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# EIGENVALUES OF SYMMETRIZABLE MATRICES *

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**Abstract.**

New perturbation theorems for matrices similar to Hermitian matrices are proved for a class of unitarily invariant norms called $Q$-norms. These theorems improve known results in certain circumstances and extend Lu's theorems for the spectral norm, see [*Numerical Mathematics: a Journal of Chinese Universities*, 16 (1994), pp. 177–185] to $Q$-norms.

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## 1 Introduction.

One of the main theorems in the theory for eigenvalue variation of Hermitian matrices is the following (see, e.g., [3], [24]):

THEOREM 1.1. *Let $A$ and $\widetilde{A}$ be two $n \times n$ Hermitian matrices. Then for any unitarily invariant norm $|||\cdot|||$ we have*

$$\left|\left|\left|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(\widetilde{A})\right|\right|\right| \le \left|\left|\left|A - \widetilde{A}\right|\right|\right|. \tag{1.1}$$

Here $\mathrm{Eig}^{\downarrow}(A)$, defined for an $n \times n$ matrix $A$ with real eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, is the diagonal matrix $\mathrm{Eig}^{\downarrow}(A) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Theorem 1.1 was proved by Weyl [28, 1912] for the spectral norm and by Loewner [21, 1934] for the Frobenius norm. Also, for the Frobenius norm it is a corollary of a theorem by Hoffman and Wielandt [14, 1953], who established, more generally, a theorem for normal matrices. For all unitarily invariant norms, (1.1) was proved by Mirsky [23, 1960]. He derived it from a theorem of Lidskii [20, 1950] and Wielandt [29, 1955].

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In the past, extensions of Theorem 1.1 have been made in the following overlapping cases:

1. $A$ and $\widetilde{A}$ are similar to Hermitian matrices, in which case they are called *symmetrizable*; see Kahan [18, 1975], Bhatia, Davis and Kittaneh [6, 1991], Bhatia, Elsner and Krause [8, 1994], Li [19, 1993], Lu [22, 1994], and most recently, Bhatia, Kittaneh and Li [11].
2. $A$ and $\widetilde{A}$ are normal matrices; see Hoffman and Wielandt [14, 1953], Bhatia [2], Bhatia and Holbrook [9] and [10], and Bhatia, Davis, and McIntosh [7, 1983].
3. $A$ and $\widetilde{A}$ are similar to normal matrices. In this case they are also called *normalizable*; see Sun [25, 1984], Zhang [30, 1986], Lu [22, 1994], and most recently, Bhatia, Kittaneh, and Li [11].
4. $A$ and $\widetilde{A}$ are unitary; see Bhatia, Davis and McIntosh [7, 1983], Bhatia and Davis [5, 1984], Bhatia and Holbrook [9, 1987], and Elsner and He [12, 1993].
5. $A$ is Hermitian, while $\widetilde{A}$ is skew-Hermitian; see Sunder [27, 1982] and Ando and Bhatia [1, 1987].
6. $A$ is Hermitian, and $\widetilde{A}$ is normalizable or even arbitrary; see Kahan [18, 1975], Lu [22, 1994] and Sun [26, 1996].

Extensions of Theorem 1.1 are also available when some information on invariant subspaces of $A$ or $\widetilde{A}$ is available. But extensions of these kinds are beyond the scope of this paper. Here we concentrate on the extensions for symmetrizable matrices. Recently Bhatia, Kittaneh and Li [11] proved the following.

THEOREM 1.2. *Suppose there are invertible matrices $X$ and $\widetilde{X}$ such that $X^{-1}AX$ and $\widetilde{X}^{-1}\widetilde{A}\widetilde{X}$ are real diagonal matrices. Then*

$$\left|\!\left|\!\left|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(\widetilde{A})\right|\!\right|\!\right| \le \left[\kappa(X)\kappa(\widetilde{X})\right]^{1/2} \left|\!\left|\!\left|A - \widetilde{A}\right|\!\right|\!\right|. \tag{1.2}$$

Here $\kappa(X) \stackrel{\mathrm{def}}{=} \|X\|_2\|X^{-1}\|_2$ is the spectral condition number of $X$, and $\|\cdot\|_2$ is the spectral norm, the largest singular value. Bhatia, Kittaneh and Li [11] also obtained bounds for matrices similar to unitary, and more generally, to normal matrices. Theorem 1.2 is the best bound in its general setting. In what follows, we shall improve it under special situations: *one of the matrices is Hermitian or almost Hermitian.* Our new results extend those of Lu [22] for the spectral norm to the class of unitarily invariant—$Q$-norm.

NOTATION: Lower case letters will be used for vectors and, rarely, scalars to follow convention; capital letters will be used for matrices. Lower case Greek letters are used exclusively for scalars. $I_k$ is the $k$-dimensional identity matrix. $A^*$ is the conjugate transpose of $A$; similarly for $x^*$. $|\!|\!|\cdot|\!|\!|$ stands for a general unitarily invariant norm, while $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm; see §2 for definitions.

## 2 Unitarily invariant norms.

A matrix norm on the space of $n \times n$ complex matrices is *unitarily invariant* (see [24, pp 74–88]) if it satisfies, besides the usual properties of any norm,

1. $\|UYV\| = \|Y\|$ for any unitary matrices $U$ and $V$;
2. $\|Y\| = \|Y\|_2$ for any $Y$ having rank one.

We denote by $\|\cdot\|$ a general *unitarily invariant norm.* The most frequently used unitarily invariant norms are the spectral norm $\|\cdot\|_2$ and the Frobenius norm $\|\cdot\|_{\mathrm{F}}$. Let $Y = (y_{ij})$ be an $n \times n$ matrix and denote its singular values by

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

Then

$$(2.1)\quad \|Y\|_2 \stackrel{\text{def}}{=} \max_{\|u\|_2=1} \|Yu\|_2 = \sigma_1,\ \|Y\|_{\mathrm{F}} \stackrel{\text{def}}{=} \left( \sum_{i,\,j=1}^{n} |y_{ij}|^2 \right)^{1/2} = \left( \sum_{i=1}^{n} \sigma_i^2 \right)^{1/2}.$$

Among all unitarily invariant norms, the *Ky Fan k-norms* (see, e.g., [16, Theorem 3.4.1 on p. 195]).

$$(2.2)\quad \|Y\|_k \stackrel{\text{def}}{=} \sum_{i=1}^{k} \sigma_i = \max_{U^*U=I_k=V^*V} \mathrm{Re}\,[\mathtt{tr}\,(U^*YV)], \quad k = 1, 2, \ldots, n,$$

play an important role in proving inequalities involving all unitarily invariant norms. Here both $U$ and $V$ are $n \times k$; $\mathtt{tr}\,(\cdot)$ is the *trace* of a matrix; $\mathrm{Re}\,[\cdot]$ is the real part of a complex number. This is because of the following proposition (see also [24, pp. 86–87]).

PROPOSITION 2.1 (KY FAN [13]). *Let $Y$ and $\widetilde{Y}$ be two $n \times n$ matrices. Then $\|Y\| \leq \|\widetilde{Y}\|$ for all unitarily invariant norms if and only if $\|Y\|_k \leq \|\widetilde{Y}\|_k$ for all $1 \leq k \leq n$.*

Special unitarily invariant norms that we are interested in here are the $Q$-norms [4]. A unitarily invariant norm $\|\cdot\|$ is a *$Q$-norm* if there exists another unitarily invariant norm $\|\cdot\|'$ such that $\|Y\| = \left(\|Y^*Y\|'\right)^{1/2}$. We follow Bhatia [4] and Ando and Bhatia [1] to denote a $Q$-norm by $\|\cdot\|_Q$. Examples of $Q$-norms include the *Ky Fan p-k norms* (see [16, Problem 3 on p. 199])

$$\|Y\|_{k;p} \stackrel{\text{def}}{=} \left( \sum_{i=1}^{k} \sigma_i^p \right)^{1/p} = \|Y^*Y\|_{k;p/2}^{1/2}$$

for $p \geq 2$ and $k = 1, 2, \ldots, n$. It is easy to see that

$$(2.3)\quad \|Y\|_{1;p} = \|Y\|_{k;\infty} = \|Y\|_2, \quad \|Y\|_{n;2} = \|Y\|_{\mathrm{F}} \quad \text{and} \quad \|Y\|_{k;1} = \|Y\|_k\,.$$

In view of the fact that the singular values of $Y^*Y$ are the squares of those of $Y$, we have, as a consequence of Proposition 2.1:

PROPOSITION 2.2. *Let $Y$ and $\widetilde{Y}$ be two $n \times n$ matrices. Then $|||Y|||_Q \leq |||\widetilde{Y}|||_Q$ for all $Q$-norms if and only if $|||Y|||_{k;2} \leq |||\widetilde{Y}|||_{k;2}$ for all $1 \leq k \leq n$.*

This proposition has been used by Bhatia [4] and Ando and Bhatia [1] in their proofs. The following characterization of $|||Y|||_{k;2}$ plays an important role in our later proofs.

PROPOSITION 2.3. *We have*

$$|||Y|||_{k;2} = \max_{U^*U=I_k} \|YU\|_{\mathrm{F}} = \max_{U^*U=I_k} \|U^*Y\|_{\mathrm{F}}. \tag{2.4}$$

PROOF. It suffices to prove the first equality in (2.4) since $|||Y|||_{k;2} = |||Y^*|||_{k;2}$. By picking the $i$th column of $U$ to be the singular vector corresponding to the singular value $\sigma_i$ of $Y$, we see that

$$|||Y|||_{k;2}^2 = \sum_{i=1}^{k} \sigma_i^2 \leq \max_{U^*U=I_k} \|YU\|_{\mathrm{F}}.$$

On the other hand, it is known that the $i$th largest singular value of $YU$ is no bigger than that of $Y$ (see, e.g., [16, Lemma 3.3.1 on p. 170]), thus we have

$$\sum_{i=1}^{k} \sigma_i^2 \geq \|YU\|_{\mathrm{F}}^2,$$

as was to be shown. □

## 3 Main results.

Our first theorem says that with $Q$-norms and when $A$ is Hermitian, one can do much better than Theorem 1.2.

THEOREM 3.1. *Suppose that $A$ is Hermitian and suppose that there is an invertible matrix $\widetilde{X}$ such that $\widetilde{X}^{-1}\widetilde{A}\widetilde{X}$ is a real diagonal matrix. Then*

$$\left|\left|\left|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(\widetilde{A})\right|\right|\right|_Q \leq \kappa(\widetilde{X})^{1/4} \left|\left|\left|A - \widetilde{A}\right|\right|\right|_Q. \tag{3.1}$$

Inequality (3.1) for the special case of the spectral norm is due to Lu [22].

THEOREM 3.2. *Under the conditions of Theorem 1.2, we have*

(3.2)
$$\left|\left|\left|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(\widetilde{A})\right|\right|\right|_Q \leq \min\{\kappa(X), \kappa(\widetilde{X})\} \cdot \frac{1}{\sqrt{2}} \left(\kappa(X)\kappa(\widetilde{X}) + \frac{1}{\kappa(X)\kappa(\widetilde{X})}\right)^{1/2} \left|\left|\left|A - \widetilde{A}\right|\right|\right|_Q,$$

(3.3)
$$\left|\left|\left|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(\widetilde{A})\right|\right|\right|_Q \leq \min\{\kappa(X), \kappa(\widetilde{X})\} \left[\kappa(X)\kappa(\widetilde{X})\right]^{1/4} \left|\left|\left|A - \widetilde{A}\right|\right|\right|_Q.$$

Inequality (3.2) for the special case of the spectral norm is also due to Lu [22].

We summarize sharpness comparisons of these inequalities and inequality (1.2) for $Q$-norms as follows.

1. When $A$ is Hermitian, inequality (3.1) and inequality (3.3) are the same. Among inequalities (1.2), (3.1) and (3.2), (1.2) is the worst in this case; and (3.1) is sharper than (3.2) provided

$$\kappa(\widetilde{X}) \geq \left(\frac{9\gamma^2+3\gamma+4}{9\gamma}\right)^2 \approx 3.383\ldots,$$

   where $\gamma = \left(\frac{19}{27}+\frac{1}{9}\sqrt{33}\right)^{1/3}$; otherwise (3.2) is sharper.

2. When $\kappa(X) \approx \kappa(\widetilde{X}) \gg 1$, asymptotically the amplification factors in front of some unitarily invariant norm of $A - \widetilde{A}$ in the inequalities are roughly

$$\text{(3.4)} \qquad \begin{array}{rcl} \frac{1}{\sqrt{2}}\kappa(X)^2 & \text{in} & (3.2), \\ \kappa(X)^{3/2} & \text{in} & (3.3), \\ \kappa(X) & \text{in} & (1.2). \end{array}$$

   So (1.2) is the best, while (3.2) is the worst.

3. Inequality (1.2) is sharper than (3.2) if

$$\text{either } \kappa(\widetilde{X}) \geq \kappa(X) \geq \frac{\sqrt{2\kappa(\widetilde{X})^2-1}}{\kappa(\widetilde{X})} \text{ or } \kappa(X) \geq \kappa(\widetilde{X}) \geq \frac{\sqrt{2\kappa(X)^2-1}}{\kappa(X)};$$

   otherwise (3.2) is sharper. If (1.2) is indeed worse, it is worse by no more than a factor $\sqrt{2}$ than (3.2), since

$$\left[\kappa(X)\kappa(\widetilde{X})\right]^{1/2} \leq \sqrt{2}\min\left\{\kappa(X), \kappa(\widetilde{X})\right\}\frac{1}{\sqrt{2}}\left(\kappa(X)\kappa(\widetilde{X}) + \frac{1}{\kappa(X)\kappa(\widetilde{X})}\right)^{1/2}$$

   always.

4. Inequality (3.3) is sharper than (1.2) if

$$\text{either} \quad 1 \leq \kappa(\widetilde{X})^3 \leq \kappa(X) \quad \text{or} \quad 1 \leq \kappa(X)^3 \leq \kappa(\widetilde{X});$$

   otherwise (1.2) is sharper.

5. Inequality (3.2) is sharper than (3.3) if

$$1 \leq \kappa(X)\kappa(\widetilde{X}) \leq \left(\frac{9\gamma^2+3\gamma+4}{9\gamma}\right)^2 \approx 3.383\ldots;$$

   otherwise (3.3) is sharper.

## 4 Proofs of Theorems 3.1 and 3.2.

The following lemma is well-known; it is implicit in the proofs in Kahan [17, 1967] and [18, 1975].

LEMMA 4.1. *Suppose that $B$ and $\widetilde{B}$ are two $n \times n$ Hermitian matrices, and suppose $\alpha$ is an eigenvalue of $B - \widetilde{B}$ with corresponding eigenvector $u$, i.e., $(B - \widetilde{B})u = \alpha u$. Let $\Gamma$ be an $n \times n$ diagonal matrix with all diagonal entries nonnegative. Then*

$$\left|u^*(B\Gamma - \Gamma\widetilde{B})u\right| \geq |\alpha| u^*\Gamma u.$$

PROOF.

$$\begin{aligned} |u^*(B\Gamma - \Gamma\widetilde{B})u| &= |u^*(B - \widetilde{B})\Gamma u + u^*(\widetilde{B}\Gamma - \Gamma\widetilde{B})u| \\ &= |\alpha u^*\Gamma u + u^*(\widetilde{B}\Gamma - \Gamma\widetilde{B})u| \\ &\geq |\alpha| u^*\Gamma u, \end{aligned}$$

since $\alpha u^*\Gamma u$ is a real number and $\widetilde{B}\Gamma - \Gamma\widetilde{B}$ is skew-Hermitian implies that $u^*(\widetilde{B}\Gamma - \Gamma\widetilde{B})u$ is a pure imaginary complex number. □

LEMMA 4.2. *Let $\Gamma$ be an $n \times n$ diagonal matrix with all diagonal entries positive, and let $u$ be an $n$-vector with $\|u\|_2 = 1$. Then*

$$u^*\Gamma u \cdot u^*\Gamma^{-1}u \geq 1.$$

PROOF.

$$1 = u^*u = u^*\Gamma^{1/2}\Gamma^{-1/2}u \leq \|u^*\Gamma^{1/2}\|_2 \|\Gamma^{-1/2}u\|_2 = \left(u^*\Gamma u \cdot u^*\Gamma^{-1}u\right)^{1/2},$$

where the inequality holds because of the Cauchy-Schwarz inequality. □

LEMMA 4.3. *Let $\Gamma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$; and let $u$ be an $n$-vector with $\|u\|_2 = 1$. Then*

$$\left(u^*\Gamma^2 u\right)^{1/2} \cdot \left(u^*\Gamma^{-2}u\right)^{1/2} \leq \frac{\sigma_1^2 + \sigma_n^2}{2\sigma_1\sigma_n}.$$

This is the well-known Kantorovich's inequality [15, Theorem 7.4.41 on p.444].

PROOF of Theorem 3.1. Since $|||\cdot|||_Q$ is unitarily invariant, we may assume that $A = \Lambda$ is real diagonal. By Proposition 2.2, it suffices to prove the inequality (3.1) for all Ky-Fan 2-$k$-norms $|||\cdot|||_{k;2}$. Write $\widetilde{X}^{-1}\widetilde{A}\widetilde{X} = \widetilde{\Lambda}$, and let $\widetilde{X} = U\Gamma V^*$ be the SVD, where $U$ and $V$ are unitary and $\Gamma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Then

$$\begin{aligned} \left|\left|\left|A - \widetilde{A}\right|\right|\right|_{k;2} &= \left|\left|\left|\Lambda - \widetilde{X}\widetilde{\Lambda}\widetilde{X}^{-1}\right|\right|\right|_{k;2} \\ &= \left|\left|\left|\Lambda - U\Gamma V^*\widetilde{\Lambda}V\Gamma^{-1}U^*\right|\right|\right|_{k;2} \\ &= \left|\left|\left|U^*\Lambda U - \Gamma V^*\widetilde{\Lambda}V\Gamma^{-1}\right|\right|\right|_{k;2} \\ &= \left|\left|\left|B - \Gamma\widetilde{B}\Gamma^{-1}\right|\right|\right|_{k;2}, \end{aligned} \tag{4.1}$$

where $B = U^*\Lambda U$ and $\widetilde{B} = V^*\widetilde{\Lambda}V$ are two Hermitian matrices. Notice that

$$\mathrm{Eig}^{\downarrow}(B) = \mathrm{Eig}^{\downarrow}(A) \quad \text{and} \quad \mathrm{Eig}^{\downarrow}(\widetilde{B}) = \mathrm{Eig}^{\downarrow}(\widetilde{A}).$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the eigenvalues of $B - \widetilde{B}$, ordered so that

$$|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|.$$

Choose orthonormal vectors $u_1, \ldots, u_n$, such that $(B - \tilde{B})u_j = \alpha_j u_j$, $j = 1, 2, \ldots, n$. Then by Theorem 1.1

$$\begin{aligned} \left(\sum_{m=1}^{k} |\alpha_m|^2\right)^{1/2} = \left|\!\left|\!\left| B - \widetilde{B} \right|\!\right|\!\right|_{k;2} &\geq \left|\!\left|\!\left| \mathrm{Eig}^{\downarrow}(B) - \mathrm{Eig}^{\downarrow}(\widetilde{B}) \right|\!\right|\!\right|_{k;2} \\ &= \left|\!\left|\!\left| \mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(\widetilde{A}) \right|\!\right|\!\right|_{k;2}. \end{aligned} \tag{4.2}$$

Set $v_i \stackrel{\mathrm{def}}{=} \Gamma u_i$. Proposition 2.3 implies

$$\begin{aligned} \left|\!\left|\!\left| B - \Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2} &\geq \left(\sum_{m=1}^{k} \left\| u_m^*(B - \Gamma\widetilde{B}\Gamma^{-1}) \right\|_2^2\right)^{1/2} \\ &\geq \left(\sum_{m=1}^{k} \frac{\left| u_m^*(B - \Gamma\widetilde{B}\Gamma^{-1})v_m \right|^2}{\|v_m\|_2^2}\right)^{1/2}. \end{aligned} \tag{4.3}$$

Now, let us bound each term within $(\cdots)^{1/2}$ from below. We have

$$\begin{aligned} \frac{|u_m^*(B - \Gamma\widetilde{B}\Gamma^{-1})v_m|}{\|v_m\|_2} &= \frac{|u_m^*(B\Gamma - \Gamma\widetilde{B})u_m|}{\|v_m\|_2} \\ &\geq \frac{|\alpha_m| u_m^*\Gamma u_m}{(u_m^*\Gamma^2 u_m)^{1/2}} \qquad \text{(by Lemma 4.1)} \\ &\geq \frac{|\alpha_m|}{\sqrt{\sigma_1}} (u_m^*\Gamma u_m)^{1/2}, \end{aligned} \tag{4.4}$$

since $x^*\Gamma^2 x \leq \sigma_1 x^*\Gamma x$. Inequalities (4.3) and (4.4) yield

$$\left|\!\left|\!\left| B - \Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2} \geq \frac{1}{\sqrt{\sigma_1}} \left(\sum_{m=1}^{k} |\alpha_m|^2 u_m^*\Gamma u_m\right)^{1/2}. \tag{4.5}$$

Similarly, setting $w_i = \Gamma^{-1}u_i$, we have

$$\begin{aligned} \left|\!\left|\!\left| B - \Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2} &\geq \left(\sum_{m=1}^{k} \left\| (B - \Gamma\widetilde{B}\Gamma^{-1})u_m) \right\|_2^2\right)^{1/2} \\ &\geq \left(\sum_{m=1}^{k} \frac{|w_m^*(B - \Gamma\widetilde{B}\Gamma^{-1})u_m|^2}{\|w_m\|_2^2}\right)^{1/2}, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\frac{|w_m^*(B-\Gamma\widetilde{B}\Gamma^{-1})u_m|}{\|w_m\|_2} &= \frac{|u_m^*(\Gamma^{-1}B-\widetilde{B}\Gamma^{-1})u_m|}{\|w_m\|_2} \\ &\geq \frac{|\alpha_m|u_m^*\Gamma^{-1}u_m}{(u_m^*\Gamma^{-2}u_m)^{1/2}} \qquad \text{(by Lemma 4.1)} \\ &\geq \sqrt{\sigma_n}|\alpha_m|\left(u_m^*\Gamma^{-1}u_m\right)^{1/2}, \end{aligned} \tag{4.7}$$

since $x^*\Gamma^{-2}x \leq \frac{1}{\sigma_n}x^*\Gamma^{-1}x$. Inequalities (4.6) and (4.7) yield

$$\left|\!\left|\!\left| B-\Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2} \geq \sqrt{\sigma_n}\left(\sum_{m=1}^{k}|\alpha_m|^2 u_m^*\Gamma^{-1}u_m\right)^{1/2}. \tag{4.8}$$

Inequalities (4.5) and (4.8) imply that

$$\begin{aligned}\left|\!\left|\!\left| B-\Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2}^2 &\geq \sqrt{\frac{\sigma_n}{\sigma_1}}\left(\sum_{m=1}^{k}|\alpha_m|^2 u_m^*\Gamma u_m\right)^{1/2}\cdot\left(\sum_{m=1}^{k}|\alpha_m|^2 u_m^*\Gamma^{-1}u_m\right)^{1/2} \\ &\geq \frac{1}{\kappa(\Gamma)^{1/2}}\sum_{m=1}^{k}|\alpha_m|^2\left(u_m^*\Gamma u_m\cdot u_m^*\Gamma^{-1}u_m\right)^{1/2} \\ &\qquad \text{(by the Cauchy-Schwarz inequality)} \\ &\geq \frac{1}{\kappa(\Gamma)^{1/2}}\sum_{m=1}^{k}|\alpha_m|^2 \qquad \text{(by Lemma 4.2)} \\ &= \frac{1}{\kappa(\widetilde{X})^{1/2}}\left|\!\left|\!\left| B-\widetilde{B} \right|\!\right|\!\right|_{k;2}^2 \\ &\geq \frac{1}{\kappa(\widetilde{X})^{1/2}}\left|\!\left|\!\left| \mathrm{Eig}^{\downarrow}(A)-\mathrm{Eig}^{\downarrow}(\widetilde{A}) \right|\!\right|\!\right|_{k;2}^2 \qquad \text{(by (4.2))}.\end{aligned}$$

Inequality (3.1) for $|\!|\!|\cdot|\!|\!|_Q = |\!|\!|\cdot|\!|\!|_{k;2}$ is a consequence of (4.1) and this. □

PROOF of Theorem 3.2. Without loss of generality, we may assume that $\kappa(\widetilde{X}) \geq \kappa(X)$. Again by Proposition 2.2, it suffices to prove the inequalities for all Ky-Fan 2-$k$-norms $|\!|\!|\cdot|\!|\!|_{k;2}$. Following notation in the proof of Theorem 3.1, we have

$$\|X^{-1}\|_2\left|\!\left|\!\left| A-\widetilde{A} \right|\!\right|\!\right|_{k;2}\|X\|_2 \geq \left|\!\left|\!\left| \Lambda - X^{-1}\widetilde{X}\widetilde{\Lambda}\widetilde{X}^{-1}X \right|\!\right|\!\right|_{k;2}. \tag{4.9}$$

Let $X^{-1}\widetilde{X} = U\Gamma V^*$ be the SVD, where $U$ and $V$ are unitary and $\Gamma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$. Then

$$\begin{aligned}\left|\!\left|\!\left| \Lambda - X^{-1}\widetilde{X}\widetilde{\Lambda}\widetilde{X}^{-1}X \right|\!\right|\!\right|_{k;2} &= \left|\!\left|\!\left| \Lambda - U\Gamma V^*\widetilde{\Lambda}V\Gamma U^* \right|\!\right|\!\right|_{k;2} \\ &= \left|\!\left|\!\left| B-\Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2},\end{aligned} \tag{4.10}$$

where $B = U^*\Lambda U$ and $\widetilde{B} = V^*\widetilde{\Lambda}V$ are Hermitian matrices. Notice that

$$\mathrm{Eig}^{\downarrow}(B) = \mathrm{Eig}^{\downarrow}(A) \quad \text{and} \quad \mathrm{Eig}^{\downarrow}(\widetilde{B}) = \mathrm{Eig}^{\downarrow}(\widetilde{A}).$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the eigenvalues of $B - \widetilde{B}$, ordered so that

$$|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|;$$

and denote the corresponding orthonormal eigenvectors of $B - \widetilde{B}$ by $u_1, u_2, \ldots, u_n$.

By Theorem 3.1, we have

$$\begin{aligned}\left[\kappa(X^{-1}\widetilde{X})\right]^{1/4} \left|\!\left|\!\left| \Lambda - X^{-1}\widetilde{X}\widetilde{\Lambda}\widetilde{X}^{-1}X \right|\!\right|\!\right|_{k;2} &\geq \left|\!\left|\!\left| \mathrm{Eig}^{\downarrow}(\Lambda) - \mathrm{Eig}^{\downarrow}(\widetilde{\Lambda}) \right|\!\right|\!\right|_{k;2} \\ &= \left|\!\left|\!\left| \mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(\widetilde{A}) \right|\!\right|\!\right|_{k;2}\end{aligned}$$

Note that

$$\kappa(\Gamma) = \kappa(X^{-1}\widetilde{X}) \leq \kappa(X)\kappa(\widetilde{X}). \tag{4.11}$$

Inequality (3.3) follows by combining the above inequality with (4.9).

To prove inequality (3.2), we notice that

$$\begin{aligned}\left|\!\left|\!\left| B - \Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2} &\geq \left[\sum_{m=1}^{k} |\alpha_m|^2 \left(\frac{u_m^*\Gamma u_m}{(u_m^*\Gamma^2 u_m)^{1/2}}\right)^2\right]^{1/2}, \\ \left|\!\left|\!\left| B - \Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2} &\geq \left[\sum_{m=1}^{k} |\alpha_m|^2 \left(\frac{u_m^*\Gamma^{-1} u_m}{(u_m^*\Gamma^{-2} u_m)^{1/2}}\right)^2\right]^{1/2};\end{aligned}$$

see (4.4) and (4.7). So

$$\begin{aligned}&\left|\!\left|\!\left| B - \Gamma\widetilde{B}\Gamma^{-1} \right|\!\right|\!\right|_{k;2}^2 \\ \geq\ & \left[\sum_{m=1}^{k} |\alpha_m|^2 \left(\frac{u_m^*\Gamma u_m}{(u_m^*\Gamma^2 u_m)^{1/2}}\right)^2\right]^{1/2} \cdot \left[\sum_{m=1}^{k} |\alpha_m|^2 \left(\frac{u_m^*\Gamma^{-1} u_m}{(u_m^*\Gamma^{-2} u_m)^{1/2}}\right)^2\right]^{1/2} \\ \geq\ & \sum_{m=1}^{k} |\alpha_m|^2 \frac{u_m^*\Gamma u_m}{(u_m^*\Gamma^2 u_m)^{1/2}} \cdot \frac{u_m^*\Gamma^{-1} u_m}{(u_m^*\Gamma^{-2} u_m)^{1/2}} \\ & \qquad \text{(by the Cauchy-Schwarz inequality)} \\ \geq\ & \sum_{m=1}^{k} |\alpha_m|^2 \frac{1}{\frac{\sigma_1^2+\sigma_n^2}{2\sigma_1\sigma_n}} \qquad \text{(by Lemmas 4.2 and 4.3)} \\ =\ & \frac{2\sigma_1\sigma_n}{\sigma_1^2+\sigma_n^2} \sum_{m=1}^{k} |\alpha_m|^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\kappa(\Gamma)+\frac{1}{\kappa(\Gamma)}} \left|\left|\left| B-\widetilde{B} \right|\right|\right|_{k;2}^{2} \\
&\geq \frac{2}{\kappa(\Gamma)+\frac{1}{\kappa(\Gamma)}} \left|\left|\left| \mathrm{Eig}^{\downarrow}(A)-\mathrm{Eig}^{\downarrow}(\widetilde{A}) \right|\right|\right|_{k;2}^{2} \qquad \text{(by Theorem 1.1).}
\end{aligned}
$$

This, together with (4.9) and (4.10), proves that

$$
\begin{aligned}
\left|\left|\left| \mathrm{Eig}^{\downarrow}(A)-\mathrm{Eig}^{\downarrow}(\widetilde{A}) \right|\right|\right|_{k;2} &\leq \frac{1}{\sqrt{2}}\left(\kappa(\Gamma)+\frac{1}{\kappa(\Gamma)}\right)^{1/2} \left|\left|\left| B-\Gamma\widetilde{B}\Gamma^{-1} \right|\right|\right|_{k;2} \\
&\leq \kappa(X)\frac{1}{\sqrt{2}}\left(\kappa(\Gamma)+\frac{1}{\kappa(\Gamma)}\right)^{1/2} \left|\left|\left| A-\widetilde{A} \right|\right|\right|_{k;2}.
\end{aligned}
$$

Now use (4.11), and the fact that for $x \geq 1$ the function $x+\dfrac{1}{x}$ is monotonically increasing, to obtain the inequality (3.2). □

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## REFERENCES

1. T. Ando and R. Bhatia, *Eigenvalue inequalities associated with the cartesian decomposition*, Linear and Multilinear Algebra, 87 (1987), pp. 133–147.
2. R. Bhatia, *Analysis of spectral variation and some inequalities*, Trans. Amer. Math. Soc., 272 (1982), pp. 323–331.
3. R. Bhatia, *Perturbation Bounds for Matrix Eigenvalues*, Pitman Research Notes in Mathematics, Longman Scientific & Technical, Harlow, Essex, 1987. Published in the USA by John Wiley.
4. R. Bhatia, *Some inequalities for norm ideals*, Comm. Math. Phys., 111 (1987), pp. 33–39.
5. R. Bhatia and C. Davis, *A bound for the spectral variation of a unitary operator*, Linear and Multilinear Algebra, 15 (1984), pp. 71–76.
6. R. Bhatia, C. Davis, and F. Kittaneh, *Some inequalities for commutators and an application to spectral variation*, Aequationes Mathematicae, 41 (1991), pp. 70–78.
7. R. Bhatia, C. Davis, and A. McIntosh, *Perturbation of spectral subspaces and solution of linear operator equations*, Linear Algebra Appl., 52–53 (1983), pp. 45–67.
8. R. Bhatia, L. Elsner, and G. M. Krause, *Spectral variation bounds for diagonalisable matrices*, Aequationes Math., 54 (1997), pp. 102–107.
9. R. Bhatia and J. A. R. Holbrook, *Unitary invariance and spectral variation*, Linear Algebra Appl., 95 (1987), pp. 43–68.
10. R. Bhatia and J. A. R. Holbrook, *A softer, stronger Lidskii theorem*, Proc. Indian Acad. Sci. (Math. Sci.), 99 (1989), pp. 75–83.
11. R. Bhatia, F. Kittaneh, and R.-C. Li, *Some inequalities for commutators and an application to spectral variation. II*, Linear and Multilinear Algebra, (1996). (to appear).

12. L. Elsner and C. He, *Perturbation and interlace theorems for the unitary eigenvalue problem*, Linear Algebra Appl., 188/189 (1993), pp. 207–229.
13. K. Fan, *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proc. Nat. Acad. Sci., 37 (1951), pp. 760–766.
14. A. J. Hoffman and H. W. Wielandt, *The variation of the spectrum of a normal matrix*, Duke Math. J., 20 (1953), pp. 37–39.
15. R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
16. R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
17. W. Kahan, *Inclusion theorems for clusters of eigenvalues of Hermitian matrices*, technical report, Computer Science Department, University of Toronto, 1967.
18. W. Kahan, *Spectra of nearly Hermitian matrices*, Proc. Amer. Math. Soc., 48 (1975), pp. 11–17.
19. R.-C. Li, *Norms of certain matrices with applications to.variations of the spectra of matrices and matrix pencils*, Linear Algebra Appl., 182 (1993), pp. 199–234.
20. V. B. Lidskii, *The proper values of the sum and product of symmetric matrices*, Doklady Akademii Nauk SSSR, 75 (1950), pp. 769–772. In Russian. Translation by C. Benster available from the National Translation Center of the Library of Congress.
21. K. Löewner, *Über monotone Matrixfunktionen*, Math. Z., 38 (1934), pp. 177–216.
22. T.-X. Lu, *Perturbation bounds of eigenvalues of symmetrizable matrices*, Numerical Mathematics: a Journal of Chinese Universities, 16 (1994), pp. 177–185. In Chinese.
23. L. Mirsky, *Symmetric gauge functions and unitarily invariant norms*, Quart. J. Math., 11 (1960), pp. 50–59.
24. G. W. Stewart and J.-G. Sun, *Matrix Perturbation Theory*, Academic Press, Boston, 1990.
25. J.-G. Sun, *On the perturbation of the eigenvalues of a normal matrix*, Math. Numer. Sinica, 6 (1984), pp. 334–336. In Chinese.
26. J.-G. Sun, *On the variation of the spectrum of a normal matrix*, Linear Algebra Appl., 246 (1996), pp. 215–223.
27. V. S. Sunder, *Distance between normal operators*, Proc. Amer. Math. Soc., 84 (1982), pp. 483–484.
28. H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann., 71 (1912), pp. 441–479.
29. H. Wielandt, *An extremum property of sums of eigenvalues*, Proc. Amer. Math. Soc., 6 (1955), pp. 106–110.
30. Z. Zhang, *On the perturbation of the eigenvalues of a non-defective matrix*, Math. Numer. Sinica, 6 (1986), pp. 106–108. In Chinese.