A note on the robustness of multivariate medians

Biman Chakraborty[a], Probal Chaudhuri[b,*]

[a]*Department of Statistics and Applied Probability, National University of Singapore, Singapore*
[b]*Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700 035, India*

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Abstract

In this note we investigate the extent to which some of the fundamental properties of univariate median are retained by different multivariate versions of median with special emphasis on robustness and breakdown properties. We show that transformation retransformation medians, which are affine equivariant, $n^{1/2}$-consistent and asymptotically normally distributed under standard regularity conditions, can also be very robust with high breakdown points. We prove that with some appropriate adaptive choice of the transformation matrix based on a high breakdown estimate of the multivariate scatter matrix (e.g. S-estimate or minimum covariance determinant estimate), the finite sample breakdown point of a transformation retransformation median will be as high as $n^{-1}[(n-d+1)/2]$, where $n=$ the sample size, $d=$ the dimension of the data, and $[x]$ denotes the largest integer smaller than or equal to x. This implies that as $n \to \infty$, the asymptotic breakdown point of a transformation retransformation median can be made equal to 50% in any dimension just like the univariate median. We present a brief comparative study of the robustness properties of different affine equivariant multivariate medians using an illustrative example.

Keywords: Affine equivariance; Asymptotic efficiency; Breakdown point; Multivariate location; Transformation retransformation estimate

1. Introduction: multivariate medians

Three fundamental properties of univariate median that have greatly contributed towards its widespread popularity as a measure of univariate location are:

(1) Its high (50%) breakdown.

(2) Its $n^{1/2}$-consistency and asymptotic normality under suitable regularity conditions and high asymptotic efficiency under distributions with heavy tails when compared with the usual mean.

(3) Its affine equivariance, which is an important geometric property of a location estimate.

There are many proposals in the literature for generalising median in multidimension (see Small, 1990; Chaudhuri, 1992 for some detailed review), and we will now give a brief review of them with special emphasis on the extent to which the above mentioned three properties of univariate median are preserved by different versions of multivariate median. Bickel (1964) studied the vector of coordinatewise medians, which is $n^{1/2}$-consistent, asymptotically normally distributed and has breakdown point 50%. However, it is not equivariant under general affine transformations of the data. As a matter of fact, it is not even equivariant under orthogonal transformations (e.g. rotations) of the data, and Bickel (1964) observed that this lack of affine equivariance has some serious negative impact on the asymptotic efficiency of co-ordinatewise median when there are high correlations among different variables in a multivariate data. Another multivariate extension of median, which is defined as a vector $\hat{\boldsymbol{\Theta}}_n$ that minimizes the sum $\sum_{i=1}^{n} \|\boldsymbol{X}_i - \boldsymbol{\Theta}\|$, where the $\boldsymbol{X}_i$'s are multivariate data points (for $\boldsymbol{x} = (x_1, \ldots, x_d)$, $\|\boldsymbol{x}\| = (x_1^2 + \cdots + x_d^2)^{1/2}$), is popularly known as *spatial median* due to Brown (1983). It is still not equivariant under general affine transformations of the data though it is equivariant under orthogonal transformations. Spatial median too is $n^{1/2}$-consistent and asymptotically normally distributed (see e.g. Chaudhuri, 1992), and it too has 50% breakdown point (see e.g. Kemperman, 1987). Nevertheless, its lack of general affine equivariance makes spatial median an unreasonable estimate when the scales of different coordinates of the data vector are widely different. The simplicial volume based median proposed by Oja (1983) is affine equivariant, $n^{1/2}$-consistent and asymptotically normal (see e.g. Arcones et al., 1994). However, Oja et al. (1990) proved that this version of multivariate median has very poor robustness property. Liu (1990) defined another version of multivariate median based on her idea of simplicial depth in a data cloud. This estimate too is affine equivariant, $n^{1/2}$-consistent and asymptotically normal (see Arcones et al., 1994). Recently, Chen (1995) has shown that it has asymptotic breakdown point less than 50% for multivariate data, and it can be quite low when the dimension of the data is large. Tukey's (1975) half-space median, which is based on the notion of half-space depth, is also an affine equivariant version of multivariate median that has asymptotic breakdown point less than 50% (see Donoho and Gasko, 1992; Chen, 1996). The asymptotic distribution and the $n^{1/2}$ rate of convergence of Tukey's median have been derived recently by Bai and He (1998).

One serious practical problem associated with all of the above-mentioned affine equivariant versions of multivariate median is that they are all quite difficult to compute for high-dimensional data. The algorithm proposed by Niinimaa (1992) and Niinimaa et al. (1992) for computing Oja's median by converting the simplicial volume minimization problem into a least absolute deviations linear regression problem is not very useful when the dimension d of the data is large since the computational complexity for this algorithm is of the order of the dth power of the sample size. Exact and efficient algorithms for computing Liu's and Tukey's median are available only in the case of bivariate data (see Rousseeuw and Ruts, 1996, 1998; Ruts and Rousseeuw, 1996). Computation of both the medians require solutions of certain difficult optimization problems in computational geometry and no good algorithms are available for them in the literature for high-dimensional data. Interestingly, though both of the coordinatewise and the spatial medians are not affine equivariant, they are both quite easy to compute even for data with fairly large dimensions. Computation of coordinatewise median requires only the ordering of the values of each of the coordinate variables and determining their middlemost order statistics. On the other hand, efficient algorithms for computing spatial median in any dimension are available in the literature (see e.g. Chaudhuri, 1996).

All these motivated Chakraborty and Chaudhuri (1996, 1998) and Chakraborty et al. (1998) to propose a transformation and retransformation procedure for converting non-equivariant cordinatewise and spatial medians into some equivariant estimates of multivariate location. They have demonstrated how this strategy can be used as an effective tool for repairing some of the deficiencies of a non-equivariant estimate and thereby improving its statistical performance. Suppose that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \in \mathbb{R}^d$, where $n > d + 1$, are n multivariate observations, and S_n denotes the collection of all subsets of size $d+1$ of $\{1, 2, \ldots, n\}$. For a fixed $\alpha = \{i_0, i_1, \ldots, i_d\} \in S_n$, consider the $d \times d$ matrix $\boldsymbol{X}(\alpha)$ whose columns are $\boldsymbol{X}_{i_1} - \boldsymbol{X}_{i_0}, \ldots, \boldsymbol{X}_{i_d} - \boldsymbol{X}_{i_0}$. We will assume that the $\boldsymbol{X}_i$'s are such that $\boldsymbol{X}(\alpha)$ is an invertible matrix. Clearly, if the $\boldsymbol{X}_i$'s are i.i.d. observations with a common absolutely continuous distribution on $\mathbb{R}^d$, the invertability of $\boldsymbol{X}(\alpha)$ is ensured with probability one for any choice of $\alpha \in S_n$. We next transform all the observations into the new coordinate system determined by the data-driven transformation matrix $\boldsymbol{X}(\alpha)$, so that for $1 \leqslant i \leqslant n$, we will write $\boldsymbol{Y}_i^{(\alpha)} = \{\boldsymbol{X}(\alpha)\}^{-1}\boldsymbol{X}_i$. Let $\hat{\boldsymbol{\Phi}}_n^{(\alpha)}$ be the coordinatewise median or the spatial median based on $\boldsymbol{Y}_i^{(\alpha)}$'s, $1 \leqslant i \leqslant n$. Then the transformation retransformation median $\hat{\boldsymbol{\Theta}}_n^{(\alpha)}$ is formed by retransforming $\hat{\boldsymbol{\Phi}}_n^{(\alpha)}$ into the original coordinate system as $\hat{\boldsymbol{\Theta}}_n^{(\alpha)} = \{\boldsymbol{X}(\alpha)\}\hat{\boldsymbol{\Phi}}_n^{(\alpha)}$. Clearly, when the dimension $d = 1$, the transformation retransformation median reduces to the usual median, which is the middle most order statistic of a univariate data set. Chakraborty and Chaudhuri (1996) and Chakraborty et al. (1998) have shown that $\hat{\boldsymbol{\Theta}}_n^{(\alpha)}$ is an affine equivariant estimate, and if the multivariate observations $\boldsymbol{X}_i$'s are i.i.d. with a common probability density function satisfying some appropriate regularity conditions, $\hat{\boldsymbol{\Theta}}_n^{(\alpha)}$ converges at $n^{1/2}$ rate and has an asymptotically normal distribution. Further, it follows from Chakraborty and Chaudhuri (1998) and Chakraborty et al. (1998) that if the common density of the $\boldsymbol{X}_i$'s is elliptically symmetric having the form $\{\det(\Sigma)\}^{-1/2} f\{(\boldsymbol{x}-\boldsymbol{\Theta})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\Theta})\}$ with $\boldsymbol{\Theta}$ as the location of elliptic symmetry and Σ as the scatter matrix, $\hat{\boldsymbol{\Theta}}_n^{(\alpha)}$ (which is a $n^{1/2}$-consistent and asymptotically normal estimate for $\boldsymbol{\Theta}$) achieves its maximum asymptotic efficiency when the subset of indices α and the associated transformation matrix $\boldsymbol{X}(\alpha)$ is chosen in such a way such that the matrix $\{\boldsymbol{X}(\alpha)\}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{X}(\alpha)$ is as close as possible to a diagonal matrix with all diagonal entries equal. They suggested an adaptive algorithm for computing $\hat{\boldsymbol{\Theta}}_n^{(\alpha)}$ by selecting that subset α for which

$$\frac{\mathrm{trace}[\{\boldsymbol{X}(\alpha)\}^{\mathrm{T}}\hat{\Sigma}^{-1}\{\boldsymbol{X}(\alpha)\}]}{(\det[\{\boldsymbol{X}(\alpha)\}^{\mathrm{T}}\hat{\Sigma}^{-1}\{\boldsymbol{X}(\alpha)\}])^{1/d}}$$

is minimized. Here $\hat{\Sigma}$ is an affine equivariant and consistent estimate of the scatter matrix Σ. We will gradually see that it is sufficient to use a high breakdown estimate of Σ in order to achieve good robustness properties for transformation retransformation medians. Note that the above mentioned minimization problem is equivalent to the problem of minimizing the ratio of the arithmetic and the geometric means of the eigenvalues of the matrix $\{\boldsymbol{X}(\alpha)\}^{\mathrm{T}}\hat{\Sigma}^{-1}\{\boldsymbol{X}(\alpha)\}$ though it does not involve explicit computation of the eigenvalues. Efficient algorithms for computing adaptive versions of transformation retransformation coordinatewise and spatial medians have been discussed in Chakraborty and Chaudhuri (1998) and Chakraborty et al. (1998), respectively. They have demonstrated numerical implementation of those algorithms using multivariate data with fairly large dimensions for which any of the affine equivariant versions of multivariate median proposed by Tukey (1975), Oja (1983) and Liu (1990) will be computationally problematic. Numerical implementation of such an algorithm will require computation of $\hat{\Sigma}$, and one can conveniently use the FAST-MCD algorithm of Rousseeuw and Van Driessen (1997), which is a very fast algorithm for obtaining an affine equivariant high breakdown estimate of the multivariate scatter matrix.

2. Robustness of transformation retransformation medians

So far not much has been reported on the robustness and the breakdown properties of such transformation retransformation medians, and that is what will be considered in details in this note. Let us begin by recalling the definition of finite sample breakdown point of an estimate following Donoho (1982). For an estimate $\boldsymbol{T}(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n)$ based on data $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$, its finite sample breakdown point $\mathscr{E}_n(\boldsymbol{T})$ is defined as

$$\mathscr{E}_n(\boldsymbol{T}) = \inf\left\{\frac{m}{n}: \sup_{\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_n} \|\boldsymbol{T}(\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_n) - \boldsymbol{T}(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n)\| = \infty\right\},$$

where the observations $\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_n$ are obtained by replacing m of the observations of the original data set $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ (see also Huber, 1981). It is quite apparent that the robustness of the adaptive transformation retransformation medians will critically depend on the robustness of the estimate $\hat{\Sigma}$ used in its construction. The following theorem describes the breakdown properties of such multivariate medians.

Theorem 2.1. *Suppose that a high breakdown estimate* $\hat{\Sigma}$ *of the scatter matrix* Σ *(e.g. the S-estimate proposed by* Davies (1987) *and* Rousseeuw and Leroy (1987) *or the minimum covariance determinant (MCD) estimate proposed by* Rousseeuw (1984)) *is used in forming the transformation matrix* $\boldsymbol{X}(\alpha)$. *Then for* $n > d+1$ *the finite sample breakdown point of the adaptive version of the transformation retransformation median will be at least* $n^{-1}[(n-d+1)/2]$, *where* $[x]$= *the largest integer smaller than or equal to* x, *and consequently its asymptotic breakdown will be* 50% *for all* $d \geqslant 1$.

Proof. Let us set $\boldsymbol{Z}(\alpha) = [\,|\det\{\boldsymbol{X}(\alpha)\}|\,]^{-1/d}\boldsymbol{X}(\alpha)$ so that $\det\{\boldsymbol{Z}(\alpha)\} = 1$. Clearly, in view of the construction of the transformation retransformation median $\hat{\boldsymbol{\Theta}}_n^{(\alpha)}$ described at the beginning of this section, this median will not be affected if one uses $\{\boldsymbol{Z}(\alpha)\}^{-1}$ instead of $\{\boldsymbol{X}(\alpha)\}^{-1}$ to transform the data points $\boldsymbol{X}_i$'s, and then the resulting $\hat{\boldsymbol{\Phi}}_n^{(\alpha)}$ (which will be altered by the change in the transformation matrix) is retransformed using $\boldsymbol{Z}(\alpha)$ instead of $\boldsymbol{X}(\alpha)$. Then the criterion for choosing the optimal subset α can be modified to the minimization of trace$[\{\boldsymbol{Z}(\alpha)\}^{\mathrm{T}}\hat{\Sigma}^{-1}\{\boldsymbol{Z}(\alpha)\}]\{\det(\hat{\Sigma})\}^{1/d}$. Suppose now that we have corrupted m of the observations such that $m \leqslant [(n-d+1)/2]$, and $\hat{\Sigma}$ is a high breakdown estimate like the S-estimate or the MCD estimate. Then $\hat{\Sigma}$ will not break, and its minimum eigenvalue will remain bounded away from zero while the maximum eigenvalue will remain bounded. Now, for the optimal choice of α according to the aforesaid criterion, trace$[\{\boldsymbol{Z}(\alpha)\}^{\mathrm{T}}\hat{\Sigma}^{-1}\{\boldsymbol{Z}(\alpha)\}]\{\det(\hat{\Sigma})\}^{1/d}$ must remain bounded even after corrupting m of the data points in view of the fact that $n > d+1$. In other words, we must have trace$[\{\boldsymbol{Z}(\alpha)\}^{\mathrm{T}}\hat{\Sigma}^{-1}\{\boldsymbol{Z}(\alpha)\}] < M\{\det(\Sigma)\}^{-1/d}$ for some $M > 0$. This implies that

$$\mathrm{trace}[\{\boldsymbol{Z}(\alpha)\}\,\{\boldsymbol{Z}(\alpha)\}^{\mathrm{T}}] \leqslant M\rho,$$

where ρ is the ratio of the maximum and the minimum eigenvalues of $\hat{\Sigma}$. Thus the maximum eigenvalue of the matrix $\{\boldsymbol{Z}(\alpha)\}\{\boldsymbol{Z}(\alpha)\}^{\mathrm{T}}$ must remain bounded in spite of corrupting m of the data points. As the determinant of this matrix is one, the smallest eigenvalue of this matrix too will be bounded away from zero.

Now, for the transformed observations $\boldsymbol{Y}_i^{(\alpha)}$'s, we have

$$\|\boldsymbol{Y}_i^{(\alpha)}\|^2 = \boldsymbol{X}_i[\boldsymbol{Z}(\alpha)\{\boldsymbol{Z}(\alpha)\}^{\mathrm{T}}]^{-1}\boldsymbol{X}_i \leqslant \{\lambda^{(\alpha)}\}^{-1}\|\boldsymbol{X}_i\|^2,$$

where $\lambda^{(\alpha)}$ is the smallest eigenvalue of $\boldsymbol{Z}(\alpha)\{\boldsymbol{Z}(\alpha)\}^{\mathrm{T}}$. This implies that with the optimal choice of the transformation matrix, a transformed data point $\boldsymbol{Y}_i^{(\alpha)}$ cannot be made to explode to ∞ without making the corresponding untransformed data point $\boldsymbol{X}_i$ to explode to ∞. Therefore, if we use the coordinatewise median vector or the spatial median vector as our $\hat{\boldsymbol{\Phi}}_n^{(\alpha)}$, in order to have $\|\hat{\boldsymbol{\Phi}}_n^{(\alpha)}\| \to \infty$, we must corrupt at least $[(n+1)/2]$ of the observations. In other words, if we corrupt only $m \leqslant [(n-d+1)/2]$ of the data points, $\|\hat{\boldsymbol{\Phi}}_n^{(\alpha)}\|$ will not explode to ∞.

Recall now that $\hat{\boldsymbol{\Theta}}_n^{(\alpha)} = \{\boldsymbol{Z}(\alpha)\}\hat{\boldsymbol{\Phi}}_n^{(\alpha)}$, which implies that $\|\hat{\boldsymbol{\Theta}}_n^{(\alpha)}\|^2 \leqslant \mu^{(\alpha)}\|\hat{\boldsymbol{\Phi}}_n^{(\alpha)}\|^2$, where $\mu^{(\alpha)}$ is the largest eigenvalue of the matrix $\{\boldsymbol{Z}(\alpha)\}^{\mathrm{T}}\{\boldsymbol{Z}(\alpha)\}$. This proves that our transformation–retransformation median $\hat{\boldsymbol{\Theta}}_n^{(\alpha)}$ will not break if we corrupt only $m \leqslant [(n-d+1)/2]$ of the observations. □

The S-estimates as well as the MCD estimate of the scatter matrix Σ are affine equivariant and known to be consistent (see Rousseeuw, 1984; Davies, 1987). Hence, the preceding theorem establishes that it is possible to construct high breakdown affine equivariant versions of multivariate median using the transformation retransformation strategy, and those multivariate medians will have good statistical efficiency in view of the results proved in Chakraborty and Chaudhuri (1998) and Chakraborty et al. (1998).

3. Discussion and concluding remarks

Transformation retransformation medians are probably the only multivariate affine equivariant medians that are known so far to possess adequately all the desired properties of the univariate median discussed in the Introduction. Among other affine equivariant multivariate medians, Tukey's half-space depth-based median is known to possess an asymptotic breakdown of $\frac{1}{3}$ when the underlying probability distribution is absolutely continuous and angularly symmetric. Chen (1996) provided some upper and lower bounds for the finite sample breakdown point of Tukey's median when the observed data is in general position. But those bounds depend on the underlying probability distributions. In the bivariate situation, Oja et al. (1990) showed that it is possible to break Oja's simplicial volume based median by corrupting only two data points, and consequently, it has a very poor (i.e. 0%) asymptotic breakdown. Chen (1995) observed that when the underlying distribution is absolutely continuous, an upper bound for the asymptotic breakdown point of Liu's simplicial depth based median is $1/(d+2)$, where d is the dimension of the data. He also obtained some upper and lower bounds of the finite sample breakdown point of Liu's median but those bounds depend on the data points. Interestingly, we have observed that one can break Liu's median by just replicating some corrupted observation only a few times.

To illustrate the breakdown properties of some of the affine equivariant multivariate medians, we have generated 50 observations from standard bivariate normal distribution (i.e. zero means and correlation and unit s.d.'s) and computed their transformation retransformation coordinatewise and spatial medians (denoted by TR1 and TR2, respectively), Liu's simplicial depth median (denoted by LIU), and Tukey's half-space depth median (denoted by TUKEY), and they are shown in Fig. 1(a). There are very little visible differences among the positions of these four versions of affine equivariant medians. In Fig. 1(b), we have plotted the same 50 bivariate normal observations with another 25 observations from bivariate normal with mean $=(10,10)$ and standard deviation of each of the two uncorrelated coordinate variables as 0.01. There we see that Tukey's half-space depth median is well outside the original data cloud, but both the transformation retransformation medians remain almost at the same positions as in Fig. 1(a). In Fig. 1(c), we have added 7 replications of the point $(10,10)$ to the set of 50 original observations. We observe in this case that Liu's median is completely shifted to the point $(10,10)$ whereas transformation retransformation medians continue to remain inside the original data cloud. In all cases, the optimal transformation matrix was chosen based on the MCD estimate of multivariate scatter computed using the FAST-MCD algorithm of Rousseeuw and Van Driessen (1997).

Among other affine equivariant high breakdown estimates of multivariate location, the minimum volume ellipsoid (MVE) estimator proposed by Rousseeuw (1984) is quite popular. But this estimator is not a multivariate generalization of univariate median – in the univariate situation, this estimator does not coincide with the middle most order statistic of the data. Apart from that MVE estimator is not $n^{1/2}$-consistent and its asymptotic distribution is non-Gaussian (see e.g. Davies, 1992). On the other hand, the minimum covariance determinant estimator of location proposed by Rousseeuw (1984), has very high breakdown point, and it is $n^{1/2}$-consistent, and asymptotically normally distributed (see Butler et al., 1993). Nevertheless it too is not a multivariate generalization of univariate median for the same reason mentioned in the case of MVE estimate. We conclude our discussion by noting that in the univariate situation, it is possible to find M-estimators with breakdown point arbitrarily close to 50%. But in higher dimensions, this fact no longer holds.

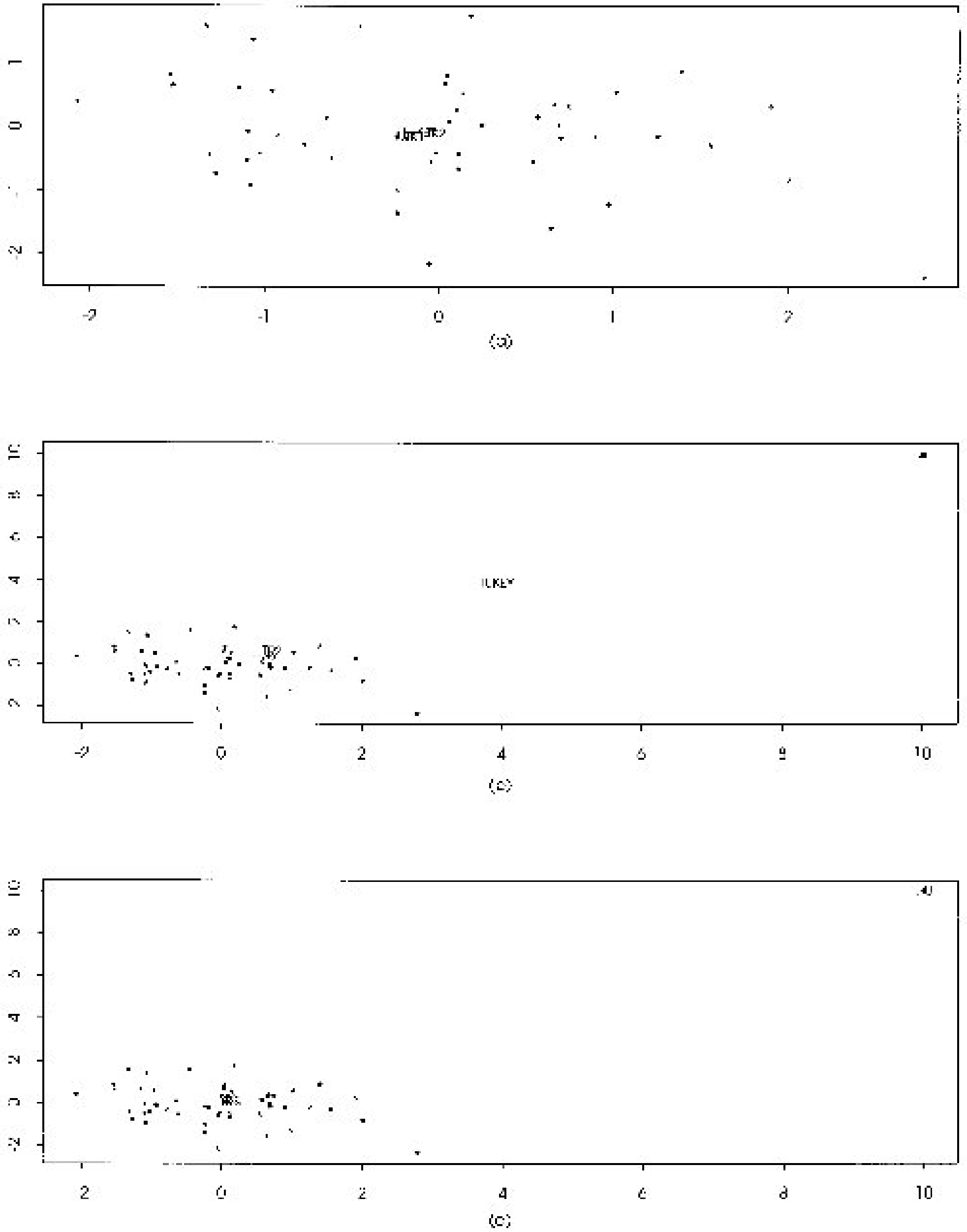

Fig. 1. Different bivariate medians for corrupted and uncorrupted data: (a) $n = 50$ bivariate standard normal data points; (b) $n = 75$ data points: 50 bivariate standard normal and 25 bivariate normal with mean = (10, 10), s.d. = (0.01, 0.01) and correltation = 0.0; (c) $n = 57$ data points: 50 bivariate standard normal and 7 replications of (10,10).

Maronna (1976) formally defined M-estimators for d-dimensional data and gave the upper bound of $1/(d+1)$ for their breakdown points (see also Huber, 1981). Other affine equivariant estimators such as those based on convex hull peeling or classical outlier rejection are also known to have breakdown points bounded by $1/(d+1)$ (Donoho, 1982).

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