Notes on matrix arithmetic–geometric mean inequalities

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Abstract

For positive semi-definite $n \times n$ matrices, the inequality $4|||AB||| \leqslant |||(A+B)^2|||$ is shown to hold for every unitarily invariant norm. The connection of this with some other matrix arithmetic–geometric mean inequalities and trace inequalities is discussed.

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1. Introduction

Some matrix versions of the classical arithmetic–geometric mean inequality (AGM) were proved in [3–5], and seem to have aroused considerable interest. See [2, Chapter IX; 6] for a discussion and further references.

In this note we prove one more inequality of this type, discuss its connection with the known results, and with some others that seem plausible but are yet unproved.

For positive real numbers a, b, the AGM says that

$$\sqrt{ab} \leqslant \frac{a+b}{2}. \tag{1.1}$$

Replacing a, b by their squares, we could write this in the form

$$ab \leqslant \frac{a^2+b^2}{2}. \tag{1.2}$$

We could also square (1.1) and write

$$ab \leqslant \left(\frac{a+b}{2}\right)^2. \tag{1.3}$$

We wish to replace the numbers a, b by positive (semi-definite) matrices A, B. Here two difficulties arise. Since A and B do not commute in general, the matrix AB is not positive. One way to get around this is to compare not the matrices themselves but their singular values and norms. The second difficulty (that makes the problem more interesting) is that the matrix square root and square functions have different monotonicity properties. Thus each of the inequalities (1.1)–(1.3) leads to different matrix versions.

We label the singular values of an $n \times n$ matrix T as $s_1(T) \geqslant \cdots \geqslant s_n(T)$. If T has real eigenvalues, we label them as $\lambda_1(T) \geqslant \cdots \geqslant \lambda_n(T)$. If T is positive, we have $s_j(T) = \lambda_j(T)$. We use the notation $|||T|||$ to denote any *unitarily invariant norm* of T. A statement like $s_j(S) \leqslant s_j(T)$ will be used to indicate that this inequality is true for all $1 \leqslant j \leqslant n$. This implies the *weak majorisation* $s_j(S) \prec_w s_j(T)$, by which we mean that the sequence $\{s_j(S)\}$ is weakly majorised by $\{s_j(T)\}$. This is equivalent to saying that $|||S||| \leqslant |||T|||$, by which we mean that any unitarily invariant norm of S is dominated by the corresponding norm of T. See [2] for details. We use the symbol $|T|$ for the operator absolute value $(T^*T)^{1/2}$.

In [4] we proved that, if A, B are positive, then

$$s_j(AB) \leqslant s_j\left(\frac{A^2+B^2}{2}\right), \tag{1.4}$$

and consequently,

$$|||AB||| \leqslant \tfrac{1}{2}|||A^2+B^2|||. \tag{1.5}$$

These are matrix versions of the AGM akin to (1.2). In [3] a generalisation of (1.5) was proved: for any matrix X

$$|||AXB||| \leqslant \tfrac{1}{2}|||A^2X+XB^2|||, \tag{1.6}$$

and it was noted that a corresponding generalisation of (1.4) fails to hold. Another proof of (1.6) was given in [5].

If instead of (1.2) we were to start with (1.1) or (1.3) as the scalar AGM, we are led to the following questions. If A, B are positive matrices, then which of the following inequalities are true:

$$s_j^{1/2}(AB) \leqslant \tfrac{1}{2}s_j(A+B), \tag{1.7}$$

$$|||\,|AB|^{1/2}\,||| \leqslant \tfrac{1}{2}|||A+B|||, \tag{1.8}$$

$$|||AB||| \leqslant \tfrac{1}{4}|||(A+B)^2|||. \tag{1.9}$$

What are the relationships between these inequalities? The inequality (1.7) implies (1.8), which in turn implies (1.9). Indeed, (1.9) is equivalent to saying that the inequality (1.8) is valid for all Q-norms (see [2] for the definition). Note also that we have written three inequalities instead of four because the inequality (1.7) is the same as $s_j(AB) \leqslant \frac{1}{4}s_j^2(A+B)$, that would have been obtained from (1.3). The difference between (1.8) and (1.9) arises because of the fact that the square function preserves weak majorisation between positive vectors but the square root function does not.

The square function on Hermitian matrices is matrix convex [2], i.e.,

$$\left(\frac{A+B}{2}\right)^2 \leqslant \frac{A^2+B^2}{2}.$$

Hence, the statement (1.7) is stronger than (1.4).

Our main result is the following.

Theorem 1. *The inequality* (1.9) *is true for all positive matrices* A, B.

This is proved in Section 2. We have remarked that this says that the inequality (1.8) is true for all Q-norms (and hence for all Schatten p-norms for $p \geqslant 2$). We will see that (1.8) is also true for the trace norm (which is not a Q-norm). This leads us to conjecture that this is true for all unitarily invariant norms. We will observe also that when $n = 2$, the inequality (1.7) is true. Again, this leads us to believe that it might be true in all dimensions.

2. Proofs

We give a proof of (1.9) for the case of the Hilbert–Schmidt (Frobenius) norm $\|\cdot\|_2$ first. As is often the case, this is simpler. For any matrix T, we have

$$\|T\|_2^2 = \sum_{i,j} |\langle e_i, Tf_j\rangle|^2, \tag{2.1}$$

where $\{e_j\}$ and $\{f_j\}$ are any two orthonormal bases. (It is routine to write the expression (2.1) with $e_j = f_j$. The one we have written follows from this since $\|T\|_2 = \|UT\|_2$ for every unitary U.) Choose $\{e_j\}$ and $\{f_j\}$ so that $Ae_j = \alpha_j e_j$, and $Be_j = \beta_j e_j$. Then

$$\begin{aligned}
\|A^2+B^2+2AB\|_2^2 &= \sum_{i,j} |\langle e_i, (A^2+B^2+2AB)f_j\rangle|^2 \\
&= \sum_{i,j} (\alpha_i^2+\beta_j^2+2\alpha_i\beta_j)^2 |\langle e_i, f_j\rangle|^2 \\
&\geqslant \sum_{i,j} [(\alpha_i^2+\beta_j^2-2\alpha_i\beta_j)^2 + 16\alpha_i^2\beta_j^2]\, |\langle e_i, f_j\rangle|^2 \\
&= \|A^2+B^2-2AB\|_2^2 + 16\|AB\|_2^2.
\end{aligned} \tag{2.2}$$

Now note that

$$\operatorname{Re}(A^2 + B^2 \pm 2AB) = (A \pm B)^2,$$
$$\operatorname{Im}(A^2 + B^2 \pm 2AB) = \pm\frac{1}{\mathrm{i}}(AB - BA).$$

Here we have used the notations $\operatorname{Re} T$ and $\operatorname{Im} T$ for the matrices $(T + T^*)/2$ and $(T - T^*)/2\mathrm{i}$, respectively. Since $\|T\|_2^2 = \|\operatorname{Re} T\|_2^2 + \|\operatorname{Im} T\|_2^2$ for all T, we obtain from (2.2)

$$\|(A + B)^2\|_2^2 \geqslant \|(A - B)^2\|_2^2 + 16\|AB\|_2^2. \tag{2.3}$$

This shows that

$$4\|AB\|_2 \leqslant \|(A + B)^2\|_2, \tag{2.4}$$

and there is equality here if and only if $A = B$.

Now for the proof of Theorem 1 in full generality. Using (1.6) we have

$$|||AB||| = |||A^{1/2}(A^{1/2}B^{1/2})B^{1/2}||| \leqslant \tfrac{1}{2}|||A^{3/2}B^{1/2} + A^{1/2}B^{3/2}|||. \tag{2.5}$$

So to prove (1.9) it suffices to prove

$$|||A^{3/2}B^{1/2} + A^{1/2}B^{3/2}||| \leqslant \tfrac{1}{2}|||(A + B)^2|||.$$

We will show more by proving

$$s_j(A^{3/2}B^{1/2} + A^{1/2}B^{3/2}) \leqslant \tfrac{1}{2}s_j(A + B)^2. \tag{2.6}$$

The arguments are similar to the ones we used in [4]; see also [2, IX.4.2].

Let X be the 2×2 block matrix

$$X = \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix},$$

and let

$$T = XX^* = \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix}.$$

Then T is unitarily equivalent to the matrix

$$X^*X = \begin{bmatrix} A + B & 0 \\ 0 & 0 \end{bmatrix}.$$

We have

$$T^2 = \begin{bmatrix} * & A^{3/2}B^{1/2} + A^{1/2}B^{3/2} \\ B^{1/2}A^{3/2} + B^{3/2}A^{1/2} & * \end{bmatrix},$$

and so if

$$U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

then the off-diagonal part of T^2 can be written as

$$\begin{bmatrix} 0 & A^{3/2}B^{1/2}+A^{1/2}B^{3/2} \\ B^{1/2}A^{3/2}+B^{3/2}A^{1/2} & 0 \end{bmatrix} = \frac{1}{2}(T^2 - UT^2U^*).$$

Now follow the arguments in [4] (repeated in [2, IX.4.2]) to see that $s_j(A^{3/2}B^{1/2}+A^{1/2}B^{3/2}) \leqslant \frac{1}{2}s_j(T^2)$. Since T^2 is unitarily equivalent to $(A+B)^2 \oplus 0$, this is the same as (2.6). This proves Theorem 1.

Next, we show that the inequality (1.8) is true for the trace norm. By a well-known result [2, Theorem IV.2.5] we have the weak majorisation

$$s_j^{1/2}(AB) \prec_w s_j^{1/2}(A)s_j^{1/2}(B). \tag{2.7}$$

By the AGM (1.1), the quantity on the right-hand side is bounded by $1/2(s_j(A)+s_j(B))$. Hence, in particular

$$\operatorname{tr}|AB|^{1/2} \leqslant \tfrac{1}{2}\operatorname{tr}(A+B). \tag{2.8}$$

This is just the inequality (1.8) for the trace norm. We observed this already in [4].

Note that in the case of the operator norm, inequalities (1.8) and (1.9) both reduce to

$$s_1^{1/2}(AB) \leqslant \tfrac{1}{2}s_1(A+B). \tag{2.9}$$

We will show that

$$s_n^{1/2}(AB) \leqslant \tfrac{1}{2}s_n(A+B). \tag{2.10}$$

This is obviously true if either A, or B is not invertible. So assume A and B are invertible. Then

$$s_n(AB) = \lambda_n^{1/2}(BA^2B) = \lambda_1^{-1/2}(B^{-1}A^{-2}B^{-1}).$$

Since

$$\lambda_1^{1/2}(B^{-1}A^{-2}B^{-1}) = s_1(A^{-1}B^{-1}) \geqslant \lambda_1(A^{-1}B^{-1}),$$

this gives

$$\begin{aligned} s_n(AB) &\leqslant \lambda_1^{-1}(A^{-1}B^{-1}) = \lambda_1^{-1}(B^{-1/2}A^{-1}B^{-1/2}) \\ &= \lambda_n(B^{1/2}AB^{1/2}) = s_n^2(A^{1/2}B^{1/2}). \end{aligned}$$

Using (1.4) now, we get from this

$$s_n(AB) \leqslant s_n^2\left(\frac{A+B}{2}\right).$$

This proves the inequality (2.10).

Thus, when $n=2$, the inequality (1.7) is valid for all values of j.

3. Remarks

1. In a recent paper Zhan [9] has proved that for positive A, B, and for all X,

$$|||AXB||| \leqslant \tfrac{1}{4}|||A^2X + 2AXB + XB^2|||. \tag{3.1}$$

This can be derived by two successive applications of (1.6). The special case

$$|||AB||| \leqslant \tfrac{1}{4}|||A^2 + 2AB + B^2|||, \tag{3.2}$$

is an inequality weaker than (1.9). This is so because

$$(A + B)^2 = \operatorname{Re}(A^2 + 2AB + B^2),$$

and hence,

$$|||(A + B)^2||| \leqslant |||A^2 + 2AB + B^2|||.$$

2. The inequality (1.7), if true, would mean that there exists a unitary matrix U (depending on A and B) such that

$$|AB|^{1/2} \leqslant \tfrac{1}{2}U(A + B)U^*. \tag{3.3}$$

This statement is stronger than a conjecture of Thompson [8] that says there exist unitary matrices U and V such that

$$|AB|^{1/2} \leqslant \tfrac{1}{2}(UAU^* + VBV^*). \tag{3.4}$$

We have proved that (3.3) is true when $n = 2$.

3. Using the polar decomposition, we can see that (1.9) implies the inequality

$$|||AB||| \leqslant \tfrac{1}{4}|||(|A| + |B^*|)^2\ |||, \tag{3.5}$$

for *all* matrices A, B. The presence of B^* instead of B on the right-hand side is not fortuitous. The example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

foils any attempt to replace B^* by B.

4. The inequality (2.4) can be written in a different form

$$\operatorname{tr} A^2B^2 \leqslant \operatorname{tr}\left(\frac{A + B}{2}\right)^4. \tag{3.6}$$

Since

$$2(AB + BA) = (A + B)^2 - (A - B)^2 \leqslant (A + B)^2,$$

we also have

$$\operatorname{tr} AB \leqslant \operatorname{tr}\left(\frac{A + B}{2}\right)^2. \tag{3.7}$$

The inequalities (3.6) and (3.7) invite the conjecture

$$\operatorname{tr} A^m B^m \leqslant \operatorname{tr}\left(\frac{A+B}{2}\right)^{2m} \tag{3.8}$$

for all $m = 1, 2 \ldots$

A well-known inequality due to Lieb and Thirring [2, p. 279; 7] says that

$$\operatorname{tr}(AB)^m \leqslant \operatorname{tr} A^m B^m. \tag{3.9}$$

So, an inequality weaker than (3.8) is

$$\operatorname{tr}(AB)^m \leqslant \operatorname{tr}\left(\frac{A+B}{2}\right)^{2m}. \tag{3.10}$$

This is true. In fact, we have a stronger inequality

$$\lambda_j(AB)^m \leqslant \lambda_j\left(\frac{A+B}{2}\right)^{2m}. \tag{3.11}$$

To see this note that

$$\lambda_j(AB)^m = [\lambda_j(B^{1/2}AB^{1/2})]^m = [s_j^2(A^{1/2}B^{1/2})]^m.$$

By (1.4) this is bounded above by

$$s_j^{2m}\left(\frac{A+B}{2}\right) = \lambda_j\left(\frac{A+B}{2}\right)^{2m}.$$

5. Since the square function is matrix convex, the $m = 1$ case of (3.11) gives the inequality

$$\lambda_j(AB) \leqslant \lambda_j\left(\frac{A^2+B^2}{2}\right). \tag{3.12}$$

When comparing this with (1.4) one should remember that $|\lambda_j(X)|$ is not always smaller than $s_j(X)$. An operator inequality that would imply (3.12) is the statement

$$B^{1/2}AB^{1/2} \leqslant \tfrac{1}{2}(A^2 + B^2).$$

This is refuted by the example

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

6. For completeness, we should mention that an elegant theory of the geometric mean $A\#B$ of two positive matrices A, B has been developed by Ando [1]. In case B is invertible

$$A\#B = B^{1/2}(B^{-1/2}AB^{-1/2})^{1/2}B^{1/2},$$

a definition given earlier by Pusz and Woronowicz. For this mean, Ando has proved the AGM in its strongest form; we have the operator inequality

$$A\#B \leqslant \tfrac{1}{2}(A+B). \tag{3.13}$$

7. Following Ando's approach, one could ask whether for positive matrices A, B the block matrix

$$\begin{bmatrix} \frac{1}{4}(A+B)^2 & AB \\ BA & \frac{1}{4}(A+B)^2 \end{bmatrix} \tag{3.14}$$

is positive. If yes, this could be another formulation of the AGM. The inequality (1.9) would follow from this, because every unitarily invariant norm is in the Lieb class $\mathscr{L}$ [2, p. 269]. The example

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix},$$

however, rules out this formulation. In this case, the matrix (3.14) is

$$\begin{bmatrix} 9 & 0 & 8 & -2 \\ 0 & 1 & 2 & 0 \\ 8 & 2 & 9 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}.$$

The 3×3 top left subdeterminant of this matrix is negative.

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Note added in proof

For $m = 3$, the conjecture (3.8) is refuted by the example

$$A = \begin{bmatrix} 4 & -5 \\ -5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix}.$$

We thank X. Zhan for this simple example.

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