Arch. Math. 71 (1998) 229–232
0003-889X/98/030229-04 $ 2.30/0

Archiv der Mathematik

Distance between commuting tuples of normal operators

By

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Abstract. A sharp bound is obtained for the distance between two commuting tuples of normal operators in terms of the distance between their joint spectra.

Introduction. Let $\boldsymbol{A} = (A_1, \ldots, A_m)$ be an m-tuple of linear operators on a Hilbert space $\mathscr{H}$. We can identify this in a natural way with an operator from $\mathscr{H}$ into the Hilbert space $\mathscr{H}^m$, the direct sum of m copies of $\mathscr{H}$, by putting $\boldsymbol{A}x = (A_1x, \ldots, A_mx)$. It is then natural to define the norm of $\boldsymbol{A}$ as

$$\|\boldsymbol{A}\| \;=\; \|\boldsymbol{A}^*\boldsymbol{A}\|^{1/2} \;=\; \left\| \sum_{j=1}^{m} A_j^* A_j \right\|^{1/2}. \tag{1}$$

When the operators A_j are pairwise commuting operators we say that $\boldsymbol{A}$ is a *commuting tuple*. In this case there is a well-known notion of a joint spectrum of $\boldsymbol{A}$ called the *Taylor joint spectrum* [11]. This is a compact subset of $\mathbb{C}^m$ and will be denoted here by the symbol $\sigma(\boldsymbol{A})$. The *joint spectral radius* of $\boldsymbol{A}$ is defined as

$$r(\boldsymbol{A}) = \max\{\|\boldsymbol{\lambda}\| : \boldsymbol{\lambda} \in \sigma(\boldsymbol{A})\}, \tag{2}$$

where $\|\boldsymbol{\lambda}\|$ stands for the Euclidean norm of $\boldsymbol{\lambda}$ as an element of $\mathbb{C}^m$. An analogue of the Gelfand-Beurling spectral radius formula for single operators has recently been established for commuting tuples [8]; see also [5].

When A_j are commuting normal operators, the Taylor spectrum of the tuple $\boldsymbol{A}$ coincides with the one obtained via the spectral theorem : $\sigma(\boldsymbol{A})$ is the support of a spectral measure P on $\mathbb{C}^m$ with respect to which the A_j have a joint spectral resolution

$$A_j = \int \lambda_j \, dP(\boldsymbol{\lambda}). \tag{3}$$

In this case we have

$$\|\boldsymbol{A}\| \;=\; r(\boldsymbol{A}). \tag{4}$$

In this note we prove the following theorem.

Mathematics Subject Classification (1991): 15A42, 47A10, 47A55.

Theorem 1.1. *Let* $\boldsymbol{A} = (A_1, \ldots, A_m)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_m)$ *be two commuting m-tuples of normal operators on a Hilbert space. Then*

$$\|\boldsymbol{A} - \boldsymbol{B}\| \leqq \sqrt{2} \max \{\|\boldsymbol{\lambda} - \boldsymbol{\mu}\| \ : \ \boldsymbol{\lambda} \in \sigma(\boldsymbol{A}), \boldsymbol{\mu} \in \sigma(\boldsymbol{B})\}. \tag{5}$$

This inequality is sharp.

When $\mathscr{H}$ is finite-dimensional and $m = 1$, this theorem says that if A, B are $n \times n$ normal matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$, and $\mu_1, \ldots, \mu_n$, respectively, then

$$\|A - B\| \leqq \sqrt{2} \max_{i,j} |\lambda_i - \mu_j|.$$

This has been proved earlier in [1] and in [9]. Such inequalities have long been of interest in perturbation theory; see [2, Ch.VI]. More recently, there has been interest in extending some of the classical perturbation bounds to commuting tuples; see [3], [4], [6], [7], [10]. This programme is carried further in this note.

2. Proof of the theorem. We will first prove a theorem about the distance between compact sets in $\mathbb{C}^m$.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{C}^m$ or $\mathbb{R}^m$. The convex hull of a subset M of a vector space will be denoted by conv (M).

Lemma 2.1. *Let* M, N *be finite subsets of* $\mathbb{R}^m$ *such that*

$$\langle u, v \rangle > 0 \quad \textit{for all} \quad u \in M, \ v \in N. \tag{6}$$

Then there exists a vector w *in* $\mathbb{R}^m$ *such that*

$$\langle w, u \rangle > 0 \quad \textit{for all} \quad u \in M \cup N. \tag{7}$$

Proof. From (6) we see that $\langle u, v \rangle > 0$ for all $u \in \operatorname{conv}(M)$ and $v \in \operatorname{conv}(N)$. Thus the vector 0 is neither in $\operatorname{conv}(M)$ nor in $\operatorname{conv}(N)$. We claim that it does not belong to $\operatorname{conv}(M \cup N)$ either. If it did, we would have

$$\sum \alpha_i u_i + \sum \beta_j v_j = 0$$

for some choice of vectors $u_1, \ldots, u_r$ from $M, v_1, \ldots, v_s$ from N, and positive numbers $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_s$ with $\sum \alpha_i + \sum \beta_j = 1$. This would give

$$0 \leqq \langle \sum \alpha_i u_i, \ \sum \alpha_i u_i \rangle = -\langle \sum \alpha_i u_i, \ \sum \beta_j v_j \rangle < 0,$$

which is not possible.

Now let w be the vector in $\operatorname{conv}(M \cup N)$ that has minimal norm. Then for $0 \leqq \varepsilon \leqq 1$

$$\|(1 - \varepsilon)w + \varepsilon u\|^2 \geqq \|w\|^2 \quad \text{for all} \quad u \in M \cup N.$$

Comparing the first order terms in ε, we get from this

$$\langle w, u \rangle \geqq \|w\|^2 > 0 \quad \text{for all} \quad u \in M \cup N. \qquad \square$$

Lemma 2.2. *Let* M, N *be finite subsets of* $\mathbb{C}^m$ *such that*

$$\operatorname{Re}\langle u, v \rangle \ > \ 0 \quad \textit{for all} \quad u \in M, \ v \in N.$$

Then there exists a vector w in $\mathbb{C}^m$ such that

$$\operatorname{Re}\langle w, u\rangle > 0 \quad \textit{for all} \quad u \in M \cup N.$$

Proof. Splitting all vectors into their real and imaginary parts, this statement can be derived from the one of Lemma 2.1. $\square$

Theorem 2.3. *Let X, Y be two compact subsets of $\mathbb{C}^m$. Then there exists a point γ in $\mathbb{C}^m$ such that*

$$\max_{\lambda\in X} \|\lambda - \gamma\| + \max_{\mu\in Y} \|\mu - \gamma\| \leqq \sqrt{2} \max_{\substack{\lambda\in X\\ \mu\in Y}} \|\lambda - \mu\|. \tag{8}$$

Proof. By an approximation argument, it is enough to prove this when X, Y are finite sets. Let f be the nonnegative real-valued function on $\mathbb{C}^m$ defined as

$$f(\gamma) = \max_{\lambda\in X} \|\lambda - \gamma\| + \max_{\mu\in Y} \|\mu - \gamma\|.$$

This function attains a minimum value, since X, Y are bounded. Assume, without loss of generality, that the minimum is attained at the point 0. Then,

$$f(0) = \max_{\lambda\in X} \|\lambda\| + \max_{\mu\in Y} \|\mu\| \leqq f(\gamma), \quad \text{for all} \quad \gamma. \tag{9}$$

Let $s = \max_{\lambda\in X} \|\lambda\|$, $t = \max_{\mu\in Y} \|\mu\|$, and let $M = \{\lambda \in X : \|\lambda\| = s\}$, $N = \{\mu \in Y : \|\mu\| = t\}$. Then M, N are finite subsets of $\mathbb{C}^m$. We claim that there exist $\lambda \in M$, $\mu \in N$ such that $\operatorname{Re}\langle \lambda, \mu\rangle \leqq 0$.

Suppose this is not the case. Then, by Corollary 2.2, there exists w in $\mathbb{C}^m$ such that $\operatorname{Re}\langle w, z\rangle > 0$ for all $z \in M \cup N$. Then for ε, a positive number close to 0, we have

$$\|z - \varepsilon w\|^2 = \|z\|^2 + \varepsilon^2\|w\|^2 - 2\varepsilon \operatorname{Re}\langle w, z\rangle < \|z\|^2 \quad \text{for all} \quad z \in M \cup N. \tag{10}$$

Also, for $\lambda \in X\backslash M$ and $\mu \in Y\backslash N$, we have for such an ε

$$\|\lambda - \varepsilon w\| \leqq \|\lambda\| + \varepsilon\|w\| < s, \tag{11}$$

$$\|\mu - \varepsilon w\| \leqq \|\mu\| + \varepsilon\|w\| < t. \tag{12}$$

The inequalities (10), (11), (12), show that, for small ε, we have

$$f(\varepsilon w) < s + t = f(0).$$

This contradicts (9).

So, we can choose $\lambda \in M$, $\mu \in N$, such that $\operatorname{Re}\langle \lambda, \mu\rangle \leqq 0$. For this pair we have

$$\|\lambda - \mu\| = (\|\lambda\|^2 + \|\mu\|^2 - 2\operatorname{Re}\langle \lambda, \mu\rangle)^{1/2} \geqq (s^2 + t^2)^{1/2} \geqq \frac{s+t}{\sqrt{2}}.$$

In other words,

$$\max_{\lambda\in X} \|\lambda\| + \max_{\mu\in Y} \|\mu\| \leqq \sqrt{2} \max_{\substack{\lambda\in X\\ \mu\in Y}} \|\lambda - \mu\|.$$

This proves the theorem. $\square$

Proof of Theorem 1.1. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are commuting m-tuples of normal operators, then so are $\boldsymbol{A} - \boldsymbol{\gamma I}$ and $\boldsymbol{B} - \boldsymbol{\gamma I}$, for every $\boldsymbol{\gamma}$ in $\mathbb{C}^m$. Here $\boldsymbol{\gamma I} = (\gamma_1 I, \dots, \gamma_m I)$. We have

$$\begin{aligned} \|\boldsymbol{A} - \boldsymbol{B}\| &\leqq \|\boldsymbol{A} - \boldsymbol{\gamma I}\| \ + \ \|\boldsymbol{B} - \boldsymbol{\gamma I}\| \\ &= \max_{\boldsymbol{\lambda} \in \sigma(\boldsymbol{A})} \|\lambda - \gamma\| \ + \ \max_{\boldsymbol{\mu} \in \sigma(\boldsymbol{B})} \|\mu - \gamma\|, \end{aligned}$$

because of (4). Now use Theorem 2.3 to conclude the proof. $\square$

The bound (5) is known to be sharp in the simplest case, $\dim \mathscr{H} = 2$ and $m = 1$. Just consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Acknowledgement. The third author thanks the Ministry of Science and Technology of Slovenia for a research grant, and the Indian Statistical Institute, New Delhi, for a visit during which this work was done.

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Eingegangen am 31. 8. 1997

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