On Lipschitzian, *INS* and Connected Matrices in Linear Complementarity Problem

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To
my mother

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CONTENTS

GLOSSARY OF NOTATION

Sets

$\bar{n}$	the set $\{1, 2, \ldots, n\}$, n is any positive integer
n^*	all nonempty subsets of $\bar{n}$
$\lvert\alpha\rvert$	cardinality of the set α
α, β, γ	denote the subsets of $\bar{n}$
α	complement of the set α relative to $\bar{n}$
$\alpha \setminus \beta$	the set $\alpha \cap \beta^c$
$\alpha \triangle \beta$	the set $(\alpha \cup \beta) \setminus (\alpha \cap \beta)$

Spaces

$\boldsymbol{R}^n$	real n-dimentional space
$\boldsymbol{R}^{m \times n}$	the space of $m \times n$ real matrices
$\boldsymbol{R}^n_+$	the nonnegative orthant of $\boldsymbol{R}^n$
$\boldsymbol{R}^n_{++}$	the positive orthant

GLOSSARY OF NOTATION (Cont.)

Vectors

z^t	transpose of z
$x^t y$	the standard inner product between x and y
$x \geq y$	$x_i \geq y_i$ for all $i \in \bar{n}$
$x > y$	$x_i > y_i$ for all $i \in \bar{n}$
$x \succ y$	x is lexicographically greater than y
$supp(z)$	support of z, i.e., $\{i \in \bar{n} : z_i \neq 0\}$
probability vector	a nonnegative vector with sum of its coordinates equal to one

Matrices

$A = (a_{ij})$	a matrix with a_{ij}s as its entries.
$A \leq B$	$a_{ij} \leq b_{ij}$ for all i, j

Matrices

$A < B$	$a_{ij} < b_{ij}$ for all $i,\ j$
$det\ A$	determinant of matrix A
I	the identity matrix
$A_{\alpha\beta}$	the submatrix of A obtained by dropping rows and columns of A corresponding to $\bar{\alpha}$ and $\bar{\beta}$ respectively
$A_{\alpha.}$	stands for $A_{\alpha\bar{n}}$, $A \in \boldsymbol{R}^{m \times n}$
$A_{.\beta}$	stands for $A_{\bar{n}\beta}$, $A \in \boldsymbol{R}^{m \times n}$
$A_{i.}$	i^{th} row of A
$A_{.j}$	j^{th} column of A
$v(A)$	value of (matrix game) A
$\wp_\alpha(A)$	principal pivotal transform of A with resepect to α

Sign Symbols

$\ominus$	nonpositive real
$\oplus$	nonnegative real

LCP Notations

(q, A)	LCP with data q and A
$F(q, A)$	set of all feasible solutions of (q, A)
$S(q, A)$	set of all solutions of (q, A)
$K(A)$	set of all q such that $S(q, A) \neq \emptyset$
$C_A(\alpha)$	complementary submatrix, $\alpha \subseteq \bar{n}$
pos A	cone generated by columns of A, $\{Ax : x \geq 0\}$

MATRIX CLASSES

Symbol	Definition
$\boldsymbol{C}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : x^tAx \geq 0 \ \forall x \in \boldsymbol{R}^n_+\}$
$\boldsymbol{C}^f_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in \bar{n}, det\ A_{\alpha\alpha} \neq 0 \Rightarrow \wp_\alpha(A) \in \boldsymbol{C}_o\}$
$\boldsymbol{C}^+_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C}_o : [x^tAx = 0,\ x \in \boldsymbol{R}^n_+] \Rightarrow (A + A^t)x = 0\}$
$\boldsymbol{D}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, det\ A_{\alpha\alpha} = 0 \Rightarrow$ columns of $A_{.\alpha}$ are linearly dependent $\}$
$\boldsymbol{E}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\ 0 \neq x \in \boldsymbol{R}^n_+\ \exists\ k \in \bar{n} \ni x_k > 0$ and $(Ax)_k > 0\}$
$\boldsymbol{E}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\ 0 \neq x \in \boldsymbol{R}^n_+\ \exists\ k \in \bar{n} \ni x_k > 0$ and $(Ax)_k \geq 0\}$
$\boldsymbol{E}^f_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in \bar{n}, det\ A_{\alpha\alpha} \neq 0 \Rightarrow \wp_\alpha(A) \in \boldsymbol{E}_o\}$
$\boldsymbol{E}_c$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : S(q,A)$ is connected $\forall q \in \boldsymbol{R}^n\}$
$\boldsymbol{E}_*$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : (q,A)$ has a unique solution $\forall\ q \geq 0, q \neq 0\}$
$\boldsymbol{INS}_k$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \|S(q,A)\| = k, \forall q \in int K(A)\}$
$\boldsymbol{INS}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : A \in \cup_{k=0}^{\infty}\boldsymbol{INS}_k\}$
$\boldsymbol{A}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, det\ A_{\alpha\alpha} \neq 0 \Rightarrow m_{ii} \geq 0$ for $M = \wp_\alpha(A)\}$
$\boldsymbol{N}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, det\ A_{\alpha\alpha} < 0\}$
$\boldsymbol{N}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, det\ A_{\alpha\alpha} \leq 0\}$
$\boldsymbol{P}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, det\ A_{\alpha\alpha} > 0\}$
$\boldsymbol{P}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, det\ A_{\alpha\alpha} \geq 0\}$
$\boldsymbol{P}_1$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{P}_o : det\ A_{\alpha\alpha} = 0,$ for exactly one $\alpha \in n^*\}$
$\boldsymbol{Q}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : S(q,A) \neq \emptyset\ \forall\ q \in \boldsymbol{R}^n\}$
$\boldsymbol{Q}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall q \in \boldsymbol{R}^n,\ F(q,A) \neq \emptyset \Rightarrow S(q,A) \neq \emptyset\}$
$\bar{\boldsymbol{Q}}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, A_{\alpha\alpha} \in \boldsymbol{Q}\}$

MATRIX CLASSES

Symbol	Definition
$\dot{\boldsymbol{Q}}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, A_{\alpha\alpha} \in \boldsymbol{Q}_o\}$
$\boldsymbol{R}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : (q, A)$ has a unique solution $\forall\ q \geq 0\}$
$\boldsymbol{R}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : (0, A)$ has a unique solution $\}$
$\boldsymbol{S}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : v(A) > 0\}$
$\dot{\boldsymbol{S}}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, v(A_{\alpha\alpha}) > 0\}$
$\boldsymbol{S}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : v(A) \geq 0\}$
$\dot{\boldsymbol{S}}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\alpha \in n^*, v(A_{\alpha\alpha}) \geq 0\}$
$\boldsymbol{U}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \lvert S(q, A)\rvert = 1\ \forall\ q \in$ interior of $K(A)\}$
$\boldsymbol{V}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\ \alpha \in n, SP(A_{\alpha\alpha}z_\alpha) = (0, 0, \ldots, 0, \ominus)^t \Rightarrow z_\alpha \not> 0\}$
$\boldsymbol{Z}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : a_{ij} \leq 0\ \forall\ i \neq j\}$

Abstract

This dissertation deals with a number of questions related to the linear complementarity problem (LCP). Given $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$ the LCP is to find a vector $z \in \boldsymbol{R}^n$ such that $Az + q \geq 0$, $z \geq 0$ and $z^t(Az + q) = 0$. There is a vast literature on LCP developed during the last four decades. LCP plays a crucial role in the study of Mathematical Programming from the point of view of algorithms as well as applications. The questions on existence and multiplicity of solutions in LCP has led researchers to introduce and study a variety of matrix classes. Most of the work of this dissertation pertains to LCP within the class of Lipschitzian matrices. Besides, various interesting results on $\boldsymbol{INS}$, adequate and connected matrices are also presented. The gist of the dissertation is presented below in a chapter-wise summary.

In Chapter 1, we introduce LCP along with some fundamental concepts and results related to it. In Section 1.1 we state the LCP with a brief historical background and discuss its importance in the applied field. In Section 1.2 we present some basic concepts and results, fundamental to the study of matrix classes in LCP. In Section 1.3 we briefly discuss the basics of game theory, as a prerequisite for Chapter 5.

In Chapter 2 results pertaining to Lipschitzian matrices are presented. Mangasarian and Shiau [25] were the first to consider Lipschitz continuity of solution maps of LCP. In this chapter, we present a number of interesting consequences of property (**) introduced by Murthy, Parthasarathy and Sriparna [36]. In [35], Murthy, Parthasarathy and Sabatini showed that Lipschitzian $\boldsymbol{Q}_0$ matrices satisfy property (**). In Section 2.3 it is shown that property (**) is sufficient for a Lipschitzian matrix to be in $\boldsymbol{Q}_0$. Further if A has this property, then A and all its PPTs must be

completely Q_o and for any q, the linear complementarity problem (q, A) can be processed by a simple principal pivoting method. We deduce, that property (**) characterizes negative N matrices and P matrices, with positive value. In Section 2.4 we study the properties of Lipschitzian matrices in general. We establish, that the Lipschitzian property is inherited by all the principal submatrices. We prove, that Lipschitzian matrices, subject to a principal rearrangement, has the *block* structure.

In Chapter 3 we mainly discuss the INS class of matrices, introduced by Stone [53], and using results on this topic we obtain some important results concerning Lipschitzian matrices. In [36], Murthy, Parthasarathy and Sriparna conjectured that nondegenerate matrices satisfying property (**) are Lipschitzian. We settle this conjecture affirmatively and deduce some necessary conditions on the nondegenerate INS class. In answer to a question raised by Stone in [54], we prove that the class of nondegenerate INS matrices is complete. We conjecture that the *block* property is sufficient for a matrix to be Lipschitzian and prove it for some special cases.

In Chapter 4, we present some results pertaining to three matrix classes, namely, (i) the class of adequate matrices introduced by Ingleton [19], (ii) the class of fully copositive matrices introduced by Murthy and Parthasarathy [32] and (iii) the class of connected matrices introduced by Cao and Ferris [3]. In the case of adequate matrices, our main result is that a column adequate matrix is in Q (in Q_o) if and only if it is completely-Q (completely-Q_o). Within the class of C_o^f-matrices, we provide a sufficient condition for a matrix to be in P_o. As a corollary to this result, we give an alternative proof of a result due to Murthy and Parthasarathy, which states that $C_o^f \cap Q_o$-matrices are in P_o. As another consequence of this result, we characterize positive semidefinite matrices within the class of bisymmetric E_0^f matrices.

In an attempt to settle the conjecture raised by Jones and Gowda [20], that $P_0 \cap Q_0$ matrices are connected, we prove the result for the special case, when the matrix is nonnegative.

Chapter 5 is somewhat expository in nature. In this chapter we show how results from two person matrix games, due to von Neumann and Kaplansky, can be effectively used to get interesting results in Linear Complementarity Problem. It is also indicated that the computation of equilibrium points, through Lemke-type algorithms for polystochastic games, is possible.

Chapter 1

Introduction

1.1 The Problem

This dissertation deals with various aspects of the linear complementarity problem (LCP). The LCP consists of finding a vector in a finite-dimensional real vector space that satisfies a certain system of linear inequalities. Specifically, given a vector $q \in R^n$ and a matrix $A \in R^{n \times n}$, the linear complementarity problem is to find a vector $x \in R^n$ satisfying the following conditions:

$$x \geq 0,\ Ax + q \geq 0$$

$$x^t(Ax + q) = 0.$$

The LCP with respect to A and q is denoted by (q, A). Although research in this field started from 1940 onwards, it gained momentum only in the sixties. This subject basically unifies linear and quadratic programs and equilibrium problems (physical or economic). Linear and quadratic programs, as well as the equilibrium problems can be formulated as an LCP.

An algorithm, known as the complementary pivot algorithm, first developed to study LCPs, has been generalized to yield efficient algorithms for computing Brouwer and Kakutani fixed points, for computing economic equilibria and for solving systems of nonlinear equations and nonlinear programming problems. A new dimension was added to the LCP after Lemke and Howson [24] developed the algorithm to solve bimatrix games. From then onwards extensive research has been going on in this field.

LCP has a varied range of applications in economics, engineering, game theory and optimization [24, 7, 11, 4, 1]. Research in LCP is carried out in two directions. One concentrates on the study of matrix classes which characterize certain properties of the LCP, the other develops algorithms to process the problem. These two directions are very much interrelated. We concentrate mainly on the study of matrix classes. In Section 1.2 we present brief discussion of the background, as a prerequisite for the study of LCP. We include some key definitions and preliminary results of LCP. In Section 1.3 we briefly discuss and give some important results related to game theory which will be useful in the later chapters.

Chapter 2 mainly deals with Lipschitzian matrices. Property (**) was introduced by Murthy, Parthasarathy and Sriparna in [36]. In [35] Murthy, Parthasarathy and Sabatini proved that Lipschitzian matrices satisfy property (**). They conjectured that a Lipschitzian matrix is in $\boldsymbol{Q}_0$ if and only if it satisfies the (**) property. This conjecture is settled affirmatively in Chapter 2. It is also shown that a negative matrix is an $\boldsymbol{N}$ matrix if and only if it satisfies property (**). Any matrix A is a $\boldsymbol{P}$ matrix if and only if it satisfies the (**) property and has positive value. Lipschitzian matrices are shown to be complete. The notion of block property is introduced and it is proved that Lipschitzian matrices have the block property.

In Chapter 3 we mainly consider the class of $\boldsymbol{INS}$ matrices. The class of $\boldsymbol{INS}$ matrices was introduced by stone [53]. To prove, that a nondegenerate matrix satisfying (**) property is Lipschitzian, we invoke certain results related to $\boldsymbol{INS}$ matrices. The $\boldsymbol{INS}$ class is shown to be complete, and it is established that within the class of $\boldsymbol{INS}$ matrices, a nondegenerate matrix has block property.

In Chapter 4 we prove that within the class of adequate matrices introduced by Ingleton [19], a matrix is $\boldsymbol{Q}_o$ if and only if it is completely $\boldsymbol{Q}_o$. Also, within the class of $\boldsymbol{C}_0^f$-matrices a matrix is $\boldsymbol{P}_o$ if and only if all its 2×2 principal minors are $\boldsymbol{P}_o$.

Chapter 4 also deals with connected matrices introduced by Cao and Ferris [3]. In [20], Jones and Gowda have shown that the class of connected matrices denoted by $\boldsymbol{E}_c$, belong to the class of $\boldsymbol{E}_0^f$- matrices. They have also shown that within the class $\boldsymbol{R}_0$, a matrix is $\boldsymbol{P}_o$ if and only if it is connected. They raised the conjecture that a $\boldsymbol{Q}_o$-matrix is in $\boldsymbol{P}_o$ if and only if it is in $\boldsymbol{E}_c$. We prove the conjecture within the class of nonnegative matrices.

In Chapter 5 we show how game theoretic results can be effectively used to get interesting results in the linear complementarity problem.

We also indicate the importance of LCP in game theory.

1.2 Matrix Classes

The LCP with respect to A and q has already been defined in Section 1.1. It can also be presented in the following form.

Given $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$, LCP(q, A) is to find w and z in $\boldsymbol{R}^n$ such that :

$$w - Az = q,$$

$$w \geq 0, \quad z \geq 0,$$

$$\text{and} \quad w^t z = 0.$$

We call the coordinates of z as primary variables, and those of w as secondary variables.

For any $q \in \boldsymbol{R}^n$ and $A \in \boldsymbol{R}^{n \times n}$, let us define $F(q, A)$ and $S(q, A)$ as follows:

$$F(q, A) = \{z \in \boldsymbol{R}^n_+ : Az + q \geq 0\}$$

$$S(q, A) = \{z \in F(q, A) : (Az + q)^t z = 0\}$$

We say that LCP(q, A) (or simply (q, A)) is feasible if $F(q, A) \neq \phi$. Each element in $F(q, A)$ is a feasible solution to (q, A). We say that (q, A) has a solution if. $S(q, A) \neq \phi$. The set $S(q, A)$ consists of all solutions to (q, A).

Complementary Cones

The notion of complementary cones was first introduced by Samelson, Thrall and Wesler [48]. Later it was studied in greater detail by Murty who obtained some useful results on the existence and multiplicity of solutions to linear complementarity problems (see [39]).

Definition 1.2.1 *Let $A \in \boldsymbol{R}^{m \times n}$. The set $\{q \in \boldsymbol{R}^m : q = Ax$ for some $x \in \boldsymbol{R}^n_+\}$ is called the convex cone generated by A and is denoted by pos A. The columns of A are called the generators of the cone.*

Definition 1.2.2 *Given $A \in \boldsymbol{R}^{n \times n}$ and any $\alpha \subseteq \bar{n}$, we define a complementary matrix B as follows:*

$$B_{\cdot j} = -A_{\cdot j} \text{ if } j \in \alpha \text{ and } \quad B_{\cdot j} = I_{\cdot j} \text{ if } j \in \alpha.$$

B is called the complementary matrix with respect to α, and is denoted by $C_A(\alpha)$.

Definition 1.2.3 *The cone generated by the complementary matrix $C_A(\alpha)$, is the complementary cone of A (or of $[I : A]$) with respect to α and is denoted by pos $C_A(\alpha)$.*

Definition 1.2.4 *If $|A_{\alpha\alpha}| \neq 0$, then $C_A(\alpha)$ is called a complementary basis. Then the complementary cone generated by it (pos $C_A(\alpha)$), is called full or nondegenerate.*

Definition 1.2.5 *The cone pos $C_A(\alpha)_{\cdot i}$ is called a facet of the complementary cone pos $C_A(\alpha)$.*

For a matrix of order n, there are 2^n (not necessarily distinct) complementary cones. The LCP(q, A) is said to have a solution if, and only if, q belongs to one of the complementary cones.

Lemma 1.2.6 *Let $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$. Then LCP(q, A) has a solution if and only if there exists an $\alpha \subseteq \bar{n}$, such that q belongs to the complementary cone with respect to α.*

Proof: Suppose LCP(q, A) has a solution, say $z \in S(q, A)$. Let $\alpha = supp(z)$ and $w = Az + q$. Then by the complementarity condition, $w_i = 0$ for $i \in \alpha$. In other words, q belongs to the complementary cone with respect to α. Conversely if q belongs to the complementary cone generated by $C_A(\alpha)$ for some $\alpha \subseteq \bar{n}$, then there exists a nonzero $x \in \boldsymbol{R}^n_+$ such that $q = C_A(\alpha)x$. Define $z \in \boldsymbol{R}^n_+$ by $z_i = x_i$ if $i \in \alpha$ and $z_i = 0$ if $i \in \alpha$. It can be checked that $z \in S(q, A)$. ■

Principal Pivot Transforms

The concept of principal pivotal transforms was introduced by Tucker [56], and developed in [6, 7, 42, 38]. Since then it has played a very important role in the study of the linear complementarity problem. In addition to the above references, the book by Cottle, Pang and Stone [9] provides a detailed account of this topic.

Definition 1.2.7 *Given $A \in R^{n\times n}$ and $\alpha \subseteq \bar{n}$, if $|A_{\alpha\alpha}| \neq 0$, then the principal pivotal transform (PPT) with respect to α is given by the matrix M defined as:*

$$M = \begin{bmatrix} (A_{\alpha\alpha})^{-1} & -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1} & (\ A/A_{\alpha\alpha}\ ) \end{bmatrix},$$

where $(\ A/A_{\alpha\alpha}\ ) = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$ is called the Schur complement of A with respect to α. We denote the PPT of A with respect to α by $\wp_\alpha(A)$.

Remark 1.2.8 *For $\alpha = \phi$, the PPT is the matrix A itself. For $\alpha = \bar{n}$, $\wp_\alpha(A) = A^{-1}$, provided $|A| \neq 0$. Further, for any $\alpha \subseteq \bar{n}$, $M = \wp_\alpha(A)$ iff $A = \wp_\alpha(M)$. The PPTs are called simple PPTs if α is a singleton set.*

Whenever we mention a PPT, we assume that it exists, that is $|A_{\alpha\alpha}| \neq 0$ for that α. The PPTs play an important role in the study of LCP, for more details on this subject see [56, 6, 7].

Definition 1.2.9 *For any $A \in R^{n\times n}$, and a permutation matrix P of the same dimension, PAP^t is called a principal rearrangement of A.*

Lemma 1.2.10 *Given any $\alpha \subseteq \bar{n}$, there exists a permutation matrix P such that*

$$PAP^t = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

Proof: Suppose $\alpha = \{k_1, k_2, \ldots k_m\}$ and $\bar{\alpha} = \{k_{m+1}, k_{m+2}, \ldots, k_n\}$. Define $P \in R^{n \times n}$ by:

$$\text{for } i \in n,\ p_{ij} = 1 \text{ if } j = k_i \text{ and } p_{ij} = 0 \text{ otherwise.}$$

Taking P as above we see that

$$PAP^t = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

■

As it was mentioned earlier, the LCP(q, A) can also be written as the problem of finding vectors w and z such that

$$\begin{bmatrix} I & -A \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = q,$$

$$w \geq 0, \quad z \geq 0,$$

$$\text{and} \quad w^t z = 0.$$

By Lemma 1.2.10, for any given $\alpha \subseteq \bar{n}$, we can rewrite the above LCP in terms of

$$\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

Note that if P is a Permutation matrix, then $PP^t = P^tP = I$. So we have,

$$\begin{bmatrix} P^tP & -P^tPAP^tP \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = q,$$

or

$$\begin{bmatrix} I & -PAP^t \end{bmatrix} \begin{bmatrix} Pw \\ Pz \end{bmatrix} = Pq$$

This can be written as,

$$I \begin{bmatrix} w_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} - \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} q_\alpha \\ q_{\bar{\alpha}} \end{bmatrix}$$

Next we present some important results connecting PPTs and LCP.

Lemma 1.2.11 *Suppose $A \in \boldsymbol{R}^{n\times n}$ and $q \in \boldsymbol{R}^n$. Let $\alpha \subseteq \bar{n}$ be such that $\wp_\alpha(A)$ exists. Then $S(q,A) \neq \phi$ iff $S(p,M) \neq \phi$, where $M = \wp_\alpha(A), p_\alpha = -(A_{\alpha\alpha})^{-1}q_\alpha$ and $p_{\bar{\alpha}} = q_{\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}q_\alpha$. Also $|S(q,A)| = |S(p,M)|$.*

Proof: Let us suppose $S(q,A) \neq \phi$, and let (w,z) be a solution of (q,A). Then we have,

$$\begin{bmatrix} w_\alpha \\ w_\alpha \end{bmatrix} - \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} q_\alpha \\ q_{\bar{\alpha}} \end{bmatrix}, \quad w \geq 0, \quad z \geq 0$$

$$\text{and} \quad w^t z = 0.$$

Premultiplying the above equation by $(C_A(\alpha))^{-1}$ and rewriting it, we get,

$$\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} - M \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} p_\alpha \\ p_{\bar{\alpha}} \end{bmatrix}$$

Also

$$\begin{bmatrix} z_\alpha \\ w_\alpha \end{bmatrix} \geq 0, \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} \geq 0, \text{ and } \begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix}^t \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} = w^t z = 0.$$

or in other words,

$$\begin{bmatrix} w_\alpha \\ z_\alpha \end{bmatrix} \in S(p,M).$$

For the proof of the converse, we assume $S(p, M) \neq \phi$ and proceed just as before. Hence we can say, there is a one-to-one correspondence between solutions of (q, A) and (p, M). Consequently $|S(q, A)| = |S(p, M)|$. ■

Remark 1.2.12 *Note that $q \in pos\ C_A(\alpha)$ iff $p \in pos\ C_M(\alpha)$ and p is called the principal pivotal transform of q with respect to α (and A). Also note that if there exists a principal pivot transform p of q such that $p \geq 0$, then it is easy to get a solution to (q, A) using this PPT.*

Remark 1.2.13 *It is evident from the previous lemma that complementary cones corresponding to A and $\wp_\alpha(A)$ are in one-to-one correspondence through the nonsingular linear transformation $q \mapsto (C_A(\alpha))^{-1}q$. Also, q is in the interior of pos $C_A(\alpha)$ iff p is in the interior of pos $C_M(\alpha)$.*

We end the discussion on PPT, by stating a theorem due to Tucker, concerning determinants of PPTs.

Theorem 1.2.14 *Suppose $A \in \mathbf{R}^{n\times n}$, and $\alpha \subseteq \bar{n}$ is such that $|A_{\alpha\alpha}| \neq 0$. If $M = \wp_\alpha(A)$ and $\beta \subseteq \bar{n}$, then*

$$det\, M_{\beta\beta} = \frac{det\, A_{\gamma\gamma}}{det\, A_{\alpha\alpha}},$$

where $\gamma = \alpha \triangle \beta$.

LCP and matrix classes

The matrix classes provide answers to various questions related to the existence and multiplicity of solutions to the linear complementarity problem. Hence, much of the research in LCP has been devoted to the study of matrix classes. The matrix classes also characterize certain properties of the linear complementarity problem. They provide answers to certain fundamental questions related to the LCP, like

What is the class of matrices $A \in \boldsymbol{R}^{n \times n}$ for which $S(q, A)$ is nonempty for all $q \in \boldsymbol{R}^n$?

For which class of matrices $A \in \boldsymbol{R}^{n \times n}$, does LCP$(q, A)$ has a unique solution for all vectors $q \in \boldsymbol{R}^n$?

What is the class of matrices A for which $S(q, A) \neq \phi$ whenever $F(q, A) \neq \phi$?

Also, from the algorithmic point of view it answers certain relevant questions like,

What are the classes of matrices which can be processed by a certain algorithm ?

These are the questions which led to the study of matrix classes, like $\boldsymbol{Q}, \boldsymbol{Q}_0, \boldsymbol{P}$, positive definite, semidefinite and consequently many others, like $\boldsymbol{P}_0, \boldsymbol{E}_0, \boldsymbol{C}_0$ and $\boldsymbol{R}_0$. There are some classes which came as a result of practical applications, like the $\boldsymbol{Z}$-matrices [4, 11] and the class of adequate matrices [19]. Other than the questions regarding multiplicity and existence of solutions, the study of matrix classes has also been initiated to answer certain questions regarding the nature of the solution set, e.g., to determine the class of matrices for which the solution set will be convex or connected say, for all q, and how the solution set behaves, when q changes. These questions gave rise to the study of connected and Lipschitzian matrices. For more details on the various matrix classes, see Cottle Pang and Stone [9] and Murty [38].

1.3 Game Theory

A two-person zero-sum matrix game can be described as follows. Player 1 chooses an integer i $(i = 1, 2, ..., m)$ and player 2 chooses an integer j $(j = 1, 2, ..., n)$ simultaneously. Then player 1 pays player 2 an amount a_{ij} (which may be positive, zero or negative). The $m \times n$ matrix $A = (a_{ij})$ is called the (player 1's) pay-off matrix.

A strategy p for player 1 is a probability vector $(p_1, p_2, ..., p_m)$. The idea is that he chooses an integer i with probability p_i. A strategy q for player 2 is defined analogously. J.von Neumann's fundamental minimax theorem asserts that there exist strategies $p = (p_1^0, p_2^0, ..., p_m^0)$, $q = (q_1^0, q_2^0, ..., q_n^0)$ and a real number v such that

$$\sum_i p_i^0 a_{ij} \leq v \quad \forall j = 1, 2, ..., n.$$

$$\sum_j q_j^0 a_{ij} \geq v \quad \forall i = 1, 2, ..., m.$$

The number v is called the minimax value associated with the matrix A, or simply, value of the game. The strategies satisfying the above inequalities are called optimal strategies for the two players. In the game described above, player 1 is the minimizer (that is, he wants to give player 2 as little as possible) and player 2 is the maximizer. We write $v(A)$ to denote the minimax value of the game A. If there is no scope of confusion, we simply write v instead of $v(A)$.

If the probability vectors are of the form $\{0, \ldots, 1, \ldots, 0\}$, then they are called *pure strategies*, otherwise they are called *mixed strategies*. A mixed strategy p is called *completely mixed* if $p > 0$.

Theorem 1.3.1 *(J. von Neumann) Every matrix game has a solution in mixed strategies.*

Theorem 1.3.2 *If the value of the game A is positive, then player 1 has a completely mixed strategy (not necessarily optimal), $p > 0$ such that $Ap > 0$.*

Theorem 1.3.3 *(Kaplansky) If player 2 has a completely mixed optimal strategy q, then $Ap = ve$ for every optimal mixed strategy p of player 1, where $e = (1, \ldots, 1)^t$.*

Theorem 1.3.4 *The value of a game A is positive (nonnegative) if and only if there exists a nonnegative nonzero vector x such that $Ax > 0$ $(Ax \geq 0)$. Similarly, the value of a game is negative (nonpositive) iff there exists a $0 \neq y \geq 0$ such that $A^t y < 0$ $(A^t y \leq 0)$.*

Theorem 1.3.5 *Let $A \in \mathbf{R}^{n \times n}$ and let M be a PPT of A with respect to some $\alpha \in n^*$. Consider the games with pay-off matrices A and M. Then $v(A) > 0$ iff $v(M) > 0$.*

Proof: It is enough to show the if part, since the 'only if' part will follow automatically. Suppose $v(A) > 0$. We can get a probability vector p such that $Ap > 0$. Let $y = Ax$. Assume without loss of generality that

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

It can be seen from Lemma 1.2.11, that

$$\begin{bmatrix} x_\alpha \\ y_{\bar{\alpha}} \end{bmatrix} - M \begin{bmatrix} y_\alpha \\ x_{\bar{\alpha}} \end{bmatrix} = 0$$

Since

$$\begin{bmatrix} x_\alpha \\ y_{\hat{\alpha}} \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} y_\alpha \\ x_{\hat{\alpha}} \end{bmatrix} > 0,$$

it follows that $v(M) > 0$. ∎

Definition 1.3.6 *For a matrix $A \in R^{n\times n}$, the game with pay-off matrix A is said to be completely mixed if, every optimal strategy (for either player) is completely mixed.*

The following fundamental theorem is due to Kaplansky [21].

Theorem 1.3.7 *A matrix game A with value zero is completely mixed iff,*
(a) A is a square matrix with rank$(A) = n - 1$, where n is the order of the matrix and
(b) all the cofactors A_{ij} of A are different from zero and have the same sign.

Chapter 2

Constructive characterization of Lipschitzian Q_0 matrices

2.1 Introduction

In this chapter we mainly study the class of Lipschitzian matrices. The Lipschitzian property is closely related to the concept of stability at a solution point. We say that LCP(q, A) is stable at x^* if x^* is an isolated solution of LCP(q, A) and there exist neighbourhoods V of x^* and U of the pair (q, A) such that for all $(\bar{q}, \bar{A}) \in U$, $S(\bar{q}, \bar{A}) \cap V$ is nonempty. If it is a singleton set for all $(\bar{q}, \bar{A}) \in U$, then x^* is said to be strongly stable. If x^* is strongly stable, then there exists a Lipschitzian function defined on a neighbourhood of the pair (q, A) and having values in a neighbourhood of x^*. The local upper Lipschitzian property of the solution set is concerned with the stability of the LCP, when the matrix A is not perturbed.

Mangasarian and Shiau [25] were the first to consider Lipschitz continuity of solution maps of LCP. The motivation of our results comes from the conjectures on Lipschitzian matrices raised by Gowda [17, 16] and Pang. In connection with LCP a number of matrix classes has been identified. Among them $\boldsymbol{Q}$ and $\boldsymbol{Q}_o$ are of fundamental interest. In [35], Murthy, Parthasarathy and Sabatini proved that Lipschitzian $\boldsymbol{Q}_o$ matrices satisfy the (**) property. As a consequence of this result, they established that a matrix $A \in \boldsymbol{R}^{n \times n}$ is a Lipschitzian $\boldsymbol{Q}$ matrix iff it is a $\boldsymbol{P}$ matrix. They conjectured that property (**) is also sufficient for a Lipschitzian matrix to be $\boldsymbol{Q}_o$. Our main goal in this chapter is to characterize $\boldsymbol{Q}_o$-matrices within the class of Lipschitzian matrices and to show that the simple pivoting methods due to Zoutendijk [59] and Bard [2], process LCP (q, A) when A is a Lipschitzian $\boldsymbol{Q}_o$-matrix.

The (**) Property was introduced by Murthy, Parthasarathy and Sriparna [36]. In Section 2.3, we present several other interesting consequences of property (**). The characterization of completely $\boldsymbol{Q}_o$-matrices, envisaged by Cottle [5], is still an open problem. In [15], Frederickson, Watson and Murty obtained a characterization of completely $\boldsymbol{Q}_o$ matrices of order less than or equal to three. Murthy and Parthasarathy in [32], characterized the nonnegative completely $\boldsymbol{Q}_o$-matrices and also proved that symmetric copositive matrices are $\boldsymbol{Q}_o$ if and only if they are completely $\boldsymbol{Q}_o$. In this section, we establish that property (**) is sufficient for a matrix to be completely $\boldsymbol{Q}_o$. Property (**) has other significant features. It characterizes $\boldsymbol{N}$ matrices of the second kind and $\boldsymbol{P}$ matrices with value positive.

So far, there is no finite characterization of Lipschitzian matrices. Mangasarian and Shiau [25] proved that $\boldsymbol{P}$ matrices are Lipschitzian, and Gowda [16] showed that negative $\boldsymbol{N}$ matrices are Lipschitzian. Stone [54] asserted that Lipschitzian matrices belong to the $\boldsymbol{INS}$ class. In general, it is difficult to decide whether a given matrix is Lipschitzian or not. In Section 2.4 we present some necessary conditions on Lipschitzian matrices. We also exhibit the decomposition structure of nonpositive Lipschitzian matrices and obtain the *block* structure of Lipschitzian matrices in general.

2.2 Preliminaries

Recall that a matrix $A \in \boldsymbol{R}^{n \times n}$ is said to be a $\boldsymbol{Q}$ matrix if, $S(q, A) \neq \emptyset$ for all $q \in \boldsymbol{R}^n$ and it is said to be in $\boldsymbol{Q}_o$ if, $S(q, A) \neq \emptyset$ whenever $F(q, A) \neq \emptyset$. The matrix A is said to be completely $\boldsymbol{Q}$ (completely $\boldsymbol{Q}_o$) if $A_{\alpha\alpha}$, the principal submatrix of A with respect to α, is in $\boldsymbol{Q}$ ($\boldsymbol{Q}_o$) for all index sets α. A matrix A is said to be a $\boldsymbol{P}$ ($\boldsymbol{N}$) matrix, if all its principal minors are positive (negative). If all its principal minors are nonzero, then it is said to be nondegenerate.
The following result is due to Murty [38].

Procedure 2.2.1 *A square matrix A of order n is nondegenerate if and only if every diagonal entry in every PPT of A is nonzero.*

A similar result on $\boldsymbol{P}$ matrices, was proved by Parson [42].

Procedure 2.2.2 *A square matrix A is a $\boldsymbol{P}$ matrix if and only if every diagonal entry of every PPT of A is positive.*

Let w and z be two vectors satisfying $w = Az + q$. Define $\bar{z}$ as

$$\bar{z} = \begin{bmatrix} w_\alpha \\ z_\beta \end{bmatrix} \text{ where } z = \begin{bmatrix} z_\alpha \\ z_\beta \end{bmatrix} \text{ and } w = \begin{bmatrix} w_\alpha \\ w_\beta \end{bmatrix}.$$

Recall (Lemma 1.2.11) that $\bar{z} \in S(\bar{q}, M)$ if and only if $z \in S(q, A)$ and $|S(\bar{q}, M)| = |S(q, A)|$. There exist constants (see Gowda [17]), C_1 and C_2

depending on A such that for $w = Az + q$ and $y = Ax + p$,

$$\|\bar{z} - \bar{x}\| \leq C_1\{\|z - x\| + \|q - p\|\}$$

and

$$\|q - p\| \leq C_2\|\bar{q} - \bar{p}\|.$$

This is the basis for some algorithms to solve LCPs. Certain classes of LCPs can be processed by using only the simple PPTs. The algorithms which use only simple PPTs are known as simple principal pivoting methods or Bard-type methods (see the book by Cottle, Pang and Stone [9], pages 239, 240 for details).

Let $B \in \mathbf{R}^{n \times n}$ be an invertible matrix with lexicographically positive rows. In Bard's algorithm, the pivot is selected with the help of a vector valued function, that strictly decreases in the lexicographic order. The function is given by

$$b^{\nu} = \frac{B^{\nu}_{r.}}{q^{\nu}_{r}} = \text{ lexico max } \{\frac{B^{\nu}_{i.}}{q^{\nu}_{i}} : q^{\nu}_{i} < 0\}.$$

Lemma 2.2.3 *Bard's Algorithm with the lexicographic pivot selection rule mentioned above preserves the property*

$$B^{\nu}_{i.} \succ q^{\nu}_{i}(\frac{B_{r.}{}^{\nu}}{q^{\nu}_{r}}) \quad where \quad i \neq r.$$

Next, we state a theorem from Cottle Pang and Stone [9], which says that the algorithm will terminate in a finite number of steps.

Theorem 2.2.4 *The algorithm due to Bard is finite, also no complementary basis will ever be used more than once.*

Remark 2.2.5 *It has been noted in [9] that if LCP(q, A) is such that at each iteration of Bard's algorithm, diagonal entries of M (PPT of A),* *corresponding to negative entries of p (PPT of q), are strictly positive, then termination occurs only when a solution has been found.*

We conclude this section with the following definition and a result due to Gowda [17].

Definition 2.2.6 *We say that $A \in \mathbf{R}^{n\times n}$ is a Lipschitzian matrix if the multivalued mapping $\Phi : q \rightarrow S(q, A)$ satisfies the following property : there exists a positive constant C such that*

$$S(q, A) \subseteq S(p, A) + C||q - p||B$$

holds for every q and p satisfying $S(q, A) \neq \emptyset$ and $S(p, A) \neq \emptyset$. Here $B = \{x \in \mathbf{R}^n : ||x|| \leq 1\}$ is the closed unit ball in $\mathbf{R}^n$.

Theorem 2.2.7 *Let A be a Lipschitzian matrix. Then every PPT of A is also Lipschitzian.*

Remark 2.2.8 *Mangasarian and Shiau [25] proved that $\mathbf{P}$-matrices are Lipschitzian. Gowda [17] established that negative $\mathbf{N}$-matrices are also Lipschitzian.*

2.3 Lipschitzian Q_0-matrices

Property (**), defined below was obtained as a necessary condition on Lipschitzian $\mathbf{Q}_o$-matrices by Murthy Parthasarathy and Sabatini in [35]. They conjectured that it is also sufficient for a Lipschitzian matrix to be in $\mathbf{Q}_o$. Besides settling this conjecture affirmatively, it will be shown that property (**) characterizes $\mathbf{N}$-matrices (of second kind) and $\mathbf{P}$-matrices (with positive value).

Definition 2.3.1 *Let $A \in R^{n\times n}$. We say that A satisfies* property (**) *if every PPT M of A satisfies the condition that the rows corresponding to nonpositive diagonal entries of M are nonpositive.*

Remark 2.3.2 *It should be noted that if A satisfies property (**), then so does every principal submatrix of A. Also every PPT of A satisfies property (**).*

The following algorithm, due to Zoutendijk and Bard, is shown to work for a special class of LCPs (see Chapter 4 of Cottle Pang and Stone [9] and Lemma 2.2.3 and Theorem 2.2.4 of this chapter). This is a simple principal pivoting method in the sense that it transforms the given LCP into its PPTs using only the diagonal entries as the pivoting blocks in each iteration.

Consider the problem (q, A), where $A \in R^{n\times n}$ and $q \in R^n$. The following algorithm, taken from [9] (page 239), is presented in a different form with a specific choice of B.

Algorithm 2.3.3 Step 0. Initialization. *Set $M = A$, $p = q$ and $B = I$, the identity matrix in $R^{n\times n}$ and $\alpha = \emptyset$.*

Step 1. Rule for termination. *If $p \geq 0$, then stop. The vector z defined by $z_\alpha = p_\alpha$ and $z_{\bar{\alpha}} = 0$ is a solution of (q, A).*

Step 2. Pivot selection. *Let r be the index such that*

$$\frac{B_{r.}}{q_r} = \text{lexico max } \{\frac{B_{i.}}{q_i} : q_i < 0\}.$$

Step 3. Pivoting. *Replace p and columns of B by their PPTs with respect to M and $\{r\}$. Replace M by PPT of M with respect to $\{r\}$. Replace α by its symmetric difference with $\{r\}$. Go to Step 1.*

Theorem 2.3.4 *Suppose $A \in \boldsymbol{R}^{n\times n}$ satisfies property (**). Then A belongs to $\boldsymbol{Q}_o$.*

Proof: Let $q \in \boldsymbol{R}^n$ be such that $F(q, A) \neq \emptyset$. We will show that if Algorithm 2.3.3 is applied to (q, A), then it will terminate in a finite number of steps with a solution to (q, A). From the discussions in Section 2.2, it follows that $F(p, M) \neq \emptyset$ in each iteration of the algorithm. This implies that, at each iteration of the algorithm, if $p_i < 0$ for any i, then $M_{i.}$ must have a positive entry. By property (**), $m_{ii} > 0$ for each i such that $p_i < 0$. From Lemma 2.2.3 and Theorem 2.2.4, it follows that the algorithm terminates in a finite number of steps with a solution to (q, A) (see also Remark 2.2.5). ∎

Corollary 2.3.5 *Suppose $A \in \boldsymbol{R}^{n\times n}$ satisfies property (**). Then A (and every PPT of A) is completely $\boldsymbol{Q}_o$.*

Proof: Follows from Remark 2.3.2. ∎

Theorem 2.3.6 *Suppose $A \in \boldsymbol{R}^{n\times n}$ is a Lipschitzian matrix. Then the following conditions are equivalent:*

(i) $A \in \boldsymbol{Q}_o$

(ii) A satisfies property (**)

(iii) A is completely $\boldsymbol{Q}_o$.

Proof: The implication (i) $\Rightarrow$ (ii) has already been proved by Murthy, Parthasarathy and Sabatini in [35]. However, for the sake of completeness, we will briefly outline the proof. Since Lipschitzian and $\boldsymbol{Q}_o$ properties are both invariant under PPT, it suffices to show that the rows corresponding to the nonpositive diagonal entries of A are nonpositive. Assume $a_{11} \leq 0$. Suppose $A_{1.}$ has a positive entry. For each positive integer k, define $p^k =$

$(0, k, \ldots, k)^t$ and $q^k = (-1/k, k, \ldots, k)^t$. Since $A_{1.}$ has a positive entry, for all large k, $F(q^k, A) \neq \emptyset$ and hence $S(q^k, A) \neq \emptyset$. Since $0 \in S(p^k, A)$ for all k, by Lipschitzian property of A, there exists a sequence $z^k \in S(q^k, A)$ such that $z^k \to 0$ as $k \to \infty$. This forces $z_j^k = 0$, $j = 2, \ldots, n$, for all large k. But then $(Az^k)_1 + q_1^k < 0$ for all large k which is a contradiction.

(ii) ⇒ (iii) follows from Corollary 2.3.5.

(iii) ⇒ (i) is obvious. ∎

We shall now illustrate Algorithm 2.3.3 applied to (q, A), where A is a Lipschitzian $\boldsymbol{Q}_0$-matrix.

Example 2.3.7 Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 0 & -1 \\ -2 & -1 & -1 & -1 \\ 3 & 2 & 1 & -1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad and\ q = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \end{bmatrix}.$$

Note that A is Lipschitzian, as the PPT of A with respect to a_{22} is a negative $\boldsymbol{N}$-matrix. For the sake of convenience, we use the following tableau.

Initialization. $\alpha = \emptyset$.

BV	z_1	z_2	z_3	z_4	p	$B_{.1}$	$B_{.2}$	$B_{.3}$	$B_{.4}$
w_1	3	2	0	-1	0	1	0	0	0
w_2	-2	-1	-1	-1	1	0	1	0	0
w_3	3	2	1	-1	-2	0	0	1	0
w_4	4	3	2	1	-2	0	0	0	1

First iteration: $r = 4$,

BV	z_1	z_2	z_3	w_4	p	$B_{.1}$	$B_{.2}$	$B_{.3}$	$B_{.4}$
w_1	7	5	2	-1	-2	1	0	0	1
w_2	2	2	1	-1	-1	0	1	0	1
w_3	7	5	3	-1	-4	0	0	1	1
z_4	-4	-3	-2	1	2	0	0	0	-1

$\alpha = \{4\}$

Second iteration: $r = 3$.

BV	z_1	z_2	w_3	w_4	p	$B_{.1}$	$B_{.2}$	$B_{.3}$	$B_{.4}$
w_1	7/3	5/3	2/3	-1/3	2/3	1	0	-2/3	1/3
w_2	-1/3	1/3	1/3	-2/3	1/3	0	1	-1/3	2/3
z_3	-7/3	-5/3	1/3	1/3	4/3	0	0	-1/3	-1/3
z_4	2/3	1/3	-2/3	1/3	-2/3	0	0	2/3	-1/3

$\alpha = \{3, 4\}$

Third iteration: $r = 4$.

BV	z_1	z_2	w_3	z_4	p	$B_{.1}$	$B_{.2}$	$B_{.3}$	$B_{.4}$
w_1	3	2	0	-1	0	1	0	0	0
w_2	1	1	-1	-2	-1	0	1	1	0
z_3	-3	-2	1	1	2	0	0	-1	0
w_4	-2	-1	2	3	2	0	0	-2	1

$\alpha = \{3\}$

Fourth iteration: $r = 2$.

BV	z_1	w_2	w_3	z_4	p	$B_{.1}$	$B_{.2}$	$B_{.3}$	$B_{.4}$
w_1	1	2	2	3	2	1	-2	-2	0
z_2	-1	1	1	2	1	0	-1	-1	0
z_3	-1	-2	-1	-3	0	0	2	1	0
w_4	-1	-1	1	1	1	0	1	-1	1

$\alpha = \{2, 3\}$.

It can be checked that $z = (0, 1, 0, 0)^t$ is a solution of (q, A).

Negative $\boldsymbol{N}$-matrices are known as $\boldsymbol{N}$-matrices of the second kind. The following theorem shows that property (**) is a necessary and sufficient condition for a negative matrix to be in $\boldsymbol{N}$.

Theorem 2.3.8 *Let $A \in \boldsymbol{R}^{n\times n}$ be a negative matrix. Then A is an $\boldsymbol{N}$-matrix if and only if it satisfies property (**).*

Proof: Since a negative $\boldsymbol{N}$-matrix is Lipschitzian and is in $\boldsymbol{Q}_o$, the 'only if part' follows from Theorem 2.3.6. We shall prove the 'if' part by induction on n. Obviously the theorem holds for $n = 1$. Assume that the theorem is true for all matrices of order $n-1$, $n > 1$. Let A be an $n \times n$ matrix satisfying property (**). Write

$$A = \begin{bmatrix} B & b \\ a^t & a_{nn} \end{bmatrix}.$$

By induction hypothesis, $B \in \boldsymbol{N}$. Let M be the PPT of A with respect to B. Then $m_{nn} = a_{nn} - a^t B^{-1} b$. Let $y^t = a^t B^{-1}$. Suppose $m_{nn} \leq 0$. By property (**), then, $y \leq 0$. This in turn implies $a^t = y^t B \geq 0$ which contradicts that $a^t < 0$. Hence $m_{nn} > 0$. Since $det\, A = m_{nn} det\, B$, $det\, A < 0$. ■

The next theorem characterizes the $\boldsymbol{P}$-matrices in terms of property (**). The value of a matrix $A \in \boldsymbol{R}^{n\times n}$ is said to be positive, if there exists a nonnegative vector $x \in \boldsymbol{R}^n$ such that $Ax > 0$. The class of matrices with positive value is known as $\boldsymbol{S}$.

Theorem 2.3.9 *Suppose $A \in \boldsymbol{R}^{n\times n}$. Then A is a $\boldsymbol{P}$-matrix if and only if A satisfies the property (**) and has value positive.*

Proof: We shall prove the 'if' part. Since value of A is positive, every PPT of A must also have value positive. If the value of a matrix is positive.

then it cannot have any nonpositive rows. Since A satisfies property (**), it follows that the diagonal entries of A and all its PPTs must be positive. From Theorem 2.2.2 it follows that A is a $\boldsymbol{P}$-matrix. ∎

Corollary 2.3.10 *If a matrix A satisfies property (**), then $A \in \boldsymbol{S}$ if and only if $A \in \boldsymbol{P}$.*

2.4 Properties of Lipschitzian matrices

In this section, we shall derive a number of properties of Lipschitzian matrices. So far, there is no finite characterization of Lipschitzian matrices. We have the following results which provide necessary conditions.

Theorem 2.4.1 *Suppose $A \in \boldsymbol{R}^{n\times n}$ is a Lipschitzian matrix. Then for any permutation matrix P and any positive diagonal matrix D, PAP^t, AD and DA are all Lipschitzian matrices.*

The above theorem states that the Lipschitzian property is invariant under principal rearrangements, positive row and column scaling. We omit the easy proof of this theorem. The next theorem shows that every Lipschitzian matrix is nondegenerate. Gowda and Sznajder [18] proved this using piecewise affine functions. We give an alternative proof.

Theorem 2.4.2 *Suppose $A \in \boldsymbol{R}^{n\times n}$ is Lipschitzian. Then A is nondegenerate.*

Proof: If we show that A has no zero diagonal entries, then from Theorem 2.2.7, it follows that the diagonal entries of every PPT of A are nonzero. From Theorem 2.2.1, it follows that A is nondegenerate. Suppose $a_{ii} = 0$ for some i, say $i = 1$. Choose $\lambda > 0$ such that $\lambda A_{\cdot 1} + e > 0$, where $e = (1, 1, \ldots, 1)^t \in \boldsymbol{R}^n$. Let $q = (0, 1, 1, \ldots, 1)^t$ and for each positive integer

k, let $q^k = (1/k, 1, 1, \ldots, 1)^t \in \mathbb{R}^n$. Since $a_{11} = 0$, $\bar{z} = (\lambda, 0, \ldots, 0)^t \in S(q, A)$. Also $S(q^k, A) \neq \emptyset \; \forall \; k \geq 1$. Since A is Lipschitzian, there exists a sequence z^k such that $z^k \in S(q^k, A)$ and $\|z^k - \bar{z}\| \to 0$ as $k \to \infty$. This implies for all large k, $z_1^k > 0$ and $z_j^k = 0$ for $j = 2, 3, \ldots, n$. This contradicts that $z^k \in S(q^k, A)$ for all large k. Hence $a_{11} \neq 0$. The theorem follows. ∎

Theorem 2.4.3 *Suppose $A \in \mathbb{R}^{n \times n}$ is Lipschitzian. Then A is completely Lipschitzian, that is, all principal submatrices of A are Lipschitzian.*

Proof: We will imitate the proof of Gowda [16], which he gave in the case of negative matrices. Let M be the leading principal submatrix of A of order $n-1$. We will show that M is Lipschitzian. Fix $\bar{p}$ and $\bar{q}$ in $\mathbb{R}^{(n-1)}$ such that $S(\bar{p}, M)$ and $S(\bar{q}, M)$ are nonempty. We will show that

$$S(\bar{p}, M) \subseteq S(q, M) + C\|\bar{p} - \bar{q}\|\bar{B}, \tag{2.4.1}$$

where C is the Lipschitzian constant with respect to A and $\bar{B}$ is the closed unit ball in $\mathbb{R}^{(n-1)}$. Let $\bar{x} \in S(\bar{p}, M)$. For each positive integer m, define $p^m = (\bar{p}^t, m)^t$ and $q^m = (q^t, m)^t$. Note that, for all large m,

$$x = (\bar{x}, 0)^t \in S(p^m, A). \tag{2.4.2}$$

This implies $S(p^m, A) \neq \emptyset$. Similarly, $S(q^m, A) \neq \emptyset$. Since A is Lipschitzian and $\|p^m - q^m\| = \|\bar{p} - \bar{q}\|$, we have for all m sufficiently large,

$$S(p^m, A) \subseteq S(q^m, A) + C\|\bar{p} - \bar{q}\|B. \tag{2.4.3}$$

From (2.4.2) and (2.4.3), it follows that there exists a sequence $z^m \in S(q^m, A)$ such that $\|z^m - x\| \leq C\|\bar{p} - \bar{q}\|$ for all large m. This implies $(Az^{m_0})_n + m_0 > 0$ for some positive integer m_0, which in turn implies that

$z_n^{mo} = 0$. This implies

$$x = \bar{z} + C\|\bar{p} - \bar{q}\|\bar{u} \text{ for some } \bar{u} \in B,$$

where $z = (z_1^{mo}, z_2^{mo}, \ldots, z_{n-1}^{mo})^t \in S(q, M)$. This completes the proof of (2.4.1). It follows that M is Lipschitzian. The rest of the proof of the theorem follows by induction. ■

Lemma 2.4.4 *Suppose $A \in \boldsymbol{R}^{2\times 2}$ has one of the following sign structures: (i)* $\begin{bmatrix} - & - \\ \oplus & - \end{bmatrix}$ *(ii)* $\begin{bmatrix} - & \oplus \\ - & - \end{bmatrix}$, *where $\oplus$ stands for $+$ or 0. Then A is not Lipschitzian.*

Proof: Note that if A has sign structure given by (i) or (ii), then $A \in \boldsymbol{Q}_o$. It is easy to see that A violates property (**) (in both the cases (i) and (ii)). From Theorem 2.3.6 it follows that A is not Lipschitzian. ■

Lemma 2.4.5 *Suppose $A \in \boldsymbol{R}^{3\times 3}$ is a Lipschitzian matrix. Then A cannot have the following sign structure:*

$$\begin{bmatrix} - & 0 & - \\ 0 & - & - \\ - & - & - \end{bmatrix}.$$

Proof: Suppose A has the above sign structure. Then the sign structure of PPT of A with respect to $\alpha = \{2, 3\}$ (exists by Theorem 2.4.2) is given by

$$\begin{bmatrix} - & + & - \\ - & + & - \\ + & - & + \end{bmatrix}$$

(note that $det\ A_{\alpha\alpha} < 0$ by Theorem 2.3.7). Clearly A is a $\boldsymbol{Q}_o$-matrix which violates property (**). This contradicts that A is a Lipschitzian matrix. It follows that A cannot have sign structure mentioned in the theorem. ■

Theorem 2.4.6 *Suppose $A \in R^{n\times n}$ is a Lipschitzian matrix. If the diagonal entries of A are negative, then $A \leq 0$.*

Proof: Note that every 2×2 matrix with negative diagonal entries is a Q_o-matrix. By property (**), it follows that every 2×2 principal submatrix is nonpositive. Hence $A \leq 0$. ■

Theorem 2.4.7 *Suppose $A \in R^{n\times n}$ is a nonpositive Lipschitzian matrix. Then there exists a permutation matrix P such that*

$$PAP^t = \begin{bmatrix} N^1 & 0 & \dots & 0 \\ 0 & N^2 & \dots & 0 \\ . & . & \dots & 0 \\ 0 & 0 & \dots & N^k \end{bmatrix},$$

where each N^j is a negative N-matrix.

Proof: We shall prove this by induction on n. From Lemma 2.4.4, the theorem is true for $n = 1$ or 2. Assume that the theorem is true for all square matrices of order less than or equal to $(n-1)$, $n > 2$. Let $\alpha = \{i : a_{in} < 0\}$. We will show that $A_{\alpha\alpha} < 0$, $A_{\alpha\bar{\alpha}} = 0$, and $A_{\bar{\alpha}\alpha} = 0$. Let $i, j \in \alpha$ and let $\beta = \{i, j, n\}$. Then a principal rearrangement of $A_{\beta\beta}$ will have the sign pattern

$$\begin{bmatrix} - & & - \\ & - & - \\ - & - & - \end{bmatrix}$$

If $a_{ij} = 0$ or $a_{ji} = 0$, then by Lemma 2.4.4, $a_{ij} = a_{ji} = 0$. This contradicts Lemma 2.4.5. It follows that $A_{\alpha\alpha} < 0$. A similar argument will show that $A_{\alpha\bar{\alpha}} = 0$ and $A_{\bar{\alpha}\alpha} = 0$. By induction hypothesis, $A_{\bar{\alpha}\bar{\alpha}}$ has the desired structure and the theorem follows. ■

Corollary 2.4.8 *Suppose $A \in R^{n\times n}$ is a Lipschitzian matrix. Then A (subject to a principal rearrangement) has the following structure:*

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 & A_{1k} \\ 0 & A_{22} & \dots & 0 & A_{2k} \\ . & . & \dots & . & . \\ 0 & 0 & \dots & A_{(k-1)(k-1)} & A_{(k-1)k} \\ A_{k1} & A_{k2} & \dots & A_{k(k-1)} & A_{kk} \end{bmatrix}$$

where A_{ii}, $i = 1, 2, \dots, k-1$, are negative N-matrices and diagonal entries of A_{kk} are positive.

Proof: Follows from Theorem 2.4.6 and Theorem 2.4.7. ■

2.5 Concluding remarks

In this chapter, we have emphasized the importance of property (**), which arose naturally as a necessary condition for a Lipschitzian matrix to be a Q_0-matrix. One of the important consequences is that if A (or any of its PPTs) is either a P-matrix or an N-matrix of second kind or is a Lipschitzian Q_0-matrix, then the corresponding LCPs can be processed by simple principal pivoting method which avoids cycling.

If in the definition of property (**), we use columns instead of rows, then Theorem 2.3.6 does not hold. Consider the following examples. $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$. It can be verified that A is a non-Q_0 Lipschitzian matrix satisfying *property* (**) with respect to columns. On the other hand, B is a Lipschitzian Q_0-matrix, but B does not satisfy property (**) with respect to columns.

Chapter 3

On Lipschitzian Q_0 and INS Matrices

3.1 Introduction

Samelson, Thrall and Wesler [48], and Ingleton [19] independently established that a matrix $A \in \boldsymbol{R}^{n\times n}$ is a $\boldsymbol{P}$ matrix, if and only if (q, A) has a unique solution for every $q \in \boldsymbol{R}^n$. The class of $\boldsymbol{U}$ matrices introduced by Cottle and Stone [8] is an extension of the class $\boldsymbol{P}$. A matrix $A \in \boldsymbol{R}^{n\times n}$ is said to be in $\boldsymbol{U}$, if (q, A) has a unique solution for all q in the interior of $K(A)$ (see Stone [52]). The $\boldsymbol{INS}$ matrices introduced by Stone [53] may be viewed as a generalization of the class $\boldsymbol{U}$.

The main purpose of this chapter is to settle a conjecture raised by Murthy, Parthasarathy and Sriparna [36], and derive certain interesting properties of nondegenerate $\boldsymbol{INS}$ matrices. The class of $\boldsymbol{INS}$ matrices was introduced by Stone [53], and he established that Lipschitzian matrices are nondegenerate $\boldsymbol{INS}$ matrices. Stone [54], proved the converse under the

Lipschitzian path-connectedness of $K(M)$.

In [36], Murthy, Parthasarathy and Sriparna conjectured that nondegenerate matrices satisfying property (**) are Lipschitzian. In an attempt to settle this conjecture, we first establish that a nondegenerate matrix satisfying property (**) belongs to the class $\boldsymbol{INS} \cap \boldsymbol{Q}_0$ and this in turn implies that it is Lipschitzian.

In the previous chapter it was shown that the class of Lipschitzian matrices is complete, that is, if a matrix is Lipschitzian then all its principal submatrices are Lipschitzian. In [54], Stone conjectured that the class of nondegenerate $\boldsymbol{INS}$ matrices is complete. We settle this conjecture affirmatively in Section 3.3.

There is no constructive characterization of Lipschitzian or $\boldsymbol{INS}$ matrices. In Section 3.4, we obtain some necessary conditions on the nondegenerate $\boldsymbol{INS}$ class. We conjecture that the *block* property (B property) is sufficient for a matrix to be Lipschitzian. We verify this for $n = 2$ and certain other special cases.

3.2 Preliminaries

The following theorem by Eaves (see [9]) establishes the equivalence between the class $\boldsymbol{Q}_0$ and the convexity of $K(M)$.

Proposition 3.2.1 *Let us suppose $M \in \boldsymbol{R}^n$, then the following conditions are equivalent:*

(i) $M \in \boldsymbol{Q}_0$.

(ii) $K(M)$ is convex.

(iii) $K(M) = pos(I, -M)$.

Definition 3.2.2 *A matrix A is said to be an INS_k-matrix if $|S(q, A)| = k$ for all $q \in intK(A)$, where $K(A)$ is the set of all p for which $S(p, A) \neq \emptyset$, and the class $INS = \cup_{k=0}^{\infty} INS_k$.*

For $A \in R^{n \times n}$, let us consider a complementary cone pos $C(J)$ relative to A, where the matrix $C(J) \in R^{n \times n}$ for $J \subset \bar{n}$ is defined as $C(J)_{.j} = -A_{.j}$ if $j \in J$ and $C(J)_{.j} = I_{.j}$ otherwise (see Section 1.2). We denote by pos $C(J)_{.\bar{i}}$ the facet relative to A for some $i \in \bar{n}$. The following definitions of proper and reflecting facets relative to A are needed in the sequel.

Definition 3.2.3 *For $A \in R^{n \times n}, \alpha \subseteq \bar{n}$ and $i \in \bar{n}$, consider the product,*

$$(detA_{\alpha\alpha})(detA_{\beta\beta}).$$

for $\beta \subseteq n$ such that $\alpha \bigtriangleup \beta = \{i\}$. The common facet pos $C(J)_{.\bar{i}}$ is proper (reflecting) if the above product is positive (negative). If the product is zero, then the common facet pos $C(J)_{.\bar{i}}$ relative to A is said to be degenerate.

If a facet $F =$ pos $C(J)_{.\bar{i}}$ is reflecting, then there exists a nonzero vector $p \in R^n$, such that $p^t F = 0$ and the columns $I_{.i}$ and $-A_{.i}$ lie on the same side of the hyperplane $H = \{x : p^t x = 0\}$

Given $A \in R^n$, the cone $K(A)$ is said to be regular if all the reflecting and degenerate facets relative to A lie on the boundary of $K(A)$.
We now state a theorem from Cottle, Pang and Stone ([9] Theorem 6.6.22, page 595), which will be useful in the next section.

Theorem 3.2.4 *Let $A \in R^{n \times n}$. Suppose $intK(A)$ is connected. Then $A \in INS$ if and only if $K(A)$ is regular.*

Definition 3.2.5 *Let $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$. A multivalued mapping $\tau : \boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$ is said to be upper semicontinuous at q if for every ϵ there exists a δ such that*

$$\tau(\bar{q}) \subseteq \tau(q) + \epsilon B$$

for all $\bar{q}$ satisfying $\|\bar{q} - q\| < \delta$.

The following theorem is taken from Cottle Pang and Stone [9].

Theorem 3.2.6 *Let $A \in \boldsymbol{R}^{n \times n}$. If $S(\bar{q}, A)$ is bounded, then the multivalued solution map $S : \boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$ is upper semicontinuous at $\bar{q}$; moreover there exists a constant $c' > 0$ and a neighbourhood V of $\bar{q}$ such that,*

$$\|z\| \leq c' \quad \textit{for all} \quad z \in S(q, A) \quad \textit{and} \quad q \in V.$$

$S(q, A)$ is said to be uniformly bounded in a set Y if,

$$\|z\| \leq K \quad \forall\, z \in S(q, A) \text{ and } \forall\, q \in Y.$$

Hence the second inclusion of the above theorem states if LCP($\bar{q}, A$) has a (nonempty) bounded solution set, then the (nonempty) solution sets of all nearby LCP(q, A), with q sufficiently close to $\bar{q}$ are uniformly bounded. In [54], Stone proved the following theorem.

Theorem 3.2.7 *If $A \in \boldsymbol{R}^{n \times n}$ is Lipschitzian, then $A \in \boldsymbol{INS}$.*

The converse of this holds under certain special conditions.

Definition 3.2.8 *A set $S \subseteq \boldsymbol{R}^n$ is said to be Lipschitz arc-connected (path-connected) if there exists a constant L such that, for all $x, y \in S$, the set S contains a polygonal arc between x and y whose length does not exceed $L\|x - y\|$.*

If $S \subseteq \boldsymbol{R}^n$ is convex, then it is clearly Lipschitz path-connected. However, many nonconvex sets are Lipschitz path-connected. The following example due to Stone from [54], illustrates this point.

Example 3.2.9 *Let* $S \equiv \{x \in \boldsymbol{R}^2 : x_1 \geq 0 \text{ or } x_2 \geq 0\}$.

This set S is not convex, but it is Lipschitz path-connected. To see it, take any $x, y \in S$, such that $x_1 < 0$ and $y_2 < 0$. The line segment between x and y may or may not lie in S. If we take $z \in \boldsymbol{R}^2$ such that $z_1 = y_1 \geq 0$ and $z_2 = x_2 \geq 0$. The line segment between x and z, and the line segment between y and z are in S. This is true for all $x, y \in S$. The length of the path thus formed will obviously be less than or equal to $\|x - y\|\sqrt{2}$. Thus S is Lipschitz path-connected.

An example of a set, which is not Lipschitz path-connected, is the following (see [54]).

Example 3.2.10 $S \equiv \{x \in \boldsymbol{R}^2 : x_2 \neq 0 \text{ or } x_1 < 0\}$

The following theorem and corollary, are due to Stone [54].

Theorem 3.2.11 *If $A \in \boldsymbol{INS}$ is a nondegenerate matrix and if $intK(A)$ is Lipschitz arc-connected, then A is Lipschitzian.*

Corollary 3.2.12 *Let $A \in \boldsymbol{Q}_o \cap \boldsymbol{R}^{n \times n}$ be given. A is Lipschitzian if and only if A is a nondegenerate $\boldsymbol{INS}$-matrix.*

Murthy et al [35] and Sridhar [51] showed the following. Let $A \in \boldsymbol{R}^{n \times n}$. If A is Lipschitzian and $\boldsymbol{Q}_o$, then A satisfies property (**). We have shown in the previous chapter that if A satisfies property (**), then A is in $\boldsymbol{Q}_o$.

3.3 On Lipschitzian Q_0 and INS matrices

For $A \in \boldsymbol{R}^{n \times n}$, if $\operatorname{int} K(A)$ is connected and all the reflecting and degenerate facets relative to A lie on the boundary of $K(A)$, then from Theorem 3.2.4, it follows that A is an INS matrix. Using this, we prove the following result.

Theorem 3.3.1 *Let $A \in \boldsymbol{R}^{n \times n}$ be a nondegenerate matrix. If A satisfies property (**), then A is in the class $\boldsymbol{INS} \cap \boldsymbol{Q_o}$.*

Proof: From Corollary 2.3.5, we note that A is a $\boldsymbol{Q_o}$-matrix. Since A is a nondenerate matrix , there are no degenerate facets relative to A. It needs to be shown that every reflecting facet relative to A lies on the boundary of $K(A)$. Suppose A satisfies property (**). Let $a_{11} < 0$. It is clear that the facet $F = \operatorname{pos}(I_{.2}, \ldots, I_{.n})$ is reflecting. Since A satisfies property (**), $a_{1j} \leq 0$ for all $j \in \{2, \ldots, n\}$. For the vector $p = I_{.1} \in \boldsymbol{R}^n$, we have $p^t F = 0$ and $p^t(-A_{.j}) \geq 0$ for all j. This implies that $p^t q \geq 0 \, \forall q \in K(A)$. Hence, $K(A)$ lies fully on one side of the hyperplane containing F. So the facet F lies on the boundary of $K(A)$.

Suppose $F = \operatorname{pos} C(J)_{.\bar{i}}$ for $J \subseteq n$, $|J| \neq 1$, is a reflecting facet for some $i \in n$. Since the complementary matrix $C(J)$ is nonsingular, we can consider a principal pivot transform with respect to J. The resulting matrix is $M = C(J)^{-1}\bar{C}(J)$, where $\bar{C}(J)$ contains columns of $[I, -A]$ not in $C(J)$. Its (i,i)th diagonal entry is negative. Since A satisfies property (**), we notice as before that the facet $F = \operatorname{pos}(I_{.1}, \ldots, I_{.i-1}, I_{.i+1}, \ldots, I_{.n})$ lies on the boundary of $K(M)$. Since there is a one-to-one correspondence between the complementary cones relative to A and those relative to M, we conclude that the reflecting facet F relative to A lies on the boundary of $K(A)$. Since this is true for any reflecting facet relative to A and $\operatorname{int} K(A)$

is connected, we conclude that A belongs to the $\boldsymbol{INS}$ class of matrices.
■

As a corollary to the above theorem we answer a question raised by Murthy Parthasarathy and Sriparna in [36], in the affirmative.

Corollary 3.3.2 *Let $A \in \boldsymbol{R}^{n\times n}$ be a nondegenerate matrix. If A satisfies property (**), then A is Lipschitzian.*

Proof: From the above theorem, it follows that A is an $\boldsymbol{INS} \cap \boldsymbol{Q}_0$ matrix. Since any $\boldsymbol{Q}_0$ matrix is Lipschitzian arc-connected (Proposition 3.2.1), the result follows from Theorem 3.2.11. ■

Combining the results given here and the known results on Lipschitzian matrices we have the following theorem.

Theorem 3.3.3 *Let $A \in \boldsymbol{R}^{n\times n}$ be a nondegenerate $\boldsymbol{Q}_0$ matrix. Then the following are equivalent:*

(i) A is Lipschitzian.

(ii) A satisfies property (**).

(iii) A is an $\boldsymbol{INS}$ matrix.

Furthermore if any of these conditions hold for A then it holds for every principal submatrix of A.

It has been observed by Murthy, Parthasarathy and Sriparna [36] that the class of Lipschitzian matrices is complete, that is, if A is Lipschitzian then all the principal submatrices of A are also Lipschitzian. In [54], Stone raises the question, whether the class of nondegenerate $\boldsymbol{INS}$ matrices is complete. From the above theorem it is clear that the class of nondegenerate $\boldsymbol{INS} \cap \boldsymbol{Q}_0$ matrices is complete.

We prove the general case in the following theorem.

Theorem 3.3.4 *The nondegenerate* $\boldsymbol{INS}$ *class is complete.*

Proof: Let $A \in \boldsymbol{INS}$ be a nondegenerate matrix. It suffices to show that B the principal submatrix of A obtained by leaving out the 1st row and 1st column, is an $\boldsymbol{INS}$ matrix. Let A be partitioned as

$$A = \begin{bmatrix} a_{11} & a^t \\ b & B \end{bmatrix} \tag{3.3.1}$$

where $a, b \in \boldsymbol{R}^{n-1}$ correspond to the first row and first column of A leaving out the diagonal entry a_{11}.

It is clear that if the facet $\bar{F} = \text{pos}\ (I_{.2}, \ldots, I_{.k-1}, -B_{.k+1}, \ldots, -B_{.n})$ is a reflecting facet relative to $K(B)$ for some $2 \leq k \leq n$, then the facet F defined by $F = \text{pos}\ (I_{.1}, \ldots, -I_{.k-1}, -A_{.k+1}, \ldots, -A_{.n})$ is a reflecting facet relative to $K(A)$. We claim that if $\bar{F} \cap int K(B) \neq \phi$, then $F \cap int K(A) \neq \phi$; this will imply that the reflecting facet F relative to $K(A)$ does not lie on the boundary of $K(A)$, contradicting that A is an $\boldsymbol{INS}$ matrix.

In order to prove our claim, let us first consider reflecting facets relative to $K(B)$ of the form $\text{pos}(I_{.2}, \ldots, I_{.k-1}, I_{.k+1}, \ldots, I_{.n})$, for some $2 \leq k \leq n$. Suppose $\bar{F} = \text{pos}(I_{.2}, \ldots, I_{.n-1})$ is reflecting and $\bar{F} \cap int K(B) \neq \phi$. Then for some $\bar{q} = (\bar{q}_2, \ldots, \bar{q}_n)^t \in \bar{F}$, there exists an $\epsilon > 0$ such that for any $\bar{q}' \in \boldsymbol{R}^{n-1}$ with $\|\bar{q}' - \bar{q}\| < \epsilon$, LCP$(\bar{q}', B)$ has a solution. We can assume without loss of generality that $\bar{q}$ is of the form $(\bar{q}_2, \ldots, \bar{q}_{n-1}, 0)^t$ where $\bar{q}_i > 0$ for $i = 2, \ldots, n-1$. Let $U = \{q : \|q - \bar{q}\| < \epsilon\}$. Since B is a nondegenerate matrix, $S(q, B)$ is uniformly bounded for q varying over a bounded set. From Theorem 3.2.6, it follows that there exists a constant $\lambda > 0$ and a neighbourhood V of $\bar{q}$ in $\boldsymbol{R}^{n-1}$, such that

$$\|z\| \leq \lambda \quad \forall \quad z \in S(q', B), \quad q' \in V.$$

We can assume without loss of generality that $U \subseteq V$. Now, let $q \in \boldsymbol{R}^n$ be defined as $q_i = q_i$ for $i = 2, \ldots, n$ and q_1 be chosen such that $q_1 > \|a\|\lambda + \epsilon$. where a is as given in (3.3.1). Clearly $q \in F$. Choose $r \in \boldsymbol{R}^n$ such that $\|r - q\| < \epsilon$. Then $\|\bar{r} - \bar{q}\| < \|r - q\|$ where $\bar{r}$ is the $n-1$ vector obtained from r by leaving out the first coordinate. Hence LCP$(\bar{r}, B)$ has a solution (u, v) where $u, v \in \boldsymbol{R}^{n-1}$. Let

$$z_i = v_{i-1},\ w_i = u_{i-1} \quad \text{for} \quad i = 2, \ldots, n$$

$$z_1 = 0 \quad w_1 = a^t z + r_1.$$

If $a^t z > 0$, then $w_1 > 0$. Otherwise, $-a^t z = |a^t z| \leq \|a\| . \|z\| \leq \|a\|\lambda < q_1 - \epsilon$. Since $r_1 \geq q_1 - \epsilon$, we have $w_1 = r_1 + a^t z \geq 0$. Hence, (w, z) is a solution to the LCP(r, A). This imples that for every $\|r - q\| < \epsilon$, $r \in K(A)$, which, in turn, implies that $F \cap intK(A) \neq \phi$. This contradicts that $A \in \boldsymbol{INS}$. Hence, the reflecting facet $\bar{F}$ relative to $K(B)$ does not intersect $intK(B)$. Now if $\bar{F}$ is any other reflecting facet relative to B, then we can consider a principal pivotal transform C of A with respect to a complementary cone relative to $K(B)$, incident on $\bar{F}$. Then C is an $\boldsymbol{INS}$ matrix, and $\bar{C}$, the principal submatrix of C leaving out its first row and first column, is a principal pivot transform of B. From the one-to-one correspondence existing between the facets relative to B and the facets relative to $\bar{C}$, we notice that $\bar{F}$ corresponds to a reflecting facet of $K(\bar{C})$ of the form, pos $(I_{.2}, \ldots, I_{.k-1}, I_{.k+1}, \ldots, I_n)$, for some $2 \leq k \leq n$. Therefore, we can appeal to our earlier argument to conclude that $\bar{F}$ does not intersect the interior of $K(B)$. This along with B being nondegenerate implies that B is an $\boldsymbol{INS}$ matrix. This concludes the proof. ∎

3.4 Nondegenerate INS and Lipschitzian matrices

Stone in [53] showed that Lipschitzian matrices are nondegenerate $\boldsymbol{INS}$-matrices and conjectured that the converse is also true. Furthermore, he showed that the conjecture is true under the additional assumption of Lipschitz *path-connectedness* (see [53] for details). Till date, no constructive characterization is known for $\boldsymbol{INS}$ matrices. Thus, there is no finite procedure to verify whether a given matrix is $\boldsymbol{INS}$ or not.

Definition 3.4.1 *Say that A has property (B) if every PPT M of A has the following block structure (subject to a principal rearrangement):*

$$M = \begin{bmatrix} M_{11} & 0 & \dots & 0 & M_{1\overline{l+1}} \\ 0 & M_{22} & \dots & 0 & M_{2\overline{l-1}} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & M_{ll} & M_{l\overline{l+1}} \\ M_{\overline{l+1}1} & M_{\overline{l+1}2} & \dots & M_{\overline{l+1}l} & M_{\overline{l+1}\,\overline{l+1}} \end{bmatrix},$$

where $M_{11}, M_{22}, \dots, M_{ll}$ are all negative $\boldsymbol{N}$-matrices (i.e., all entries and all principal minors are negative) and the diagonal entries of $M_{\overline{l+1}\,\overline{l+1}}$ are positive.

In the previous chapter we have shown that a Lipschitzian matrix satisfies property (B). In this section, we show that every nondegenerate $\boldsymbol{INS}$-matrix also satisfies property (B).

Note that if a matrix A has property (B), then it must be nondegenerate as all the diagonal entries of every PPT of A is nonzero (see Proposition 2.2.1). From the definition, it follows that property (B) is invariant under PPTs and is inherited by all the principal submatrices.

Theorem 3.4.2 *Suppose $A \in \boldsymbol{R}^{n\times n}$ is a nondegenerate $\boldsymbol{INS}$-matrix. Then A has property (B).*

Proof: Let $\alpha = \{i : a_{ii} < 0\}$. By Theorem 3.3.4, $A_{\alpha\alpha}$ is a nondegenerate $\boldsymbol{INS}$-matrix. Also, for $i, j \in \alpha$, $i \neq j$, $A_{\beta\beta} \in \boldsymbol{INS}$, where $\beta = \{i, j\}$. It is easy to check that if $A_{\beta\beta}$ has a positive entry, then $A_{\beta\beta} \notin \boldsymbol{INS}$. It follows that $A_{\alpha\alpha}$ is nonpositive and hence is in $\boldsymbol{Q}_o$. From Corollary 3.2.12, $A_{\alpha\alpha}$ is Lipschitzian. From Theorem 2.4.7 of the previous chapter we get,

$$A_{\alpha\alpha} = \begin{bmatrix} N^1 & 0 & \dots & 0 \\ 0 & N^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & N^l \end{bmatrix} \quad \text{for some } l \geq 1,$$

where each N^i is a negative $\boldsymbol{N}$-matrix. Since all the PPTs of a nondegenerate $\boldsymbol{INS}$-matrix are also nondegenerate $\boldsymbol{INS}$, we conclude that A has property (B). ■

We conjecture that property (B) is also sufficient for a matrix to be Lipschitzian. We verify this conjecture in certain special cases.

Theorem 3.4.3 *Suppose $A \in \boldsymbol{R}^{n\times n}$. Assume that any one of the following conditions holds:*

(i) $n = 2$

(ii) $A \leq 0$

(iii) *A is completely-$\boldsymbol{Q}$*

(iv) $A \geq 0$.

Then the following statements are equivalent:

(a) *A is nondegenerate* $\boldsymbol{INS}$

(b) *A is Lipschitzian*

(c) *A has property (B).*

Proof: In view of Stone's result that (b) $\Rightarrow$ (a) (Theorem 3.2.7), it suffices to show that (c) implies (b). So assume that (c) holds.

(i) If the diagonal entries of A are negative, then property (B) implies that either A is a negative $\boldsymbol{N}$-matrix or $A \simeq \begin{bmatrix} - & 0 \\ 0 & - \end{bmatrix}$. In the first case, when A is a negative $\boldsymbol{N}$-matrix, A is Lipschitzian by Remark 2.2.8. In the other case A is a nondegenerate matrix satisfying property (**), hence it is Lipschitzian by Corollary 3.3.2. If the diagonal entries of A are positive, then either A is a $\boldsymbol{P}$-matrix or A^{-1} is a negative $\boldsymbol{N}$-matrix. Once again A is Lipschitzian (see Remark 2.2.8). Consider the last case $a_{11} < 0$ and $a_{22} > 0$, without loss of generality. It is easy to check (graphically) that A is $\boldsymbol{INS}$ and that $K(A)$ is Lipschitz path-connected (see Stone [53] for details and Example 3.2.9). From Theorem 3.2.11, we conclude that A is Lipschitzian.

(ii) By property (B), A can be decomposed into a block diagonal matrix where each submatrix on the diagonal is a negative $\boldsymbol{N}$-matrix. As negative $\boldsymbol{N}$-matrices are Lipschitzian one can easily verify that A is also Lipschitzian.

(iii) In this case we actually show that A is a $\boldsymbol{P}$-matrix and this we do by induction on the order of the matrix. Obviously, the result is true for $n = 1$. Assume the result for all matrices of order $n-1$, $n > 1$. Suppose $A \in \boldsymbol{R}^{n\times n}$ satisfies the hypothesis. Then all the proper principal minors of A are positive. If $A \notin \boldsymbol{P}$, then *det* $A < 0$ and that the diagonal entries of A^{-1} are negative. By property (B), A^{-1} must be nonpositive. But this contradicts that $A \in \boldsymbol{Q}$. Hence $A \in \boldsymbol{P}$ (So by Mangasarian and Shiau [25], A is Lipschitzian).

(iv) From the hypothesis and (c), $a_{ii} > 0$ for all i. Since $A \geq 0$, A is completely-$\boldsymbol{Q}$. Therefore $A \in \boldsymbol{P}$. ■

Proposition 3.4.4 *Suppose $A \in \boldsymbol{R}^{n \times n}$. Assume that for some index set α, $A_{\alpha\alpha}$ is Lipschitzian and $A_{\bar{\alpha}\bar{\alpha}} \in \boldsymbol{P}$. If $A_{\bar{\alpha}\alpha} = 0$ or $A_{\alpha\bar{\alpha}} = 0$, then A is Lipschitzian.*

Proof: Assume $A_{\bar{\alpha}\alpha} = 0$. Let p, $q \in K(A)$. Let λ_1 and λ_2 be the Lipschitzian constants corresponding to $A_{\alpha\alpha}$ and $A_{\bar{\alpha}\bar{\alpha}}$, respectively. Take any arbitrary $x \in S(p, A)$. We will exhibit a $z \in S(q, A)$ such that $\|z - x\| \leq \lambda\|p - q\|$, where λ, to be chosen later, depends only on λ_1, λ_2 and A. Since $S(q, A) \neq \emptyset$, choose any $\bar{z} \in S(q, A)$. Let $y = Ax + p$ and $\bar{w} = A\bar{z} + q$. Note that $x_\alpha \in S(p'_\alpha, A_{\alpha\alpha})$ and $\bar{z}_\alpha \in S(q'_\alpha, A_{\alpha\alpha})$, where $p'_\alpha = p_\alpha + A_{\alpha\bar{\alpha}}x_{\bar{\alpha}}$ and $q'_\alpha = q_\alpha + A_{\alpha\bar{\alpha}}\bar{z}_{\bar{\alpha}}$. Since $A_{\alpha\alpha}$ is Lipschitzian, there exists a $z_\alpha \in S(q'_\alpha, A_{\alpha\alpha})$ such that

$$\begin{aligned} \|x_\alpha - z_\alpha\| &\leq \lambda_1\|p'_\alpha - q'_\alpha\| \\ &\leq \lambda_1\|p_\alpha - q_\alpha\| + \lambda_1\|B\|\|x_{\bar{\alpha}} - z_{\bar{\alpha}}\| \end{aligned}$$

Since $z_\alpha \in S(q'_\alpha, A_{\alpha\alpha})$, $w_\alpha = A_{\alpha\alpha}z_\alpha + q_\alpha + A_{\alpha\bar{\alpha}}\bar{z}_{\bar{\alpha}}$ and $w^t_\alpha z_\alpha = 0$. This implies $z = (z^t_\alpha, \bar{z}^t_{\bar{\alpha}})^t \in S(q, A)$. As $A_{\bar{\alpha}\bar{\alpha}} \in \boldsymbol{P}$, $x_{\bar{\alpha}}$ and $z_{\bar{\alpha}}$ are the unique solutions of $(p_{\bar{\alpha}}, A_{\bar{\alpha}\bar{\alpha}})$ and $(q_{\bar{\alpha}}, A_{\bar{\alpha}\bar{\alpha}})$. Therefore, $\|x_{\bar{\alpha}} - z_{\bar{\alpha}}\| \leq \lambda_2\|p_{\bar{\alpha}} - q_{\bar{\alpha}}\|$.

Combining this with the above inequality, we get

$$\begin{aligned}
\|x - z\| &\leq \|x_\alpha - z_\alpha\| + \|x_{\bar\alpha} - z_{\bar\alpha}\| \\
&\leq \lambda_1\|p_\alpha - q_\alpha\| + (\lambda_1\lambda_2\|B\| + \lambda_2)\|p_{\bar\alpha} - q_{\bar\alpha}\| \\
&\leq \lambda_1\|p - q\| + (\lambda_1\lambda_2\|B\| + \lambda_2)\|p - q\| \\
&\leq \lambda\|p - q\|, \text{ where } \lambda = \lambda_1 + \lambda_2 + \lambda_1\lambda_2\|B\|.
\end{aligned}$$

It follows that A is Lipschitzian. The case $A_{\alpha\bar\alpha} = 0$ can be tackled in a similar fashion. ∎

The above Proposition is not valid if we simply assume that $A_{\alpha\alpha}$ and $A_{\bar\alpha\bar\alpha}$ are Lipschitzian. As a counter example, consider $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. It is clear that $A \notin \boldsymbol{INS}$ and hence A is not Lipschitzian.

Chapter 4

Some results on C_0^f, Adequate and Connected matrices

4.1 Introduction

As discussed earlier, the characterization of completely Q_0 matrices [5] in general is still an open problem. Murthy and Parthasarathy [31, 32] have shown that nonnegative matrices, symmetric copositive matrices and fully copositive matrices are in Q_0 if and only if they are completely Q_0. We establish, that a column adequate matrix, introduced by Ingleton [19], is in Q (Q_0) if and only if it is completely Q (completely Q_0).

The class of C_0^f matrices was introduced by Murthy and Parthasarathy in [31]. Within the class of C_0^f matrices, we provide a sufficient condition under which a given matrix will be in P_0. As a corollary to this result we give an alternative proof of a result by Murthy and Parthasarathy [31], which states that $C_0^f \cap Q_0 \subseteq P_0$. As another consequence of this result, we deduce that a bisymmetric E_0^f-matrix A is positive semidefinite if and only if the rows and columns of $A + A^t$ corresponding to the zero diagonal entries are zero.

In Section 4.4 we consider the class of connected matrices. The class of connected matrices are those for which the solution set is (topologically) connected for all q. We settle a conjecture raised by Jones and Gowda [20], subject to some additional assumptions. The original problem however still remains unsolved.

There are three papers dealing with connectedness of $S(q, A)$. In [47], Rapcsak gives a sufficient condition for the connectedness of certain subsets of the solution set of an LCP corresponding to a symmetric matrix. But Cao and Ferris [3] were the first to study explicitly the class of connected matrices. They denoted this class as P_c-matrices, and proved the following.

(i) The matrices $P_c \cap Q_0$ are processed by Lemke's algorithm [23].

(ii) $P_0 \subseteq P_c \subseteq E_0$ for 2×2 matrices.

(iii)If $A \in P_0$, then for all q except those in a set of measure zero (which depends on A), LCP(q, A) has a connected solution set.

They conjectured that (ii) holds for matrices of general order n. Stone gave an example to show that the claim $P_0 \subseteq P_c$ is false for matrices of general order. The second inclusion, that is $P_c \subseteq E_0$, was proved by Jones and Gowda, and they denote this class as E_c. They have in fact shown that $E_c \subseteq E_0^f$ [20] .

The following results given below are due to Jones and Gowda [20].

(i) If A is a P_0 matrix then for any $q \in R^n$, if $S(q, A)$ has a bounded connected component, then $S(q, A)$ is connected.

(ii) The following are equivalent for an $\boldsymbol{R}_0$-matrix:

(a) $A \in \boldsymbol{P}_0$.

(b) A is connected.

They conjectured that $\boldsymbol{P}_0 \cap \boldsymbol{Q}_0 = \boldsymbol{E}_c \cap \boldsymbol{Q}_0$. In an attempt to settle this conjecture, we show in Section 3, that if all the entries of the matrix are nonnegative, then $\boldsymbol{P}_0 \cap \boldsymbol{Q}_0 = \boldsymbol{E}_c \cap \boldsymbol{Q}_0$. We also prove that if $A \in \boldsymbol{R}^{n\times n}$ is a $\boldsymbol{P}_1$-matrix, then $S(q, A)$ is connected.

4.2 Preliminary results.

Adequate and fully copositive matrices

We recall the definitions of some of the matrix classes, relevant to this chapter. $A \in \boldsymbol{R}^{n\times n}$, is said to be a $\boldsymbol{P}$-matrix ($\boldsymbol{P}_0$-matrix) if all its principal minors are positive (nonnegative); if all principal minors of A are nonzero, then A is called a nondegenerate matrix; A is semimonotone ($\boldsymbol{E}_0$) if (q, A) has a unique solution for every $q > 0$; A is fully semimonotone ($\boldsymbol{E}_0^f$) if every PPT of A is in $\boldsymbol{E}_0$; A is copositive ($\boldsymbol{C}_0$) if $x^t A x \geq 0$ for every $x \geq 0$; A is fully copositive ($\boldsymbol{C}_0^f$) if every PPT of A is in $\boldsymbol{C}_0$.
The following results are due to Murthy and Parthasarathy [32].

Theorem 4.2.1 *Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C}_0^f$. Then $a_{ii} > 0$ for all $i \in \bar{n}$, implies that $A \in \boldsymbol{P}_0$.*

Theorem 4.2.2 *Let $A \in \boldsymbol{R}^{n\times n}$. If A is a nonnegative matrix with $n \geq 2$, then A belongs to $\boldsymbol{Q}_0$ iff the following condition holds:*

$$\text{for every} \quad i \in \bar{n}, \quad A_{i.} \neq 0 \Rightarrow a_{ii} > 0.$$

Theorem 4.2.3 *Suppose $A \in R^{n \times n} \cap Q_o \cap P_o$. Assume for some $i_0, j_0 \in \bar{n}, a_{i_0 i_0} = 0$ and $a_{i_0 j_0} > 0$. Then there exists a $k \in \bar{n}$ such that $a_{k i_0} < 0$.*

Theorem 4.2.4 *Suppose $A \in R^{n \times n} \cap E_o \cap N_o$. Assume that A is nonsingular. Then there exists a principal rearrangement*

$$\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A such that $\alpha \neq \phi$, $\alpha \neq \bar{n}$, $A_{\alpha\alpha}$, and $A_{\bar{\alpha}\bar{\alpha}}$ are nonpositive strict upper triangular matrices, and $A_{\alpha\bar{\alpha}}$, and $A_{\bar{\alpha}\alpha}$ are nonnegative matrices.

The following results on C_o^f matrices are due to Murthy and Parthasarathy [31].

Theorem 4.2.5 *Suppose $A \in R^{n \times n} \cap C_o^f$. Then $A \in Q_o$ if and only if the following condition holds:*

$$\text{for any } PPT\ M \text{ of } A,\ m_{ii} = 0 \Rightarrow m_{ij} + m_{ji} = 0 \quad \forall \quad i, j \in \bar{n}.$$

A matrix A is said to be bisymmetric if for some index set α, $A_{\alpha\alpha}$ and $A_{\bar{\alpha}\bar{\alpha}}$ are symmetric, and $A_{\bar{\alpha}\alpha} = -A^t_{\alpha\bar{\alpha}}$. It is easy to check that PPTs of bisymmetric matrices are bisymmetric.

Theorem 4.2.6 *Suppose $A \in R^{n \times n} \cap E_o^f$ is a bisymmetric matrix. Then A is fully copositive.*

A number of matrix classes are invariant under principal pivoting, i.e., if a matrix is in class C, then all its PPTs are also in C. The matrix classes Q, Q_o, P, P_o, E_o^f, C_o^f, INS and Lipschitzian matrices all fall in this category. In the definition below we consider another class of matrices which is invariant under PPTs.

Definition 4.2.7 *A real square matrix $A \in \boldsymbol{A}$ if for every PPT M of A the diagonal entries are nonnegative.*

Remark 4.2.8 *Note that $\boldsymbol{E}_0^f$, which contains the classes $\boldsymbol{P}_0$ and $\boldsymbol{C}_0^f$ (see [9, 31, 32]), is a subclass of $\boldsymbol{A}$. However, $\boldsymbol{A} \setminus \boldsymbol{E}_0^f$ is nonempty as $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is an example of this kind. Furthermore, it is easy to check that if $A \in \boldsymbol{A}$, then $A^t \in \boldsymbol{A}$.*

Definition 4.2.9 *Any real square matrix A is said to have property $(\boldsymbol{D})$, if for every index set α the following holds:*

$$\det A_{\alpha\alpha} = 0 \;\Rightarrow\; \text{columns of } A_{\cdot\alpha} \text{ are linearly dependent.}$$

Let $\boldsymbol{D}$ denote the class of matrices satisfying property $(\boldsymbol{D})$. Note that if $A \in \boldsymbol{A}$ ($A \in \boldsymbol{D}$), then $A_{\alpha\alpha} \in \boldsymbol{A}$ ($A_{\alpha\alpha} \in \boldsymbol{D}$) for every α. An interesting property of $\boldsymbol{D}$ is that if $A \in \boldsymbol{D}$, then (q, A) has a solution with a complementary basis for any q with $S(q, A) \neq \emptyset$ (see [32]). Another interesting property of $\boldsymbol{D}$, which is a direct consequence of the definition, is the following.

Proposition 4.2.10 *If $A \in \boldsymbol{D}$ is nonsingular, then A is nondegenerate.*

Song Xu in [50], introduced the class of column competent matrices and established an equivalence between the class $\boldsymbol{D}$ and column competent matrices.

Definition 4.2.11 *A matrix $A \in \boldsymbol{R}^{n \times n}$ is said to be column competent if*

$$z_i(Mz)_i = 0 \text{ for all } i = 1, 2, \ldots, n \Rightarrow Mz = 0$$

Theorem 4.2.12 *The following two statements are equivalent.*

(i) $A \in \boldsymbol{R}^{n \times n}$ *is column competent.*

(ii)A belongs to the class $\boldsymbol{D}$.

The following result by Song Xu (see [50]) gives an interesting characterization of the class $\boldsymbol{D}$.

Theorem 4.2.13 *The following statements are equivalent.*

(i) A is column competent.

(ii) For all vector q, the LCP(q, A) has a finite number of w-solutions.

(iii) For all vector q, any w-solution of the LCP(q, A), if it exists, must be locally w-unique.

Any w-solution, $\hat{w}$, of LCP(q, A), is said to be locally unique if there exists a neighborhood of $\hat{w}$ within which $\hat{w}$ is the only w-solution.

The following theorem is due to Murthy and Parthasarathy [32].

Theorem 4.2.14 *Suppose* $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{Q}_o$. *Let* $i \in \bar{n}$ *and* $\alpha = \bar{i}$. *Suppose A has property* $\boldsymbol{D}$. *Then either* $A_{\alpha\alpha} \in \boldsymbol{Q}_o$ *or there exists a* $\beta \in n^*$ *satisfying:*

(a) $i \in \beta$.

(b) $\det A_{\beta\beta} \neq 0$

(c) $M_{\cdot i} \leq 0$, *where* $M = \wp_{\beta} A$

(d) $u \in S(e_i, A)$, *where* $u_{\beta} = -M_{\beta i}$ *and* $u_{\bar{\beta}} = 0$ *and*

(e) $(Au)_i = -1$

A matrix A is said to be a column (row) adequate matrix if A (A^t) is in $\boldsymbol{D} \cap \boldsymbol{P}_o$. Ingleton [19] introduced the class of adequate matrices (i.e. both row and column adequate) and showed that if A is adequate, then for every q with $S(q, A) \neq \emptyset$, $Az + q$ is constant over $S(q, A)$.

Connected matrices

A matrix $A \in \boldsymbol{R}^{n\times n}$ is said to be a $\boldsymbol{P}_1$-matrix, if $A \in \boldsymbol{P}_0$ and exactly one of its principal minors is equal to zero. It is said to be a $\boldsymbol{U}$ matrix if LCP(q, A) has a unique solution for all $q \in \text{int}K(A)$. The following theorem is from Cottle Pang and Stone ([9] page 235).

Theorem 4.2.15 *A $\boldsymbol{P}_1$ matrix is also a $\boldsymbol{Q}_0$ matrix.*

The next result is due to Cottle and Stone (see [8] Theorem 6).

Theorem 4.2.16 *If $A \in (\boldsymbol{P}_1\backslash\boldsymbol{Q}) \cap \boldsymbol{R}^{n\times n}$, then $A \in \boldsymbol{U}$, and $K(A)$ is a half-space. If, in addition, $|A| = 0$, then the normal to the hyperplane $\partial K(A)$ can be chosen as a positive vector.*

The following theorem is due to Murthy and Parthasarathy [32].

Theorem 4.2.17 *Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{E}_0^f \cap \bar{\boldsymbol{Q}}_0$. Then A belongs to $\boldsymbol{P}_0$.*

Remark 4.2.18 *We know that a nonnegative matrix with all the diagonal entries positive, is a $\boldsymbol{Q}$-matrix, and if $A \in \boldsymbol{R}^{n\times n}$ is a $\boldsymbol{P}_0 \cap \boldsymbol{Q}$ matrix, then A is an $\boldsymbol{R}_0$ matrix (see Cottle Pang and Stone [9]).*

We know if A is an $\boldsymbol{R}_0$ matrix then the solution set is bounded for all $q \in \boldsymbol{R}^n$.

Next, we discuss certain concepts related to connectedness.

Definition 4.2.19 *A set S in a topological space Y is said to be disconnected if, it can be written in the form, $S = S_1 \cup S_2$ where S_1, S_2 are closed sets and $S_1 \cap S_2 = \phi$. A set S is said to be connected if, it is not disconnected.*

A component $C(y)$ in Y is the maximal connected subset of Y containing y, that is there is no connected subset of Y that properly contains $C(y)$. We state a fundamental theorem (see Dugundji [12]), on connected sets.

Theorem 4.2.20 *Let Y be a topological space. The union of any family of connected subsets having at least one point in common is also connected.*

A path in Y is a continuous mapping $f : I \to Y$, where I is the unit interval. If $f : I \to Y$ is a path in Y, we call $f(0) \in Y$ the initial (or the starting) point of Y and $f(1) \in Y$ the terminal (or end) point, of the path f, and say that f runs from $f(0)$ to $f(1)$ or joins $f(0)$ to $f(1)$.

Definition 4.2.21 *A topological space Y is path-connected (or pathwise connected) if each pair of its points can be joined by a path.*

Analogously, path connected components are defined as maximal path-connected subsets of the topological space Y.
Each path-connected space is connected. But a connected space need not be path-connected.

Lemma 4.2.22 *Suppose $S_1, S_2, \ldots, S_k$ are closed convex sets. If $\cup S_j$ is connected, then it is path connected.*

Proof: Let us take the set S_1 say. Then $S_1 \cap (\cup_{i=2}^{k} S_i) \neq \phi$, since by our hypothesis $\cup S_j$ is connected. Hence, there exists a $j \in \{2, 3, \ldots, k\}$, such that $S_1 \cap S_j \neq \phi$. Without loss of generality, let this be S_2. Since S_1, S_2 are convex sets (hence path-connected), and $S_1 \cap S_2 \neq \phi$, $S_1 \cup S_2$ is path-connected. Now instead of S_1, we can take $S_1 \cup S_2$ and similarly show that there exists a $j \in \{3, \ldots, k\}$, say $j = 3$ such that $(S_1 \cup S_2) \cap S_3 \neq \phi$. Since $S_1 \cup S_2$ is path-connected and S_3 is convex, we conclude that $S_1 \cup S_2 \cup S_3$ is path-connected. Proceeding in this way, we can conclude that $\cup S_j$ is path-connected.

Definition 4.2.23 *A matrix $A \in R^{n \times n}$ is said to be connected if for every $q \in R^n$, $S(q, A)$ is connected.*

The following theorem is due to Jones and Gowda [20].

Theorem 4.2.24 *Suppose $A \in \mathbf{R}^{n\times n}$ and $q \in \mathbf{R}^n$. Then the following statements hold.*

(i) If A is connected, then $A \in \mathbf{E}_0^f$.

(ii) If A is a $\mathbf{P}_0$ matrix then for any $q \in \mathbf{R}^n$ if $S(q, A)$ has a bounded connected component, then $S(q, A)$ is connected.

(iii) The following are equivalent for an $\mathbf{R}_0$-matrix:

(a) $A \in \mathbf{P}_0$.

(b) A is connected.

4.3 On $C_0{}^f$ and Adequate matrices

We now present our main results on column adequate matrices.

Theorem 4.3.1 *If $A \in \mathbf{A} \cap \mathbf{D}$, then $A \in \mathbf{P}_0$.*

Proof: We prove this by induction on n. Obviously the theorem is true if $n = 1$. Assume that the theorem is true for all $(n-1)\times(n-1)$ matrices. Let $A \in \mathbf{R}^{n\times n} \cap \mathbf{A} \cap \mathbf{D}$. By the previous observations, $A_{\alpha\alpha} \in \mathbf{P}_0$ for all α such that $|\alpha| = n-1$. Suppose $A \notin \mathbf{P}_0$. Then $det\, A < 0$. Note that A is almost $\mathbf{P}_0$. Since $A \in \mathbf{A}$, diagonal entries of A^{-1} by Theorem 1.2.14, are equal to zero. This means that $det\, A_{\alpha\alpha} = 0$ for all α with $|\alpha| = n-1$. Since $A \in \mathbf{D}$, this implies that columns of A are linearly dependent which contradicts that A is nonsingular. It follows that $A \in \mathbf{P}_0$. ■

Corollary 4.3.2 *Suppose $A \in \mathbf{R}^{n\times n}$. The following conditions are equivalent:*

(a) $A \in \mathbf{P}_0 \cap \mathbf{D}$.

(b) $A \in \boldsymbol{\Lambda} \cap \boldsymbol{D}$.

It is known that nondegenerate $\boldsymbol{E}_0^f$-matrices are $\boldsymbol{P}$-matrices.

Corollary 4.3.3 *If $\boldsymbol{E}_0^f \cap \boldsymbol{D}$, then $A \in \boldsymbol{P}_0$.*

A matrix A is said to be completely-$\boldsymbol{Q}$ if all its principal submatrices including A are $\boldsymbol{Q}$-matrices. Cottle introduced this class in [5] and characterized completely-$\boldsymbol{Q}$ matrices as the class of strictly semimonotone matrices. One of the problems posed by Cottle [5] is the characterization of completely-$\boldsymbol{Q}_0$ matrices which is still an open problem. In Chapter 2 we give a characterization of completely-$\boldsymbol{Q}_0$ matrices, satisfying property (**) (see [31, 32]). The following result gives a characterization of completely $\boldsymbol{Q}_0$ matrices within the class of column adequate matrices.

Theorem 4.3.4 *Suppose $A \in \boldsymbol{\Lambda} \cap \boldsymbol{D}$. Then*

(a) *$A \in \boldsymbol{Q}_0$ if and only if A is completely-$\boldsymbol{Q}_0$.*

(b) *$A \in \boldsymbol{Q}$ if and only if A is completely-$\boldsymbol{Q}$.*

Proof: (a) It suffices to show the 'only if' part. Suppose $A_{\alpha\alpha} \notin \boldsymbol{Q}_0$, say, for $\alpha = \{1, 2, \ldots, n-1\}$. By Theorem 4.2.14, there exists a β such that $n \in \beta$, $det\, A_{\beta\beta} \neq 0$ and $M_{\cdot n} \leq 0$, where $M = \wp_\beta(A)$. Since $A \in \boldsymbol{P}_0$ (Theorem 4.3.1), $M_{nn} = \frac{det\, A_{\gamma\gamma}}{det\, A_{\beta\beta}} = 0$, where $\gamma = \beta \setminus \{n\}$. This implies $det\, A_{\gamma\gamma} = 0$ which in turn implies $det\, A_{\beta\beta} = 0$ as $A \in \boldsymbol{D}$. From this contradiction, it follows that $A_{\alpha\alpha} \in \boldsymbol{Q}_0$. By induction it follows that A is completely-$\boldsymbol{Q}_0$.

(b) We will again show the 'only if' part. Note that the conclusions of Theorem 4.2.14 remain valid even if we replace $\boldsymbol{Q}_0$ by $\boldsymbol{Q}$ in the statement of that theorem (almost the same proof can be repeated). Hence it follows (from the proof of part (a) here) that A is completely-$\boldsymbol{Q}$. ■

Corollary 4.3.5 *Every column adequate matrix is in $\boldsymbol{Q}$ if and only if it is strictly semimonotone.*

We now turn our attention to the results on C_o^f-matrices. In [31], using the concept of *incidence*, Murthy and Parthasarathy established that $C_o^f \cap \boldsymbol{Q}_o \subseteq \boldsymbol{P}_o$. We recapture this result as a consequence of our results here.

Theorem 4.3.6 *Suppose $A \in \boldsymbol{R}^{n\times n} \cap C_o^f$, $n \geq 2$. If the rows and columns of $A + A^t$ corresponding to the zero diagonal entries of A are zero, then $A \in \boldsymbol{P}_o$.*

Proof: From the hypothesis and Theorem 4.2.1, it is clear that every 2×2 principal submatrix of A is in $\boldsymbol{P}_o$. Assuming that every $(k-1)\times(k-1)$, $k \geq 2$, principal submatrix of A is in $\boldsymbol{P}_o$, we will show that every $k \times k$ principal submatrix of A is also in $\boldsymbol{P}_o$. Let B be any $k \times k$ principal submatrix of A such that all its proper principal minors are nonnegative. Suppose $det\, B < 0$. Then B is an almost $\boldsymbol{P}_o$-matrix and $B^{-1} \in C_o^f \cap \boldsymbol{N}_o$. Let $E = B^{-1}$. By Theorem 4.2.4, there exists a subset α of $\bar{n}$ such that

$$\phi \neq \alpha \neq \bar{n}, \quad E_{\alpha\alpha} \leq 0, \quad E_{\bar{\alpha}\bar{\alpha}} \leq 0, \quad E_{\alpha\bar{\alpha}} \geq 0 \text{ and } E_{\bar{\alpha}\alpha} \geq 0.$$

Since $E \in \boldsymbol{C}_o$, we must have $E_{\alpha\alpha} = E_{\bar{\alpha}\bar{\alpha}} = 0$. Without loss of generality we can assume that

$$E = B^{-1} = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix},$$

where C and D are nonnegative square matrices of same order. It follows that C and D are nonsingular, and $B = \begin{bmatrix} 0 & D^{-1} \\ C^{-1} & 0 \end{bmatrix}$. From the hypothesis, it follows that $C^{-1} + (D^{-1})^t = 0$ and hence $D^{-1} = -(C^{-1})^t$. This in turn implies that $D = -C^t$. This contradicts that D is nonnegative. Hence $det\, B \geq 0$ and the theorem follows. ■

Corollary 4.3.7 *Suppose $A \in R^{n\times n} \cap C_o^f \cap Q_o$. Then $A \in P_o$.*

Proof: If $n = 1$, there is nothing to prove. Assume $n \geq 2$. We will show that every 2×2 principal submatrix of A is in P_o. Suppose, to the contrary, assume that $A_{\alpha\alpha} \notin P_o$ for some α with $|\alpha| = 2$. Without loss of generality, we may take $\alpha = \{1,2\}$. Then $A_{\alpha\alpha} \simeq \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ (this notation means $a_{11} = a_{22} = 0$ and a_{12}, a_{21} are positive). Since $A_{\alpha\alpha} \notin Q_o$ we must have $n > 2$ and a $j \in \bar{\alpha}$ such that $a_{j1} < 0$ (follows from Theorem 4.2.3). Note that if $a_{1j} \leq 0$, then $A \notin C_o^f$. But if $a_{1j} > 0$, then also $A \notin C_o^f$. This is because if $a_{1j} > 0$, then taking $\alpha = \{1,2,j\}$ we get,

$$A_{\alpha\alpha} \simeq \begin{bmatrix} 0 & + & + \\ + & 0 & * \\ - & * & \oplus \end{bmatrix}.$$

If we take PPT with respect to $\alpha = \{1,2\}$, and if we denote it as $M = \wp_\alpha(A)$, then it can be easily checked that $M_{\alpha\alpha}$ is of the form,

$$M_{\alpha\alpha} \simeq \begin{bmatrix} 0 & + & * \\ + & 0 & - \\ * & - & * \end{bmatrix}.$$

This is not a C_o matrix, hence it follows that every 2×2 principal submatrix of A is in P_o. Suppose $B = \begin{bmatrix} 0 & b \\ c & a \end{bmatrix} \in R^{2\times 2} \cap C_o^f \cap P_o$. Then from [31] (by Murthy and Parthasarathy) we get that for every i, such that $a_{ii} = 0$, $a_{ij} + a_{ji} = 0$ for all j. For the sake of completeness we outline the proof here. If $bc \neq 0$, then $bc < 0$ and $B^{-1} = \begin{bmatrix} \frac{-a}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}$. Since B is copositive, $b + c \leq 0$ and since B^{-1} is copositive, $\frac{b+c}{bc} \geq 0$ or $b + c \leq 0$.

Hence $b + c = 0$. Let $a_{ii} = 0$. If $a_{ij} < 0$ then a_{ji} has to be strictly positive, since A is copositive. Then $a_{ij} + a_{ji} = 0$. If $a_{ij} > 0$, then from Theorem 4.2.3, there exists a k such that $a_{ki} < 0$. If $a_{ji} = 0$, then $k \neq j$ and $a_{ik} > 0$ (since A is copositive). Let $\alpha = \{i, j, k\}$.

$$A_{\alpha\alpha} \simeq \begin{bmatrix} 0 & + & + \\ 0 & * & * \\ - & * & * \end{bmatrix} \text{ and } M_{\alpha\alpha} \simeq \begin{bmatrix} * & * & - \\ * & * & 0 \\ + & - & 0 \end{bmatrix}.$$

Here, $M_{\alpha\alpha}$ is not copositive. This contradicts that $A \in C_o^f$. It follows that $a_{ji} \neq 0$ and hence $a_{ij} + a_{ji} = 0$. If $a_{ij} = 0$, then if $a_{ji} \neq 0$, $a_{ji} > 0$, by the assumption of copositivity of A. By the previous argument, $a_{jj} > 0$. But then, by taking PPT with respect to $\alpha = \{i, j\}$, we see that $M_{\alpha\alpha}$ does not belong to C_o. Hence for every i, such that $a_{ii} = 0$, we have $a_{ij} + a_{ji} = 0$ for all j. Hence the rows and columns corresponding to zero diagonal entries of A are zero. From Theorem 4.3.6, it follows that $A \in P_o$. ■

In [31], it was shown by Murthy and Parthasarathy that a C_o^f-matrix is in Q_o if, and only if, it is completely-Q_o. The arguments used to prove this can be extended to obtain the following result.

Theorem 4.3.8 *Suppose $A \in R^{n\times n} \cap C_o^f$. If $A \in Q_o$, then A^t and all its PPTs are completely-Q_o.*

Proof: It can be verified that if a matrix $B \in A$ satisfies the condition : every PPT C of B satisfies

$$c_{ii} = 0 \Rightarrow c_{ij} + c_{ji} = 0 \text{ for all } i \text{ and } j,$$

then B and all its PPTs are completely-Q_o matrices. This is because if, B has this property, then Graves' algorithm processes (q, B) for any q and terminates either with a solution or with the conclusion that $F(q, B) =$

$\emptyset$ (see Chapter 4 of the book by K.G.Murthy [38] and Theorem 4.2.5). Therefore we will show, that any PPT of A^t satisfy the above condition. Let $D = \wp_\alpha(A^t)$ for some α. Observe that $\wp_\alpha(A)$ exists. Let $M = \wp_\alpha(A)$. It can be checked that, $M = SD^tS$, where $S = \begin{bmatrix} I_{\alpha\alpha} & 0 \\ 0 & -I_{\bar\alpha\bar\alpha} \end{bmatrix}$. Hence for each i, j, either $d_{ij} + d_{ji} = m_{ij} + m_{ji}$ or $d_{ij} + d_{ji} = -(m_{ij} + m_{ji})$. If $d_{ii} = 0$ for some i, then $m_{ii} = 0$ and by Theorem 4.2.5, $m_{ij} + m_{ji} = 0$. From this it follows that if for some i, $d_{ii} = 0$, then $d_{ij} + d_{ji} = 0$. ■

The converse, is however not true. The problem arises from the fact that transpose of a C_0^f-matrix need not be in C_0^f. As a counterexample, consider the C_0^f-matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. It can be checked from the definition, or using Theorem 4.2.2, that A^t and its PPTs are completely-Q_0, but $A \notin Q_0$.

Theorem 4.3.9 *Suppose $A \in R^{n\times n}$ is a bisymmetric E_0^f-matrix. Then the following are equivalent:*

(a) $A \in Q_0$

(b) *A is positive semidefinite*

(c) *for any i, j, $a_{ii} = 0 \Rightarrow a_{ij} + a_{ji} = 0$*

(d) *every 2×2 principal submatrix of A is in P_0.*

Proof: We first observe that every bisymmetric E_0^f-matrix is in C_0^f (Theorem 4.2.6). Implication (a) $\Rightarrow$ (b) was already established in [31]. The implication

(b) $\Rightarrow$ (c) is a well known fact about positive semidefinite matrices. The implication (c) $\Rightarrow$ (d) is a direct consequence of Theorem 4.3.6. To complete the proof of the theorem, we will show that (d) $\Rightarrow$ (a). Assume that

A satisfies (d). Using the fact that every 2×2 principal submatrix of A is in $C_o^f \cap P_o$, it is easy to show that A satisfies (c). Hence, by Theorem 4.3.6, $A \in P_o$. Let M be any PPT of A. Suppose $m_{ii} = 0$ for some i. As A is bisymmetric, so is M. So for any j, either $m_{ij} = -m_{ji}$ or $m_{ij} = m_{ji}$. If $m_{ij} = -m_{ji}$, then $m_{ij} + m_{ji} = 0$. If $m_{ij} = m_{ji}$, then, as $M \in P_o$ and $m_{ii} = 0$, we must have $m_{ij} = m_{ji} = 0$. Thus for any j, $m_{ij} + m_{ji} = 0$. By Theorem 4.2.5, it follows that $A \in Q_o$. ■

4.4 Sufficient conditions for connected matrices

In this section we have settled a conjecture raised by Jones and Gowda [20], for the special case, when the matrix is nonnegative.

Theorem 4.4.1 *If $A \in R^{n\times n}$ is a nonnegative $P_o \cap Q_o$ matrix, then the matrix A is a connected matrix.*

Definition 4.4.2 *Say that $A \in R^{n\times n}$ has property* **(B)** *if for every bounded set $H \in R^n$ the set $\cup_{q\in H} S(q, A)$ is bounded.*

Remark 4.4.3 *If a matrix A has property* (**B**), *then $A \in R_o$ (see Cottle Pang and Stone [9]).*

Proposition 4.4.4 *If $A \in R^n$ is a nonnegative Q-matrix, then A has property* (**B**).

Proof: Since A is nonnegative Q, $a_{ii} > 0$ for all i. Let $H \subseteq R^n$ be bounded. Suppose $T = \cup_{q\in H} S(q, A)$ is unbounded. Then there exist sequences $q^k \in H$ and $z^k \in S(q^k, A)$, such that for some $i \in n$, $z_i^k \to \infty$

as $k \to \infty$. From complementarity, we must have

$$A_{i.}z^k + q_i^k = 0 \text{ for all large } k$$

Since $A_{i.} \geq 0$, $a_{ii} > 0$, $z_i^k \to \infty$ and $\{q^k\}$ is bounded, the left hand side of the equation is positive for all k sufficiently large. From this contradiction we conclude that T is bounded. ∎

The following theorem is due to Jones and Gowda [20].

Theorem 4.4.5 *Let $A \in \mathbf{R}^{n\times n}$. If A is in $\mathbf{Q} \cap \mathbf{P}_0$ (or equivalently in $\mathbf{R}_0 \cap \mathbf{P}_0$), then $S(q, A)$ is connected for all $q \in \mathbf{R}^{n\times n}$.*

Theorem 4.4.6 *Let $A \in \mathbf{R}^{n\times n}$ and $q \in \mathbf{R}^n$. Then the following conditions hold:*

(i) $S(q, A)$ is a finite union of closed convex sets.

(ii) If $S(q, A)$ is connected then it is path connected.

Proof: (i) For each $\beta \subseteq \bar{n}$, define $S(\beta) = \{z \in S(q, A) : A_{\beta.}z + q_\beta = 0, z_{\bar{\beta}} = 0\}$. Clearly, S_β is a closed convex set and $S(q, A) = \cup_\beta S(\beta)$. Thus $S(q, A)$ is a finite union of closed convex sets.

(ii) This follows from Lemma 4.2.22 ∎

Theorem 4.4.7 *Let $A \in \mathbf{R}^{n\times n} \cap \mathbf{P}_0$. If A has property (B), then $S(q, A)$ is connected for every q.*

Proof: Since A has property **(B)** it is an $\mathbf{R}_0$-matrix, by Theorem 4.4.3. Since $A \in \mathbf{P}_0 \cap \mathbf{R}_0$, it follows (from Theorem 3.9.22 of Cottle, Pang and Stone), $A \in \mathbf{Q}$. Now invoking Theorem 4.4.5, we get the desired result. ∎

Theorem 4.4.8 *Suppose $A \in \mathbf{R}^{n\times n}$ is a nonnegative $\mathbf{P}_o \cap \mathbf{Q}_o$-matrix. Then $S(q, A)$ is connected.*

We shall prove this by induction on n. In fact we will show that $S(q, A)$ is path-connected, which will in turn imply connectedness. Observe that A is completely $\mathbf{Q}_o$ from Theorem 4.2.2. Write $\alpha = \{i : a_{ii} > 0\}$. Then from Theorem 4.2.2 it follows that $A_{\bar{\alpha}.} = 0$. If $\bar{\alpha} = \phi$, then $A \in \mathbf{R}_o \cap \mathbf{P}_o$ and consequently $S(q, A)$ is connected from Theorem 4.2.24. If $\alpha = \phi$, then A is a null matrix and trivially $S(q, A)$ is connected. So we will assume $\alpha \neq \phi$ and $\alpha \neq \bar{n}$. If $q_i > 0$ for some $i \in \bar{\alpha}$, then induction hypothesis gives the desired result. So we assume $q_i = 0$ for every $i \in \bar{\alpha}$. From Theorem 4.4.6, write

$$S(q, A) = S_1 \cup S_2 \cup \ldots \cup S_h,$$

where each S_j is a closed convex set. Define

$$S_0 = \{z \in S(q, A) : z_{\bar{\alpha}} = 0\}.$$

Note that $S_0 = S(q_\alpha, A_{\alpha\alpha})$, and as $A_{\alpha\alpha} \in \mathbf{P}_o \cap \mathbf{R}_o$, S_0 is a nonempty connected set. By Theorem 4.4.6, S_0 is path connected.

Fix a $z^0 \in S_0$. We will show that every element $\bar{z}$ of $S(q, A)$ has a path connecting it to z_0. That is, we will exhibit a continuous function

$$F(t) : [0, 1] \to S(q, A), \text{ such that } F(0) = z^0 \text{ and } F(1) = \bar{z}.$$

This will imply that $S(q, A)$ is path connected.

So fix any $\bar{z} \in S(q, A)$. Let V be the path component of $S(q, A)$ containing $\bar{z}$. Since S_js are convex, there exists a $J \subseteq \{1, 2, \ldots, h\}$ such that $V = \cup_{j\in J} S_j$. Consequently, V is a closed set.

If $V \cap S_0 \neq \phi$, there is a path connecting z^0 and $\bar{z}$. So assume

$$V \cap S_0 = \phi.$$

Define

$$\lambda_0 = \inf_{z \in V} p^t z,$$

where p is a vector whose first k coordinates are equal to 0 and the last $n-k$ coordinates are equal to 1 ($p^t z$ is the sum of last $n-k$ coordinates of z).

Claim: There exists a $z^* \in V$ such that $p^t z^* = \lambda_0$. From the definition of λ_0, there exists a sequence $\{z^m\}$ in V such that $p^t z^m$ converges to λ_0. Note that $z^m_{\bar{\alpha}}$ is bounded. Also, for each m, $z^m_{\alpha} \in S(q^m_{\alpha}, A_{\alpha\alpha})$, where $q^m_{\alpha} = q_{\alpha} + A_{\alpha\bar{\alpha}} z_{\bar{\alpha}}{}^m$. As $\{z^m_{\bar{\alpha}}\}$ is bounded, $\{q^m_{\alpha}\}$ is bounded. By property **(B)** of $A_{\alpha\alpha}$, $\{z^m_{\alpha}\}$ is bounded. Thus, $\{z^m\}$ is a bounded sequence. Without loss of generality, we may assume $z^m \to z^*$. As V is closed, $z^* \in V$. This proves the claim.

Since $V \cap S_0 = \phi$, $\lambda_0 > 0$ and $z^*_{\bar{\alpha}} \neq 0$. Define $\beta = \{i \in \bar{\alpha} : z^*_i > 0\}$. Let m_0 be such that $z^*_i - \frac{1}{m_0} > 0$ for all $i \in \beta$. For each m, define z^m as

$$z^m_{\alpha} = x^m,\ z^m_i = z^*_i - \frac{1}{m} \text{ for } i \in \beta, \text{ and } z^m_i = 0 \text{ for } i \in \bar{\alpha}\backslash\beta,$$

where x^m is any particular solution of $S(q^m_{\alpha}, A_{\alpha\alpha})$ ($S(q^m_{\alpha}, A_{\alpha\alpha}) \neq \phi$ since $A_{\alpha\alpha}$ is a $\boldsymbol{Q}$-matrix) and $q^m_{\alpha} = q_{\alpha} - \sum_{i\in\beta}(z^*_i - \frac{1}{m})A_{\alpha i}$. Then for each $m \geq m_0$, $z^m \in S(q, A)$. Without loss of generality, we may assume $z^m \in S_1$ for all m.

Note that $p^t z^{m_0} < \lambda_0$. From the definitions of λ_0 and V, it follows that

$$V \cap S_1 = \phi.$$

Since $\{q^m_{\alpha}\}$ is bounded, by property **(B)** of $A_{\alpha\alpha}$, $\{x^m\}$ is bounded. Hence, $\{z^m\}$ is bounded. Without loss of generality, assume $z^m \to u$. Since S_1 is closed, $u \in S_1$.

Define $q_\alpha^0 = q_\alpha + \sum_{i \in \beta} z_i^* A_{\alpha i}$. Observe that u_α and z_α^* belong to $S(q_\alpha^0, A_{\alpha\alpha})$. Since $A_{\alpha\alpha} \in \boldsymbol{P}_o \cap \boldsymbol{R}_o$, $S(q_\alpha^0, A_{\alpha\alpha})$ is path connected. Note that $x \in S(q_\alpha^0, A_{\alpha\alpha})$ if and only if $z \in S(q^0, A)$ where $q^0 = (q_\alpha^0, q_{\bar{\alpha}}^0)^t$ with $q_{\bar{\alpha}}^0 = 0$ and $z_\alpha = x$ and $z_{\bar{\alpha}} = z_\alpha^*$. Let

$$p(t) : [0,1] \to S(q_\alpha^0, A_{\alpha\alpha}) \text{ such that } p(0) = u_\alpha \text{ and } p(1) = z_\alpha^*$$

be a path connecting u_α and z_α^*. Then, $F(t) : [0,1] \to S(q^0, A)$ defined by

$$F(t)_\alpha = p(t) \text{ and } F(t)_{\bar{\alpha}} = z_{\bar{\alpha}}^*$$

is a path between u and z^*. From the definition of V, $u \in V$. But this contradicts that $V \cap S_1 = \phi$. It follows that $\lambda_0 = 0$ and $S_0 \subseteq V$. Therefore there is a path connecting z^0 and $\bar{z}$. This completes the proof of the theorem. ∎

Theorem 4.4.9 *Let $A \in \boldsymbol{R}^{n \times n}$ be a nonnegative matrix. If $A \in \boldsymbol{E}_c \cap \boldsymbol{Q}_o$ then $A \in \boldsymbol{P}_o$.*

Proof: Since $A \in \boldsymbol{E}_c$, from Jones and Gowda's result (Theorem 4.2.24), $A \in \boldsymbol{E}_0^f$. Since $A \in \boldsymbol{Q}_o$ and the matrix A is nonnegative, $A \in \bar{\boldsymbol{Q}}_o$. Hence from Theorem 4.2.17, we conclude that $A \in \boldsymbol{P}_o$. ∎

Thus we have proved the conjecture for the special case when the matrix A is nonnegative. Nevertheless the conjecture raised by Jones and Gowda, $\boldsymbol{P}_o \cap \boldsymbol{Q}_o = \boldsymbol{E}_c$ still remains open. The problem $\boldsymbol{E}_c \cap \boldsymbol{Q} \subseteq \boldsymbol{P}_o$ also remains open.

Next theorem establishes the class $\boldsymbol{P}_1$ to be connected matrices.

Theorem 4.4.10 *If $A \in \boldsymbol{P}_1$, then $S(q, A)$ is connected.*

Proof: Since $A \in \boldsymbol{P}_1$ from Theorem 4.2.15 we get, $A \in \boldsymbol{Q}_o$.

Case I: $A \in \boldsymbol{Q}$.

If $A \in \boldsymbol{Q}$ then since $A \in \boldsymbol{P}_o$, $A \in \boldsymbol{R}_o$. From Theorem 4.2.24, we conclude that $S(q, A)$ is connected.

Case II: $A \in \boldsymbol{Q}_o \backslash \boldsymbol{Q}$.

Since $A \in \boldsymbol{P}_1$, A is a $\boldsymbol{P}_o$-matrix, with exactly one principal minor equal to zero. Without loss of generality, we can take $|A| = 0$. Also, since $A \in \boldsymbol{P}_o \cap (\boldsymbol{Q}_o \backslash \boldsymbol{Q})$, the value of the game with pay-off matrix A is zero (v(A)=0). Since $|A| = 0$ all the proper principal submatrices of A are $\boldsymbol{P}$ matrices. Hence the game with pay-off matrix A is completely mixed. Hence, there exists $\pi > 0$ such that $\pi^t A = 0$, where π is an optimal strategy of the minimizer. Since $A \in \boldsymbol{P}_1 \cap (\boldsymbol{Q}_o \backslash \boldsymbol{Q})$, $A \in \boldsymbol{U}$ (see Theorem 4.2.16). Hence for all $q \in \text{int}K(A)$, LCP(q, A) has a unique solution. Also, $K(A) = \{q : \pi^t q \geq 0\}$ (see pages 211, 212 of [8]). Let us consider a $q \in \partial K(A)$, for this q we have $\pi^t q = 0$.

Claim. $S(q, A) = \{x \geq 0 : Ax + q = 0\}$

Suppose not, then there exist an $x \in S(q, A)$ such that $Ax + q \geq 0$, and $(Ax + q)_i > 0$ for some i. Then

$$\pi^t(Ax + q) > 0$$

$$\text{i.e. } \pi^t Ax + \pi^t q > 0$$

But since $\pi^t A = \pi^t q = 0$, this is a contradiction. Thus we have proved our claim. Hence, $S(q, A)$ is connected. ■

Chapter 5

LCP in Static and Dynamic Games

5.1 Introduction

In this chapter we briefly indicate, how LCP and minimax theorem complement each other.This chapter is somewhat expository in nature. We show how results from two person matrix games, due to von Neumann and Kaplansky, can be effectively used to get interesting results in LCP. This is done in Section 5.2. Here we show that a completely mixed game A with $v(A) = 0$ is a $\boldsymbol{Q}_0$-matrix. We also show a sort of converse to the next theorem due to Cottle and Stone [8].

Theorem 5.1.1 *If $A \in \boldsymbol{U} \cap \boldsymbol{Q}_0$ then A is a $\boldsymbol{P}_0$ matrix.*

Recall A is a $\boldsymbol{P}_1$ matrix if exactly one principal minor is zero and all the other principal minors are positive. In other words, $A \in \boldsymbol{P}_1$ if $A \in \boldsymbol{P}_0$ and exactly one principal minor of A is zero. Without loss of generality we will assume $\det A = 0$, whenever $A \in \boldsymbol{P}_1$.

In Section 1.3 we have briefly discussed the basics of a two-person zero-sum game. We denote the minimax value of the game with pay-off matrix A as $v(A)$. A generalized version of the following proposition is proved in Chapter 1 (Theorem 1.3.5).

Proposition 5.1.2 *Let A be a square matrix of order n. Suppose $v(A) > 0$. If A is nonsingular then $v(A^{-1}) > 0$, where A^{-1} stands for the inverse of A.*

Recall that a strategy is pure if it is of the form $(0, 0, 1, 0, \ldots, 0)$, otherwise it is mixed. In case $p_i > 0 \,\forall\, i$, we call the strategy $p = (p_1, p_2, \ldots, p_m)$ completely mixed. We say that the matrix game A is completely mixed if every optimal strategy of either player is completely mixed. The following results, given in Section 1.3, are due to Kaplansky [21].

Theorem 5.1.3 *Let A denote the pay-off matrix of order $m \times n$ of a two person game. We then have the following:*

(i) If player 1 has a completely mixed optimal strategy, then any optimal strategy $q^0 = (q_1^0, q_2^0, \ldots, q_n^0)$ for player 2 satisfies $\sum_j a_{ij} q_j^0 = v(A) \,\forall\, i = 1, 2, \ldots, m$.

(ii) If $m = n$ and if the game is not completely mixed then both players have optimal strategies that are not completely mixed.

(iii) A game with $v(A) = 0$ is completely mixed if and only if (a) $m = n$ and the rank of $A = n - 1$ and (b) all the cofactors A_{ij} of A are different from zero and have the same sign.

(iv) Suppose A is a completely mixed game. Then $v(A) = \frac{|A|}{\sum_i \sum_j A_{ij}}$ where $|A|$ stands for the determinant of A and A_{ij}'s are the cofactors.

(v) Let $V = (v_{ij})$ denote the matrix of order $m \times n$ where v_{ij} is the value of a game whose pay-off matrix is obtained from A by omitting its ith row

and jth column. Then the game with pay-off matrix A is not completely mixed if and only if the game with pay-off matrix V has a pure saddle point, that is, if and only if there exists a pair (i_0, j_0) such that,

$$v_{i_0 j} \leq v_{i_0 j_0} \leq v_{i j_0} \quad \forall i, j \quad where \quad v_{i_0 j_0} = v(A).$$

If the game A is completely mixed then the game with pay-off matrix A^t (transpose of A) is also completely mixed and $v(A) = v(A^t)$. If A is a nonsingular matrix with $A^{-1} < 0$ (every entry in A^{-1} is negative), then the game with pay-off matrix A is completely mixed. A is an $\boldsymbol{N}$ matrix if every principal minor of A is negative. Call S a signature matrix if S is a diagonal matrix and the diagonals are either 1 or -1.
Parthasarathy and Ravindran (see [43]) proved the following results.

Theorem 5.1.4 *Let $A \in \boldsymbol{R}^{n \times n}$ be such that $a_{ij} < 0$ for all i, j. Then the following statements are equivalent:*
(i) A is an $\boldsymbol{N}$-matrix.
(ii) For every signature matrix $S \neq +I$ or $-I$, $v(SAS)$ is positive.
(iii) A does not reverse the sign of any non-unisigned vector, that is,

$$(Ax)_i(x_i) \leq 0 \quad for\ all\ i \quad implies \quad x \leq 0\ or\ x \geq 0.$$

Theorem 5.1.5 *Let A be a nonsingular matrix. Then the following statements are equivalent:*
(i) A is an almost $\boldsymbol{P}$-matrix.
(ii) $v(SAS) > 0$ for all except one, say $S = S_0$ (as well as $-S_0$), and for that S_0, $v(S_0AS_0) < 0$ and $S_0A^{-1}S_0 < 0$.

We now state the "Theorem of alternatives" due to Farkas', which we will use shortly.

Theorem 5.1.6 *Let $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^n$. Then exactly one of the following conditions holds.*

(i) $Ax = b$ has a nonnegative solution.

(ii) The system $y^tA \leq 0$, $y^tb > 0$ has a solution.

In Section 5.4 it is indicated that the computation of equilibrium points, through Lemke-type algorithms for polystochastic games, is possible.

5.2 Game Theoretic Results in LCP

In this section, we demonstrate the usefulness of minimax theorem and Kaplansky's results in LCP.

Proposition 5.2.1 (i) *If $A \in \mathbf{Q}$, then $v(A) > 0$.*

(ii) *If $v(A^t) \geq 0$ and $Ax = 0$ for some $x > 0$, then $A \notin \mathbf{Q}$.*

Proof: of (i) : Since $A \in \mathbf{Q}$, LCP (A, q) will have a solution for every $q \in \mathbf{R}^n$. Let $q^* = (-1, -1, \ldots -1)$. Then $Az + q^* = w$ where $z \geq 0$, $w \geq 0$ and $z^tw = 0$. Since $q^* < 0$, z cannot be a zero vector. Hence $Az = w - q^* > 0$ and consequently the "maximizer" can assure himself a positive income. Thus $v(A) > 0$.

Analogously, (ii) can be proved. Let A be a square matrix of order n. Our next result gives a characterization of $\mathbf{N}$ matrices through LCP.

Theorem 5.2.2 *Let $A < 0$. Then the following statements are equivalent.*

(i) A is an $\mathbf{N}$-matrix.

(ii) For every signature matrix $S \neq I$ or $-I$, there exists a nonnegative vector x such that $SASx > 0$, that is, $v(SAS) > 0$.

(iii) A does not reverse the sign of any non unisigned vector, that is

$(Ax)_i x_i \le 0 \, \forall \, i$ *implies that* $x \ge 0$ *or* $x \le 0$.

(iv) For all $q > 0$, *LCP* (A, q) *has exactly two solutions.*

(v) $SAS \in \boldsymbol{Q}$ *for every* $S \ne I$ *or* $-I$.

Proof: Equivalence of (i) to (iv) can be seen through Theorem 5.1.4. We now prove $(ii) \Rightarrow (v)$, as $(v) \Rightarrow (ii)$ follows from Proposition 5.2.1. Note (ii) implies A is an $\boldsymbol{N}$-matrix. Since $A < 0$, $v(SAS) > 0$ whenever $S \ne I$. Hence from Proposition 5.1.2, we have that $v(SA^{-1}S) > 0$ whenever $S \ne I$. Write $M = SA^{-1}S$. To complete the proof we will verify Karamardian's sufficient condition for $M \in \boldsymbol{Q}$ [22]. That is, we need to check that the system

$$0 \ne x \ge 0, \quad t \ge 0,$$

$$x_i > 0 \Rightarrow \quad (Mx)_i + t = 0,$$

$$x_i = 0 \Rightarrow \quad (Mx)_i + t \ge 0,$$

is inconsistent. [Here x is a vector, t is a scalar]. On the contrary, suppose there is a solution $x \ge 0$ with at least one coordinate positive and $t \ge 0$ to the above system. If $x_i > 0$ for every i then $v(M) \le 0$ and it leads to a contradiction. If $x_{i_0} = 0$ for some i_0, then the principal submatrix omitting the i_0th row and the i_0th column from $M(= SA^{-1}S)$ is a $\boldsymbol{P}$-matrix and this will imply $x_i = 0$ for all $i \ne i_0$ or $x = 0$ which leads to a contradiction. Thus $SA^{-1}S \in \boldsymbol{Q}$ or $SAS \in \boldsymbol{Q}$ for $S \ne I$.

A similar result can be given for $\boldsymbol{P}$-matrices. Interested readers can consult [38]. The following result is due to Mohan and Eagambaram [13]. However our proof is game theoretic in nature and is different from theirs.

Theorem 5.2.3 *Let* $A \in \boldsymbol{R}^{n \times n}$. *Suppose* $\pi^t A = 0 = A\delta$ *where* $\pi > 0$, $\delta > 0$ *and rank of* $A = n - 1$. *Then* $K(A) = \{q : \pi^t q \ge 0\}$. *Furthermore, if* $\pi^t q = 0$ *then the number of solutions for LCP(q, A) is infinite. That is* *if $q \in \partial K(A)$ (= boundary of $K(A)$), then the number of solutions for LCP(q, A) is infinite.*

Proof: First we show that $A \in \boldsymbol{Q}_o$. If q is feasible then there exists $x \geq 0$ such that $Ax + q \geq 0$. This implies $\pi^t q \geq 0$. We will prove $q \in K(A)$. If $\pi^t q = 0$ then $Ax + q \geq 0$ implies $Ax + q = 0$, since π is a strictly positive vector. Hence $q \in K(A)$. If, $\pi^t q > 0$ and $Ax + q \geq 0$, we will produce a solution. From Kaplansky's results (Theorem 5.1.3) , the game with pay-off matrix A is completely mixed with value zero. Also $det A = 0$. We also know that cofactors A_{ij} are different from zero and are of the same sign. Assume without loss of generality that $A_{ij}s$ are positive.
Let

$$B_k = \begin{bmatrix} a_{11} + \frac{1}{k} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

We can take k sufficiently large so that the cofactors of B_k are also different from zero and is strictly positive. This along with the fact that $|B_k| > 0$, gives ${B_k}^{-1} > 0$. Hence ${B_k}^{-1} \in \boldsymbol{Q}$ or $B_k \in \boldsymbol{Q}$.

$$\begin{bmatrix} a_{11} + \frac{1}{k} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^k \\ \vdots \\ x_n^k \end{bmatrix} + \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = w^{(k)}. \quad (*)$$

If $\{x^k\}$ is bounded, then clearly LCP(q, A) has a solution. If $\{x^k\}$ is unbounded, assume w.l.g, $\frac{x^k}{\|x^k\|} \to x^* \neq 0$ and $x^* \geq 0$. It follows from (*), $Ax^* \geq 0$. Another application of Kaplansky's theorem yields x^* to be strictly positive. Hence, $x^k > 0$ for large k. Consequently,

$$Ax^k + \begin{pmatrix} q_1 + \frac{x_1^k}{k} \\ \vdots \\ q_n \end{pmatrix} = 0.$$

This is impossible for,

$$\pi^t(Ax^k + \begin{pmatrix} q_1 + \frac{x_1^k}{k} \\ \vdots \\ q_n \end{pmatrix}) = \pi^t Ax^k + \pi^t q + \pi_1 \frac{x_1^k}{k} > 0$$

[Since $\pi^t q > 0$]. Thus we have shown that $A \in \boldsymbol{Q}_o$. The fact that $K(A) = \{q : \pi^t q \geq 0\}$, follows from the following lemma.

Lemma 5.2.4 *Let $M \in \boldsymbol{R}^{n \times n}$, $M\delta = 0$ for some $\delta > 0$ and rank of $M = n - 1$. If $M \in \boldsymbol{Q}_o \backslash \boldsymbol{Q}$, then $K(M)$ is a half-space. In fact, $K(M) = \{q : y^t q \geq 0\}$ where y is an optimal strategy for the minimizer.*

Proof of the lemma can be given using Kaplansky's theorem and a "Theorem of alternatives" (Theorem 5.1.6). We omit the details. We continue the proof of Theorem 5.2.3. We have shown that $A \in \boldsymbol{Q}_o$ and all the conditions of Lemma 5.2.4 are satisfied. Consequently, $K(A) = \{q : \pi^t q \geq 0\}$. For the second part, if $\pi^t q = 0$, then $Ax + q = 0$ and this means $A(x + \delta) + q = 0$. In other words, LCP(q, A) has infinite number of solutions. Cottle and Stone have shown in [8], that if $A \in \boldsymbol{P}_1$ and if $A \notin \boldsymbol{Q}$ then $A \in \boldsymbol{U}$ and $K(A)$ is a half-space. Since we are assuming $|A| = 0$ whenever $A \in \boldsymbol{P}_1$ and all other principal minors are positive, it follows that A is completely mixed with $v(A) = 0$. Hence Cottle and Stone's result can be rephrased as follows: If $A \in \boldsymbol{P}_1$ with $|A| = 0$, then $A \in \boldsymbol{U}$. Is the converse true, that is, if $A \in \boldsymbol{U}$ and if A is completely mixed with $v(A) = 0$, then can we say that $A \in \boldsymbol{P}_1$? The answer is yes as the following theorem shows.

Theorem 5.2.5 *Let A be a completely mixed game with $v(A) = 0$. Then the following two statements are equivalent.*

(i) $A \in P_1$, that is, $\det A = 0$ and all other principal minors are positive.

(ii) $A \in U$, that is, for every $q \in$ interior of $K(A)$, $LCP(q, A)$ has a unique solution.

Proof: Cottle and Stone have shown that $(i) \Rightarrow (ii)$, since *(i)* implies that $A \in Q_o \backslash Q$. We will show $(ii) \Rightarrow (i)$. Since A is completely mixed with $v(A) = 0$, it follows from Theorem 5.1.3 (*iv*), $\det A = 0$. To complete the proof we need to show every proper principal minor of A is positive. Proof of this depends heavily on Theorem 5.2.3. We will make use of the same notation as in Theorem 5.2.3.

Since $A \in Q_o$ and $A \in U$, it follows (See Theorem 5.1.1) that $A \in P_o$. Suppose $a_{11} = 0$. Consider,

$$\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = w,$$

Then,

$$\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 2x \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = w'.$$

We can choose $q_2, \ldots q_n$, large enough with $\pi^t q > 0$, where $q = (0, q_2, \ldots, q_n)^t$. Clearly, such a $q \in$ interior of $K(A) =$ interior of $\{q : \pi^t q \geq 0\}$ and LCP(q, A) will have two solutions contradicting our hypothesis, namely $A \in U$. Hence, we conclude $a_{ii} > 0$ for every i. We will now show that

2×2 principal minors are positive. Suppose $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$.

Also there exist t_1, t_2 (reals) not both zero such that $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} =$ 0. Choose q_1 and q_2 so that $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0$.

We can choose $q_3, \ldots, q_n$ so large satisfying (1) $\pi^t q > 0$ and (2) LCP(q, A) has at least two solutions namely$(1, 1, 0 \ldots, 0)$ and $(1 + t_1, 1 + t_2, 0 \ldots, 0)$. This leads to a contradiction to the hypothesis, namely $A \in \boldsymbol{U}$. Thus, every 2×2 principal minors are positive. We can continue this process for every $k \times k$ principal minor with $k \leq n - 1$. Thus $A \in \boldsymbol{P}_1$. ■

Let $\boldsymbol{E}_*$ denote the class of all square matrices A with the property, LCP(q, A) has a unique solution for every non-zero non-negative vector q and LCP$(0, A)$ has at least two solutions. Danao attributes the following conjecture to Cottle. Let $A \in \boldsymbol{P}_0 \cap \boldsymbol{R}^{n \times n}$. Then $A \in \boldsymbol{E}_*$ if and only if $A \in \boldsymbol{P}_1 \backslash \boldsymbol{Q}$. One can verify that if $A \in \boldsymbol{P}_1 \backslash \boldsymbol{Q}$, then $A \in \boldsymbol{E}_*$ (See Danao[10]). Converse is not true as the following example shows. In other words, $A \in \boldsymbol{E}_* \cap \boldsymbol{P}_0$ need not imply $A \in \boldsymbol{P}_1$.

We now give an example of a matrix A with the following property. $A \in \boldsymbol{P}_0 \cap \boldsymbol{E}_*$ but $A \notin \boldsymbol{U}$ and consequently by Theorem 5.2.5, $A \notin \boldsymbol{P}_1$.

Example 5.2.6 *Let*

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & 1 & -4 \\ -3 & -4 & 2 & 5 \end{bmatrix}$$

Here $K(A) = \{q = (q_1, q_2, q_3, q_4) : q_1 + q_2 + q_3 + q_4 \geq 0\}$.

Let $q_0 = (-4, -4, 20, 100)$. Then $q_0 \in$ interior of $K(A)$. LCP(q_0, A) has (at least) two solutions namely $x^1 = (4,0,0,0)^t$ and $x^2 = (0,4,0,0)^t$ with $Ax^i + q^0 \geq 0$ and $x^i(Ax^i + q_0) = 0$ for $i = 1,2$. Thus $A \notin \boldsymbol{U}$. Also observe that column sums and row sums are zero. Rank of A is 3. Thus invoking Theorem 5.2.5, we infer $A \notin \boldsymbol{P_1}$. It is also not hard to check that $A \in \boldsymbol{E_*} \cap \boldsymbol{P_0}$.

We will end this section by another nice application of Theorem 5.1.3. Recall that $A \in \boldsymbol{R}^{n \times n}$ is said to be an E_0 matrix if LCP(q, A) has a unique solution for every positive vector $q \in \boldsymbol{R}^n$. This implies and is implied by the fact that $v(A_{\alpha\alpha}) \geq 0$ for every nonempty index set $\alpha \subseteq \bar{n}$ (see [9]).

Theorem 5.2.7 *Let $A \in \boldsymbol{R}^{n \times n}$. Then the following two conditions are equivalent.*

$(i) A \in E_0$.

$(ii) A^t \in E_0$.

Proof: $(i) \Rightarrow (ii)$. Let $A \in E_0$ then $v(A_{\alpha\alpha}) \geq 0 \,\forall \alpha$. It is enough to show that $v({A_{\alpha\alpha}}^t) \geq 0 \,\forall \alpha$. We prove this by induction on n. If $n = 1$, the result is immediate. So, we will assume the result to be true for all α with cardinality of $\alpha \leq n-1$. That is, we have $v(A_{\alpha\alpha}) \geq 0 \,\forall \alpha$ and $v(A^t_{\alpha\alpha}) \geq 0 \,\forall \alpha$ with $|\alpha| \leq n-1$. We need to show that $v(A^t) \geq 0$. Suppose $v(A^t) < 0$. It means the game A^t is completely mixed and consequently $v(A^t) = v(A) < 0$ leading to a contradiction. Hence $A^t \in E_0$.

Proof of $(ii) \Rightarrow (i)$ follows from the fact that $(A^t)^t = A$. ∎

In the next section we will indicate the importance of LCP in stochastic game theory.

5.3 LCP in stochastic Game Theory

When Nash introduced the concept of equilibrium points for noncooperative games [40], one of the major open problems in the bimatrix games was to compute the equilibrium points. Original proof by Nash is existential in nature and it uses either Brouwer's fixed point theorem or Kakutani's fixed point theorem. It was Lemke and Howson who gave a remarkable (finite) algorithm to get an equilibrium point for bimatrix games. Historically, LCP was conceived as a unifying formulation for the linear and quadratic programming problems as well for the bimatrix game. A rich account of LCP dealing with bimatrix games in all its depth is discussed in the excellent monograph by Cottle, Pang and Stone [9].

Since we do not make any original contributions with reference to stochastic games, we will briefly discuss the recent works of Mohan, Neogy and Parthasarathy [27, 28]. We now introduce the notion of polystochastic games.

A polystochastic game is defined by the objects,
$(N, S, N_i(s), A_{ij}(s), q(t/s, a_n),$ for $s \in S,\ i \neq j,\ i, j \in N)$.
Here, $N = \{1, 2, \dots n\}$ denotes the set of players, $S = \{1, 2, \dots n\}$ denotes the states of a system , $N_i(s)$ stands for the actions available to player i in state s, the matrix $A_{ij}(s)$ denotes the matrix of partial costs incurred by player i depending on the actions of players i and j, $i \neq j$ and $q(t/s, a_n)$ is the probability that the game moves to state t given the game is played in state s and player n chooses action $a_n \in N_n(s)$. Suppose $x_i(s)$ denotes the vector of probabilities over $N_i(s)$ used by player i. Then the total expected cost incurred by player i on any given day is given by $x_i'(s)(\sum_{i \neq j} A_{ij}(s) x_j(s))$. Here prime refers to the transpose. Suppose the

game is played over the infinite horizon. Let $f_i = \{x_i(s) : s \in S\}$ be a sequence of mixed strategies played by player i. Such strategies f_i are called stationary strategies.

We will consider two pay-off criteria. They are,

(i) β-discounted pay-off and

(ii) undiscounted pay-off.

(i) β- discounted pay-off : For the n players, β discounted pay-off vector is given by where $\beta \in [0,1)$,

$$I_\beta^i(f_1, f_2, \dots f_n) = \sum_{r=0}^{\infty} \beta^r Q^r(f_1, f_2, \dots f_n)c,$$

where c stands for the cost vector, $(f_1, f_2, \dots f_n)$ for the strategies used by the n players, $Q^0 = I$ (Identity matrix) $Q^r = Q \times Q \times \cdots \times Q$ (product of Q taken r times) and Q is the Markov matrix whose (t,s)th element given $f_1, f_2, \dots f_n$ is given by $q(t/s.\, f_n(s))$ [These are one player controlled transitions. We assume nth player controls the transition probabilities].

(ii) Undiscounted pay-off : For the ith player's pay-off is given by

$$\Phi(f_1, f_2, \dots f_n) = Q^* \mathbf{C}.$$

where $Q^* = \lim_{i \to \infty}(\frac{1}{k} \sum_{i=1}^{k} Q^i)$ and c is the cost vector.

Objective: Every player wants to minimize his cost.

Call $(f_1^*, f_2^*, \dots f_n^*)$ an equilibrium point if

$$I_\beta^1(f_1^*, f_2^*, \dots f_n^*)(s) \leq I_\beta^1(f_1, f_2^*, \dots f_n^*)(s) \quad \forall \quad f_1 \text{ and } s. \quad (5.3.1)$$

$$I_\beta^2(f_1^*, f_2^*, \dots f_n^*)(s) \leq I_\beta^2(f_1^*, f_2, \dots f_n^*)(s) \quad \forall \quad f_2 \text{ and } s, \quad (5.3.2)$$

$$\vdots \qquad \vdots \tag{5.3.3}$$

$$I_\beta^n(f_1^*, f_2^*, \dots f_n^*)(s) \quad \le \quad I_\beta^n(f_1^*, \dots f_{n-1}^*, f_n)(s) \quad \forall \quad f_n \text{ and } s \tag{5.3.4}$$

A similar definition can be given for the undiscounted case.

Existence of equilibrium points is known. See [55, 14]. Is it possible to give an efficient algorithm to get an equilibrium point to polystochastic games? It was shown recently that indeed one can compute an equilibrium point through Lemke-type algorithm. The details can be found in [27, 28]. In the case of undiscounted pay-off, we are able to give Lemke-type algorithm under the additional assumption that the transition probability matrix is irreducible. The general problem remains open.

For references on stochastic games, see [41, 44, 46, 27, 28].

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