Sankhyā : The Indian Journal of Statistics
1993, Volume 55, Series A, Pt. 2, pp. 226–232

ON THE DISTRIBUTION OF ECCENTRICITY

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SUMMARY. Displacement of the centre of a circular cross-section of a processed job from its ideal position is termed as eccentricity. For a particular type of processing viz. shaping of jobs by metal removal, it is shown that the distribution of eccentricity is an extreme value distribution unlike the popular belief of a chi-distribution (+ve square root of chi-square distribution) with 2 d.f. Some data sets are also analysed.

1. Introduction

Consider a circular piece of job where centre should be at a preassigned point. Due to several factors the centre of the job deviates from its ideal position when the processing of the job is over. The distance of the centre of the job from the ideal position is termed as eccentricity.

For example, consider a metal bar which is to be given a cylindrical shape. It has a preassigned axis before the processing starts. When the processing is over the new axis of the job may not be the same as the earlier one due to random fluctuation of the job, bad machining tool etc. For a particular cross-section, the distance of the centre from the original axis of the job where the original centre of the cross-section lies is the measure of eccentricity.

If the eccentricity is high then the job will be deformed resulting in malfunction of the job.

There are other instances of eccentricity too, e.g. the base of a ceiling fan may not be concentric with the suspending rod, or the tyred wheel of a bicycle may not be concentric with the free wheel effecting the smooth running of the cycle.

2. The model

So far chi-distribution with 2 d.f. is used to explain the behaviour of eccentricity under appropriate assumptions e.g. displacements in two orthogonal directions x, y are i.i.d. $N(0, 1)$ and $r = \sqrt{(x^2+y^2)}$ is a chi-distribution.

However in some cases the empirical data is negatively skew (e.g. data set 2) which clearly do not match with the feature of a chi-distribution.

Paper received. November 1989; revised April 1991.

AMS subject classification. Primary 62E20; secondary 62N10.

Key words and phrases. Eccentricity, extreme value distribution, chi-distribution, ovality, dial gauge.

In Dasgupta, Ghosh and Rao (1981), the distribution of ovality was derived, postulating a simple cutting model. This model may also be used to explain the distribution of eccentricity where jobs are given circular cross-section by metal cutting.

The operation is described in Dasgupta, Ghosh and Rao (1981). The unfinished metal bar is gripped by a jaw chuck at one end and usually supported by some other means, say by fixing a pin point at the other end of the axis of the job. Usually for a heavy job, holding only at one end may not be enough to balance the job. The job is rotated by a motor and the cutting tool moves from one cross section to another removing the metal, We use the notations of Dasgupta, Ghosh and Rao (1981).

As the job rotates, the centre of a particular cross section also fluctuates from its rest position. Let ϵ_i be the amount of displacement of the centre towards the cutting tool at i-th rotation $i = 1 \ldots n$. These random displacements ϵ_i may be considered to be iid random variables assuming that the rotations are mutually independent. Now, for the 1st rotation, a particular cross section moves forward by an amount ϵ_1 towards the cutting tool. So, if the cutting tool is very hard and the job is made of relatively scft material then the amount cut by the tool will be same as displacement towards the tool, i.e., ϵ_1. In the second revolution the displacement is ϵ_2. Now the job will come in contact with the tool if $\epsilon_2 \geqslant \epsilon_1$ in which case there will be a further cut of the amount $(\epsilon_2 - \epsilon_1)$. So the total cut upto second stage is $\epsilon_1 + (\epsilon_2 - \epsilon_1) = \epsilon_2$ if $\epsilon_2 \geqslant \epsilon_1$. Otherwise i.e., if $\epsilon_2 < \epsilon_1$ the cut upto second stage is ϵ_1. Combining these we can write, cut upto second stage is max (ϵ_1, ϵ_2). Arguing similarly cut upto n-th stage is $\max\limits_{1 \leqslant i \leqslant n} \epsilon_i$.

If the job is heavy, while suspending it horizontally for processing, the suspending axis may be sightly curved in the middle where there is no support. Even the glass which is considered to be nearly rigid deviates from rigidity when a large square sheet of glass is suspended horizontally while supporting it only at two diagonally opposite corners.

The quantities ϵ_i's should be looked upon as a net displacement between the cutting tool and the point under processing from its position at rest. Because of high pressure at the point of contact with the cutting tool, centre of that cross-section may shift while two end points of the original axis remain undisturbed. It is also possible that the cutting tool fluctuates its position because of vibration of the machine under operation. All these accumulated effects we attribute to the fluctuation of the position of the centre of that particular cross-section, pretending that the cutting tool, two end points of the original axis and the machine remain undisturbed throughout the operation.

Since the centre of a particular cross-section had a displacement $\max_{1 \leq i \leq n} \epsilon_i$ and the cutting tool did cut the total amount displaced towards it, the two end points of the original axis remaining undisturbed, the centre of that cross-section will also be shifted by the same amount from its original position. So the eccentricity $\max_{1 \leq i \leq n} \epsilon_i$ has an extreme value distribution after suitable standardisation,

If the tool cuts only a fraction of the displacement of the cross section towards it then the cut upn n-th stage T_n takes the form, vide Dasgupta, Ghosh and Rao (1981).

$$T_n = T_{n-1}+(\epsilon_n - T_{n-1})^+ c_n,\ T_1 = c_1 \epsilon_1^+ \quad 0 < c_n \leqslant 1, n \geqslant 1, \epsilon^+ = \max(\epsilon,\ 0),$$

c_i depends on the hardness of the job and the cutting tool e.g. $c_i = 1$ if the tool is too hard and the job is soft. One may assume that max ϵ_i has a limiting distribution i.e. $(\max \epsilon_i - a_n)/b_n \sim$ an extreme value distribution. In Dasgupta, Ghosh and Rao (1981), it is also assumed

$$c_i = 1 - o(|a_i^{-1} b_i|),\ c_i \uparrow 1 \text{ and } \overline{\lim_{i \to \infty}}\ f(ik)/f(i) < \infty \ \forall\ k$$

where $f(i) = |a_i^{-1} b_i|$ is non decreasing in i. These imply that

$$T_n = \max_{1 \leq i \leq n} \epsilon_i + o_p(b_n) \quad \text{(see } A3,\ p\ 201 \text{ of Dasgupta, Ghosh and Rao (1981).}$$

In the general form of the cut derived in Dasgupta, Ghosh and Rao (1981) p. 188, (3.13). we have

$$d(\theta) = d - 2 \max_{1 \leq i \leq n} \epsilon_i - 2(b^2 + (a^2 - b^2) \cos^2 \theta)^{1/2} + \eta(\theta) + o_p(b_n).$$

The second term in the r.h.s contributes to eccentricity being twice the random shift of the centre. Other terms contribute to ovality. Here d is the diameter of the job to be processed to a first approximation ; a and b are the max and min span of the radius of the overall bearing eccentricity in the machine ; θ is the particular angle (of the point being processed) made with the axis of the spans a and b ; $\eta(\theta)$ is the residual random part varying from point to point (i.e, over θ)

3. Technique of measurements

The measurements of eccentricity is obtained by rotating the finished job around the preassigned axis and then observing the deflection on a dial gauge. If there is a deviation of the pointer, then that implies presence of eccentricity for that particular cross-section when there is no ovality. The dial reading equals twice the amount of eccentricity in absence of ovality. The maximum and the minimum of the dial reading occur in opposite direction i.e. at 180° if only eccentricity is present. On the other hand if there is no eccentricity but ovality is present then these reading occur at orthogonal direction i.e. at 90°. Therefore the directions at which minimum and maximum readings occur may indicate the presence of ovality/eccentricity or both.

Next consider the problem of finding the amount of eccentricity by dial reading when both ovality and eccentricity are present. In such a case, observe the position of the job when dial shows maximum deflection then note the dial reading at opposite direction i.e. at 180°. The difference of the two readings will clearly be

$$((b_1^2+(a_1^2-b_1^2)\cos^2\theta)^{1/2}+r)-((b_1^2+(a_1^2-b_1^2)\cos^2\theta)^{1/2}-r) = 2r$$

where r is the eccentricity, a_1, b_1 are the max. and min. radius of the elliptical job (with ovality $2(a_1-b_1)$) and θ is the angle made by the presassigned centre with the pair of the axis of the elliptical cross-section.

One may of course find the amount of ovality taking observation at different positions of the job, when it is at rest by a slide calliperse. The difference of min. and max. reading will provide the value of ovality. The presence of eccentricity will not effect the readings since the job is at rest.

4. Fitting the model and analysing the data

Under appropriate assumptions the distributions of eccentricity is an extreme value distribution and may be either of the following types after standardisation.

$$\begin{array}{lll} \text{I} & F(x) = \exp(-(-x)^\alpha), & x < 0 \ \alpha > 0 \\ & \quad = 1 & x \geqslant 0 \\ \text{II} & F(x) = 0 & x < 0 \\ & \quad = \exp(-x^{-\alpha}) & x \geqslant 0, \alpha > 0 \\ \text{III} & F(x) = \exp(-e^{-x}), & -\infty < x < \infty \end{array}$$

See Dasgupta *et al.*, p. 190, as mentioned the orein we decide the appropriate type to be fitted considering the skewness of the empirical distribution based on the particular data. The centering constant is usually estimated by sample extremum. The other parameters are approximated by equating the theoretical distribution function with empirical distributon function at convenient points. χ^2 test of significance provides a conservative test.

In all the sets of data presented below the magnitude of ovality in jobs were negligible.

Data Set 1. This relates to the concentricity of the ring of the back cover 80 table fans measures with respect to the axis of the job. The rings were given circular shape by metal cutting operation.

The grouped data are given below.

eccentricity (in 10^3 inch)	frequency	expected frequency
(0—1.5]	12	9.66
(1.5—2.5]	21	25.80
(2.5—3.5]	19	16.38
(3.5—4.5]	9	9.16
(4.5—6.5]	7	8.00
> 6.5	12	10.20
total	80	80

Type of distribution fitted $F(x) = \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right)$; $x \geqslant \mu, \mu = 0$. Equation solved for α and σ are $\exp(-(1.881/\sigma)^{-\alpha}) = 20/80$, $\exp(-(4.5/\sigma)^{-\alpha}) = 61/80$ giving $\sigma = 2.239$, $\alpha = 1.869$. $\chi^2 = 3.12$, $\chi^2_{.05,2} = 5.99$, calculated χ^2 is insignificant and the fit is satisfactory.

Data Set 2. These observations relates to a Journal subjected to vertical boring operation. This component is used in Kaplan runner assembly to be used in a hydro-turbine. The observations are taken at different cross-section of the Journal. The maximum eccentricity allowed was 30 microns. Two observations falling above 30 microns are not considered.

eccentricity (in microns)	frequency	expected frequency
$\leqslant 10$	6	6
(10—20]	8	6.724
(20—25]	5	6.276
(25—30]	10	10
total	29	29

Type of distribution fitted : $F(x) = \exp\left(-\left(\frac{30-x}{\sigma}\right)^{\alpha}\right)$, $x \leqslant 30$, Equation solved for α and σ : $\exp(-(20/\sigma) = 6/29$ and $\exp(-(5/\sigma^{\alpha})^{\alpha} = 19/29$ giving $\alpha = .9354$, $\alpha = 12.302$, $\chi^2 = 0.502$, $\chi^2_{.05,1} = 3.84$ calculated χ^2 is insignificant and the fit is satisfactory.

Observe that unlike chi-distribution the empirical distribution is negatively shew.

Data Set 3. This set relates to the eccentricity measurements of a hexagonal bolt used for packing purpose.

eccentricity (in micron)	observed frequency	expected frequency
$\leqslant 24$	4	2.56
(24—26]	4	4.07
(26—28]	6	7.37
(28—30]	11	10.28
(30—32]	10	10.72
(32—34]	7	7.63
(34—36]	3	2.37
total	45	45

The empirical distribution is negatively skew. Type of distribution fitted : $F(x) = \exp\left(-\left(\frac{\mu-x}{\sigma}\right)^{\alpha}\right)$; $x \leqslant \mu, \mu = 36$. Equation solved for α and σ : $\exp(-(8/\sigma^{\alpha})^{\alpha}) = 14/45$, $\exp(-(4/\sigma)^{\alpha}) = 35/45$ giving $\alpha = 2.216$, $\sigma = 7.46$. $\chi^2 = 1.384$, $\chi^2_{.05,3} = 7.81$ calculated χ^2 is insignificant and the fit is satisfactory.

Data Set 4. This set relates to the eccentricity of the short journal with respect to the long journal of the rotar of table fan in the units of 10^{-4} inch.

class interval	frequency	expected frequency
[1—1.5]	7	7.00
(1.5—2]	7	8.82
(2—2.5]	6	4.37
(2.5—3.5]	5	3.98
> 3.5	5	5.83
total	30	30

Type of distribution fitted : $F(x) = \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right)$, $x \geqslant \mu, \mu = 1$. Equation solved for α and σ are $\exp(-(.5/\sigma)^{-\alpha}) = 7/30$ and $\exp(-2.1/\sigma)^{-\alpha}) = 23/30$ giving $\alpha = 1.185$ and $\sigma = 0.686$ $\chi^2 = 1.36$, $\chi^2_{.05,1} = 3.84$. Calculated χ^2 is insignificant and the data fits the model.

These models may be used to find upper and lower control limits for eccentricity to check whether the process is under control.

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