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Some Generalizations of Locally Uniform Rotundity

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Recently P. Kenderov and W. Moors [9, 10] proved that every weakly locally uniformly rotund (WLUR) Banach space is σ-fragmentable in the sense of J. Jayne *et al*. [8]. This result was generalized in [15] where it is proved that every WLUR Banach space is locally uniformly rotund (LUR) renormable. However, a careful analysis of [5, 9, 10] (see also [11]) shows that a Banach space X is σ-fragmentable if X has the property that $x = x^{**}$ whenever $x \in X$, $x^{**} \in X^{**}$, and $\|x\| = \|x^{**}\| = \|(x + x^{**})/2\|$. It is easy to see that every WLUR Banach space has this property [17]. In

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[5, p. 420] the question of how to characterize this property is raised. In this paper, we show that this property is strictly weaker than WLUR. We characterize the property in terms of the initial space X without referring to the bidual of X and in terms of unbounded nested sequence of balls in the dual space X^*.

DEFINITION 1. An "unbounded nested sequence of balls" in a Banach space X is an increasing sequence $\{B_n = B(x_n, r_n)\}$ of open balls in X with $r_n \uparrow \infty$.

DEFINITION 2. Let x be a point of the unit sphere S_X of a Banach space X. We say that x is

(a) [4] a rotund point of the unit ball B_X of X (or, X is rotund at x) if $\|y\| = \|(x+y)/2\| = 1$ implies $x = y$;

(b) an LUR (resp. WLUR) point of B_X if for any $\{x_n\} \subseteq B_X$, the condition

$$\lim_n \left\| \frac{x_n + x}{2} \right\| = 1$$

implies

$$\lim_n \|x_n - x\| = 0 \qquad \left(\text{resp. w-}\lim_n (x_n - x) = 0\right);$$

(c) an almost LUR (ALUR) (resp. weakly almost LUR (WALUR)) point of B_X if for any $\{x_n\} \subseteq B_X$ and $\{x_m^*\} \subseteq B_{X^*}$, the condition

$$\lim_m \lim_n x_m^* \left(\frac{x_n + x}{2} \right) = 1$$

implies

$$\lim_n \|x_n - x\| = 0 \qquad \left(\text{resp. w-}\lim_n (x_n - x) = 0\right);$$

(d) a midpoint LUR (MLUR) (resp. weakly midpoint LUR (WMLUR)) point of B_X if $\lim_n \|x \pm x_n\| = 1$ implies

$$\lim_n \|x_n\| = 0 \qquad \left(\text{resp. w-}\lim_n x_n = 0\right).$$

We say that a Banach space X has one of the above properties if every point of S_X has the same property.

Remark 3. In our terminology, the result of [5] mentioned above says that if every $x \in S_X$ is a rotund point of $B_{X^{**}}$, then X is σ-fragmentable.

Clearly, every rotund point of B_X is an extreme point, indeed an exposed point of B_X. But the converse is not generally true. For example, no extreme point of B_{l_∞} or B_{l_1} is a rotund point of B_{l_∞} or B_{l_1}. However, if every point of S_X is an extreme point of B_X, then every point of S_X is a rotund point of B_X and the space X is rotund.

We have the following diagram for all Banach spaces:

$$\begin{array}{ccccc} \text{LUR} & \to & \text{ALUR} & \to & \text{MLUR} \\ \downarrow & & \downarrow & & \downarrow \\ \text{WLUR} & \to & \text{WALUR} & \to & \text{WMLUR} \end{array}$$

It is necessary to prove only ALUR → MLUR and WALUR → WMLUR. Indeed let $x \in S_X$ be an ALUR (resp. WALUR) point of B_X and $\lim_n \|x \pm x_n\| = 1$. Then for every $x^* \in S_{X^*}$ such that $x^*(x) = 1$, we have $\lim_n x^*(x_n) = 0$. So $\lim_n x^*((x + (x_n + x))/2) = 1$, hence $\lim_n \|x_n\| = 0$ (resp. w-lim $x_n = 0$).

In the study of continuity of metric projections, L. P. Vlasov [19] (see [16, Theorem 2]) showed that X^* is rotund if and only if the union of any unbounded nested sequence of balls in X is either the whole of X or an open half-space.

In Theorem 6, we show that this result follows from the properties of rotund points rather than extreme points, by obtaining a localization of this property to characterize rotund points of B_{X^*}. A variant of this was obtained in [4, Theorem 2] and seems to have been overlooked since then. Our proof is simpler.

Later F. Sullivan [17] introduced a stronger property, called Property (V) (called the Vlasov Property in [2]) and showed that a Banach space X has the Property (V) if and only if X^* is rotund and X is Hahn–Banach smooth. The following reformulation of the definition comes from [2, Proposition 3.1].

Definition 4. A Banach space X is said to have the Vlasov Property, if for every unbounded nested sequence $\{B_n\}$ of balls and $x^*, y_n^* \in S_{X^*}$, if x^* is bounded below on $\cup B_n$, and the sequence $\{\inf y_n^*(B_n)\}$ is bounded below, or, specifically, if there exists $c \in \mathbb{R}$ such that

$$x^*(b) > c \qquad \text{for all } b \in \cup B_n,$$

$$y_k^*(b) > c, \qquad \text{for all } b \in B_n,\ n \le k$$

then w-lim $y_n^* = x^*$.

In [2], it is shown that variants of the Vlasov Property characterize the w*-Asymptotic Norming Properties (w*-ANPs). See [2] for the details.

Here we also show that a weaker version of the Vlasov Property can be localized to characterize rotund points of B_{X^*}.

In Corollary 8, we show that for any Banach space X, a point $x \in S_X$ is a WALUR point of B_X if and only if x is a rotund point in $S_{X^{**}}$. It is known [1] that l_∞ has no equivalent WMLUR norm. Hence the norm

$$\|x\| = \|x\|_\infty + \left(\sum_{i=1}^{\infty} 2^{-i} x_i^2 \right)^{1/2}, \qquad x = (x_i) \in l_\infty,$$

is a rotund norm in l_∞ which fails to be WALUR. In [6], it is shown that every separable Banach space has an equivalent norm with at most countably many MLUR points and by the construction of the norm, none of the MLUR points are WALUR points. In Corollary 12 below, we show that there exists an ALUR Banach space which fails to be WLUR. We raise the following two questions:

QUESTION 1. *Does every separable Banach space admit an equivalent norm with property* α, *introduced in* [12], *without rotund points in the unit ball*?

Let us remark that evidently l_1 has property α and its unit ball has no rotund point. In [18] it is proved that if X is weakly compactly generated then there exists in X an equivalent norm with respect to which X has property α. In [7] this result is extended for Banach spaces with a fundamental biorthogonal system. In [13] it was shown that any Banach space with property α fails to have LUR points of its unit ball.

QUESTION 2. *Does every WALUR space admit an equivalent LUR norm*?

A negative answer to this question will give an example of a σ-fragmentable Banach space X such that (X, weak) endowed with the metric of the norm has no countable cover by sets of small local diameter in the sense of [8]. It is known that a WMLUR space with countable cover by sets of small local diameter is LUR renormable [14].

DEFINITION 5. Let X be a Banach space. We say that $x^* \in S_{X^*}$ is a w*-ALUR point of B_{X^*} if for any $\{x_n^*\} \subseteq B_{X^*}$ and $\{x_m\} \subseteq B_X$, the condition

$$\lim_m \lim_n \left(\frac{x_n^* + x^*}{2} \right)(x_m) = 1$$

implies w*-lim $x_n^* = x^*$.

We now present the main results of this paper.

THEOREM 6. *Let X be a Banach space. For $x^* \in S_{X^*}$, the following are equivalent*:

(a) *x^* is a rotund point of B_{X^*}*;

(b) *x^* is a w*-ALUR point of B_{X^*}*;

(c) *for every unbounded nested sequence* $\{B_n\}$ *of balls such that* x^* *is bounded below on* $\cup B_n$, *if for any* $\{y_n^*\} \subseteq S_{X^*}$, *the sequence* $\{\inf y_n^*(B_n)\}$ *is bounded below, then* $\text{w}^*\text{-lim } y_n^* = x^*$;

(d) *for every unbounded nested sequence* $\{B_n\}$ *of balls such that* x^* *is bounded below on* $\cup B_n$, *if* $y^* \in S_{X^*}$ *is also bounded below on* $\cup B_n$, *then* $y^* = x^*$;

(e) *for every unbounded nested sequence* $\{B_n\}$ *of balls such that* x^* *is bounded below on* $\cup B_n$, $\cup B_n$ *is an affine half-space determined by* x^*.

Proof. (a) $\Rightarrow$ (b). Let $\{x_n^*\} \subseteq B_{X^*}$ and $\{x_m\} \subseteq B_X$ such that

$$\lim_m \lim_n \left(\frac{x_n^* + x^*}{2} \right)(x_m) = 1.$$

As $\{x_n^*\} \subseteq B_{X^*}$, $\{x_n^*\}$ has w*-cluster points in B_{X^*}. Let y^* be a w*-cluster point of $\{x_n^*\}$. Then $\|y^*\| \leq 1$. It follows that

$$\lim_m \left(\frac{x^* + y^*}{2} \right)(x_m) = 1$$

and hence, $\|(x^* + y^*)/2\| = 1$. By (a), we conclude that $y^* = x^*$. This implies the sequence $\{x_n^*\}$ has a unique w*-cluster point x^*, that is, $\text{w}^*\text{-lim } x_n^* = x^*$.

(b) $\Rightarrow$ (c). Let $\{B_n\}$ be an unbounded nested sequence of balls such that x^* is bounded below on $\cup B_n$. Let $\{y_n^*\} \subseteq S_{X^*}$, such that $\{\inf y_n^*(B_n)\}$ is bounded below. Suppose $c \in \mathbb{R}$ is a common lower bound. Let $B_n = B(x_n, r_n)$. We may assume without loss of generality that $0 \in B_1$. If we now put $y_n = x_n/r_n$, it follows that $\|y_n\| < 1$. The fact that $\{B_n\}$ is nested implies that $y_n^*(y_m) \geq 1 + c/r_m$ for all $n \geq m$. It follows that

$$\left(\frac{x^* + y_n^*}{2} \right)(y_m) \geq 1 + c/r_m$$

for all $n \geq m$. Since $r_m \to \infty$, we conclude

$$\lim_m \lim_n \left(\frac{x^* + y_n^*}{2} \right)(y_m) = 1.$$

Hence, $\text{w}^*\text{-lim } y_n^* = x^*$.

(c) $\Rightarrow$ (d). Apply (c) to the constant sequence $\{y_n^* = y^*\}$.

(d) $\Rightarrow$ (e). Suppose $\{B_n\}$ is an unbounded nested sequence of balls such that x^* is bounded below on $B = \cup B_n$. Put $\alpha = \inf x^*(B)$. Then

$$B \subseteq \{x : x^*(x) > \alpha\} = H.$$

If $B \neq H$, there exist $z \in H$ and $y^* \in S_{X^*}$ such that $\inf y^*(B) > y^*(z) = \beta$ (say). Thus, y^* is also bounded below on B. By (d), $x^* = y^*$. But then,

$$\beta = y^*(z) = x^*(z) > \alpha = \inf x^*(B) = \inf y^*(B) > \beta,$$

a contradiction.

(e) $\Rightarrow$ (a). Suppose there exists $y^* \in S_{X^*} \setminus \{x^*\}$ such that $(x^* + y^*)/2 \in S_{X^*}$. Let $\{x_n\} \subseteq B_X$ such that $(x^* + y^*)(x_n) \to 2$. Then, in fact, $x^*(x_n) \to 1$ and $y^*(x_n) \to 1$. Choose a sequence $\{\delta_n\}$ such that $\delta_n > 0$ for all n and $\sum_{n=1}^{\infty} \delta_n < 1$. Passing to a subsequence if necessary, we may assume, $x^*(x_n) > 1 - \delta_n$ and $y^*(x_n) > 1 - \delta_n$.

Let $B_n = B(\sum_{i=1}^{n} x_i, n + \sum_{i=1}^{n} \delta_i)$. Clearly $\{B_n\}$ is an unbounded nested sequence of balls. For any $n \in \mathbb{N}$,

$$\begin{aligned}\inf x^*(B_n) &= x^*\left(\sum_{i=1}^{n} x_i\right) - n - \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \left[x^*(x_i) - 1 - \delta_i\right] \\ &= -\sum_{i=1}^{n} \left[1 - x^*(x_i) + \delta_i\right] > -\sum_{i=1}^{n} 2\delta_i > -2.\end{aligned}$$

Similarly, $\inf y^*(B_n) > -2$. That is, both x^* and y^* are bounded below on $B = \cup B_n$. Now, if $B = \{x : x^*(x) > \inf x^*(B)\}$, then

$$\{x : x^*(x) > \inf x^*(B)\} \subseteq \{x : y^*(x) > \inf y^*(B)\}.$$

It follows that $\ker x^* \subseteq \ker y^*$, and hence, $x^* = \lambda y^*$ for some $\lambda \in \mathbb{R}$ and since $x^*, y^* \in S_{X^*}$, $x^* = y^*$. ∎

As an immediate corollary, we get the quoted result of Vlasov with a much easier proof.

COROLLARY 7 (Vlasov). *X^* is strictly convex if and only if the union of any unbounded nested sequence of balls is either the whole of X or an open half-space.*

Proof. Since X^* is strictly convex if and only if every $x^* \in S_{X^*}$ is a rotund point of B_{X^*}, this is immediate from Theorem 6 (a) $\Leftrightarrow$ (e). ∎

COROLLARY 8. *Let X be a Banach space. For $x \in S_X$, the following are equivalent:*

(a) *x is a rotund point of $B_{X^{**}}$;*

(b) *x is a w^*-ALUR point of $B_{X^{**}}$;*

(b′) *x is a wALUR point of B_X;*

(c) *for every unbounded nested sequence $\{B_n^*\}$ of balls in X^* such that x is bounded below on $\cup B_n^*$, if for any $\{y_n^{**}\} \subseteq S_{X^{**}}$, the sequence $\{\inf y_n^{**}(B_n^*)\}$ is bounded below, then* w*-lim $y^{**}{}_n = x$;

(c′) *for every unbounded nested sequence* $\{B_n^*\}$ *of balls in* X^* *such that* x *is bounded below on* $\cup B_n^*$, *if for any* $\{y_n\} \subseteq S_X$, *the sequence* $\{\inf y_n(B_n^*)\}$ *is bounded below*, *then* w-lim $y_n = x$;

(d) *for every unbounded nested sequence* $\{B_n^*\}$ *of balls in* X^* *such that* x *is bounded below on* $\cup B_n^*$, *if any* $x^{**} \in S_{X^{**}}$ *is also bounded below on* $\cup B_n^*$, *then* $x = x^{**}$;

(e) *for every unbounded nested sequence* $\{B_n^*\}$ *of balls in* X^* *such that* x *is bounded below on* $\cup B_n^*$, $\cup B_n^*$ *is an affine half-space determined by* x.

Proof. The equivalence of (a) to (e) is Theorem 6, while (b) ⇒ (b′) and (c) ⇒ (c′) are immediate. And (b′) ⇒ (c′) follows similarly as (b) ⇒ (c) of Theorem 6.

(c′) ⇒ (d). Let $\{B_n^* = B(x_n^*, r_n)\}$ be an unbounded nested sequence of balls in X^* such that x and some $x^{**} \in S_{X^{**}}$, $x^{**} \neq x$, are both bounded below on $\bigcup B_n^*$, by some $c \in \mathbb{R}$. It follows that for all $n \geq 1$,

$$\inf x(B_n^*) = x_n^*(x) - r_n \geq c$$
$$\inf x^{**}(B_n^*) = x^{**}(x_n^*) - r_n \geq c.$$

Assume without loss of generality that $0 \in B_1^*$. Let $x_0^* \in S_{X^*}$ be such that $x_0^*(x - x^{**}) = \varepsilon$ (say) > 0. By Goldstein's Theorem, there exists $\{x_n\} \in B_X$ such that

$$\left| x_0^*(x_n - x^{**}) \right| < \varepsilon/2$$

and for all $n \geq 1$,

$$\left| x_n^*(x_n - x^{**}) \right| < 1.$$

Then

$$1 \geq \|x_n\| \geq \left(\frac{x_n^*}{r_n}\right)(x_n) > x^{**}\left(\frac{x_n^*}{r_n}\right) - \frac{1}{r_n} \geq 1 + \frac{c-1}{r_n}.$$

Hence, $\|x_n\| \to 1$. Putting $y_n = x_n/\|x_n\|$, it follows that $\{y_n\} \subseteq S_X$. Moreover,

$$\begin{aligned}\inf x_n(B_n^*) &= x_n^*(x_n) - r_n\|x_n\| \\ &> x_n^*(x^{**}) - 1 - r_n\|x_n\| \\ &\geq c - 1 + r_n - r_n\|x_n\| = (c-1) + r_n(1 - \|x_n\|) \\ &\geq c - 1.\end{aligned}$$

It follows that

$$\inf y_n(B_n^*) > \frac{c-1}{\|x_n\|}$$

and so, $\{\inf y_n(B_n^*)\}$ is also bounded below. By (c′), w-lim $y_n = x$, and therefore, w-lim $x_n = x$ as well. In particular,

$$\lim_n x_0^*(x_n - x) = 0$$

and therefore, for sufficiently large n,

$$|x_0^*(x - x^{**})| < |x_0^*(x_n - x)| + |x_0^*(x_n - x^{**})| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

a contradiction. ∎

LEMMA 9. *Let $(X, \|\cdot\|_1)$ be a Banach space and let $\|\cdot\|_2$ be a seminorm on X^{**} such that*

(a) $|\cdot| = \|\cdot\|_1 + \|\cdot\|_2$ *is an equivalent norm on X;*

(b) $\|\cdot\|_2|_B$ *is w*-continuous for any bounded subset B in X^{**}.*

Then $|x^{**}| = \|x^{**}\|_1 + \|x^{**}\|_2$ for all x^{**} in X^{**}.

Proof. First we show that for every $x^{**} \in X^{**}$,

$$|x^{**}| \leq \|x^{**}\|_1 + \|x^{**}\|_2. \tag{1}$$

Let $x^{**} \in X^{**}$. By Goldstein's Theorem, there exists a net $\{x_\alpha\}$ in X such that w*-$\lim_\alpha x_\alpha = x^{**}$ and $\lim_\alpha \|x_\alpha\|_1 = \|x^{**}\|_1$ for all α. Since $\|\cdot\|_2$ is w*-continuous on every bounded subset in X^{**}, we get $\lim_\alpha \|x_\alpha\|_2 = \|x^{**}\|_2$. Hence

$$|x^{**}| \leq \lim_\alpha |x_\alpha| = \lim_\alpha \|x_\alpha\|_1 + \lim \alpha \|x_\alpha\|_2 = \|x^{**}\|_1 + \|x^{**}\|_2.$$

So (1) is proved. Put

$$B = \{x^{**} \in X^{**} : |x^{**}| \leq 1\},$$
$$C = \{x^{**} \in X^{**} : \|x^{**}\|_1 + \|x^{**}\|_2 \leq 1\},$$
$$D = \{x \in X : |x| \leq 1\}.$$

From the definition of $|\cdot|$ and (1), we get $D \subseteq C \subseteq B$. Taking into account that $\overline{D}^{w^*} = B$ and C is w*-closed, we get $B = C$. ∎

DEFINITION 10. A Banach space X is said to have the Kadec–Klee (KK) property if the weak and norm convergent sequences in S_X coincide.

PROPOSITION 11. *Let X be an infinite-dimensional Banach space such that X^* is separable. Then there exists an equivalent norm $|\cdot|$ on X such that $(X, |\cdot|)$ has the KK property and $(X, |\cdot|)^{**}$ is rotund, but $(X, |\cdot|)$ fails to be WLUR.*

Proof. Since X is separable, from a theorem of M. Kadec (see, e.g., [3, Proposition 2.1.4, Theorem 2.2.6]), there exists an equivalent norm $\|\cdot\|$ on X with the KK property.

Pick $x_0 \in X$ and $x^* \in X^*$ such that $x^*(x_0) = \|x_0\| = \|x^*\| = 1$. Let

$$\|x\|_1 = |x^*(x)| + \|x - x^*(x)x_0\|, \qquad x \in X.$$

We claim that $(X, \|\cdot\|_1)$ also has the KK property. Indeed, let $\|x_n\|_1 = \|x\|_1 = 1$ and w-lim $x_n = x$. Then $\lim_n x^*(x_n) = x^*(x)$. So

$$\text{w-}\lim_n \left(x_n - x^*(x_n)x_0\right) = x - x^*(x)x_0$$

and

$$\begin{aligned}\lim_n \|x_n - x^*(x_n)x_0\| &= \lim_n \left(\|x_n\|_1 - |x^*(x_n)|\right) = \|x\|_1 - |x^*(x)| \\ &= \|x - x^*(x)x_0\|.\end{aligned}$$

Since $(X, \|\cdot\|)$ has the KK property, we get

$$\lim_n \left\|\left(x_n - x^*(x_n)x_0\right) - \left(x - x^*(x)x_0\right)\right\| = 0.$$

Hence $\lim_n \|x_n - x\| = 0$. Thus $(X, \|\cdot\|_1)$ has the KK property.

Pick a bounded sequence $\{x_k^*\}$ which spans X^*. For $x^{**} \in X^{**}$, define

$$\|x^{**}\|_2 = \left(\sum_{k=1}^{\infty} 2^{-k}\left(x^{**}(x_k^*)\right)^2\right)^{1/2}$$

and for $x \in X$, set $|x| = \|x\|_1 + \|x\|_2$. Since $(X, \|\cdot\|_1)$ has the KK property, we get that $(X, |\cdot|)$ also has the KK property. Since $\|\cdot\|_2$ is w*-continuous on bounded subsets in X^{**}, by Lemma 9, for all $x^{**} \in X^{**}$,

$$|x^{**}| = \|x^{**}\|_1 + \|x^{**}\|_2. \tag{2}$$

Now, we show that $(X, |\cdot|)^{**}$ is rotund. Indeed, let x^{**}, $y^{**} \in X^{**}$ and $|x^{**}| = |y^{**}| = |(x^{**} + y^{**})/2| = 1$. This and (2) imply that

$$\|x^{**}\|_2 + \|y^{**}\|_2 = \|x^{**} + y^{**}\|_2.$$

Hence there exists $c \geq 0$ such that

$$x^{**}(x_k^*) = cy^{**}(x_k^*), \qquad k = 1, 2, \dots .$$

Since $\{x_k^*\}$ spans X^*, we get $x^{**} = cy^{**}$ which implies that $c = 1$. So $x^{**} = y^{**}$ and hence $(X, |\cdot|)^{**}$ is rotund.

It remains to show that $(X, |\cdot|)$ is not WLUR.

Since X is infinite-dimensional for each $n \in \mathbb{N}$, there exists

$$x_n \in (\ker x^*) \cap \left(\bigcap_{k=1}^{n} \ker x_k^* \right), \qquad \text{and} \qquad |x_n| = |x_0|.$$

From the choice of x_n, we get w-$\lim_n x_n = 0$. Hence

$$\lim_n \|x_n\|_2 = 0 \qquad \text{and} \qquad \lim_n \|x_0 + x_n\|_2 = \|x_0\|_2. \tag{3}$$

Since $\|x_n\| = \|x_n\|_1 = |x_n| - \|x_n\|_2 = |x_0| - \|x_n\|_2$, it follows from (3) that

$$\lim_n \|x_n\| = |x_0|. \tag{4}$$

Since

$$\begin{aligned} |x_0 + x_n| &= \|x_0 + x_n\|_1 + \|x_0 + x_n\|_2 \\ &= |x^*(x_0)| + \|x_0 + x_n - x^*(x_0 + x_n) x_0\| + \|x_0 + x_n\|_2 \\ &= |x^*(x_0)| + \|x_n\| + \|x_0 + x_n\|_2, \end{aligned}$$

from (3) and (4), we get

$$\lim_n |x_0 + x_n| = |x^*(x_0)| + |x_0| + \|x_0\|_2 = 2|x_0|.$$

Hence $(X, |\cdot|)$ is not WLUR. ∎

COROLLARY 12. *There exists an ALUR Banach space which fails to be WLUR.*

Proof. Let $(X, |\cdot|)$ be the space from Proposition 11 which fails to be WLUR. By Corollary 8, $(X, |\cdot|)$ is WALUR. Since $(X, |\cdot|)$ has the KK property, we get that $(X, |\cdot|)$ is ALUR. ∎

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