Oscillating Multipliers for some Eigenfunction Expansions

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ABSTRACT. Let P be a non-negative, self-adjoint differential operator of degree d on $\mathbb{R}^n$. Assume that the associated Bochner–Riesz kernel s_R^δ satisfies the estimate $|s_R^\delta(x,y)| \le C\, R^{n/d}(1+R^{1/d}|x-y|)^{-\alpha\delta+\beta}$ for some fixed constants $\alpha > 0$ and β. We study L^p boundedness of operators of the form $m(P)$, m coming from the symbol class S_ρ^{-a}. We prove that $m(P)$ is bounded on L^p if $a > \frac{n(1-\rho)}{\alpha}|\frac{1}{p}-\frac{1}{2}|$. We also study multipliers associated to the Hermite operator H on $\mathbb{R}^n$ and the special Hermite operator L on $\mathbb{C}^n$ given by the symbols $m_\alpha(\lambda) = \lambda^{-\alpha/2} J_\alpha(t\sqrt{\lambda})$. As a special case we obtain L^p boundedness of solutions to the Wave equation associated to H and L.

1. Introduction

Suppose P is a differential operator of degree d on a Riemannian manifold M, which is self adjoint and formally non-negative. Let

$$Pf = \int_0^\infty \lambda\, dE_\lambda f$$

be the spectral resolution of P. Given a bounded function $m(\lambda)$ we can define the operator $m(P)$ by

$$m(P) = \int_0^\infty m(\lambda)\, dE_\lambda .$$

Such operators are always bounded on $L^2(M)$. However some smoothness assumptions are needed on $m(\lambda)$ to ensure that $m(P) : L^p(M) \to L^p(M)$ is bounded for $p \neq 2$. There is a universal multiplier theorem due to Stein [19], which guarantees that $m(P)$ is bounded on $L^p(M)$, $1 < p < \infty$. His condition on $m(\lambda)$ requires that $m(\lambda)$ is in the symbol class $S_1^0(R)$.

Recall that the symbol classes $S_\rho^\alpha(R)$, $\alpha \in R$, $0 \le \rho \le 1$ are defined to be the class of all C^∞ functions on R which satisfy the estimates

$$\left|m^{(j)}(\lambda)\right| \le c_j(1+|\lambda|)^{\alpha-\rho j}$$

for $j = 0, 1, \ldots$. Sharp results under weaker regularity assumptions are known in many particular cases. When $P = -\Delta$ on R^n the corresponding result is the classical theorem of Marcinkiewicz–Mihlin–Hörmander and one requires the above estimate to hold only for $j = 0, 1, \ldots N$ where N is the smallest integer bigger than $\frac{n}{2}$. The case of compact Riemannian manifolds has been studied by Seeger and Sogge [15]. In most of the particular cases, optimal conditions on $m(\lambda)$ that guarantee boundedness of $m(P)$ on a given L^p have been obtained.

Operators of the form $m(P)$ with m coming from $S_\rho^{-\alpha}(R), 0 \le \rho < 1, \alpha > 0$ are also important as they occur naturally in applications. For example, the solution to the Cauchy problem

$$\partial_t^2 u(x,t) + P\, u(x,t) = 0,\ u(x,0) = 0,\ \partial_t u(x,0) = f(x)$$

is given by

$$u(x,t) = \frac{\sin t\sqrt{P}}{\sqrt{P}} f(x)$$

and the function $m(\lambda) = \frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}$ comes from the symbol class $S_{\frac{1}{2}}^{-\frac{1}{2}}(R)$. The boundedness properties of this operator have been investigated in various contexts.

When $P = -\Delta$ on R^n, Miyachi [9] and Peral [12] have shown that $\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$ is bounded on $L^p(R^n)$ for $|\frac{1}{p} - \frac{1}{2}| \le \frac{1}{n-1}$. The case of the sublaplacian on the Heisenberg group H^n has been recently settled by Müller and Stein [11]. Operators of the form $P = -\Delta + V(x)$ where V is a non-negative potential have been studied in the thesis of Zhong [6]. More generally, multipliers of the form

$$m_{\alpha\beta}(\lambda) = |\lambda|^{-\beta} e^{i|\lambda|^\alpha} \psi(\lambda), \qquad \operatorname{Re}\beta \ge 0,\ \alpha \ge 0$$

where ψ denotes a $C^\infty(\mathbb{R})$ function which vanishes for $|\lambda| \le \frac{1}{2}$ and equals 1 if $|\lambda| \ge 1$ have attracted much interest. For the Euclidean case see the works of Hirschman [3], Wainger [24], Miyachi [10], Schonbek [14], and others. Multipliers of the above type on non-compact symmetric spaces have been studied by Giulini and Meda [2]. Results for the sublaplacian on stratified groups have been obtained by Mauceri and Meda [8].

In this article we are mainly concerned with operators P for which the associated Bochner–Riesz kernel $s_R^\delta(x,y)$ satisfy a good pointwise estimate for large values of δ. We first recall the definition of the Bochner–Riesz means associated to P. These means are defined for Re $\delta \ge 0$ by the equation

$$S_R^\delta f = \int_0^R \left(1 - \frac{\lambda}{R}\right)^\delta dE_\lambda f\,.$$

We assume that the operators S_R^δ are integral operators. Let $s_R^\delta(x,y)$ be the kernel of S_R^δ defined by the equation

$$S_R^\delta f(x) = \int s_R^\delta(x,y)\, f(y)\, dy\,.$$

We consider operators P for which the kernels $s_R^\delta(x,y)$ satisfy an estimate of the form

$$\left|s_R^\delta(x,y)\right| \le CR^{\frac{n}{2}}\left(1 + R^{\frac{1}{2}}|x-y|\right)^{-\alpha\delta+\beta} \tag{1.1}$$

where $\alpha > 0$ and β is a fixed constant, for all large δ.

Estimates of the type (1.1) are known in various special cases such as the Laplacian $-\Delta$ on R^n, the Hermite operator $H = -\Delta + |x|^2$ on R^n, the special Hermite operator $L = -\Delta + \frac{1}{4}|z|^2 - i\sum(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j})$ on C^n and so on. Another class of operators for which estimates of the form (1.1) are known is the class of Rockland operators on stratified nilpotent Lie groups. If L is a Rockland operator of homogeneous degree d on a stratified nilpotent group G, then the estimate

$$\left|s_R^\delta(x)\right| \le C R^{\frac{Q}{d}} \left(1 + R^{\frac{1}{d}}|x|\right)^{\frac{\delta}{3}+\beta}$$

has been proved in Hulanicki [5]. Here Q is the homogeneous dimension of G.

Using a heat kernel estimate proved in [1] and following an idea of Hulanicki and Jenkins [4] one can prove the following. If L is a Rockland operator of homogeneous degree 2, then the Riesz kernel associated to L satisfies

$$\left|s_R^\delta(x)\right| \le C R^{\frac{Q}{2}} \left(1 + R^{\frac{1}{2}}|x|\right)^{-\delta+\beta} .$$

In [7] Mauceri studied operators of the form $p(iT, \mathcal{L})$ on the Heisenberg group H^n where $T = \partial_t$, $\mathcal{L}$ is the sublaplacian and p is a homogeneous polynomial of degree d with certain properties. We remark that they fall under the category of Rockland operators. Among other things he has proved that the Riesz kernel satisfies the estimate

$$\left|s_R^\delta(g)\right| \le C R^{\frac{Q}{d}} \left(1 + R^{\frac{1}{d}}|g|\right)^{-\delta-1} .$$

We now state a general multiplier theorem valid for operators of the above kind.

Theorem 1.

Let $m \in S_\rho^{-\alpha}(\mathbb{R})$ $0 \le \rho \le 1$ and $1 < p < \infty$. Assume that the spectral measure of P has no mass at the origin. If the Bochner–Riesz kernel $s_R^\delta(x, y)$ associated to P satisfies the estimates (1.1) *then $m(P) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded whenever $\alpha > \frac{n(1-\rho)}{a}|\frac{1}{p} - \frac{1}{2}|$.*

Next we specialize to some particular cases and study certain multipliers in detail. First we consider the special Hermite operator L on $\mathbb{C}^n$. We remark that L is related to the sublaplacian on the Heisenberg group H^n. The Heisenberg group H^n is the Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$, with the group operation

$$(z, t).(w, s) = \left(z + w, t + s + \frac{1}{2}\operatorname{Im} z.\overline{w}\right) .$$

The sublaplacian $\mathcal{L}$ is explicitly given by

$$\mathcal{L} = -\Delta - \frac{1}{4}|z|^2\,\partial_t^2 - N\partial_t$$

where N is the rotation operator $\sum_{j=1}^n (x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j})$. The special Hermite operator L and the sublaplacian $\mathcal{L}$ are related by $\mathcal{L}(e^{it} f(z)) = e^{it} L f(z)$. For this reason L is called the twisted Laplacian. Spectral decomposition associated to L is given by the special Hermite expansions, namely

$$Lf(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} (2k + n) f \times \varphi_k(z) .$$

Here $\varphi_k(z)$ are the Laguerre functions of type $(n - 1)$,

$$\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2} ,$$

and the twisted convolution $f \times g$ of two functions is given by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z - w) g(w) e^{\frac{i}{2}\operatorname{Im} z.\overline{w}}\, dw .$$

We remark that this is related to the group convolution on H^n.

Given a bounded function m on $\mathbb{R}$ define the operator

$$m(L)f = (2\pi)^{-n} \sum_{k=0}^{\infty} m(2k+n) f \times \varphi_k .$$

Multipliers for the special Hermite expansions have been extensively studied in recent times, see the works [21] and [22] and the references given there. Consider the multipliers given by

$$m_\alpha(\lambda) = t\,\lambda^{-\frac{\alpha}{2}} J_\alpha\left(t\sqrt{\lambda}\right) .$$

Observe that when $\alpha = \frac{1}{2}$

$$m_\alpha(\lambda) = \sqrt{\frac{2}{\pi}} \frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}$$

and so, $\sqrt{\frac{\pi}{2}}\, tL^{-\frac{1}{4}} J_{\frac{1}{2}}(t\sqrt{L}) f(z) = u(z,t)$ solves the wave equation

$$\left(\partial_t^2 + L\right) u(z,t) = 0,\ u(z,0) = 0,\ \partial_t u(z,0) = f(z) . \tag{1.2}$$

The function m_α belongs to $S_{\frac{1}{2}}^{-\frac{\alpha}{2}-\frac{1}{4}}(\mathbb{R})$ and so Theorem 1 shows that

$$L^{-\frac{\alpha}{2}} J_\alpha\left(t\sqrt{L}\right) : L^p\left(\mathbb{C}^n\right) \to L^p\left(\mathbb{C}^n\right)$$

for $\alpha > 2n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. However, this result can be improved.

Theorem 2.

The operators $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$ satisfy the estimates

$$\left\| L^{-\frac{\alpha}{2}} J_\alpha\left(t\sqrt{L}\right) f \right\|_p \le C_t \|f\|_p \quad , f \in L^p\left(\mathbb{C}^n\right)$$

provided $\alpha > (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$ and $1 \le p \le \infty$.

Corollary 1.

The solution $u(z,t) = \frac{\sin t\sqrt{L}}{\sqrt{L}} f(z)$ of the Cauchy problem (1.2) *satisfies*

$$\|u(.,t)\|_p \le C_t \|f\|_p$$

provided $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n-1}$.

Theorem 2 will be proved by studying an analytic family of operators. Let

$$\psi_k^\alpha(\lambda) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha\left(\frac{1}{2}\lambda^2\right) e^{-\frac{1}{4}\lambda^2}$$

be the Laguerre functions of type α defined for all Re$\alpha > -\frac{1}{2}$. Consider the family

$$T_t^\alpha f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \psi_k^\alpha(t) f \times \varphi_k(z) .$$

We will use the facts that $T_t^\alpha : L^1 \to L^1$ if Re $\alpha > n-1$ and $T_t^\alpha : L^2 \to L^2$ if Re $\alpha > -\frac{1}{2}$. Analytic interpolation will prove that $T_t^\alpha : L^p \to L^p$ if $\alpha > (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. Using a Hilb type asymptotic formula for the Laguerre functions $\psi_k^\alpha(t)$, we will compare the operators $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$ and T_t^α, which will prove Theorem 2.

The above results have analogues for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. We remark that H and the sublaplacian $\mathcal{L}$ on the Heisenberg group are related via the group Fourier transform. Recall that for each $\lambda \in \mathbb{R}$ there is an irreducible unitary representation π_λ of H^n realized on $L^2(\mathbb{R}^n)$ which is explicitly given by

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\phi(\xi + y)\,,$$

$\phi \in L^2(\mathbb{R}^n)$. The group Fourier transform of a function $f \in L^1(H^n)$ is defined to be the operator valued function

$$\hat{f}(\lambda) = \int_{H^n} \pi_\lambda(z,t) f(z,t)\, dz\, dt\,.$$

Then it is known that (see [22])

$$(\mathcal{L}f)\hat{}(\lambda) = \hat{f}(\lambda)H(\lambda)$$

where $H(\lambda) = -\Delta + \lambda^2|x|^2$. In particular, $(\mathcal{L}f)\hat{}(1) = \hat{f}(1)H$.

The spectral decomposition of H is given by the Hermite expansions. Let $\Phi_\alpha(x)$, $\alpha \in \mathbb{N}^n$ be the normalized Hermite functions which are eigenfunctions of H with eigenvalues $(2|\alpha| + n)$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $P_k f$ be the projections defined by

$$P_k f(x) = \sum_{|\alpha|=k} (f, \Phi_\alpha)\, \Phi_\alpha(x)\,.$$

Then the spectral decomposition of H is given by

$$Hf = \sum_{k=0}^{\infty} (2k+n)\, P_k f\,.$$

For various properties of the Hermite expansions we refer to the monograph of Thangavelu [21].

In order to study the boundedness properties of $H^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{H})$ we look at the following analytic family of operators. Let

$$S_t^\alpha f(x) = \sum_{k=0}^{\infty} \psi_k^\alpha(t) P_k f(x)\,.$$

The operators S_t^α and T_t^α are related to each other via the Weyl transform. The representation π_1 of H^n defines a projective representation π of $\mathbb{C}^n$ by the prescription $\pi(z) = \pi_1(z, 0)$. The integrated representation

$$W(f) = \int_{\mathbb{C}^n} \pi(z)\, f(z)\, dz$$

is then called the Weyl transform, which takes functions f on $\mathbb{C}^n$ into bounded operators acting on $L^2(\mathbb{R}^n)$. For $f \in L^1(\mathbb{C}^n)$ we have the relation

$$W\left(T_t^\alpha f\right) = S_t^\alpha W(f)\,.$$

Using this connection we will prove the following.

Theorem 3.

Let S_t^α be defined as above. Then $S_t^\alpha : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded for $1 \le p \le \infty$ whenever $\alpha > n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$.

As before comparing $H^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{H})$ with S_t^α we obtain corresponding results for the operators $H^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{H})$ and by taking $\alpha = \frac{1}{2}$ we get the following estimate for the solution $u(x,t)$ of the Cauchy problem:

$$\left(\partial_t^2 + H\right) u(x,t) = 0, u(x,0) = 0, \partial_t u(x,0) = f(x) .$$

Corollary 2.

Let u be the solution of the above problem given by $u(x,t) = \frac{\sin t\sqrt{H}}{\sqrt{H}} f(x)$. Then,

$$\|u(.,t)\|_p \le C_t \|f\|_p , \quad \left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{n} .$$

Unlike the case of L or the standard Laplacian $-\Delta$, the above is the best one can get. That is the above estimate for the solution $u(x,t)$ to the wave equation cannot be extended to the bigger range $|\frac{1}{p} - \frac{1}{2}| \le \frac{1}{n-1}$. This has already been observed in [6] where such estimates for the operators $-\Delta + V(x)$ have been obtained. However if we consider only radial functions it is possible to improve the above result. We have the following result.

Theorem 4.

Assume that $f \in L^p(\mathbb{R}^n)$ is radial. Then the estimate

$$\left\|\frac{\sin t\sqrt{H}}{\sqrt{H}} f\right\|_p \le C_t \|f\|_p$$

holds provided $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{n-1}$.

The operators S_t^α have been studied in [13] where among other things it was proved that

$$\lim_{t\to 0} S_t^{n-1} f(x) = f(x), \quad a.e\ x \in \mathbb{R}^n ,$$

for $f \in L^p(\mathbb{R}^n)$, $p > \frac{2n}{2n-1}$. This result can be improved and as a consequence we can get an almost everywhere convergence result for the Riesz means

$$v^\alpha(x,t) = \int_0^t \left(1 - \frac{s^2}{t^2}\right)^{\alpha-\frac{1}{2}} v(x,s)\, ds$$

of the solution $v(x,t)$ to the Cauchy problem

$$\left(\partial_t^2 + H\right) v(x,t) = 0, v(x,0) = f(x), \partial_t v(x,0) = 0 .$$

Corollary 3.

The maximal operator $\sup_{0<t\le 1} |S_t^\alpha f|$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ for $\alpha = \frac{n}{2}$. Consequently, for $f \in L^p(\mathbb{R}^n)$,

$$\lim_{t\to 0} S_t^{\frac{n}{2}} f(x) = f(x)$$

for almost every $x \in \mathbb{R}^n$.

As in [13] we can compare $S_t^{\frac{n}{2}} f(x)$ with $\frac{v^{\frac{n}{2}}(x,t)}{t}$ and prove that in one and two dimensions

$$\lim_{t\to 0} \frac{v^{\frac{n}{2}}(x,t)}{t} = f(x)$$

for almost every x when $f \in L^1 \cap L^p(\mathbb{R}^n)$. We remark that Riesz means for the solutions of the Schrödinger equation for the standard Laplacian has been studied by Sjostrand [16] . See also Miyachi [10].

2. A General Multiplier Theorem

We now proceed to prove Theorem 1. We start with a simple proposition. Throughout this section we assume that the spectral measure of P has no mass at the origin.

Proposition 1.

Let m be a smooth compactly supported function on $\mathbb{R}$ and $1 \le p \le \infty$. If the Bochner–Riesz kernel $s_R^\delta(x, y)$ associated to P satisfies the estimates (1.1) then $m(P) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded.

Proof. Let $K(x, y)$ be the kernel of the operator $m(P)$. Thus,

$$K(x, y) = \int_0^\infty m(\lambda)\, dE_\lambda(x, y) = \int_0^\infty m(\lambda)\, \partial_\lambda s_\lambda^0(x, y) .$$

Integrating by parts and making use of the identity

$$\frac{d}{d\lambda}\left(\lambda^l s_\lambda^l(x, y)\right) = l\lambda^{l-1} s_\lambda^{l-1}(x, y)$$

we get

$$K(x, y) = C_l \int_0^\infty m^{(l-1)}(\lambda)\, \lambda^l\, s_\lambda^l(x, y)\, d\lambda .$$

Now using the estimate (1.1) we have

$$|K(x, y)| \le C_l \int_0^\infty m^{(l+1)}(\lambda)\, \lambda^{l+\frac{n}{a}} \left(1 + \lambda^{\frac{1}{a}}(|x - y|\right)^{-\alpha l+\beta} .$$

From the above expression it is clear that, if l is large enough then we have

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)|\, dx < \infty$$

which proves the proposition. □

In view of the above proposition, to prove Theorem 1 it is enough to prove the following.

Theorem 5.

Let $m \in S_\rho{}^\alpha(\mathbb{R})$, $0 \le \rho \le 1$ be such that $m(\lambda) = 0$ for $|\lambda| \le 1$ and $1 < p < \infty$. If the Bochner–Riesz kernel $s_R^\delta(x, y)$ associated to P satisfies the estimates (1.1) then $m(P) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded whenever $\alpha > \frac{n(1-\rho)}{a}|\frac{1}{p} - \frac{1}{2}|$.

Proof. Let $\varphi \in C_0^\infty(\frac{1}{2} \le t \le 2)$ be such that $\sum_{j=-\infty}^{\infty} \varphi(2^{-j}t) = 1$ for every $t > 0$. Let $m_j(t) = m(t)\varphi(2^{-j}t)$ and $m_j(P)$ be the corresponding multiplier transform, that is

$$m_j(P)f = \int_0^\infty m_j(\lambda) dE_\lambda f .$$

We then have $m(P) = \sum_{j=0}^{\infty} m_j(P)$ since $m(\lambda)$ vanishes for $|\lambda| \le 1$. Under the hypothesis of Theorem 5 we will show that there exists a $\delta > 0$ such that

$$\left\| m_j(P) f \right\|_p \le C\, 2^{-\delta j} \|f\|_p \tag{2.1}$$

for all $f \in L^p(\mathbb{R}^n)$. Theorem 5 will then follow by summing a geometric series.

In order to get the estimate (2.1) we look at the kernel $k_j(x, y)$ of $m_j(P)$ which is given by

$$k_j(x, y) = \int_0^\infty m_j(\lambda) dE_\lambda(x, y) .$$

Let $1 < p \leq 2$. Since $\alpha > \frac{n(1-\rho)}{a}(\frac{1}{p} - \frac{1}{2})$ we can choose $\epsilon > 0$ such that $\alpha > n(\frac{1-\rho}{a} + \epsilon)(\frac{1}{p} - \frac{1}{2})$. Let $\gamma = \frac{1-\rho}{a} + \epsilon - \frac{1}{d}$ so that $\alpha > n(\gamma + \frac{1}{d})(\frac{1}{p} - \frac{1}{2})$. We write

$$k_j(x, y) = k_{j,1}(x, y) + k_{j,2}(x, y)$$

where $k_{j,1}(x, y) = k_j(x, y)$ if $|x - y| \leq 2^{\gamma j}$ and 0 elsewhere. We first consider the operator given by

$$K_{j,2} f(x) = \int_{\mathbb{R}^n} k_{j,2}(x, y) f(y)\, dy .$$

Proposition 2.

Under the hypothesis of the Theorem 5 the following holds. For some $\delta > 0$

$$\int_{\mathbb{R}^n} \left|K_{j,2} f(x)\right|^p dx \leq C 2^{-\delta j p} \int_{\mathbb{R}^n} |f(x)|^p\, dx$$

for all $f \in L^p(\mathbb{R}^n)$ and $j = 1, 2, \ldots$.

In order to prove the above proposition we will make use of the following estimate on the kernel $k_j(x, y)$.

Proposition 3.

Let m be as in Theorem 5. Then we have

$$\left|k_j(x, y)\right| \leq C_l\ 2^{j[l(1-\rho-\frac{a}{d})+\frac{n+\beta+\alpha}{d}]}\ |x - y|^{-al+\beta+\alpha}$$

for any integer l.

We will assume this for a moment and complete the proof of Proposition 2. We only need to show that

$$\sup_y \int_{\mathbb{R}^n} \left|k_{j,2}(x, y)\right| dx \leq C 2^{-\delta j}$$

for some $\delta > 0$. In view of the above estimate on the kernel $k_j(x, y)$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \left|k_{j,2}(x, y)\right| dx &\leq C_l\, 2^{j[l(1-\rho-\frac{a}{d})+\frac{n+\beta+\alpha}{d}]} \int_{2^{\gamma j}}^\infty t^{-al+\beta+\alpha+n-1}\, dt \\ &\leq C_l\, 2^{jl[1-\rho-a(\gamma+\frac{1}{d})]}\, 2^{j(n+\beta+\alpha)(\gamma+\frac{1}{d})} .\end{aligned}$$

Since $1 - \rho - a(\gamma + \frac{1}{d}) < 0$ choosing l large enough we can get the required decay.

To prove Proposition 3 we need to use the estimate (1.1). Since

$$k_j(x, y) = \int_0^\infty m_j(\lambda) dE_\lambda(x, y)$$

and $E_\lambda(x, y) = s_\lambda^0(x, y)$, integrating by parts and making use of the identity

$$\frac{d}{d\lambda}\left(\lambda^m s_\lambda^m(x, y)\right) = m\lambda^{m-1} s_\lambda^{m-1}(x, y)$$

we get

$$k_j(x, y) = C_l \int_0^\infty \lambda^{l-1} s_\lambda^{l-1}(x, y) \partial_\lambda^l \left(m_j(\lambda)\right) d\lambda .$$

As $m_j \in S_\rho^{-a}$ and is supported in $2^{j-1} \le t \le 2^{j+1}$ we have

$$\left|k_j(x,y)\right| \le C_l \int_{2^{j-1}}^{2^{j+1}} \lambda^{l-1}\lambda^{-\rho l}\lambda^{\frac{a}{d}}\left(1+\lambda^{\frac{1}{d}}|x-y|\right)^{-al+\alpha-\beta} d\lambda$$
$$\le |x-y|^{-al+\alpha+\beta}\, 2^{j[l(1-\rho-\frac{a}{\rho})+\frac{n+\beta+\alpha}{d}]}\,.$$

This completes the proof of the Proposition 3. □

Thus we have taken care of $k_{j,2}(x,y)$. To deal with $k_{j,1}$, we proceed as follows. First we prove the following analogue of the Hardy–Littlewood–Sobolev theorem for the operator P which has been proved in [23]. However for the sake of completeness we state and prove it here.

Theorem 6.

Let $0 < \alpha < n$, $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then we have

$$\left\|(1+P)^{-\frac{\alpha}{d}} f\right\|_q \le C\|f\|_p\,.$$

Proof. By spectral theorem

$$(1+P)^{-\frac{\alpha}{d}} f = \int_0^\infty (1+\lambda)^{-\frac{\alpha}{d}}\, dE_\lambda f$$

and so the kernel of $(1+P)^{-\frac{\alpha}{d}}$ is given by

$$k_\alpha(x,y) = \int_0^\infty (1+\lambda)^{-\frac{\alpha}{d}}\, dE_\lambda(x,y)\,.$$

As before let φ be a smooth function supported in $(\frac{1}{2}, 2)$ such that $\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\lambda) = 1$ for every $\lambda > 0$. Let $k_{\alpha,j}(x,y)$ be the kernel of $\varphi(2^{-j}P)(1+P)^{-\frac{\alpha}{d}}$. Then

$$k_{\alpha,j}(x,y) = \int_{2^{j-1}}^{2^{j+1}} m_{\alpha,j}(\lambda)\, dE_\lambda(x,y)$$

where $m_{\alpha,j}(\lambda) = \varphi(2^{-j}\lambda)(1+\lambda)^{-\frac{\alpha}{d}}$.

Integrating by parts we get

$$k_{\alpha,j}(x,y) = c_{\alpha,l} \int_{2^{j-1}}^{2^{j+1}} \partial_\lambda^l\left(m_{\alpha,j}\right)(\lambda)\lambda^{l-1} s_\lambda^{l-1}(x,y)\, d\lambda\,.$$

It is easy to see that $|\partial_\lambda^l(m_{\alpha,j})(\lambda)| \le C\,\lambda^{-l}$ with C depending only on α and l. We use the estimate (1.1) to get

$$\left|k_{\alpha,j}(x,y)\right| \le C \int_{2^{j-1}}^{2^{j+1}} \lambda^{-\frac{\alpha}{d}+\frac{n}{d}-1}\left(1+\lambda^{\frac{1}{d}}|x-y|\right)^{-al+\alpha+\beta} d\lambda$$

which is bounded by

$$C|x-y|^{\alpha-n}\int_{2^{j-1}|x-y|^d}^{2^{j-1}|x-y|^d} \lambda^{-\frac{\alpha}{d}+\frac{n}{d}-1}\left(1+\lambda^{\frac{1}{d}}\right)^{-al+\alpha+\beta} d\lambda\,.$$

Since for any $t > 0$ at most two of the intervals of the type $(2^{j-1}t, 2^{j+1}t)$ can intersect we have,

$$|k_\alpha(x,y)| \le C\sum_{j=-\infty}^{\infty}\left|k_{\alpha,j}(x,y)\right| \le C\,|x-y|^{\alpha-n}\int_0^\infty \lambda^{-\frac{\alpha}{d}+\frac{n}{d}-1}\left(1+\lambda^{\frac{1}{d}}\right)^{-al+\alpha+\beta}\,.$$

The last integral converges if l is large enough since $0 < \alpha < n$ and we have

$$|k_\alpha(x, y)| \leq C|x - y|^{\alpha - n} .$$

Now it is a routine matter to prove the proposition. See, for example, the proof of the Hardy–Littlewood–Sobolev theorem in Stein [18]. $\square$

Using the above theorem,we prove the following result. Let B be any ball of radius $2^{\gamma j}$.

Proposition 4.
There is a $\delta > 0$ such that

$$\left(\int_B |m_j(P)f(x)|^p\, dx\right)^{\frac{1}{p}} \leq C2^{-\delta j} \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx\right)^{\frac{1}{p}} .$$

for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$.

Proof. By Hölder's inequality,

$$\left(\int_B |m_j(P)f(x)|^p\, dx\right)^{\frac{1}{p}} \leq |B|^{\frac{1}{p}-\frac{1}{2}} \left(\int_{\mathbb{R}^n} |m_j(P)f(x)|^2\, dx\right)^{\frac{1}{2}} .$$

Now by spectral theorem

$$\|m_j(P)f\|_2^2 = \int_{2^{j-1}}^{2^{j+1}} |m_j(\lambda)|^2\, d\,(E_\lambda f, f) .$$

Since $m_j \in S_\rho^{-\alpha}$, the above is bounded by

$$\begin{aligned}
&\int_{2^{j-1}}^{2^{j+1}} (1+\lambda)^{-2\alpha} d\,(E_\lambda f, f) \\
&\leq \int_{2^{j-1}}^{2^{j+1}} \lambda^{-2\alpha+\frac{2n}{d}(\frac{1}{p}-\frac{1}{2})} (1+\lambda)^{-\frac{2n}{d}(\frac{1}{p}-\frac{1}{2})} d\,(E_\lambda f, f) \\
&\leq C2^{j[-2\alpha+\frac{2n}{d}(\frac{1}{p}-\frac{1}{2})]} \left\|(1+P)^{-\frac{s}{d}} f\right\|_2^2 .
\end{aligned}$$

with $s = n(\frac{1}{p} - \frac{1}{2})$. Using the result of Theorem 6 we obtain

$$\begin{aligned}
\left(\int_B |m_j(P)f(x)|^p\, dx\right)^{\frac{1}{p}} &\leq C2^{n\gamma j(\frac{1}{p}-\frac{1}{2})} 2^{j[-\alpha+\frac{n}{d}(\frac{1}{p}-\frac{1}{2})]} \|f\|_p \\
&= 2^{-j[\alpha-(\gamma+\frac{1}{d})n(\frac{1}{p}-\frac{1}{2})]} \|f\|_p
\end{aligned}$$

which completes the proof by the choice of γ. $\square$

We are now in a position to complete the proof of Theorem 5. Let $\mathcal{K}_{j,1}$ be the operator defined by

$$\mathcal{K}_{j,1} f(x) = \int_{\mathbb{R}^n} k_{j,1}(x, y)\, f(y)\, dy .$$

To deal with this operator we decompose f into three parts. Let $\xi \in \mathbb{R}^n$ and define

$$f_1(x) = f(x)\chi\left(|x - \xi| \leq \frac{3}{4}2^{j\gamma}\right)$$

$$f_2(x) = f(x)\chi\left(\frac{3}{4}2^{j\gamma} < |x - \xi| \leq \frac{5}{4}2^{j\gamma}\right)$$

and $f_3 = f - f_1 - f_2$. Let $B(\xi)$ be the ball $|x-\xi| \le \frac{1}{4}2^{j\gamma}$. We will show that

$$\int_{B(\xi)} \left|\mathcal{K}_{j,1}f(x)\right|^p dx \le C2^{-\epsilon jp} \int_{|x-\xi|\le\frac{3}{4}2^{j\gamma}} |f(x)|^p dx$$

for some $\epsilon > 0$. Integration with respect to ξ will prove

$$\int_{\mathbb{R}^n} \left|\mathcal{K}_{j,1}f(x)\right|^p dx \le C2^{-\epsilon jp}\|f\|_p^p .$$

When $|x-\xi| \le \frac{1}{4}2^{j\gamma}$ and y belonging to the support of f_3 it follows that $|x-y| > 2^{j\gamma}$ and consequently $\mathcal{K}_{j,1}f_3 = 0$. When $|x-\xi| \le \frac{1}{4}2^{j\gamma}$ and y belonging to the support of f_2 one has $|x-y| > \frac{1}{2}2^{j\gamma}$ and we can repeat the proof of the Proposition 2 to conclude that

$$\int_{B(\xi)} \left|\mathcal{K}_{j,1}f_2(x)\right|^p dx \le C2^{-\epsilon jp} \int_{\mathbb{R}^n} |f_2(x)|^p \, dx .$$

Finally, applying Proposition 4 we obtain the estimate

$$\int_{B(\xi)} \left|\mathcal{K}_{j,1}f_1(x)\right|^p dx \le C2^{-\epsilon jp} \int_{\mathbb{R}^n} |f_1(x)|^p \, dx .$$

Putting together all the above estimates we prove Theorem 5.

3. Special Hermite Expansions

In this section we take up the special Hermite operator L on $\mathbb{C}^n$ and prove Theorem 2. If $s_R^\delta(z)$ is the kernel of the Bochner-Riesz operator then

$$s_R^\delta(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \left(1 - \frac{2k+n}{R}\right)_+^\delta \varphi_k(z) .$$

Before we proceed some remarks are in order. Bochner–Riesz kernel $s_R^\delta(z)$ associated to L satisfies the estimate,

$$\left|s_R^\delta(z)\right| \le CR^n \left(1 + R^{\frac{1}{2}}|z|\right)^{-\delta-n-\frac{1}{3}} ,$$

see Proposition 2.5.1 in [21]. Therefore Theorem 1 will imply that operators $m(L)$, m coming from $S_\rho^{-\alpha}(\mathbb{R})$ are bounded on $L^p(\mathbb{C}^n)$ provided $\alpha > 2n(1-\rho)|\frac{1}{p} - \frac{1}{2}|$, $1 < p < \infty$. We remark that above can be improved to include the case $p = 1$ as well. To prove this we proceed as follows. A close examination of the proof of Theorem 1 reveals that we need only to prove the following.

Let B be any ball of radius $2^{\gamma j}$ where γ is as in the proof of Theorem 1 and let k_j be the kernel of $m_j(L)$.

Proposition 5.

There is a $\delta > 0$ such that

$$\left(\int_B \left|f \times k_j(z)\right| dz\right) \le C2^{-\delta j} \int_{\mathbb{C}^n} |f(z)| dz$$

for all $f \in L^1(\mathbb{C}^n)$.

Proof. By Holder's inequality

$$\int_B |f \times k_j(z)|\, dz \le C|B|^{\frac{1}{2}} \left(\int_{\mathbb{C}^n} |f \times k_j(z)|^2\, dz \right)^{\frac{1}{2}} .$$

By the Plancherel theorem for the special Hermite expansion, we have

$$\int_{\mathbb{C}^n} |f \times k_j(z)|^2\, dz = (2\pi)^{-2n} \sum_{k=0}^{\infty} |m_j(2k+n)|^2 \, \|f \times \varphi_k\|_2^2$$

which is dominated by

$$\sum_{2^{j-1} \le 2k+n \le 2^{j+1}} (2k+n)^{-2\alpha} \, \|f \times \varphi_k\|_2^2 \, .$$

Since

$$\|f \times \varphi_k\|_2 \le \|\varphi_k\|_2 \, \|f\|_1 \le C k^{\frac{n-1}{2}} \|f\|_1$$

(see [21]) the above is dominated by $C2^{j(-2\alpha+n)}\|f\|_1^2$. Now the proof can be completed as in the previous section. □

Next we turn our attention towards a proof of Theorem 2. As we stated in the introduction we consider the analytic family of operators defined for Re $\alpha > -\frac{1}{2}$ by,

$$T_t^\alpha f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \psi_k^\alpha(t) f \times \varphi_k(z) \, .$$

Here ψ_k^α are the Laguerre functions

$$\psi_k^\alpha(t) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha \left(\frac{1}{2} t^2 \right) e^{-\frac{1}{4}t^2} \, .$$

We require the following estimates

$$\sup_{0<t\le 1} |\psi_k^\alpha(t)| \le C \quad \text{for} \quad \alpha \ge -\frac{1}{2} \tag{3.1}$$

for which we refer to Szegö [20]. We also make use of the following proposition.

Proposition 6.

Let μ_t be the normalized surface measure on the sphere $S_t = \{|z| = t\}$ in $\mathbb{C}^n$. Then

$$f \times \mu_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!} \varphi_k(t) \, f \times \varphi_k(z) \, .$$

Proof. See Theorem 2.4.4 in [22]. □

Using analytic interpolation theorem we establish the following result.

Theorem 7.

Let $1 \le p \le \infty$ and $\alpha > (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$ then

$$\|T_t^\alpha f\|_p \le C\|f\|_p \quad \textit{for} \quad 0 < t \le 1 \, .$$

Proof. In view of Proposition 6 we know that T_t^{n-1} is bounded on $L^p(\mathbb{C}^n)$, $1 \le p \le \infty$. From the estimate (3.1) it follows that $T_t^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{C}^n)$. For the analytic interpolation we need to consider T_t^α when α is complex. We make use of the following formula connecting Laguerre polynomials of different type:

$$L_k^{\alpha+\beta}(t) = \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(\beta)\,\Gamma(k+\alpha+1)} \int_0^1 s^\alpha (1-s)^{\beta-1} L_k^\alpha(st)\, ds$$

which is valid for Re $\alpha > -1$ and Re $\beta > 0$.

Given $\alpha = n-1+\delta+i\sigma, \delta > 0$ we can write

$$\psi_k^\alpha(t) = \frac{\Gamma(n+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\Gamma(n)} \int_0^1 s^{n-1}\,(1-s)^{\delta+i\sigma-1}\, e^{-\frac{1}{4}(1-s)t^2}\, \psi_k^{n-1}\left(t\sqrt{s}\right)\, ds\,,$$

so that T_t^α is expressible in terms of T_t^{n-1} as

$$T_t^\alpha = \frac{\Gamma(n+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\Gamma(n)} \int_0^1 s^{n-1}\,(1-s)^{\delta+i\sigma-1}\, e^{-\frac{1}{4}(1-s)t^2}\, T_{t\sqrt{s}}^{n-1}\, ds\,.$$

From this it follows that T_t^α is bounded on $L^p(\mathbb{C}^n) 1 \le p \le \infty$ for Re $\alpha > n-1$. Similarly, when $\alpha = -\frac{1}{2}+\delta+i\sigma$ it can be shown that T_t^α is bounded on $L^2(\mathbb{C}^n)$. Using estimates for the gamma functions, we can easily check that the family T_t^α is admissible in the sense of Stein [17]. Applying Stein's analytic interpolation theorem we obtain Theorem 7. □

We will now consider operators of the form $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$, with the corresponding multiplier $(2k+n)^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{2k+n})$. We first remark that it is enough to consider the multiplier $(2k+\alpha+1)^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{2k+\alpha+1})$. To see this let us assume $t=1$ and look at their difference

$$m(k) = (2k+n)^{-\frac{\alpha}{2}} J_\alpha\left(\sqrt{2k+n}\right) - (2k+\alpha+1)^{-\frac{\alpha}{2}} J_\alpha\left(\sqrt{2k+\alpha+1}\right)\,.$$

Writing $F(t) = t^{-\alpha} J_\alpha(t)$ we have

$$\begin{aligned} m(k) &= F\left(\sqrt{2k+n}\right) - F\left(\sqrt{2k+\alpha+1}\right) \\ &= \int_{\alpha+1}^n \frac{F'\left(\sqrt{t+2k}\right)}{2\sqrt{t+2k}}\, dt \end{aligned}$$

since $F'(t) = -t^{-\alpha} J_{\alpha+1}(t)$ we have the expression

$$m(k) = -\frac{1}{2}\int_{\alpha+1}^n \frac{J_{\alpha+1}\left(\sqrt{t+2k}\right)}{\left(\sqrt{t+2k}\right)^{\alpha+1}}\, dt$$

which clearly shows that $m \in S_{\frac{1}{2}}^{-\frac{2\alpha+3}{2}}(\mathbb{R})$. Therefore $m(k)$ will define an L^p multiplier provided $\frac{\alpha}{2}+\frac{3}{4} > n(\frac{1}{p}-\frac{1}{2})$ which is clearly satisfied when $\alpha > (2n-1)(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$.

Thus it is enough to consider the multiplier $(2k+\alpha+1)^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{2k+\alpha+1})$. Recall that we are assuming $0 < t \le 1$. We compare this multiplier with $\psi_k^\alpha(t)$ using a Hilb type asymptotic formula for the Laguerre polynomials, see Szegö [20]. More precisely formula (8.64.3) on p. 217 of [20] gives,

$$\psi_k^\alpha(t) = 2^\alpha\,\Gamma(\alpha+1)\,\frac{J_\alpha\left(t\sqrt{2k+\alpha+1}\right)}{\left(t\sqrt{2k+\alpha+1}\right)^\alpha} + m(k,\alpha,t) \tag{3.2}$$

where

$$m(k,\alpha,t) = \frac{C(\alpha)}{\sin\alpha\pi}\, t^4 \int_0^1 \left\{ J_\alpha\left(t\sqrt{N}\right) J_{-\alpha}\left(ts\sqrt{N}\right) - J_{-\alpha}\left(t\sqrt{N}\right) J_\alpha\left(ts\sqrt{N}\right) \right\} s^{\alpha+3}\, \psi_k^\alpha(ts)\, ds$$

where $N = 2k+\alpha+1$ and $C(\alpha)$ a constant which depends only on α. In the above formula, if α is an integer, $J_{-\alpha}$ must be replaced by the modified Bessel function Y_α and $\sin\alpha\pi$ by -1. Now define $m_\alpha(\lambda) = \lambda^{-\frac{\alpha}{2}} J_\alpha(\sqrt{\lambda})$ and

$$a_\alpha(\lambda,t,s) = \left(J_\alpha\left(t\sqrt{\lambda}\right) J_{-\alpha}\left(ts\sqrt{\lambda}\right) - J_{-\alpha}\left(t\sqrt{\lambda}\right) J_\alpha\left(ts\sqrt{\lambda}\right) \right) s^{\alpha+3} t^4 .$$

For the symbols a_α we prove the following estimates.

Lemma 1.

For $0 \le r\,,s \le 1$ we have the estimates

$$\left|\partial_\lambda^k\, a_\alpha(\lambda,r,s)\right| \le C_k\, (1+\lambda)^{-\frac{k}{2}-\frac{1}{2}}$$

valid for all $\lambda > 0$, $k \ge 0$. More precisely,

$$\left|\partial_\lambda^k a_\alpha(\lambda,r,s)\right| \le C\, r^3\, s^{\frac{5}{2}} (1+\lambda)^{-\frac{1}{2}(k+1)} \left\{ \left(1+r^2\lambda\right)^{-\frac{\alpha}{2}} \left(1+r^2s^2\lambda\right)^{\frac{\alpha}{2}} + s^{2\alpha}\left(1+r^2\lambda\right)^{\frac{\alpha}{2}} \left(1+r^2s^2\lambda\right)^{-\frac{\alpha}{2}} \right\}$$

Proof. Let $B_\alpha(\lambda) = \lambda^{-\frac{1}{2}\alpha} J_\alpha(\sqrt{\lambda})$ and when α is a negative integer replace J_α by Y_α. Then B_α satisfies the equation

$$\frac{d}{d\lambda} B_\alpha(\lambda) = -\frac{1}{2} B_{\alpha+1}(\lambda) .$$

The asymptotic properties of the Bessel function give us the estimates

$$\left| \left(\frac{d}{d\lambda}\right)^k B_\alpha(\lambda) \right| \le C(1+\lambda)^{-\frac{1}{2}(\alpha+k+\frac{1}{2})} .$$

Consider the first term in $a_\alpha(\lambda,t,s)$ which is equal to $B_\alpha(r^2\lambda)B_{-\alpha}(r^2s^2\lambda)s^3r^4$. The k^{th} derivative of this term is a linear combination of terms of the form

$$r^{2j+4}\, B_{\alpha+j}\left(r^2\lambda\right) \left(r^2 s^2\right)^{k-j} B_{-\alpha+k-j}\left(r^2 s^2 \lambda\right) s^3$$

which is bounded by a constant times

$$r^{2k+4}\, s^{2k-2j+3} \left(1+r^2\lambda\right)^{-\frac{1}{2}(\alpha+j+\frac{1}{2})} \left(1+r^2 s^2 \lambda\right)^{-\frac{1}{2}(-\alpha+k-j+\frac{1}{2})} .$$

As $0 \le r\,,s \le 1$, the above is bounded by a constant times

$$r^3 s^{\frac{5}{2}}\, (1+\lambda)^{-\frac{1}{2}(k+1)} \left(1+r^2\lambda\right)^{-\frac{\alpha}{2}} \left(1+r^2s^2\lambda\right)^{\frac{\alpha}{2}}$$

which is bounded by $C\,(1+\lambda)^{-\frac{1}{2}(k+1)}$. Similarly, the k^{th} derivative of the second term is bounded by

$$C\, r^3\, s^{2\alpha+\frac{5}{2}}\, (1+\lambda)^{-\frac{1}{2}-\frac{k}{2}} \left(1+r^2\lambda\right)^{\frac{\alpha}{2}} \left(1+r^2s^2\lambda\right)^{-\frac{\alpha}{2}}$$

which in turn is bounded by $C\,(1+\lambda)^{-\frac{1}{2}(k+1)}$. This proves the lemma. □

Iterating in the formula (3.2) we have

$$\psi_k^\alpha(t) = 2^\alpha\,\Gamma(\alpha+1)\frac{J_\alpha\left(t\sqrt{2k+\alpha+1}\right)}{\left(t\sqrt{2k+\alpha+1}\right)^\alpha} + m_1\left(\sqrt{2k+\alpha+1},t\right) + e\left(\sqrt{2k+\alpha+1},t\right)$$

where

$$m_1\left(\sqrt{2k+\alpha+1},t\right) = C_1(\alpha)\int_0^1 a_\alpha(N,t,s)m_\alpha\left(t^2s^2N\right)\,ds\,.$$

From the above expression it is easy to check that $m_1(\sqrt{\lambda},t) \in S_{\frac{1}{2}}^{-\frac{\alpha}{2}-\frac{3}{4}}$ and so when $\alpha > (2n-1)(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$, $m_1(\sqrt{2k+\alpha+1},t)$ defines an L^p multiplier. Further iteration produces better and better terms. We can write

$$\psi_k^\alpha(t) = 2^\alpha\,\Gamma(\alpha+1)\frac{J_\alpha\left(t\sqrt{2k+\alpha+1}\right)}{\left(t\sqrt{2k+\alpha+1}\right)^\alpha} + \sum_{j=1}^{l} m_j\left(\sqrt{2k+\alpha+1},t\right) + e_l\left(\sqrt{2k+\alpha+1},t\right)$$

where the error term $e_l(\sqrt{2k+\alpha+1},t)$ can be written as

$$\begin{aligned} &e_l\left(\sqrt{2k+\alpha+1},t\right)\\ &= C_l(\alpha)\int_0^1\cdots\int_0^1 a_\alpha\left(N,t,s_1\right)\,a_\alpha\left(N,ts_1,s_2\right)\dots a_\alpha\left(N,ts_1s_2\dots s_{l-1},s_l\right)\\ &\qquad \psi_k^\alpha\left(ts_1\dots s_l\right)\,ds_1\,\dots ds_l\,. \end{aligned}$$

Here $N = 2k+\alpha+1$.

All the sequences $m_j(\sqrt{2k+\alpha+1},t)$, $j=1,2,\dots l$ will define bounded multipliers on $L^p(\mathbb{C}^n)$ when $\alpha > (2n-1)|\frac{1}{p}-\frac{1}{2}|-\frac{1}{2}$. Now it follows from Lemma 1 that the multiplier $a_\alpha(\lambda,t,s_1)a_\alpha(\lambda,ts_1,s_2)\dots a_\alpha(\lambda,ts_1s_2\dots s_{l-1},s_l)$ belongs to $S_{\frac{1}{2}}^{-\frac{l}{2}}$ with estimates uniform in $s_1,s_2,\dots s_l$. Hence using Theorem 1 and Theorem 7 we get that the operator defined by the sequence $e_l(\sqrt{2k+\alpha+1},t)$ is also bounded on the same L^p if $\frac{\alpha}{2}+\frac{3}{4}\le\frac{l}{2}$. Thus the difference

$$\psi_k^\alpha(t) - 2^\alpha\,\Gamma(\alpha+1)\left(t\sqrt{2k+\alpha+1}\right)^{-\alpha}J_\alpha\left(t\sqrt{2k+\alpha+1}\right)$$

defines a bounded L^p multiplier for $\alpha > (2n-1)(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$. As $\psi_k^\alpha(t)$ defines an L^p multiplier this implies that $(2k+\alpha+1)^{-\frac{\alpha}{2}}J_\alpha(t\sqrt{2k+\alpha+1})$ also defines a bounded L^p multiplier which completes the proof of Theorem 2.

By taking $\alpha=\frac{1}{2}$ in the theorem and noting that $\sqrt{\frac{2}{\pi}}\frac{\sin\sqrt{t}}{\sqrt{t}} = \frac{J_{\frac{1}{2}}(\sqrt{t})}{(\sqrt{t})^{\frac{1}{2}}}$ we infer that $u(z,t) = \frac{\sin t\sqrt{L}}{\sqrt{L}}f(z)$ satisfies the estimate

$$\|u(.,t)\|_p \le C_t\|f\|_p \qquad \text{for} \quad \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{2n-1}\,.$$

This proves Corollary 1.

4. Hermite Expansions

In this section we proceed to prove analogous results for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. Bochner–Riesz kernels associated to H are given by the expression

$$s_R^\delta(x,y) = \sum_{k=0}^{\infty}\left(1-\frac{2k+n}{R}\right)_+^\delta \Phi_k(x,y)$$

where

$$\Phi_k(x,y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y).$$

They satisfy the following estimate (see [21, Section 3]):

$$|s_R^\delta(x,y)| \le CR^{\frac{n}{2}}\left(1+R^{\frac{1}{2}}|x-y|\right)^{-\delta+\beta+n+2}.$$

So Theorem 1 implies that $m(H)$ is bounded on $L^p(\mathbb{R}^n)$ for m coming from $S_\rho^{-\alpha}(\mathbb{R})$ provided $\alpha > n(1-\rho)|\frac{1}{p}-\frac{1}{2}|$, $1 < p < \infty$. We remark that as in the previous section,using the estimate $\|P_k f\|_2 \le Ck^{\frac{n}{2}-\frac{1}{4}}\|f\|_1$, the above can be improved to include the case $p = 1$ as well. We now proceed to prove Theorem 3.

We start by considering the analytic family of operators defined for Re $\alpha > -\frac{1}{2}$ by

$$S_t^\alpha f(x) = \sum_{k=0}^{\infty}\psi_k^\alpha(t)P_k f(x)$$

where

$$\psi_k^\alpha(t) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}L_k^\alpha\left(\frac{1}{2}t^2\right)e^{-\frac{1}{4}t^2}.$$

When $\alpha = n-1$, S_t^{n-1} is precisely the Weyl transform of the surface measure μ_t on the sphere $\{z : |z| = t\}$ in $\mathbb{C}^n$. That is,

$$S_t^{n-1} f = \int_{|z|=t}\pi(z)f\,d\mu_t.$$

First we express S_t^α for Re $\alpha > n-1$ as the Weyl transform of a function. Let

$$g_t^\alpha(z) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(n)}w_{2n-1}^{-1}t^{-2n}\left(1-\frac{|z|^2}{t^2}\right)_+^{\alpha-1}e^{-\frac{1}{4}(t^2-|z|^2)}$$

where w_{2n-1} is the measure of $\{z : |z| = t\}$. Then for Re $\alpha > 0$, g_t^α is an integrable function.

Proposition 7.

Let S_t^α and g_t^α be defined as above. Then for Re $\alpha > 0$ *we have the relation* $S_t^{\alpha+n-1}f = W(g_t^\alpha)f$, $f \in L^2(\mathbb{R}^n)$.

Proof. We use the following formula which connects Laguerre polynomials of different types:

$$\frac{\Gamma(k+1)\Gamma(\mu+\nu+1)}{\Gamma(k+\mu+\nu+1)} L_k^{\mu+\nu}\left(\frac{t^2}{2}\right)e^{-\frac{t^2}{4}}$$
$$= \frac{\Gamma(k+1)\Gamma(\mu+\nu+1)}{\Gamma(\nu)\Gamma(k+\mu+1)} \int_0^1 s^\mu(1-s)^{\nu-1}e^{-\frac{t^2}{4}(1-s)} L_k^\mu\left(\frac{st^2}{2}\right)e^{-\frac{st^2}{4}}\,ds$$

which is valid for Re $\mu > -1$, Re $\nu > 0$. In the above take $\mu = n-1$ and $\nu = \alpha$. Then after a change of variables we have

$$\frac{\Gamma(k+1)\Gamma(\alpha+n)}{\Gamma(k+\alpha+n)} L_k^{\alpha+n-1}\left(\frac{t^2}{2}\right)e^{-\frac{t^2}{4}}$$
$$= \frac{\Gamma(k+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(k+n)} \int_0^1 s^{2n-1}\left(1-s^2\right)^{\alpha-1} e^{-\frac{t^2}{4}(1-s^2)} L_k^{n-1}\left(\frac{s^2t^2}{2}\right)e^{-\frac{s^2t^2}{4}}\,ds$$
$$= w_{2n-1}^{-1}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n)}\frac{k!\,(n-1)!}{(k+n-1)!}\int_{\mathbb{C}^n}\left(1-|z|^2\right)_+^{\alpha-1} e^{-\frac{t^2}{4}(1-|z|^2)}\varphi_k(tz)\,dz$$
$$= \frac{k!\,(n-1)!}{(k+n-1)!}\int_{\mathbb{C}^n} g_t^\alpha(z)\,\varphi_k(z)\,dz\,.$$

Thus we have the relation

$$\psi_k^{\alpha+n-1}(t) = \frac{k!\,(n-1)!}{(k+n-1)!}\int_{\mathbb{C}^n} g_t^\alpha(z)\,\varphi_k(z)\,dz\,.$$

Since $g_t^\alpha(z)$ is a radial function it can be expanded in terms of $\varphi_k(z)$ and we have the expansion

$$g_t^\alpha(z) = (2\pi)^{-n}\sum_{k=0}^\infty \frac{k!\,(n-1)!}{(k+n-1)!}\left(\int_{\mathbb{C}^n} g_t^\alpha(w)\,\varphi_k(w)\,dw\right)\varphi_k(z)$$

which leads to the formula

$$g_t^\alpha(z) = (2\pi)^{-n}\sum_{k=0}^\infty \psi_k^{\alpha+n-1}(t)\,\varphi_k(z)\,.$$

Taking Weyl transform of both sides and noting that $W(\varphi_k) = (2\pi)^n P_k$ we get our proposition. □

As noted above g_t^α is integrable and hence $W(g_t^\alpha)$ is a bounded operator on $L^2(\mathbb{R}^n)$ whenever Re $\alpha > 0$. We will express $W(g_t^\alpha)$ as an integral operator with an explicit kernel K_t^α which has an analytic continuation for Re $\alpha > -\frac{n}{2}-\frac{1}{2}$. Using this we will analytically continue $W(g_t^\alpha)$ agreeing with S_t^α on a bigger range of α.

Recalling the definition of the Weyl transform given in the introduction, we have the explicit formula

$$W(g)\phi(\xi) = \int_{\mathbb{C}^n} e^{i(x.\xi+\frac{1}{2}x.y)}\,g(x,y)\,\phi(\xi+y)\,dx\,dy$$

where $g(x,y)$ stands for $g(x+iy)$. Thus $W(g)$ is an integral operator with kernel

$$K_g(\xi,y) = \int_{\mathbb{R}^n} g(x,y-\xi)\,e^{\frac{i}{2}(\xi+y).x}\,dx\,.$$

In view of this formula, the kernel K_t^α of $W(g_t^\alpha)$ is given by

$$K_t^\alpha(\xi, y) = \int_{\mathbb{R}^n} g_t^\alpha(x, y-\xi)\, e^{\frac{i}{2}(\xi+y)\cdot x}\, dx .$$

In order to evaluate this integral we expand the exponential factor in the definition of g_t^α into an infinite series getting

$$g_t^\alpha(z) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(n)}\, w_{2n-1}^{-1}\, t^{-2n} \sum_{j=0}^{\infty} \frac{\left(-\frac{t^2}{4}\right)^j}{j!} \left(1 - \frac{|z|^2}{t^2}\right)^{\alpha+j-1}$$

We now define k_t^α to be the kernel

$$k_t^\alpha(\xi, y) = \frac{t^{-2n}}{\Gamma(\alpha)} \int_{\mathbb{R}^n} \left(1 - \frac{|x|^2}{t^2} - \frac{|y-\xi|^2}{t^2}\right)_+^{\alpha-1} e^{\frac{i}{2}x\cdot(\xi+y)}\, dx$$

so that K_t^α is expressed as

$$K_t^\alpha(\xi, y) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(n)} w_{2n-1}^{-1} \sum_{k=0}^{\infty} \frac{\left(-\frac{t^2}{4}\right)^j}{j!}\, \Gamma(\alpha+j)\, k_t^{\alpha+j}(\xi, y) .$$

Note that $k_t^\alpha(\xi, y)$ vanishes for $|y-\xi| \geq t$. Therefore, by putting $s^2 = 1 - \frac{|y-\xi|^2}{t^2}$ and making a change of variables in the definition of k_t^α we get

$$\begin{aligned} k_t^\alpha(\xi, y) &= \frac{t^{-2n}}{\Gamma(\alpha)} \int_{\mathbb{R}^n} \left(s^2 - \frac{|x|^2}{t^2}\right)_+^{\alpha-1} e^{\frac{i}{2}x\cdot(\xi-y)}\, dx \\ &= \frac{t^{-n}}{\Gamma(\alpha)}\, s^{2\alpha+n-2} \int_{\mathbb{R}^n} \left(1 - |x|^2\right)_+^{\alpha-1} e^{\frac{i}{2}tsx\cdot(y+\xi)}\, dx . \end{aligned}$$

The last integral is a constant multiple of the Bessel function (see Theorem 4.15 in Stein–Weiss [17])

$$\int_{\mathbb{R}^n} \left(1 - |x|^2\right)_+^{\alpha-1} e^{ix\cdot\xi}\, dx = \pi^{\frac{n}{2}}\, 2^{\alpha+\frac{n}{2}-1}\, \Gamma(\alpha)\, J_{\alpha+\frac{n}{2}-1}(|x|)\, |x|^{-\alpha+\frac{n}{2}-1} .$$

Therefore we have the formula

$$k_t^\alpha(\xi, y) = \pi^{\frac{n}{2}}\, t^{-n} \left(1 - \frac{|y-\xi|^2}{t^2}\right)_-^{\alpha+\frac{n}{2}-1} G_{\alpha+\frac{n}{2}-1}\left(\frac{1}{2}ts|y+\xi|\right)$$

where we have set $G_\alpha(r) = 2^\alpha r^{-\alpha} J_\alpha(r)$. Putting this back in the expression for K_t^α we obtain

$$K_t^\alpha(\xi, y) = c_n \Gamma(\alpha+n) t^{-n}$$

$$\sum_{k=0}^{\infty} \frac{\left(-\frac{t^2}{4}\right)^j}{j!} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \left(1 - \frac{|y-\xi|^2}{t^2}\right)_+^{\alpha+\frac{n}{2}-j-1} G_{\alpha+\frac{n}{2}+j-1}\left(\frac{1}{2}ts|y-\xi|\right) .$$

Note that for fixed y and ξ each term in the sum is holomorphic in α as long as Re $\alpha > -(\frac{n-1}{2})$. We also note that

$$\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots\cdots(\alpha+j-1)$$

is an entire function of α.

For the kernel K_t^α we now prove the following estimate.

Proposition 8.

Assume that Re $\alpha > -(\frac{n-1}{2})$. *Then*

$$\left|K_t^\alpha(\xi, y)\right| \le C_\alpha\, e^{\frac{t^2}{4}}\, t^{-n} \left(1 - \frac{|y-\xi|^2}{t^2}\right)_+^{-\frac{1}{2}}$$

where C_α is of admissible growth as a function of Im α.

Proof. We only need to check that

$$\sup_{j\ge 0}\left|\frac{\Gamma(\alpha+n)\,\Gamma(\alpha+j)}{\Gamma(\alpha)}\, G_{\alpha+\frac{n}{2}+j-1}(r)\right| \le C_\alpha$$

for all values of $r > 0$, when Re $\alpha > -(\frac{n-1}{2})$. The Bessel function $J_\alpha(r)$, for Re $\alpha > -\frac{1}{2}$ is defined by the integral

$$J_\alpha(r) = \frac{2^{-\alpha}\, r^\alpha}{\Gamma\left(\alpha+\frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^{1} e^{irs}\left(1-s^2\right)^{\alpha-\frac{1}{2}}\, ds\,.$$

Therefore, $|G_\alpha(r)| \le A|\Gamma(\alpha+\frac{1}{2})|^{-1}$ where A depends only on Re α. So we have to show that

$$\left|\frac{\Gamma(\alpha+n)\,\Gamma(\alpha+j)}{\Gamma(\alpha)\,\Gamma(\alpha+j+\frac{n-1}{2})}\right| \le C_\alpha\,.$$

When $n = 2m+1$, the left hand side reduces to $\frac{\alpha(\alpha+1)\cdots\cdots(\alpha+n-1)}{(\alpha+j)(\alpha+j+1)\cdots\cdots(\alpha+j+n-1)}$ which is certainly bounded by a constant C_α of admissible growth. When n is even we can use Stirling's formula to arrive at the same conclusion.

We can now complete the proof of Theorem 3. Consider the family of operators

$$K_t^\alpha f(x) = \int_{\mathbb{R}^n} K_t^\alpha(x,y)\, f(y)\, dy\,.$$

Note that in view of Proposition 8 this is an admissible analytic family of operators for Re $\alpha > -(\frac{n-1}{2})$ which are bounded on $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$ uniformly in $0 < t \le 1$. By the result of Proposition 7 we know that S_t^α agrees with $K_t^{\alpha-(n-1)}$ for Re $\alpha > n-1$. But S_t^α is analytic in the bigger range Re $\alpha > -\frac{1}{2}$ and so we can think of S_t^α as an analytic continuation of $K_t^{\alpha-(n-1)}$.

As in the proof of Theorem 2 we now have the estimate $\|S_t^\alpha f\|_2 \le C\|f\|_2$ for Re $\alpha > -\frac{1}{2}$, C being independent of t, for $0 < t \le 1$. For $\alpha = \frac{n-1}{2} + \delta + i\gamma$, $S_t^\alpha = K_t^{-(\frac{n-1}{2})+\delta+i\gamma}$ is bounded on $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. By analytic interpolation we get

$$\left\|S_t^\alpha f\right\|_p \le C\|f\|_p, \qquad \alpha > n\left|\frac{1}{p}-\frac{1}{2}\right| - \frac{1}{2}$$

where C is independent of t, $0 < t \le 1$. This completes the proof of Theorem 3. □

Next we proceed to show that,we can improve the above result in the case of radial functions. We need the following facts about the Hermite expansions. We refer to the monograph [21] for details. When f is a radial function Hermite expansion reduces to a Laguerre expansion. More precisely we have the following.

Theorem 8.
If f is a radial function then $P_{2k+1} f = 0$ and

$$P_{2k} f = R_k^{\frac{n}{2}-1}(f)\, L_k^{\frac{n}{2}-1}\left(r^2\right)\, e^{-\frac{1}{2}r^2} ,$$

where

$$R_k^{\frac{n}{2}-1}(f) = 2\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{n}{2}\right)} \int_0^\infty f(s)\, L_k^{\frac{n}{2}-1}\left(s^2\right)\, e^{-\frac{1}{2}s^2}\, s^{n-1}\, ds .$$

Proof. See [21, (Theorem 3.4.1)].

We also need the following facts about the Laguerre translations. The Laguerre translations $T_x^\alpha f(y)$ of a function f on $\mathbb{R}_+ = [0, \infty)$ for $\alpha \geq 0$ is defined by

$$T_x^\alpha f(y) = \frac{2^\alpha \Gamma(\alpha+1)}{\sqrt{2\pi}} \int_0^\pi f\left(\left(x^2+y^2+2xy\cos\theta\right)^{\frac{1}{2}}\right) j_{\alpha-\frac{1}{2}}(xy\sin\theta)\, \sin^{2\alpha}\theta\, d\theta ,$$

where we have written $j_\alpha(t) = t^{-\alpha} J_\alpha(t)$. The following results are proved in [21] (see p. 139 and 141). ☐

Lemma 2.
If $\psi_k^\alpha(x) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(x^2)\, e^{-\frac{1}{2}x^2}$ then $T_x^\alpha \psi_k^\alpha(y) = \psi_k^\alpha(x)\psi_k^\alpha(y)$.

Lemma 3.
For $\alpha \geq 0$ and $1 \leq p \leq \infty$ we have $\|T_x^\alpha f\|_{p,\mu} \leq \|f\|_{p,\mu}$ where $\|f\|_{p,\mu}^p = \int_0^\infty |f(x)|^p x^{2\alpha+1}\, dx$.

We refer the reader to the monograph [21] for more properties of Laguerre translations. Now define the analytic family of operators

$$R_t^\alpha f = \sum_{k=0}^\infty \psi_k^\alpha(t) P_{2k} f .$$

Here we are assuming f is a radial function. For this family we have the following results.

Proposition 9.
i) $\|R_t^\alpha f\|_1 \leq C(\alpha)\|f\|_1$, for $\mathrm{Re}\ \alpha > \frac{n}{2} - 1$
ii) $\|R_t^\alpha f\|_2 \leq C(\alpha)\|f\|_2$, for $\mathrm{Re}\ \alpha > -\frac{1}{2}$.

Proof. Since f is a radial function we have by Theorem 8

$$R_t^\alpha f(x) = \sum_{k=0}^\infty \psi_k^\alpha(t)\, R_k^{\frac{n}{2}-1}(f)\, L_k^{\frac{n}{2}-1}\left(|x|^2\right)\, e^{-\frac{1}{2}|x|^2} .$$

Hence by Lemma 2

$$R_t^{\frac{n}{2}-1} f(x) = T_t^{\frac{n}{2}-1} f(|x|)$$

and by Lemma 3 we have

$$\left\| R_t^{\frac{n}{2}-1} \right\|_{L^1(\mathbb{R}^n)} = \left\| R_t^{\frac{n}{2}-1} f \right\|_{1,\mu} \leq C\|f\|_{1,\mu} = C\|f\|_{L^1(\mathbb{R}^n)} .$$

Now as in the earlier sections we can prove (i). Similarly we can prove (ii) as the sequence $\psi_k^{-\frac{1}{2}}(t)$ form a bounded sequence and so the operator $R_t^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}^n)$. It can be checked that R_t^α

forms an admissible analytical family of operators in the sense of Stein. By analytic interpolation theorem we get the following result. ☐

Theorem 9.

Assume that $f \in L^p(\mathbb{R}^n)$ is radial. Then we have the estimate $\|R_t^\alpha f\|_p \le C\|f\|_p$, for $\alpha > (n-1)|\frac{1}{p}-\frac{1}{2}|-\frac{1}{2}$.

Again as before comparing $J_\alpha(t)\,t^{-\alpha}$ with $\psi_k^\alpha(t)$ we get the same result for the multiplier $\frac{J_\alpha(t\sqrt{2k+n})}{(\sqrt{2k+n})^\alpha}$; that is

$$\left\| \frac{J_\alpha\left(t\sqrt{H}\right)}{\sqrt{H}^\alpha} f \right\|_p \le C\|f\|_p$$

for all radial functions when $\alpha > (n-1)|\frac{1}{p}-\frac{1}{2}|-\frac{1}{2}$. Taking $\alpha = \frac{1}{2}$ we have the following corollary.

Corollary 4.

Assume $f \in L^p(\mathbb{R}^n)$ is radial. Then

$$\left\| \frac{\sin t\sqrt{H}}{\sqrt{H}} f \right\|_p \le C\|f\|_p \quad \textit{for} \quad \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{n-1} .$$

We conclude this section by proving the almost everywhere convergence result stated as Corollary 3 in the introduction. When $\alpha > \frac{n-1}{2}$, S_t^α is given by the kernel $K_t^{\alpha-(n-1)}$. Proceeding as in the proof of Proposition 8 it is easy to show that

$$\sup_{0<t\le 1} \left| S_t^{\frac{n}{2}} f(x) \right| \le C \sup_{0<t<\infty} t^{-n} \int_{|x-y|\le t} |f(x)|\, dx .$$

The right-hand side is just the Hardy–Littlewood maximal function and hence

$$\int \sup_{0<t\le 1} \left| S_t^{\frac{n}{2}} f(x) \right|^p dx \le C \int |f(x)|^p\, dx$$

for $1 < p < \infty$. Hence $S_t^{\frac{\alpha}{2}} f(x)$ converges to $f(x)$ almost everywhere as $t \to 0$.

References

[1] Dziubanski, J., Hebisch, W., and Zienkiewicz, J. (1994). Note on semi groups generated by positive Rockland operators on graded homogeneous groups, *Studia Math.*, **110**, (1994), 115–126.

[2] Giulini, S. and Meda, S. (1990). Oscillating multipliers on noncompact symmetric spaces, *J. Reine. Angew. Math.*, **409**, 93–105.

[3] Hirschman, I.I. (1959). On multiplier transformations, *Duke Math. J.*, **26**, 221–242.

[4] Hulanicki, A. and Jenkins, J. (1983). Almost everywhere summability on nilmanifolds, *Trans. Am. Math. Soc.*, **278**, 703–715.

[5] Hulanicki, A. (1984). A functional calculus for Rockland operators on nilpotent groups, *Studia Math.*, **78**, 253–266.

[6] Zhong, J. (1993). *Harmonic Analysis for Some Schrödinger Operators*, Princeton University thesis.

[7] Mauceri, G. (1981). Riesz Means for the eigenfunction expansions for a class of Hypoelliptic differential operators, *Ann. Inst. Fourier*, 31(4), 115–140.

[8] Mauceri, G. and Meda, S. (1990) Vector-valued multipliers on stratified groups, *Revist. Mat. Ibero.*, **6**, 141–154.

[9] Miyachi, A. (1980). On some estimates for the wave equation in L^p and H^p, *J. Fac. Sci. Tokyo, Sci. IA*, 27, 331–354.

[10] Miyachi, A. (1980). On some Fourier Multipliers for $H^p(\mathbb{R}^n)$, *J. Fac. Sci. Univ. Tokyo, Sect IA*, **27**, 157–179.

[11] Müller, D. and Stein, E.M. (1999). L^p estimates for the wave equation on the Heisenberg group. *Revist. Mat. Ibero.*, **15**(2), 297–334.

[12] Peral, J. (1980). L^p estimates for the wave equation, *J. Funct. Anal.*, **36**, 114–145.

[13] Ratnakumar, P.K. and Thangavelu, S. (1998). Spherical means, Wave equation and Hermite–Laguerre expansions, *J. Funct. Anal.*, **154**(2), 253–290.

[14] Schonbek, T.P. (1988). L^p multipliers: a new proof of an old theorem, *Proc. Am. Math. Soc.*, **102**, 361–364.

[15] Seeger, A. and Sogge, C.D. (1986). On the boundedness of functions of (pseudo) differential operators on compact manifolds, *Duke Math. J.*, **59**, 709–736.

[16] Sjöstrand, S. (1970). On the Riesz means of the solutions of the Schrödinger equation, *Ann. Scuola Norm Sup. Pisa*, **24**, 331–348.

[17] Stein, E.M. and Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press.

[18] Stein, E.M. (1971). *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press.

[19] Stein, E.M. (1971). Topics in harmonic analysis related to Littlewood–Paley theory. *Ann. Math. Stud.*, Princeton University Press.

[20] Szegö, G. (1967). Orthogonal polynomials, *Am. Math. Soc. Colloq. Pub.* 23, Providence, RI.

[21] Thangavelu, S. (1993). Lectures on Hermite and Laguerre expansions, *Mathematical Notes*, 42, Princeton University Press.

[22] Thangavelu, S. (1998). Harmonic analysis on the Heisenberg group, *Progress in Math.*, Vol. 159, Birkhäuser.

[23] Thangavelu, S. (2000). Some remarks on Bochner–Riesz means, *Colloq. Math.*, **83**(2), 217–230.

[24] Wainger, S. (1965). Special trigonometric series in *k* dimensions, *Mem. Am. Math. Soc.*, **59**.

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