References

[1] R. Bhatia and P. Šemrl, *Approximate isometries on Euclidean spaces*, Amer. Math. Monthly 104 (1997), 497–504.
[2] A. A. Jafarian and A. R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal. 66 (1986), 255–261.
[3] M. Jerison, *The space of bounded maps into a Banach space*, Ann. of Math. 52 (1950), 307–327.
[4] S. Mazur et S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, C. R. Acad. Sci. Paris 194 (1932), 946–948.
[5] P. Šemrl, *Linear maps that preserve the nilpotent operators*, Acta Sci. Math. (Szeged) 61 (1995), 523–534.

Indian Statistical Institute
New Delhi 110 016, India
E-mail: rbh@isid.ernet.in

Department of Mathematics and Statistics
University of Victoria
Victoria, B.C.
Canada V8W 3P4
E-mail: sourour@math.uvic.ca

Department of Mathematics
University of Ljubljana
Jadranska 19
1000 Ljubljana, Slovenia
E-mail: peter.semrl@fmf.uni-lj.si

Received June 27, 1997 (3908)

STUDIA MATHEMATICA 134 (2) (1999)

Perturbation theorems for Hermitian elements in Banach algebras

by

RAJENDRA BHATIA (New Delhi) and DRISS DRISSI (Kuwait)

Abstract. Two well-known theorems for Hermitian elements in C^*-algebras are extended to Banach algebras. The first concerns the solution of the equation $ax - xb = y$, and the second gives sharp bounds for the distance between spectra of a and b when a, b are Hermitian.

1. Introduction. Let $\mathcal{A}$ be a complex unital Banach algebra. An element a of $\mathcal{A}$ is said to be *Hermitian* (or *conservative*) if $\|e^{ita}\| = 1$ for all real numbers t. This notion is a natural generalization of self-adjoint elements in a C^*-algebra, and has been of considerable interest in the theory of Banach algebras. See, e.g., [7].

Several properties of self-adjoint elements of C^*-algebras remain true for Hermitian elements of Banach algebras, while many others do not. For example, if a is Hermitian then $\|a\| = r(a)$, the spectral radius of a. This was proved, almost at the same time, by Browder [8], Katsnelson [11] and Sinclair [16]. All the three proofs depended on Bernstein's inequality for entire functions; in fact this theorem about Hermitian elements is *equivalent* to Bernstein's inequality [13]. Among the properties that are strikingly different from the corresponding fact in C^*-algebras is the following. If a is Hermitian and invertible, then a^{-1} need not be Hermitian. In this case, an interesting inequality has been proved by Partington [15]: if a is Hermitian and invertible, then

$$\|a^{-1}\| \le \frac{\pi}{2} r(a^{-1}), \tag{1}$$

and the inequality is sharp. Partington's proof used Kolmogorov's inequalities [12] for derivatives of functions. A different proof and a generalization were given by Haagerup and Zsidó [10].

1991 *Mathematics Subject Classification*: 46H30, 47B10.
Key words and phrases: Banach algebra, Hermitian element, spectral radius.
The second author acknowledges support from Kuwait University, Research Grant SM 163.

In this note we prove for Hermitian elements of Banach algebras analogues of two theorems that are important in the perturbation theory of spectra in C^*-algebras.

Our first theorem is about the equation

$$ax - xb = y \tag{2}$$

in Banach algebras. This equation has been studied by several authors; see [6] for a recent survey. It is well known that if $\sigma(a)$, the spectrum of a, is disjoint from $\sigma(b)$, then the equation (2) has a unique solution x for every y. Motivated by some questions in perturbation of spectral subspaces of self-adjoint operators, Bhatia, Davis and McIntosh [5] obtained a particular form of the solution of (2) when a and b are self-adjoint operators in a Hilbert space. We will prove an analogue for general Banach algebras.

THEOREM 1.1. *Let a and b be Hermitian elements of a Banach algebra with $\sigma(a)\cap\sigma(b) = \emptyset$. Let f be any function in $L^1(\mathbb{R})$ whose Fourier transform $\widehat{f}$ defined as*

$$\widehat{f}(s) = \int_{-\infty}^{\infty} e^{-its} f(t)\,dt$$

has the property that $\widehat{f}(s) = 1/s$ whenever $s \in \sigma(a)-\sigma(b)$. Then the solution of the equation (2) can be expressed as

$$x = \int_{-\infty}^{\infty} e^{-ita} y e^{itb} f(t)\,dt. \tag{3}$$

As a corollary we obtain the following.

COROLLARY 1.2. *Let a and b be Hermitian elements of a Banach algebra such that $\operatorname{dist}(\sigma(a), \sigma(b)) = \delta > 0$. Then the solution x of the equation (2) is bounded by*

$$\delta\|x\| \le \frac{\pi}{2}\|y\|. \tag{4}$$

The constant $\pi/2$ in this inequality is sharp.

This inequality follows from (3) upon using the solution of a *minimal extrapolation problem* for the Fourier transform obtained by Sz.-Nagy and A. Strausz [18] (see also [17]). This argument was used in [4] and [5]. Haagerup and Zsidó [10] use the same result to prove the inequality (1); they also obtain a significant generalization of Sz.-Nagy's result.

Our second theorem concerns perturbation of spectra.

THEOREM 1.3. *Let a and b be Hermitian elements of a Banach algebra and let $\triangle(\sigma(a), \sigma(b))$ denote the Hausdorff distance between their spectra. Then*

$$\triangle(\sigma(a), \sigma(b)) \le \frac{\pi}{2}\|a-b\|. \tag{5}$$

The constant $\pi/2$ in this inequality cannot be improved. We also have

$$\|a-b\| \le \max\{|\lambda-\mu| : \lambda \in \sigma(a),\ \mu \in \sigma(b)\}. \tag{6}$$

We should remark that for C^*-algebras the factor $\pi/2$ in (5) can be replaced by 1. This is true more generally when a, b are normal; see [1, p. 119], [3, p. 161].

The proofs of these theorems are given in Section 2. They are followed by some remarks in Section 3.

2. Proofs. Each element a of $\mathcal{A}$ induces two operators on $\mathcal{A}$, the left multiplication L_a and the right multiplication R_a, defined as $L_a(x) = ax$ and $R_a(x) = xa$ for every $x \in \mathcal{A}$. Given a, b in $\mathcal{A}$ we denote by $C_{a,b}$ the operator $C_{a,b} = L_a - R_b$. It is called a *generalized commutator*. The following facts are easy to verify:

$$\sigma(L_a) = \sigma(R_a) = \sigma(a), \tag{7}$$

$$\|L_a^n\| = \|R_a^n\| = \|a^n\| \quad \text{for all } n \ge 0, \tag{8}$$

$$\|e^{itL_a}\| = \|e^{itR_a}\| = \|e^{ita}\|. \tag{9}$$

So, if a is a Hermitian element of $\mathcal{A}$ then L_a and R_a are Hermitian elements of the Banach algebra $\mathcal{L}(\mathcal{A})$. If a and b are Hermitian, then so is $C_{a,b}$.

We will need some facts from local spectral theory; for more details see [2] and references there.

Let $T \in \mathcal{L}(X)$ and $x \in X$. We define Ω_x to be the set of $\alpha \in \mathbb{C}$ for which there exists a neighbourhood V_α of α with u analytic on V_α having values in X, such that $(\lambda - T)u(\lambda) = x$ on V_α. This set is open and contains the complement of the spectrum of T. The function u is called a *local resolvent* of T on V_α. By definition the *local spectrum* of T at x, denoted by $\sigma_x(T)$, is the complement of Ω_x, so it is a compact subset of $\sigma(T)$. The *local spectral radius* of T at x is defined as $r_x(T) = \sup\{|\lambda| : \lambda \in \sigma_x(T)\}$.

Proof of Theorem 1.1. Since L_a and R_b commute, we have

$$\sigma(C_{a,b}) = \sigma(L_a) - \sigma(R_b) = \sigma(a) - \sigma(b).$$

Since this set does not contain the point 0, $C_{a,b}$ is invertible. For each y the local spectrum $\sigma_y(C_{a,b})$ is contained in $\sigma(C_{a,b}) = \sigma(a) - \sigma(b)$. Since a and b are Hermitian, the generalized commutator $C_{a,b}$ is also Hermitian. By the characterization of Hermitians given by G. Lumer in [7, p. 46], $C_{a,b}$ is the generator of a uniformly continuous group of isometries. Using a well known result (which goes back to Colojoară–Foiaş) claiming the equality of the spectrum of a Hermitian operator with the Arveson spectrum of the generated isometry group, that is, the support of the canonical functional calculus with the Fourier transforms for integrable functions on the line (see [9] for more details), for $f \in L^1(\mathbb{R})$ with $\widehat{f}(s) = 1/s$ for all s in $\sigma(a) - \sigma(b)$ we obtain

$$C_{a,b}^{-1}(y) = \int_{-\infty}^{\infty} e^{-itC_{a,b}} y f(t)\, dt.$$

Since L_a and R_b commute, we have

$$e^{-itC_{a,b}} y = e^{-itL_a} e^{itR_b} y = e^{-ita} y e^{itb}.$$

Since the solution of the equation (2) can be written as $x = C_{a,b}^{-1}(y)$, this proves the theorem.

We should remark that the proof of the special case of this theorem for Hilbert space operators given in [4] depends on the spectral resolution of self-adjoint operators. This argument is not available in the setting of Banach algebras.

Proof of Corollary 1.2. Since $\|e^{ita}\| = \|e^{itb}\| = 1$ for all t, from (3) we get $\|x\| \le \|f\|_{L^1}\|y\|$, where f is any function that satisfies the hypothesis of Theorem 1.1. Hence $\|x\| \le (c_1/\delta)\|y\|$, where

(10) $$c_1 = \inf\{\|f\|_{L^1} : \widehat{f}(s) = 1/s \text{ for } |s| \ge 1\}.$$

By an old theorem of Sz.-Nagy and Strausz [18] (see also [17]), $c_1 = \pi/2$.

We should point out that the same argument leads to a proof of Partington's inequality (1). Since a is Hermitian, $\sigma(a)$ is a subset of $\mathbb{R}$; since it is invertible the number $\delta := \inf\{|\lambda| : \lambda \in \sigma(a)\}$ is positive. If we choose a function f_δ in $L^1(\mathbb{R})$ such that $\widehat{f_\delta}(t) = 1/t$ for $|t| \ge \delta$, then by the holomorphic functional calculus,

$$a^{-1} = \int_{-\infty}^{\infty} e^{-ita} f_\delta(t)\, dt.$$

From this we obtain the inequality (1) again by the theorem of Sz.-Nagy and Strausz. (This appears already in the paper by Haagerup and Zsidó [10].)

Proof of Theorem 1.3. The idea of the proof is essentially the same as that in [3, p. 161].

Let $\varepsilon = \frac{\pi}{2}\|a - b\|$. We have to show that if β is any point of $\sigma(b)$ then there exists a point α in $\sigma(a)$ such that $|\beta - \alpha| \le \varepsilon$. Applying a translation, we may assume that $\beta = 0$. Suppose $\delta := \inf\{|\alpha| : \alpha \in \sigma(a)\} > \varepsilon$. Then a must be invertible and $r(a^{-1}) < 1/\varepsilon$. So, using (1) we get

$$\|a^{-1}\| \le \frac{\pi}{2} r(a^{-1}) < \frac{\pi}{2\varepsilon}.$$

Hence

$$\|a^{-1}(b - a)\| \le \|a^{-1}\| \cdot \|b - a\| < 1.$$

This shows that $1 + a^{-1}(b - a)$ is invertible, and therefore, so is $b = a(1 + a^{-1}(b - a))$. This is not possible if $\sigma(b)$ contains the point 0. Hence, we must have $\delta \le \varepsilon$. Interchanging the roles of a and b, we obtain the inequality (5).

To prove (6) first note that if E and F are compact subsets of $\mathbb{R}$, then we can find a point γ such that

$$\max_{\lambda\in E} |\lambda - \gamma| + \max_{\mu\in F} |\mu - \gamma| = \max_{\lambda\in E,\, \mu\in F} |\lambda - \mu|.$$

Now, if a and b are Hermitian elements of $\mathcal{A}$ and if e is the unit element, then for any real number γ,

$$\begin{aligned}\|a - b\| &\le \|a - \gamma e\| + \|b - \gamma e\| = r(a - \gamma e) + r(b - \gamma e)\\ &= \max_{\lambda\in\sigma(a)} |\lambda - \gamma| + \max_{\mu\in\sigma(b)} |\mu - \gamma| = \max_{\lambda\in\sigma(a),\, \mu\in\sigma(b)} |\lambda - \mu|.\end{aligned}$$

This proves (6). This idea, essentially due to L. Elsner, is also used in [3, Problem VI. 8.4].

It remains to show that the factor $\pi/2$ occurring in (4) and (5) cannot be replaced by anything smaller. This follows from a construction in McEachin [14]. He has demonstrated the existence of $n \times n$ matrices A_n, B_n and Q_n such that A_n and B_n are Hermitian, $\triangle(\sigma(A_n), \sigma(B_n)) = 1$, and

$$\lim_{n\to\infty} \|A_n Q_n - Q_n B_n\|/\|Q_n\| = 2/\pi.$$

So, if a_n and b_n are the Hermitian elements in the space of operators on $n \times n$ matrices corresponding to left multiplication by A_n and right multiplication by B_n, respectively, then

$$\triangle(\sigma(a_n), \sigma(b_n)) = 1 \quad \text{and} \quad \|a_n - b_n\| \to 2/\pi.$$

This shows that the inequality (5) is sharp.

The same example shows that (4) is sharp for the algebra $A = \mathcal{L}(\mathcal{L}(\mathcal{H}))$, where $\mathcal{H}$ is a Hilbert space. Indeed, the motivation for McEachin was to show the sharpness of this inequality in this special case considered in [4] and [5].

This example can also be used to show the sharpness of (1), for which a different example was constructed by Partington [15].

3. Remarks

3.1. A local version of inequality (1) that could be useful in operator theory can be proved using the ideas in Section 2. This says that if A is an invertible Hermitian operator on a Banach space X, then for every x in X,

(11) $$\|A^{-1}x\| \le \frac{\pi}{2} r_x(A^{-1}),$$

where $r_x(T)$ denotes the local spectral radius of T at x.

3.2. A. R. Sourour has pointed out to us that if a is positive in addition to being Hermitian, then the factor $\pi/2$ occurring in (1) can be dropped. This can be proved as follows. Assume, without loss of generality, that $\sigma(a) \subset [\delta, 1]$. Let $b = 1 - a$. Then $\sigma(b) \subset [0, 1-\delta]$, and hence, $\|b\| = 1 - \delta$. Since $a^{-1} = (1-b)^{-1} = 1 + b + b^2 + \ldots$, it follows that

$$\|a^{-1}\| \leq 1 + (1-\delta) + (1-\delta)^2 + \ldots = 1/\delta = r(a^{-1}).$$

3.3. An element a of $\mathcal{A}$ is said to be *normal* if $a = h + ik$ where h and k are two commuting Hermitian elements of $\mathcal{A}$. There has been some interest in this notion [7]. However, normal elements do not have as pleasing properties as Hermitian elements. For example, the norm of a normal element need not be equal to its spectral radius. (This is always so in C^*-algebras.)

In [5] Bhatia, Davis and McIntosh have studied the equation (2) in C^*-algebras with a and b normal. The idea that we have used in the proof of Theorem 1.1 can be used to extend Theorem 4.2 of [5] (see also Theorem 9.5 in [6]) to the setting of Banach algebras. Now, the solution of (2) is expressed as a two-dimensional Fourier transform. This leads to a minimal extrapolation problem for the two-dimensional Fourier transform: evaluate the constant

$$c_2 = \inf \left\{ \|f\|_{L^1(\mathbb{R}^2)} : \widehat{f}(s_1, s_2) = \frac{1}{s_1 + is_2} \text{ whenever } s_1^2 + s_2^2 \geq 1 \right\}. \tag{12}$$

It has been shown in [4] that $c_2 < 2.91$. However, the problem of finding the exact value of c_2 remains open.

The inequalities proved in this paper for Hermitian elements can thus be extended to normal elements with larger constants replacing $\pi/2$. The interested reader can easily work out the details.

Acknowledgements. We are grateful to Professor J. Zemánek for providing several references and encouraging us to write this paper, and to Professor A. R. Sourour for helpful conversations.

References

[1] B. Aupetit, *A Primer on Spectral Theory*, Springer, 1991.

[2] B. Aupetit and D. Drissi *Local spectrum and subharmonicity*, Proc. Edinburgh Math. Soc. 39 (1996), 571–579.

[3] R. Bhatia, *Matrix Analysis*, Springer, 1997.

[4] R. Bhatia, C. Davis and P. Koosis, *An extremal problem in Fourier analysis with applications to operator theory*, J. Funct. Anal. 82 (1989), 138–150.

[5] R. Bhatia, C. Davis and A. McIntosh, *Perturbation of spectral subspaces and solution of linear operator equations*, Linear Algebra Appl. 52-53 (1983), 45–67.

[6] R. Bhatia and P. Rosenthal, *How and why to solve the operator equation $AX - XB = Y$*, Bull. London Math. Soc. 29 (1997), 1–21.

[7] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, Cambridge Univ. Press, 1971.

[8] A. Browder, *On Bernstein's inequality and the norm of Hermitian operators*, Amer. Math. Monthly 78 (1971), 871–873.

[9] D. E. Evans, *On the spectrum of a one-parameter strongly continuous representation*, Math. Scand. 39 (1976), 80–82.

[10] U. Haagerup and L. Zsidó, *Resolvent estimate for Hermitian operators and a related minimal extrapolation problem*, Acta Sci. Math. (Szeged) 59 (1994), 503–524.

[11] V. E. Katsnelson, *A conservative operator has norm equal to its spectral radius*, Mat. Issled. 5 (1970), 186–189 (in Russian).

[12] A. N. Kolmogorov, *On inequalities between upper bounds of the successive derivatives of an arbitrary function on an infinite interval*, Uchen. Zap. Moskov. Gos. Univ. Mat. 30 (1939), 3–16 (in Russian); English transl.: Amer. Math. Soc. Transl. 4 (1949), 233–243.

[13] B. Ya. Levin, *Lectures on Entire Functions*, Transl. Math. Monographs 150, Amer. Math. Soc., 1996.

[14] R. McEachin, *A sharp estimate in an operator inequality*, Proc. Amer. Math. Soc. 115 (1992), 161–165.

[15] J. R. Partington, *The resolvent of a Hermitian operator on a Banach space*, J. London Math. Soc. (2) 27 (1983), 507–512.

[16] A. M. Sinclair, *The norm of a Hermitian element in a Banach algebra*, Proc. Amer. Math. Soc. 28 (1971), 446–450.

[17] B. Sz.-Nagy, *Über die Ungleichung von H. Bohr*, Math. Nachr. 9 (1953), 255–259.

[18] B. Sz.-Nagy and A. Strausz, *On a theorem of H. Bohr*, Mat. Termész. Értes. 57 (1938), 121–133 (in Hungarian).

Indian Statistical Institute
New Delhi 110 016, India
E-mail: rbh@isid.ernet.in

Department of Mathematics
Kuwait University
P.O. Box 5969
Safat 13060, Kuwait
E-mail: drissi@math-1.sci.kuniv.edu.kw

Received July 2, 1997
Revised version October 13, 1998 (3912)