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SOME RECENT RESULTS ON THE LINEAR COMPLEMENTARITY PROBLEM*

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Abstract. In this article we present some recent results on the linear complementarity problem. It is shown that (i) within the class of column adequate matrices, a matrix is in $\boldsymbol{Q_0}$ if and only if it is completely $\boldsymbol{Q_0}$ (ii) for the class of $\boldsymbol{C_0^f}$-matrices introduced by Murthy and Parthasarathy [*SIAM J. Matrix Anal. Appl.*, 16 (1995), pp. 1268–1286], we provide a sufficient condition under which a matrix is in $\boldsymbol{P_0}$ and as a corollary of this result, we give an alternative proof of the result that $\boldsymbol{C_0^f} \cap \boldsymbol{Q_0} \subseteq \boldsymbol{P_0}$ (iii) within the class of INS-matrices introduced by Stone [Department of Operations Research, Stanford University, Stanford, CA, 1981], a nondegenerate matrix must necessarily have the block property introduced by Murthy, Parthasarathy, and Sriparna [G. S. R. Murthy, T. Parthasarathy, and B. Sriparna, *Linear Algebra Appl.*, 252 (1997), pp. 323–337]. Furthermore, we conjecture that if a matrix has block property, then it must be Lipschitzian. This problem is an important one from two angles: if the conjecture is true, it provides a finite test to check whether a given matrix is Lipschitzian or nondegenerate *INS*; and it settles an open problem posed by Stone. It is shown that the conjecture is true in the cases of 2×2-matrices, nonnegative and nonpositive matrices of general order.

Key words. linear complementarity problem, adequacy, matrix classes, principal pivoting

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1. Introduction.

Given a matrix $A \in \boldsymbol{R}^{n\times n}$ and $q \in \boldsymbol{R}^n$ the linear complementarity problem (LCP) is to find a vector $z \in \boldsymbol{R}^n$ such that

$$Az + q \geq 0, \quad z \geq 0, \text{ and } \quad z^t(Az + q) = 0. \tag{1.1}$$

LCP has numerous applications, both in theory and in practice, and is treated by a vast literature (see [2, 10]). Let $F(q, A) = \{ z \in \boldsymbol{R}^n_+ : Az + q \geq 0 \}$ and $S(q, A) = \{ z \in F(q, A) : (Az + q)^t z = 0 \}$. A number of matrix classes have been defined in connection with LCP, the fundamental ones being $\boldsymbol{Q}$ and $\boldsymbol{Q_0}$. The class $\boldsymbol{Q}$ consists of all real square matrices A such that $S(q, A) \neq \phi$ for every $q \in \boldsymbol{R}^n$ [11], and $\boldsymbol{Q_0}$ consists of all real square matrices A such that $S(q, A) \neq \phi$ whenever $F(q, A) \neq \phi$ [9].

For any positive integer n, write $\bar{n} = \{1, 2, \ldots, n\}$, and for any subset α of $\bar{n}$, write $\bar{\alpha} = \bar{n} \setminus \alpha$. For any $A \in \boldsymbol{R}^{n\times n}$, $A_{\alpha\alpha}$ is obtained by dropping rows and columns corresponding to $\bar{\alpha}$ from A. For any $x \in \boldsymbol{R}^n$, x_α is obtained from x by dropping coordinates corresponding to $\bar{\alpha}$, and x_i denotes the ith coordinate of x. Consider $A \in \boldsymbol{R}^{n\times n}$. If $\alpha \subseteq \bar{n}$ is such that $\det A_{\alpha\alpha} \neq 0$, then the matrix M defined by

$$M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}, \ M_{\alpha\bar{\alpha}} = -M_{\alpha\alpha}A_{\alpha\bar{\alpha}}, \ M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}M_{\alpha\alpha}, \ M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}A_{\alpha\bar{\alpha}}$$

is known as the principal pivotal transform (PPT) of A with respect to α and will be denoted by $\wp_\alpha(A)$. Note that a PPT is defined only with respect to those α for

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which $det\, A_{\alpha\alpha} \neq 0$. By convention, when $\alpha = \emptyset$, $det\, A_{\alpha\alpha} = 1$ and $M = A$ (see [2]). Whenever we refer to PPTs, we mean the ones which are well defined.

We shall recall the definitions of some matrix classes that are relevant to this paper. Let $A \in \boldsymbol{R}^{n\times n}$. Then A is said to be a $\boldsymbol{P}$-matrix ($\boldsymbol{P_0}$-matrix) if all its principal minors are positive (nonnegative); if all principal minors of A are nonzero, then A is called a nondegenerate matrix; A is semimonotone ($\boldsymbol{E_0}$) if (q, A) has a unique solution for every $q > 0$; A is fully semimonotone ($\boldsymbol{E_0^f}$) if every PPT of A is in $\boldsymbol{E_0}$; A is copositive ($\boldsymbol{C_0}$) if $x^t Ax \geq 0$ for every $x \geq 0$; A is fully copositive ($\boldsymbol{C_0^f}$) if every PPT of A is in $\boldsymbol{C_0}$. For the definition of INS and Lipschitzian matrices see section 3.

In this article, we present some new results pertaining to three matrix classes, namely, (i) the class of adequate matrices introduced by Ingleton [4], (ii) the class of fully copositive matrices introduced by Murthy and Parthasarathy [7], and (iii) the class of INS-matrices introduced by Stone [13].

In the case of adequate matrices (see section 2), our main result is that a column adequate matrix is in $\boldsymbol{Q}$ (in $\boldsymbol{Q_0}$) if and only if it is completely-$\boldsymbol{Q}$ (completely-$\boldsymbol{Q_0}$). Characterization of completely-$\boldsymbol{Q_0}$ matrices in general is a complex problem [1]. Murthy and Parthasarathy [6, 7, 8] have shown that nonnegative matrices, symmetric copositive matrices, $\boldsymbol{C_0^f}$-matrices and Lipschitzian matrices are in $\boldsymbol{Q_0}$ if and only if they are completely-$\boldsymbol{Q_0}$.

Within the class of $\boldsymbol{C_0^f}$-matrices, we provide a sufficient condition under which a matrix will be in $\boldsymbol{P_0}$. As a corollary to this result, we provide an alternative proof of a result due to Murthy and Parthasarathy which states that $\boldsymbol{C_0^f} \cap \boldsymbol{Q_0}$-matrices are in $\boldsymbol{P_0}$. As another consequence of this result, we deduce that a bisymmetric $\boldsymbol{E_0^f}$-matrix A is positive semidefinite if, and only if, the rows and columns of $A + A^t$ corresponding to the zero diagonal entries are zero.

Last, we consider the class of INS-matrices and show that a nondegenerate INS-matrix must necessarily satisfy the *block* property. There are no constructive characterizations of Lipschitzian or INS-matrices. In [8], the authors showed that Lipschitzian matrices must necessarily satisfy the block property, and Stone [14] showed that Lipschitzian matrices are nondegenerate INS-matrices. We conjecture that block property is a characterization of Lipschitzian matrices. It is proven that the conjecture is true in the cases of nonnegative or nonpositive matrices and 2×2 matrices.

The results on adequate and $\boldsymbol{C_0^f}$-matrices are presented in section 2, and the results on INS- and Lipschitzian matrices are presented in section 3.

2. Results on adequate and $\boldsymbol{C_0^f}$-matrices. A number of matrix classes are invariant under principal pivoting; i.e., if a matrix is in class $\mathcal{C}$, then all its PPTs are also in $\mathcal{C}$. The matrix classes $\boldsymbol{Q}$, $\boldsymbol{Q_0}$, $\boldsymbol{P}$, $\boldsymbol{P_0}$, $\boldsymbol{E_0^f}$, $\boldsymbol{C_0^f}$, INS- and Lipschitzian matrices all fall in this category. In the definition below we consider another class of matrices which is also invariant under PPTs.

DEFINITION 2.1. *Say that a real square matrix $A \in \Lambda$ if for every PPT M of A the diagonal entries are nonnegative.*

Remark 2.2. Note that $\boldsymbol{E_0^f}$, which contains the classes $\boldsymbol{P_0}$ and $\boldsymbol{C_0^f}$ (see [2, 6, 7]), is a subclass of Λ. However, $\Lambda \setminus \boldsymbol{E_0^f}$ is nonempty as $\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right]$ is an example of this kind. Furthermore, it is easy to check that if $A \in \Lambda$, then $A^t \in \Lambda$.

Another class of matrices that is required for our results is the following.

DEFINITION 2.3. *Say that a real square matrix A has property (D) if for every index set α the following holds:*

$$det\, A_{\alpha\alpha} = 0 \;\Rightarrow\; \text{columns of } A_{\cdot\alpha} \text{ are linearly dependent.}$$

Let $\boldsymbol{\mathcal{D}}$ denote the class of matrices satisfying property (D). Note that if $A \in \Lambda$ ($A \in \boldsymbol{\mathcal{D}}$), then $A_{\alpha\alpha} \in \Lambda$ ($A_{\alpha\alpha} \in \boldsymbol{\mathcal{D}}$) for every α. An interesting property of $\boldsymbol{\mathcal{D}}$ is that if $A \in \boldsymbol{\mathcal{D}}$, then (q, A) has a solution with a complementary basis for any q with $S(q, A) \neq \phi$ (see [7]). Another interesting property of $\boldsymbol{\mathcal{D}}$, which is a direct consequence of the definition, is the following.

PROPOSITION 2.4. *If $A \in \boldsymbol{\mathcal{D}}$ is nonsingular, then A is nondegenerate.*

A matrix A is said to be a column (row) adequate matrix if A (A^t) is in $\boldsymbol{\mathcal{D}} \cap \boldsymbol{P_0}$. Ingleton [4] introduced the class of adequate matrices (i.e., both row and column adequate) and showed that if A is adequate, then, for every q with $S(q, A) \neq \phi$, $Az + q$ is unique over $S(q, A)$. We now present our main results on column adequate matrices.

THEOREM 2.5. *If $A \in \Lambda \cap \boldsymbol{\mathcal{D}}$, then $A \in \boldsymbol{P_0}$.*

Proof. We prove this by induction on n. Obviously the theorem is true if $n = 1$. Assume that the theorem is true for all $(n-1) \times (n-1)$ matrices. Let $A \in \boldsymbol{R}^{n\times n} \cap \Lambda \cap \boldsymbol{\mathcal{D}}$. By above observations, $A_{\alpha\alpha} \in \boldsymbol{P_0}$ for all α such that $|\alpha| = n - 1$. Suppose $A \notin \boldsymbol{P_0}$. Then $det\, A < 0$. Note that A is almost $\boldsymbol{P_0}$. Since $A \in \Lambda$, diagonal entries of A^{-1} are equal to zero. This means that $det\, A_{\alpha\alpha} = 0$ for all α with $|\alpha| = n - 1$. Since $A \in \boldsymbol{\mathcal{D}}$, this implies that columns of A are linearly dependent which contradicts that A is nonsingular. It follows that $A \in \boldsymbol{P_0}$. ∎

COROLLARY 2.6. *Suppose $A \in \boldsymbol{R}^{n\times n}$. The following conditions are equivalent:*

(a) $A \in \boldsymbol{P_0} \cap \boldsymbol{\mathcal{D}}$;

(b) $A \in \Lambda \cap \boldsymbol{\mathcal{D}}$.

It is known that nondegenerate $\boldsymbol{E_0^f}$-matrices are $\boldsymbol{P}$-matrices.

COROLLARY 2.7. *If $\boldsymbol{E_0^f} \cap \boldsymbol{\mathcal{D}}$, then $A \in \boldsymbol{P_0}$.*

A matrix A is said to be completely-$\boldsymbol{Q}$ (completely-$\boldsymbol{Q_0}$) if all its principal submatrices including A are $\boldsymbol{Q}$-matrices ($\boldsymbol{Q_0}$-matrices). Cottle introduced these classes in [1] and characterized completely-$\boldsymbol{Q}$ matrices as the class of strictly semimonotone matrices (A is said to be strictly semimonotone if (q, A) has a unique solution for every nonnegative q). One of the problems posed by Cottle [1] is the characterization of completely-$\boldsymbol{Q_0}$ matrices which is still an open problem. Murthy and Parthasarathy have characterized completely-$\boldsymbol{Q_0}$ matrices in certain special cases (see [6, 7, 8]). The following result augments these special cases with column adequate matrices.

THEOREM 2.8. *Suppose $A \in \Lambda \cap \boldsymbol{\mathcal{D}}$. Then*

(a) *$A \in \boldsymbol{Q_0}$ if and only if A is completely-$\boldsymbol{Q_0}$;*

(b) *$A \in \boldsymbol{Q}$ if and only if A is completely-$\boldsymbol{Q}$.*

Proof. (a) It suffices to show the "only if" part. Suppose $A_{\alpha\alpha} \notin \boldsymbol{Q_0}$, say, for $\alpha = \{1, 2, \ldots, n-1\}$. By Theorem 2.19 of [7], there exists a β such that $n \in \beta$, $det\, A_{\beta\beta} \neq 0$ and $M_{\cdot n} \leq 0$, where $M = \wp_\beta(A)$. Since $A \in \boldsymbol{P_0}$ (Theorem 2.5 above), $M_{nn} = \frac{det\, A_{\gamma\gamma}}{det\, A_{\beta\beta}} = 0$, where $\gamma = \beta \setminus \{n\}$. This implies $det\, A_{\gamma\gamma} = 0$, which in turn implies $det\, A_{\beta\beta} = 0$ as $A \in \boldsymbol{\mathcal{D}}$. From this contradiction, it follows that $A_{\alpha\alpha} \in \boldsymbol{Q_0}$. By induction it follows that A is completely-$\boldsymbol{Q_0}$.

(b) Once again, we will show the "only if" part. Note that the conclusions of Theorem 2.19 of [7] remain valid even if we replace $\boldsymbol{Q_0}$ by $\boldsymbol{Q}$ in the statement of that theorem (almost the same proof can be repeated). Hence it follows (from the proof of part (a) here) that A is completely-$\boldsymbol{Q}$. ∎

COROLLARY 2.9. *Every column adequate matrix is in* $\boldsymbol{Q}$ *if and only if it is strictly semimonotone.*

We now turn our attention to the results on $\boldsymbol{C_0^f}$-matrices. In [6], using the concept of *incidence*, it was shown that $\boldsymbol{C_0^f} \cap \boldsymbol{Q_0} \subseteq \boldsymbol{P_0}$. We recapture this result as a consequence of our results here.

THEOREM 2.10. *Suppose* $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C_0^f}$, $n \geq 2$. *If the rows and columns of* $A + A^t$ *corresponding to the zero diagonal entries of* A *are zero, then* $A \in \boldsymbol{P_0}$.

Proof. From the hypothesis and Theorem 3.17 of [7], it is clear that every 2×2 principal submatrix of A is in $\boldsymbol{P_0}$. Assuming that every $(k-1) \times (k-1)$, $k \geq 2$, principal submatrix of A is in $\boldsymbol{P_0}$, we will show that every $k \times k$ principal submatrix of A is also in $\boldsymbol{P_0}$. Let B be any $k \times k$ principal submatrix of A such that all its proper principal minors are nonnegative. Suppose $det\, B < 0$. Arguing as in Theorem 3.17 of [7], we can show that

$$B^{-1} = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix},$$

where C and D are nonnegative square matrices of the same order. It follows that C and D are nonsingular and that $B = \left[\begin{smallmatrix} 0 & D^{-1} \\ C^{-1} & 0 \end{smallmatrix}\right]$. From the hypothesis, it follows that $C^{-1} + (D^{-1})^t = 0$ and hence $D^{-1} = -(C^{-1})^t$. This in turn implies that $D = -C^t$. This contradicts that D is nonnegative. Hence $det\, B \geq 0$. The theorem follows. ◻

COROLLARY 2.11. *Suppose* $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C_0^f} \cap \boldsymbol{Q_0}$. *Then* $A \in \boldsymbol{P_0}$.

Proof. If $n = 1$, there is nothing to prove. Assume $n \geq 2$. We will show that every 2×2 principal submatrix of A is in $\boldsymbol{P_0}$. Suppose, to the contrary, assume that $A_{\alpha\alpha} \notin \boldsymbol{P_0}$ for some α with $|\alpha| = 2$. Without loss of generality, we may take $\alpha = \{1, 2\}$. Then $A_{\alpha\alpha} \simeq \left[\begin{smallmatrix} 0 & + \\ + & 0 \end{smallmatrix}\right]$ (this notation means $a_{11} = a_{22} = 0$ and a_{12}, a_{21} are positive). Since $A_{\alpha\alpha} \notin \boldsymbol{Q_0}$, we must have $n > 2$ and a $j \in \bar{\alpha}$ such that $a_{j1} < 0$ (follows from Theorem 2.9 of [7]). Note that if $a_{1j} \leq 0$, then $A \notin \boldsymbol{C_0^f}$. But if $a_{1j} > 0$, then also $A \notin \boldsymbol{C_0^f}$ (follows from Theorem 4.1 of [8]). It follows that every 2×2 principal submatrix of A is in $\boldsymbol{P_0}$ and hence $A \in \boldsymbol{P_0}$. Arguing as in Lemma 3.2 of [6], we can show that for every i such that $a_{ii} = 0$, we have $a_{ij} + a_{ji} = 0$ for all j. Notice that in the proof of Lemma 3.2 of [6] we need only that every 2×2 principal submatrix of A is in $\boldsymbol{P_0}$. Hence the rows and columns of $A + A^t$ corresponding to zero diagonal entries of A are zero. From Theorem 2.10, it follows that $A \in \boldsymbol{P_0}$. ◻

In [6], it was shown that a $\boldsymbol{C_0^f}$-matrix is in $\boldsymbol{Q_0}$ if and only if it is completely-$\boldsymbol{Q_0}$. The arguments used to prove this can be extended to obtain the following result.

THEOREM 2.12. *Suppose* $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C_0^f}$. *If* $A \in \boldsymbol{Q_0}$, *then* A^t *and all its PPTs are completely-*$\boldsymbol{Q_0}$.

Proof. It can be verified that if a matrix $B \in \Lambda$ satisfies the condition that for every PPT C of B satisfies

$$c_{ii} = 0 \Rightarrow c_{ij} + c_{ji} = 0 \text{ for all } i \text{ and } j,$$

then B and all its PPTs are completely-$\boldsymbol{Q_0}$ matrices. This is because, if B has this property, then Graves's algorithm processes (q, B) for any q and terminates either with a solution or with the conclusion that $F(q, B) = \emptyset$ (see Chapter 4 of [10] and Theorem 3.4 of [6]). Therefore, we will show that any PPT of A^t will satisfy the above condition. Let $D = \wp_\alpha(A^t)$ for some α. Observe that $\wp_\alpha(A)$ exists. Let $M = \wp_\alpha(A)$. It can be checked that, $M = SD^tS$, where $S = \left[\begin{smallmatrix} I_{\alpha\alpha} & 0 \\ 0 & -I_{\bar{\alpha}\bar{\alpha}} \end{smallmatrix}\right]$. Hence for each i, j, either $d_{ij} + d_{ji} = m_{ij} + m_{ji}$ or $d_{ij} + d_{ji} = -(m_{ij} + m_{ji})$. If $d_{ii} = 0$ for some i, then $m_{ii} = 0$, and by Theorem 3.4 of [6], $m_{ij} + m_{ji} = 0$. From this it follows that if for some i, $d_{ii} = 0$, then $d_{ij} + d_{ji} = 0$. ∎

One may ask whether the converse of the above theorem is true. That is, if $A \in \boldsymbol{C_0^f}$ and A^t and all its PPTs are completely-$\boldsymbol{Q_0}$, then is it true that $A \in \boldsymbol{Q_0}$? The answer to this question is "no." The problem arises from the fact that transpose of a $\boldsymbol{C_0^f}$-matrix need not be in $\boldsymbol{C_0^f}$. As a counter example, consider the $\boldsymbol{C_0^f}$-matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. It can be checked, directly or using Theorem 2.5 of [7], that A^t and its PPT are completely-$\boldsymbol{Q_0}$ but $A \notin \boldsymbol{Q_0}$.

A matrix A is said to be bisymmetric if, for some index set α, $A_{\alpha\alpha}$ and $A_{\bar{\alpha}\bar{\alpha}}$ are symmetric and $A_{\bar{\alpha}\alpha} = -A^t_{\alpha\bar{\alpha}}$. It is easy to check that PPTs of bisymmetric matrices are bisymmetric.

THEOREM 2.13. *Suppose $A \in \boldsymbol{R}^{n\times n}$ is a bisymmetric $\boldsymbol{E_0^f}$-matrix. Then the following are equivalent:*

(a) $A \in \boldsymbol{Q_0}$;
(b) *A is positive semidefinite*;
(c) *for any i, j, $a_{ii} = 0 \;\Rightarrow\; a_{ij} + a_{ji} = 0$*;
(d) *every 2×2 principal submatrix of A is in $\boldsymbol{P_0}$.*

Proof. We first observe that every bisymmetric $\boldsymbol{E_0^f}$-matrix is in $\boldsymbol{C_0^f}$ (Theorem 4.7 of [6]). Implication (a) ⇒ (b) was already established in [6]. The implication (b) ⇒ (c) is a well-known fact about positive semidefinite matrices. The implication (c) ⇒ (d) is a direct consequence of Theorem 2.10. To complete the proof of the theorem, we will show that (d) ⇒ (a). Assume that A satisfies (d). Using the fact that every 2×2 principal submatrix of A is in $\boldsymbol{C_0^f} \cap \boldsymbol{P_0}$, it is easy to show that A satisfies (c). Hence, by 2.10, $A \in \boldsymbol{P_0}$. Let M be any PPT of A. Suppose $m_{ii} = 0$ for some i. As A is bisymmetric, so is M. So for any j, either $m_{ij} = -m_{ji}$ or $m_{ij} = m_{ji}$. If $m_{ij} = -m_{ji}$, then $m_{ij} + m_{ji} = 0$. If $m_{ij} = m_{ji}$, then, as $M \in \boldsymbol{P_0}$ and $m_{ii} = 0$, we must have $m_{ij} = m_{ji} = 0$. Thus for any j, $m_{ij} + m_{ji} = 0$. By Theorem 3.4 of [6], it follows that $A \in \boldsymbol{Q_0}$. ∎

3. Block property.

Stone [13] introduced the class of INS-matrices. A matrix A is said to be an INS_k-matrix if $|S(q, A)| = k$ for all $q \in \text{int}K(A)$, where $K(A)$ is the set of all p for which $S(p, A) \neq \emptyset$; and $INS = \cup_{k=0}^{\infty} INS_k$. Next we say that A is Lipschitzian matrix if there exists a positive number λ, called the Lipschitzian constant, such that for any $p, q \in K(A)$, the following holds: given any $x \in S(p, A)$, there exists a $z \in S(q, A)$ such that $\|x - z\| \;\leq\; \lambda\|p - q\|$. Stone [14] showed that Lipschitzian matrices are nondegenerate INS-matrices and conjectured that the converse is also true. Furthermore, he showed that the conjecture is true with an additional assumption of Lipschitz *path-connectedness* (see [14] for details). To date, no constructive characterizations are known for INS and Lipschitzian matrix classes. Thus, there is no finite procedure to verify whether a given matrix is INS or Lipschitzian.

DEFINITION 3.1. *Say that A has property (B) if every PPT M of A has the following block structure (subject to a principal rearrangement):*

$$M = \begin{bmatrix} M_{11} & 0 & \dots & 0 & M_{1\overline{l+1}} \\ 0 & M_{22} & \dots & 0 & M_{2\overline{l+1}} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & M_{ll} & M_{l\overline{l+1}} \\ M_{\overline{l+1}1} & M_{\overline{l+1}2} & \dots & M_{\overline{l+1}l} & M_{\overline{l+1}\,\overline{l+1}} \end{bmatrix},$$

where $M_{11}, M_{22}, \ldots, M_{ll}$ are all negative $\boldsymbol{N}$-matrices (i.e., all entries and all principal minors are negative) and the diagonal entries of $M_{\overline{l+1}\ \overline{l+1}}$ are positive.

In [8], the authors showed that every Lipschitzian matrix must have property (B). In this section, we will show that every nondegenerate INS-matrix also must have property (B).

Note that if a matrix A has property (B), then it must be nondegenerate as every PPT of A has no zero diagonal entries (see Corollary 3.5, p. 204 of [10]). From the definition, property (B) is invariant under PPTs and is inherited by all the principal submatrices.

THEOREM 3.2. *Suppose $A \in \boldsymbol{R}^{n\times n}$ is a nondegenerate INS-matrix. Then A has property (B).*

Proof. Let $\alpha = \{i : a_{ii} < 0\}$. By Theorem 5 of [12], $A_{\alpha\alpha}$ is a nondegenerate INS-matrix. Also, for $i, j \in \alpha,\ i \neq j,\ A_{\beta\beta} \in INS$, where $\beta = \{i, j\}$. It is easy to check that if $A_{\beta\beta}$ has a positive entry, then $A_{\beta\beta} \notin INS$. It follows that $A_{\alpha\alpha}$ is nonpositive and hence in $\boldsymbol{Q_0}$. From Corollary 3.5 of [14], $A_{\alpha\alpha}$ is Lipschitzian. From Theorem 4.7 of [8],

$$A_{\alpha\alpha} = \begin{bmatrix} N^1 & 0 & \ldots & 0 \\ 0 & N^2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & N^l \end{bmatrix} \text{ for some } l \geq 1,$$

where each N^i is a negative $\boldsymbol{N}$-matrix. Since every PPT of nondegenerate INS-matrix is also nondegenerate INS, we conclude that A has property (B). ∎

Our conjecture is that property (B) is also sufficient condition for a matrix to be Lipschitzian. Below we verify this conjecture in certain special cases.

THEOREM 3.3. *Suppose $A \in \boldsymbol{R}^{n\times n}$. Assume that any one of the following conditions holds:*

(i) $n = 2$;

(ii) $A \leq 0$;

(iii) *A is completely-$\boldsymbol{Q}$*;

(iv) $A \geq 0$.

Then the following statements are equivalent:

(a) *A is nondegenerate INS*;

(b) *A is Lipschitzian*;

(c) *A has property (B).*

Proof. In view of Stone's result that (b) $\Rightarrow$ (a) (Theorem 3.2 of [14]), it suffices to show that (c) implies (b). So assume that (c) holds.

(i). If the diagonal entries of A are negative, then property (B) implies that either A is a negative $\boldsymbol{N}$-matrix or $A \simeq \left[\begin{smallmatrix} - & 0 \\ 0 & - \end{smallmatrix}\right]$. In either case, A is Lipschitzian (see [3]). If the diagonal entries of A are positive, then either A is a $\boldsymbol{P}$-matrix or A^{-1} is a negative $\boldsymbol{N}$-matrix. Once again A is Lipschitzian (see [5]). Consider the last case $a_{11} < 0$ and $a_{22} > 0$, without loss of generality. It is easy to check (graphically) that A is INS and that $K(A)$ is Lipschitz path-connected (see [14] for details and the example following Definition 3.3 in [14]). From Theorem 3.4 of [14], we conclude A is Lipschitzian.

(ii). By property (B), A can be decomposed into a block diagonal matrix where each submatrix on the diagonal is a negative $\boldsymbol{N}$-matrix. As negative $\boldsymbol{N}$-matrices are Lipschitzian, one can easily verify that A is also Lipschitzian.

(iii). In this case we actually show that A is a $\boldsymbol{P}$-matrix and this we do by induction on the order of the matrix. Obviously the result is true for $n = 1$. Assume the result for all matrices of order $n-1,\ n>1$. Suppose $A \in \boldsymbol{R}^{n\times n}$ satisfies the hypothesis. Then all the proper principal minors of A are positive. If $A \notin \boldsymbol{P}$, then $det\, A < 0$ and the diagonal entries of A^{-1} are negative. By property (B), A^{-1} must be nonpositive. But this contradicts that $A \in \boldsymbol{Q}$. Hence $A \in \boldsymbol{P}$.

(iv). From the hypothesis and (c), $a_{ii} > 0$ for all i. Since $A \geq 0$, A is completely-$\boldsymbol{Q}$. Therefore $A \in \boldsymbol{P}$. □

Proposition 3.4. *Suppose $A \in \boldsymbol{R}^{n\times n}$. Assume that for some index set α, $A_{\alpha\alpha}$ is Lipschitzian and $A_{\bar\alpha\bar\alpha} \in \boldsymbol{P}$. If $A_{\bar\alpha\alpha} = 0$ or $A_{\alpha\bar\alpha} = 0$, then A is Lipschitzian.*

Proof. Assume $A_{\bar\alpha\alpha} = 0$. Let $p, q \in K(A)$. Let λ_1 and λ_2 be the Lipschitzian constants corresponding to $A_{\alpha\alpha}$ and $A_{\bar\alpha\bar\alpha}$ respectively. Take any arbitrary $x \in S(p, A)$. We will exhibit a $z \in S(q, A)$ such that $\|z - x\| \leq \lambda\|p - q\|$, where λ, to be chosen later, depends only on λ_1, λ_2, and A. Since $S(q, A) \neq \phi$, choose any $\bar z \in S(q, A)$. Let $y = Ax + p$ and $\bar w = A\bar z + q$. Note that $x_\alpha \in S(p'_\alpha, A_{\alpha\alpha})$ and $\bar z_\alpha \in S(q'_\alpha, A_{\alpha\alpha})$, where $p'_\alpha = p_\alpha + A_{\alpha\bar\alpha}x_{\bar\alpha}$ and $q'_\alpha = q_\alpha + A_{\alpha\bar\alpha}\bar z_{\bar\alpha}$. Since $A_{\alpha\alpha}$ is Lipschitzian, there exists a $z_\alpha \in S(q'_\alpha, A_{\alpha\alpha})$ such that

$$\begin{aligned}\|x_\alpha - z_\alpha\| &\leq \lambda_1\|p'_\alpha - q'_\alpha\| \\ &\leq \lambda_1\|p_\alpha - q_\alpha\| + \lambda_1\|B\|\|x_{\bar\alpha} - z_{\bar\alpha}\|.\end{aligned}$$

Since $z_\alpha \in S(q'_\alpha, A_{\alpha\alpha})$, $w_\alpha = A_{\alpha\alpha}z_\alpha + q_\alpha + A_{\alpha\bar\alpha}\bar z_{\bar\alpha}$ and $w^t_\alpha z_\alpha = 0$. This implies $z = (z^t_\alpha, \bar z^t_{\bar\alpha})^t \in S(q, A)$. As $A_{\bar\alpha\bar\alpha} \in \boldsymbol{P}$, $x_{\bar\alpha}$ and $z_{\bar\alpha}$ are the unique solutions of $(p_{\bar\alpha}, A_{\bar\alpha\bar\alpha})$ and $(q_{\bar\alpha}, A_{\bar\alpha\bar\alpha})$. Therefore, $\|x_{\bar\alpha} - z_{\bar\alpha}\| \leq \lambda_2\|p_{\bar\alpha} - q_{\bar\alpha}\|$. Combining this with the above inequality, we get

$$\begin{aligned}\|x - z\| &\leq \|x_\alpha - z_\alpha\| \\ &\leq \lambda_1\|p_\alpha - q_\alpha\| + (\lambda_1\lambda_2\|B\| + \lambda_2)\|p_{\bar\alpha} - q_{\bar\alpha}\| \\ &\leq \lambda_1\|p - q\| + (\lambda_1\lambda_2\|B\| + \lambda_2)\|p - q\| \\ &\leq \lambda\|p - q\|, \text{ where } \lambda = \lambda_1 + \lambda_2 + \lambda_1\lambda_2\|B\|.\end{aligned}$$

It follows that A is Lipschitzian, and the case $A_{\alpha\bar\alpha} = 0$ can be tackled in a similar fashion. □

Proposition 3.4 is not valid if we simply assume that $A_{\alpha\alpha}$ and $A_{\bar\alpha\bar\alpha}$ are Lipschitzian. As a counter example, consider $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. It is clear that $A \notin INS$, and hence A is not Lipschitzian.

The following is an example of a matrix with property (B).

Example 3.5.

$$A = \begin{bmatrix} -1 & -2 & -2 \\ -2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

It is not known whether A is Lipschitzian or not.

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