THE THEORY OF FRACTIONAL REPLICATION IN FACTORIAL EXPERIMENTS

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INTRODUCTION

The general theory of symmetrical factorial experiments has been considered in detail by Bose and Kishen (1940), Fisher (1943, 1945) and Bose (1947). It was shown by the author (Rao, 1946, 1947) that the designs for factorial experiments can be made to depend on the existence of combinatorial arrangements known as arrays of strength d. These arrangements are also useful in the construction of multifactorial designs suggested by Plackett and Burman (1946) and the fractional replicated designs developed by Finney (1945). In this article the general theory of fractional replicated designs has been developed using the properties of arrays of strength d. Some principles underlying this theory have been given by Kempthorne (1947) and Kishen (1947).

When the levels of a factor and/or the number of factors are large, there results a large number of treatment combinations so that even a single replication of the factorial experiment becomes unwieldy. From a single replication all the main effects and interactions with the exception of those confounded with the blocks can be measured but the error cannot be estimated. From practical experience it has been found that some higher order interactions are absent or negligible so that the sum of squares due to these interactions could be combined to build up an estimate of error. It has been shown elsewhere (Rao, 1947) that when the higher order interactions can be neglected the main effects and some lower order interactions can be measured from only a subset of the treatment cominations. In such a case it is economical to use only a subset of the treatment combinations in the actual experiment.

As a practical requirement of the general problem we may state that the design should

(i) make use of only a subset of the assemblies (treatment combinations) and

(ii) admit the measurability of main effects and first order interactions unaffected by the interactions of order less than $(d-1)$ and lead to unbiased tests of singificance

2. HYPERCUBES OF STRENGTH d AND MEASUREMENT OF CONTRASTS

Let there be n factors $F_1, \ldots, F_n$ each of which can assume s values, those corresponding to the i-th being represented by $i_1, i_2, \ldots, i_s$. An ordered set

$$1_a\ 2_b \ldots n_k$$

may be called a combination or an assembly. This can, without any ambiguity, be represented by $(a\, b \ldots k)$. There are altogether s^n assemblies of which a subset of N assemblies may be called an array. This array is said to be of strength d if all the s^d assemblies corresponding to any d factors chosen out of n occur an equal number of times. Such an array, represented by (N, n, s, d), is called a hypercube of strength d.

It was shown elsewhere (Rao, 1947) that from an array of strength $(d+k-1)$,

(i) all the main effects and interactions up to order $(k-1)$ can be measured when interactions of order equal to and greater than $(d-1)$ are absent,

(ii) the expressions for main effects and interactions are simply obtained from the usual definitions by retaining only those treatment combinations present in the array, and

(iii) that these expressions belonging to different contrasts are orthogonal.

From this it follows that if a fractional replicated design consists of a subset of treatment combinations forming an array of strength $(d+1)$, then we can obtain independent estimates of main effects and first order interactions unaffected by the presence of interactions less than the order $(d-1)$.

If these treatment combinations forming an array of strength $(d+1)$ are arranged in b blocks of equal size then a sufficient condition that the main effects and first order interactions are preserved in such a design is that each block is an array of strength 2.

If b is the number of blocks of k plots each, N the number of assemblies or treatment combinations used, n the number of factors and s the levels of each factor, then the following relationships hold.

$$N = bk \qquad \dots \quad (2.1)$$

$$N-1 \geqslant {}^{n}c_{1}(s-1)+ \dots + {}^{n}c_{\frac{d+1}{2}}(s-1)^{\frac{d+1}{2}} \text{ if } (d+1) \text{ is even} \qquad \dots \quad (2.2)$$

$$N-1 \geqslant {}^{n}c_{1}(s-1)+ \dots + {}^{n}c_{\frac{d}{2}}(s-1)^{\frac{d}{2}} + {}^{n-1}c_{\frac{d}{2}}(s-1)^{\frac{d+2}{2}} \text{ if } (d+1) \text{ is odd} \dots \quad (2.3)$$

These are the inequalities given by Rao (1947) for a set of N assemblies to form an array of strength $(d+1)$.

Since each block of k plots is an array of strength 2 it follows that

$$k-1 \geqslant {}^{n}c_{1}\,(s-1) \qquad \dots \quad (2.4)$$

The main effects and first order interactions together have ${}^{n}c_{1}(s-1)$ and ${}^{n}c_{2}(s-1)^{2}$ degrees of freedom. Since all these are estimable from the above design it follows that the above degrees of freedom together with those for between blocks are less than N$-$1, the total number of degrees of freedom available.

$$N-1 \geqslant (b-1) + {}^{n}c_{1}(s-1) + {}^{n}c_{2}(s-1)^{2} \qquad \dots \quad (2.5)$$

In general, if main effects and interactions up to the order p are estimable from a fractional replicated design, when the interactions of order equal to and greater than $(d-1)$ are absent, the subset of assemblies used need be an array of strength $(d+p)$ or n whichever is smaller and each block an array of strength $(p+1)$.

3. Construction of Designs when s is a Prime or a Prime Power

The s levels of a factor may be identified with the s elements of a Galois field GF(s) represented by $\alpha_0=0, \alpha_1, \ldots, \alpha_{s-1}$. A combination of n factors can be represented by $(a\, b \ldots k)$ where the letters can assume all values in GF(s). The s^n ordered sets arising out of the combinations of n factors correspond to assemblies. The equation

$$a_1x_1+ \ldots +a_nx_n = 0 \qquad \ldots \quad (3.1)$$

in GF(s) defines a subset of s^{n-1} combinations. This contains all combinations equally repeated of any set of m factors provided m is less than the number of non-zero coefficients in the equation (3.1). If the equation

$$b_1x_1+ \ldots +b_nx_n = 0 \qquad \ldots \quad (3.2)$$

is different from (3.1), then (3.1) and (3.2) both define s^{n-2} combinations. These combinations automatically satisfy an equation of the form

$$(\lambda a_1+\mu b_1)x_1+ \ldots +(\lambda a_n+\mu b_n)x_n = 0 \qquad \ldots \quad (3.3)$$

where λ and μ are any two elements of GF(s). In general any $(n-r)$ independent homogeneous equations define s^r combinations each of which satisfies an equation obtained as a linear combination of the $(n-r)$ equations. *The necessary and sufficient condition that these s^r combinations form an array of strength d is that no equation derivable as a linear combination from the $(n-r)$ independent equations determining the subset contains less than $(d+1)$ non-null coefficients.*

If the above condition is satisfied then any combination of any set of d factors satisfies the system of equations. Also the number of solutions corresponding to any combination is the same. Hence the complete set of solutions to the above equations forms an array of strength d, which establishes the sufficiency of the condition.

If an array of strength d is derived as a solution of a set of equations then this set of equations satisfies the condition stated in the above theorem. If not let there exist $\lambda_1, \ldots, \lambda_{n-r}$ such that the linear combination of the equations with the λ's as compounding coefficients gives an equation of the form.

$$a_1x_a+ \ldots +a_ix_k = 0$$

where $i < d+1$. All combinations of the i factors $F_a, \ldots, F_k$ do not satisfy the above equation which is contrary to the assumption. Hence the necessity of the condition.

The above result gives us a method of constructing a hypercube of strength d with s^r assemblies, if one exists, involing n factors. We have to find $(n-r)$ homogen-geneous linear equations such that no equation derivable from them contains less than $(d+1)$ non-null coefficients.

If we are using a single block as a design satisfying the requirements of the problem stated in the introduction we need chose an array of strength $(d+1)$. This supplies the solution to multifactorial designs treated more fully by Plackett and Burman (1946) and Rao (1947). Let this be determined by

$$a_{i1}x_1 + \ldots + a_{in}x_n = 0, \; i = 1, 2, \ldots, m \qquad \ldots \quad (3.4)$$

If there exist t more equations

$$a_{i1}x_1 + \ldots + a_{in}x_n = 0, \; i = m+1, \ldots, m+t \qquad \ldots \quad (3.5)$$

such that the equations (3.4) and 3.5) together supply an array of strength 2, then this forms a part of the array of strength $(d+1)$ defined by eqnations (3.4) only. In fact, the array of strength $(d+1)$ can be constructed by combining the s^t sets of arrays of strength 2 obtained as solutions of the equations

$$\begin{aligned} a_{i1}x_1 + \ldots + a_{in}x_n &= 0, \quad i = 1, \ldots, m \\ a_{i1}x_1 + \ldots + a_{in}x_n &= \alpha_j, \quad i = m+1, \ldots, m+t \\ j &= 0, 1, \ldots, s-1. \end{aligned} \qquad \ldots \quad (3.6)$$

The array of strength $(d+1)$ arranged in s^t sub arrays of strength 2 can be obtained in a simple way as follows. First obtain the solutions (*i.e.*, all combinations of $x_1, \ldots, x_n$) satisfying the equations (3.4) and (3.5). There are s^{n-m-t} assemblies satisfying these equations and they form an array of strength 2. This may be called the key array in relation to the above problem. Now consider any assembly satisfying the equations (3.4) but not occuring in the key array and obtain a second array of s^{n-m-t} assemblies by adding this particular assembly (like a vector) to each of the assemblies in the key array. A third array can be obtained in a similar way by using the key array and considering an assembly satisfying the equation (3·4) but not occuring in the first two and so on. By this method s^t arrays each of strength 2 are generated. Since all the assemblies involved satisfy the equation (3.4), all the s^t sub arrays put together will be an array of strength $(d+1)$. This simple method is available because all solutions of non-homogeneous equations in (3.6) can be obtained from the solutions of the corresponding homogeneous equations by adding to them (like a vector) any particular solution of the non-homogeneous equations.

If we choose these s^t groups each containing s^{n-m-t} assemblies, then we get a design preserving the main effects and first order interactions unaffected by interactions of order less than $(d-1)$. The total number of assemblies used is s^{n-m}.

The above result gives us a method of deriving a fractional replicated design by omitting some blocks from a design containing all the assemblies. Consider a complete factorial design in which the main effects and first order interactions are preserved. In this each block will be an array of strength 2. Let at least one interaction of order greater than $(d+1)$ be confounded in the design. If there exists a design confoundding only those interactions of order greater than $(d+1)$, then each block of this design will be an array of strength $(d+1)$. The assemblies in any block of this design may be chosen for the fractional replicated design. These assemblies will be distributed in a certain number of blocks in the original design. This set of blocks supplies the fractional replicated design.

The method of construction for designs, admitting the measurability of main effects and first order interactions when all interactions of order greater than the first are absent, is illustrated in two cases. In these cases the subset of assemblies is an array of strength 4 and each block an array of strength 2.

Consider the 2^8 design in 4 blocks confounding the interactions *ABCDE, DEFGH, ABCFGH*. Each block of this design containing 2^6 assemblies is an array of strength 4. If two more idependent interactions ABF and BDG are to be confounded the total number of blocks is 2^4 with 2^4 plots in a block. The totality of the confounded interactions is

ABF	*ADFG*	*ABCDE*	*ABCFGH*
BDG	*BEFH*	*BCEFG*	*ABDEGH*
AEH	*CDEF*	*ACDFH*	
CGH	*BCDH*	*DEFGH*	
	AEEG		

From this design we need choose four blocks which together constitute a single block of the previous design. This supplies a design in 4 blocks using only a quarter of the number of assemblies.

Consider the case $s = 4$. The levels of a factor in this case are represented by 0, 1, α and α^2 satisfying the relationships, $1 = \alpha+\alpha^2$, $1+\alpha = \alpha^2$, $1+\alpha^2 = \alpha$. If $n = 5$, then the equation

$$x_1 + \alpha x_2 + x_3 + x_4 + \alpha x_5 = 0 \qquad \ldots \quad (3.7)$$

defines 4^4 assemblies which form an array of strength 4. This together with the equations

$$\begin{aligned} x_1 + x_4 + x_5 &= 0 \\ x_2 + \alpha x_4 + x_5 &= 0 \end{aligned} \qquad \ldots \quad (3.8)$$

define 4^2 assemblies which can be easily shown to form an array of strength 2. Using this as a key block we can generate, on the total, 4^2 blocks covering all the 4^4 assemblies which are solutions of (3.7) by the process of addition explained before. This supplies a design in a quarter replicate. It may be observed that the complete factorial design preserving main effects and first order interactions can be obtained by starting with the above key block and developing the other blocks till all the the assemblies are covered.

4. Summary

It is observed that when some higher order interactions are absent the lower order interactions and main effects can be defined by using only a subset of treatment combinations. This admits the possibility of experimenting with only a subset and thus reducing the cost.

When the subset satisfies the requirements of an array of strength d (introduced earlier in 1946 by the author) the actual expressions defining main effects and

desired interactions can be easily obtained The procedure is first to consider the general expression suitable when none of the higher order interactions vanish and then retain only those treatment combinations that occur in the array of strength d. An additional advantage of using such an array is that the estimates of main effects and the desired interactions all become mutually independent thus introducing great simplicity in the analysis.

The designs using only subsets of the treatment combinations can be derived from the general confounded designs, containing all the treatment combinations, by selecting only a few of the blocks. The problems of fractional replicated designs is thus linked with the general problem of confounded designs of factorial experiments. In future it may be made a rule that, whenever the general confounded designs are listed, all possible fractional replications corresponding to each design may be indicated.

The analysis of a fractional replicated design deserves some attention. The main components of the analysis of variance table are the sum of squares due to main effects and first order interactions. The rest of the sum of squares can be atributed to error except when some of the second and higher order interactions could be isolated. If these interactions are not negligible the estimate of the error is slightly enhanced but this keeps us on the safe side so far as the tests of significance are concerned.

A list of useful designs involving more than 5 factors is being prepared elsewhere and will be published shortly.

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