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LINEAR ALGEBRA AND ITS APPLICATIONS

Orthogonality of matrices and some distance problems

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Abstract

If A and B are matrices such that $\|A+zB\| \geqslant \|A\|$ for all complex numbers z, then A is said to be orthogonal to B. We find necessary and sufficient conditions for this to be the case. Some applications and generalisations are also discussed.

Keywords: Birkhoff – James orthogonality; Derivative; Norms; Distance problems

Let A and B be two $n \times n$ matrices. The matrix A will be identified with an operator acting on an n-dimensional Hilbert space H in the usual way. The symbol $\|A\|$ stands for the norm of this operator. A is said to be orthogonal to B (in the Birkhoff–James sense [7]) if $\|A+zB\| \geqslant \|A\|$ for every complex number z. In Section 1 of this note we give a necessary and sufficient condition for A to be orthogonal to B. The special case when $B = I$ can be applied to get some distance formulas for matrices as well as a simple proof of a well-known result of Stampfli on the norm of a derivation. In Section 2 we consider the analogous problem when the norm $\|.\|$ is replaced by the Schatten p-norm. The special case $A = I$ of this problem has been studied by Kittaneh [8], and used to

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characterise matrices whose trace is zero. In Section 3 we make some remarks on how to extend some results from Section 1 to infinite-dimensional Hilbert spaces, and formulate a conjecture about orthogonality with respect to induced matrix norms.

1. The operator norm

Theorem 1.1. *A matrix A is orthogonal to B if and only if there exists a unit vector $x \in H$ such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx\rangle = 0$.*

Proof. If such a vector x exists then

$$\|A + zB\|^2 \geqslant \|(A + zB)x\|^2 = \|Ax\|^2 + |z|^2\|Bx\|^2 \geqslant \|Ax\|^2 = \|A\|^2.$$

So, the sufficiency of the condition is obvious.

Before proving the converse in full generality we make a remark that serves three purposes. It gives a proof in a special case, indicates why the condition of the theorem is a natural one, and establishes a connection with the theorem in Section 2.

It is well-known that the operator norm $\|.\|$ is not Fréchet differentiable at all points. However, if A is a point at which this norm is differentiable, then there exists a unit vector x, unique upto a scalar multiple, such that $\|Ax\| = \|A\|$, and such that for all B

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \|A + tB\| = \operatorname{Re}\left\langle \frac{A}{\|A\|}x, Bx\right\rangle.$$

See Theorem 3.1 of [1]. Using this, one can easily see that the statement of the theorem is true for all matrices A that are points of differentiability of the norm $\|.\|$.

Now let A be any matrix and suppose A is orthogonal to B. Let $A = UP$ be a polar decomposition of A with U unitary and P positive. Then we have

$$\|P + zU^*B\| \geqslant \|P\| = \|A\|$$

for all z. In other words, the distance of P to the linear span of U^*B is $\|P\|$. Hence, by the Hahn–Banach theorem, there exists a linear functional ϕ on the space of matrices such that $\|\phi\| = 1$, $\phi(P) = \|P\|$, and $\phi(U^*B) = 0$. We can find a matrix T such that $\phi(X) = \operatorname{tr}(XT)$ for all X. Since $\|\phi\| = 1$ the trace norm (the sum of singular values) of T must be 1. So, T has a polar decomposition

$$T = \left(\sum_{j=1}^{n} s_j u_j u_j^*\right) V,$$

where s_j are singular values of T in decreasing order, $\sum_{j=1}^{n} s_j = 1$, the vectors u_j form an orthonormal basis for H, and V is unitary. We have

$$\|P\| = \mathrm{tr}(PT) = \sum_{j=1}^{n} s_j \,\mathrm{tr}[Pu_j(V^*u_j)^*]$$
$$= \sum_{j=1}^{n} s_j \langle Pu_j, V^*u_j\rangle \leqslant \sum_{j=1}^{n} s_j\|Pu_j\| \leqslant \sum_{j=1}^{n} s_j\|P\| = \|P\|.$$

Hence, if k is the rank of T (i.e., $s_k \neq 0$, but $s_{k+1} = 0$), then $\|Pu_j\| = \|P\|$ for $j = 1, \ldots, k$; and hence $Pu_j = \|P\|u_j$. From the conditions for the Cauchy–Schwarz inequality to be an equality we conclude that V^*u_j is a scalar multiple of Pu_j, $j = 1, \ldots, k$. Obviously, these scalars must be positive, and so, $V^*u_j = u_j$ for all $j = 1, \ldots, k$. It follows that T is of the form

$$T = \sum_{j=1}^{k} s_j u_j u_j^*,$$

where u_j belong to the eigenspace K of P corresponding to its maximal eigenvalue $\|P\|$. Then $\phi(U^*B) = 0$ implies

$$\sum_{j=1}^{k} s_j \langle B^*Uu_j, u_j\rangle = 0.$$

If Q is the orthoprojector on the linear span of the u_j, then this equality can be rewritten as

$$\sum_{j=1}^{k} s_j \langle QB^*UQu_j, u_j\rangle = 0.$$

Since the numerical range of any operator is a convex set, there exists a unit vector $x \in K$ such that

$$0 = \langle QB^*UQx, x\rangle = \langle B^*Ux, x\rangle = \langle Ux, Bx\rangle.$$

So,

$$\langle Ax, Bx\rangle = \langle UPx, Bx\rangle = \|P\|\langle Ux, Bx\rangle = 0. \qquad \square$$

Notice that orthogonality is not a symmetric relation. The special cases when A or B is the identity are of particular interest [3,4,8,10].

Theorem 1.1 says that I is orthogonal to B if and only if $W(B)$, the numerical range of B, contains 0. For another proof of this see Remark 4 of [8].

The more complicated case when $B = I$ has been important in problems related to derivations and operator approximations. In this case the theorem (in infinite dimensions) was proved by Stampfli ([10], Theorem 2). A different proof attributed to Ando [3] can be found in [4] (p. 206). It is this proof that we have adopted for the general case.

Problems of approximating an operator by a simpler one have been of interest to operator theorists [4], numerical analysts [6], and statisticians [9]. The second special result gives a formula for the distance of an operator to the class of scalar operators. We have, by definition,

$$\operatorname{dist}(A, \mathbb{C}I) = \min_{z\in\mathbb{C}} \|A + zI\|. \tag{1.1}$$

If this minimum is attained at $A_0 = A + z_0 I$ then A_0 is orthogonal to the identity. Theorem 1.1 then says that

$$\begin{aligned}\operatorname{dist}(A, \mathbb{C}I) &= \|A_0\| = \max\{|\langle A_0x, y\rangle| \colon \|x\| = \|y\| = 1 \text{ and } x \perp y\}\\ &= \max\{|\langle Ax, y\rangle| \colon \|x\| = \|y\| = 1 \text{ and } x \perp y\}.\end{aligned} \tag{1.2}$$

This result is due to Ando [3]. We will use it to calculate the diameter of the unitary orbit of a matrix.

The *unitary orbit* of a matrix A is the set of all matrices of the form UAU^* where U is unitary. The diameter of this set is

$$\begin{aligned}d_A &= \max\{\|VAV^* - UAU^*\| \colon U, V \text{ unitary }\}\\ &= \max\{\|A - UAU^*\| \colon U \text{ unitary}\}.\end{aligned} \tag{1.3}$$

Notice that this diameter is zero if and only if A is a scalar matrix. The following theorem is, therefore, interesting.

Theorem 1.2. *For every matrix A we have*

$$d_A = 2 \operatorname{dist}(A, \mathbb{C}I). \tag{1.4}$$

Proof. For every unitary U and scalar z we have

$$\|A - UAU^*\| = \|(A - zI) - U(A - zI)U^*\| \leqslant 2\|A - zI\|.$$

So,

$$d_A \leqslant 2 \operatorname{dist}(A, \mathbb{C}I).$$

As before we choose $A_0 = A + z_0I$ and an orthogonal pair of unit vectors x and y such that

$$\operatorname{dist}(A, \mathbb{C}I) = \|A_0\| = \langle A_0x, y\rangle.$$

By the condition for equality in the Cauchy–Schwarz inequality we must have $A_0x = \|A_0\|y$. We can find a unitary U satisfying $Ux = x$ and $Uy = -y$. Then $UA_0U^*x = -\|A_0\|y$. We have

$$d_A = d_{A_0} \geqslant \|A_0x - UA_0U^*x\| = 2\|A_0\| = 2 \operatorname{dist}(A, \mathbb{C}I). \qquad \square$$

From (1.3) and (1.4) we have

$$\max\{\|AU - UA\|: U \text{ unitary}\} = 2\,\mathrm{dist}(A, \mathbb{C}I). \tag{1.5}$$

If X is any operator with $\|X\| = 1$, then X can be written as $X = \frac{1}{2}(V + W)$ where V and W are unitary. (Use the singular value decomposition of X, and observe that every positive number between 0 and 1 can be expressed as $\frac{1}{2}(e^{i\theta} + e^{-i\theta})$.) Hence we have

$$\max_{\|X\|=1} \|AX - XA\| = 2\,\mathrm{dist}(A, \mathbb{C}I). \tag{1.6}$$

Recall that the operator $\delta_A(X) = AX - XA$ on the space of matrices is called an inner derivation. The preceding remark shows that the norm of δ_A is 2 $\mathrm{dist}(A, \mathbb{C}I)$. This was proved (for operators in a Hilbert space) by Stampfli [10]. The proof we have given for matrices is simpler. In Section 4 we will show how to prove the result for infinite-dimensional Hilbert spaces.

A trivial upper bound for d_A is $2\|A\|$. This bound can be attained. For example, any block diagonal matrix of the form

$$\begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix}$$

is unitarily similar to

$$\begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix}.$$

A simple lower bound for d_A is given in our next proposition.

Proposition 1.3. *Let A be any matrix with singular values $s_1(A) \geqslant \cdots \geqslant s_n(A)$. Then*

$$d_A \geqslant s_1(A) - s_n(A). \tag{1.7}$$

Proof. Let z be any complex number with polar form $z = re^{i\theta}$. Let $A = UP$ be a polar decomposition of A. Then

$$\begin{aligned} \|A - zI\| = \|P - zU^*\| &\geqslant \inf\{\|P - zV\|: V \text{ unitary}\} \\ &= \inf\{\|P - rV\|: V \text{ unitary}\}. \end{aligned}$$

By a theorem of Fan and Hoffman, the value of the last infimum is $\|P - rI\|$ (see [5], p. 276). So

$$\begin{aligned} \min_{z\in\mathbb{C}} \|A - zI\| \geqslant \min_{r\geqslant 0} \|P - rI\| &= \min_{r\geqslant 0} \max_j |s_j - r| \\ &= \tfrac{1}{2}(s_1(A) - s_n(A)). \end{aligned}$$

The proposition now follows from Theorem 1.2. □

If A is a Hermitian matrix then there is equality in (1.7).

2. The Schatten norms

For $1 \leqslant p < \infty$, the Schatten p-norm of A is defined as

$$\|A\|_p = \left[\sum_{j=1}^{n} (s_j(A))^p\right]^{1/p},$$

where $s_1(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of A.

If $1 < p < \infty$, then the norm $\|.\|_p$ is Fréchet differentiable at every A. In this case

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \|A + tB\|_p^p = p \operatorname{Re} \operatorname{tr} |A|^{p-1} U^* B, \tag{2.1}$$

for every B, where $A = U|A|$ is a polar decomposition of A. Here $|A| = (A^*A)^{1/2}$. If $p = 1$ this is true if A is invertible. See [2] (Theorem 2.1) and [1] (Theorems 2.2 and 2.3).

As before, we say that A is orthogonal to B in the Schatten p-norm (for a given $1 \leqslant p < \infty$) if

$$\|A + zB\|_p \geqslant \|A\|_p \quad \text{for all } z. \tag{2.2}$$

The case $p = 2$ is special. The quantity

$$\langle A, B \rangle = \operatorname{tr} A^* B$$

defines an inner product on the space of matrices, and the norm associated with this inner product is $\|.\|_2$. The condition (2.2) for orthogonality is then equivalent to the usual Hilbert space condition $\langle A, B \rangle = 0$. Our next theorem includes this as a very special case.

Theorem 2.1. *Let A have a polar decomposition $A = U|A|$. If for any $1 \leqslant p < \infty$ we have*

$$\operatorname{tr} |A|^{p-1} U^* B = 0, \tag{2.3}$$

then A is orthogonal to B in the Schatten p-norm. The converse is true for all A, if $1 < p < \infty$, and for all invertible A, if $p = 1$.

Proof. If (2.3) is satisfied, then for all z

$$\operatorname{tr} |A|^p = \operatorname{tr} |A|^{p-1}(|A| + zU^*B).$$

Hence, by Hölder's Inequality ([5], p. 88),

$$\begin{aligned}\mathrm{tr}\,|A|^p &\leqslant \| |A|^{p-1}\|_q \| |A| + zU^*B\|_p = \| |A|^{p-1}\|_q \|A + zB\|_p \\ &= [\mathrm{tr}\,|A|^{(p-1)q}]^{1/q} \; \|A + zB\|_p = (\mathrm{tr}\,|A|^p)^{1/q} \|A + zB\|_p,\end{aligned}$$

where q is the index conjugate to p (i.e., $1/p + 1/q = 1$). Since

$$(\mathrm{tr}\,|A|^p)^{1-1/q} = (\mathrm{tr}\,|A|^p)^{1/p} = \|A\|_p,$$

this shows that

$$\|A\|_p \leqslant \|A + zB\|_p \quad \text{for all } z.$$

Conversely, if (2.2) is true, then

$$\|\mathrm{e}^{\mathrm{i}\theta}A + tB\|_p \geqslant \|\mathrm{e}^{\mathrm{i}\theta}A\|_p$$

for all real t and θ. Using the expression (2.1) we see that this implies

$$\mathrm{Re}\;\mathrm{tr}(|A|^{p-1} \; \mathrm{e}^{-i\theta}U^*B) = 0,$$

for all A if $1 < p < \infty$, and for invertible A if $p = 1$. Since this is true for all θ, we get (2.3). □

The following example shows that the case $p = 1$ is exceptional. If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$\|A + zB\|_1 \geqslant \|A\|_1 \quad \text{for all } z.$$

However,

$$\mathrm{tr}\; U^*B = \mathrm{tr}\; B \neq 0.$$

The ideas used in our proof of Theorem 2.1 are adopted from Kittaneh [8] who restricted himself to the special case $A = I$.

3. Remarks

Remark 3.1. Theorem 1.1 can be extended to the infinite-dimensional case with a small modification. Let A and B be bounded operators on an infinite-dimensional Hilbert space H. Then A is orthogonal to B if and only if there exists a sequence $\{x_n\}$ of unit vectors such that $\|Ax_n\| \to \|A\|$, and $\langle Ax_n, Bx_n\rangle \to 0$. Indeed, if such a sequence $\{x_n\}$ exists then

$$\|A+zB\|^2 \geqslant \|(A+zB)x_n\|^2 = \|Ax_n\|^2 + |z|^2\|Bx_n\|^2 + 2\,\mathrm{Re}(\bar{z}\langle Ax_n, Bx_n\rangle).$$

So,

$$\|A+zB\|^2 \geqslant \limsup \|(A+zB)x_n\|^2 \geqslant \|A\|^2.$$

To prove the converse we first note that Theorem 1.1 can be reformulated in the following way: if A and B are operators acting on a finite-dimensional Hilbert space H then

$$\min \|A+zB\| = \max\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1 \text{ and } y \perp Bx\}.$$

It follows that for operators A and B acting on an infinite-dimensional Hilbert space H we have

$$\min \|A+zB\| = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1 \text{ and } y \perp Bx\}.$$

This implication was proved in the special case when $B = I$ in [4] (p. 207). A slight modification of the proof yields the general case. Assume now that A is orthogonal to B. Then $\min \|A+zB\| = \|A\|$. Therefore we can find sequences of unit vectors $\{x_n\}, \{y_n\} \in H$ such that $\langle Ax_n, y_n\rangle \to \|A\|$ and $y_n \perp Bx_n$. It follows that $\|Ax_n\| \to \|A\|$, and consequently

$$y_n - \frac{Ax_n}{\|Ax_n\|} \to 0$$

and

$$\lim_{n\to\infty} \langle Ax_n, Bx_n\rangle = \lim_{n\to\infty} \|Ax_n\|\langle y_n, Bx_n\rangle = 0.$$

This completes the proof.

Remark 3.2. The statement following (1.6) about norms of derivations can also be proved for infinite-dimensional Hilbert spaces by a limiting argument.

Let H be an infinite-dimensional separable Hilbert space, and let A be a bounded operator on H. Let $\{P_n\}$ be a sequence of finite rank projections increasing to the identity. Denote by A_n the finite rank operator P_nA restricted to the range of P_n. Let $\min_{z\in\mathbb{C}} \|A_n - zI\| = \|A_n - z_nI\|$. For each n we have

$$\begin{aligned}\sup_{\|X\|\leqslant 1} \|AX - XA\| &\geqslant \sup_{\|X\|\leqslant 1} \|AP_nXP_n - P_nXP_nA\| \\ &\geqslant \sup_{\|X\|\leqslant 1} \|P_n(AP_nXP_n - P_nXP_nA)P_n\| \\ &= \sup_{\|X\|\leqslant 1} \|(P_nAP_n)(P_nXP_n) - (P_nXP_n)(P_nAP_n)\| \\ &= 2\|A_n - z_nI\|.\end{aligned}$$

Passing to a subsequence, if necessary, assume that $z_n \to z_0$. Then

$$\lim_{n\to\infty} \|A_n - z_n I\| = \|A - z_0 I\| \geqslant \operatorname{dist}(A, \mathbb{C}I).$$

Hence,

$$\sup_{\|X\|\leqslant 1} \|AX - XA\| \geqslant 2\operatorname{dist}(A, \mathbb{C}I).$$

Thus the norm of the derivation δ_A is equal to 2 $\operatorname{dist}(A, \mathbb{C}I)$.

Remark 3.3. In view of Theorem 1.1 we are tempted to make the following conjecture. Let $\|.\|$ now represent any norm on the vector space $\mathbb{C}^n$, and also the norm it induces on the space of $n \times n$ matrices acting as linear operators on $\mathbb{C}^n$. We conjecture that

$$\|A + zB\| \geqslant \|A\| \quad \text{for all } z$$

if and only if there exists a unit vector x such that $\|Ax\| = \|A\|$ and

$$\|Ax + zBx\| \geqslant \|Ax\| \quad \text{for all } z.$$

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