Optimality of a class of efficiency-balanced designs

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Abstract

A class of binary efficiency-balanced (called GB-EB) designs are shown to be D- and E_f-optimal among all block designs with given replication numbers. These designs have very high A- and D-efficiencies among designs with given v, b and k. Under certain conditions, it is shown that the GB-EB designs can be used to obtain optimal row-column designs.

Keywords: BIB; PBIB; EB; GB-EB designs; D-, E_f-optimality; A-, D-efficiency; Block designs; Row-column designs

1. Introduction

We consider a block design d with v treatments and b blocks of size k each, which is said to be *proper*. For it, let $R = \text{diag}(r_1, \ldots, r_v)$, $\boldsymbol{r} = (r_1, \ldots, r_v)'$ and $N_d = (n_{dij})$, the usual $v \times b$ incidence matrix of d, where n_{dij} is the number of times the ith treatment occurs in the jth block and r_i is the replication number of the ith treatment, for $i = 1, \ldots, v$ and $j = 1, \ldots, b$. Under the usual additive homoscedastic linear model, the coefficient matrix of the reduced normal equation is given by $C_d = R - k^{-1} N_d N_d'$ which has zero row sums. A design is said to be connected if rank $(C) = v - 1$. For a connected block design d the non-zero eigenvalues of $R^{-1} C_d$ are called the *canonical efficiency-factors* of d. A connected block design d is said to be *efficiency-balanced* (EB) if and only if the canonical efficiency-factors of d are all equal. A necessary and sufficient condition, due to Williams (1975), for a proper connected design d to be EB is that its incidence matrix satisfies the equality $N_d N_d' = k(1 - e)R + (e/b)\boldsymbol{r}\boldsymbol{r}'$, where $0 < e \leqslant 1$ is the unique canonical efficiency-factor of d. The simplicity in the analysis of EB designs has been noted by various authors but, with the availability of computers, this property alone does not seem to be very attractive and further statistical justification, through optimality considerations, is called for. It, however, appears that not much work has been reported on the optimality aspects of EB designs, especially in the non-equireplicate case, except for two recent papers – one by Mukerjee and Saha (1990), in which the optimality of EB designs has been studied in classes of competing designs with unequal replications and unequal block sizes and the other paper by Das and Dey (1991) studies optimality of some non-binary proper EB designs. The present paper investigates optimality of (generalized) binary proper EB designs.

A proper block design such that n_{dij}, $i=1,\ldots,v$; $j=1,\ldots,b$, can take one of two possible values, x and $x+1$, where $x=[k/v]$, i.e. the largest integer not exceeding k/v, is called binary when $x=0$ (or $k<v$) and *generalized binary* when $x>0$ (or $k\geqslant v$). Such designs have a significant role in theory of optimal designs since they maximize the trace of the C-matrix (see Shah and Sinha, 1989). Here the generalized binary proper EB design is denoted by a GB-EB design. A binary and proper EB design is a balanced incomplete block (BIB) design. The optimality of such a design is well known (cf. Kiefer, 1975). Kageyama and Das (1991) showed that a GB-EB design has at most two distinct replication numbers. There is a large number of equireplicate GB-EB designs which are in fact balanced block designs (BBD). Again the optimality of BBD is well established (cf. Kiefer, 1975). However, the optimality of GB-EB designs with two different replication numbers has been an open problem since their characterizations and construction were obtained by Angelis et al. (1994) and Das and Kageyama (1994).

For the sake of completeness, in Section 2, we reproduce some results on construction of GB-EB designs from Das and Kageyama (1994). In Section 3, first, optimality of GB-EB designs is shown. Then for GB-EB designs we obtain the A- and D-efficiency lower bounds, considering a broader class of competing designs. In Section 4 we give some remarks and provide a table of GB-EB designs along with their A- and D-efficiency lower bounds. Finally, in Section 5 we show how EB row-column designs can be obtained from GB-EB designs and establish the optimality of such EB row-column designs.

2. GB-EB designs

We consider GB-EB designs with $v=v_1+v_2$ ($v_i>0$, $i=1,2$) treatments in which v_1 treatments are replicated r_1 times and v_2 treatments are replicated r_2 times, $r_1\neq r_2$. Without loss of generality, let $r_1<r_2$. Since we are in a generalized binary set-up, $r_1\geqslant bx$ and $r_2\leqslant b(x+1)$. Let $r_1=bx+s, r_2=b(x+1)-t, s\geqslant 0, t\geqslant 0, s+t<b$, $x>0$.

Here, we denote by $\otimes$ the Kronecker product of matrices, I_n the identity matrix of order n, and by $J_{n,m}$ the $n\times m$ matrix of ones. $1_n=J_{n,1}$ and $J_n=J_{n,n}$. A BIB design with parameters (v,b,k) is denoted by BIB(v,b,k) and a 2-associate class partially balanced incomplete block (PBIB) design with parameters $v,b,k,n_1,n_2,\lambda^{(1)}$ and $\lambda^{(2)}$ based on any association scheme is denoted by PBIB $(v,b,k,n_1,n_2,\lambda^{(1)},\lambda^{(2)})$.

For definition of BIB and PBIB designs we refer to Shah and Sinha (1989).

The GB-EB designs can be classified into the three classes as (i) EB designs with $v\geqslant 2$ and $e=1$; (ii) EB designs with $v=2$ and $e<1$; and (iii) EB designs with $v\geqslant 3$ and $e<1$. The EB designs of (i) are orthogonal designs and their incidence matrix is given by $(x1'_{v_1}:(x+1)1'_{v_2})'1'_b$ with $v_1=v(x+1)-k$ and $v_2=k-vx$. The EB design of (ii) is given by the incidence matrix as

$$\begin{bmatrix} x1'_{b-s} & (x+1)1'_s \\ (x+1)1'_{b-s} & x1'_s \end{bmatrix} \quad \text{with } e=\frac{b^2x(x+1)}{\{b^2x(x+1)+s(b-s)\}}.$$

The EB designs of (iii) are based on certain BIB and PBIB designs. We first give a necessary condition for the existence of such EB designs.

Lemma 2.1. *A GB-EB design with v ($\geqslant 3$) treatments, b blocks of size k each and $e<1$ exists only if there are $r_1=bx+s$ and $r_2=b(x+1)-t$ such that*

(i) $b\geqslant v$,

(ii) $1<s<\min\{(b-1)/2, b(k/v-x), b(x+1-k/v)-1\}$,

(iii) $s<t<\min\{2s, b-s, b(x+1-k/v)\}$,

(iv) $bx(k-x)(t-s)=s(b-t)(k-2x-1)$,

(v) $(vr_2 - bk)/(r_2 - r_1) = v_1$, an integer,
(vi) for $v_1 \neq 1, bx(x+1)r_1/r_2 = \lambda_1$, an integer,
(vii) for $v_1 \neq v-1$. $bx(x+1)r_2/r_1 = \lambda_2$, an integer.

The three theorems below give construction methods for the EB designs of class (iii) along with an infinite series of designs for $v_1 \neq 1$ or $v-1$.

Theorem 2.1. *Let $v(=v_1+v_2), b, k, r_1$ and r_2 satisfy the necessary conditions in Lemma 2.1. Then the incidence matrix of a GB-EB design with parameters $v_1 = 1, v_2 = v-1, b, k, r_1 = bx+s, r_2 = b(x+1)-t$ and $e = b^2x(x+1)/(r_1r_2)$ is given by*

$$\begin{bmatrix} x1'_{b_1} & (x+1)1'_{b_2} \\ N_1 + xJ & N_2 + xJ \end{bmatrix} \quad \text{with } b_1 = b-s \text{ and } b_2 = s,$$

where N_i is the $(v-1)\times b_i$ incidence matrix $(i=1,2)$ of either a BIB design such that N_1 corresponds to BIB $(v-1, b-s, k-vx)$ and N_2 corresponds to BIB $(v-1, s, k-vx-1)$ or a PBIB design such that N_1 corresponds to PBIB $(v-1, b-s, k-vx, n_1, n_2, \lambda^{(1)}, \lambda^{(2)})$ and N_2 corresponds to PBIB $(v-1, s, k-vx-1, n_1, n_2, \lambda-\lambda^{(1)}, \lambda-\lambda^{(2)})$ with $\lambda = \lambda_2 - x(2b-2t+xb), \lambda_2 = bx(x+1)r_2/r_1$ based on the same association scheme, for some integers $n_1, n_2, \lambda^{(1)}$ and $\lambda^{(2)}$.

Theorem 2.2. *Let $v(=v_1+v_2), b, k, r_1$ and r_2 satisfy the necessary conditions in Lemma 2.1. Then the incidence matrix of a GB-EB design with parameters $v_1 = v-1, v_2 = 1, b, k, r_1 = bx+s, r_2 = b(x+1)-t$ and $e = b^2x(x+1)/(r_1r_2)$ is given by*

$$\begin{bmatrix} N_1 + xJ & N_2 + xJ \\ x1'_{b_1} & (x+1)1'_{b_2} \end{bmatrix} \quad \text{with } b_1 = t \text{ and } b_2 = b-t.$$

where N_i is the $(v-1)\times b_i$ incidence matrix $(i=1,2)$ of either a BIB design such that N_1 corresponds to BIB $(v-1, t, k-vx)$ and N_2 corresponds to BIB $(v-1, b-t, k-vx-1)$ or a PBIB design such that N_1 corresponds to PBIB $(v-1, t, k-vx, n_1, n_2, \lambda^{(1)}, \lambda^{(2)})$ and N_2 corresponds to PBIB $(v-1, b-t, k-vx-1, n_1, n_2, \lambda-\lambda^{(1)}, \lambda-\lambda^{(2)})$ with $\lambda = \lambda_1 - x(xb+2s), \lambda_1 = bx(x+1)r_1/r_2$ based on the same association scheme, for some integers $n_1, n_2, \lambda^{(1)}$ and $\lambda^{(2)}$.

Theorem 2.3. *For a pair of positive integers x and n, the following N is the incidence matrix of a GB-EB design with parameters:*

$$v_1 = (x+1)n+1, \quad v_2 = xn+1, \quad b = 2(xn+1)\{(x+1)n+1\},$$
$$k = 2x\{(x+1)n+1\}+1, \quad r_1 = (xn+1)[2x\{(x+1)n+1\}+1],$$
$$r_2 = \{(x+1)n+1\}[2x\{(x+1)n+1\}+1],$$
$$e = \frac{4x(x+1)(xn+1)\{(x+1)n+1\}}{[2x\{(x+1)n+1\}+1]^2},$$
$$N = \begin{bmatrix} 1'_{v_2} \otimes (I_{v_1} + xJ_{v_1}) & xJ_{v_1, v_1v_2} \\ \{(x+1)J_{v_2} - I_{v_2}\} \otimes 1'_{v_1} & (x+1)J_{v_2, v_1v_2} \end{bmatrix}.$$

Apart from the above series, recently Kherwa and Prasad (1995) have given a new series of GB-EB designs. An exhaustive list of possible GB-EB designs with $e<1$ in the parametric range of $2<v<k\leqslant 20$ and $b\leqslant 100$, which satisfies the necessary conditions in Lemma 2.1, originally includes 121 designs. After accounting for

multiples of existing smaller designs, we have a list of 45 designs and their multiples. Such information is given in Section 4 in a tabular form.

3. Optimality of GB-EB designs

In practical situations, when the use of non-equireplicate design is contemplated, it often happens that there are restrictions on the availability of the treatment and, as such, a particular replication pattern has to be followed. In such situations, in addition to the number of treatments, the number of blocks and the block sizes being fixed, the replication numbers may also be fixed a priori from a practical standpoint. Hence, while studying the optimality aspects of a non-equireplicate EB design d^*, it is often natural to restrict attention to a class of designs having the same number of treatments, the same number of blocks, the same block sizes and the same replication numbers as d^*. We denote by $D(v,b,k,\mathbf{r})$ the class of all connected block designs having v treatments, b blocks each of size k and replication vector $\mathbf{r}$. For designs having two distinct replication numbers, we denote by $D(v_1,v_2,b,k,r_1,r_2)$, the class of all connected block designs having v_1+v_2 treatments, b blocks each of size k and $v_1(v_2)$ treatments replicated $r_1(r_2)$ times. We first state two results which follows from Mukerjee and Saha (1990). For the definition of various optimality criteria, one may refer to the monograph of Shah and Sinha (1989).

Lemma 3.1. *Let d^* be an EB design and suppose*

$$\sum_{i=1}^{v} r_i^{-1} \sum_{j=1}^{b} n_{dij}^2 \geqslant \sum_{i=1}^{v} r_i^{-1} \sum_{j=1}^{b} n_{d^*_{ij}}^2 \tag{3.1}$$

for any $d \in D(v,b,k,\mathbf{r})$. Then d^* is D-optimal in $D(v,b,k,\mathbf{r})$ for the estimation of every complete set of $v-1$ linearly independent treatment contrasts.

Let ϕ_d denote the minimum efficiency (or equivalently the minimum non-zero eigenvalue of $R^{-1}C$) among the efficiencies of the treatment contrasts in d, where efficiency is relative to the corresponding (unblocked) completely randomized design with the same replication numbers as in the block design d. We will call a design d^* to be E_f-optimal if $\phi_{d^*} \geqslant \phi_d$ for any other competing d. In other words, an E_f-optimal design maximizes the minimum canonical efficiency-factor among all competing designs. Note that the E_f-optimality criterion is analogous to the E-optimality criterion and they are equivalent under the equireplicate class of designs.

Lemma 3.2. *Let d^* be an EB design and suppose the condition* (3.1) *holds for any $d \in D(v,b,k,\mathbf{r})$. Then, d^* is E_f-optimal in $D(v,b,k,\mathbf{r})$.*

We now consider GB-EB designs with exactly two distinct replication numbers. Let G_1, G_2 be nonempty sets which provide a disjoint partition of $\{1,2,\ldots,v\}$. Also let $|G_1| = v_1$ and $|G_2| = v - v_1 = v_2$ where $|W|$ denotes the cardinality of the set W. Let the treatments in $G_1(G_2)$ be replicated $r_1(r_2(>r_1))$ times. Before we come to our main theorem, we need the following lemma which is easy to prove.

Lemma 3.3. *Let $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ be integers such that $\sum_{i=1}^{n} x_i = t$ where n and t are some given positive integers. Then the minimum of $\sum_{i=1}^{n} x_i^2$ is obtained when $n(m+1)-t$ of x_i's are equal to m and $t-nm$ of x_i's are equal to $m+1$ where $m=[t/n]=$ largest integer contained in t/n. The minimum value attained is $t(2m+1)-nm(m+1)$.*

We now show the optimality of GB-EB designs.

Theorem 3.1. *Let d^* be a GB-EB design in $D(v_1, v_2, b, k, r_1, r_2)$. Then d^* is D- and E_j-optimal in $D(v_1, v_2, b, k, r_1, r_2)$.*

Proof. For every $d \in D(v_1, v_2, b, k, r_1, r_2)$, we have using Lemma 3.3:

$$\begin{aligned}\sum_{i=1}^{v} r_i^{-1} \sum_{j=1}^{b} n_{dij}^2 &= r_1^{-1} \sum_{i \in G_1} \sum_{j} n_{dij}^2 + r_2^{-1} \sum_{i \in G_2} \sum_{j} n_{dij}^2 \\ &\geqslant r_1^{-1} v_1 \{r_1(2[r_1/b]+1) - b[r_1/b]([r_1/b]+1)\} \\ &\quad + r_2^{-1} v_2 \{r_2(2[r_2/b]+1) - b[r_2/b]([r_2/b]+1)\}. \end{aligned} \tag{3.2}$$

Now since $r_1 \geqslant bx$, $r_2 \leqslant b(x+1)$ and $r_1 < r_2$, it follows that $[r_1/b]=x$ and $[r_2/b]=x$ or $x+1$ depending on $r_2 < b(x+1)$ or $r_2 = b(x+1)$. Hence, from (3.2) we have, for $r_2 < b(x+1)$,

$$\begin{aligned}\sum_{i=1}^{r} r_i^{-1} \sum_{j=1}^{b} n_{dij}^2 &\geqslant r_1^{-1} v_1 \{r_1(2x+1) - bx(x+1)\} + r_2^{-1} v_2 \{r_2(2x+1) - bx(x+1)\} \\ &= r_1^{-1} \sum_{i \in G_1} \sum_{j} n_{d^*ij}^2 - r_2^{-1} \sum_{i \in G_2} \sum_{j} n_{d^*ij}^2 \end{aligned}$$

and, for $r_2 = b(x+1)$

$$\begin{aligned}\sum_{i=1}^{v} r_i^{-1} \sum_{j=1}^{b} n_{dij}^2 &\geqslant r_1^{-1} v_1 \{r_1(2x+1) - bx(x+1)\} + r_2^{-1} v_2 \{b(x+1)^2\} \\ &= r_1^{-1} \sum_{i \in G_1} \sum_{j} n_{d^*ij}^2 + r_2^{-1} \sum_{i \in G_2} \sum_{j} n_{d^*ij}^2. \end{aligned}$$

Now appealing to Lemmas 3.1 and 3.2 we get the desired result. □

It may be of further interest to see how the GB-EB designs d^* perform in a broader class of designs where the replication vector is arbitrary. Let $D(v, b, k)$ denote the class of connected block designs with v treatments, b blocks of each size k. For a block design $d \in D(v, b, k)$, let $0 = z_{d0} < z_{d1} \leqslant z_{d2} \leqslant \cdots \leqslant z_{d,v-1}$ be the eigenvalues of C_d. Also, let $\phi_A(d) = \sum_{i=1}^{v-1} z_{di}^{-1}$ (= A-value of d) and $\phi_D(d) = \prod_{i=1}^{v-1} z_{di}^{-1}$ (= D-value of d). A design is A-optimal (D-optimal) if it minimizes $\phi_A(d)(\phi_D(d))$ over all the designs in $D(v, b, k)$. The A- and D-efficiency of a design d is defined as

$$e_A(d) = \phi_A(d_A)/(\phi_A(d))$$

and

$$e_D(d) = \{\phi_D(d_D)/(\phi_D(d))\}^{1/(v-1)},$$

where $d_A(d_D)$ is the A-optimal (D-optimal) design in $D(v, b, k)$. Cheng and Wu (1981) have obtained the A- and D-efficiency lower bounds for designs with given parametres $v, b, k(\leqslant v)$. Their bounds were attained only by BIB designs. An improved bound taking into account designs with $k > v$ can be obtained on lines similar to Cheng and Wu (1981). The following lemma gives the A- and D-efficiency lower-bounds for any design $d \in D(v, b, k)$.

Lemma 3.4. *The A- and D-efficiency lower bounds for a design $d \in D(v,b,k)$ is given by*

$$e'_A(d) = \frac{k(v-1)^2}{b\{k(k-x)-(k-vx)(x+1)\}\phi_A(d)},$$

$$e'_D(d) = \frac{k(v-1)}{b\{k(k-x)-(k-vx)(x+1)\}\{\phi_D(d)\}^{1/(v-1)}},$$

where $n = \lfloor k/v \rfloor$.

The efficiency lower-bounds of Lemma 3.4 are attained by BBDs.

We now give the $\phi_A(d^*)$ and $\phi_D(d^*)$ values for an EB design d^*.

Lemma 3.5. *Let d^* be an EB design with parameters $v, b, k, \mathbf{r}$ and efficiency factor e. Then*

$$\phi_A(d^*) = \frac{v-1}{ve}\sum_{i=1}^{v} r_i^{-1}$$

and

$$\phi_D(d^*) = \frac{bk}{ve^{v-1}\prod_{i=1}^{v} r_i}.$$

Proof. First note that the C-matrix of an EB design d^* is $C_{d^*} = e(R - \mathbf{r}\mathbf{r}'/bk)$. The $\phi_A(d^*)$ is now obtained using the fact that the sum of the variances of all elementary treatment contrasts equals $\sigma^2 v \sum_{i=1}^{v-1} z_{d^*i}^{-1}$, where σ^2 is the per observation variance. The $\phi_D(d^*)$ is obtained by working out the product of the non-zero eigenvalues of $R - \mathbf{r}\mathbf{r}'/b$ as follows:

It can be shown that the characteristic polynomial of $R - \mathbf{r}\mathbf{r}'/bk$ is

$$f(\mu) = \det(\mu I - R + \mathbf{r}\mathbf{r}'/bk) = \prod_{i=1}^{v}(\mu - r_i)\left(1 + \sum_{i=1}^{v}\frac{r_i^2}{bk(\mu - r_i)}\right).$$

Or,

$$f(\mu) = \prod_{i=1}^{v}(\mu - r_i) - \sum_{i=1}^{v}\frac{r_i^2}{bk}\prod_{j \neq i}(\mu - r_j).$$

Since 0 is an eigenvalue of $R - \mathbf{r}\mathbf{r}'/bk$ we can write $f(\mu)$ as

$$f(\mu) = \mu g(\mu) = \mu(\mu - \mu_1)\cdots(\mu - \mu_{v-1})$$

where $\mu_1, \ldots, \mu_{v-1}$ are the non-zero eigenvalues of $R - \mathbf{r}\mathbf{r}'/bk$. Since $f'(\mu) = g(\mu) + \mu g'(\mu)$ we get $f'(0) = g(0) = (-1)^{v-1}\prod_{i=1}^{v-1}\mu_i$ or $\prod_{i=1}^{v-1}\mu_i = (-1)^{v-1}f'(0)$. Differentiating $f(\mu)$ w.r.t. μ and putting $\mu = 0$ gives the result. □

The A- and D-efficiency lower bounds for a GB-EB design then follows from Lemmas 3.4 and 3.5. Thus we have

Theorem 3.2. *The A- and D-efficiency lower-bounds for a GB-EB design d^* with parameters $v_1, v_2(= v - v_1)$, b, k, r_1, r_2 and $e = b^2x(x+1)/(r_1 r_2)$ is*

$$e'_A(d^*) = \frac{bkv(v-1)x(x+1)}{\{k(k-x)-(k-vx)(x+1)\}\{v(r_1+r_2)-bk\}}$$

and

$$e'_D(d^*) = \frac{bk(v-1)x(x+1)\{r_1^{v_1} r_2^{v_2} v/(bk)\}^{1/(v-1)}}{\{k(k-x)-(k-vx)(x+1)\}r_1 r_2}.$$

These lower bounds to A- and D-efficiency for the GB-EB design d^* with $e<1$ have been computed and presented in the table. It is apparent from the table that the designs, apart from being D- and E_f-optimal in $D(v_1, v_2, b, k, r_1, r_2)$ have high A- and D-efficiencies (among designs in $D(v, b, k)$) as well.

For any given $v, b, k(\geqslant v)$ there always exists a GB-EB design with $e=1$. It follows from Theorem 3.2 that these GB-EB designs d^* with parameters $v_1 = v(x+1)-k$, $v_2 = k - vx, b, k$, $r_1 = xb$ and $r_2 = (x+1)b$ have the A- and D-efficiency lower-bounds as $e'_A(d^*) = kv(v-1)x(x+1)/\{(k(k-x)-(k-vx)(x+1))(v(2x+1)-k)\}$ and $e'_D(d^*) = k(v-1)\{x^{v_1}(x+1)^{v_2}v/k\}^{1/(v-1)}/\{k(k-x)-(k-vx)(x+1)\}$.

4. Remarks and tabulation

Das and Kageyama (1994) observed that for given v, b, k if there exists two GB-EB designs (say d_i) with different sets of replications (r_{1d_i}, r_{2d_i}), $i=1,2$, then efficiency-factor $e_{d_1} > e_{d_2}$ provided $r_{1d_1} + r_{2d_1} > r_{1d_2} + r_{2d_2}$. Also, from Theorem 3.2 it follows that $e'_A(d_1) > e'_A(d_2)$ provided $r_{1d_1} + r_{2d_1} < r_{1d_2} + r_{2d_2}$. A completely reverse trend. This implies that with the increase in the value of the efficiency-factor, the lower bound to the A-efficiency decreases and vice versa. This is a clear indication that the efficiency factor is unable to throw any light for comparing designs – at least in the above setup. It follows that for given v, b, k, non-orthogonal GB-EB designs with $e<1$ is always A- and D-better than orthogonal GB-EB designs with $e=1$.

In Table 1, an exhaustive list of GB-EB designs with $e<1$ are arranged in the ascending order of k within the parametric range $2<v<k\leqslant 20$ and $b\leqslant 100$. The designs marked with asterisk has additional property which is mentioned in the end of Section 5. The table has columns corresponding to the parameters $v, b, k, v_1, v_2, r_1, r_2, e'_A(d^*)$ and $e'_D(d^*)$. The entries under reference column refer to the solution of designs. Here a BIB $(v, {}^vc_k, k)$ is denoted by (v, k). $p(v, k)$ refers to p copies of the BIB design. pBIBq refer to $p(\geqslant 1)$ copies of the BIB design with serial number q listed in Hall (1986; Appendix). BIB$^+$ stands for the complement of a BIB design. $Rq, R'q, Tq, T'q$ and Sq refer to PBIB designs or their complements in Clatworthy (1973). $A(x, n)$ stands for designs obtained in Theorem 2.3 belonging to the series with parameters x and n. KP stands for design series of Kherwa and Prasad (1995). Finally, the figure a in the last column shows a copies of the design within the same scope of parameters. They have the same efficiency bounds. Only one design has an unknown solution indicated by a dash (–) and is retained in the table to inspire researchers for attempting a solution.

5. EB row-column designs derived from GB-EB designs

The row-column designs considered here have bk experimental units arranged in a rectangular array of b columns and k rows such that each unit receives only one of the v treatments being studied. For an arbitrary row-column design d, the "C-matrix", under an appropriate model is given by

$$C_d^{RC} = R - k^{-1}N_d N_d' - b^{-1}M_d M_d' + (bk)^{-1}rr' = R - k^{-1}N_d N_d' - b^{-1}M_d(I - k^{-1}J)M_d', \tag{5.3}$$

where R, r are as defined earlier, and $N_d(M_d)$ is the $v\times b$ treatment-column ($v\times k$ treatment-row) incidence matrix. We denote by $\bar{D}(v, b, k)$, the class of all connected row-column designs with v treatments, k rows and b columns.

Table 1
GB-EB designs with $e<1$ in the parametric range of $2<v<k\leqslant 20$ and $b\leqslant 100$

v	b	k	v_1	v_2	r_1	r_2	$e'_A(d^*)$	$e'_D(d^*)$	Reference	Multiple
4*	9	6	1	3	12	14	0.9969	0.9985	2(3,2) + (3,1)	2 ~ 11
5*	12	7	3	2	14	21	0.9717	0.9854	A(1,1)	2 ~ 8
5	24	7	4	1	33	36	0.9991	0.9995	2(4,2) + 3(4,1)	2 ~ 4
5	30	8	1	4	36	51	0.9846	0.9926	6(4,3) + (4,2)	2, 3
6*	25	8	5	1	32	40	0.9947	0.9972	(5,2) + 3(5,1)	2 ~ 4
6	30	9	1	5	40	46	0.9978	0.9989	2(5,3) + (5,2)	2, 3
7	45	9	6	1	55	75	0.9907	0.9951	(6,2) + 5(6,1)	2
4*	15	10	1	3	36	38	0.9996	0.9998	3(3,2) + 2(3,1)	2 ~ 6
6	70	10	1	5	80	124	0.9775	0.9895	12(5,4) + (5,3)	
7	40	11	1	6	50	65	0.9927	0.9966	2(6,4) + BIB4	2
8*	30	11	5	3	33	55	0.9502	0.9738	A(1,2)	2, 3
8	42	11	7	1	57	63	0.9991	0.9995	3BIB1 + (7,2)	2
5*	14	12	4	1	33	36	0.9991	0.9995	(4,2) + 2(4,1)	2 ~ 7
8*	21	12	1	7	28	32	0.9983	0.9992	2BIB'1 + BIB1	2 ~ 4
5	48	13	1	4	108	129	0.9961	0.9981	9(4,3) + 2(4,2)	2
10	60	13	9	1	76	96	0.9957	0.9978	2BIB2 + (9,2)	
4	21	14	1	3	72	74	0.9999	1.0000	4(3,2) + 3(3,1)	2 ~ 4
6	30	14	5	1	68	80	0.9972	0.9985	(5,2) + 4(5,1)	2, 3
9*	22	14	1	8	28	35	0.9956	0.9979	R136 + S6	2 ~ 4
6	50	15	1	5	120	126	0.9997	0.9999	3(5,3) + 2(5,2)	2
10	54	15	1	9	72	82	0.9986	0.9993	2BIB'20 + BIB20	
11*	20	15	10	1	27	30	0.9992	0.9996	T33 + T9	2 ~ 5
11*	56	15	7	4	60	105	0.9383	0.9672	A(1,3)	
12*	33	15	11	1	40	55	0.9932	0.9963	KP	2, 3
6	55	16	1	5	120	152	0.9936	0.9969	9(5,4) + (5,3)	
10*	39	16	1	9	48	64	0.9932	0.9968	R'68 + R137	
10*	78	16	1	9	96	128	0.9932	0.9968	5BIB'2 + BIB'20	
11*	78	16	5	6	96	128	0.9817	0.9909	KP	
12*	78	16	9	3	96	128	0.9862	0.9928	–	
5	60	17	4	1	201	216	0.9994	0.9997	4(4,2) + 9(4,1)	
7*	24	17	4	3	51	68	0.9834	0.9916	A(2,1)	2 ~ 4
7	60	17	6	1	145	150	0.9999	0.9999	3BIB4 + 2(6,2)	
10	84	17	1	9	96	148	0.9845	0.9930	2(9,7) + BIB'2	
11	63	17	1	10	81	99	0.9970	0.9986	3BIB'8 + BIB27	
13	77	17	12	1	99	121	0.9974	0.9986	BIB48 + BIB47	
4	27	18	1	3	120	122	1.0000	1.0000	5(3,2) + 4(3,1)	2, 3
5*	22	18	1	4	72	81	0.9983	0.9992	4(4,3) + (4,2)	2 ~ 4
7	65	18	1	6	150	170	0.9984	0.9992	3(6,4) + (6,3)	
8*	91	18	7	1	198	252	0.9947	0.9972	(7,2) + 10(7,1)	
11*	25	18	1	10	30	42	0.9913	0.9960	T'12 + T57	2 ~ 4
12	33	18	1	11	44	50	0.9989	0.9994	2BIB'5 + BIB5	2, 3
14	78	19	13	1	105	117	0.9993	0.9996	BIB93 + 3BIB3	
14*	90	19	9	5	95	171	0.9309	0.9630	A(1,4)	
6	65	20	5	1	212	240	0.9983	0.9991	2(5,2) + 9(5,1)	
8	35	20	1	7	84	88	0.9998	0.9999	3BIB'1 + 2BIB1	2

With each row-column design d are associated the block designs d_N and d_M with incidence matrices N_d and M_d respectively, i.e., $d_N(d_M)$ is the block design obtained by treating the {columns} ({rows}) of d as blocks. Then, from (5.3), it follows that

$$C_d^{(RC)} = C_d - b^{-1}M_d(I - k^{-1}J)M'_d, \tag{5.4}$$

where

$$C_d = R - k^{-1} N_d N_d' \tag{5.5}$$

is the C-matrix of d_N.

From (5.4), it is clear that $C_d^{(RC)} \leqslant C_d$, where for a pair of nonnegative definite matrices A and B, $A \geqslant B$ indicates that $A - B$ is nonnegative definite.

Now, we quote some results and definitions from Das and Dey (1989).

Definition 5.1. A $k \times b$ array containing entries from a finite set $\Omega = \{1, 2, \ldots, v\}$ of v treatment symbols is called a Youden Type (YT) row-column design if the ith treatment symbol occurs in each row of the array m_i times, for $i = 1, 2, \ldots, v$, where $m_i = r_i/k$ and r_i is the replication of the ith treatment symbol in the array.

Theorem 5.1. *Consider a block design with v treatments, b blocks each of size k, and suppose the ith treatment is replicated r_i times for $i = 1, 2, \ldots, v$. If we write the b blocks of this design as columns, then a necessary and sufficient condition for converting (by rearranging treatments within columns) it into a YT row-column design is that r_i/k is an integer, for $i = 1, 2, \ldots, v$.*

Theorem 5.2. *A necessary and sufficient condition for $C_d^{(RC)} = C_d$ is that $d \in \bar{D}(v, b, k)$ is a YT design.*

Remark 5.1. In view of Theorem 5.2, it is clear that if the block design d_N corresponding to a row-column design $d \in \bar{D}(v, b, k)$ is ϕ-optimal according to some non-increasing optimality criterion ϕ, then d is also ϕ-optimal, provided d is a YT design (An optimality criterion ϕ is non-increasing if $\phi(A) \leqslant \phi(B)$ whenever $A - B$ is nonnegative definite). Thus, in the case of YT designs, the search for optimal designs in a row-column setting reduced to that in a block design setup.

A row-column design d is said to be EB if and only if $C_d^{(RC)} = e(R - \mathbf{r}\mathbf{r}'/bk)$, with $0 < e \leqslant 1$ as the unique canonical efficiency-factor of d.

Like in case of block designs, for row-column designs having two distinct replication numbers, we denote by $\bar{D}(v_1, v_2, b, k, r_1, r_2)$, the class of all connected row-column designs with $v_1 + v_2$ treatments, k rows, b columns and $v_1(v_2)$ treatments replicated $r_1(r_2)$ times.

From Definition 5.1, Theorems 5.1 and 5.2, and Remark 5.1, the following result is obvious.

Theorem 5.3. *The block contents of the GB-EB design d^* can be rearranged to yield a YT design, provided r_1 and r_2 are divisible by k. Further, in such a case, the YT design is a EB row-column design, and is D- and E_f-optimal in $\bar{D}(v_1, v_2, b, k, r_1, r_2)$.*

Note that GB-EB designs with $e = 1$ and $b = tk$, for any positive integer t, the conditions in Theorem 5.3 are satisfied and hence, these designs can be used to obtain D- and E_f-optimal EB row-column designs. Among GB-EB designs with $e < 1$ listed in the table in Section 4, the designs (for all or some multiples) marked with asterisk can be converted to a YT design and hence yield EB row-column designs.

The A- and D-efficiency lower bounds obtained for a block design hold for a row-column design as well. The lower bound to A- and D-efficiency for a row-column design d is identical to those in Lemma 3.4 and are attained by Youden designs. Note that the EB row-column design, derived from GB-EB design, have the same C-matrix as the GB-EB design. Therefore, the A- and D-efficiency lower bounds for the EB row-column design are the same as the bounds for GB-EB design from which it is derived.

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