times E)$ with $\Lambda 1 = 1$. Suppose that there exist (countably additive) probability measures μ_1, μ_2 on E such that*

$$\Lambda(F_f) = \int f(x)\mu_1(dx)$$

$$\Lambda(G_g) = \int g(y)\mu_2(dy)$$

for $f, g \in C_b(E)$, where $F_f(x,y) = f(x), G_g(x,y) = g(y)$.

Then there exists a countably additive probability measure ν on $E \times E$ such that for all $F \in C_b(E \times E)$

$$\Lambda(F) = \int_{E\times E} F d\nu. \tag{2.1.4}$$

Proof. First, note that there exists a unique finitely additive measure ν on the Borel field of $E \times E$ satisfying (2.1.4). (See [14, Theorem II.5.7]). Fix $\varepsilon > 0$. Since E is a complete separable metric space, we can choose K, a compact subset of E with $\mu_i(K) \ge 1 - \varepsilon$, $i = 1, 2$. Then

$$\begin{aligned} \nu((K \times K)^c) &\le \nu(K^c \times E) + \nu(E \times K^c) \\ &= \mu_1(K^c) + \mu_2(K^c) \le 2\varepsilon. \end{aligned}$$

Let

$$\tilde{K} = K \times K \subset E \times E.$$

and $\{F_m\} \subset C_b(E \times E)$ be decreasing to zero. Then $\|F_m\| \leq \|F_1\|$. For $\delta > 0$, let B_m^δ be defined by $B_m^\delta = \{F_m \geq \delta\}$. Then B_m^δ is a decreasing sequence of closed sets and $B_m^\delta \cap \tilde{K}$ is compact in $E \times E$. Since $\cap_{m=1}^\infty \{B_m^\delta \cap \tilde{K}\} = \emptyset$, $\exists\ m_0$ such that $B_m^\delta \cap \tilde{K} = \emptyset$ whenever $m \geq m_0$. Hence for $m \geq m_0$, we have

$$\begin{aligned} \Lambda F_m &= \int_{E\times E} F_m d\nu \\ &\leq \int_{E\times E} I_{\{\|F_m\|<\delta\}} F_m d\nu + \int_{E\times E} I_{\{\tilde{K}^c\}} F_m d\nu \\ &\leq \delta + 2\|F_1\|\varepsilon. \end{aligned}$$

Since this holds for all δ and ε, we get,

$$\Lambda F_m \to 0 \ \text{ as } \ m \to \infty.$$

Hence by Daniell's Theorem (See [13, Proposition II.7.1]) there exists a unique σ-additive probability measure, again denoted by ν, defined on the Borel σ-field of $E \times E$, satisfying (2.1.4) . ■

2.2 The Main Result

We need the following lemma.

Lemma 2.2.1 *Let A be an operator on $C_b(E)$. Suppose that for each $x \in E$ the martingale problem for (A, δ_x) has a solution with sample paths in $D([0,\infty), E)$. Suppose that ϕ is continuously differentiable and convex on $G \subset \mathbb{R}^m$. Let $f_1, f_2, \ldots, f_m \in \mathcal{D}(A)$ satisfy $(f_1, f_2, \ldots, f_m) : E \to G$ and $\phi(f_1, f_2, \ldots, f_m) \in \mathcal{D}(A)$. Then*

$$A\phi(f_1, f_2, \ldots, f_m) \geq \nabla\phi(f_1, f_2, \ldots, f_m) \cdot (Af_1, Af_2, \ldots, Af_m). \tag{2.2.1}$$

Proof. Let $x \in E$ and X be a solution to the $D([0,\infty), E)$-martingale problem for (A, δ_x). We have

$$\begin{aligned} &\mathbb{E}[\phi(f_1(X(t)), f_2(X(t)), \ldots, f_m(X(t)))] - \phi(f_1(x), f_2(x), \ldots, f_m(x)) \\ &\qquad = \mathbb{E}[\int_0^t (A\phi(f_1, f_2, \ldots, f_m))(X(s))ds]. \end{aligned}$$

Using convexity of ϕ, we get

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^t (A\phi(f_1,\ldots,f_m))(X(s))ds\Big] \\
&\geq \nabla\phi(f_1(x),\ldots,f_m(x))\cdot \mathbb{E}[f_1(X(t))-f_1(x),\ldots,f_m(X(t))-f_m(x)] \\
&= \nabla\phi(f_1(x),\ldots,f_m(x))\cdot \mathbb{E}\Big[\int_0^t Af_1(X(s))ds,\ldots,\int_0^t Af_m(X(s))ds\Big].
\end{aligned}
$$

This holds for all $t>0$. Dividing by t and letting $t\to 0$ gives (2.2.1) . ∎

Theorem 2.2.2 *Let $\mathcal{D}(A)$ be an algebra that separates points and vanishes nowhere. Suppose A satisfies the separability condition (I) and that for all $\nu\in\mathcal{P}(E)$, there exists a solution to the $D([0,\infty),E)$ -martingale problem for (A,ν). Suppose that $\mu\in\mathcal{P}(E)$ satisfies*

$$\int_E Af\,d\mu = 0 \quad \forall\, f\in\mathcal{D}(A). \tag{2.2.2}$$

Then on some probability space, there exists a filtration $(\mathcal{G}_t)_{t\geq 0}$ and a $(\mathcal{G}_t)_{t\geq 0}$-progressively measurable process X such that X is a stationary process and that X is a solution to the martingale problem for (A,μ) w.r.t. $(\mathcal{G}_t)_{t\geq 0}$.

Proof. For $n\geq 1$ define A_n on $\mathcal{R}(I-n^{-1}A)$ by

$$A_n f = n[(I-n^{-1}A)^{-1}-I]f$$

Note that since the martingale problem for (A,δ_x) admits a solution for all $x\in E$, by Lemma 1.1.7 A is dissipative *i.e.* (1.1.17) holds. As a consequence, $(I-n^{-1}A)$ is one to one and hence A_n is well defined. Also for $f\in\mathcal{D}(A)$, $f_n := (I-n^{-1}A)f$ satisfy

$$A_n f_n = Af \quad \forall\, n; \quad \|f_n - f\|\to 0 \quad \text{as } n\to\infty. \tag{2.2.3}$$

And for $g=(I-n^{-1}A)f$, $f\in\mathcal{D}(A)$, we have

$$\int_E A_n g\,d\mu = \int_E Af\,d\mu = 0. \tag{2.2.4}$$

Note that A_n satisfies all the conditions of the Theorem. We now proceed as follows. We construct stationary solutions X_n to the martingale problem for

(A_n, μ) and then using (2.2.3) show that X_n's converge in a suitable sense to get a stationary solution to the martingale problem for (A, μ).

Step 1: Construction of stationary solution to the martingale problem for A_n:

Fix n. Let $M \subset C_b(E \times E)$ be the linear space of functions of the form

$$F(x,y) = \sum_{i=1}^{m} f_i(x) g_i(y) + g(y) \tag{2.2.5}$$

$f_1, ..., f_m, g \in C_b(E)$; $g_1, ... g_m \in \mathcal{R}(I - n^{-1}A); m \geq 1$. Define Λ on M by

$$\Lambda F = \int_E \Big[\sum_{i=1}^{m} f_i(x)(I - n^{-1}A)^{-1} g_i(x) + g(x) \Big] \mu(dx) \tag{2.2.6}$$

for F as in (2.2.5) . Then $\Lambda 1 = 1$.

Let $h_1, h_2, \ldots, h_m \in \mathcal{D}(A)$, let $\alpha_k = \|(I - n^{-1}A)h_k\|$, and let ϕ be a polynomial on $\mathbb{R}^m$ that is convex on $\prod_{i=1}^{m}[-\alpha_i, \alpha_i]$. Since $\mathcal{D}(A)$ is an algebra, $\phi(h_1, h_2, \ldots, h_m) \in \mathcal{D}(A)$, and by Lemma 2.2.1,

$$A\phi(h_1, h_2, \ldots, h_m) \geq \nabla \phi(h_1, h_2, \ldots, h_m) \cdot (Ah_1, Ah_2, \ldots, Ah_m).$$

Consequently,

$$\begin{aligned} &\phi((I - n^{-1}A)h_1, \ldots, (I - n^{-1}A)h_m) \\ &\geq \phi(h_1, \ldots, h_m) - \frac{1}{n} \nabla \phi(h_1, \ldots, h_m) \cdot (Ah_1, \ldots, Ah_m) \\ &\geq \phi(h_1, \ldots, h_m) - \frac{1}{n} A\phi(h_1, \ldots, h_m), \end{aligned} \tag{2.2.7}$$

and using (2.2.2)

$$\int_E \phi((I - n^{-1}A)h_1, \ldots, (I - n^{-1}A)h_m) d\mu \geq \int_E \phi(h_1, \ldots, h_m) d\mu,$$

or equivalently

$$\int_E \phi(g_1, \ldots, g_m) d\mu \geq \int_E \phi((I - n^{-1}A)^{-1} g_1, \ldots, (I - n^{-1}A)^{-1} g_m) d\mu \tag{2.2.8}$$

for $g_1, g_2, \ldots, g_m \in \mathcal{R}(I - n^{-1}A)$. Since all convex functions on $\mathbb{R}^m$ can be approximated uniformly on any compact set $K \subset \mathbb{R}^m$ by a polynomial that is convex on K, (2.2.8) holds for all ϕ convex on $\mathbb{R}^m$.

Let F be given by (2.2.5) , and define $\phi : \mathbb{R}^m \to \mathbb{R}$ by

$$\phi(u) = \sup_x \sum_{i=1}^m f_i(x) u_i. \tag{2.2.9}$$

Note that ϕ is convex. Hence it follows from (2.2.8) that

$$\begin{aligned} \Lambda F &= \int_E \sum_{i=1}^m f_i(x)(I - n^{-1}A)^{-1} g_i(x)\mu(dx) + \int_E g(x)\mu(dx) \\ &\leq \int_E \phi((I-n^{-1}A)^{-1}g_1, \ldots, (I-n^{-1}A)^{-1}g_m) d\mu + \int_E g d\mu \\ &\leq \int_E \phi(g_1, \ldots, g_m) d\mu + \int_E g d\mu \\ &= \int_E \sup_x \Big[\sum_{i=1}^m f_i(x) g_i(y) + g(y) \Big] \mu(dy) \\ &\leq \|F\|. \end{aligned} \tag{2.2.10}$$

Similarly $-\Lambda F = \Lambda(-F) \leq \| -F\| = \|F\|$. Together we get $|\Lambda F| \leq \|F\|$. This holds for all $F \in M$. Also note that if $F \geq 0$, then $\| \, \|F\| - F\| \leq \|F\|$ and from (2.2.10) , we get $\|F\| - \Lambda F = \Lambda(\|F\| - F) \leq \|F\|$. *i.e.* $\Lambda F \geq 0$. Using Hahn-Banach theorem we now extend Λ to a bounded, positive linear functional on $C_b(E \times E)$. From the construction it is clear that for $f, g \in C_b(E), F_f(x,y) = f(x), G_g(x,y) = g(y)$, we have

$$\Lambda F_f = \int_E f(x)\mu(dx) \tag{2.2.11}$$

$$\Lambda G_g = \int_E g(y)\mu(dy) \tag{2.2.12}$$

Then by Theorem 2.1.2, we get a probability measure ν on $E \times E$ satisfying

$$\Lambda F = \int_{E\times E} F d\nu, \quad \forall\, F \in M. \tag{2.2.13}$$

Since E is a complete separable metric space, there exists a transition probability function (see *e.g.* [7, appendix]) $\eta : E \times \mathcal{B}(E) \to [0,1]$ satisfying

$$\nu(B_1 \times B_2) = \int_{B_1} \eta(x, B_2)\mu(dx) \quad \forall B_1, B_2 \in \mathcal{E}.$$

It follows from (2.2.12) that

$$\int_E \eta(x, B)\mu(dx) = \nu(E \times B) = \mu(B) \quad \forall\, B \in \mathcal{E}. \tag{2.2.14}$$

Now, (2.2.6) and (2.2.13) imply that for all $f \in C_b(E)$, $g \in \mathcal{R}(I - n^{-1}A)$

$$\int_{E\times E} f(x)g(y)\nu(dx, dy) = \int_E f(x)(I - n^{-1}A)^{-1}g(x)\mu(dx).$$

Hence we have

$$\int_E g(y)\eta(x, dy) = (I - n^{-1}A)^{-1}g(x) \quad \mu - \text{ a.s.} \tag{2.2.15}$$

for all $g \in \mathcal{R}(I - n^{-1}A)$.

Let $Y(0)$, $Y(1)$,...,$Y(k)$,... be an E-valued Markov chain with initial distribution μ and transition function η. Define

$$M(k) := g(Y(k)) - \sum_{i=0}^{k-1} n^{-1}A_n g(Y(i)) \tag{2.2.16}$$

Then $M(k) - M(k-1) = g(Y(k)) - (I - n^{-1}A)^{-1}g(Y(k-1))$ and (2.2.15) gives

$$\begin{aligned}
&\mathbb{E}[M(k) - M(k-1)|Y(0), \ldots, Y(k-1)] \\
&\quad = \mathbb{E}[g(Y(k))|Y(0), \ldots, Y(k-1)] - (I - n^{-1}A)^{-1}g(Y(k-1)) \\
&\quad = \mathbb{E}[g(Y(k))|Y(k-1)] - (I - n^{-1}A)^{-1}g(Y(k-1)) \\
&\quad = \int_E g(x)\eta(Y(k-1), dx) - (I - n^{-1}A)^{-1}g(Y(k-1)) \\
&\quad = (I - n^{-1}A)^{-1}g(Y(k-1)) - (I - n^{-1}A)^{-1}g(Y(k-1)) \\
&\quad = 0.
\end{aligned}$$

Hence $\{M(k) : k \geq 0\}$ is a $\sigma(Y(0), \ldots, Y(k))$ - martingale. Also (2.2.14) implies that $\{Y(k) : k \geq 0\}$ is stationary.

Let V be a Poisson process with parameter n, which is independent of Y. Define

$$X_n(t) := Y(V(t)).$$

Then X_n is a stationary Markov Process with initial distribution μ. The fact that (2.2.16) is a martingale implies that

$$g(X_n(t)) - \int_0^t A_n g(X_n(s))ds$$

is a martingale for all $g \in \mathcal{D}(A_n)$. Thus X_n is a stationary solution to the martingale problem for A_n. This completes step 1.

Step 2: Convergence of finite dimensional distributions of (a subsequence of) X_n:

For $f \in \mathcal{D}(A)$, let f_n be as in (2.2.3) . Then conditions of Theorem 1.2.2 are satisfied and we get relative compactness of $(f_1 \circ X_n, f_2 \circ X_n, \dots, f_i \circ X_n)$ in $D([0,\infty), \mathbb{R}^i)$, for $f_1, f_2, \dots, f_i \in \mathcal{D}(A), i \geq 1$.

Let $\mathcal{D}_0 = \{g_k\}_{k=1}^{\infty}$ be the countable subset of hypothesis (I). Let $\|g_k\| = a_k$ and $\hat{E} = \prod_{k=1}^{\infty}[-a_k, a_k]$. Since $\mathcal{D}(A)$ separates points and vanishes nowhere, so does $\mathcal{D}_0$. It now follows that $(g_1(X_n(.)), g_2(X_n(.)), ..., g_k(X_n(.)), ...)$ is relatively compact in $D([0,\infty), \hat{E})$. Thus we get a subsequence, which we relabel as X_n, such that $(g_1(X_n(.)) g_2(X_n(.)), ..., g_k(X_n(.)), ...)$ converges weakly to a $D([0,\infty), \hat{E})$ valued random variable, say, $Z(.) = (Z_1(.), ..., Z_k(.), ...)$, *i.e.*

$$(g_1(X_n(.)), ...g_k(X_n(.)), ...) \xrightarrow{\mathcal{L}} Z(.) \text{ as } n \to \infty. \tag{2.2.17}$$

Define $\underline{g} : E \to \hat{E}$ by

$$\underline{g}(x) = (g_1(x), ..., g_k(x), ...). \tag{2.2.18}$$

Then $\underline{g}$ is a one to one, continuous function. This implies that $\underline{g}(E)$ is a Borel subset of $\hat{E}$. Also $\underline{g}^{-1}$ defined on $\underline{g}(E)$ is measurable. (See [14, Corollary I.3.3.]). We extend the definition of $\underline{g}^{-1}$ to all of $\hat{E}$ by setting $\underline{g}^{-1}(z) = e$ for $z \notin \underline{g}(E)$, where e is a fixed point in E. We now use Skorokhod representation to get a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $D([0,\infty), \hat{E})$ valued random variables $\tilde{\xi}_n$ and $\tilde{Z}$ defined on it satisfying

$$\mathcal{L}(\tilde{\xi}_n) = \mathcal{L}(\underline{g}(X_n)) \quad \forall\, n \tag{2.2.19}$$

$$\mathcal{L}(\tilde{Z}) = \mathcal{L}(Z) \tag{2.2.20}$$

$$\tilde{\xi}_n \to \tilde{Z} \quad a.s. \text{ as } n \to \infty. \tag{2.2.21}$$

Now,

$$\mathcal{L}(\underline{g}(X_n(t))) = \mu \circ \underline{g}^{-1} := \tilde{\mu} \quad \forall\, n, t.$$

Hence

$$\mathcal{L}(\tilde{\xi}_n(t)) = \tilde{\mu} \quad \forall\, n, t \tag{2.2.22}$$

which implies $\tilde{\xi}_n(t) \in \underline{g}(E)$ a.s. Then defining

$$\tilde{X}_n(t) = \underline{g}^{-1}(\tilde{\xi}_n(t)), \tag{2.2.23}$$

it follows that $\tilde{X}_n$ is a measurable process. Since $(X_n(t))_{t\geq 0}$ is a stationary process, it follows that $(\tilde{\xi}_n(t))_{t\geq 0}$ is a stationary process and hence $(\tilde{Z}(t))_{t\geq 0}$ is a $\hat{E}$ valued r.c.l.l. stationary process. Hence $(\tilde{Z}(t))_{t\geq 0}$ does not have any fixed points of discontinuity, *i.e.* $P(\tilde{Z}(t) = \tilde{Z}(t-)) = 1 \ \forall\ t$. Thus $\tilde{\xi}_n(t) \to \tilde{Z}(t)$ a.s. for all t. Since $\mathcal{L}(\tilde{\xi}_n(t)) = \tilde{\mu}$ for all n, it follows that $\mathcal{L}(\tilde{Z}(t)) = \tilde{\mu}$. Hence

$$P(\tilde{Z}(t) \in \underline{g}(E)) = 1 \quad \forall \quad t$$

and defining

$$\tilde{X}(t) = \underline{g}^{-1}(\tilde{Z}(t)) \tag{2.2.24}$$

we get a stationary $(\mathcal{G}_t)_{t\geq 0}$-progressively measurable process $\tilde{X}$, where $\mathcal{G}_t = \mathcal{F}_t^{\tilde{Z}}$. Further

$$\underline{g}(\tilde{X}_n(t)) \to \underline{g}(\tilde{X}(t)) \quad a.s. \quad \forall \quad t. \tag{2.2.25}$$

This and Lemma 2.1.1 imply that $\tilde{X}_n(t)$ converges to $\tilde{X}(t)$ as $n \to \infty$ in E in $\tilde{P}$ probability for each t. This completes step 2.

To complete the proof, we will show that $\tilde{X}$ is a solution to the martingale problem for (A, μ) w.r.t. $(\mathcal{G}_t)_{t\geq 0}$. Recall that we have already proved that $\tilde{X}$ is a stationary process and is $(\mathcal{G}_t)_{t\geq 0}$- progressively measurable. Note that $(\tilde{X}_n(t))_{t\geq 0}$ has the same finite dimensional distributions as $(X_n(t))_{t\geq 0}$, and hence by Lemma 1.1.1 $\tilde{X}_n$ is a solution to the martingale problem for A_n, *i.e.* for all $f \in \mathcal{D}(A_n)$

$$f(\tilde{X}_n(t)) - \int_0^t A_n f(\tilde{X}_n(s))ds \tag{2.2.26}$$

is a $\tilde{P}$ martingale. Now for $g \in C_b(E)$

$$g(\tilde{X}_n(t)) \to g(\tilde{X}(t)) \text{ as } n \to \infty \tag{2.2.27}$$

in $\tilde{P}$ probability. This holds for all t. An application of DCT gives for $g \in C_b(E)$

$$\mathbb{E}|g(\tilde{X}_n(s)) - g(\tilde{X}(s))| \to 0 \text{ as } n \to \infty$$

and hence using Fubini's theorem we get

$$\mathbb{E}\int_0^t |g(\tilde{X}_n(s)) - g(\tilde{X}(s))|ds \to 0 \text{ as } n \to \infty.$$

As a consequence, we have

$$\int_0^t g(\tilde{X}_n(s))ds \to \int_0^t g(\tilde{X}(s))ds \tag{2.2.28}$$

in $\tilde{P}$ probability for all t.
From (2.2.26) it follows that for $0 \le t_1 < t_2 < \ldots < t_{m+1}$, $h_1, \ldots, h_m \in C_b(E)$ and f_n as in (2.2.3)

$$\mathbb{E}[(f_n(\tilde{X}_n(t_{m+1})) - f_n(\tilde{X}_n(t_m)) - \int_{t_m}^{t_{m+1}} Af(\tilde{X}_n(s))ds) \prod_{k=1}^{m} h_k(\tilde{X}_n(t_k))] = 0$$

and since for $f \in \mathcal{D}_0$, $\|f_n - f\| \to 0$ as $n \to \infty$ we get,

$$\mathbb{E}[(f(\tilde{X}_n(t_{m+1})) - f(\tilde{X}_n(t_m)) - \int_{t_m}^{t_{m+1}} Af(\tilde{X}_n(s))ds) \prod_{k=1}^{m} h_k(\tilde{X}_n(t_k))] \to 0$$

as $n \to \infty$. Now (2.2.27) , (2.2.28) and an application of DCT gives

$$\mathbb{E}[(f(\tilde{X}(t_{m+1})) - f(\tilde{X}(t_m)) - \int_{t_m}^{t_{m+1}} Af(\tilde{X}(s))ds) \prod_{k=1}^{m} h_k(\tilde{X}(t_k))] = 0$$

Applying Lemma 1.1.1 once again, we get that $(\tilde{X}(t))_{t\ge 0}$ is a solution to the martingale problem for (A, μ) with respect to $(\mathcal{F}_t^{\tilde{X}})_{t\ge 0}$. Since $\tilde{X}(t) = \tilde{Z}(t)$ a.s. for all t, it follows that $\tilde{X}$ is a solution to the martingale problem for (A, μ) with respect to $(\mathcal{G}_t)_{t\ge 0}$. ∎

It should be noted that the stationary solution constructed above may not have r.c.l.l. paths. Thus even when the $D([0,\infty), E)$ - martingale problem for (A, ν) is well posed for all $\nu \in \mathcal{P}(E)$ in addition to the conditions in Theorem 2.2.2 above, it does not follow that μ is an invariant measure for the Markov process associated with A. This can be deduced if we assume that every solution to the martingale problem for (A, μ) admits an r.c.l.l. modification. This is our next result.

Theorem 2.2.3 *Let $\mathcal{D}(A)$ be an algebra that separates points and vanishes nowhere. Suppose A satisfies the separability condition (I). Suppose that the $D([0,\infty), E)$- martingale problem for (A, δ_x) is well posed for all $x \in E$. Let $(T_t)_{t\ge 0}$ be the semigroup associated with the martingale problem for A in Theorem 1.1.2.*

Further suppose that

(II) Every progressively measurable solution to the martingale problem for (A,μ) admits an r.c.l.l. modification.

If $\mu \in \mathcal{P}(E)$ satisfies

$$\int_E Af\,d\mu = 0 \quad \forall\, f \in \mathcal{D}(A)$$

then μ is an invariant measure for the semigroup $(T_t)_{t\geq 0}$.

Proof. Let X be the stationary solution constructed in Theorem 2.2.2. In view of hypothesis (II), we can assume that X is r.c.l.l. If Q is the law of X, then it follows that Q is a solution to the $D([0,\infty),E)$ martingale problem for (A,μ) and that $Q\circ(\theta(t))^{-1}=\mu$ for all t. Using (1.1.4) and (1.1.9) it now follows that

$$\int_E (T_t f)d\mu = \int_E f d\mu.$$

for all t and hence that μ is an invariant measure for $(T_t)_{t\geq 0}$. ∎

Remark 2.2.1 : It follows from Theorem 1.1.5 that when E is a compact metric space, (II) always holds. If A is a diffusion operator on $\mathbb{R}^m$ with bounded coefficients, (II) is satisfied. (See Example 1 of section 1.3). In fact the same proof as in that case shows that when E is a locally compact separable metric space and A is an operator on $\hat{C}(E)$ (continuous functions vanishing at infinity), then also (II) holds if A is conservative, *i.e.* (1,0) is in the bp-closure of $\{(f, Af) : f \in \mathcal{D}(A)\}$. It may be noted that in [5] Echeverria proved this result without assuming that A is conservative in the locally compact case.

When $E = H$, a real,separable Hilbert space and A is the operator corresponding to a diffusion on H, then also we have seen in Example 2 of section 1.1 that (II) holds.

Chapter 3

Evolution equations: uniqueness

3.1 The Evolution Equation

In this chapter we consider operators A on $C_b(E)$. We continue to assume condition (I).

As in the last chapter we *imbed* the martingale problem into a compact set, as follows.

Let $\{g_k\}$ be as in condition (I), *i.e.* $\{g_k : k \geq 1\} \subset \mathcal{D}(A)$ and $\{(f, Af) : f \in \mathcal{D}(A)\} \subset$ bp-closure $\{(g_k, Ag_k) : k \geq 1\}$. Let $\|g_k\| = a_k$ and define $\hat{E} = \prod_{k=1}^{\infty}[-a_k, a_k]$ and $\underline{g} : E \to \hat{E}$ by

$$\underline{g}(x) = (g_1(x), ..., g_k(x), ...) \tag{3.1.1}$$

Let $\mathcal{U}$ be the algebra generated by

$$\{u_k \in C(\hat{E}) : u_k((z_1, ..., z_k, ...)) = z_k\}.$$

Define $\mathcal{A}$ with domain $\mathcal{U}$ as follows.

$$\mathcal{A}(cu_{i_1}u_{i_2}...u_{i_k})(z) = \begin{cases} cAg_{i_1}g_{i_2}...g_{i_k}(x) & \text{if } z = \underline{g}(x) \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.2}$$

Again note that

$$u_k(\underline{g}(x)) = g_k(x) \text{ and } \mathcal{A}u_k(\underline{g}(x)) = Ag_k(x).$$

The next lemma shows that the martingale problems for A and $\mathcal{A}$ are equivalent in a certain sense.

Lemma 3.1.1 *Let X be a solution to the martingale problem for A and let*

$$Z(t) = \underline{g}(X(t)) \quad \forall \ t \geq 0. \tag{3.1.3}$$

Then Z is a solution to the martingale problem for $\mathcal{A}$. Conversely, if Z is a solution to the martingale problem for $\mathcal{A}$ with

$$P(Z(t) \in \underline{g}(E)) = 1 \quad \forall \ t \geq 0 \tag{3.1.4}$$

then

$$X(t) = \underline{g}^{-1}(Z(t))$$

defines a solution to the martingale problem for A. Thus if the martingale problem for A is well - posed, then there exists a unique solution Z to the martingale problem for $\mathcal{A}$ satisfying (3.1.4) .

Proof: Let X be a solution to the martingale problem for A and let Z be defined by (3.1.3) . Then Z is a $\underline{g}(E)$ valued process. Also for $u \in \mathcal{U}$ with $u(\underline{g}(x)) = g(x)$, we have that for $0 \leq t_1 < t_2 < \cdots < t_m < t < r$, $\underline{h_1}, \ldots, \underline{h_m} \in B(\hat{E})$,

$$\begin{aligned} & \mathbb{E}[(u(Z(r)) - u(Z(t)) - \int_t^r \mathcal{A}u(Z(s))ds) \prod_{i=1}^m \underline{h_i}(Z(t_i))] \\ = \ & \mathbb{E}[(g(X(r)) - g(X(t)) - \int_t^r Ag(X(s))ds) \prod_{i=1}^m h_i(X(t_i))] \\ = \ & 0 \end{aligned} \tag{3.1.5}$$

where $h_i = \underline{h_i} \circ \underline{g}$. It follows from Lemma 1.1.1 that Z is a solution to the martingale problem for $\mathcal{A}$. The second assertion in the lemma can be proved similarly. The last part follows from the earlier statements. ∎

Remark 3.1.1 : One can deduce from this lemma that if $\mathcal{D}(A)$ is an algebra that separates points and if condition (I) holds, then well - posedness in the class of progressively measurable processes implies well - posedness in the class of all measurable processes.

For, if X is a measurable solution to the martingale problem for A, then Z given by (3.1.3) is a solution to the martingale problem for $\mathcal{A}$. Since $\hat{E}$ is compact, Z has a r.c.l.l. modification $\hat{Z}$, (see Theorem 1.1.5). Clearly $\hat{Z}$ continues to satisfy (3.1.4) . It follows that $Y(t) = \underline{g}^{-1}(\hat{Z}(t))$ is a progressively measurable solution to the martingale problem for A and has the same finite dimensional distributions as those of X.

We are interested in the following perturbation of the operator A. Let $\lambda \in C_b(E)$. Consider the operator $A - \lambda(\cdot)$.

We will henceforth denote by $\mathcal{M}(E)$ the set of all positive finite measures on $(E, \mathcal{E})$. We want to look at the measure valued evolution equation

$$\int_E f d\nu_t = \int_E f d\nu_0 + \int_0^t (\int_E (Af - \lambda(\cdot)f) d\nu_s) ds, \quad f \in \mathcal{D}(A) \tag{3.1.6}$$

where A and λ are as above and $\{\nu_t\}_{t\geq 0} \subset \mathcal{M}(E)$ satisfy

$$t \mapsto \nu_t(B) \text{ is measurable } \forall\, B \in \mathcal{E}. \tag{3.1.7}$$

Note that if $(X(t))_{t\geq 0}$ is a solution to the $D([0,\infty), E)$ - martingale problem for (A, ν_0) then

$$f(X(t)) \exp\{-\int_0^t \lambda(X(s)ds\} - \int_0^t \Big[\exp\{-\int_0^s \lambda(X(u))du\} \\ \big(Af(X(s)) - \lambda(X(s))f(X(s))\big)\Big] ds$$

is a martingale. Define ν_t by $\nu_t(B) := \mathbb{E}(I_B(X(t)) \exp\{-\int_0^t \lambda(X(s))ds\})$. Then it can be easily seen that $(\nu_t)_{t\geq 0}$ satisfies (3.1.7) and is a solution to (3.1.6) .

We are interested in the question of uniqueness of (measure valued) solutions to (3.1.6) . Note that when L is the generator of a Markov process uniqueness in the class of measures satisfying (3.1.7) is proved in [11] . We have already seen that if A satisfies the conditions of Theorem 1.1.2, then it determines a Markov process given by transition probability $P(\cdot,\cdot,\cdot)$ and corresponding to a one parameter semigroup $(T_t)_{t\geq 0}$ on $B(E)$. Further if L is the generator of the semigroup, then A is the restriction of L to $\mathcal{D}(A)$. Our aim is to prove uniqueness of solutions to (3.1.6) in such a case.

Also note that if $\{\mu_t\}_{t\geq 0}$ satisfy (3.1.6) then $\mu_t' = \mu_t e^{\alpha t}$ satisfy

$$\int_E f d\mu_t' = \int_E f d\mu_0' + \int_0^t (\int_E Af - \lambda(\cdot)f + \alpha f) d\mu_s') ds \qquad f \in \mathcal{D}(A). \tag{3.1.8}$$

Conversely if $\{\mu_t'\}_{t\geq 0} \subset \mathcal{M}(E)$ satisfy (3.1.8) then $\mu_t = \mu_t' e^{-\alpha t}$ satisfy (3.1.6) . So without loss of generality we consider $\lambda \geq 0$ throughout this section.

We will assume that A satisfies the conditions of Theorem 2.2.3. In particular, the martingale problem for A is well posed. Without loss of generality we will assume that $1 \in \mathcal{D}(A)$ with $A1 = 0$. Choose a point Δ, $\Delta \notin E$. Let

$$E^\Delta = E \cup \{\Delta\}. \tag{3.1.9}$$

Define a metric d' on E^Δ by $d'(\Delta, \Delta) = 0$, $d'(\Delta, x) = d'(x, \Delta) = 1$ and $d'(x, y) = d(x, y) \wedge 1$ for $x, y \in E$.. We consider the martingale problem on E^Δ. Extend λ to E^Δ by defining

$$\tilde{\lambda}(\Delta) = 0. \tag{3.1.10}$$

Define operators A^Δ and C on $C_b(E^\Delta)$ by

$$\mathcal{D}(A^\Delta) = \{f \in C_b(E^\Delta) : f|_E \in \mathcal{D}(A)\} \tag{3.1.11}$$

and for $f \in \mathcal{D}(A^\Delta)$

$$\begin{aligned} A^\Delta f(x) &= Af(x) \ \ \forall\ x \in E \\ A^\Delta f(\Delta) &= 0. \end{aligned} \tag{3.1.12}$$

For $f \in C_b(E^\Delta), x \in E^\Delta$

$$Cf(x) = \tilde{\lambda}(x)(f(\Delta) - f(x)). \tag{3.1.13}$$

We will now show that the martingale problem for $A^\Delta + C$ is well posed. It is easy to see that the martingale problem for A^Δ is well - posed. Also existence of a solution to the $D([0, \infty), E)$ - martingale problem for $(A^\Delta + C, \delta_x)$ follows from theorem 1.1.8. Heuristically, this solution can be described as follows. If $X(0) = x \in E$, then it evolves as the Markov process corresponding to A until it is *killed* at which time it jumps to Δ and stays there. Further, if $X(t) = y$, the process is killed at time t with intensity $\lambda(y)$. The next result shows that this is the only solution.

Theorem 3.1.2 *Let $A, A^\Delta, \lambda, \tilde{\lambda}$ and C be defined by (3.1.10) – (3.1.13) . Suppose that A satisfies the conditions of Theorem 2.2.3. Then so does $A^\Delta + C$.*

Proof. Clearly $\mathcal{D}(A^\Delta + C)$ is an algebra that separates points in E^Δ and vanishes nowhere. Also $A^\Delta + C$ satisfies condition (I). Existence of r.c.l.l. soltuion to the martingale problem has been noted above.

We have seen that the martingale problem for A^Δ is well - posed. Existence of $D([0,\infty),E^\Delta)$ solution for the martingale problem for (A^Δ,π) for any $\pi\in\mathcal{P}(E^\Delta)$ follows easily. Hence A^Δ satisfies the conditions of Theorem 1.1.2. Let $(T_t^\Delta)_{t\geq 0}$ be the associated semigroup.

Let X be a measurable solution to the martingale problem for $A^\Delta+C$. We will show that the one dimensional distributions of X are uniquely determined. In view of Theorem 1.1.3 this will show well - posedness of the martingale problem for $A^\Delta+C$ completing the proof.

Since $I_E\in\mathcal{D}(A^\Delta+C)$ and $(A^\Delta+C)I_E=-\tilde{\lambda}I_E=-\tilde{\lambda}$, we get

$$M(t):=I_E(X(t))+\int_0^t\tilde{\lambda}(X(s))ds \tag{3.1.14}$$

is a martingale. Non-negativity of $\tilde{\lambda}$ implies that $I_E(X(t))$ is a supermartingale. The filtration may not be right continuous. Hence to get an r.c.l.l. modification of $I_E(X(t))$ we proceed as follows.

Using $(I_E(X(t)))^2=I_E(X(t))$, a simple calculation gives that

$$(M(t))^2-\int_0^t\tilde{\lambda}(X(s))ds$$

is a martingale. (See [9, pg. 446] . Also [7, Problem II.29]). Similarly

$$(M(t)-M(s))^2-\int_s^t\tilde{\lambda}(X(u))du,\quad t\geq s$$

is a martingale. This implies that the map $t\mapsto M(t)$ is continuous in probability. Hence $t\mapsto I_E(X(t))$ is continuous in probability and since it is a supermartingale it has an r.c.l.l. modification, say $(N(t))_{t\geq 0}$. N can be taken to be $\{0,1\}$ valued. Let

$$\tau=\inf\{t>0:N(t)=0\}. \tag{3.1.15}$$

Then $N(u)=0$ for $u\geq\tau$ a.s. since N is a positive supermartingale. Thus $N(t)\equiv I_{\{\tau>t\}}$ and

$$I_E(X(t))=I_{\{\tau>t\}}\text{ a.s.} \tag{3.1.16}$$

Hence using (3.1.14) and integration by parts we get

$$I_{\{\tau>t\}}\exp\{\int_0^t\tilde{\lambda}(X(s))ds\} \tag{3.1.17}$$

is a martingale. Let $\{g_k\}_{k=1}^{\infty}$ be the countable set satisfying the separability condition (I) with $\|g_k\| = a_k$. Without loss of generality assume that $g_1 = g_2 = 1$. We will continue to denote by the same symbol g_k, the extension of g_k with $g_k(\Delta) = 0$ for $k \geq 2$ and $g_1(\Delta) = 1$. Then $\{g_k\}_{k=1}^{\infty}$ separates points in E^{Δ}. Let

$$\hat{E} = \prod_{k=1}^{\infty} [-a_k, a_k]$$

and $\underline{g} : E^{\Delta} \to \hat{E}$ be defined by

$$\underline{g}(x) = (g_1(x), \ldots, g_k(x), \ldots). \tag{3.1.18}$$

Define for $z \notin \underline{g}(E^{\Delta})$, $\underline{g}^{-1}(z) = e$ for some fixed point e in E^{Δ}. Let $\hat{\lambda} \in B(\hat{E})$ be defined by $\hat{\lambda} = \tilde{\lambda} \circ \underline{g}^{-1}$. Then

$$\hat{\lambda}(\underline{g}(x)) = \tilde{\lambda}(x) \quad \forall\, x \in E^{\Delta}. \tag{3.1.19}$$

Define operator $\mathcal{A}$ as in (3.1.2) with A replaced by A^{Δ} in the definition. Now, on $\mathcal{D}(\mathcal{A})$ define operator $\mathcal{C}$ by

$$\mathcal{C}u(z) = \mathcal{A}u(z) + \hat{\lambda}(z)(u(\underline{g}(\Delta)) - u(z)). \tag{3.1.20}$$

Then $Z(t) = \underline{g}(X(t))$ is a solution to the martingale problem for $\mathcal{C}$. Theorem 1.1.5 implies that Z has a r.c.l.l. modification, say $\hat{Z}$. Arguing as in (3.1.5) and using (3.1.17) , we get that

$$I_{\{\tau > t\}} \exp\{\int_0^t \hat{\lambda}(\hat{Z}(s))ds\} \tag{3.1.21}$$

is a non-negative mean one martingale.

Fix $T > 0$. Define Q on $D([0,\infty), \hat{E})$ by

$$Q(\hat{\theta}(t_1) \in \Gamma_1, \ldots, \hat{\theta}(t_m) \in \Gamma_m) = \\ \mathbb{E}\Big[\prod_{i=1}^{m} I_{\Gamma_i}(\hat{Z}(t_i)) I_{\{\tau > t_m\}} \exp\{\int_0^{t_m} \hat{\lambda}(\hat{Z}(s))ds\}\Big] \tag{3.1.22}$$

for all $0 \leq t_1 < \ldots < t_m \leq T$ and all choices of Borel sets $\Gamma_1, \ldots, \Gamma_m$. Here $\hat{\theta}$ is the co-ordinate process on $D([0,\infty), \hat{E})$. (3.1.22) defines a probability measure on $D([0,\infty),\hat{E})$ since $\hat{Z}$ is an r.c.l.l. process. Since X is a solution to the martingale problem for $A^\Delta + C$, we get for $f \in \mathcal{D}(A^\Delta + C)$ with $f(\Delta) = 0$

$$f(X(t)) - \int_0^t (Af(X(s)) - \lambda(X(s))f(X(s)))ds$$

is a martingale and hence using integration by parts, we get

$$f(X(t))\exp\{\int_0^t \tilde{\lambda}(X(s))ds\} - \int_0^t Af(X(s))\exp\{\int_0^s \tilde{\lambda}(X(u))du\}ds$$

is a martingale. Since $f(\Delta) = 0$, using (3.1.16) , we get

$$f(X(t))I_{\{\tau>t\}}\exp\{\int_0^t \lambda(X(s))ds\} \\ - \int_0^t Af(X(s))I_{\{\tau>s\}}\exp\{\int_0^s \lambda(X(u))du\}ds$$

is a martingale. Or, arguing as in (3.1.5) , for $u \in \mathcal{D}(\mathcal{C})$

$$u(\hat{Z}(t))I_{\{\tau>t\}}\exp\{\int_0^t \hat{\lambda}(\hat{Z}(s))ds\} \\ - \int_0^t \mathcal{A}u(\hat{Z}(s))I_{\{\tau>s\}}\exp\{\int_0^s \hat{\lambda}(\hat{Z}(r))dr\}ds \tag{3.1.23}$$

is a martingale. Hence, using Lemma 1.1.1, we get for $0 \le t_1 < \dots < t_{m+1} \le T, h_1, \dots, h_m \in C(\hat{E})$,

$$\begin{aligned} &\mathbb{E}^Q\Big[\Big(u(\hat{\theta}(t_{m+1})) - u(\hat{\theta}(t_m)) - \int_{t_m}^{t_{m+1}} \mathcal{A}u(\hat{\theta}(s))ds\Big)\prod_{k=1}^m h_k(\hat{\theta}(t_k))\Big] \\ &= \mathbb{E}^P\Big[\Big(u(\hat{Z}(t_{m+1}))I_{\{\tau>t_{m+1}\}}\exp\{\int_0^{t_{m+1}} \hat{\lambda}(\hat{Z}(r))dr\} \\ &\quad -u(\hat{Z}(t_m))I_{\{\tau>t_m\}}\exp\{\int_0^{t_m} \hat{\lambda}(\hat{Z}(r))dr\} \\ &\quad - \int_{t_m}^{t_{m+1}} \mathcal{A}u(\hat{Z}(s))I_{\{\tau>s\}}\exp\{\int_0^s \hat{\lambda}(\hat{Z}(r))dr\}ds\Big)\prod_{k=1}^m h_k(\hat{Z}(t_k))\Big] \\ &= 0. \end{aligned} \tag{3.1.24}$$

It follows that under Q, $\hat{\theta}$ is a solution to the martingale problem for $\mathcal{A}$ satisfying

$$Q(\hat{\theta}(t) \in \underline{g}(E^\Delta)) = 1 \quad \forall\, t. \tag{3.1.25}$$

Hence, $X'(t) := \underline{g}^{-1}(\hat{\theta}(t))$ is a solution to the martingale problem for A^{Δ}. Recalling that the A^{Δ} detemines a Markov process corresponding to a semigroup $(T_t^{\Delta})_{t\geq 0}$, we get for $u \in B(\hat{E})$

$$\begin{aligned} \mathbb{E}^Q[u(\hat{\theta}(t))] &= \mathbb{E}^Q[u \circ \underline{g}(X'(t))] \\ &= \mathbb{E}^Q[[T_t^{\Delta}(u \circ \underline{g})](X'(0))] \\ &= \mathbb{E}^P[[T_t^{\Delta}(u \circ \underline{g})](X(0))] \end{aligned}$$

This can be rephrased as

$$\mathbb{E}^P[u(\hat{Z}(t))\exp\{\int_0^t \hat{\lambda}(\hat{Z}(r))dr\}I_{\{\tau>t\}}] = \mathbb{E}^P[T_t^{\Delta}(u \circ \underline{g})(X(0))] \tag{3.1.26}$$

for all $0 \leq t \leq T$. Similarly, if for $s > 0$, fixed, we define $\tilde{Q}$ on $D([0.\infty), \hat{E})$ by

$$\begin{aligned} &\tilde{Q}(\hat{\theta}(t_1)) \in \Gamma_1, ..., \hat{\theta}(t_m) \in \Gamma_m) \\ &= \frac{\mathbb{E}[I_F \prod_{i=1}^m I_{\Gamma_i}(\hat{Z}(s+t_i))I_{\{\tau>s+t_m\}}\exp\{\int_s^{s+t_m}\hat{\lambda}(\hat{Z}(r))dr\}]}{P(\{\tau > s\}\cap F)} \end{aligned} \tag{3.1.27}$$

for all $0 \leq t_1 < ... < t_m \leq T$ for all choices of Borel sets $\Gamma_1, ..., \Gamma_m$ and $F \in \mathcal{F}_s^{\hat{Z}}$ with $P(F) > 0$, then, $\tilde{Q}$ is a solution to the martingale problem for $\mathcal{A}$ and we get

$$\mathbb{E}^P[I_F u(\hat{Z}(t))\exp\{\int_s^t \hat{\lambda}(\hat{Z}(r))dr\}I_{\{\tau>t\}}] = \mathbb{E}^P[I_F[T_{t-s}^{\Delta}(u \circ \underline{g})](X(s))]$$

for all $s \leq t \leq s+T$. Since $F \in \mathcal{F}_s^{\hat{Z}}$ was arbitrary, we get

$$\mathbb{E}^P[u(\hat{Z}(t))\exp\{\int_s^t \hat{\lambda}(\hat{Z}(r))dr\}I_{\{\tau>t\}}|\mathcal{F}_s^{\hat{Z}}] = T_{t-s}^{\Delta}(u \circ \underline{g})(X(s)) \text{ a.s.} \tag{3.1.28}$$

for all $s \leq t \leq s+T$. Further since $T > 0$ was arbitrary (3.1.26) and (3.1.28) hold for all $t \geq 0$ and $t \geq s$ respectively. Let $f \in B(E^{\Delta})$ with $f(\Delta) \equiv 0$ and $u := f \circ \underline{g}^{-1}$. Then note that $u(\underline{g}(x)) = f(x)$ for all $x \in E^{\Delta}$. Using $u(\hat{Z}(t)) \equiv 0$ if $\tau \leq t$, we get

$$\begin{aligned} &\mathbb{E}^P[u(\hat{Z}(t))] - \mathbb{E}^P[T_t^{\Delta}f(X(0))] \\ &\quad = \mathbb{E}^P[u(\hat{Z}(t))(1 - \exp\{\int_0^t \hat{\lambda}(\hat{Z}(r))dr\}I_{\{\tau>t\}})] \end{aligned}$$

$$= \mathbb{E}^P[u(\hat{Z}(t))(1 - \exp\{\int_0^t \hat{\lambda}(\hat{Z}(r))dr\})I_{\{\tau>t\}}]$$
$$= -\mathbb{E}^P[\int_0^t u(\hat{Z}(t))\hat{\lambda}(\hat{Z}(s))\exp\{\int_s^t \hat{\lambda}(\hat{Z}(r))dr\}I_{\{\tau>t\}}ds]$$
$$= -\mathbb{E}^P[\int_0^t \mathbb{E}^P[u(\hat{Z}(t))\exp\{\int_s^t \hat{\lambda}(\hat{Z}(r))dr\}I_{\{\tau>t\}}|\mathcal{F}_s^{\hat{Z}}]\hat{\lambda}(\hat{Z}(s))ds]$$
$$= -\mathbb{E}^P[\int_0^t T_{t-s}^{\Delta}f(X(s))\hat{\lambda}(\hat{Z}(s))ds] \tag{3.1.29}$$

Or

$$\begin{aligned}\mathbb{E}^P[f(X(t))] - \mathbb{E}^P[T_t^{\Delta}f(X(0))]& \\ = \quad & -\mathbb{E}^P[\int_0^t T_{t-s}^{\Delta}f(X(s))\lambda(X(s))ds] \\ = \quad & \int_0^t \mathbb{E}^P[CT_{t-s}^{\Delta}f(X(s))]ds. \end{aligned} \tag{3.1.30}$$

Hence iterating, we get

$$\begin{aligned}\mathbb{E}^P[f(X(t))] &= \mathbb{E}^P[T_t^{\Delta}f(X(0))] \\ &\quad + \int_0^t \mathbb{E}^P[T_s^{\Delta}CT_{t-s}^{\Delta}f(X(0))]ds \\ &\quad + \int_0^t \int_0^s \mathbb{E}^P[CT_{s-r}^{\Delta}CT_{t-s}^{\Delta})f(X(r))]drds\end{aligned}$$

and so on. Thus the distribution of $X(t)$ is determined by $C, (T_s^{\Delta})_{s\geq 0}$ and $X(0)$. Hence the distribution of $X(t)$ is determined for every $t \geq 0$. As noted earlier this completes the proof. ∎

We first prove that when the operator A satisfies the conditons of Theorem 2.2.3 then there exists a unique solution to the measure valued evolution equation (3.1.6) for $\lambda = 0$.

Theorem 3.1.3 *Let $\mathcal{D}(A)$ be an algebra that separates points in E and vanishes nowhere. Suppose that A satisfies the separability condition (I) and that for all $x \in \mathcal{P}(E)$, there exists a solution to the $D([0,\infty), E)$-martingale problem for (A, δ_x). Further suppose that (II) is satisfied.*

If $\{\nu_t\} \subset \mathcal{P}(E)$ and $\{\mu_t\} \subset \mathcal{P}(E)$ satisfy for every Borel set U in E

$$t \mapsto \rho_t(U) \text{ is measurable.} \tag{3.1.31}$$

and the equation

$$\int_E f d\rho_t = \int_E f d\rho_0 + \int_0^t (\int_E A f d\rho_s) ds \quad f \in \mathcal{D}(A) \tag{3.1.32}$$

with $\nu_0 = \mu_0$*, then* $\nu_t = \mu_t$ *for all* $t \geq 0$.

Proof. Let $E_0 = E \times \{-1, 1\}$, $\beta > 0, \nu_0 \in \mathcal{P}(E)$ and B be the operator on $C_b(E_0)$ with domain $\mathcal{D}(B)$ which is the linear span of

$$\{f_1 f_2 : f_1 \in \mathcal{D}(A), f_2 \in C(\{-1, 1\})\}$$

and

$$B f_1 f_2(x, v) = f_2(v) A f_1(x) + \beta(f_2(-v) \int_E f_1 d\nu_0 - f_1(x) f_2(v)). \tag{3.1.33}$$

By definition of B, it is clear that $\mathcal{D}(B)$ is an algebra that separates points in E_0 and that B satisfies the separability condition (I).

Existence of $D([0, \infty), E_0)$ valued solutions to the martingale problem for $(B, \delta_{(x,v)})$, for every $(x, v) \in E_0$ follows from Theorem 1.1.8. We will prove that the martingale problem for (B, μ) is well - posed for every $\mu \in \mathcal{P}(E^0)$. Clearly it suffices to consider μ of the form $\mu_1 \times \delta_v$, $\mu_1 \in \mathcal{P}(E)$, $v \in \{-1, 1\}$.

Step I. Let (Y, V) be a progressively measurable solution to the martingale problem for B with $V(0) = v$. Let

$$Z(t) = \underline{g}(Y(t)) \quad \forall\, t \geq 0$$

and $\mathcal{B}$ be an operator defined on

$$\mathcal{D}(\mathcal{B}) = \{u f_2 : u \in \mathcal{U}, f_2 \in C(\{-1, 1\})\} \tag{3.1.34}$$

by

$$\mathcal{B} u f_2(z, v) = f_2(v) \mathcal{A} u(z) + \beta(f_2(-v) \int_{\hat{E}} u d\tilde{\nu}_0 - f_2(v) u(z)) \tag{3.1.35}$$

where

$$\tilde{\nu}_0 = \nu_0(\underline{g}^{-1}(\Gamma \cap \underline{g}(E))). \tag{3.1.36}$$

Then arguing as in (3.1.5) (Z, V) is a solution to the martingale problem for $\mathcal{B}$. Since $\mathcal{D}(\mathcal{B})$ is an algebra that separates points in $\overline{\underline{g}(E)} \times \{-1, 1\}$, it is a measure determining set. Further, since, $\overline{\underline{g}(E)} \times \{-1,1\}$ is compact, it follows from Theorem 1.1.5 that (Z,V) has a r.c.l.l. modification, say $(\hat{Z},\hat{V})$, in $\overline{\underline{g}(E)} \times \{-1,1\}$.

We now consider an operator $\mathcal{C}$ on $C_b([0,1]^\infty \times \mathbb{Z}^+)$ defined by

$$\mathcal{D}(\mathcal{C}) = \{uh : u \in \mathcal{U}, h \in C_b(\mathbb{Z}^+)\} \tag{3.1.37}$$

and

$$\mathcal{C}uh(z,n) = h(n)\mathcal{A}u(z) + \beta(h(n+1)\int_{\hat{E}} u d\tilde{\nu}_0 - h(n)u(z)) \tag{3.1.38}$$

Define

$$\begin{aligned} \tau_0 &\equiv 0 \\ \tau_k &\equiv \inf\{t > \tau_{k-1} : V(t) = (-1)^k v\} \quad ; \quad k \geq 1 \end{aligned}$$

and

$$N(t) = k \quad \text{if } \tau_k \leq t < \tau_{k+1}.$$

Then $(\hat{Z},N)$ is a r.c.l.l. solution to the martingale problem for $\mathcal{C}$. This can be seen as follows. Fix $u \in \mathcal{U}$. It suffices to show

$$\begin{aligned} u(\hat{Z}(t))I_{\{N(t)=k\}} - \int_0^t \Big(\mathcal{A}u(\hat{Z}(s))I_{\{N(s)=k\}} \\ + \quad \beta[(\int_{\hat{E}} u d\tilde{\nu}_0)I_{\{N(s)+1=k\}} - u(\hat{Z}(s))I_{\{N(s)=k\}}]\Big)ds \end{aligned} \tag{3.1.39}$$

is a martingale. Now for all $f \in C(\{-1,1\})$

$$u(\hat{Z}(t))f(\hat{V}(t)) - \int_0^t \mathcal{B}uf(\hat{Z}(s),\hat{V}(s))ds$$

is a martingale, and hence

$$\begin{aligned} u(\hat{Z}(t \wedge \tau_{k+1}))f(\hat{V}(t \wedge \tau_{k+1})) - u(\hat{Z}(t \wedge \tau_{k-1}))f(\hat{V}(t \wedge \tau_{k-1})) \\ - \int_{t\wedge\tau_{k-1}}^{t\wedge\tau_{k+1}} \mathcal{B}uf(\hat{Z}(s),\hat{V}(s))ds \end{aligned} \tag{3.1.40}$$

is a martingale. Now (3.1.39) is just (3.1.40) for $f(V(s)) = I_{\{V(s)=(-1)^k v\}}$.

Now proceeding exactly as in the proof of Theorem 3.1.2, we get that the one dimensional distributions of $(\hat{Z}(t),N(t))_{t\geq 0}$ are uniquely determined by $Y(0), (T_s)_{s\geq 0}$, β and $\tilde{\nu}_0$ where $(T_s)_{s\geq 0}$ is the semigroup corresponding to the martingale problem for A. Note that N is a Poisson process with jump points same as that of V. In fact $V(t) = (-1)^{N(t)}V(0)$. It follows that the one dimensional distributions of V are uniquely determined by $Y(0), (T_s)_{s\geq 0}$, β and $\tilde{\nu}_0$. Also, since $Y(t) = \underline{g}^{-1}(Z(t))$, and since $\hat{Z}$ is a modifcation of Z, it follows that the one dimensional distributions of Y are uniquely determined by $Y(0), (T_s)_{s\geq 0}$, β and $\tilde{\nu}_0$. Now Theorem 1.1.3 implies that the martingale problem for B is well - posed.

Step II. Let $(\nu_t)_{t\geq 0}$ satisfy (3.1.31) – (3.1.32) . Define

$$\nu = (\beta \int_0^\infty e^{-\beta t}\nu_t dt) \times (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}). \tag{3.1.41}$$

Then for $f(x,v) = f_1(x)f_2(v) \in \mathcal{D}(B)$, using that $e^{-\beta t} = \beta \int_t^\infty e^{-\beta s}ds$, we have

$$\begin{aligned}\int_{E_0} Bf d\nu &= c\beta \int_E \int_0^\infty Af_1(x)e^{-\beta t}d\nu_t dt \\ &\quad + c\beta \int_E f_1 d\nu_0 - c\beta^2 \int_E \int_0^\infty f_1 e^{-\beta t} d\nu_t dt \\ &= c\beta^2 \int_0^\infty e^{-\beta t}[\int_0^t \int_E (Af_1 d\nu_s)ds \\ &\quad + \int_E f_1 d\nu_0 - \int_E f_1 d\nu_t]dt \\ &= 0.\end{aligned}$$

where $c = \frac{1}{2}(f_2(-1) + f_2(1))$. The last equality follows from (3.1.32) . Clearly this holds for all $f \in \mathcal{D}(B)$. Now B satisfies conditions of Theorem 2.2.3 . Hence ν is an invariant measure for the Markov process characterised by B. We now claim that there exists only one stationary distribution for B.

Let $\{\gamma_t\}$ be the one dimensional distributions for the solution of the $D([0,\infty), E_0)$-martingale problem for $(B, \nu_0 \times \delta_1)$. Let (Y, V) be any other solution to the $D([0,\infty), E_0)$-martingale problem for B. Define $\sigma_0 = \inf\{t > 0 : V(t) = -1\}$ and $\sigma_1 = \inf\{t > 0 : V(t) = 1\}$. Note that $\mathcal{L}((Y(\sigma_1), V(\sigma_1))) = \nu_0 \times \delta_1$. Hence using strong Markov property (Theorem 1.1.4) we get

$$P\{(Y(t), V(t)) \in \Gamma\} = P\{(Y(t), V(t)) \in \Gamma, t < \sigma_1\} + \mathbb{E}[\gamma_{t-\sigma_1}(\Gamma)I_{\{\sigma_1 \leq t\}}].$$

Hence

$$\begin{aligned}\lim_{t\to\infty} t^{-1}\int_0^t P(Y(s),V(s)\in\Gamma)ds &= \lim_{t\to\infty} t^{-1}\int_0^t \Big[P\{(Y(s),V(s))\in\Gamma, s<\sigma_1\} \\ &\qquad + E[\gamma_{s-\sigma_1}(\Gamma)I_{\{\sigma_1\le s\}}]\Big]ds \\ &= \lim_{t\to\infty} t^{-1}\int_0^t E[\gamma_{s-\sigma_1}(\Gamma)I_{\{\sigma_1\le s\}}]ds \\ &= \lim_{t\to\infty} t^{-1}\int_0^t \gamma_s(\Gamma)ds.\end{aligned}$$

Thus if (Y,V) is a stationary solution we get

$$P(Y(s),V(s)\in\Gamma) = \lim_{t\to\infty} t^{-1}\int_0^t \gamma_u(\Gamma)du \ \forall\ s.$$

Hence uniqueness of stationary distribution follows.

Thus if $(\mu_t)_{t\ge 0}\subset \mathcal{P}(E)$ satisfy (3.1.31) – (3.1.32) and μ is defined by (3.1.41) with ν_t replaced by μ_t, it is a stationary distribution and uniqueness implies

$$\int_0^\infty e^{-\beta t}\nu_t dt = \int_0^\infty e^{-\beta t}\mu_t dt.$$

Since $\beta > 0$ was arbitrary, we get $\nu_t = \mu_t\ \forall\ t\ge 0$.

∎

Theorem 3.1.4 *Suppose A is an operator on $C_b(E)$ such that $\mathcal{D}(A)$ is an algebra that separates points in E and vanishes nowhere. Suppose conditions (I) and (II) are satisfied. Suppose that the $D([0,\infty),E)$-martingale problem for (A,δ_x) is well - posed for every $x\in E$.*

Let $\lambda\in C_b(E)$. If $\{\mu_t\}_{t\ge 0}\subset \mathcal{M}(E)$ and $\{\nu_t\}_{t\ge 0}\subset \mathcal{M}(E)$ satisfy (3.1.6) and (3.1.7) with $\mu_0=\nu_0$, then $\mu_t=\nu_t$ for all $t\ge 0$.

Proof. As remarked earlier we will assume that $\lambda\ge 0$. Define E^Δ, A^Δ, C as before. Clearly $\mathcal{D}(A^\Delta + C)$ is an algebra that separates points in E^Δ and vanishes nowhere. It is easy to see that $A^\Delta + C$ satisfies (I). Existence of solution to the $D([0,\infty),E)$-martingale problem for $(A^\Delta + C,\delta_x)$ for every $x\in E^\Delta$, follows from Theorem 1.1.8. And Theorem 3.2.2 implies that $A^\Delta + C$ satisfies condition (II). Hence $A^\Delta + C$ satisfies the condition of Theorem 2.2.3.

Hence, applying Theorem 3.1.3 for the operator $A^\Delta + C$, we get uniqueness of solution to the measure valued equation

$$\int_{E^\Delta} f d\gamma_t = \int_{E^\Delta} f d\tilde{\nu}_0 + \int_0^t (\int_{E^\Delta} (A^\Delta + C) f d\gamma_s) ds. \tag{3.1.42}$$

Let $\{\nu_t\}_{t\geq 0}$ be a solution to (3.1.6) (satisfying (3.1.31)). Since $1 \in \mathcal{D}(A)$ with $A1 = 0$, we get

$$\nu_t(E) = \nu_0(E) - \int_0^t \int_E \lambda d\nu_s ds$$

and hence $\nu_t(E) \leq 1$. Set $\tilde{\nu}_t(U) = \nu_t(U \cap E) + (1 - \nu_t(E)) I_U(\Delta)$ for U Borel in E^Δ. Then it is easy to see from (3.1.6) that for $f \in \mathcal{D}(A^\Delta + C)$,

$$\begin{aligned}\int_{E^\Delta} f d\tilde{\nu}_t - \int_{E^\Delta} f d\tilde{\nu}_0 - \int_0^t (\int_{E^\Delta} (A^\Delta + C) f d\tilde{\nu}_s) ds \\ = f(\Delta)[(1 - \nu_t(E)) - \int_0^t \int_E \lambda d\nu_s ds] = 0.\end{aligned}$$

Hence $\tilde{\nu}_t$ is a solution to (3.1.42) . Thus, uniqueness of solution to (3.1.42) implies the required uniqueness of solution to (3.1.6) . ∎

3.2 The Time Dependent Case

We have the following version of Theorem 3.1.4 in the time inhomogeneous case. See section 1.3 for the relavent definitions and terminology. Let $\lambda \in C_b(E^0)$.

Theorem 3.2.1 *Suppose that the operator A^0 defined by (1.1.30) satisfies the conditions of Theorem 3.1.4.*

If $\{\nu_t\} \subset \mathcal{M}(E)$ and $\{\mu_t\} \subset \mathcal{M}(E)$ satisfy (3.1.7) and

$$\int_E f(x)\rho_t(dx) = \int_E f(x)\rho_0(dx) + \int_0^t (\int_E (A_s f(x) - \lambda_s(x) f(x))\rho_s(dx)) ds \quad f \in \mathcal{D} \tag{3.2.1}$$

with $\nu_0 = \mu_0$, then $\nu_t = \mu_t$ for all $t \geq 0$.

Proof. We will use Theorem 3.1.4. Let $(\nu_t)_{t\geq 0}$ be a solution to (3.2.1) and define $(\nu_t^0)_{t\geq 0}$ by $\nu_t^0 = \delta_t \times \nu_t$. Note that, for $f \in \mathcal{D}, h \in C_0^1([0,\infty))$,

$$\int_{E^0} f h d\nu_t^0 = h(t) \int_E f d\nu_t.$$

Hence

$$\begin{aligned}\int_{E^0} fh d\nu_t^0 &= \int_{E^0} fh d\nu_0^0 + \int_0^t \frac{\partial}{\partial s}[h(s)\int_E f d\nu_s]ds \\ &= \int_{E^0} fh d\nu_0 + \int_0^t (\int_{E^0}(A^0 fh - \lambda(\cdot,\cdot)fh)d\nu_s^0)ds. \end{aligned} \quad (3.2.2)$$

It is clear from the linearity of A^0 that (3.2.1) holds for all $g \in \mathcal{D}'$. Now applying Theorem 3.1.4 we get that ν_t^0 is uniquely determined by $\nu_0^0 = \delta_0 \times \nu_0$. It follows that ν_t is uniquely determined by ν_0. ∎

3.3 Application to filtering theory

In this section, we will give an application of the results in the previous section to filtering theory. We recall here briefly, the white noise model of filtering.

Suppose that the signal process (*i.e.* the process of interest) $(X(t))_{t\geq 0}$ is a Markov process and that $(X(t))_{t\geq 0}$ is not directly observable. Instead, one can observe a function $h_t(X(t))$ of the signal corrupted by additive noise $(e(t))_{t\geq 0}$ - assumed to be *white noise* . In other words the observation process $(y(t))_{t\geq 0}$ is

$$y(t) = h_t(X(t)) + e(t) \quad (3.3.1)$$

where $\mathcal{H}$ is a separable Hilbert space, $h : [0,T] \times E \longrightarrow \mathcal{H}$ is a measurable function such that $\int_0^T \|h_s(X(s))\|^2 ds < \infty$ and $(e(t))_{t\geq 0}$ is $\mathcal{H}$ valued white noise. The norm in $\mathcal{H}$ is denoted by $\|\cdot\|$ and the innerproduct by $(\cdot,\cdot)$. In the framework of countably additive probability theory, white noise $(e(t))_{t\geq 0}$ does not exist as a process and to formalise this model one has to proceed differently. (See [11, appendix and references]).

However on a finitely additive probability space, one can construct white noise $(e(t))_{t\geq 0}$ and then the model (3.3.1) can be given a formal meaning. The sample space for $(e(t))_{t\geq 0}$ and $(y(t))_{t\geq 0}$ is $L^2([0,T],\mathcal{H})$. The quantity of interest in the filtering theory is the conditional distribution $F_t(y)$ of $(X(t))$ given $(y_s : 0 \leq s \leq t)$, *i.e.*

$$F_t(y)(B) = E[I_B(X(t)) \mid y_s : 0 \leq t]$$

for $B \in \mathcal{E}$. We now state a result from [11, p. 363-366] . For the meaning of conditional expectation in this setup and related matters, we refer the reader to chapter 6 in the reference cited above. This result is also given in [10] .

Let $c_s^y(x) := (h_s(x), y_s) - \frac{1}{2}\|h_s(x)\|$. Then

$$F_t(y)(B) = \Gamma_t(y)(B).[\Gamma_t(y)(E)]^{-1}$$

where

$$\Gamma_t(y)(B) = E[I_B(X(t))\exp(\int_0^t c_s^y(X(s))ds)]. \qquad (3.3.2)$$

$\Gamma_t(y)$ is called the unnormalised conditional distribution of $(X(t))$ given $(y_s : 0 \leq s \leq t)$. We can now deduce the following result from Theorem 3.2.1.

Theorem 3.3.1 *Suppose that the signal process $(X(t))_{t\geq 0}$ is the unique solution to the martingale problem for $((A_t)_{t\geq 0}, \nu)$ where $(A_t)_{t\geq 0}$ is as in section 3.2. Suppose that the operator A^0 defined by (1.1.30) with domain $\mathcal{D}'$ satisfies the conditions of Theorem 3.1.4.*

Also suppose that h is a bounded continuous function. Then for all $y \in C([0,T], \mathcal{H})$ the unnormalised conditional distribution $\Gamma_t(y)$ is the unique solution to the equation

$$\langle g, \Gamma_t(y)\rangle = \langle g, \nu\rangle + \int_0^T \langle A_s g + c_s^y g, \Gamma_s(y)\rangle ds \quad g \in \mathcal{D}. \qquad (3.3.3)$$

We can equivalently state the above conclusion as $\Gamma_t(y)$ is the unique solution to the equation

$$\langle f(t,.), \Gamma_t(y)\rangle = \langle f(0,.), \nu\rangle + \int_0^T \langle (A^0 f)(s,.) + c_s^y(.)f(s,.), \Gamma_s(y)\rangle ds \qquad (3.3.4)$$

$f \in \mathcal{D}'$.

It may be noted that in [11], $\Gamma_t(y)$ has been characterised as the unique solution to (3.3.4) , with A^0 replaced by the generator L of the Markov process $(t, X(t))$ and $\mathcal{D}'$ replaced by the domain $\mathcal{D}_L$ of L. In that case, h is not required to be bounded.

Though Theorem 3.3.1 requires h to be bounded, for $y \in C([0,T], \mathcal{H})$, it yields $\Gamma_t(y)$ as the unique solution to (3.3.3) or equivalently (3.3.4) . This is a significant improvement since $\mathcal{D}_L$ can be very large and we have no control over it whereas we can choose $\mathcal{D}'$ and in most cases we can choose it to be much smaller. When the signal process is an infinite dimensional diffusion (as in example 2 of section 1.3), $\mathcal{D}$ can be taken to consisit of cylinder functions, *i.e.* functions depending upon finitely many coordinates, but $\mathcal{D}_L$ will contain functions which are not cylinder functions.

Even though Theorem 3.3.1 gives a charachterisation of $\Gamma_t(y)$ for $y \in C([0,T],\mathcal{H})$, it is enough because it is known that $y \longrightarrow \Gamma_t(y)$ is Lipschitz continuous (see [11, p 479]) and $C([0,T],\mathcal{H})$ is dense in $L^2([0,T],\mathcal{H})$.

Chapter 4

Some Results On Weak Convergence

4.1 The Results

It is known that if X (X^n) is an E-valued process which is a solution to a well posed martingale problem for A (respectively A_n) and if A_n converges to A in the following sense: for all f in domain of A, there exists f_n in domain of A_n such that $f_n \to f$ and $A_n f_n \to Af$ uniformly; then the following conditions are sufficient to give us weak convergence of X_n to X (see [7, p.236]).

(i) *Domain A is an algebra that separates points in E.*

(ii) *For every $\epsilon > 0$ and $T < \infty$, $\exists$ compact set K in E such that*

$$\inf_{n \geq 1} P(X_n(t) \in K, \text{ for all } t, 0 \leq t \leq T) \geq 1 - \epsilon.$$

The latter condition is known as the *Compact containment condition.* When the underlying space E is not locally compact, *e.g.* an infinite dimensional linear space, this may be difficult to verify. This condition also appears in a recent article ([6]) by X. Fernique where he studies the weak convergence of processes taking values in infinite dimensional spaces.

Using the imbedding used in Chapter 3 we are able to replace the compact containment conditon by the tightness of one dimensional marginals and the condition that the martingale problem for A is well - posed in the class of progressively measurable solutions.

Recall that $\mathcal{U}$ is said to *strongly separate points* if for $x_n, x \in E$ $f(x_n) \to f(x)$ for all $f \in \mathcal{U}$ implies $x_n \to x$.

We will consider the time inhomogenous case. The corresponding martingale problem has been defined in section 1.3 and the related terminology developed there. In this context the *separability* condition reads as follows. Here, $(f, A_\bullet f)$ stands for the function $(t,x) \to (f(x), [A_t f](x))$, which is the *graph* of (A_t).

$(I)'$ There exists $\{g_k : k \geq 1\} \subset \mathcal{D}$ such that $\{(f, A_\bullet f) : f \in \mathcal{D}\}$ is contained in the bp-closure of $\{(g_k, A_\bullet g_k) : k \geq 1\}$.

We imbed the martingale problem into a compact space as before.

Let $\{g_k\}$ be as in $(I)'$ above and let $\|g_k\| = a_k$. Let $\hat{E} = \prod_{k=1}^{\infty}[-a_k, a_k]$ and $\underline{g} : E \to \hat{E}$ be defined by

$$\underline{g}(x) = (g_1(x), ..., g_k(x), ...) \tag{4.1.1}$$

Since $\mathcal{D}$ separates points, it follows from $(I)'$ that $\{g_k\}$ also separates points and hence $\underline{g}$ is a one to one mapping. Thus $\underline{g}^{-1}$ is well-defined on $\underline{g}(E)$. We define $\underline{g}^{-1}$ on all of $\hat{E}$ by defining $\underline{g}^{-1}(z) = e$ for $z \notin \underline{g}(E)$ where e is a fixed point in E. Note that $\underline{g}(E)$ is a Borel subset of $\hat{E}$ (see [14, Corollary I.3.3]). Hence $\underline{g}^{-1}$ is a measurable mapping.

Define operators $\mathcal{A}_t$, $t \geq 0$ with common domain $\mathcal{U}$ as follows. Let $\mathcal{U}$ be the algebra generated by

$$\{u_k \in C(\hat{E}) : u_k((z_1, ..., z_k, ...)) = z_k\} \tag{4.1.2}$$

and

$$\mathcal{A}_t(cu_{i_1}u_{i_2}...u_{i_k})(z) = \begin{cases} cA_t g_{i_1} g_{i_2} ... g_{i_k}(x) & \text{if } z = \underline{g}(x) \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.3}$$

Note that

$$u_k(\underline{g}(x)) = g_k(x) \quad \text{and} \quad \mathcal{A}_t u_k(\underline{g}(x)) = A_t g_k(x).$$

We state the analogue of Lemma 3.1.1.

Lemma 4.1.1 *Let $(X(t))$ be a progressively measurable solution to the martingale problem for (A_t) and let*

$$Z(t) = \underline{g}(X(t)) \quad \forall \ t \geq 0. \tag{4.1.4}$$

Then $(Z(t))$ is a solution to the $(\mathcal{A}_t)$ martingale problem. Conversely, if Z is a progressively measurable solution to the martingale problem for $\mathcal{A}_t$ with

$$P(Z(t) \in \underline{g}(E)) = 1 \quad \forall \ t \geq 0 \tag{4.1.5}$$

then

$$X(t) = \underline{g}^{-1}(Z(t))$$

defines a progressively measurable solution to the martingale problem for A_t. Thus if the martingale problem for (A_t) is well - posed, then there exists a unique solution Z to the martingale problem for (A_t) satisfying (4.1.5) .

The proof is similar to that of Lemma 3.1.1.

Theorem 4.1.2 *Suppose that A satisfies the separability condition $(I)'$. Suppose that $\mathcal{D}$ is an algebra that separates points in E and vanishes nowhere, that there exists a countable subset of $\mathcal{D}$ which strongly separates points and that the martingale problem for $((A_t)_{t\geq 0}, \mu)$ admits an r.c.l.l. solution, X. Also suppose that the martingale problem for $(A_t)_{t\geq 0}$ is well posed in the class of progressively measurable solutions.*

Let $X_n \in D([0,\infty), E)$ be a sequence of processes. Suppose that

(III) $\forall\ f \in \mathcal{D}, 0 \leq t_1 < \ldots < t_{m+1}, h_1, \ldots h_m \in C_b(E), m \geq 1$, we have

$$\lim_{n\to\infty} \mathbb{E}[(f(X_n(r)) - f(X_n(t)) - \int_t^r A_s f(X_n(s))ds) \prod_{i=1}^m h_i(X_n(t_i))] = 0.$$

Suppose $\{X_n(t) : n \geq 1\}$ is tight for all $t \geq 0$ and $X_n(0) \xrightarrow{\mathcal{L}} \mu$. If for every $f \in \mathcal{D}$ there exist progressively measurable processes ξ_n, ϕ_n satisfying, for every $T < \infty$

$$\xi_n(t) - \int_0^t \phi_n(s)ds \text{ is a martingale} \tag{4.1.6}$$

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in\mathbb{Q}\cap[0,T]} |\xi_n(t) - f(X_n(t)|] = 0 \tag{4.1.7}$$

$$\sup_{n\geq 1} \mathbb{E}[\|\phi_n\|_{p,T}] < \infty \text{ for some } p \in (1,\infty] \tag{4.1.8}$$

then $X_n \xrightarrow{\mathcal{L}} X$.

Proof. Let $\{g_k\}_{k\geq 1}$ be as in $(I)'$. Without loss of generality assume that $\{g_k\}_{k\geq 1}$ strongly separates points. Let $\hat{E}$ and $\underline{g}$ be as in Lemma 4.1.1 above. Since $\{g_k\}$ strongly separates points, $\underline{g}$ is a homeomorphism onto $\underline{g}(E)$, *i.e.* $\underline{g}$ is a one to one continuous function and $\underline{g}^{-1} : \underline{g}(E) \to E$ is also continuous.

Let Z_n and Z be defined by $Z_n = \underline{g}(X_n)$, $Z = \underline{g}(X)$. Then by Lemma 4.1.1 we get that Z is a solution to the martingale problem for $(\mathcal{A}_t)_{t\geq 0}$.

The fact that $\mathcal{D}$ is an algebra along with (4.1.6) - (4.1.8) and Theorem 1.2.3 implies that $Z_n = \underline{g}(X_n)$ is tight in $D([0,\infty),\hat{E})$. Let $Z_{n_k} \xrightarrow{\mathcal{L}} \tilde{Z}$. Then arguing as in (3.1.5) , for $f \in \mathcal{U}, \underline{h_i} \in C(\hat{E})$, defining $f = u \circ \underline{g}$, $h_i = \underline{h_i} \circ \underline{g}$, and using condition (III), we get

$$\begin{aligned}
&I\!E[(u(\tilde{Z}(r)) - u(\tilde{Z}(t)) - \int_t^r \mathcal{A}_s u(\tilde{Z}(s))ds) \prod_{i=1}^m \underline{h_i}(\tilde{Z}(t_i))] \\
&= \lim_{k\to\infty} I\!E[(u(\tilde{Z}_{n_k}(r)) - u(\tilde{Z}_{n_k}(t)) - \int_t^r \mathcal{A}_s u(\tilde{Z}_{n_k}(s))ds) \prod_{i=1}^m \underline{h_i}(\tilde{Z}_{n_k}(t_i))] \\
&= \lim_{k\to\infty} I\!E[(f(X_{n_k}(r)) - f(X_{n_k}(t)) - \int_t^r \mathcal{A}_s f(X_{n_k}(s))ds) \prod_{i=1}^m h_i(X_{n_k}(t_i))] \\
&= 0. \qquad (4.1.9)
\end{aligned}$$

Hence $\tilde{Z}$ is a solution to the martingale problem for $(\mathcal{A}_t)_{t\geq 0}$.

For $u \in \mathcal{U}$ since $u^2 \in \mathcal{U}$, it can be shown that for the martingale

$$M^u(t) := u(\tilde{Z}(t)) - \int_0^t \mathcal{A}_s u(\tilde{Z}(s))ds$$

the predictable quadratic variation process is given by

$$\langle M^u, M^u\rangle(t) = \int_0^t (\mathcal{A}_s u^2 - 2u\mathcal{A}_s u)(\tilde{Z}(s))ds.$$

(See [9]). The latter is a continuous process implying that $(M^u(t))_{t\geq 0}$ and hence $(u(\tilde{Z}(t)))_{t\geq 0}$ is continuous in probability. In particular $(u(\tilde{Z}(t)))_{t\geq 0}$ has no fixed points of discontinuity for all $u \in \mathcal{U}$. Thus $(\tilde{Z}(t))_{t\geq 0}$ cannot have any fixed points of discontinuity and hence for every t

$$\tilde{Z}_n(t) \xrightarrow{\mathcal{L}} \tilde{Z}(t). \qquad (4.1.10)$$

Since $\{X_n(t) : n \geq 1\}$ is tight, for every $\varepsilon > 0$, there exists a compact set $K_t \subset E$ such that for all $n \geq 1$,

$$P\{X_n(t) \in K_t\} \geq 1 - \varepsilon.$$

Recalling that $Z_n(t) = \underline{g}(X_n(t))$, we get

$$P\{Z_n(t) \in \underline{g}(K_t)\} \geq 1 - \varepsilon, \ \ \forall\, n \geq 1. \qquad (4.1.11)$$

Since $\underline{g}(K_t)$ is compact in $\underline{g}(E)$, (4.1.10) and (4.1.11) imply

$$P\{\tilde{Z}(t) \in \underline{g}(E)\} = 1 \quad \forall\, t \geq 0. \tag{4.1.12}$$

Thus Z as well $\tilde{Z}$ are solutions to the martingale problem for $((\mathcal{A}_t)_{t\geq 0}, \mu \circ \underline{g}^{-1})$ satisfying (4.1.5) and hence have the same law by Lemma 4.1.1. So we have

$$\underline{g}(X_n) \xrightarrow{\mathcal{L}} \underline{g}(X). \tag{4.1.13}$$

Since $\underline{g}$ is an homeomorphism onto $\underline{g}(E)$, this implies $X_n \xrightarrow{\mathcal{L}} X$. ∎

Remark 4.1.1 : In many cases, a choice of ξ_n, ϕ_n to try is

$$\xi_n(t) = \epsilon_n{}^{-1} \int_0^{\epsilon_n} \mathbb{E}[f(X_n(t+s))|\sigma(X_n(u) : u \leq t)]ds \tag{4.1.14}$$

$$\phi_n(t) = \epsilon_n{}^{-1}\mathbb{E}[f(X_n(t+\epsilon_n)) - f(X_n(t)|\sigma(X_n(u) : u \leq t)] \tag{4.1.15}$$

for a sequence $\epsilon_n \to 0$. It can be verified that here

$$\xi_n(t) - \int_0^t \phi_n(s)ds$$

is a martingale. See [7, p.227] .

Remark 4.1.2 : If ξ_n, ϕ_n in Theorem 4.1.2 can be chosen to satisfy

$$\lim_{n\to\infty} \mathbb{E}|A_t f(X_n(t)) - \phi_n(t)| = 0 \tag{4.1.16}$$

then it can be easily seen that the sequence $\{X_n\}$ satisfies (III). And if it is given that the finite dimensional distributions of the process $X_n(t)$ converge to those of $X(t)$, then again it can be shown that (III) holds. See [7, p.235] .

In Theorem 4.1.2 above, if $\{g_k\}$ does not strongly separate points, then $\underline{g}$ is no longer a homeomorphism onto its range and we cannot conclude that $X_n \xrightarrow{\mathcal{L}} X$ in $D([0,\infty), E)$. However we still get convergence of finite dimensional distributions of X_n to those of X. This is our next result.

Theorem 4.1.3 *Suppose A satisfies condition $(I)'$. Suppose that $\mathcal{D}$ is an algebra that separates points in E and vanishes nowhere. Also suppose that the martingale problem for $(A_t)_{t\geq 0}$ is well - posed in the class of progressively measurable solutions. Let X_n, X be E - valued progressively measurable processes* *such that $X_n(0) \xrightarrow{\mathcal{L}} X(0)$. Assume $\{X_n(t) : n \geq 1\}$ is tight for all $t \geq 0$ and that X_n satisfy condition (III). If for every $f \in \mathcal{D}$, there exist progressively measurable real valued processes ξ_n, ϕ_n satisfying (4.1.6) - (4.1.8) then the finite dimensional distributions of X_n converge to those of X.*

Proof. Proceeding exactly as in the proof of Theorem 4.1.2 we get $\underline{g}(X_n) \xrightarrow{\mathcal{L}} \underline{g}(X)$. Now (4.1.10) implies that

$$(\underline{g}(X_n(t_1)), \ldots, \underline{g}(X_n(t_j))) \xrightarrow{\mathcal{L}} (\underline{g}(X(t_1)), \ldots, \underline{g}(X(t_j))) \tag{4.1.17}$$

for all $t_1, \ldots, t_j$, for all j. Since $\{X_n(t) : n \geq 1\}$ is tight and $\mathcal{D}$ is a measure determining class, (4.1.17) implies

$$(X_n(t_1), \ldots, X_n(t_j)) \xrightarrow{\mathcal{L}} (X(t_1), \ldots, X(t_j)).$$

∎

The following theorem gives conditions when the processes X_n are solutions to a martingale problem for $(A_t^n)_{t\geq 0}$.

Theorem 4.1.4 *Suppose that A satisfies the separability condition $(I)'$, that $\mathcal{D}$ is an algebra that separates points in E and vanishes nowhere and that there exists a countable subset of $\mathcal{D}$ which strongly separates points. Also suppose that the martingale problem for (A_t) is well posed in the class of progressively measurable solutions.*

For $n \geq 1$ let $(A_t^n)_{t\geq 0}$ be operators on $C_b(E)$ with domain $\mathcal{D}_n$. Let $X_n, X \in D([0,\infty), E)$ be solutions to the martingale problem for (A_t^n), (A_t) respectively with $X_n(0) \xrightarrow{\mathcal{L}} X(0)$. Further suppose that $\{X_n(t) : n \geq 1\}$ is tight for all $t \geq 0$.

If for every $f \in \mathcal{D}$ there exists $f_n \in \mathcal{D}_n$ such that for every compact subset K and $T > 0$, we have for a constant C_T,

$$\|f_n - f\| \to 0 \text{ as } n \to \infty \tag{4.1.18}$$

$$\sup_{t\in[0,T]} \|A_t^n f_n\| \leq C_T \quad \forall\, n \geq 1 \tag{4.1.19}$$

$$\sup_{x\in K} |A_t^n f_n(x) - A_t f(x)| \to 0 \text{ as } n \to \infty. \tag{4.1.20}$$

Then $X_n \xrightarrow{\mathcal{L}} X$ in $D([0,\infty), E)$.

Proof. Set

$$\begin{aligned}\xi_n(t) &= f_n(X_n(t)) \\ \phi_n(t) &= A_t^n f_n(X_n(t))\end{aligned}$$

Then clearly ξ_n, ϕ_n satisfy conditions (4.1.6) - (4.1.8) with say $p = 2$. It remains to show that X_n satisfy the asymptotic solution condition (III). Then we can invoke Theorem 4.1.2 to complete the proof.

Since X_n is a solution to the martingale problem for $(A_t^n)_{t\geq 0}$, we have for $g \in \mathcal{D}_n$, $0 \leq t_1 < \ldots < t_m = t < r, h_1 < \ldots h_m \in C_b(E), m \geq 1$,

$$\mathbb{E}[(g(X_n(r)) - g(X_n(t)) - \int_t^r A_s^n g(X_n(s))ds) \prod_{i=1}^m h_i(X_n(t_i))] = 0.$$

And hence for $f \in \mathcal{D}$, $f_n \in \mathcal{D}_n$ as in the statement of the theorem, we have

$$\begin{aligned}&\mathbb{E}[(f(X_n(r)) - f(X_n(t)) - \int_t^r A_s f(X_n(s))ds) \prod_{i=1}^m h_i(X_n(t_i))] \\ &= \mathbb{E}[((f - f_n)(X_n(r)) - (f - f_n)(X_n(t)) \\ &\quad - \int_t^r (A_s f - A_s^n f_n)(X_n(s))ds) \prod_{i=1}^m h_i(X_n(t_i))] \\ &\leq 2\|f_n - f\| \prod_{i=1}^m \|h_i\| + \mathbb{E} \int_t^r |(A_s f - A_s^n f_n)(X_n(s))|ds \prod_{i=1}^m h_i(X_n(t_i))\end{aligned} \tag{4.1.21}$$

By (4.1.18) the first term tends to zero as $n \to \infty$. Now,

$$\begin{aligned}&\mathbb{E}|(A_s f - A_s^n f_n)(X_n(s))| \\ &\quad \leq \sup_{x\in K} |A_s f(x) - A_s^n f_n(x)| + (\|A_s f\| + \|A_s^n f_n\|) P(X_n(s) \in K^c)\end{aligned}$$

Using (4.1.19) , (4.1.20) and the fact that $\{X_n(s) : n \geq 1\}$ is tight, the expression on the right above can be made arbitrarily small for large enough n. Hence

$$\mathbb{E}|(A_s f - A_s^n f_n)(X_n(s))| \to 0 \text{ as } n \to \infty.$$

Now an application of DCT implies that the R.H.S. in (4.1.21) goes to zero as $n \to \infty$. This completes the proof as remarked earlier. ∎

Here is a version of the preceeding result, when $\mathcal{D}$ may not strongly separate points. It can be deduced from Theorem 4.1.3.

Theorem 4.1.5 *Suppose A satisfies condition $(I)'$. Suppose that $\mathcal{D}$ is an algebra that separates points in E and vanishes nowhere. Also suppose that the martingale problem for $(A_t)_{t\geq 0}$ is well - posed in the class of progressively measurable solutions. Let $(A_t^n)_{t\geq 0}$ be operators with domain $\mathcal{D}_n$. Let X_n, X be progressively measurable solutions to the martingale problems for $(A_t^n)_{t\geq 0}$, $(A_t)_{t\geq 0}$ respectively. Suppose that $X_n(0) \xrightarrow{\mathcal{L}} X(0)$. Further, suppose that $\{X_n(t) : n \geq 1\}$ is tight for all $t \geq 0$.*

If for all $f \in \mathcal{D}$ there exist $f_n \in \mathcal{D}_n$ such that for every compact subset K and $T > 0$, (4.1.18) - (4.1.20) hold, then the finite dimensional distributions of the proceess X_n converge to those of X.

Remark 4.1.3 : The time independent version of this result was implicit in the proof of Theorem 2.2.2. There, for every $f \in \mathcal{D}$, f_n's were so chosen that $A^n f_n = Af$ for all $n \geq 1$. Further $\|f_n - f\| \to 0$ as $n \to \infty$.

4.2 Examples

In this section we will consider processes taking values in a real, separable Hilbert space H. The inner product on H will be denoted by $(\cdot,\cdot)$ and the norm by $\|\cdot\|$. $\mathcal{L}(H,H)$ (respectively $\mathcal{L}_1(H,H)$) will denote the space of linear (respectively trace-class) operators on H and for $\Sigma \in \mathcal{L}_1(H,H)$, $\|\Sigma\|_1$) will denote the trace norm of Σ. $\mathcal{L}_1^+(H,H)$ is the set of positive trace-class operators on H.

1. Donsker's invariance principle: the ∞ - dimensional case

The first example is illustrative, and it shows the power of Theorem 4.1.2 by deducing Donsker's invariance principle for Hilbert space valued random variables (See [12]) from the Central limit theorem via some simple computations.

Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. H -valued random variables with $\mathbb{E}Y_1 = 0$ and $\mathbb{E}\|Y_1\|^2 < \infty$. Then it can be shown that

$$\mathbb{E}(Y_1,\phi_1)(Y_1,\phi_2) = (\Sigma\phi_1,\phi_2)$$

for a trace class operator Σ. Let

$$X_n(t) = \frac{1}{\sqrt{n}}(Y_1 + Y_2 + \ldots + Y_k) \quad \text{for } \frac{k}{n} \le t < \frac{k+1}{n}.$$

Let $(W(t))_{t\ge 0}$ be an H -valued Wiener process with covariance operator Σ. Then we have,

Theorem 4.2.1 *$X_n \xrightarrow{\mathcal{L}} W$ in $D([0,\infty), H)$.*

Proof. Fix a CONS $\{\psi_k : k \ge 1\}$ in H and let $P_m : H \to \mathbb{R}^m$ be defined by $P_m(h) = ((h,\psi_1),\ldots,(h,\psi_m))$.

Let $\mathcal{D}$ be the algebra generated by

$$\{g \circ P_m : g \in C_0^2(\mathbb{R}^m); m \ge 1\} \cup \{u_f : u_f(h) = f(\|h\|^2), f \in C_0^2(\mathbb{R})\}.$$

Let A be an operator on H with domain $\mathcal{D}$ defined by

$$A(g \circ P_m)(h) = \frac{1}{2}\sum_{i,j=1}^{m}(\Sigma\psi_i,\psi_j)g_{ij}(P_m(h)) \tag{4.2.1}$$

$$A(u_f)(h) = f_1(\|h\|^2)\|\Sigma\|_1 + 2f_{11}(\|h\|^2)(\Sigma h, h) \tag{4.2.2}$$

and for functions of the form $F(h) = u_f(h)g \circ P_m(h)$ where $f \in C_0^2(\mathbb{R})$ and $g \in C_0^2(\mathbb{R}^m)$,

$$\begin{aligned} AF(h) = {} & u_f(h)[A(g \circ P_m)](h) + g \circ P_m(h)[Au_f](h) \\ & + 2f_1(\|h\|^2)\sum_{j=1}^{m}(\Sigma h,\psi_j)(g_j \circ P_m)(h) \end{aligned} \tag{4.2.3}$$

where $g_i = \frac{\partial}{\partial x_i}g$; $g_{ij} = \frac{\partial}{\partial x_j}g_i$. It is easy to prove that W is a solution to the martingale problem for (A,δ_0) and that the martingale problem for A is well posed in the class of all progressively measurable processes. It is clear that $\mathcal{D}$ is an algebra, that $\mathcal{D}$ strongly separates points and that A and $\mathcal{D}$ satisfy the separability condition (I). By the central limit theorem for H - valued random variables, finite dimensional distributions of X_n converge to those of W. See ([14]). Thus $X_n(t)$ is tight for each t fixed and as noted in Remark 4.1.2, the condition (III) also holds. We will now prove that for $f \in \mathcal{D}$, (ξ_n, ϕ_n) defined by (4.1.14) , (4.1.15) for $\epsilon_n = n^{-1}$ satisfy (4.1.7) and (4.1.8) for $p = 2$. Once this is proved, we could deduce from Theorem 4.1.2 that $X_n \xrightarrow{\mathcal{L}} W$ in $D([0,\infty), H)$.

An easy computation shows that here

$$\begin{aligned} \xi_n(t) &= f(X_n(t)) + (\frac{nt - [nt]}{n})e_n(X_n(t)) \\ \phi_n(t) &= e_n(X_n(t)) \end{aligned}$$

where

$$e_n(h) = n\mathbb{E}[f(h + \frac{1}{\sqrt{n}}Y_1) - f(h)].$$

Since $\|X_n(t)\|^2$ is a submartingale,

$$\mathbb{E} \sup_{0 \le t \le T} \|X_n(t)\|^2 \le 4\mathbb{E}\|X_n(T)\|^2 \le 4T\mathbb{E}\|Y_1\|^2. \qquad (4.2.4)$$

We will verify that $|e_n(h)| \le C_f(1 + \|h\|^2)$ for a constant depending on f which along with (4.2.4) will imply that (4.1.7) , (4.1.8) are satisfied for $p = \infty$.

First let $f(h) = g((h,\psi_1),\ldots,(h,\psi_m))$ for $g \in C_0^2(\mathbb{R}^m)$. Using Taylor's expansion, we get

$$\begin{aligned} e_n(h) &= n\sum_{i=1}^m g_i(P_m(h))\frac{1}{\sqrt{n}}\mathbb{E}(Y_1,\psi_i) \\ &+ \frac{1}{2}n\sum_{i,j=1}^m \frac{1}{n}\mathbb{E}g_{ij}(P_m(h + \eta_n\frac{1}{\sqrt{n}}Y_1))(Y_1,\psi_i)(Y_1,\psi_j) \end{aligned}$$

where $0 \le \eta_n \le 1$. Using $\mathbb{E}(Y_1,\psi_i) = 0$, we get

$$\begin{aligned} |e_n(h)| &\le (\sup_{i,j}\|g_{ij}\|)\sum_{i,j=1}^m \mathbb{E}|(Y_1,\psi_i)||(Y_1,\psi_j)| \\ &= (\sup_{i,j}\|g_{ij}\|)m\sum_{i=1}^m \mathbb{E}|(Y_1,\psi_i)|^2 \\ &\le (\sup_{i,j}\|g_{ij}\|)m\|\Sigma\|_1. \end{aligned}$$

On the other hand, if $f = g(\|h\|^2)$, using Taylor's expansion again, we get for some $c_n, c_n' \in \mathbb{R}$,

$$\begin{aligned}
e_n(h) &= n\mathbb{E}\Big\{g(\|h+\frac{1}{\sqrt{n}}Y_1\|^2)-g(\|h\|^2)\Big\}\\
&= n\mathbb{E}\Big\{g(\|h\|^2+\frac{2}{\sqrt{n}}(h,Y_1)+\frac{1}{n}\|Y_1\|^2)-g(\|h\|^2+\frac{2}{\sqrt{n}}(h,Y_1))\Big\}\\
&\quad +n\mathbb{E}\Big\{g(\|h\|^2+\frac{2}{\sqrt{n}}(h,Y_1))-g(\|h\|^2)\Big\}\\
&= n\mathbb{E}\Big\{g_1(c_n)\cdot\frac{1}{n}\|Y_1\|^2\Big\}+n\mathbb{E}\Big\{g_1(\|h\|^2)\cdot\frac{2}{\sqrt{n}}(h,Y_1)\\
&\qquad +g_{11}(c_n')(\frac{2}{\sqrt{n}}(h,Y_1))^2\Big\}\\
&\le C_1(\mathbb{E}\|Y_1\|^2+\mathbb{E}(h,Y_1)^2)\\
&\le C_2(1+\|h\|^2)
\end{aligned}$$

for suitable constants C_1, C_2. We have again used the fact that $\mathbb{E}(h,Y_1)=0$.

The case $F(h)=u_f(h)g\circ P_m(h)$ can be deduced from the above analysis by noting that

$$\begin{aligned}
e_n(h) &= n\mathbb{E}\Big[f(\|h+\frac{1}{\sqrt{n}}Y_1\|^2)g\circ P_m(h+\frac{1}{\sqrt{n}}Y_1)-f(\|h\|^2)g\circ P_m(h)\Big]\\
&\le n\mathbb{E}\Big[\|f\|(g\circ P_m(h+\frac{1}{\sqrt{n}}Y_1)-g\circ P_m(h))\\
&\quad +\|g\|(f(\|h+\frac{1}{\sqrt{n}}Y_1\|^2)-f(\|h\|^2))\Big]\\
&\le C_F(1+\|h\|^2).
\end{aligned}$$

As noted earlier, this completes the proof. ∎

2. Hilbert space valued diffusions

We will now show that an Hilbert space valued diffusion depends continuously on its drift and diffusion coefficients.

Let $a:[0,\infty)\times H\to\mathcal{L}_1^+(H,H)$ and $b:[0,\infty)\times H\to H$ be bounded measurable functions, continuous in the second variable.

Fix a CONS $\{\psi_k : k\ge 1\}$ in H and let $P_m : H\to\mathbb{R}^m$ be defined by $P_m(h)=((h,\psi_1),\dots,(h,\psi_m))$. Let $\mathcal{D}$ be the algebra generated by

$$\{g\circ P_m : g\in C_0^2(\mathbb{R}^m), m\ge 1\}\cup\{u_f : u_f(h)=f(\|h\|^2), f\in C_0^2(\mathbb{R})\}.$$

For every $t \geq 0$, define an operator A_t on $\mathcal{D}$ by

$$[A_t(g \circ P_m)](h) = \sum_{i=1}^{m}(b(t,h),\psi_i)g_i(P_m(h)) + \frac{1}{2}\sum_{i,j=1}^{m}(a(t,h)\psi_i,\psi_j)g_{ij}(P_m(h)) \tag{4.2.5}$$

and

$$[A_t(u_f)](h) = f_1(\|h\|^2)[\|a(t,h)\|_1 + 2(b(t,h),h)] + 2f_{11}(\|h\|^2)(a(t,h)h,h) \tag{4.2.6}$$

and for functions of the form $F(h) = u_f(h)g \circ P_m(h)$ where $f \in C_0^2(\mathbb{R})$ and $g \in C_0^2(\mathbb{R}^m)$,

$$\begin{aligned} A_tF(h) = u_f(h)[A_t(g \circ P_m)](h) + g \circ P_m(h)[A_tu_f](h) \\ +2f_1(\|h\|^2)\sum_{j=1}^{m}(a(t,h)h,\psi_j)(g_j \circ P_m)(h) \end{aligned} \tag{4.2.7}$$

where $f_i = \frac{\partial}{\partial x_i}f$; $f_{ij} = \frac{\partial}{\partial x_j}f_i$. Then the martingale problem for $(A_t)_{t\geq 0}$ is well - posed in the class of progressively measurable processes. (This can be seen as in example 2 of section 1.3, where the time homogeneous case was considered. Here the domain, $\mathcal{D}$, is so chosen that it strongly separates points). Further every solution has a continuous modification. The solution is a Hilbert space valued diffusion as defined in [16] . Also it is easy to see that $\mathcal{D}$ satisfies the separability condition and that $\mathcal{D}$ has a countable subset that strongly separates points. And of course $\mathcal{D}$ is an algebra. Thus $(A_t)_{t\geq 0}, \mathcal{D}$ satisfy the conditions of Theorem 4.1.4. As an application of Theorem 4.1.4, we will show that the diffusion process depends continuously on a, b. Results of this type are well-known in the finite dimensional case even when the coefficients are not assumed to be bounded. (See [15]). Now, for every n, let a_n, b_n be bounded measurable functions as above and define operators $(A_t^n)_{t\geq 0}$ by (4.2.5) and (4.2.6) by replacing a, b by a_n, b_n respectively.

Theorem 4.2.2 *Assume that a_n, b_n satisfy*

$$\sup_{n\geq 1}\sup_{0\leq s\leq T}\sup_{h\in H}\|a_n(s,h)\|_1 + \|b_n(s,h)\| < \infty \tag{4.2.8}$$

$$\int_0^T \sum_{i=1}^{\infty}[\sup_{\|h\|\leq R} a^{ii}(s,h)]ds < \infty \tag{4.2.9}$$

where $a^{ii}(s,h) = (a(s,h)\xi_i, \xi_i)$ *and*

$$\lim_{n\to\infty}\int_0^T \sup_{\|h\|\le R}(\|a_n(s,h)-a(s,h)\|_1 + \|b_n(s,h)-b(s,h)\|)ds = 0 \qquad (4.2.10)$$

for all $T, R > 0$. *Let* X_n, X *be solutions to* $(A_t^n)_{t\ge 0}$, $(A_t)_{t\ge 0}$ *martingale problems respectively such that* $X_n(0) \xrightarrow{\mathcal{L}} X(0)$. *Then* $X_n \xrightarrow{\mathcal{L}} X$ *as processes in* $C([0,\infty), H)$.

Proof: For every $f \in \mathcal{D}$ taking $f_n = f$, it can be seen that (4.1.19) , (4.1.20) are satisfied. Thus if we show that $X_n(t)$ is tight for evary t, we can invoke Theorem 4.1.4, since as mentioned earlier, all the other conditions of the theorem are satisfied. We then conclude that $X_n \xrightarrow{\mathcal{L}} X$ in $D([0,\infty), H)$. Since $X_n, X \in C([0,\infty), H)$, we get the desired convergence in $C([0,\infty), H)$.

Using that X_n is a solution to the $(A_t^n)_{t\ge 0}$ martingale problem, it can be proved that

$$(X_n(t) - X_n(0), \psi_i) - \int_0^t b_n^i(s, X_n(s))ds$$

and

$$[(X_n(t) - X_n(0), \psi_i) - \int_0^t b_n^i(s, X_n(s))ds]^2 - \int_0^t a_n^{ii}(s, X_n(s))ds$$

are martingales, where $a_n^{ii} = (a_n\psi_i, \psi_i)$ and $b_n^i = (b_n, \psi_i)$. Thus,

$$\begin{aligned}\mathbb{E}\sum_{i=N}^{\infty}(X_n(t) - X_n(0), \psi_i)^2 \;\le\; & \sum_{i=N}^{\infty} 2\mathbb{E}\int_0^t a_n^{ii}(s, X_n(s))ds \\ & +2t\sum_{i=N}^{\infty}\mathbb{E}\int_0^t (b_n^i(s, X_n(s)))^2 ds. \quad (4.2.11)\end{aligned}$$

Now,

$$\begin{aligned}&\sum_{i=N}^{\infty}\mathbb{E}\int_0^t a_n^{ii}(s, X_n(s))ds \\ &\le \int_0^t \sum_{i=N}^{\infty}\sup_{\|h\|\le R} a_n^{ii}(s,h)ds \\ &\quad + \sup_{h\in H}\sup_{0\le s\le t}\|a_n(s,h)\|_1 \int_0^t P(\|X_n(s)\| > R)ds \\ &\le \int_0^t [\sum_{i=N}^{\infty}\sup_{\|h\|\le R} a^{ii}(s,h) + \sup_{\|h\|\le R}\|a_n(s,h) - a(s,h)\|_1 ds] \\ &\quad + \sup_{h\in H}\sup_{0\le s\le t}\|a_n(s,h)\|_1 \int_0^t P(\|X_n(s)\| > R)ds. \qquad (4.2.12)\end{aligned}$$

For $0 \le s \le t$

$$
\begin{aligned}
\sup_{n\ge 1} P(\|X_n(s)\| > R) &\le \sup_{n\ge 1} P(\|X_n(s) - X_n(0)\| \ge \frac{R}{2}) + \sup_{n\ge 1} P(\|X_n(0)\| \ge \frac{R}{2}) \\
&\le \frac{4}{R^2}\sup_{n\ge 1} \mathbb{E}\|X_n(s) - X_n(0)\|^2 + P(\|X_n(0)\| \ge \frac{R}{2}) \\
&\le \frac{C_t}{R^2}\sup_{n\ge 1}\{\sup_{0\le s\le t}\sup_{h\in H}\|a_n(s,h)\|_1 + \sup_{0\le s\le t}\sup_{h\in H}\|b_n(s,h)\|^2\} \\
&\qquad + \sup_{n\ge 1} P(\|X_n(0)\| \ge \frac{R}{2}) \\
&\to 0 \qquad \text{as } R \to \infty,
\end{aligned}
$$

where C_t is a constant depending only on t. Now (4.2.8) – (4.2.10) imply

$$
\lim_{N\to\infty}\limsup_{n\to\infty}\sum_{i=N}^{\infty}\mathbb{E}\int_0^t a_n^{ii}(s, X_n(s))ds = 0. \tag{4.2.13}
$$

Doing a similar analysis for the second term in (4.2.11) , we get

$$
\lim_{N\to\infty}\limsup_{n\to\infty}\mathbb{E}\sum_{i=N}^{\infty}(X_n(t) - X_n(0), \psi_i)^2 = 0 \tag{4.2.14}
$$

Also (4.2.8) implies

$$
\sup_n \mathbb{E}\|X_n(t) - X_n(0)\|^2 < \infty. \tag{4.2.15}
$$

By slightly modifying the proof of Lemma 2.2 in [14, p 157], one can show that (4.2.14) and (4.2.15) together imply that $\{X_n(t) - X_n(0)\}_{n\ge 1}$ is tight. Since $X_n(0) \xrightarrow{\mathcal{L}} X(0)$, $X_n(0)$ is tight and hence we get tightness of $\{X_n(t) : n \ge 1\}$ for each t. This completes the proof. ∎

3. Continuous dependence of solution to SEE on its coefficients

The last application is about Hilbert space valued processes arising as solutions to a Stochastic evolution equation which have been discussed in section 1.4. Here again, we will show continuous dependence of finite dimensional distributions of the solution on the coefficients.

Let L be a self-adjoint (unbounded) operator on H with dense domain. Suppose that L^{-1} is a bounded, compact operator. Let $\{\lambda_k^{-1} : k \ge 1\}$ be the eigenvalues of L^{-1} and let $\{\psi_k : k \geq 1\}$ be the corresponding eigen vectors. Note that $\{\psi_k : k \geq 1\}$ is a CONS in H. We will assume that $\lambda_k > 0$ for all k. Let $T > 0$, and

$$\sigma : [0,T] \times H \to \mathcal{L}(H,H) \tag{4.2.16}$$

$$b : [0,T] \times H \to H \tag{4.2.17}$$

be continuous functions satisfying

$$\|\sigma^*(s,h)\psi_k\| \leq \rho_k \tag{4.2.18}$$

$$(b(s,h),\psi_k) \leq \rho_k \tag{4.2.19}$$

$$\|(\sigma^*(s,h_1) - \sigma^*(s,h_2))\psi_k\| \leq \tilde{\rho}_k\|h_1 - h_2\| \tag{4.2.20}$$

$$(b(s,h_1) - b(s,h_2),\psi_k) \leq \tilde{\rho}_k\|h_1 - h_2\| \tag{4.2.21}$$

$\forall\ h, h_1, h_2 \in H, s \in [0,T], \forall\ k$ where $\rho_k, \tilde{\rho}_k$ satisfy

$$\sum_{k=1}^{\infty} \frac{(\rho_k)^2}{\lambda_k} \leq C_1, \sum_{k=1}^{\infty} \frac{(\tilde{\rho}_k)^2}{\lambda_k} < \infty. \tag{4.2.22}$$

Let $(W(t))_{t\geq 0}$ be a Cylindrical Brownian Motion on H defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Consider the evolution equation

$$dX(t) = -LX(t)dt + \sigma(t,X(t))dW(t) + b(t,X(t))dt \tag{4.2.23}$$

We have seen in section 1.4 that (4.2.23) admits a unique solution for every initial $X(0)$ which is independent of W. Further, the law of X is uniquely determined.

Let

$$\mathcal{D} = \{g : g(h) = f((h,\psi_1),\ldots,(h,\psi_m)); f \in C_0^2(\mathbb{R}^m), m \geq 1\} \tag{4.2.24}$$

and define operators $(A_t)_{t\geq 0}$ on $\mathcal{D}$ by

$$\begin{aligned}(A_t g)(h) &= \sum_{i=1}^{m}(-\lambda_i h + b(t,h),\psi_i)f_i((h,\psi_1),\ldots,(h,\psi_m)) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m}(\sigma^*(t,h)\psi_i, \sigma^*(t,h)\psi_j)f_{ij}((h,\psi_1),\ldots,(h,\psi_m))\end{aligned} \tag{4.2.25}$$

We have seen in Theorem 1.4.10 and Remark 1.4.2 that the martingale problem for $(A_t)_{t\geq 0}$ is well - posed in the class of progressively measurable processes. Note that under the above conditions their may not exist a continuous modification of the solution.

Let for $n \geq 1$, σ_n and b_n be continuous functions as in (4.2.16) and (4.2.17) respectively satisfying (4.2.18) – (4.2.21) with $\rho_k, \tilde{\rho}_k$ replaced by $\rho_k^n, \tilde{\rho}_k^n$ respectively where $\rho_k^n, \tilde{\rho}_k^n$ satisfy

$$\sum_{k=1}^{\infty} \frac{(\rho_k^n)^2}{\lambda_k} \leq C_1 \qquad \sum_{k=1}^{\infty} \frac{(\tilde{\rho}_k^n)^2}{\lambda_k} < \infty \tag{4.2.26}$$

and

$$\lim_{m\to\infty} [\sup_{n\geq 1} \sum_{k=m}^{\infty} \frac{(\rho_k^n)^2}{\lambda_k}] = 0 \tag{4.2.27}$$

Assume further that for every compact subset $K \subset H$,

$$\sup_{h\in K} \int_0^T \|[\sigma_n(s,h) - \sigma(s,h)]\psi_k\|^2 ds \to 0 \tag{4.2.28}$$

$$\sup_{h\in K} \int_0^T ([b_n(s,h) - b(s,h)], \psi_k)^2 ds \to 0 \tag{4.2.29}$$

Define operators $(A_t^n)_{t\geq 0}$ on domain $\mathcal{D}$ as in (4.2.25) with b_n and σ_n in place of b and σ respectively. Let X_n be the unique solution to the martingale problem for $((A_t^n)_{t\geq 0}, \mu_n)$ where μ_n converge weakly, say, to μ. Let X be the unique solution to the martingale problem for $((A_t)_{t\geq 0}, \mu)$. We then have

Theorem 4.2.3 *Finite dimensional distributions of the process X_n converge to the corresponding distributions of X.*

Proof: We will show that for every fixed t, $\{X_n(t) : n \geq 1\}$ is tight. As noted earlier, a set of sufficient conditions for a sequence $\{Y_n : n \geq 1\}$ of H valued random variables to be tight is

$$\lim_{m\to\infty} \sup_{n\geq 1} \mathbb{E} \sum_{i=m}^{\infty} (Y_n, \psi_i)^2 = 0 \tag{4.2.30}$$

and

$$\sup_{n\geq 1} \mathbb{E}\|Y_n\|^2 < \infty. \tag{4.2.31}$$

Here, it is easy to see that

$$(X_n(t), \psi_i) = e^{-\lambda_i t}(X_n(0), \psi_i) + \int_0^t \langle e^{-\lambda_i(t-s)}(\sigma_n^*(s, X_n(s))\psi_i, dW(s)\rangle + \int_0^t e^{-\lambda_i(t-s)}(b(s, X_n(s)), \psi_i)ds \quad (4.2.32)$$

Since $X_n(0) \xrightarrow{\mathcal{L}} X(0)$ the first term in (4.2.32) is tight. We will verify (4.2.30) for the last two terms in (4.2.32) . Using (4.2.18) and (4.2.19) , we get

$$\begin{aligned} \mathbb{E}[\int_0^t \langle e^{-\lambda_i(t-s)}\sigma_n^*(s, X_n(s))\psi_i, dW(s)\rangle]^2 &= \mathbb{E}\int_0^t e^{-2\lambda_i(t-s)}(\sigma_n^*(s, X_n(s))\psi_i, \sigma_n^*(s, X_n(s))\psi_i)ds \\ &\le \int_0^t e^{-2\lambda_i(t-s)}(\rho_i^n)^2 ds \\ &\le \frac{(\rho_i^n)^2}{2\lambda_i} \end{aligned} \quad (4.2.33)$$

and

$$\begin{aligned} \mathbb{E}[\int_0^t e^{-\lambda_i(t-s)}(b(s, X_n(s)), \psi_i)ds]^2 &\le 2T\mathbb{E}\int_0^t e^{-2\lambda_i(t-s)}(b(s, X_n(s)), \psi_i)^2 ds \\ &\le 2T\frac{(\rho_i^n)^2}{2\lambda_i}. \end{aligned} \quad (4.2.34)$$

From (4.2.27) and (4.2.33) – (4.2.34) it is clear that (4.2.30) is satisfied for each of the last two terms in (4.2.32) .Similarly we can verify that (4.2.31) holds. Hence we get tightness of $\{X_n(t) : n \ge 1\}$.

In order to apply Theorem 4.1.5, remains to show that for every $f \in \mathcal{D}$, we can get $f^n \in \mathcal{D}_n$ such that (4.1.18) – (4.1.20) is satisfied. It is easy to see that the rest of the conditions of that theorem are satisfied.

Let $f^n \equiv f$. This takes care of (4.1.18) . Also using bounds on a_n and b_n and the asumption (4.2.26) one has

$$\begin{aligned} \|A_t^n f^n\| &\le \sum_{i=1}^m (\lambda_i + \rho_i^n)\|f_i\| + \frac{1}{2}\sum_{i,j=1}^m (\rho_i^n)^2\|f_{ij}\| \\ &\le \sum_{i=1}^m (\lambda_i + \sqrt{C_1\lambda_i})\|f_i\| + \frac{1}{2}\sum_{i,j=1}^m (C_1\lambda_i\|f_{ij}\| \end{aligned}$$

and clearly (4.1.19) is satisfied, while (4.1.20) follows immediately from (4.2.28) and (4.2.29) .

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