Chaotic group actions on manifolds

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Abstract

We construct chaotic actions of certain finitely generated infinite abelian groups on even-dimensional spheres, and of finite index subgroups of $SL_n(\mathbb{Z})$ on tori. We also study chaotic group actions via compactly supported homeomorphisms on open manifolds.

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1. Introduction

Cairns et al. [2] introduced the notion of a chaotic group action as a generalization of chaotic dynamical systems (see definition below). They showed that a group G acts chaotically on a compact Hausdorff space if and only if G is residually finite. They constructed chaotic action of the infinite cyclic group on the 2-disk and observed that the special linear group $SL_n(\mathbb{Z})$ acts chaotically the n-dimensional torus.

In the present paper, we construct many chaotic group actions on (even-dimensional) spheres and tori using the examples of Cairns et al. as building blocks. More precisely we show, among other things, that (i) many finitely generated infinite abelian groups act chaotically on even-dimensional spheres; (ii) any finite index subgroup of $SL_n(\mathbb{Z})$, $n \geqslant 2$, acts chaotically on the nk-dimensional torus for any $k \geqslant 1$. For precise statements see Section 2.

We also consider chaotic group actions on (connected) *open* manifolds where each element of the group is a compactly supported self homeomorphism. For a (self)-homeomorphism f of space X, $\mathrm{supp}(f)$, the support of f, is defined to be the set $\{x \in X \mid f(x) \neq x\}$. We say that a homeomorphism f of X is *compactly supported* if $\mathrm{supp}(f)$ is relatively compact, i.e., the closure of $\mathrm{supp}(f)$ is compact. A basic result is that any compactly supported homeomorphism of a connected open manifold is of infinite order (see [6]). The requirement that the group act chaotically (and effectively) via *compactly supported* homeomorphisms, leads to many interesting results of group theoretic nature. The results we obtain are by no means exhaustive. However, we do not know of a single chaotic group action via compactly supported homeomorphism on any open manifold. We conjecture that no such action can possibly exist on the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$. The strongest evidence for this conjecture is that it is true when one restricts attention to such classes of groups as solvable groups, groups with non-trivial center, groups which decompose as a direct product with one of the factors being finitely generated (see Section 3). However a complete resolution of the conjecture has eluded us. (It is not hard to show that no group can act chaotically on any 1-dimensional manifold, the case of the circle having been covered in [2].) Most of our results regarding compactly supported chaotic actions on open manifolds are valid for any noncompact, locally compact space and are treated in that generality.

For the convenience of the reader, we reproduce below the definition of a chaotic action.

Definition [2]. Let G act effectively and continuously on a Hausdorff topological space X. The group G is said to act chaotically on X if

(i) (topological transitivity) given non-empty open sets $U, V \subseteq X$, there exists $g \in G$ such that $gU \cap V \neq \emptyset$, and
(ii) the set of periodic points of X, i.e., those points with finite G-orbits, is dense in X.

Convention. All group actions are assumed to be effective.

2. Chaos on compact manifolds

In this section we shall construct chaotic group actions on specific manifolds such as spheres and tori. The building blocks of our constructions are the chaotic $\mathbb{Z}$-action on the 2-disk constructed in [2], and the chaotic action of the group $SL_n(\mathbb{Z})$ on the n-torus. We now state the main results of this section.

Theorem 2.1. *Let $G = \mathbb{Z}^l \oplus F$ where $l \geqslant 1$, and $F = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, $k \geqslant 0$. Then G acts chaotically on*

(i) *the $2d$-dimensional sphere if $k = l = d$ or if $0 \leqslant k < l \leqslant d$,*
(ii) *the $2d$-dimensional torus if $t = l = d$ or if $1 \leqslant t < l \leqslant d$ where $k = 2t$ or $2t - 1$.*

Let $s_0, \ldots, s_{m-1}$ denote a set of generators of $\mathbb{Z}^m$. The group $\mathbb{Z}_m$ acts on $\mathbb{Z}^m$ by cyclically permuting the generators s_i. We denote the semidirect product of $\mathbb{Z}^m$ with $\mathbb{Z}_m$

by H_m. Similarly let $e_{i,j}$, $0 \leqslant i < m$, $0 \leqslant j < n$, denote a basis for a free abelian group of rank mn. The group $\mathbb{Z}_m \oplus \mathbb{Z}_n = \langle r, s \mid r^m, s^n, rsr^{-1}s^{-1} \rangle$ acts on $\mathbb{Z}^{mn}$ where the action is defined by $r.e_{i,j} \mapsto e_{i+1,j}$, and $s.e_{i,j} = e_{i,j+1}$, where $e_{m,j} = e_{0,j}$, $e_{i,n} = e_{i,0}$. Let $H_{m,n}$ denote the semidirect product $\mathbb{Z}^{mn} \rtimes (\mathbb{Z}_m \oplus \mathbb{Z}_n)$.

Theorem 2.2.

(i) *The group $H_m = \mathbb{Z}^m \rtimes \mathbb{Z}_m$ acts chaotically on the 2-sphere for any $m \geqslant 2$.*

(ii) *The group $H_{m,n} = \mathbb{Z}^{mn} \rtimes (\mathbb{Z}_m \oplus \mathbb{Z}_n)$ acts chaotically on the 2-torus for any $m, n \geqslant 2$.*

Theorem 2.3. *Let $n \geqslant 2$, $k \geqslant 1$ be integers. Every finite index subgroup of $SL_n(\mathbb{Z})$ acts chaotically on the nk-dimensional torus.*

Before we prove the main theorems of this section, we make a preliminary observation. Recall that the action of G on a space X is said to be topologically mixing if, given any two non-empty open sets U and V, there exists a $g \in G$ and an integer $n > 0$, such that

$$g^k(U) \cap V \neq \emptyset \quad \text{for all } k \geqslant n.$$

Lemma 2.4.

(i) *Let G_1 and G_2 act chaotically on X_1 and X_2. Then product action of $G = G_1 \times G_2$ on $X_1 \times X_2$ is chaotic. If $x_i \in X_i$ is fixed by G_i, $i = 1, 2$, then G acts chaotically on the smash product $X_1 \wedge X_2 = X_1 \times X_2/(X_1 \times x_2 \cup x_1 \times X_2)$.*

(ii) *Let X be acted on by $\mathbb{Z}$ such that the action is topologically mixing, then the diagonal action of $\mathbb{Z}$ on X^d is topologically mixing for any $d \geqslant 1$.*

Proof. (i) Note that $(y_1, y_2) \in X_1 \times X_2$ is a periodic point for the G-action if and only if each $y_i \in X_i$ is a periodic point for the action of G_i on X_i. This implies that the G-periodic points on $X_1 \times X_2$ form a dense subset.

Let U, V be arbitrary open sets in $X_1 \times X_2$. Let U_i, V_i be open subsets of X_i, $i = 1, 2$, such that $U_1 \times U_2 \subset U$, $V_1 \times V_2 \subset V$. Let $g_i \in G_i$ be such that $g_i(U_i) \cap V_i \neq \emptyset$. Then $g(U) \cap V \neq \emptyset$ for $g = (g_1, g_2) \in G$. Thus the G-action is topologically transitive.

The second assertion of Lemma 2.4(i) follows readily from the first.

(ii) Let $U = \prod_{1 \leqslant i \leqslant d} U_i$, $V = \prod_{1 \leqslant i \leqslant d}$ be basic open sets of X^d. Let f be a generator of the $\mathbb{Z}$-action. Choose n so large that $f^k(U_i) \cap V_i \neq \emptyset$ for all $k \geqslant n$, $1 \leqslant i \leqslant d$. Then it is clear that $f^k(U) \cap V \neq \emptyset$ for all $k \geqslant n$. This proves (ii). □

Remark. There are several other well-known results concerning topological transitivity and related concepts such as topological mixing, weak mixing etc. A theorem of Furstenberg [5] says that if $f : X \to X$ is the generator of a $\mathbb{Z}$-action which is weakly mixing, then the diagonal action of $\mathbb{Z}$ on X^d, $d \geqslant 1$, is topologically transitive. Suppose $\mathbb{Z}$ acts on X chaotically such that the restriction of the action to each subgroup $n\mathbb{Z} \subset \mathbb{Z}$, $n \geqslant 1$ is transitive, then Banks [1] has shown that the action of $\mathbb{Z}$ is weakly mixing.

Proof of Theorem 2.1. We first show that even-dimensional spheres and even-dimensional tori admit chaotic $\mathbb{Z}$-actions. Indeed, the chaotic $\mathbb{Z}$-action on the 2-disk D constructed by Cairns et al. [2] is topologically mixing. See Theorem A of [4]. Lemma 2.4(ii) implies that the diagonal action of $\mathbb{Z}$ on the Cartesian product D^d is chaotic. The desired action of $\mathbb{Z}$ on the sphere S^{2d} is obtained by semiconjugation since S^{2d} is obtained by pinching the boundary of D^d to a point. Again the chaotic $\mathbb{Z}$-action on the 2-torus considered by Cairns et al. [2] is topologically mixing. By Lemma 2.4(ii) again, we obtain a chaotic $\mathbb{Z}$-action on any even-dimensional torus.

Case 1: $F = 0$. Now, let $G = \mathbb{Z}^d$ act on $X = (S^2)^d$ where the action of the ith factor $\mathbb{Z}$ of G on the ith coordinate is the chaotic $\mathbb{Z}$-action on S^2 constructed in [2]. Note that the "north pole", $*$, of the sphere S^2 is fixed under this action. Hence the subspace $Y = \{z = (z_1, \ldots, z_d) \mid z_i = * \text{ for some } i\}$ is stable under the action of G. Hence we obtain an action of G on the quotient space $X/Y \cong S^{2d}$. The action of G remains chaotic on S^{2d} by Lemma 2.4(i). Note that the "north pole" $Y \in S^{2d}$ is a G-fixed point.

Since the restriction of the above action to the diagonal copy $\Delta \cong \mathbb{Z}$ is the same as the chaotic $\mathbb{Z}$-action on S^{2d} constructed at the beginning of the proof, we see that if $\Delta \subset H \subset \mathbb{Z}^d = G$, then the restriction of G-action to H is also chaotic on S^{2d}. In particular, any free abelian group of rank l, $1 \leqslant l \leqslant d$, acts chaotically on S^{2d}. This prove Theorem 2.1(i) when $F = 0$.

Before considering the case $k \geqslant 1$, we let $m \geqslant 2$, and construct a chaotic $\mathbb{Z} \oplus \mathbb{Z}_m$-action on the unit disk D in $\mathbb{R}^2$. We then use this to construct a $\mathbb{Z} \oplus \mathbb{Z}_m$-action on the S^2, which is thought of the space obtained from D by collapsing its boundary, ∂D, to a point.

The $\mathbb{Z}_m$-action on D is generated by a $2\pi/m$ rotation, r, about the center of the disk. Let D_0 denote a sector of the disk having angle $2\pi/m$ at the center of the disk. Denote by D_j the image of D_0 under the map r^j, $1 \leqslant j < m$. Our generator g of the infinite cyclic group is the homeomorphism of D which is f on D_0 and, on D_j, it is the map $r^j f r^{-j}$, where f is the chaotic homeomorphism of the 2-disk constructed by Cairns et al. [2]. Note that since the map f is identity on the boundary of D_0, g is a well-defined self-homeomorphism of D. Note that rgr^{-1} is the same as the map g. Thus we have obtained an action of $\mathbb{Z} \oplus \mathbb{Z}_m$ on D. Now the boundary ∂D of D is stable under the action of $\mathbb{Z} \oplus \mathbb{Z}_m$ and hence we obtain an action of $\mathbb{Z} \oplus \mathbb{Z}_m$ on $D/\partial D = S^2$. The resulting action is chaotic on S^2. Note that the "north pole", namely the point determined by ∂D is fixed under this action.

Case 2: Let $F \neq 0$, so that $k \geqslant 1$. Let G be as in Theorem 2.1(i), with $k \geqslant 1$. Then, write G as $\prod_{0 \leqslant i \leqslant k} G_i$, where $G_0 = \mathbb{Z}^{l-k}$, $G_i = \mathbb{Z} \oplus \mathbb{Z}_{n_i}$, $1 \leqslant i \leqslant k$. The group G_i acts chaotically on the sphere $S_i := S^2$ for $i \geqslant 1$. The group G_0 acts chaotically on $S_0 := S^{2(d-k)}$ if $l > k$. When $l = k$, one has $k = l = d$ by hypothesis. In this case, it is still true that $G_0 = \{1\}$ acts chaotically on the one point space, $\{*\}$, again denoted S_0. Thus we see that G acts chaotically on the product $X := \prod_{0 \leqslant i \leqslant k} S_i$. Note that since the north pole, $*$, of each of the spaces S_i are fixed by G_i for each $i \geqslant 0$, we see that the subspace $Y = \{z = (z_0, \ldots, z_k) \subset X \mid z_i = * \text{ for some } i \geqslant 0\}$ is stable under the action of G. Hence G acts chaotically on the quotient space $X/Y \cong S^{2d}$. This completes the proof of (i).

(ii) We first construct a chaotic $\mathbb{Z} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_n$-action on the 2-torus T. This is an obvious generalization of the $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$-action on T given in [2]. We regard T as space obtained from the unit square $D = I \times I$ with opposite sides identified in the usual manner. Now subdivide the square D into mn squares $D_{i,j}$, $0 \leqslant i < m$, $0 \leqslant j < n$, where $D_{i,j} = [i/m, (i+1)/m] \times [j/n, (j+1)/n]$. Note that $(x, y) \mapsto (x + 1/m, y)$ and $(x, y) \mapsto (x, y + 1/n)$ define commuting homeomorphisms r, s of T of orders m and n, respectively. The generator g of the infinite cyclic group acts on T as follows: Let $\widetilde{g}$ restricted to $D_{0,0}$ be the chaotic homeomorphism f on the 2-disk constructed by Cairns et al. [2]. $\widetilde{g}$ restricted to $D_{i,j}$ defined to be the composition $r^i s^j \circ f \circ s^{-j} r^{-i}$, $0 \leqslant i < m$, $0 \leqslant j < n$. It is straightforward to check that $\widetilde{g}$ defines a homeomorphism g on T and that g commutes with r and s. Thus we obtain an action of $\mathbb{Z} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_n$. That this action is chaotic follows easily from the fact that f acts chaotically on $D_{0,0}$.

Observe that a similar construction leads to a chaotic $\mathbb{Z} \oplus \mathbb{Z}_m$-action on the 2-torus T.

Using these actions on 2-tori, and the chaotic action of $\mathbb{Z}$ on any even-dimensional torus as building blocks constructed at the beginning of the proof, one constructs the required G-action on the $2d$-dimensional torus just as in the proof of (i). □

Proof of Theorem 2.2. (i) We use the notation introduced in the proof of Theorem 2.1 where we constructed a chaotic action of $\mathbb{Z} \oplus Z_m$ on the 2-sphere. First we "expand" the $\mathbb{Z} \oplus \mathbb{Z}_m$-action on disk D as follows. Consider the homeomorphism s of the disk defined as $s(x) = f(x)$, $x \in D_0$, and $s(x) = x$, $x \notin D_0$ using the notation of proof of Theorem 2.1. Let $s_j = r^j s r^{-j}$, $0 \leqslant j \leqslant m$, $s_m = s_0 = s$. Note that $\mathrm{supp}(s_j) \subset D_j$. Since $D_i \cap D_j$ has empty interior for $0 \leqslant i < j < m$, it follows $s_i s_j = s_j s_i$ for any $i, j \geqslant 1$. Thus the s_i generate a free abelian group of rank m and that the group $\mathbb{Z}_m = \langle r | r^m \rangle$ acts on this group by cyclically permuting the generators s_i. Hence we obtain an action of H_m on D. Clearly the elements $g = s_1 \cdots s_m$ and r commute. The subgroup of H_n generated by g, r is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_m$ and its action on D is the same as that constructed in Theorem 2.1. It follows that the H_m-action is chaotic since the periodic points for the H_m-action is the same as for the action of $\mathbb{Z} \oplus \mathbb{Z}_m \cong \langle g, r \rangle \subset H_m$. Since the boundary of D is stable under the action of H_m, we obtain an action of H_m on $D/\partial D \equiv S^2$. Clearly this action is chaotic on S^2.

(ii) Using the notation introduced in the proof of Theorem 2.1(ii), let $e_{i,j}$ be the homeomorphism of the 2-torus T defined by $e_{i,j}(z) = r^i s^j f s^{-j} r^{-i}(z)$, for $z \in D_{i,j}$, $e_{i,j}(z) = z$, for $z \notin D_{i,j}$, $0 \leqslant i < m$, $0 \leqslant j < n$. Then the $e_{i,j}$ generate a free abelian group of rank mn. Conjugation by the homeomorphisms r and s permute the $e_{i,j}$ in such a manner that the group of homeomorphisms of T generated by the $e_{i,j}$, $0 \leqslant i < m$, $0 \leqslant j < n$, r, s, is the semidirect product $\mathbb{Z}^{mn} \rtimes (\mathbb{Z}_m \oplus \mathbb{Z}_n) = H_{m,n}$. The proof that $H_{m,n}$ acts chaotically is exactly similar to part (i) above. □

Consider the usual $SL_n(\mathbb{Z})$-action on the torus $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$. As observed by Cairns et al. [2], this action is chaotic. Our proof of Theorem 2.3 involves a well-known property of a set of reals which are $\mathbb{Q}$-linearly independent.

Notation. The group $SL_n(\mathbb{Z})$ acts on the left of $\mathbb{R}^n$ and hence on the left of the n-torus $\mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}$. The element of $\mathbb{T}$ determined by an $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ will be written as

$[x] = [x_1, \ldots, x_n]$. For $g \in SL_n(\mathbb{Z})$, $x \in \mathbb{R}^n$, the element $g(x)$ is the product of g with the column vector x^t.

Proof of Theorem 2.3. Let G be any finite index subgroup of $SL_n(\mathbb{Z})$. Consider the G-action on the kn-dimensional torus $X = \mathbb{T}^k$, $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$, obtained from restricting the diagonal action of $SL_n(\mathbb{Z})$ on X. We claim that this action of G on X is chaotic. Indeed we need only verify topological transitivity. Replacing G by a (finite index) subgroup of G if necessary, we may as well assume that G is normal in $SL_n(\mathbb{Z})$. For $1 \leqslant i, j \leqslant n$, $i \neq j$ let $E_{i,j}$ denote the $n \times n$ matrix whose entry in kth row, lth column is $\delta_{i,k}\delta_{j,l}$. Since the element $1 + E_{1,2}$ generates an infinite cyclic group, and since G is of finite index in $SL_n(\mathbb{Z})$, there exists a $d \geqslant 2$ such that $1 + dE_{1,2} \in G$. Since G is normal, conjugating $1 + dE_{1,2}$ by suitable (signed) permutation matrices, it is seen that $1 + dE_{i,j} \in G$ for all $i \neq j$, $1 \leqslant i, j \leqslant n$.

Let $x = [x_1, \ldots, x_k] \in X$, $x_i = [x_{i,1}, \ldots, x_{i,n}] \in \mathbb{T}$, where the elements $x_{i,j} \in \mathbb{R}$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$ and 1 form a $\mathbb{Q}$-linearly independent subset of the reals.

Claim. *The G-orbit of x is dense in X.*

To prove the claim, let $U = \prod_{1 \leqslant i \leqslant k} U_i$, where $U_i = \prod_{1 \leqslant j \leqslant n} U_{i,j} \subset T$, $U_{i,j}$ being any basic open set in $\mathbb{R}/\mathbb{Z}$. We need to find an element $g \in G$ such that $g(x) \in U$. Since $1, x_{1,2}, \ldots, x_{k,2}$ are linearly independent over $\mathbb{Q}$, the set $\{[rx_{1,2}, \ldots, rx_{k,2}] \mid r \in d\mathbb{Z}\}$ is a dense subset of $\mathbb{R}^k/\mathbb{Z}^k$. It follows that translation of this set by $[x_{1,1}, \ldots, x_{k,1}]$ is also dense in $\mathbb{R}^k/\mathbb{Z}^k$. Therefore we can find an $r_1 \in \mathbb{Z}$ such that $y_{i,1} := x_{i,1} + dr_1 x_{i,2} \in U_{i,1}$ for $1 \leqslant i \leqslant k$. Having obtained $r_{j-1} \in \mathbb{Z}$ such that $y_{i,j-1} = x_{i,j-1} + dr_j x_{i,j} \in U_{i,j-1}$, $1 \leqslant i \leqslant k$, $j < n$, inductively, we use $\mathbb{Q}$-linear independence of $1, x_{1,j}, \ldots, x_{k,j}$, to obtain an $r_j \in \mathbb{Z}$ such that $y_{i,j} := x_{i,j} + dr_j x_{i,j+1} \in U_{i,j}$, $1 \leqslant i \leqslant k$. Finally, since $1, y_{1,1}, \ldots, y_{k,1}$ are $\mathbb{Q}$-linearly independent, we can find an $r_n \in \mathbb{Z}$ such that $y_{i,n} := dr_n y_{i,1} + x_{i,n} \in U_{i,n}$, $1 \leqslant i \leqslant k$. Now the element $y = (y_1, \ldots, y_k) \in U \subset \mathbb{T}^k = X$, where $y_i = [y_{i,1}, \ldots, y_{i,n}] \in \mathbb{T}$.

Write $e_{i,j} = 1 + dE_{i,j} \in G$. Then, for $1 \leqslant i \leqslant k$,

$$\begin{aligned} & e_{n,1}^{r_n} e_{n-1,n}^{r_{n-1}} \cdots e_{1,2}^{r_1}(x_i) \\ & \quad = e_{n,1}^{r_n} e_{n-1,n}^{r_{n-1}} \cdots e_{2,3}^{r_2}\big([y_{i,1}, x_{i,2}, \ldots, x_{i,n}]\big), \\ & \quad = \\ & \quad \vdots \\ & \quad = e_{n,1}^{r_n}\big([y_{i,1}, \ldots, y_{i,n-1}, x_n]\big) \\ & \quad = [y_{i,1}, \ldots, y_{i,n}]. \end{aligned}$$

Hence writing $g = e_{n,1}^{r_n} e_{n-1,n}^{r_{n-1}} \cdots e_{1,2}^{r_1}$, we obtain $g(x) = y \in U$, completing the proof. □

Remarks.

(i) It follows from Theorem 2.3 that any free group of finite rank acts chaotically on the $2k$-dimensional torus as such a group can be imbedded as a finite index subgroup of $SL_2(\mathbb{Z})$. A result of Cairns and Kolganova says that any countably generated (nontrivial) free group can be made to act chaotically on any compact connected triangulable manifold of dimension at least two. (See Remark (i) following Theorem 2.5 below.)

(ii) Let $k \geqslant 2$. Denote by $X^{(k)}$ the kth symmetric power of X; thus it is the quotient of X^k by the group permutation group $Perm(k)$ which on X^k acts by permuting the coordinates. Note that if X is a 2-manifold then $X^{(k)}$ is a $2k$-dimensional manifold. If G acts on X, then the diagonal action of G on X^k induces an action on $X^{(k)}$.

Theorem 2.5. *There exists a chaotic $\mathbb{Z}$-action on the complex projective space $\mathbb{C}P^k$ for any $k \geqslant 1$.*

Proof. One has the chaotic $\mathbb{Z}$-action on $\mathbb{C}P^1 \cong S^2 =: X$ constructed by Cairns et al. As observed in the proof of Theorem 2.1, this action is topologically mixing and hence, by applying Lemma 2.4(ii), the diagonal action of $\mathbb{Z}$ on X^k is chaotic. The induced $\mathbb{Z}$-action on the symmetric product $\mathbb{C}P^k \cong (S^2)^{(k)}$ is easily seen to be chaotic. □

Remarks.

(i) Cairns and Kolganova [3] have constructed $\mathbb{Z}$-actions, indeed, any countably generated free group actions, on any compact connected triangulable manifold M of dimension $d \geqslant 2$. This is achieved by first obtaining such an action on the d-dimensional disk on whose boundary the group acts as identity. The required action on M is obtained via semiconjugacy as M can be realized as the space obtained from the d-dimensional disk by suitable boundary identifications. (See proof of our Theorem 2.1 for chaotic $\mathbb{Z}$-action on even-dimensional disks.) However, it would be interesting to have a more direct, "explicit", description of chaotic group actions on specific manifolds.

(ii) We do note know if an arbitrary finitely generated infinite abelian group can be made to act chaotically (and effectively) on a sphere of suitable dimension.

3. Chaos on open manifolds

Let X be a *non*-compact, locally compact, locally connected Hausdorff space. For a homeomorphism f of X, the support $\mathrm{supp}(f)$ of f is defined to be the set $\{x \in X \mid f(x) \neq x\}$. Denote by $G_0(X)$, the subgroup of those homeomorphisms f of X which are compactly supported, i.e., the closure of $\mathrm{supp}(f)$ is a compact subset of X. In this section, we obtain some general results concerning chaotic group actions on X. The most interesting example of such a space is an open manifold. Their importance justifies our choice of the title for this section.

We begin with some elementary observations.

Lemma 3.1. *Let G be any group that acts effectively on a topological space X. Then*

(i) $\mathrm{supp}(fgf^{-1}) = f(\mathrm{supp}(g))$, *for any* $f, g \in G$.

(ii) $\mathrm{supp}(fg) \subset \mathrm{supp}(f) \cup \mathrm{supp}(g)$ *for any* $f, g \in G$.

(iii) *If* $S \subset G$, *then* $\mathrm{supp}(S) = \mathrm{supp}(H)$, *where* H *is the subgroup generated by* S. *Here* $\mathrm{supp}(S)$ *denotes the set* $\bigcup_{g\in S} \mathrm{supp}(g)$.
(iv) *If* H *is normal in* G, *then* $\mathrm{supp}(H)$ *is a* G*-invariant open subset of* H.

Proof. Statements (i) and (ii) are trivial to prove. (iii) follows from (ii) and the obvious fact that $\mathrm{supp}(f) = \mathrm{supp}(f^{-1})$. (iv) follows from (i). □

Theorem 3.2. *Let* X *be a locally connected, locally compact, non-compact, Hausdorff topological space and let* G *a subgroup of* $G_0(X)$. *Suppose that* G *acts topologically transitively on* X. *Then*
(i) *If the set of periodic points is dense in* X, *then* X *is connected.*
(ii) $\mathrm{supp}(S) = \bigcup_{g\in S} \mathrm{supp}(g)$ *is dense in* X *for any set* $S \subset G$ *that generates* G. *In particular the group* G *is not finitely generated.*
(iii) *Nontrivial normal subgroups of* G *are not finitely generated.*
(iv) *The center of* G *is trivial. In particular,* G *is not nilpotent.*

Proof. (i) Since X is locally connected, its connected components are open in X. Suppose that X is not connected. Using topological transitivity of the G-action, it is easy to see that given any two distinct connected components U and V of X, there exists a $g \in G$ such that $g(U) = V$. Since g is compactly supported, it follows that $U \subset \mathrm{supp}(g)$ is relatively compact. Since U is also closed, it follows that U must be compact. Thus every component of X is compact. Since X itself is non compact, X must have infinitely many connected components. On the other hand, since G acts transitively on the set of connected components of X, it follows that no element of X can have finite orbit. This contradicts our hypothesis that periodic points for the G-action is dense in X. Hence X must be connected.

(ii) By Lemma 3.1 (iii) and (iv), we see that $\mathrm{supp}(S)$ is a G-invariant open subset of X. Since the action of G is topologically transitive, $\mathrm{supp}(S)$ must be dense. Now, if S were finite, then X would be a finite union of compact subsets of X, namely the closure of the supports of elements of S, contradicting the assumption that X is not compact. Hence S has to be finite.

(iii) Proof is similar to (ii), using Lemma 3.1(iv).

(iv) To show that the center of G is trivial, let, if possible, $g \in G$, $g \neq 1$ be in the center of G. Since $g \neq 1$, there must an open set $U \subset \mathrm{supp}(g)$ such that $U \cap g(U) = \emptyset$. Let V be an open set disjoint from $\mathrm{supp}(g)$. By topological transitivity, we can find $h \in G$ such that $hU \cap V \neq \emptyset$. Let $y \in hU \cap V$ so that $u := h^{-1}(y) \in U$. Now, $g = hgh^{-1}$ implies $y = g(y) = hgh^{-1}(y) = hg(u)$. Hence $h^{-1}(y) = g(u) \in g(U)$. This is a contradiction since $U \cap g(U) = \emptyset$. □

As an immediate consequence abelian subgroups of $G_0(X)$ cannot act chaotically on X. We shall later see that if a subgroup G of $G_0(X)$ acts chaotically on X, then G cannot even have a finite index solvable subgroup. In other words, virtually solvable subgroups of $G_0(X)$ cannot act chaotically on X.

Let G be group acting chaotically and effectively on a Hausdorff space Y. Let H be a subgroup of G. Then H, being residually finite, acts chaotically and faithfully on some

Hausdorff space. It is however not true that the action of H obtained by restriction of the G-action on X is chaotic. For example, there exists a chaotic action of the group $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on the 2-torus such that no cyclic subgroup acts chaotically on the torus (see [2], cf. Theorem 2.1(ii)). However, in our case we can prove the following.

Theorem 3.3. *Let X be locally compact, non compact, Hausdorff space. Let G act chaotically on X and let H be a subgroup of G such that for some set S of distinct left coset representatives of H in G, the set $\mathrm{supp}(S) = \bigcup_{s \in S} \mathrm{supp}(s)$ is not dense in X. Then the H-action on X by restriction is chaotic.*

Proof. We need only check topological transitivity for the H-action on X. Let U be any non-empty open set in X which does not meet $A := \mathrm{supp}(S)$. Set $W_s = \bigcup_{h \in H} sh(U)$, $s \in S$. (We shall assume that $1 \in S$.) We claim that $W_1 - A = W_s - A$ for any $s \in S$. To see this note that $s(W_1) = W_s$ and $s(A) = A$ for all $s \in S$. It follows that $W_1 - A = W_s - A$ for all $s \in S$.

We claim that W_1 is dense in X. Note that this would imply topological transitivity of the H-action. To prove the claim, observe that $W := \bigcup_{s \in S} W_s$ is dense in X since G acts chaotically on X. In particular, $W - A = \bigcup_{s \in S}(W_s - A) = W_1 - A$ is dense in $X - A$. Therefore we need only show, for any open set $V \subset A$, that $W_1 \cap V$ is non-empty. Let $g \in G$ be such that $V \cap g(U) \neq \emptyset$. Writing $g^{-1} = sh$ with $s \in S$ and $h \in H$, we see that $V \cap hs(U) \neq \emptyset$. Since $U \cap A = \emptyset$ and $\mathrm{supp}(s) \subset A$, we have $s(U) = U$ so that $V \cap h(U) \neq \emptyset$. It follows that $V \cap W_1 \neq \emptyset$ as $h(U) \subset W_1$. □

Corollary 3.4. *Let $G \subset G_0(X)$ act chaotically on X and let H be a finite index subgroup of G. Then H acts chaotically on X. If H' is any subgroup of G which acts chaotically on X, then the centralizer of H' in G is trivial.*

Proof. Let H be a finite index subgroup of G and let S be a (finite) set of district left coset rerpresentatives for H in G. Then $A = \bigcup_{s \in S} \mathrm{supp}(s)$ is relatively compact as S is finite and each $g \in G$ is compactly supported. Since X is non-compact, it follows that A is not dense in X. Hence H acts chaotically.

Let $g \in G$, $g \neq 1$ commute with every element in H'. Then the subgroup $K = \langle H', g \rangle \subset G$ acts chaotically on M. For, topological transitivity of H' implies the topological transitivity of K, the densitity of periodic points for the K-action follows from the corresponding property for the G-action. As $g \in K$ is in the center of K this contradicts Theorem 3.2. □

Proposition 3.5. *Let H and H' be normal subgroups of $G \subset G_0(X)$. Suppose G, H and H' act chaotically on X. Then the group $[H, H'] \subset G$ acts chaotically on X. In particular, no finite index subgroup of G can be solvable.*

Proof. We need only show that the $K := [H, H']$-action is topologically transitive. Let U and V be any two disjoint open sets in X. Let $h \in H$ be such that $h(U) \cap V$ is non-empty.

Let $U' = h^{-1}(V) \cap U$. Let W be any open set outside the support of $h \in H$. Let $h' \in H'$ be such that $U' \cap h'(W)$ is non-empty. Let $U'' = U' \cap h'(W)$, so that $h'^{-1}(U'') \subset W$. Now $hh'h^{-1}h'^{-1}(U'') = h(U'') \subset V$ since $h(w) = w$ for all $w \in W$. Hence $[h, h'](U) \cap V$ is non-empty.

If K is a finite index subgroup of G, then K-action (via restriction) on X is chaotic. It follows that $K' = [K, K] \subset K$ again acts chaotically on X. Repeated application of the same argument shows that K cannot be solvable. □

Proposition 3.6. *Let $G \subset G_0(X)$ act chaotically on X. Suppose that G decomposes as a nontrivial product $H \times H'$. Then neither H nor H' acts topologically transitively on X.*

Proof. Let, if possible, the H-action be topologically transitive. Then so is the action of $K = \langle H, h' \rangle$ for any $1 \neq h' \in H'$. Since $K \subset G$, the periodic points for the K-action is dense in M. It follows that K acts chaotically on M. Clearly the h' is in the center of K which contradicts Theorem 3.2. □

Definition. We shall say that the group G satisfies property $\mathcal{C}$ if there exists a countable collection $\{N_i\}_{i \geqslant 1}$ of finite index normal subgroups N_i, such that $\bigcap_i N_i = \{1\}$.

Clearly G is residually finite if has property $\mathcal{C}$. We state without proof the following theorem, which implies that any group that acts chaotically on a connected open manifold must have property $\mathcal{C}$. Proof that (ii) implies (i) uses the observation that if G acts chaotically on a second countable Hausdorff space X, then there exists a *countable* family of finite orbits whose union is dense in X.

Theorem 3.7. *For a group G the following are equivalent.*

(i) *G satisfies $\mathcal{C}$.*

(ii) *G acts chaotically and faithfully on a second countable Hausdorff X.*

Note added in proof

Theorem 2.1 can be improved to show:

Theorem. *Any finitely generated infinite abelian group can be made to act chaotically on an even-dimensional sphere.*

Proof. We use the notation of Theorem 2.1 and its proof. Let $G = \mathbb{Z}^l \oplus F$ where $F = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, $l \geqslant 1$, $k \geqslant 0$. We assume $1 \leqslant l < k$. Let $\widetilde{G} = \mathbb{Z}^k \oplus F$. Consider the chaotic $\widetilde{G}$-action on the k-fold product $\prod_{1 \leqslant i \leqslant k} D_i$, where each D_i is a copy of the 2-disk D, obtained as the product of the chaotic $\mathbb{Z} \oplus \mathbb{Z}_{n_i}$-actions on the D_i, $1 \leqslant i \leqslant k$, constructed in the proof of Theorem 2.1. One has the induced action of the diagonal copy $\mathbb{Z} \cong \Delta \subset \mathbb{Z}^k \oplus 0 \subset \widetilde{G}$ on $\prod_{1 \leqslant i \leqslant k} D_{i,0} \subset \prod_{1 \leqslant i \leqslant k} D_i$, where $D_{i,0} \subset D_i$ is the $2\pi/n_i$-sector used in the construction of the chaotic $\mathbb{Z} \oplus \mathbb{Z}_{n_i}$-action on D_i. Using the fact that the $\mathbb{Z}$-action on the 2-disk D constructed by Cairns et al. [2] is topologically mixing and applying Lemma 2.4(ii) we see that the Δ-action on $\prod D_{i,0}$ is chaotic. Since the translates of $\prod D_{i,0}$ by elements of F cover $\prod D_i$, it follows that the action of $\Delta \oplus F$ got by restricting the $\widetilde{G}$-action on $\prod D_i$ is chaotic. Imbed G in $\widetilde{G}$ such that $\Delta \oplus F \subset G \subset \widetilde{G}$. The resulting G-action on $\prod D_i$ is then chaotic. The required chaotic G-action on the sphere $S^{2k} \cong \prod D_i / \partial(\prod D_i)$ is obtained by semiconjugacy. $\square$

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