CONVOLUTION PROPERTIES OF DISTRIBUTIONS ON TOPOLOGICAL GROUPS

By

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PREFACE

This thesis is being submitted to the Indian Statistical Institute in support of the author's application for the degree of Doctor of Philosophy. The thesis embodies research carried out by the author during the period 1959-1962 under the supervision of Prof. C.R.Rao at the Indian Statistical Institute.

The thesis is concerned with the study of convolution properties of distributions on topological groups. Some of the results, in Chapters II and III, have been obtained in collaboration with Dr. R. Rangarao and Dr.K.R.Parthasarathy to whom the author is indebted for several discussions, suggestions and criticisms. These results have been submitted elsewhere for publication under the joint authorship. The bulk of the thesis consists of author's contributions. The author records his gratefulness to the Research and Training School of the Indian Statistical Institute for providing facilities for research. Thanks are due to Mr.G.M.Das for his efficient typing of the thesis.

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Chapter I

INTRODUCTION

The collection of probability distributions on the real line which is a semigroup under the convolution operation and a topological space under weak convergence has attracted, during the last quarter of a century, the attention of such great mathematicians as Khintchine, Kolmogorov, Bawly, Levy and others. Their results constitute a deep analysis of the factorisation problems on the one hand and the asymptotic behaviour of infinite convolutions on the other. However, a complete and systematic investigation of these problems is not available in considerable generality. The aim of the present thesis is to investigate the topological and algebraic properties and the inter relationship between them in the space of probability distributions on general topological groups.

The starting point of the investigation is a result of Khintchine according to which any distribution on the real line is the convolution of two distributions, one of which is the convolution of a countable or a finite number of indecomposable distributions and the other is an infinitely divisible distribution without any indecomposable factor. This gives an indication of the existence of indecomposable distributions in the real line. However, this does not say whether there exist non-atomic or absolutely continuous indecomposable distributions.

In the second chapter, we show that, under very general conditions, any infinite group contains a large class of non-atomic and absolutely continuous indecomposable distributions. The main tool in proving this result is the following theorem which reveals an interesting structural property of the semigroup of distributions: if $\lambda_n = \alpha_n * \beta_n$ for each n and λ_n is weakly compact, then there exist translates α_n', β_n' of α_n and β_n respectively, such that $\{\alpha_n'\}$ and $\{\beta_n'\}$ are compact.

In chapter III we study the structure of infinitely divisible distributions on a locally compact abelian separable metric group and get a representation for their characteristic functionals. We prove further that limit distributions of 'sums' of uniformly infinitesimal random variables are infinitely divisible and can be obtained as limits of certain accompanying infinitely divisible distributions. We remark here that these results are well known and due to Levy, Khintchine and Bowly in the case of real line. The case of the multiplicative group of complex numbers of modulus unity was studied by Levy [12] and the finite dimensional vector space by Takano [20]. Recently Kloss considered some special types of compact groups. Hunt [7] has considered the same problem in the case of Lie groups but from a slightly different angle.

The representation of the characteristic functional of an infinitely divisible distribution that we obtain here turns out to be non-unique in general. However, we are able to get a unique canonical representation of processes with independent increments.

In proving the results mentioned in the earlier paragraph the local compactness of the space plays a crucial role in the sense that we make use of the continuity relationships between the distributions and their characteristic functionals. If however, some suitable conditions for the weak compactness of distributions are known in terms of the characteristic functionals, then we can generalise them to complete and separable metric groups. Hilbert Space happens to be one such case because of the availability of such a compactness criterion due to Prohorov [16]. Chapter IV is devoted to the extension of all the results of chapter III in the case of a Hilbert Space.

In the last chapter we generalise Khintchine's theorem (which we have mentioned earlier) to the case of locally compact groups with a slight modification necessitated by the presence of idempotent distributions which in turn correspond to normalised Haar measures of compact subgroups. We also investigate how far these results can be extended to a general complete separable metric group.

Chapter II

Indecomposable Distributions

2.1. Introduction

According to a theorem of A.I. Khintchine [9], any distribution on the real line can be written as the convolution of two distributions, one of which is the convolution of a finite or countable number of indecomposable distributions and the other is infinitely divisible with out indecomposable factors. Further any distribution which is not infinitely divisible has at least one indecomposable factor. This result gives an indication of the existence of a large class of indecomposable distributions. It is however not clear from this result alone that there exists a non-atomic or absolutely continuous indecomposable distribution. This question was raised by H. Cramer [3] and an answer in the affirmative was given by P. Levy [13]. However what is available is only a few examples. In this connection there arises naturally the question of the 'size' of the class $\mathcal{M}_1$, of indecomposable distributions among the class $\mathcal{M}$ of all distributions. More precisely what is the category of $\mathcal{M}_1$ in $\mathcal{M}$?

In the present chapter these questions are answered under the framework of a complete separable metric group G. We consider three classes of distributions: (1) all indecomposable distributions, (2) all non-atomic indecomposable distributions (3) all indecomposable distributions that are absolutely continuous with respect to the Haar measure when the group G is locally compact abelian. It is shown that under suitable conditions the three classes are of the second category.

A theorem concerning convergence of distributions when suitably centered is proved in the present chapter which will serve as a very useful tool in the following chapters.

The analysis carried out also throws some light on the existence of non-atomic measures on separable and complete metric spaces.

2.2. Preliminaries. Throughout the chapter we suppose that G denotes a complete separable metric group. Additional assumptions on G will be specially mentioned as and when necessary. We employ the customary notation of denoting the group operation as xy, $x, y \in G$ in the case of general groups and as $x+y$ if G is abelian. e always denotes the unit in G. For any two subsets A, B of G we write $AB = [z : z = xy, x \in A, y \in B]$ and $A^{-1} = [z : z^{-1} \in A]$ (in case the group is abelian we use instead the symbols $A+B$ and $-A$ respectively).

The convolution operation. By a measure (or distribution) we mean a probability measure defined on the σ-field $\mathcal{B}$ of Borel subsets of G. Let $\mathcal{M}$ denote the collection of all probability measures on $\mathcal{B}$ For $\mu, \nu \in \mathcal{M}$ the convolution $\mu * \nu$ is defined as follows

$$(2.1) \qquad (\mu * \nu)(A) = \int \mu(Ax^{-1}) d\,\nu(x).$$

With this operation $\mathcal{M}$ becomes a semigroup which is abelian if and only if G is so. It should be noted that $\mu * \nu$ in (2.1) can be written in the equivalent form

$$(\mu * \nu)(A) = \int \nu(x^{-1} A) d\mu(x)$$

For each $g \in G$ and $\mu \in \mathfrak{M}$, $\mu * g$ denotes the right translate of μ by g i.e., the measure $\mu(Eg^{-1})$. $g * \mu$ is defined similarly. By a translate of μ we mean a measure of the form $\mu * g$ or $g * \mu$.

Definition 2.2.1. A measure λ is decomposable if and only if there exist two nondegenerate measures μ and ν such that $\lambda = \mu * \nu$. In the contrary case λ is said to be indecomposable.

Definition 2.2.2. A nondegenerate measure α is said to be a factor of a measure β if and only if there exists a measure γ such that either $\beta = \alpha * \gamma$ or $\beta = \gamma * \alpha$.

We shall denote by $\mathfrak{M}_0$ the set of all decomposable measures and $\mathfrak{M}_1$ the set of all indecomposable measures.

Definition 2.2.3. The spectrum of a measure μ is the smallest closed set $A \subset G$ such that $\mu(A) = 1$.

The existence of the spectrum is well known and it is also easy to show that if A, B, and C are the spectra of the measures μ, ν and $\mu * \nu$ respectively then C = closure of (AB).

Topologies in $\mathfrak{M}$. In the sequel we shall be mainly concerned with the weak topology in $\mathfrak{M}$. It is defined through convergence as follows:

Definition 2.2.4. A sequence of measures μ_n converges weakly to a measure μ if and only if, for every real valued bounded continuous function f defined on G, $\int f d\mu_n \to \int f d\mu$.

It is clear that the class of subsets of $\mathfrak{M}$ of the form

$$[\mu : | \int f_i d\mu - \int f_i d\mu_0 | < \varepsilon_i, \ i = 1, 2, \dots k]$$

where $(f_1, \ldots, f_k)$ is any finite set of bounded continuous functions and $(\varepsilon_1, \ldots, \varepsilon_k)$ is any finite set of positive numbers forms a neighbourhood system for the weak topology in $\mathcal{M}$. It is useful to note that the sets of the type

$$[\mu : \mu(V_i) > \mu_o(V_i) - \varepsilon_i, \; i = 1, 2, \ldots, k]$$

where $\varepsilon_i > 0$ for all i and V_i are open subsets of G, are open in the weak topology.

Now we shall gather a few results about the weak topology in $\mathcal{M}$ which we need in the sequel.

Theorem 2.2.1. (Prohorov [16], Varadarajan [21]). If G is a complete separable metric space, the space $\mathcal{M}$ of measures on G becomes a complete separable metric space under the weak topology.

Theorem 2.2.2. (Prohorov [16]). If G is a complete separable metric space, a subset $M \subset \mathcal{M}$ is conditionally compact in the weak topology if and only if, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset G$ such that $\mu(K_\varepsilon) > 1 - \varepsilon$ for every $\mu \varepsilon M$.

Theorem 2.2.3. (Ranga Rao [17]). In a complete separable metric space G a sequence $\mu_n \varepsilon \mathcal{M}$ converges weakly to $\mu \varepsilon \mathcal{M}$ if and only if the following holds: For every class $\mathcal{A}$ of continuous functions on G such that

i) $\mathcal{A}$ is uniformly bounded,

ii) $\mathcal{A}$ is compact in the topology of uniform convergence on compacta,

$$\lim_{n \to \infty} \sup_{f \varepsilon \mathcal{A}} \left| \int f d\mu_n - \int f d\mu \right| = 0.$$

All topological notions in $\mathfrak{M}$ used in sections 3-6 refer to the weak topology. Only in the last section, we find it necessary to consider the strong topology induced by the norm $||\mu|| = \sup_{A \in \mathfrak{B}} |\mu(A)|$, $\mu \in \mathfrak{M}$.

Indecomposable distributions on the real line. We now state two results due to P. Levy [13] concerning absolutely continuous indecomposable distributions on the real line. We shall have occasion to use the latter one in the last section.

Let Z be any real-valued random variable which takes values in a bounded interval. We denote by $[Z]$ the integral part of Z and the conditional distribution of Z given that $[Z] = n$, by μ_n.

Theorem 2.2.4. Let Z be any real-valued random variable taking values in a bounded interval and satisfying the following properties:

(a) $[Z]$ is even with probability one,

(b) the distribution of $[Z]$ is indecomposable,

(c) the family of distributions μ_n, n running over possible values of $[Z]$, has no common factor.

Then the distribution of Z is indecomposable.

Theorem 2.2.5. Let μ_1, μ_2 denote the uniform distributions on the intervals $[a, b]$ and $[c, d]$ respectively. If $(b-a)/(d-c)$ is irrational then μ_1 and μ_2 have no common factor.

2.3. Shift compactness in $\mathfrak{M}$. In the theory of sums of independent random variables we often come across the situation where a sequence of distributions fails to converge to any limit but actually does converge when suitably centered (see Gnedenko and Kolmogorov [4])

In this section we shall make an analysis of this phenomenon in relation to the convolution operation between distributions on groups. To this end, it is convenient to introduce the following

Definition 2.3.1. A family $\mathcal{N}$ is said to be shift compact if, for every sequence $\mu_n \varepsilon \mathcal{N}$ $(n = 1, 2, \ldots)$, there is a sequence of measures ν_n such that (1) ν_n is a translate of μ_n and (2) ν_n has a convergent subsequence.

The main result to this section is the following theorem which reveals an important structural property of the topological semigroup in relation to the notion of shift compactness. This will be used often in the following chapters.

Theorem 2.3.1. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, be three sequences of measures on G such that

$$\lambda_n = \mu_n * \nu_n \qquad (n = 1, 2, \ldots) \tag{3.1}$$

If the sequence $\{\lambda_n\}$ is conditionally compact, then each one of the sequences $\{\mu_n\}$ and $\{\nu_n\}$ is shift compact.

As an immediate consequence we have the following

Corollary 2.3.1. For any $\lambda \varepsilon \mathcal{M}$, the family $\mathcal{F}(\lambda)$ of all factors of λ is shift compact.

Before proceeding to the proof of Theorem 2.3.1 we shall establish the following

Lemma 2.3.1. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ be three sequences of measures on G such that $\lambda_n = \mu_n * \nu_n$ for each n. If the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are conditionally compact then so is the sequence $\{\nu_n\}$.

Proof. Since the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are conditionally compact it follows from Theorem 2.2.2 that given $\varepsilon > 0$, there exists a compact set K_ε such that

$$\lambda_n(K_\varepsilon) > 1 - \varepsilon, \quad \mu_n(K_\varepsilon) > 1 - \varepsilon$$

for all n. Then we have

$$(3.2) \qquad 1 - \varepsilon < \lambda_n(K_\varepsilon) = \int \nu_n(x^{-1}K_\varepsilon)d\mu_n(x)$$
$$\leq \int_{K_\varepsilon} \nu_n(x^{-1} K_\varepsilon)d\mu_n(x) + \varepsilon$$

or

$$(3.3) \qquad \int_{K_\varepsilon} \nu_n(x^{-1} K_\varepsilon)d\mu_n(x) > 1 - 2\varepsilon$$

(3.3) implies the existence of a point $x_n \varepsilon K_\varepsilon$ with the property

$$\nu_n(x_n^{-1} K_\varepsilon) > 1 - 3\varepsilon$$

and consequently we have

$$\nu_n(K_\varepsilon^{-1} K_\varepsilon) > 1 - 3\,\varepsilon$$

for all n. Since $K_\varepsilon^{-1} K_\varepsilon$ is compact and independent of n, another application of Theorem 2.2.2 leads to the fact that the sequence $\{\nu_n\}$ is conditionally compact. This completes the proof of the lemma.

Proof of Theorem 2.3.1. We choose a sequence ε_r of positive numbers such that $\sum \varepsilon_r < \frac{1}{4}$. Then the conditional compactness of the sequence $\{\lambda_n\}$ implies, by Theorem 2.2.2, that there exists a sequence of compact sets K_r such that

$$\lambda_n(K_r) > 1 - \varepsilon_r, \qquad r = 1, 2, \ldots$$

for all n. Now choose a positive sequence η_n descending to zero and satisfying

$$\sum_{r=1}^{\infty} \varepsilon_r \eta_n^{-1} \leq \frac{1}{2}.$$

Let

$$E_{nr} = [x : \mu_n(K_r x^{-1}) > 1 - \eta_n]$$

(3.4)

$$F_n = \bigcap_{r=1}^{\infty} E_{nr} .$$

Then, from (3.1) and (3.4), we have

$$1 - \varepsilon_r \leq \lambda_n(K_r) = \int_{E_{nr}} \mu_n(K_r x^{-1}) d\, \nu_n(x) + \int_{E_{nr}'} \mu_n(K_r x^{-1}) d\, \nu_n(x)$$

$$\leq \nu_n(E_{nr}) + (1 - \eta_n)\ \nu_n(E_{nr}')$$

where E_{nr}' denotes the complement of the set E_{nr}. Thus we obtain

$$\nu_n(E_{nr}') \leq \frac{\varepsilon_r}{\eta_n}$$

and consequently

$$\nu_n(F_n') \leq \sum_n \varepsilon_r \eta_n^{-1} \leq \frac{1}{2} .$$

Hence $F_n \neq \emptyset$. Let x_n be any element in F_n. Then, from the definition of F_{nr} we have

(3.5)

$$\mu_n(K_r x_n^{-1}) > 1 - \eta_n$$

for all n and all r.

We now write $\alpha_n = \mu_n * x_n$ (the right translate of μ_n by x_n) and $\beta_n = x_n^{-1} * \nu_n$. Then, obviously, $\lambda_n = \alpha_n * \beta_n$ and from (3.5) and Theorem 2.2.2 it follows that both the sequences $\{\lambda_n\}$ and $\{\alpha_n\}$ are conditionally compact. Lemma 2.3.1 now implies that $\{\beta_n\}$ is conditionally compact. The fact that α_n and β_n are translates of μ_n and ν_n respectively completes the proof of the theorem.

2.4. The class $\mathcal{M}_1$ is a G_δ. The purpose of this section is to prove the following

Theorem 2.4.1. Let G be a complete separable metric group Then the class $\mathcal{M}_1$ of all indecomposable distributions forms a G_δ.

Before proceeding to the proof of this theorem we make a digression in order to pick up a few auxiliary facts.

Let $f_1, f_2, \ldots$, be a sequence of bounded functions on G with the following properties

(a) for each j, $f_j(x)$ is uniformly continuous in both the right and left uniformities of G.

(b) the sequence $\{f_j\}$ separates points of G.

The existence of such a sequence of functions may be seen as follows. Since G is a separable metric group there is a sequence of neighbourhoods $\{N_i\}$ of e such that $\bigcap_{i=1}^{\infty} N_i = \{e\}$. Then by a well known result (cf. A. Weil [23, pp. 13-14]) there exists a sequence of functions $f_i(x)$ $(i = 1, 2, \ldots)$ which are uniformly continuous in the two-sided uniformity (i.e., in both the right and left uniformities) and such that $f_i(e) = 0$ and $f_i(x) = 1$

for $x \notin N_i$. Let $\{x_n\}$ be a sequence dense in G. Then the countable family $S = \{\phi_{ij}\}$ where $\phi_{ij}(x) = f_i(xx_j)$ possesses both the properties. It is only necessary to prove that the family S separates points of G. If not let a and b be two distinct elements of G such that $\phi_{ij}(a) = \phi_{ij}(b)$ for all i and j or equivalently $f_i(ax_j) = f_i(bx_j)$ for all i and j. Since x_j is dense it follows that $f_i(ab^{-1}) = f_i(e) = 0$ for all i. But $ab^{-1} \notin N_i$ for some i and hence $f_i(ab^{-1}) = 1$ for some i. This contradiction shows that $S = (\phi_{ij})$ separates points of G.

In all that follows, $S = \{f_j\}$ is a fixed sequence with the above properties.

It is then clear that a measure μ on G is degenerate if and only if the induced measure μf_j^{-1} on the real line is degenerate for each j. For any real valued bounded continuous function f and any measure μ, we write

$$(4.1)\qquad \begin{aligned} V_1(f,\mu) &= \sup_{a\in G} \left[\int f^2(xa)d\mu - \left(\int f(xa)d\mu\right)^2\right], \\ V_2(f,\mu) &= \sup_{a\in G} \left[\int f^2(ax)d\mu - \left(\int f(ax)d\mu\right)^2\right] \end{aligned}$$

It is obvious that a measure μ is degenerate if and only if $V_1(f_j,\mu) = 0$ for all j or equivalently $V_2(f_j,\mu) = 0$ for all j.

Lemma 2.4.1. If f is bounded and uniformly continuous in both the right and left uniformities of the group G and μ_n is a sequence of measures converging weakly to μ, then

$$\lim_{n\to\infty} V_1(f,\mu_n) = V_1(f, \mu)$$

$$\lim_{n\to\infty} V_2(f, \mu_n) = V_2(f, \mu)$$

Proof. For each f which is bounded and uniformly continuous in the right as well as the left uniformity, it is clear that each one of the families of functions $\{f(xa), a \in G\}$, $\{f^2(xa), a \in G\}$, $\{f(ax), a \in G\}$, $\{f^2(ax), a \in G\}$ is uniformly bounded and equi-continuous at each point of G. Consequently, they are conditionally compact in the topology of uniform convergence on compacta. The lemma is then an immediate consequence of Theorem 2.2.3.

Let Now $E_{ij}(\varepsilon)$ be defined as follows

$$E_{ij}(\varepsilon) = [\mu : \mu = \alpha * \beta,\ V_1(f_i, \alpha) \geq \varepsilon,\ V_2(f_j, \beta) \geq \varepsilon] \tag{4.2}$$

where f_i and f_j are any two functions from S. Then we have

Lemma 2.4.2. For any $\varepsilon > 0$ and each i, j the set $E_{ij}(\varepsilon)$ is closed.

Proof. Let μ_n be a sequence of measures in $E_{ij}(\varepsilon)$ converging to some measure μ. Then by (4.2) there exist measures α_n and β_n such that

$$\begin{gathered} \mu_n = \alpha_n * \beta_n \\ V_1(f_i, \alpha_n) \geq \varepsilon, \quad V_2(f_j, \beta_n) \geq \varepsilon \end{gathered} \tag{4.3}$$

From Theorem 2.3.1, it follows that there exists a sequence $a_n \in G$ such that the sequences of measures $\alpha_n * a_n$ and $a_n^{-1} * \beta_n$ are conditionally compact. Thus we can choose subsequences $\alpha_{n_k} * a_{n_k}$ and $a_{n_k}^{-1} * \beta_{n_k}$ converging to some measures α_0 and β_0 respectively.

Since μ_n converges to μ and $\mu_{n_k} = \alpha_{n_k} * a_{n_k} * a^{-1}_{n_k} * \beta_{n_k}$ we have $\mu = \alpha_0 * \beta_0$. It is clear from the definition of $V_1(f, \mu)$ (see (4.1) that

$$(4.4) \qquad V_1(f, \alpha_{n_k} * a_{n_k}) = V_1(f, \alpha_{n_k}).$$

From Lemma 4.1 it follows immediately that

$$\lim_{k \to \infty} V_1(f_i, \alpha_{n_k}) = V_1(f_i, \alpha_0).$$

Similarly

$$\lim_{k \to \infty} V_2(f_j, \beta_{n_k}) = V_2(f_j, \beta_0).$$

Thus from (4.3) we have

$$V_1(f_i, \alpha_0) \geq \varepsilon, \qquad V_2(f_j, \beta_0) \geq \varepsilon$$

or $\mu \varepsilon E_{ij}(\varepsilon)$. This completes the proof.

We shall now prove Theorem 2.4.1 by showing that the set of all decomposable measures is an F_σ. In fact

$$(4.5) \qquad \mathcal{M}_0 = \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{r=1}^{\infty} E_{ij}(r^{-1}).$$

It is clear that any measure belonging to the right side of (4.5) is decomposable and hence belongs to $\mathcal{M}_0$. Let now μ be any measure in $\mathcal{M}_0$. Then there exist two nondegenerate measures α and β such that $\mu = \alpha * \beta$. Since α and β are nondegenerate it follows from the remarks made earlier that there exist two functions f_i and f_j belonging to S, with the property

$$V_1(f_i, \alpha) > 0, \quad V_2(f_j, \beta) > 0.$$

Let $\varepsilon = \min [V_1(f_i, \alpha), V_2(f_j, \beta)]$. Then for $r > 1/\varepsilon$, $\mu \varepsilon E_{ij}(r^{-1})$. Thus $\mathcal{M}_0$ is contained in the right side of (4.5). An application of Lemma 2.4.2 completes the proof of the theorem.

Remarks. 1. Let $\mathcal{M}_c$ denote the class of nonatomic measures. It may be noted that the class of all indecomposable nonatomic measures is a G_δ in $\mathcal{M}_c$ under the relative topology. At the moment, it is not clear whether even a single nonatomic measure exists in G. These points will be clarified in section 6.

2. Let G be a locally compact abelian separable metric group and $\mathcal{A}(G)$ the class of all absolutely continuous distributions in G. Since the norm topology in $\mathcal{A}(G)$ is stronger than the weak topology it is clear that the set of all indecomposable absolutely continuous distributions is a G_δ in $\mathcal{A}(G)$ (the relevant topology being the norm topology).

To determine the category of the various classes of indecomposable distributions, it is thus sufficient to find their closures. We note one case where the class $\mathcal{M}_1$ is of first category. This is the situation when the group G is finite, as is implied in the work of Vorobev [22].

In the rest of the chapter we will study the closures of three classes of indecomposable distributions

(1) the general case - the class $\mathcal{M}_1$ itself;

(2) the nonatomic case - the class of all nonatomic measures in $\mathcal{M}_1$

(3) the absolutely continuous case - the class of indecomposable distributions absolutely continuous with respect to the

Haar measure in a locally compact abelian group G.

<u>2.5. The general case.</u> Before we state the main theorem of this section we begin with some lemmas, the purpose of which is to construct indecomposable distributions in G. To this end we introduce the following definition.

Definition 2.5.1. A subset $A \subset G$ is said to be decomposable if there exist two sets $A_1, A_2 \subset G$ such that (a) each of A_1, A_2 contains at least two elements and (b) $A_1A_2 = A$; a set $A \subset G$ is said to be indecomposable if it is not decomposable.

Lemma 2.5.1. Let A be any countable indecomposable set and μ a measure such that $\mu(A) = 1$ and $\mu(\{g\}) > 0$ for every $g \in A$. Then μ is indecomposable.

Proof. Let us suppose that μ is decomposable. Then $\mu = \mu_1 * \mu_2$ where μ_1 and μ_2 are nondegenerate measures with mass concentrated at a countable or finite number of points. Let

$$A_i = [g : g \in G,\ \mu_i(g) > 0], \qquad i = 1, 2.$$

From the conditions of the lemma it follows that $A = A_1A_2$, which contradicts the fact that A is indecomposable.

Lemma 2.5.2. Let B be an infinite countable set $\{g_1, g_2, \ldots, \}$ with the following property: $g_r g_s^{-1} \neq g_t g_u^{-1}$ for every set of distinct intergers r, s, t, u no two of which are equal. If F is any finite subset of G then the set $B \cup F$ is indecomposable.

Proof. Suppose the lemma is not true. Then there exist two sets A_1, A_2, at least one of which contains an infinite number of elements and such that $B \cup F = A_1 A_2$. Let $A_1 = (x_1, x_2 \ldots x_n \ldots,)$ and $y_1, y_2 \in A_2$. Since the elements $x_r y_1 r = 1, 2, \ldots$ are all distinct all but a finite number of them belong to B. Thus there exists a finite set N of integers such that

$$x_r y_1 \in B \qquad\qquad \text{for} \quad r \notin N$$

Take any integer $m \notin N$. For at most one integer s, say $s = k_1$, $x_s y_1$ can be equal to $x_m y_2$. Similarly, or at most one integer, say $s = k_2$, $x_s y_2$ can be equal to $x_m y_1$. Choose any integer $n \notin N$ and different from k_1 and k_2. Then $x_m y_1$, $x_m y_2$, $x_n y_1$ and $x_n y_2$ are all distinct and belong to B. But

$$x_m y_1 (x_n y_1)^{-1} = x_m y_2 (x_n y_2)^{-1}$$

which contradicts the defining property of B.

Lemma 2.5.3. If G is an infinite group, then there exists a set B with the property described in Lemma 2.5.2.

Proof. Let g_1, g_2, g_3 be any three distinct elements of the group G. Suppose $g_1, g_2, \ldots, g_n$ have been chosen. Consider the set A_n of all elements of the form $g_{i_1}^{\pm 1} \; g_{i_2}^{\pm 1} \; g_{i_3}^{\pm 1}$ where i_1, i_2, i_3 are any three positive integers less than or equal to n. Since A_n is finite and the group G is infinite A_n' is nonempty. Choose any element g_{n+1} from A_n'. The sequence $g_1, g_2, \ldots$ chosen in this way has the required property.

Theorem 5.1. If the group G is infinite, then $\mathcal{M}_1$ is a dense G_δ

Proof. Any measure in G is a weak limit of measures concentrated at a finite number of points. From Lemma 2.5.1-2.5.3 it is clear that any measure with a finite spectrum is a weak limit of indecomposable distributions. Thus indecomposable distributions are dense in $\mathcal{M}$. In view of Theorem 2.4.1, this completes the proof.

2.6. The monatomic case. To start with we shall investigate the existence of a non-atomic measure in an arbitrary complete separable metric space. This, in itself, is not with out interest. (cf. [19]).

Theorem 2.6.1. Let X be any complete separable metric space without any isolated points. Then there exists a non-atomic measure on X.

Proof. Let $\mathcal{M}$ be the class of all probability measures on X. Then by Theorem 2.2.1 $\mathcal{M}$ is a complete separable metric space under the weak topology. For any given $\varepsilon > 0$, we denote by $C(\varepsilon)$ the class of all measures which have at least one atom of mass greater than or equal to ε. Then the class of all measures with atomic components can be represented as $\bigcup_{r=1}^{\infty} C(1/r)$. If there does not exist any non-atomic measure, we have

$$\mathcal{M} = \bigcup_{r=1}^{\infty} C\left(\frac{1}{r}\right).$$

It is not difficult to verify by making use of Theorem 2.2.2 that $C(\varepsilon)$ is closed in the weak topology. Thus by Baire category theorem, at least one $C(1/r)$ has interior. Hence there exists a measure μ_0 with an atom of positive mass $> \delta > 0$ such that, whenever a sequence of measures μ_n

converges weakly to μ_o, μ_n has an atom of mass at least δ for sufficiently large n. Since measures with finite spectrum (i.e., for which spectrum is a finite set) are everywhere dense we can, without loss of generality, assume that μ_o is a measure with masses $p_1, p_2, \ldots, p_k$ at the points $x_1, x_2, \ldots, x_k$ respectively.

Let $N_n(x_1), N_n(x_2), \ldots, N_n(x_k)$ be sequences of neighbourhoods shrinking to $x_1, x_2, \ldots, x_k$ respectively. We can and do assume that these neighbourhoods are disjoint for each fixed n. Since by assumption X has no isolated points each of these neighbourhoods contains an infinite number of points. We distribute the mass p_i among the points of $N_n(x_i)$ such that the mass at each point is less than $\delta/2$. By doing this for every i and every n we obtain a sequence of measures μ_n converging weakly to μ_o and such that the mass of μ_n at any point is $\leq \delta/2$. This contradicts the defining property of μ_o and shows that $C(1/r)$ has no interior for any r. The proof is complete.

Corollary 2.6.1. Let X be any complete separable metric space with an uncountable number of points. Then there exists a non-atomic measure on X.

Proof. Let Y denote the set of all accumulation points of X. Then it is well known (cf. Hausdorff [6 pp. 146]) that X can be written in the form $X = Y \cup N$ where

(a) Y is closed and dense in itself,

(b) N is countable.

By Theorem 2.6.1 there exists a non-atomic measure on Y and hence on X.

The proof of Theorem 2.6.1 actually yields something more. In fact we have

Corollary 2.6.2. Let X be a complete separable metric space without isolated points. Then the set of all non-atomic measures which give positive mass to each open subset of X is a dense G_δ in $\mathcal{M}$.

If G is any nondiscrete complete metric group, then it is clear that it is necessarily uncountable and cannot have any isolated point. Consequently we have the following:

Theorem 6.2. Let G be any nondiscrete complete separable metric group. Then the set of all non-atomic indecomposable distributions which give positive mass to each open set is a dense G_δ in $\mathcal{M}$.

2.7. The absolutely continuous case. In this section we suppose that G is in addition locally compact abelian and consider measures absolutely continuous with respect to the Haar measure on G. Let $\mathcal{A} = \mathcal{A}(G)$ denote the collection of these measures. The convergence notion that is appropriate for $\mathcal{A}$ is the norm convergenced of measures or the L_1 convergence of their densities. The main object here is to show that in the sense of this convergence the indecomposable measures in $\mathcal{A}$ are dense.

In the first instance we develop in the following lemmas a general method of constructing absolutely continuous indecomposable distributions in G.

Lemma 2.7.1. Let A_1, A_2, A_3 be three closed disjoint subsets of G satisfying the following conditions :

(1) $(A_i - A_i) \cap (A_j - A_k) = \emptyset$ for $i = 1, 2, 3$ and $j \neq k$

(2) $(A_1 - A_2) \cap (A_2 - A_3) = (A_2 - A_3) \cap (A_3 - A_1)$

$= (A_3 - A_1) \cap (A_1 - A_2) = \emptyset$.

Let μ_1, μ_2, μ_3 be measures with $\mu_i(A_i) = 1$ and $\lambda = p_1\mu_1 + p_2\mu_2 + p_3\mu_3$ where $p_i > 0$ $(i = 1, 2, 3)$ and $p_1 + p_2 + p_3 = 1$. Then λ is decomposable if and only if μ_1, μ_2, μ_3 have a nondegenerate common factor.

Proof. If μ_1, μ_2, μ_3 have a nondegenerate common factor, it is obvious that λ is decomposable. Conversely let us suppose that λ is decomposable. Then there exist two nondegenerate measures α and β such that $\lambda = \alpha * \beta$. Let C and D denote the spectra of α and β respectively. It is obvious that

(7.1) $$C + D \subset A_1 \cup A_2 \cup A_3 = A.$$

For each $c \in G$, we write

(7.2) $$D_i(c) = [d : d \in D \text{ and } c + d \in A_i] = D \cap (A_i - c)$$

for $i = 1, 2, 3$. The rest of the proof depends on an analysis of the nature of decomposition $\{ D_i(c) \}$ of D. It is convenient to divide it into three steps.

1. The sets $D_i(c)$ possess the following properties.

(i) $\bigcup_{i=1}^{3} D_i(c) = D$ for each c

(ii) $D_i(c) \cap D_j(c) = \emptyset$ for $i \neq j$.

(iii) For any two distinct c_1 and c_2, $D_i(c_1) = D_i(c_2)$ for some i, implies that $D_j(c_1) = D_j(c_2)$ for $j = 1, 2, 3$.

(iv) if $c_1 \neq c_2$, $D_i(c_1) \cap D_j(c_2) \neq \emptyset$ implies that $D_i(c_1) = D_j(c_2)$.

The first three properties are very simple. We shall prove (iv). Let us suppose that $D_i(c)$ and $D_j(c)$ have a common point d and $D_i(c_1) \neq D_j(c_2)$. Then there exists a point $d' \in D_i(c_1)$ which is not in $D_j(c_2)$. From (i) it follows that there is a $k(\neq j)$ such that $d' \in D_k(c_2)$. From these facts we have

(7.3)
$$c_1 + d \in A_i, \qquad c_1 + d' \in A_i \quad \text{and}$$
$$c_2 + d \in A_j, \qquad c_2 + d' \in A_k.$$

Consequently

$$d - d' \in (A_i - A_i) \cap (A_j - A_k).$$

This contradicts the assumption (1) of the lemma and proves (iv). It should be noted that the property (iv) implies that the decompositions $\{D_i(c)\}$ for $c \in C$ are only permutations of each other.

II. One of the following relations is always satisfied. Either

(a) for each $c \in C$, all but one of $D_i(c)$ are empty, i.e., $D_i(c) = D$ for some i, or

(b) for any two $c_1, c_2 \in C$, $D_i(c_1) = D_i(c_2)$ for $i = 1, 2, 3$.

The proof of this is quite straightforward and is similar to that of (iv) above.

III. Now we suppose that case (a) obtains. Let $C_i = [c : D_i(c) \neq \emptyset]$. It is then easily verified that (1) C_i's are mutually disjoint and their union is C, (2) $C_i + D \subset A_i$ for each i. Let the measures α_i $(i = 1, 2, 3)$ be defined as follows:

$$\alpha_i(E) = \alpha(E \cap C_i)/\alpha(C_i)$$

(Note that $\alpha(C_i) > 0$). It is then easy to see that

$$\alpha_i * \beta = \mu_i \qquad \text{for } i = 1, 2, 3.$$

Thus for each i, β is a factor of μ_i.

In case (b), let $D_i = D_i(c)$. Obviously D_i's are mutually disjoint and $C + D_i \subset A_i$ for each i. Writing $\beta_i(E) = \beta(E \cap D_i)/\beta(D_i)$ we get as before

$$\alpha * \beta_i = \mu_i \qquad \text{for } i = 1, 2, 3.$$

In this case α is the required common factor. This completes the proof of the lemma.

Lemma 2.7.2. Let G be a noncompact group. Then for any given compact set K, there exist elements $g, h \in G$ such that the sets K, $K + g$, $K + h$ satisfy the conditions (1) and (2) of Lemma 2.7.1.

Proof. It may be verified that conditions (1) and (2) of Lemma 2.7.1 in this case reduce to choosing g and h such that none of the elements g, h, $g - h$, $g + h$, $2g + h$, $2h - g$ belong to the compact set $C = (K - K) - (K - K)$. Let

$$F = [x : x = 2y,\ y \in G].$$

Then there are two possibilities.

Case 1. F has compact closure. In this case we can choose an element g such that $g \in C$ and $\bar{F} \cap (C + g) = \emptyset$. Since G is noncompact such elements exist. Let h be any element such that $h \notin C \cup (C+g) \cup (C-g) \cup (C+2g)$. The pair g, h satisfies our requirements.

Case 2. The closure of F is not compact. Let $g \notin C$ be arbitrary. Since $\bar{F}$ is not compact we can find an $h \in G$ such that $2h \notin C+g$ and $h \notin C \cup (C+g) \cup (C-g) \cup (C-2g) \cup (C+2g)$. As is easily verified the pair g, h serves our purpose. This completes the proof.

Lemma 2.7.3. Let G be an infinite compact metric abelian group. Let A be a subset such that

(1) $$0 < \lambda(A) < 1$$

(2) $$\int_A \chi_j(x)\,d\lambda(x) \neq 0 \qquad \text{for } j = 0, 1, 2, \ldots$$

where λ is the normalized Haar measure on G and $\chi_0, \chi_1, \ldots$ are the characters of G. If λ_1, λ_2 are defined by

$$\lambda_1(E) = \lambda(E \cap A)/\lambda(A)$$

$$\lambda_2(E) = \lambda(E \cap A')/\lambda(A'),$$

then λ_1 and λ_2 do not have a common factor.

Proof. Let χ_0 be the identity character. Since for the Haar measure $\int_G \chi_j d\lambda = 0$ for $j \neq 0$ and $0 < \lambda(A) < 1$, we have

(7.4) $$\int_{A'} \chi_j \, d\lambda \neq 0 \quad \text{for every } j.$$

We shall now prove that the measures λ_1 and λ_2 cannot have a common factor. If this is not true, then let μ be a common factor. Then there exist measures α_1, α_2 such that

$$\lambda_1 = \alpha_1 * \mu, \quad \lambda_2 = \alpha_2 * \mu. \tag{7.5}$$

From the definitions of λ_1 and λ_2 and (7.5) we have

$$\lambda(A)\lambda_1 + \lambda(A')\lambda_2 = \lambda = (\lambda(A)\alpha_1 + \lambda(A')\alpha_2)*\mu. \tag{7.6}$$

Taking the characteristic functionals on both sides of (7.5) and (7.6) we get

$$\begin{aligned} \int \chi_j \, d\lambda_1 &= (\int \chi_j d\alpha_1)(\int \chi_j d\mu) \\ \int \chi_j \, d\lambda_2 &= (\int \chi_j d\alpha_2)(\int \chi_j d\mu) \end{aligned} \tag{7.7}$$

$$(\lambda(A) \int \chi_j d\alpha_1 + \lambda(A') \int \chi_j d\alpha_2)(\int \chi_j d\mu) = 0 \qquad \text{for } j \neq 0.$$

From condition (2) of the lemma and (7.7) we deduce that

$$\int \chi_j d\mu \neq 0 \qquad \text{for all } j.$$

Thus from (7.7) we have

$$\lambda(A) \int \chi_j d\alpha_1 + \lambda(A') \int \chi_j d\alpha_2 = 0 \quad \text{for all } j \neq 0$$

which is the same as saying

$$\lambda(A)\alpha_1 + \lambda(A')\alpha_2 = \lambda. \tag{7.8}$$

From (7.5) and the definition of λ_1 and λ_2 we get

$$\int \alpha_1(A' - x)d\mu(x) = \lambda_1(A') = 0$$

$$\int \alpha_2(A - x)d\mu(x) = \lambda_2(A) = 0.$$

Consequently

$$\alpha_1(A' - x) = 0 \text{ a.e}(\mu)$$

$$\alpha_2(A - x) = 0 \text{ a.e}(\mu).$$

Thus there exists a point x_o such that

$$(7.9) \qquad \alpha_1(A' - x_o) = \alpha_2(A - x_o) = 0.$$

(7.8) and (7.9) imply that

$$\alpha_1(E) = \frac{\lambda(E \cap [A - x_o])}{\lambda(A)} = [\lambda_1 * (- x_o)](E)$$

$$\alpha_2(E) = \frac{\lambda(E \cap [A' - x_o])}{\lambda(A')} = [\lambda_2 * (- x_o)](E).$$

Thus from (7.5) and (7.10) we obtain

$$(7.11) \qquad \lambda_1 = \lambda_1 * (- x_o) * \mu, \quad \lambda_2 = \lambda_2 * (- x_o) * \mu.$$

Taking characteristic functionals on both the sides of (7.11) we have

$$\int \chi_j d((- x_o) * \mu) = 1 \qquad \text{for all } j.$$

Thus μ is degenerate at the point x_o. The proof of the lemma is complete.

Lemma 2.7.4. In any infinite compact group G there exists a set A possessing the properties (1) and (2) of Lemma 2.7.3.

Proof. Let $S(\lambda)$ be the measure ring obtained by considering the space of Borel subsets of G modulo λ-null sets. This is a complete metric space with the distance $d(E, F) = \lambda(E \Delta F)$ where E and F belong to $S(\lambda)$ (cf. [5 pp. 165-169]). Let $\chi_0, \chi_1, \ldots$ be the characters of G, χ_0 being the identity. We consider the following mapping from $S(\lambda)$ to the complex plane. For any $E \in S(\lambda)$, we write

$$f_j(E) = \int_E \chi_j d\lambda.$$

The mapping f_j is obviously continuous. Hence the sets

$$V_j = [E: \int_E \chi_j d\lambda \neq 0]$$

are open in $S(\lambda)$. We shall now prove that each V_j is dense in $S(\lambda)$. Let $A \in S(\lambda)$ and

$$\int_A \chi_j d\lambda = 0$$

Let $\lambda(A) = c > 0$, Since λ is non-atomic, for any $0 < \varepsilon < c$ there exists a set $B \subset A$ such that $\varepsilon/2 < \lambda(B) < \varepsilon$. Let C be any subset of B for which

$$\int_C \chi_j d\lambda \neq 0.$$

Such a C exists, for otherwise χ_j will vanish in B almost everywhere but at the same time $|\chi_j| = 1$. The set $A \cap C'$ has the property

$$d(A \cap C', A) = \lambda((A \cap C') \Delta A) = \lambda(C) < \varepsilon.$$

Since this is true for any sufficiently small ε it is possible to get A as a limit of elements belonging to V_j. Since the class of sets A with $\lambda(A) > 0$ is dense in the ring $S(\lambda)$ it follows that the sets V_j are actually dense in $S(\lambda)$. By the Baire category theorem it follows that $\bigcap_{j=1}^{\infty} V_j$ is dense in $S(\lambda)$. Thus there exist Borel sets with the required properties.

Lemma 2.7.5. In any locally compact separable metric abelian group G there exist two absolutely continuous measures with compact supports which do not have a common factor.

We shall prove this lemma in two steps. First of all let us assume that G is a finite dimensional vector space. Let A_1 and A_2 be two cubes in G such that the ratio of the lengths of their sides is irrational.

Then the uniform distributions μ_1 and μ_2, concentrated in A_1 and A_2 respectively, cannot have a common factor. For, if they have, then at least one of the one-dimensional marginal distributions of μ_1 must have a common factor with the corresponding marginal distribution of μ_2. Since the corresponding marginal distributions of μ_1 and μ_2 are rectangular distributions in the real line with the ratio of the lengths of their supports irrational, it follows from Theorem 2.5.5. that they cannot have a common factor. This proves the lemma in the case when G is a vector space.

If G is an infinite compact group, we have by Lemma 2.7.3 two absolutely continuous measures which do not have a common factor.

Now a result of Pontrjagin [15] states that for any general locally compact group G there exists an open subgroup H such that

$$H = V \oplus Z$$

where V is a vector group, Z a compact group and $\oplus$ denotes the direct sum. In V we take any two absolutely continuous measures μ_1 and μ_2 without any common factor. If Z is infinite we take two absolutely continuous measures ν_1 and ν_2 in Z without common factor. If Z is finite we take ν_1 and ν_2 to be any two degenerate measures. We form the product measures

$$\lambda_1 = \mu_1 \times \nu_1 \quad , \qquad \lambda_2 = \mu_2 \times \nu_2$$

in H. Since H is open λ_1 and λ_2 are absolutely continuous with respect to the Haar measure in G. Since none of the marginals of λ_1 and λ_2 have a common factor we conclude that λ_1 and λ_2 themselves cannot have a common factor. This completes the proof of the lemma.

Theorem 2.7.1. In any locally compact noncompact complete separable metric abelian group G the set of all absolutely continuous indecomposable distributions is a dense G_δ in $\mathcal{M}(G)$.

That the set under consideration is a G_δ follows from the remarks made in section 4. It remains to be proved that it is dense. It is clear that the set of all absolutely continuous measures with compact supports is everywhere dense. Thus it remains only to prove that any absolutely continuous measure μ with compact support is a limit of a sequence of absolutely continuous indecomposable measures. Let the support of μ be K_0. Let μ_1 and μ_2 be two absolutely continuous measures with compact supports K_1 and K_2 and having no common factor. Such measures exist because of Lemma 2.7.5. Let

$$K = K_0 \cup K_1 \cup K_2$$

By using Lemma 2.7.3 we choose two points $g, h \in G$ such that K, $K+g$, $K+h$ satisfy conditions (1) and (2) of Lemma 2.7.1. We write

$$\alpha_1 = \mu, \quad \alpha_2 = \mu_1 * g, \quad \alpha_3 = \mu_2 * h$$

and

$$\mu_n = (1 - \frac{2}{n})\, \alpha_1 + \frac{1}{n} \alpha_2 + \frac{1}{n} \alpha_3, \qquad n \geq 2.$$

From Lemma 2.7.1 it follows that μ_n is indecomposable. It is obvious that μ_n converges in norm to α_1 which is the same as μ. This completes the proof of theorem.

Remark. In the above theorem the assumption of noncompactness of G has played a crucial role. The question arises - is this assumption necessary ? Or, more precisely, if G is an infinite compact group, is the collection of indecomposable distributions in $\mathcal{A}(G)$ dense in $\mathcal{A}(G)$? The answer is not known.

Chapter III

THE CASE OF A LOCALLY COMPACT GROUP

3.1. Introduction

The celebrated Levy Khintchine representation for the characteristic function $\phi(t)$ of an infinitely divisible distribution on the real line is given by

$$(1.1) \qquad \phi(t) = \exp\left\{ irt - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{\infty} \left[e^{itx} - 1 - \frac{itx}{1+x^2}\right] \frac{1+x^2}{x^2}\, dG(x) \right\}$$

where r and σ are real constants, $\sigma \geq 0$ and $G(x)$ is a bounded non-decreasing function of x which is continuous at the origin. r, σ and G are uniquely determined by $\phi(t)$. Conversely any function of the type (1.1) is the characteristic function of an infinitely divisible distribution. Khintchine and Bawly went further and proved that the limit distributions of sums of uniformly infinitesimal random variables are infinitely divisible and that they can be obtained as limits of certain accompanying laws which are infinitely divisible. A historical and complete account can be found in Gnedenko and Kolmogorov [4].

Severals attempts have been made to extend these results to other situations. K. Takano [20] has generalised both the representation and the theorem on accompanying laws to finite dimensional vector spaces. Hunt [7] has given representation of one parametric semigroups in Lie groups. Very recently Kloss [10] has extended these results to certain compact groups.

Bochner [2] has studied this problem from an entirely new angle, by axiomatising the concept of characteristic functions.

In the present chapter we generalise both the representation and the accompanying laws to the case of a locally compact group. The method although similar to the classical method in the case of the real line is slightly different and more systematic. Some of the methods are powerful enough to be capable of application in nonlocally compact groups as well.

3.2. Preliminaries and Notations.

All groups considered in the present chapter are locally compact, abelian and separable metric. Let X denote such a group and Y be its character group. For $x \in X$ and $y \in Y$, let (x,y) denote the value of the character y at the point x. By duality theory the relation between X and Y is perfectly symmetric, that is, X is the character group of Y. Further if G is a closed subgroup of X and H is the annihilator of G in Y defined by the relation

$$H = [y : (x, y) = 1 \quad \text{for all} \quad x \in G]$$

then G and Y/H are character groups of each other. These facts and some well known results concerning the structure of locally compact abelian groups will be used freely. These results are contained in A. Weil [23].

By a measure on X we shall mean a non-negative countably additive set function defined on the Borel σ-field of subsets of X. We shall refer to probability measures as distributions. Let $\mathcal{M}$ denote the class of all distributions. Then $\mathcal{M}$ becomes a semigroup under convolution. (See section 2.2). Since the group is abelian ~~so~~ is $\mathcal{M}$. The convolution $\lambda_1 * \ldots, \lambda_n$

of the n distributions will be denoted by $\prod_{j=1}^{n} \lambda_j$ and if $\lambda_1 = \lambda_2 = \ldots \lambda_n = \lambda$ this will be denoted by λ^n.

These definitions can also be extended to the case of totally finite measures also and we will have occasion to use them.

For any measure μ on X we denote by $\bar{\mu}$ the measure defined by $\bar{\mu}(A) = \mu(-A)$ where $-A$ is the set of inverses of A. Then $\bar{\mu}$ is also a measure. We denote by $|\mu|^2$ the measure $\mu * \bar{\mu}$.

The topology in $\mathcal{M}$ will be the weak topology which was defined in section 2.2. All the properties of this topology which we need have already been mentioned in section 2.2. Convergence of a sequence μ_n in this topology to μ will be denoted by $\mu_n \Rightarrow \mu$.

For $\mu \in \mathcal{M}$, the characteristic function $\mu(y)$ is a function on the character group Y, defined as follows

$$\mu(y) = \int_X (x, y) d\, \mu(x)$$

Some of the well known properties of the function $\mu(y)$ are given below.

1) $\mu(y)$ is a uniformly continuous function of y in the group uniformity
2) $\mu(y)$ determines μ uniquely
3) $(\mu * \lambda)(y) = \mu(y)\, \lambda(y)$ for all $y \in Y$ and $\mu, \lambda \in \mathcal{M}$
4) $\bar{\mu}(y) = \overline{\mu(y)}$
5) $\mu_n \Rightarrow \mu$ if and only if $\mu_n(y) \to \mu(y)$ uniformly over compact subsets of Y,
6) if $\mu_n(y)$ converges uniformly over compact subsets of Y then there is a $\mu \in \mathcal{M}$ such that $\mu_n(y) \to \mu(y)$ and hence $\mu_n \Rightarrow \mu$.

A distribution μ is said to idempotent if $\mu^2 = \mu * x$ for some $x \in X$. If we write $\lambda = \mu * (-x)$ then it is clear that $\lambda^2 = \lambda$ so that $\lambda(y) = 0$ or 1. From the inequality

$$(2.1) \qquad [1-R(x,y_1+y_2)]+[1-R(x,y_1-y_2)] \leq 2[2-R(x,y_1)-R(x,y_2)]$$

(R denoting the real part) it is clear that the set of all y for which $\lambda(y) = 1$ is an open and closed subgroup of Y. The annihilator G of this subgroup is compact and λ is the normalised Haar measure of G. Thus μ is a translate of the Haar distribution of a compact subgroup.

It follows from theorem 2.3.1 that a family μ_α of distributions is shift compact if and only if the family $|\mu_\alpha|^2$ is compact.

From theorem 2.3.1 can also be deduded the following

<u>Theorem 3.2.1.</u>

Let $\{\alpha_n\}$ be a sequence of distributions such that α_n is a factor of α_{n+1} for each n. Then if $\{\alpha_n\}$ is shift compact, one can find translates α'_n of α_n such that α'_n converges weakly.

<u>Proof.</u> Let β and β' be any two limits of shifts of subsequences from α_n. Then β is a factor β' and β' is a factor of β. Consequently β and β' are shifts of one another and hence we can choose suitable translates α'_n of α_n such that α'_n converges.

In an exactly the same way is proved

<u>Theorem 3.2.2.</u>

If $\{\alpha_n\}$ is a sequence of distribution such that α_n contains α_{n+1} as a factor for each n, shift compactness being automatically ensured, there exist translates α_n' of α_n such that α_n' converges weakly

Remark

These theorems were proved by Itô [8] for the real line and extended very recently by Kloss [10]. They are however valid in complete separable metric groups as can be seen from the above proof.

3.3. Some auxiliary Lemmas

In the present section we will prove some lemmas to be used later.

Lemma 3.3.1. For each compact set $C \subset Y$ there is a neighbourhood N_C of the identity in X and a finite set $E \subset C$ such that

$$\sup_{y \in C} [1 - R(x, y)] \leq M \sup_{y \in E} [1 - R(x, y)]$$

for all $x \in N_C$. Here M is a finite constant depending on C and R denotes the real part.

Proof. From the inequality.

$$1 - R(x_1 + x_2, y) \leq 2[1 - R(x_1, y) + 1 - R(x_2, y)]$$

it is clear that if the lemma is valid in two groups X_1 and X_2 it is valid for their direct sum $X_1 \oplus X_2$. Let now Y' be the closed subgroup generated by C and ϕ its annihilator in X. If τ denotes the cannonical map from X on to $X' = X/\phi$ it is obvious that $R(x, y) = R(\tau(x), y)$ for all $x \in X$ and $y \in Y'$. It is thus sufficient to prove the lemma when the groups concerned are X' and Y' instead of X and Y. Since Y' is compactly generated it is of the form $V \oplus G \oplus I^r$ where V is a finite dimensional vector group, G is a compact group and I^r the product of the integer group taken r times. Hence X' is of the form $V \oplus D \oplus K^r$ where D is a discrete group and K^r the r-dimensional torus. Since the lemma is trivially valid for the real line, the discrete group and the compact group the proof of the lemma is complete.

<u>Lemma 3.3.2.</u> For any $y \in Y$ there is a continuous function $hy(x)$ on X with the following properties:

1) $|hy(x)| \leq \pi$ for all $x \in X$

2) $(x, y) = \exp(i\, hy(x))$ for all $x \in Ny$

where

$$Ny = [x : |(x, y) - 1| \leq \tfrac{1}{2}]$$

<u>Proof:</u> Let $(x, y) = \exp(i\, \phi(x))$ where $-\pi \leq x < \pi$. Then $\phi(x)$ will be a continuous function of x when restricted to the closed set Ny and can be extended to the whole of X as a continuous function $hy(x)$ satisfying the first condition. Any such extension will serve the purpose of the lemma.

<u>Lemma 3.3.3.</u> There is a function $g(x, y)$ defined on the product space $X \times Y$ satisfying the following properties.

1) $g(x, y)$ is a jointly continuous function of x and y

2) $\sup\limits_{x \in X} \sup\limits_{y \in C} |g(x,y)| < \infty$ for each compact subset $C \subset Y$.

3) $g(x, y_1+y_2) = g(x, y_1) + g(x, y_2)$ for each $x \in X$ and $y_1, y_2 \in Y$

4) If C is any compact subset of y then there is a neighbourhood N_C of the identity in X such that $(x, y) = \exp[i\, g(x,y)]$ for all $x \in N_C$ and $y \in C$.

5) If C is any compact subset of Y then $g(x, y)$ tends to zero uniformly over $y \varepsilon C$ as x tends to the identity in X.

<u>Proof</u>. We will reduce the proof of the proposition to the case of simple groups by making use of the structure theory. Suppose that the lemma is true for an open subgroup G of X. Let H and Y be the character groups of G and X respectively. Since H can be obtained as a quotient group of Y by taking quotient with respect to the annihilator of G in Y there is a canonical homomorphism τ from Y to H. Suppose $g(x, h)$ has been defined for $x \varepsilon G$ and $h \varepsilon H$ with the required properties. We extend the definition of g as follows. For $x \varepsilon G$ and $y \varepsilon Y$ we define

$$g(x, y) = g(x, \tau(y))$$

For $x \notin G$, we define

$$g(x, y) = 0 \quad \text{for all } y \varepsilon Y$$

Since an open subgroup is closed the continuity of $g(x, y)$ follows immediately. The rest of the properties is an immediate consequence of their validity in $G \times H$.

In the case of a general group X we take G to be the group generated by a compact neighbourhood of the identity. This is both open and closed in X. This group G has the simple structure $V \oplus C \oplus I^r$ where V is the vector group, C a compact group and I^r the product of r copies of the integer group. We now observe that if the functions $g_1(x, y)$ and $g_2(u,v)$ with the properties mentioned in the lemma exist in groups X and U with character groups Y and V, then a function $g(\xi, \eta)$, with the same properties exists for $\xi \in X \oplus U$ and $\eta \in Y \oplus V$. We only have to choose

$$g(\xi, \eta) = g_1(x, y) + g_2(u, v)$$

where x and u are projections of ξ in X and U respectively and y and v are projections of η in Y and V respectively. Thus it is enough to construct $g(x, y)$ in the case of real line, compact group and the integer group. In the case of the integer group we can take $g(x, y)$ to be identically zero. In the case of the real line we can take $g(x, y) = \theta(x)y$ where

$$\theta(x) = \begin{cases} x & \text{for } x \in [-1, 1] \\ 1 & \text{for } x > 1 \\ -1 & \text{for } x < -1 \end{cases}$$

Here we take $(x, y) = e^{ixy}$.

Thus in order to complete the proof of the lemma it is enough to consider the case of a compact group X. Let X_e be the component of the identity in X, Y_1 the annihilator of X_e in Y, $X_1 = X/X_e$ and $Y_e = Y/Y_1$. Then Y_e is the character group of X_e and Y_1 is the character group of X_1. Since X_e is compact and connected Y_e is a discrete torsion free group. Let $\{d_\alpha\}$ be a maximal family of mutually independent elements in Y_e. Then for $d \in Y_e$ there exist elements $d_{\alpha_1}, \ldots d_{\alpha_k}$ from the maximal family and integers $n, n_1, \ldots, n_k$ such that

$$nd = n_1 d_{\alpha_1} + \ldots n_k d_{\alpha_k} \tag{3.1}$$

This representation is unique except for a multiplication by an integer on both the sides.

Each element of Y_o is a coset of Y_1 in Y. We take the coset d_α and pick out an element y_α of Y from this coset. We fix this element y_α and define

$$g(x, y_\alpha) = h_{y_\alpha}(x)$$

as in lemma 3.3.2. This is defined corresponding to each d_α from $\{d_\alpha\}$. Let now $y \in Y$ be arbitrary. Then y belongs to some coset of Y_1 which is an element of Y_o. If this element is denoted by d, then there exist integers $n, n_1, \ldots, n_k$ and elements $d_{\alpha_1}, \ldots, d_{\alpha_k}$ from d_α such that (3.1) is satisfied. We define

$$g(x,y) = \frac{n_1}{n} g(x, y_{\alpha_1}) + \ldots \frac{n_k}{n} g(x, y_{\alpha_k})$$

We shall now prove that the $g(x, y)$ constructed in this way has all the required properties.

Since $g(x, y)$ is continuous in x for each fixed y and Y is discrete the continuity of $g(x, y)$ in the both the variables follows immediately. Properties (2) and (3) are obvious from the nature of construction. Since compact sets in Y are finite sets it is enough to prove property (4) for each $y \in Y$. For any $y \in Y_1$ let $[y]$ denote the coset of Y_1 to which y belongs. Then $[y]$ is an element of Y_o. (3.1) can be rewritten as

$$(3.2) \qquad n[y] = n_1 [y_{\alpha_1}] + \cdots n_k [y_{\alpha_k}]$$

For any two elements $y_1, y_2 \in [y]$ it is clear that $y_1 - y_2 \in Y_1$. Since Y_1 is the character group of X/X_0 which is compact and totally disconnected, every element of Y_1 is of finite order. Hence for every $y \in Y_1$ there exists a neighbourhood of the identity in X where $(x, y) = 1$. This for any two elements $y_1, y_2 \in [y]$ there is a neighbourhood of the identity in X where $(x, y_1) = (x, y_2)$.

Making use of the remarks made in the previous paragraph we will complete the proof of the lemma. From the construction of $g(x, y)$ and lemma 3.3.2 it is clear that for each y_α there exists a neighbourhood of the identity in X where $\exp\{ig(x, y_\alpha)\} = (x, y_\alpha)$. Let now $y \in Y$ be arbitrary. From (3.2) it is clear that there exist elements $y_{\alpha_j 1} \ldots y_{\alpha_j n_j}$ in $[y_{\alpha_j}]$ for $j = 1, 2 \ldots k$, such that

$$(3.3) \qquad ny = \sum_{j=1}^{k} (y_{\alpha_j 1} + \ldots \; y_{\alpha_j n_j})$$

From the remarks made earlier it follows that there exists a neighbourhood of the identity in X (depending on $y_{\alpha_j r}$ and y_{α_j}) where

$$(x, y_{\alpha_j r}) = (x, y_{\alpha_j})$$

Denoting by N the intersection of all the neighbourhoods corresponding to $y_{\alpha_j r}$ $(r = 1, 2, \ldots n_j, \; j = 1, 2, \ldots k)$ we have

$$(3.4) \qquad (x, y_{\alpha_j r}) = (x, y_{\alpha_j}) \quad \text{for } x \in N, \; r = 1,2,\ldots n_j, \; j=1,2,\ldots k$$

From (3.3) and (3.4) we obtain

$$(x, y)^n = (x, ny) = \prod_{j=1}^{k} (x, y_{\alpha_j})^{n_j} \quad \text{for} \quad x \in N$$

Since there are neighbourhoods of the identity where $(x, y_{\alpha_j}) = e^{ig(x,y_{\alpha_j})}$ it follows that there exists a neighbourhood of the identity where

$$(x, y)^n = e^{ing(x,y)}$$

Since $(x, y)^n$ and $e^{ing(x,y)}$ are both continuous and non vanishing at the identity there exists a neighbourhood of the identity in X, where

$$(x, y) = e^{ig(x, y)}$$

Property (5) is an immediate consequence of (1) and the fact that $g(x, y)$ is identically zero in y when x is the identity of X.

<u>3.4. Infinitely dividible distributions</u>.

We will in this section introduce infinitely divisible distributions and study some of their elementary properties. The results of the section are valid for complete separable metric groups.

<u>Definition 3.4.1.</u> A distribution μ is said to be infinitely divisible if for each integer n, there are elements $x_n \in X$ and $\nu_n \in \mathcal{M}$ such that $\mu = \nu_n^n * x_n$.

We remark that this definition is slightly different from the classical definition in the case of the real line. This modified definition is needed only if the group is not divisible. For divisible groups both are obviously equivalent. It will be shown later that if μ is infinitely divisible according to definition 3.4.1 then there is a translate of μ which is infinitely divisible according to the classical sense.

<u>Theorem 3.4.1</u>: The infinitely divisible distributions form a closed subsemigroup of $\mathfrak{M}$.

<u>Proof</u>: If λ and μ are infinitely divisible it is then obvious that $\lambda * \mu$ is also infinitely divisible. Let now μ_k, $k = 1, 2, \ldots$ be a sequence of infinitely divisible distributions weakly converging to μ. For any fixed integer n, let

$$(4.1) \qquad \mu_k = \lambda_{kn}^n * x_{kn}$$

From theorem 2.3.1 it is clear that there is a subsequence of λ_{kn} which converges after suitable shifting to λ_n. Since $\mu_k \Rightarrow \mu$ it follows from (4.1) that there exists an element x_n such that $\mu = \lambda_n^n * x_n$. This completes the proof.

<u>Theorem 3.4.2</u>. Let $\hat{\mu}(y)$ be the characteristic function of an infinitely divisible distribution μ. If $\hat{\mu}(y_o) = 0$ for some character y_o then μ has a non trivial idempotent factor.

<u>Proof</u>. Since μ is infinitely divisible, there exist, for each n, an element $x_n \in X$ and a distribution λ_n such that $\mu = \lambda_n^n * x_n$. Since $\hat{\mu}(y_o) = 0$, $\hat{\lambda}_n(y_o)$ also vanishes for each n. By theorem 2.3.1. λ_n is shift compact. Let λ be a limit of shifts of λ_n. Then $\hat{\lambda}(y_o)=0$ and hence λ is non-degenerate. It is also clear that $\lambda, \lambda^2, \ldots \lambda^n, \ldots$

are all factors of μ. Thus the sequence $\{\lambda^n\}$ are all factors of μ and hence λ^n is shift compact. But any limit of shifts of λ^n will be an idempotent factor of μ. Since λ is non-degenerate, this proves the theorem.

Remark: If the group is not locally compact it may not have any characters. For the characters that are available the theorem is true. The same is meant in the following when even we say 'character'.

Definition 3.4.2: If F is any totally finite measure on X the distribution $e(F)$ associated with F is defined as follows

$$e(F) = e^{-F(X)} \left[1 + F + \frac{F^2}{2!} + \dots \frac{F^n}{n!} + \dots \right]$$

where 1 denotes the measure with unit mass degenerate at the identity.

$e(F)$ is obviously infinitely divisible since $e(F) = [e(\frac{F}{n})]^n$ for each n. Its characteristic function is given by

$$e(F)(y) = \exp \left[\int [x,y)-1]dF\right]$$

The normalised Haar measure of a compact subgroup is another example of an infinitely divisible distribution.

Suppose F_n is a sequence of totally finite measures and we form the sequence $e(F_n)$. We will find out some necessary conditions on F_n in order that $e(F_n)$ may be shift compact.

Theorem 3.4.3. Let $\mu_n = e(F_n)$ where F_n is a sequence of totally finite measures. Then in order that

(a) $\{\mu_n\}$ be shift compact

(b) if μ is any limit of shifts of $\{\mu_n\}$ then μ has no idempotent factors

the following conditions are necessary.

i) For each neighbourhood N of the identity the family $\{F_n\}$ restricted to N' is weakly conditionally compact

ii) for each $y \in Y$

$$\sup_n \int [1 - R(x, y)] \, dF_n < \infty$$

$R(x, y)$ denoting the real part of (x, y).

Before proving this theorem we will prove a lemma

<u>Lemma 3.4.1.</u> Let F_n be as in theorem 3.4.3 and inaddition let $\sup_n F_n(N') \leq k$, where N' is the complement of a neighbourhood N of the identity. If the sequence $e(F_n)$ is shift compact then F_n restricted to N' is weakly conditionally compact.

<u>Proof</u>. Let G_n denoted the restriction of F_n to N'. Then $e(G_n)$ is a factor of $e(F_n)$. By theorem 2.3.1 $e(G_n)$ is also shift compact. Let $H_n = G_n + \bar{G}_n$. Then $e(H_n) = e(G_n) * e(\bar{G}_n) = e(G_n) * \overline{e(G_n)} = | e(G_n)|^2$. $e(H_n)$ is therefore compact and hence tight. An application of theorem 2.2.2 shows that for any $\varepsilon > 0$ there is a compact set C such that

$$e(H_n)(C') < \varepsilon$$

Since $e(H_n) = e^{-H_n(X)} \sum_{r=0}^{\infty} \frac{H_n^r}{r!}$, we have

$$\varepsilon > e(H_n)(C') \geq e^{-H_n(X)} H_n(C') \geq e^{-2k} H_n(C')$$
$$\geq G_n(C') e^{-2k}$$

for all n. Since k is a constant independent of ε and n, from the arbitrariness of ε follows the tightness of $\{G_n\}$.

<u>Proof of theorem 3.4.3.</u> Let the conditions (a), (b) of theorem 3.4.3 be valid. Let if possible $\sup_n F_n(N') = \infty$ for some neighbourhood N of the identity. We can choose a subsequence for which

$$F_{n_k}(N') \geq 2k \quad \text{for } k = 1, 2, \ldots \tag{4.1}$$

Let L_k, $k = 1, 2, \ldots$ be measures such that

$$\begin{aligned} &L_k(A) \leq \frac{1}{k} F_{n_k}(A) \quad \text{for every Borel set } A \\ &L_k(N) = 0 \\ &L_k(N') = 1 \end{aligned} \tag{4.2}$$

The distribution $\lambda_k = e(L_k)$ is a factor of $e(F_{n_k})$ and since $e(F_n)$ is shift compact, so is λ_n by theorem 2.3.1. Let λ be any limit of shifts of λ_k. From (4.1) and (4.2) it follows that any power of λ is a factor of μ where μ is some limit of shifts of a further subsequence chosen from $\mu_{n_k} = e(F_{n_k})$. Thus the sequence λ^n is shift compact and any limit of shifts of λ^n will be an idempotent factor of μ. Since

μ has no idempotent factors it follows that λ has to be degenerate. Thus the sequence $|\lambda_k|^2$ converges to the distribution degenerate at the identity. Hence $e(L_k + \bar{L}_k)(N') \to 0$ as $k \to \infty$.

But

$$e(L_k + \overline{L_k})(N') = e^{-(L_k+\bar{L}_k)(X)} \sum_{r=0}^{\infty} \frac{(L_k + \bar{L}_k)^r}{r!}$$

$$\geq e^{-2} L_k (N') \geq e^{-2}$$

which is a contradiction. An application of lemma 3.4.1 shows that condition (i) of the theorem is necessary.

In order to prove (ii) we observe that $e(F_n + \bar{F}_n) = |e(F_n)|^2$ is a compact sequence and an application of theorems 3.4.1 and 3.4.2 shows

$$\underline{\operatorname{Lim}}_{n \to \infty} \exp \int [R(x, y) - 1] dF_n > 0$$

which implies (ii).

3.5. General limit theorems for infinitesimal summands:

In the case of the real line a well known result due to Bawly and Khèntchine (See [4], chapter 4) asserts that the limit of sums of infinitesimal random variables is infinitely divisible and it can be obtained as a limit of a certain accompanying sequence of infinitely divisible distributions. The purpose of this section is to extend these results to the case of a locally compact group when the limiting distribution is free of idempotent factors.

<u>Definition 3.5.1.</u> A triangular sequence α_{nj} $j = 1, 2, \dots k_n$, $n = 1, 2, \dots$ of distributions is said to be uniformly infinitesimal if

$$\lim_{n \to \infty} \sup_{1 \leq j \leq k_n} \sup_{y \in K} | \alpha_{nj}(y) - 1 | = 0$$

for each compact subset K of Y.

Before proceeding to the statement of the main result of the section we will prove a lemma which will be used often in this section.

<u>Lemma 3.5.1.</u> Let $\mu_n = \prod_{j=1} \alpha_{nj}$ where the sequence α_{nj} is uniformly infinitesimal. If the distribution μ is a limit of shifts of μ_n then the set of characters $[y : \mu(y) \neq 0]$ is an open subgroup of Y and consequently the normalised Haar measure of the annihilator of this subgroup in X is a factor of μ.

<u>Proof</u>. Since μ is a limit of shifts of μ_n it is clear that for a subsequence (which we shall denote by μ_n itself) $| \mu_n |^2 \to | \mu |^2$ and hence

$$(5.1) \qquad \lim_{n \to \infty} \prod_{j=1}^{k_n} | \alpha_{nj}(y) |^2 = | \mu(y) |^2$$

If $\mu(y) \neq 0$ it clear that $\mu(-y) \neq 0$ (5.1) implies that the necessary and sufficient condition that $\mu(y) \neq 0$ is that

$$(5.2) \qquad \sup_n \sum_{j=1}^{k_n} (1 - | \alpha_{nj}(y) |^2) < \infty.$$

If we make use of the inequality

$$1 - \phi(y_1 + y_2) \leq 2[1 - \phi(y_1) + 1 - \phi(y_2)]$$

valid for any real valued characteristic function it is clear that the validity of (5.2) for y_1 and y_2 implies its validity for $y_1 \pm y_2$. The continuity of $\mu(y)$ implies that $[y : \mu(y) \neq 0]$ is an open subgroup of Y.

In what follows we choose and fix a function $g(x, y)$ according to the specifications of lemma 3.3.3. The main theorem of the present section can be stated as follows:

<u>Theorem 3.5.1.</u> Let α_{nj} be a uniformly infinitesimal sequence of distributions and let

$$\mu_n = \prod_{j=1}^{k_n} \alpha_{nj} .$$

Define x_{nj} as that element of the group X defined by the equality

$$(x_{nj}, y) = \exp\left[-i\int g(x,y)d\alpha_{nj}(x)\right] \quad \text{for all} \quad y \in Y.$$

Let further λ_n be defined by the formula

$$\lambda_n = \prod_{j=1}^{k_n} \beta_{nj} * x_n$$

where

$$\beta_{nj} = e(\alpha_{nj} * x_{nj}),$$

$$x_n = -\sum_{j=1}^{k_n} x_{nj}.$$

If either λ_n or μ_n is shift compact and no limit of shifts of them has an idempotent factor then

$$\lim_{n \to \infty} \sup_{y \in K} | \lambda_n(y) - \mu_n(y) | = 0$$

for each compact subset K of Y

Proof: During the course of the proof of the theorem we shall adopt the following conventions: by $C_1, C_2, \ldots$ we denote constants not depending upon n, but may depend on the compact set K. All the statements made are for sufficiently large n. By N we will denote a sufficiently small neighbourhood of the origin.

Let us first observe that as $n \to \infty$ the elements x_{nj} tend to the identity in X uniformly over j. This and the fact that x_{nj} is well defined as an element of X is an immediate consequence of the properties of $g(x, y)$ and the uniform infinitesimality of α_{nj}. Hence $\alpha_{nj} * x_{nj}$ as well as β_{nj} are uniformly infinitesimal. Thus $\lambda_n(y)$ and $\mu_n(y)$ will be non vanishing over K after certain stage (Here K is a compact subset of Y which is arbitrary but fixed) We can therefore use the logarithms freely. Let us first suppose that μ_n is shift compact and no limit of shifts of μ_n has an idempotent factor. Then it follows from Lemma 3.5.1 that the sequence $|\mu_n(y)|$ is bounded away from zero. It will therefore suffice to prove that

$$\sup_{y \in K} | \log \lambda_n(y) - \log \mu_n(y) | = 0$$

for some version of the logarithm.

We have

$$\log \lambda_n(y) = \sum_j \log \beta_{nj}(y) - \sum_j \log (x_{nj}, y)$$

$$= \sum_j [(\alpha_{nj} * x_{nj})(y) - 1] - \sum_j \log (x_{nj}, y)$$

and

$$\log \mu_n(y) = \sum_j \log \alpha_{nj}(y)$$

writing $\theta_{nj} = \alpha_{nj} * x_{nj}$ we obtain

$$| \log \lambda_n(y) - \log \mu_n(y) | = | \sum_j [\theta_{nj}(y) - 1] - \sum_j \log (x_{nj}, y) - \log \theta_{nj}(y) + \sum_j \log (x_{nj}, y) |$$

$$= | \sum_j [\theta_{nj}(y) - 1 - \log \theta_{nj}(y)] |$$

$$\leq C_1 \sum_j | 1 - \theta_{nj}(y) |^2$$

$$\leq C_1 (\sum_j | 1 - \theta_{nj}(y) |)(\sup_j | 1 - \theta_{nj}(y) |)$$

Since $\{ \theta_{nj} \}$ is uniformly infinitesimal it is clear from the above inequality that it is enough to prove that

$$(5.3) \qquad \sup_n \sup_{y \in K} [\sum_j | 1 - \theta_{nj}(y) |] < \infty .$$

We have for any neighbourhood N of the identity in X

$$(5.4) \qquad |1-\Theta_{nj}(y)| \leq \left|\int_N [1-(x,y)]d\Theta_{nj}\right| + \left|\int_{N'} [1-(x,y)]d\Theta_{nj}\right|$$

$$\leq \left|\int_N [1-(x,y)]d\Theta_{nj}\right| + 2\,\Theta_{nj}(N')$$

From property (4) of $g(x,y)$ in lemma 3.3.3 it is clear that there is a neighbourhood N of the identity such that for $y \in K$ we have

$$(x, y) = e^{ig(x,y)}$$

and consequently

$$(5.5) \qquad |(x, y) - 1 - ig(x, y)| \leq C_2 g^2(x, y)$$

(5.4) and (5.5) imply

$$(5.6) \qquad |1-\Theta_{nj}(y)| \leq \left|\int_N g(x,y)d\Theta_{nj}\right| + C_2\left|\int_N g^2(x,y)d\Theta_{nj}\right| + 2\Theta_{nj}(N')$$

for all $y \in K$. By property (2) of $g(x, y)$ in lemma 3.3.3.

$$(5.7) \qquad \left|\int_X g(x,y)d\Theta_{nj}\right| = \left|\int g(x+x_{nj}, y)d\alpha_{nj}(x)\right|$$

$$\leq \left|\int_N g(x+x_{nj}, y)d\alpha_{nj}(x)\right| + C_3\,\alpha_{nj}(N')$$

Since all the x_{nj} will be in any neighbourhood of the identity after a certain stage and since $e^{ig(x,y)} = (x, y)$ for $x \in N$ and $y \in K$ we conclude by making use of property (5) of lemma 3.3.3 that

$$(5.8) \qquad g(x + x_{nj}, y) = g(x, y) + g(x_{nj}, y)$$

for all $x \in N$ and $y \in K$. Further

$$e^{ig(x_{nj}, y)} = (x_{nj}, y) = e^{-i\int g(x,y)d\alpha_{nj}}$$

for all $y \in K$ and sufficiently large n. By property (5) of $g(x,y)$ in lemma 3.3.3 we get

$$(5.9) \qquad g(x_{nj}, y) = -\int g(x, y)d\alpha_{nj}$$

(5.8) and (5.9) imply

$$\begin{aligned} \left|\int_N g(x+x_{nj},y)d\alpha_{nj}\right| &= \left|\int_N [g(x,y) + g(x_{nj},y)]d\alpha_{nj}\right| \\ &= \left|\int_N g(x,y)d\alpha_{nj} - \alpha_{nj}(N)\int g(x,y)d\alpha_{nj}\right| \\ &= \left|\alpha_{nj}(N')\int_N g(x,y)d\alpha_{nj} - \alpha_{nj}(N)\int_{N'} g(x,y)d\alpha_{nj}\right| \\ &\leq C_4\,\alpha_{nj}(N'). \end{aligned}$$

The above inequality together with (5.7) implies that

$$(5.10) \qquad \left|\int g(x,y)d\theta_{nj}\right| \leq C_5\,\alpha_{nj}(N') \quad \text{for } y \in K.$$

(5.6), (5.10) and property (2) of lemma 3.3.3 give

$$|1-\theta_{nj}(y)| \leq c_2 \int g^2(x,y)d\theta_{nj} + c_6\theta_{nj}(N') + c_7\ \alpha_{nj}(N')$$

for $y \in K$. Thus inorder to complete the proof of the theorem we have only to show that

$$(5.11) \qquad \sup_n \sum_j \theta_{nj}(N') < \infty$$

$$(5.12) \qquad \sup_n \sum_j \alpha_{nj}(N') < \infty$$

$$(5.13) \qquad \sup_n \sup_{y \in K} \sum_j \int g^2(x, y)\, d\theta_{nj} < \infty.$$

To this end we consider the distribution

$$|\mu_n|^2 = \prod_{j=1}^{k_n} |\alpha_{nj}|^2$$

Since $|\mu_n|^2$ is compact and no limit of $|\mu_n|^2$ has an idempotent factor according to lemma 3.5.1 $|\mu_n(y)|^2$ is bounded away from zero uniformly for $y \in K$ and in n. Thus

$$\sup_n \sup_{y \in K} \sum_j (1 - |\alpha_{nj}(y)|^2) < \infty$$

This is the same as (5.3) with $|\alpha_{nj}|^2$ replacing θ_{nj} and hence

$$\lim_{n \to \infty} \sup_{y \in K} \left[\exp \sum_j (|\alpha_{nj}(y)|^2 - 1) - |\mu_n(y)|^2\right] = 0$$

Thus the sequence $e(\sum_j |\alpha_{nj}|^2)$ is compact. We now appeal to theorem 3.4.3.

Then

$$(5.14) \qquad \sup_n \sum_j |\alpha_{nj}|^2(N') < \infty$$

$$(5.15) \qquad \sup_n \sum_j \int [1 - R(x,y)]d\, |\alpha_{nj}|^2 < \infty$$

We now choose a neighbourhood V of the identity such that $V + V \subset N$. Then

$$\sum_j \alpha_{nj}(N') \leq \sum \alpha_{nj}[(V+V)']$$

$$\leq \sum \inf_{x \varepsilon V} \alpha_{nj} [(V + x)']$$

$$= \sum \inf_{x \varepsilon V} \alpha_{nj} (V' + x)$$

$$\leq \sum [\alpha_{nj} (V)]^{-1} \int_V \alpha_{nj} (V' + x) d\, \alpha_{nj}$$

$$\leq \sum [\alpha_{nj}(V)]^{-1} \int \alpha_{nj} (V' + x) d\, \alpha_{nj}$$

$$\leq \sup_j [\alpha_{nj}(V)]^{-1} \sum_j |\alpha_{nj}|^2(V')$$

Since $\{\alpha_{nj}\}$ is uniformly infinitesimal $\sup_j [\alpha_{nj} (V)]^{-1} < 1 + \varepsilon$ for any $\varepsilon > 0$ and all sufficiently large n. The above inequality and the validity of (5.14) for any heighbourhood N of the identity imply (5.12) Since $|\alpha_{nj}|^2 = |\theta_{nj}|^2$ and θ_{nj} is also uniformly infinitesimal the same argument leads to (5.11).

From (5.15) we have for any heighbourhood V of the identity in X

$$(5.16) \qquad \sup_n \sum_j \int_V \int_V [1 - R(x_1 - x_2, y)] d\theta_{nj}(x_1) d\theta_{nj}(x_2) < \infty$$

We now choose V such that $V - V \subset N$. Then

$$R(x_1 - x_2, y) = \cos \; g(x_1 - x_2, y)$$

Since $1 - \cos\theta \geq \frac{\theta^2}{4}$ for all suitably small θ, we have from property (5) of $g(x, y)$ in lemma 3.3.3

$$1 - R(x_1 - x_2, y) \geq \frac{1}{4} \; g^2(x_1 - x_2, y)$$

for $y \varepsilon K$. Since $e^{ig(x,y)} = (x,y)$ for $x \varepsilon N$ and $y \varepsilon K$ the same property of $g(x, y)$ gives

$$g(x_1 - x_2, y) = g(x_1, y) - g(x_2, y) \quad \text{for} \quad x_1, x_2 \,\varepsilon\, V \quad y \,\varepsilon\, K$$

Thus for $x_1, x_2 \,\varepsilon\, V \quad y \,\varepsilon\, K$

$$(5.17) \qquad 1-R(x_1-x_2,y) \geq \frac{1}{4} [g^2(x_1,y) + g^2(x_2,y) - 2g(x_1,y)g(x_2,y)]$$

(5.16) and (5.17) imply.

$$(5.18) \qquad \sup_n \sup_{y \varepsilon K} \sum_j [\int_V g^2(x,y) d\theta_{nj}(x) - (\int_V g(x_1,y) d\theta_{nj})^2] < \infty$$

(5.10), (5.11) and (5.18) imply (5.13). This completes the proof of the theorem.

Let us now suppose that λ_n is shift compact and no limit of shifts of λ_n has an idempotent factor. Then by theorem 3.4.3

$$(5.19) \qquad \sup_n \sum_{j=1}^{k_n} \theta_{nj}(N') < \infty \text{ for every neighbourhood } N \text{ of the identity}$$

$$(5.20) \qquad \sup_{y \in K} \sup_n \sum_{j=1}^{k_n} \int [1 - \cos (x,y)] d\, \theta_{nj} < \infty$$

Since $\{ x_{nj} \}$ are uniformly infinitesimal all the x_{nj}'s belong to V for any neighbourhood V of the identity for sufficiently large n. Hence if $V - V \subset N$

$$\theta_{nj} (V) = \alpha_{nj} (V - x_{nj})$$

$$\leq \alpha_{nj} (V - V)$$

$$\leq \alpha_{nj} (N)$$

or $\sum_j \alpha_{nj} (N') \leq \sum_j \theta_{nj} (V')$ for large n.

Hence from (5.19) follows (5.12). Since (5.20) implies (5.13) it follows that (5.19) and (5.20) imply (5.11), (5.12) and (5.13). This completes the theorem.

<u>Corollary</u>: If $\{ \alpha_{nj} \}$, λ_n, μ_n are as in theorem 3.5.1 then inorder that $\{ \mu_n \}$ after a suitable shift may converge to a limit μ without idempotent factors it is necessary and sufficient that λ_n, after

the same shift, converges to μ.

This is an immediate consequence of the theorem.

Theorem 3.5.2.: If $\{\alpha_{nj}\}$ is uniformly infinitesimal, $\mu_n = \prod_{j=1}^{k_n} \alpha_{nj}$, and μ the limit of shifts of μ_n then μ is infinitely divisible.

Proof: If μ has no idempotent factor then μ is also a limit of shifts of λ_n. Since each λ_n is infinitely divisible μ is also infinitely divisible. By Lemma 3.5.1 if $\mu(y)$ has an idempotent factor it is the normalised Haar measure of the annihilator G of the open subgroup

$$H = [y : \mu(y) \neq 0]$$

Let τ be canonical homomorphism from X to $X/\tilde{G}$. Then the sequence $\{\alpha_{nj}\tau^{-1}\}$ is uniformly infinitesimal in X/G and $\mu_n\tau^{-1} \Rightarrow \mu\tau^{-1}$ has no idempotent factors and hence is infinitely divisible. But $\mu(y) = \mu\tau^{-1}(y)$ for $y \in H$ and $= 0$ otherwise. Therefore μ is also infinitely divisible.

Remark: In the statement of theorem 3.5.2 we assumed that $\{\alpha_{nj}\}$ is uniformly infinitesimal. However it is enough to assume the existence of a sequence $\{x_{nj}\}$ of elements from the group with the property that $\{\alpha_{nj} * x_{nj}\}$ is uniformly infinitesimal. This is equivalent to the statement that any limit of shifts of $\{\alpha_{nj}\}$ is degenerate.

3.6. Gaussian distributions:

Suppose F_n is a sequence of finite measures in the group X with the properties: 1) $F_n \to 0$ outside every neighbourhood of the identity and 2) $e(F_n)$ converges to a distribution after suitable shift. If the total mass of F_n is not uniformly bounded $e(F_n)$ may actually converge to a non-degenerate distribution. We shall now find out the possible limit laws which occur in this manner.

Definition 3.6.1: A distribution μ is said to be Gaussian if it has the following properties

1) μ is infinitely divisible

2) If $\mu = e(F) * \alpha$ and α is infinitely divisible then F vanishes out side the identity.

Theorem 3.6.1. A distribution μ is Gaussian if and only if $\hat{\mu}(y)$ is of the form

$$\hat{\mu}(y) = (x, y)\, e^{-\phi(y)}$$

where x is a fixed element of X and $\phi(y)$ a non-negative continuous function satisfying the condition

(6.1) $$\phi(y_1 + y_2) + \phi(y_1 - y_2) = 2[\phi(y_1) + \phi(y_2)]$$

for each $y_1, y_2 \in Y$.

Proof: Let μ be Gaussian. Then μ cannot have a non-degenerate idempotent factor. For, otherwise the Haar measure of some compact sub-group will be a factor of μ and hence if F is any measure concentrated in that subgroup then $\mu = \mu * e(F)$. This contradicts the property (ii) of definition 3.6.1. From the definition of infinite divisibility it follows, that for each n there exists a distribution α_n and an element g_n of X such that

$$\mu = \alpha_n^n * g_n$$

Since μ has no idempotent factors any limit of shifts of α_n is degenerate. Hence α_n can be so shifted that the shifted α_n converges to the distribution degenerate at the identity. Since these shifts can be absorbed in g_n we will suppose that α_n converges to the distribution degenerate at the identity. We now write as in theorem 3.5.1

$$\theta_n = \alpha_n * x_n$$

$$\beta_n = e(\theta_n)$$

$$\lambda_n = \beta_n^n * (-n\, x_n) * g_n$$

where x_n is that element of X defined by

$$(x_n, y) = \exp\left[-i \int g(x, y) d\, \alpha_n\right]$$

The absense of idempotent factors for μ implies that

$$\lim_{n \to \infty} \sup_{y \in K} | \lambda_n(y) - \mu(y) | = 0$$

by theorem 3.5.1

Thus $| \mu(y) | = \lim_{n \to \infty} \exp \left(n \int [R(x, y) - 1] d\, \theta_n\right)$. We shall first show that the function $\phi(y) = \lim_{n \to \infty} \left(n \int [1 - R(x, y)] d\theta_n\right)$ satisfies (6.1).

We write $P_n = n\theta_n$. Then $e(P_n)$ is a shift of λ_n and hence $e(P_n)$ shift compact. Now theorem 3.4.3 implies that P_n restricted to N' is tight for every neighbourhood N of the identity. But any limit P of P_n restricted to N' will be such that $\mu = e(P) * \alpha$ where α is also infinitely divisible. From condition (ii) of definition 3.6.1 it follows that the mass of P_n outside every neighbourhood of the identity tends to zero. Thus

$$(6.2) \qquad \phi(y) = \lim_{n \to \infty} \int_N [1 - R(x, y)] d\, P_n$$

for every neighbourhood N of the identity. (6.2) and the identity

$$\lim_{x \to e} \frac{[1 - R(x, y_1 + y_2)] + [1 - R(x, y_1 - y_2)]}{2[1 - R(x, y_1)] + 2[1 - R(x, y_2)]} = 1$$

(e denotes the identity and the limit to be interpreted in the sense that the numerator lies between $(1 - \varepsilon)$ and $(1 + \varepsilon)$ times the denominator for a suitable neighbourhood)

imply that

$$\phi(y_1 + y_2) + \phi(y_1 - y_2) = 2[\phi(y_1) + \phi(y_2)]$$

Thus inorder to complete the proof it suffices to show that $\frac{\mu(y)}{|\mu(y)|}$ is a character on Y. Let us denote this by $\lambda(y)$. It can be verified that for every neighbourhood N of the identity

$$(6.3)\quad \lambda(y_1+y_2)[\lambda(y_1)\lambda(y_2)]^{-1} = \exp\Big(\lim_{n\to\infty}\int_N [I(x,y_1+y_2)-I(x,y_1)-I(x,y_2)]dP_n\Big)$$

Where $I(x,y)$ denotes the imaginary part of (x, y), provided the limit on the right hand side exists. For any $\varepsilon > 0$ there is a neighbourhood N of the identity such that

$$|I(x, y_1)| < \varepsilon, \quad |I(x, y_2)| < \varepsilon \quad \text{for } x \in N.$$

Since

$$\Big|I(x,y_1+y_2)-I(x,y_1)-I(x,y_2)\Big| \leq \Big|I(x,y_1)\Big|\Big|1-R(x,y_2)\Big| + \Big|I(x,y_2)\Big|\Big|1 - R(x, y_1)\Big|$$

and $\sup_n \int [1 - R(x, y)]dP_n < \infty$ (by theorem 3.4.3)

we have

$$\overline{\lim_{n\to\infty}}\ \Big|\int_N [I(x,y_1+y_2) - I(x,y_1)-I(x,y_2)]dP_n\Big| \leq C\varepsilon$$

where C depends only on y_1 and y_2. This together with the fact that the expression in (6.3) is independent of N, show that

$$\lambda(y_1 + y_2) = \lambda(y_1)\cdot\lambda(y_2)$$

Since $\lambda(y)$ is obviously continuous, there is an element $x \in X$ such that $\lambda(y) = (x, y)$. Hence

$$\mu(y) = (x, y)\exp[-\phi(y)]$$

This proves the sufficiency.

Conversely let $\mu(y) = (x, y) \exp[-\phi(y)]$ where $\phi(y)$ is a non-negative continuous function of y satisfying (6.1). Let $y_1, \dots y_k$ be some k characters. Then it is easily verified that $\exp[-\phi(n_1y_1 + \dots n_ky_k)]$ considered as a function of the integers $n_1, n_2, \dots n_k$ is positive definite in the product of integer group taken k times. This implies that $\exp[-\phi(y)]$ itself is positive definite and since it is continuous it is the characteristic function of some measure. Since $\mu(y) = 1$ at the identity of Y the measure is a distribution. Infinite divisibility is obvious since $\frac{\phi}{n}$ possesses the same properties as ϕ We shall now prove property (ii) of definition 3.6.1. Let, if possible $\mu(y) = \mu_1(y) \cdot \mu_2(y)$, $\mu_1 = e(F)$ and μ_2 is infinitely divisible. Since $\mu(y)$ does not vanish at any y $\mu_n(y)$ can not vanish at any y. Hence by theorem 3.5.1 μ_2 is a limit of shifts of distributions of the type $e(H)$. From (2.1) it is clear that

$$(6.4) \quad -\log | e(H)(y_1+y_2)| - \log|e(H)(y_1-y_2)|$$

$$\leq 2[-\log | e(H)(y_1)| - \log | e(H)(y_2)|].$$

Thus (6.4) is valid when $e(H)$ replaced by either μ_2 or μ_1 Substituting μ_2 and μ_1 in (6.4) and adding we have

$$\phi(y_1 + y_2) + \phi(y_1 - y_2) \leq 2[\phi(y_1) + \phi(y_2)]$$

Since equality takes place here, we must have

$$\int([1 - R(x, y_1 + y_2)] + [1 - R(x, y_1 - y_2)]dF$$

$$= 2 \int([1 - R(x, y_1)] + [1 - R(x, y_2)]dF$$

for each y_1, y_2 ε Y.

This is the same as

$$\int[1 - R(x, y_1)][1 - R(x, y_2)]dF = 0$$

Since F is a measure it should vanish outside the identity.

3.7. Representation of Infinitely divisible distributions.

As already mentioned in the introduction, we will obtain in the present section a representation for the infinitely divisible distributions in a locally compact group, similar to the representation (1.1) for the real line

Definition 3.7.1: An infinitely divisible distribution λ is called a proper factor of another infinitely divisible distribution μ if $\mu = \lambda * \alpha$ where α is also infinitely divisible.

Lemma 3.7.1. The set of proper factors of an infinitely divisible distribution is closed.

Theorem 3.7.1. If μ is an infinitely divisible distribution without idempotent factors, then $\mu(y)$ has a representation

$$(7.0) \qquad \mu(y) = (x_o, y) \exp\left[\int[(x,y)-1-ig(x,y)]dF(x) - \phi(y)\right]$$

where x_o is a fixed element of X, $g(x, y)$ is a function on $X \times Y$ which is independent of μ and is chosen according to lemma 3.3.3, F is a σ-finite measure with finite mass outside every neighbourhood of the identity X which satisfies

$$\int [1 - R(x, y)]\, dF < \infty \text{ for every } y \in Y$$

and $\phi(y)$ is a non-negative continuous function

Satisfying

$$\phi(y_1 + y_2) + \phi(y_1 - y_2) = 2[\phi(y_1) + \phi(y_2)] ;$$

for every $y_1, y_2 \in Y$. Conversely any function of the type (7.0) is the characteristic function of an infinitely divisible distribution.

<u>Proof</u>: Let μ be any infinitely divisible distribution without an idempotent factor. Choose and fix a sequence N_k of neighbourhoods in X descending to the identity. Let μ_1 be that proper factor of μ which is of the type $e(F)$ and for which $F(N_1) = 0$, $F(N_1')$ is the maximum. Such an $e(F)$ exists because of theorems 2.3.1, 3.4.3 and lemma 3.7.1. Let the maximum be attained at F_1, $\mu = \mu_1 * \lambda_1$ where $\mu_1 = e(F_1)$. Since λ_1 is infinitely divisible and without idempotent factors the same argument can be applied to λ_1 and the neighbourhood N_2. Thus there exists a measure F_2 for which $F_2(N_2) = 0$, $F_2(N_2')$ is a maximum, $\mu_2 = e(F_2)$ and $\lambda_1 = \mu_2 * \lambda_2$ with λ_2 infinitely divisible.

Repeating this procedure we can write

(7.1) $$\mu = \mu_1 * \mu_2 * \cdots \mu_n * \lambda_n$$

(7.2) $$\lambda_{n-1} = \mu_n * \lambda_n$$

(7.3) $$\mu_n = e(F_n), \quad F_n(N_n) = 0,$$

λ_n is infinitely divisible and $F_n(N_n')$ is maximum in the sense explained earlier. Thus by theorem 3.2.1 there exist shifts of $\mu_1, * \mu_2, * \cdots * \mu_n$

and λ_n converging to ν and λ respectively and $\mu = \nu * \lambda$. We now assert that λ cannot have any proper factor of the type $e(F)$. Suppose on the contrary $e(F)$ is a proper factor of λ. Then F should have a positive mass outside some N_k. Since λ_n is a sequence such that λ_{n+1} is a factor of λ_n for each n, λ will be a factor of λ_n for each n. λ will actually be a proper factor since each λ_{n+1} is a proper factor of λ_n. Let $\lambda_k = e(F) * \Theta$ where Θ is infinitely divisible. If F' is the restriction of F to N_k' then (7.2) and (7.3) imply that $\lambda_{k-1} = e(F_k + F') * e(F - F') * \Theta$. This is a contradiction since the total mass of $F_k + F'$ exceeds that of F_k. Thus λ has no proper factors of the type $e(F)$ and hence is a Gaussian distribution. An application of theorem 3.6.1. implies that

$$\lambda(y) = (x_1, y) \exp \left\{ - \phi(y) \right\}$$

Now let us write $H_n = F_1 + F_2 + \dots F_n$. From the construction of the distribution $\nu_n = \mu_1 * \dots \mu_n$ it is clear that

$$\nu_n(y) = \exp \left[\int [(x,y)-1] dH_n \right]$$

Since $\exp \left\{ \int ig(x,y) dH_n \right\}$ is a character, there are $x_n \in X$ such that

$$\nu_n(y) = (x_n, y) \exp \left[\int [(x, y)-1 - ig(x,y)] dH_n \right]$$

Absorbing the shift in x_n we can suppose that $\nu_n \to \nu$. Since $e(H_n)$ is a factor of μ and H_n increases as $n \to \infty$ it follows from the shift compactness of $e(H_n)$, theorem 3.4.3 and lemma 3.3.1 that H_n increases to a σ-finite measure H, giving finite mass out side each neighbourhood of the identity and

$$\int \sup_{y \varepsilon K} [1 - R(x, y)] \, dH < \infty$$

for each compact set $K \subset Y$. Since $(x, y) - 1 - ig(x, y)$ is bounded uniformly in $y \varepsilon K$ by property (2) of lemma 3.3.3 for every neighbourhood N of the identity

$$\lim_{n \to \infty} \int_{N'} [(x,y) - 1 - ig(x,y)] dH_n = \int_{N'} [(x,y) - 1 - ig(x,y)] dH$$

uniformly for $y \varepsilon K$. When N is sufficiently small we have by the properties (4) and (5) of $g(x, y)$ in lemma 3.3.3.

$$(x, y) = e^{ig(x,y)} \quad \text{for } x \varepsilon N, \; y \varepsilon K$$

$$g^2(x, y) \leq C_1[1 - R(x, y)] \quad \text{for } x \varepsilon N, \; y \varepsilon K$$

where C_1 depends only on K.

Thus

$$(7.6) \qquad |(x,y)-1-ig(x,y)| \leq C_2[1-R(x,y)] \quad x \varepsilon N, \; y \varepsilon K,$$

where C_2 is a constant depending only on K. Thus

$$\int_N \sup_{y \varepsilon K} |(x,y)-1-ig(x,y)| dH_n \leq C_2 \int_N \sup_{y \varepsilon K} [1-R(x,y)] dH$$

The above inequality implies that $\int[(x,y)-1-ig(x,y)]dH_n$ converges to

$\int [(x,y) - 1 - ig(x,y)]dH$ uniformly in $y \varepsilon K$.

Since

$$\nu_n(y) = (x_n,y) \exp [\int [(x,y) - 1 - ig(x, y)dH_n]$$

and $\nu_n \Rightarrow \nu$, it follows that (x_n, y) converges uniformly over compact sets to some (x_o, y) or

$$\nu(y) = (x_2, y) \exp [\int[(x, y) - 1 - ig(x,y)]dH]$$

Since $\lambda(y) = (x_1,y) \exp [- \phi(y)]$ and $\mu = \nu * \lambda$ (7.0) follows.

To prove the converse we first notice that if F is a totally finite measure then $\exp [i \int g(x,y)dF]$ is a character and hence $\mu(y)$ given by (7.0) is an infinitely divisible characteristic function. In the general case we consider a sequence F_n of totally finite measures increasing to F. If K is any compact subset of Y then according to lemma 3.3.1, $1 - R(x,y)$ is uniformly integrable with respect to F for $y \varepsilon K$.

(7.6) implies the uniform integrability of $(x, y) - 1 - ig(x,y)$ for $y \varepsilon K$. This shows that the functions

$$(x_o, y) \exp (\int [(x,y)-1-ig(x,y)]dF_n - \phi(y))$$

converge uniformly over compact sets to $\mu(y)$. Thus by theorem 3.4.1 $\mu(y)$ is the characteristic function of an infinitely divisible distribution

3.8. Uniqueness of the representation.

In the case of the real line it is well known that the canonical representation of an infinitely divisible distribution is unique. However such a result fails to be valid in a general group. We will show that the non-uniqueness is essentially due to the presence of compact subgroups in the original group or equivalently due to the disconnectedness of the character group.

Before proceeding ta the statement of the main result of this section we shall explain a few conventions and prove a lemma. If μ is any infinitely divisible distribution without idempotent factors we say that μ has the representation $(x_o, F\ \phi)$ where x_o, F and ϕ are as in theorem 3.7.1.

If F is any signed measure we denote by F_y the measure given by

$$F_y(A) = \int_A [1 - R(x, y)]dF.$$

<u>Lemma 3.8.1.</u> Let μ be a totally finite signed measure. If $\mu(y)$ is constant on the cosets of a closed subgroup Y_o of Y, then μ vanishes completely on the complement of the annihilator of Y_o in X.

<u>Proof:</u> Let $Y_1 = Y/Y_o$ and X_1 the annihilator of Y_o in X. $\mu(y)$ being constant on cosets of Y_o, can be considered as a function on Y_1. Then $\mu(y_1)$ for $y_1 \in Y_1$ is the characteristic function of a signed measure on X_1. Since, for $x \in X_1$,(x, y) remains constant on cosets of Y_1 and $\mu(y)$ has the same property we can write

$$\mu(y) = \int_{X_1} (x,y_1)d\lambda = \int_{X_1} (x,y)d\lambda$$

where y_1 denotes that coset of Y_1 to which y belongs. This shows that the signed measure μ and λ are identical and hence μ vanishes identically on the complement of X_1.

<u>Theorem 3.8.1.</u> If (x_1, F_1, ϕ_1) and (x_2, F_2, ϕ_2) are two representations of the same infinitely divisible distribution without idempotent factors then (i) $\phi_1 = \phi_2$ and (ii) the signed measure $(F_1 - F_2)$ vanishes completely on the complement of the annihilator of the component of identity of the character group Y.

<u>Proof:</u> Writing $F = F_1 - F_2$ $\phi = \phi_1 - \phi_2$ and $x_o = x_2 - x_1$ we have

(8.1) $\exp \int [(x,y)-1-ig(x,y)]dF = (x_o,y) \exp [\phi(y)],$

(8.2) $\phi(y_1+y_2) + \phi(y_1-y_2) - 2[\phi(y_1)+\phi(y_2)] = 0$, for $y_1, y_2 \in Y$

Equating the logarithm of the absolute value on both sides of (8.1) we obtain

(8.3) $\phi(y) = \int [R(x, y) - 1]dF$

Substituting the values of the above expression at y_1+y_2, y_1-y_2, y_1 and y_2 in (8.2) we get after simplification

(8.4) $\int [1 - R(x, y_1)][1 - R(x,y_2)]dF = 0$

(8.4) can be rewritten as

$$\int [1 - R(x, y_1)]dF_{y_2} = 0 \text{ for every } y_1, y_2 \in Y$$

Since F_{y_2} is totally finite we have

$$(8.5) \qquad \int (x,y)\,d(F_{y_2} + \bar{F}_{y_2}) = 2\, F_{y_2}(X)$$

Since the right hand side of (8.5) is independent of y for each fixed y_2 the signed measure $F_{y_2} + \bar{F}_{y_2}$ is degenerate at the identity. But the mass of F_{y_2} at the identity is zero. Thus $F_{y_2} + \bar{F}_{y_2} = 0$. In particular $F_{y_2}(X) = 0$, i.e.

$$(8.6) \qquad \int [1 - R(x, y)]dF = 0 \qquad \text{for every } y$$

(8.3) and (8.6) imply the equality of ϕ_1 and ϕ_2.

In order to prove the second part of the theorem we make use of the equality of ϕ_1 and ϕ_2 and rewrite (8.1) in the form

$$(8.7) \qquad \exp\left[\int [(x,y)-1- ig(x,y)]dF\right] = (x\ ,y)$$

substituting $y = y_1 + y_2$, $y_1 - y_2$ and y_1 successively in (8.7) and dividing the product of the first two by the square of the third we obtain

$$(8.8) \qquad \exp \int (x, y_1)[1 - R(x, y_2)]dF = 1 \qquad y_1, y_2 \in Y$$

or equivalently

$$\int (x, y_1)[1-R(x,y_2)]dF = 2\pi i\, n(y_1,y_2)$$

where $n(y_1, y_2)$ is an integer valued continuous function of y_1 and y_2.

We fix y_2 for the present. Then $n(y_1, y_2)$ remains constant on every connected subset of Y and inparticular on the cosets of the component of identity in Y. This implies by lemma 3.8.1 that the signed measure F_{y_2} vanishes identically on the complement of the annihilator in X of the component of the identity of Y. Since this is true for each y_2 it follows that F itself vanishes identically outside this annihilator. This proves the theorem.

<u>Remark 1:</u> The annihilator of the component of the identity ~~by~~ of Y being the character group of a totally disconnected group has a sequence of compact subgroups increasing to the whole group. This reflects on the role of compact subgroups in making the representation non-unique. In particular if the group X has no compact subgroups then the representation is unique.

<u>Remark 2 :</u> It was shown in the course of the proof of theorem 3.8.1 that the measure F_y is antisymmetric for each character y i.e. $F_y(A) = - F_y(-A)$ for every Borel set. But if every element of the group were of order two then such a measure would be identically zero, coupled with remark 1 we may say that if the group X is such that every compact subgroup contains only elements of order two then the representation is unique.

<u>Remark 3:</u> Conversely if X is any compact group such that not all elements are of order two then the representation is not always unique as can be seen by the following example. We take an element y_0 in the character group which is not of order two and consider the function

$$f(x) = 2\pi i\,[\overline{(x, y_0)} - (x, y_0)]$$

$f(x)$ is real and not identically zero. If h denotes the normalised Haar measure of x, we have

$$\begin{aligned} \int (x, y) f(x)\, dh(x) &= 2\pi i \quad \text{for } y = y_0 \\ &= -2\pi i \quad \text{for } y = -y_0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

If $f^{+} = \max(f, 0)$ and $\bar{f} = -\min(f, 0)$, we define the two measures

$$F_1(A) = \int_A f^{+}(x)\, dh(x)$$

$$F_2(A) = \int_A \bar{f}(x)\, dh(x)$$

Then $F_1 \neq F_2$, but

$$\exp\left[\int [(x,y)-1-ig(x,y)]dF_1\right] = (x_0,y)\exp\left[\int[(x,y)-1-ig(x,y)]dF_2\right]$$

where $(x_0, y) = \exp\left[i\int g(x,y)d(F_2-F_1)\right]$. Thus $(e, F_1, 0)$ and $(x_0, F_2, 0)$ are two representations of the same infinitely divisible distribution.

3.9. Compactness criteria:

When the group is the real line necessary and sufficient conditions are known for a sequence of infinitely divisible distributions to converge to another infinitely divisible distribution. These results are given in Gnedenko and Kolmogorov [4]. Since the representation is not unique it is not possible to give such a condition for the convergence. However it is possible to give simple conditions for compactness.

Before proceeding to state the main result of the section we shall investigate what happens to the representation when we pass over from a group to a quotient group. Let $G \subset X$ be some closed subgroup of X and $X' = X/G$ the quotient group. Let τ be the canonical homomorphism from X on to X'. If Y' is the character group of X' we choose and fix a $g'(x',y')$ defined on $X' \times Y'$ according to lemma 3.3.3. We observe that Y' is the annihilator of G in Y' and hence a subgroup of Y. Any infinitely divisible distribution μ' on X' without idempotent factors has a representation (x', F', ϕ') (with g replaced by g') according to theorem 3.7.1. We shall now prove the following

<u>Lemma 3.9.1.</u> Let μ be an infinitely divisible distribution on X with a representation (x, F, ϕ). If $\mu' = \mu\tau^{-1}$, $x' = \tau x$, $F' = F\tau^{-1}$ and ϕ' the restriction of ϕ to Y', then μ' is an infinitely divisible distribution on X' and is a shift of the distribution represented by (x', F', ϕ').

<u>Proof:</u> Since $e(F)\tau^{-1} = e(F\tau^{-1})$ for every finite measure F, any σ-finite measure is a limit of an increasing sequence of finite measures and the Gaussian component is unique the lemma is proved.

We shall now prove the following

<u>Theorem 3.9.1.</u> Let $\{\mu_\alpha\}$ be a family of infinitely divisible distributions without idempotent factors and with representations $\{(x_\alpha, F_\alpha, \phi_\alpha)\}$. The necessary and sufficient conditions that $\{\mu_\alpha\}$ be
shift compact and that any limit of shifts of $\{\mu_\alpha\}$ be
devoid of idempotent factors are that

1) The family $\{F_\alpha\}$ of measure is weakly conditionally compact when restricted outside any neighbourhood of the identity

2) $\sup\limits_\alpha [\int [1 - R(x,y)] dF_\alpha] < \infty$ for each $y \in Y$

3) $\sup\limits_\alpha \phi_\alpha(y) < \infty$ for each $y \in Y$

<u>Proof:</u> The necessity of the above two conditions is obvious in view of theorem 3.4.3. Regarding sufficiency we first observe that if $\{\mu_\alpha\}$ is shift compact then conditions (2) and (3) ensure that no limit has an idempotent factor. So it enough to prove shift compactness. Or equivalently one should prove the compactness of the family $|\mu_\alpha|^2$.

We now observe that if (X, F, ϕ) is a representation of an infinitely divisible distribution μ in a group X and τ is a continuous homomorphism of X on to another group X' then $\mu\tau^{-1}$ is a shift of the distribution represented by $(\tau x, F\tau^{-1}, \phi')$ where ϕ' is the restriction of ϕ to the character group of X' (a subgroup of Y). Further if $\{(x_\alpha, F_\alpha, \phi_\alpha)\}$ satisfies conditions (1), (2) and (3) so does the family $\{(\tau x_\alpha, F_\alpha\tau^{-1}, \phi'_\alpha)\}$. Making use of these remarks we will reduce the proof of the general case to some simple groups.

In order that a family of distributions on a group X may be compact it is sufficient if there exists a compact subgroup C such that the induced distributions in X/C are compact. But any group X has a compact subgroup C such that

$$X/C = V \oplus D \oplus K^r$$

where V is the vector group, D a discrete group and K^r the r dimensional torus. Since a family in the product space is compact as soon as the marginals are so, it is enough to prove the sufficiency of (1), (2), (3) in the case of the real line, the circle group and the discrete group. In the case of the real line the boundedness of $\int [1 - R(x,y)]dF_\alpha$ implies the boundedness of $\int_{|X| < \varepsilon} x^2 \, \alpha F_\alpha$ for a suitable ε which together with condition (1) implies the equicontinuity of $\exp[-\int [1-R(x,y)]dF_\alpha]$. Since $\phi_\alpha(y) = \sigma_\alpha^2 y^2$ (3) implies the boundedness of $\{\sigma_\alpha^2\}$ and it is clear that $|\mu_\alpha|^2$ is compact. In the case of a compact group the space of all distributions is compact. In the case of a discrete group identity itself is open. $\{F_\alpha\}$ themselves are therefore weakly conditionally compact and $\phi \equiv 0$. This proves the theorem.

<u>Corollary 3.9.1.</u> In addition to the conditions (1),(2) and (3) of theorem 3.9.1 the condition that $\{x_\alpha\}$ are conditionally compact is necessary and sufficient to ensure the compactness of $\{\mu_\alpha\}$ represented by $\{(x_\alpha, F_\alpha, \phi_\alpha)\}$.

<u>Proof:</u> From (7.6) we have

$$|(x,y) - 1 - ig(x,y)| \leq C\,[1 - R(x, y)]$$

for all $x \in N$, $y \in K$ where C is a constant depending only on K, N a suitable neighbourhood of e and K a given compact set of. This implies the equicontinuity of the functions

$$\exp\left[\int [(x, y) - 1 - ig(x,y)]dF_\alpha - \phi_\alpha(y)\right].$$

Hence x_α is conditionally compact if and only if μ_α is so. Proceeding along the same lines as in the proof of theorem 3.6.1. it is possible to prove the following

Theorem 3.9.2. Let μ_α be a sequence of infinitely divisible distributions with representations (x_n, F_n, ϕ_n). Let μ_n converge to μ after a suitable shift and F_n to F outside each continuity neighbourhood of the identity. Then μ has a representation (x, F, ϕ) for a suitable choice of x and ϕ.

3.10. Representation for convolution semigroups.

We have observed earlier that the representation of an infinitely divisible distribution is not unique. We shall now consider the representation problem for a one parameter convolution semigroup of distributions. By such a semigroup we mean a family $\{\mu_t\}$ of distributions indexed by $t \geq 0$ such that $\mu_t * \mu_s = \mu_{t+s}$. We shall further suppose that μ_t converges weakly to the distribution degenerate at the identity when $t \to 0$. Obviously for such semigroups $\mu_t(y) \neq 0$ for any $t > 0$ and $y \in Y$. That such a semigroup has a unique canonical representation is the content of the following theorem.

Theorem 3.10.1. Let $\{\mu_t\}$ be a one parameter convolution semigroup of distributions such that μ_t converges weakly to the distribution degenerate at the identity as $t \to 0$. Then $\mu_t(y)$ has the representation.

$$\mu_t(y) = (x_t, y) \exp \left[t \int [(x,y) - 1 - ig(x,y)] dF - t\,\phi(y) \right]$$

where F and ϕ are as in theorem 3.7.1 and x_t a continuous one para-

parametric semigroup in X. $\{x_t\}$, F, and ϕ are unique.

<u>Proof</u>. Since $\mu_t = (\mu_{t/n})^n$, μ_t is infinitely divisible. As remarked in the beginning $\mu_t(y) \neq 0$ for any y and hence μ_t has no idempotent factors. By theorem 3.7.1, μ_t has a representation (Z_t, F_t, ϕ_t). The uniqueness of ϕ_t implies that $\phi_t = t\,\phi_1$ we write $\phi = \phi_1$ and $\lambda_t(y) = \mu_t(y)\, e^{t\phi(y)}$. Then λ_t is a weakly continuous convolution semigroup and λ_t has neither Gaussian nor idempotent factors. The distribution with representation $(e, n!\, F_{1/n!}, 0)$ is a shift of λ_1 for every n and in view of theorem 3.9.1 we can choose a subsequence $n_k!\ F_{1/n_k!}$ such that it converges weakly to F outside each continuity neighbourhood of F. Thus by theorem 3.9.2 λ_1 has a representation $(Z, F, 0)$ for some $Z \in X$. If $t = p/q$ is a rational then a shift of $\lambda_{p/q}$ has the representation $(e, \frac{p}{q}\, n!\, F_{1/n!}, 0)$ for all sufficiently large n and hence $\lambda_{p/q}$ is a shift of $P_{p/q}$ where

$$P_t(y) = \exp\left[t \int \left[(x, y) - 1 - ig(x,y)\right] dF\right]$$

By the continuity of the semigroups it is clear that for each t, λ_t is a shift of P_t. Thus

$$\lambda_t(y) = (x_t, y) * P_t(y)$$

and x_t becomes automatically a one parametric continuous semigroup in X.

If now $(x_t, t\,F, t\,\phi)$ and $(x_t', t\,F', t\,\phi')$ are two representations of $\mu_t(y)$, we have $\phi = \phi'$ and further proceeding along the same lines as the proof of theorem 3.8.1 we obtain

$$\exp\left(t \int (x, y_1)[1 - R(x, y)]\, d(F - F')\right) = 1$$

for every t. This will imply that $F - F' = 0$. Hence the representation is always unique.

Chapter IV

DISTRIBUTIONS ON THE HILBERT SPACE

4.1. Introduction

In the present chapter the results of the earlier chapter regarding the representation of infinitely divisible distributions and the theorem on accompanying laws are proved when the group is a Hilbert Space. Conditions for the compactness of a sequence of infinitely divisible distributions are obtained in terms of the quantities occurring in the representation.

4.2. Preliminaries

X is a real separable Hilbert Space. (x, y) denotes the inner-product and $\| x \|$ the norm. With vector addition as group operation X becomes a complete separable metric group. We will denote by $\mathcal{M}$ the semigroup of all distributions.

For every $\mu \in \mathcal{M}$ its characteristic function is defined on X by the formula

$$\hat{\mu}(y) = \int e^{i(x,y)} \, d\mu(x)$$

We will, in this section, mention some results obtained by Prohorov [16] concerning compactness criteria for distributions on X.

<u>Definition 4.2.1.</u> A positive semidefinite Hermitian operator A is called an S-operator if it has finite trace. The class of sets of the type $[x : (Sx, x) < t]$ where S runs over S-operators and t over positive numbers forms a neighbourhood system at the origin for a certain topology which is called the S-topology. A net $\{ x_\alpha \}$ converges to zero

in S-topology if and only if (Sx_α, x_α) converges to zero for every S-operator S.

We have the following theorem concerning characteristic functions and S-topology which was obtained by Sazanov [18].

<u>Theorem 4.2.1.</u> In order that a function $\mu(y)$ may be the characteristic function of a distribution on X it is necessary and sufficient that $\mu(0) = 1$, $\mu(y)$ be positive definite and continuous at $y = 0$ in the S-topology. (Here and elsewhere in the chapter 0 will denote the null element of X or the identity of the group and will be called the origin).

We also have the following theorem of Prohorov [16].

<u>Theorem 4.2.2.</u> In order that a positive definite function $\mu(y)$ with $\mu(0) = 1$ be the characteristic function of a distribution on X it is necessary and sufficient that for every $\varepsilon > 0$ there exists an S-operator S_ε such that

$$1 - R\mu(y) \leq (S_\varepsilon y, y) + \varepsilon$$

where R denotes the real part.

<u>Definition 4.2.2.</u> Let μ be a distribution on X for which $\int (x, y)^2 d\mu < \infty$ for each y. Then the covariance operator S of μ is that Hermitian operator for which

$$(Sy, y) = \int (x, y)^2 d\mu(x)$$

This operator S will be positive semidefinite and will be an S-operator if and only if

$$\int \| x \|^2 d\mu(x) < \infty$$

<u>Definition 4.2.3.</u> A sequence S_n of S-operators will be called compact if and only if the following two conditions are satisfied

$$1) \quad \sup_n \text{Trace}(S_n) < \infty$$

$$2) \quad \lim_{N \to \infty} \sup_n \sum_{j=N}^{\infty} (S_n e_j, e_j) \to 0$$

for some orthonormal sequence $e_1, e_2, \ldots, e_j, \ldots$

When S is the covariance operator of a distribution μ on X for which

$$\int \| x \|^2 \, d\mu(x) < \infty$$

we have the relation

$$\sum_{j=N}^{\infty} (S e_j, e_j) = \int r_N^2(x) \, d\mu(x)$$

where

$$r_N^2(x) = \sum_{j=N}^{\infty} (x, e_j)^2$$

and $e_1, e_2, \ldots$ any orthonormal basis.

The following theorems concerning conditions for compactness of a sequence μ_n of distributions on X were obtained by Prohorov [16].

<u>Theorem 4.2.3.</u> In order that a sequence μ_n of distributions on X be weakly conditionally compact it is necessary and sufficient that for every $\varepsilon > 0$, there exist a compact sequence S_n^{ε} of

S-operators such that

$$1 - R\,\mu_n(y) \leq (S_n^{\varepsilon}\, y\, y) + \varepsilon$$

for all n and y. Here $\mu_n(y)$ is the characteristic function of μ_n.

<u>Theorem 4.2.4</u>. Let μ_n be a sequence of distributions for which the covariance operators S_n exist and are S-operators. Let further S_n be compact. Then μ_n is weakly conditionally compact.

<u>Theorem 4.2.5.</u> In order that a sequence μ_n of distributions on X may be weakly conditionally compact it is necessary that for any $\varepsilon > 0$

$$\lim_{N \to \infty} \sup_n \mu_n \left[r_N^2 (x) > \varepsilon \right] = 0.$$

In defining the covariance operator in definition 4.2.2 we assumed that μ is a distribution. Actually if M is any σ-finite measure for which

$$\int \| x \|^2 \, dM < \infty$$

then S is well defined as an S-operator by the relation

$$(S\, y,\, y) = \int (x,\, y)^2 \, dM(x).$$

<u>4.3. An estimate of the variance.</u>

Let $X_1, X_2, \ldots X_n$, be n symmetric independent random variables in the Hilbert Space X. We will give in this section an estimate for the variance $E \| X_1 + \ldots X_n \|^2$ when each X_j is bounded uniformly in norm by a constant C independent of j.

To this end we introduce the concentration function $Q_\mu(\ell)$ following Levy ([24] pp.138)

Definition 4.3.1. The concentration function $Q_\mu(\ell)$ of a distribution μ in the Hilbert Space X is defined for $0 < \ell < \infty$ as

$$Q_\mu(\ell) = \sup_{x \in X} \mu(S_\ell + x)$$

where S_ℓ denotes the sphere $[x: \|x\| \leq \ell]$ and $S_\ell + x$ its translate by the element x of X.

We now list a few elementary properties of these functions.

Theorem 4.3.1.

1) $Q_\mu(\ell)$ is a nondecreasing function of ℓ and $\lim_{\ell \to \infty} Q_\mu(\ell) = 1$

2) If $\mu_1 * \mu_2 = \mu$ then, for every ℓ

$$Q_\mu(\ell) \leq \operatorname{Min}[Q_{\mu_1}(\ell), \; Q_{\mu_2}(\ell)].$$

3) If μ_n is shift compact then $\lim_{\ell \to \infty} \inf_n Q_{\mu_n}(\ell) = 1$

Proof: 1) and 2) are obvious and (3) is a consequence of tightness.

Theorem 4.3.2. Let $X_1, X_2, \ldots X_n$ be n mutually independent symmetric random variables. Let $S_j = X_1 + \ldots X_j$. Further let $Q(\ell)$ denote the concentration function of the sum $S_n = X_1 + \ldots X_n$. If T is defined as

$$T = \sup_{1 \leq j \leq n} \|S_j\|$$

then one has for any $\ell > 0$

$$P\{T > 4\ell\} \leq 2[1 - Q(\ell)].$$

<u>Proof:</u> This remarkable result of Levy, which is a refinement of Kolmogorov's inequality in terms of the variance, although proved by him for the real line, offers no difficulty for generalisation. Let the events E_k be defined as follows:

$$E_k = [\|S_1\| \leq 4\ell, \ldots, \|S_{k-1}\| \leq 4\ell, \|S_k\| > 4\ell]$$

$\{T > 4\ell\} = E_1 \cup E_2 \cup \ldots \cup E_n$, $E_i \cap E_j = \emptyset$. By $P_r\{\ \}$ we will denote the probability of the event with in the brackets given that the event E_r has occurred. We then have

$$(3.0) \quad P_r\{\|S_n\| \leq 2\ell\} \leq P_r\{\|S_n - S_r\| > 2\ell\} = P\{\|S_n - S_r\| > 2\ell\}$$

This is because E_r and $\|S_n\| \leq 2\ell$ imply that $\|S_n - S_r\| > 2\ell$ and $S_n - S_r$ is distributed independently of E_r. Let us further suppose that $Q(\ell) > \frac{1}{2}$ We now consider the distribution μ_{rn} of $S_n - S_r$ whose concentration function is denoted by $Q_{rn}(\ell)$. Since μ_{rn} is a factor of μ we have $Q_{rn}(\ell) > \frac{1}{2}$. This implies the existence of a point x in the space X such that

$$(3.1) \qquad \mu_{rn}(S_\ell + x) > Q_{rn}(\ell) - \varepsilon > \tfrac{1}{2}$$

Since μ_{rn} is a symmetric distribution

$$\mu_{rn}(S_\ell - x) = \mu_{rn}(-S_\ell + x) = \mu_{rn}(S_\ell + x) > \tfrac{1}{2}$$

Therefore

$$S_\ell - x \cap S_\ell + x \neq \emptyset ,$$

In other words there exists a point y such that

$$\| x+y \| \leq \ell ; \quad \| x-y \| \leq \ell .$$

These two imply that $\| x \| \leq \ell$ and hence

$$(3.2) \qquad x + S_\ell \subset S_{2\ell}$$

From (3.1) and (3.2) follows

$$\mu_{rn}(S_{2\ell}) > Q_{rn}(\ell) - \varepsilon$$

Since ε is arbitrary we have

$$(3.3) \qquad \mu_{rn}(S_{2\ell}) \geq Q_{rn}(\ell)$$

which is the same as

$$P\left\{\| S_n - S_r \| > 2\ell\right\} \leq 1 - Q_{rn}(\ell)$$
$$\leq 1 - Q(\ell)$$

We have further, using (3.0) and (3.4)

$$P\left\{T > 4\epsilon,\ \|S_n\| \le 2\epsilon\right\} = \sum_{r=1}^{n} \Pr\left\{\|S_n\| \le 2\right\} P[E_r]$$

$$(3.5) \qquad \le \left(\sum_{r=1}^{n} P[E_r]\right)(1 - Q(\epsilon))$$

$$= P\left\{T > 4\epsilon\right\}(1 - Q(\epsilon))$$

We also have

$$(3.6) \qquad P\left\{T \le 4\epsilon,\ \|S_n\| \le 2\epsilon\right\} \le P\left\{T \le 4\epsilon\right\} = 1 - P\left\{T > 4\epsilon\right\}$$

Adding (3.5) and (3.6) we get

$$(3.7) \qquad P\left\{\|S_n\| \le 2\epsilon\right\} \le 1 - P\left\{T > 4\epsilon\right\}\left\{Q(\epsilon)\right\}$$

From (3.3) putting $r = 0$, we obtain

$$(3.8) \qquad P\left\{\|S_n\| \le 2\epsilon\right\} \ge Q(\epsilon)$$

(3.7) and (3.8) imply since $Q(\epsilon) > \frac{1}{2}$

$$(3.9) \qquad P\left\{T > 4\epsilon\right\} \le \frac{1 - Q(\epsilon)}{Q(\epsilon)} \le 2[1 - Q(\epsilon)]$$

However if $Q(\epsilon) \le \frac{1}{2}$ the theorem is trivially true. Thus the proof of the theorem is complete.

We have the following theorem which gives an estimate of the variance This is well known for the real line. We reproduce the proof for the real line, which can be found in Halmos [5] page 198, replacing however $|x|$ by $\|x\|$.

Theorem 4.3.3. Let $X_1, X_2, \ldots X_n$ be n independent random variables in a Hilbert Space, uniformly bounded by a constant c in norm. Let each X_i have zero expectation. In addition let

$$P\left\{ \sup_{1 \leq j \leq n} \| S_j \| \leq d \right\} \geq \varepsilon > 0$$

where $S_j = X_1 + \ldots X_j$. Then

$$E \| S_n \|^2 \leq \frac{d^2 + (c + d)^2}{\varepsilon}$$

Proof: Let the sets E_k be defined as follows

$$E_k = \left\{ \sup_{1 \leq j \leq k} \| S_j \| \leq d \right\}$$

Then $E_1 \supset E_2 \ldots \supset E_n$ and $P(E_n) \geq \varepsilon > 0$. By P we mean the product measure of $X_1, X_2, \ldots X_n$. Let us define

$$F_k = E_{k-1} - E_k$$

and

$$\alpha_k = \int_{E_k} \| S_k \|^2 \, dP$$

We will take E_o as the whole space and α_o as zero. We then have

$$\alpha_k - \alpha_{k-1} = \int_{E_k} ||S_k||^2 dP - \int_{E_{k-1}} ||S_{k-1}||^2 \, dP$$

$$= \int_{E_{k-1}} ||S_k||^2 \, dP - \int_{F_k} ||S_k||^2 dP - \int_{E_{k-1}} ||S_{k-1}||^2 dP$$

(3.10)

$$= \int_{E_{k-1}} ||S_{k-1}||^2 dP - 2\int_{E_{k-1}} (S_{k-1}, X_k) dP + \int_{E_{k-1}} ||X_k||^2 dP -$$

$$\int_{F_k} ||S_k||^2 \, dP - \int_{E_{k-1}} ||S_{k-1}||^2 \, dP$$

$$= \int_{E_{k-1}} || X_k ||^2 \, dP - \int_{F_k} || S_k ||^2 \, dP$$

$$\geq P(E_{k-1}) \; E \; ||X_k||^2 \; P - (c+d)^2 P(F_k)$$

We notice here that $\int_{E_{k-1}} (S_{k-1}, X_k) \, dP = 0$ since X_k is independent of S_{k-1}, has zero expectation and is independent of E_{k-1} as well. Moreover $F_k \subset E_{k-1}$ and hence $|| S_k || \leq ||S_{k-1}|| + || X_k || \leq (c+d)$ over F_k. Since $P(E_n) \leq P(E_k)$ for any k we have

$$(3.11) \qquad \alpha_k - \alpha_{k-1} \geq P(E) \; E \; || X_k ||^2 - (c+d)^2 P(F_k) \quad \text{for } k = 1, 2, \ldots$$

Since $F_1, F_2, \ldots F_n$ are all disjoint adding (3.11) for $k=1,2,\ldots n$, we have

$$(3.12) \qquad \alpha_n \geq P(E) \; E \; ||S_n||^2 - (c+d)^2$$

However since $||S_n|| \leq d$ on E_n

As we have already remarked in the beginning of section 3.4 the results obtained in that section are valid for the Hilbert Space and we will restate them. We will keep in mind also that the Hilbert Space has no nontrivial compact subgroups and hence there are no idempotent distributions.

Theorem 4.4.1. The infinitely divisible distributions form a closed sub-semigroup among all distributions.

Theorem 4.4.2. If μ is an infinitely divisible distribution and $\mu(y)$ its characteristic function then $\mu(y)$ is nonvanishing.

For every finite measure F the infinitely divisible distribution $e(F)$ is associated in the same way as in definition 3.4.2. We then have

Theorem 4.4.3. Let $\mu_n = e(F_n)$. In order that μ_n may be shift compact it is necessary that

1) For each neighbourhood N of the identity F_n restricted to N' is weakly conditionally compact.

2) $\sup\limits_n \int [1 - \cos(x, y)] d F_n < \infty$ for each y.

Theorem 4.4.4. Let $\mu_n \Rightarrow \mu$. Then $\mu_n(y) \to \mu(y)$ uniformly over every bounded sphere.

Proof: This follows at once from theorem 2.2.3 since the set of functions (x, y) as y varies over a bounded sphere form a equicontinuous family of functions in x.

Theorem 4.4.5. Let μ_n be shift compact and $\mu_n(y) \to \mu(y)$ uniformly over bounded spheres. Then $\mu_n \Rightarrow \mu$.

<u>Proof:</u> Since μ_n is shift compact let x_n in X be chosen such that $\mu_n * x_n$ is compact. We will now show that x_n in X is compact. If x_n is not compact then we can produce a subsequence from x_n which has no further convergent subsequence. We will denote the subsequence by x_n itself. Since $\mu_n^n * x_n$ is weakly compact it has a subsequence $\mu_{n_j} * x_{n_j}$ converging weakly. Thus $\mu_{n_j}(y)\, e^{i(x_{n_j}, y)}$ as well as $\mu_{n_j}(y)$ converge uniformly over bounded spheres. Since from the compactness of $|\mu_n|^2$ one can conclude the existence of a sphere S such that $\operatorname*{Inf}_n |\mu_n(y)| \geq \epsilon > 0$ for all y in S, it follows that $e^{i(x_{n_j}, y)}$ converges uniformly over S and hence x_{n_j} converges in norm. This proves the theorem.

Before obtaining the representation we will show that if $e(F_n)$ is shift compact then

$$\sup_n \int_{||X|| \leq 1} ||x||^2 \, dF_n < \infty$$

To this end we consider the following lemma.

<u>Lemma 4.4.1.</u> Let $f(y)$ be a non-negative function on X such that $f(2y) \leq 4f(y)$ for all values of y. If $f(y) \leq \epsilon$ when ever $(Sy, y) \leq \delta$ where S is some S-operator then

$$f(y) \leq (S_1 y, y) + \epsilon \qquad \text{for all } y$$

where $S_1 = 4\,\epsilon\,\delta^{-1} S$.

<u>Proof:</u> Defining $S_e = \epsilon\,\delta^{-1} S$, we see that

(4.1) $\qquad f(y) \leq \varepsilon$ whenever $(S_o y, y) \leq \varepsilon$

Further if $(S_o y, y) \leq 4^n \varepsilon$ where n is a positive integer, then denoting by y_n the element $2^{-n} y$, we have

$$(S_o y_n, y_n) = 4^{-n}(S_o y, y) \leq \varepsilon$$

consequently $f(y_n) \leq \varepsilon$. But since $f(2y) \leq 4f(y)$, we have

$$f(y) \leq 4^n f(y_n) \leq 4^n \varepsilon .$$

So from (4.1) we have

(4.2) $\qquad f(y) \leq 4^n \varepsilon$ whenever $(S_o y, y) \leq 4^n \varepsilon$.

Let y be any element of X and let $(S_o y, y) = t$.

Case i. Let $t > \varepsilon$. If n is a non-negative integer such that

(4.3) $\qquad 4^n \varepsilon < t \leq 4^{n+1} \varepsilon$

We have since $(S_o y, y) = t \leq 4^{n+1} \varepsilon$, using (4.2) and (4.3)

(4.4) $\qquad f(y) \leq 4^{n+1} \varepsilon \leq 4t = 4(S_o y, y)$

Case ii. Let $t \leq \varepsilon$. Then from (4.1) we have

(4.5) $\qquad f(y) \leq \varepsilon$

(4.4) and (4.5) give at once

$$f(y) \leq \text{Max}\,[\,\varepsilon,\ (4\,S_o y,\ y)]$$
$$\leq \varepsilon + (S_1 y,\ y)$$

We shall need while proving the next theorem the following inequality. If $a_1, \ldots a_m$ are any m real numbers such that $|a_j| \leq 1$ for $1 \leq j \leq m$, then

$$1 - a_1 a_2 \ldots a_m \leq \sum_{j=1}^{m} (1 - a_j) \tag{4.6}$$

This inequality is proved by induction if all the a's are positive. If at least one of them say a_r is negative

$$\begin{aligned} 1 - a_1 \ldots a_m &\leq 1 + |\,a_1 \ldots a_m\,| \\ &\leq 1 + |\,a_r| \\ &= 1 - a_r \\ &\leq \sum_{j=1}^{m} (1 - a_j). \end{aligned}$$

We will now prove

<u>Theorem 4.4.6</u>. Let F_n be a sequence of finite measures such that $e(F_n)$ is shift compact. Then

$$\sup_n \int_{||X|| \leq 1} ||x||^2 \, dF_n < \infty.$$

<u>Proof:</u> We assume without any loss of generality that each F_n vanishes outside the unit sphere. Otherwise we can consider the restriction of F_n to the unit sphere instead of F_n. Let $M_n = F_n + \bar{F}_n$.

Then $e(M_n) = |e(F_n)|^2 = \lambda_n$ is compact. We will show that $\int ||x||^2 dM_n$ is uniformly bounded. To this end we assume that the total mass of M_n is an integer for every n. If this were not so we can write $M_n = M_n^{(1)} + M_n^{(2)}$ where $M_n^{(1)}$ is symmetric with an integral total mass and $M_n^{(2)}$ has total mass less than unity. Consequently

$$\int_{||X|| \leq 1} ||x||^2 \, dM_n^{(2)} \leq 1$$

Since our aim is to prove that

$$\sup_n \int_{||X|| \leq 1} ||x||^2 \, dM_n < \infty$$

it suffices to show that

$$\sup_n \int_{||X|| \leq 1} ||x||^2 \, dM_n^{(1)} < \infty$$

Now since the total mass of M_n is an integer say k_n we will write

$$M_n = F_{n1} + \ldots \quad F_{nk_n}$$

where F_{nj} for $j = 1, 2, \ldots k_n$, $n = 1, 2, \ldots$

is a symmetric probability measure in the unit sphere. Let us now denote by μ_n the convolution

$$\mu_n = F_{n1} * F_{n2} * \ldots F_{nk_n}$$

Since each F_{nj} is symmetric and has zero expectation

$$\int ||x||^2 \, d\,\mu_n = \sum_{j=1}^{k_n} \int ||x||^2 dF_{nj} = \int ||x||^2 dM_n$$

Hence it suffices to show that

$$\sup_n \int ||x||^2 \, d\,\mu_n < \infty$$

If $Q_n(\ell)$ denotes the concentration function of μ_n from theorem 4.3.4 we have

$$\int ||x||^2 \, d\,\mu_n \leq \frac{16\ell^2 + (4\ell+1)^2}{2Q_n(\ell) - 1}$$

whenever $Q_n(\ell) > \frac{1}{2}$. Therefore it is enough to prove that

$$\operatorname*{Inf}_n \; Q_n(\ell) \geq \frac{3}{4} \quad \text{for some } \ell.$$

which will follow from theorem 4.3.1 if we prove that μ_n is weakly conditionally compact.

Since each $F_{nj}(y)$ is a real characteristic function and

$$\mu_n(y) = \prod_{j=1}^{k_n} F_{nj}(y) \tag{4.7}$$

it follows from (4.6) that

$$\begin{aligned} 1 - \mu_n(y) &\leq \sum_{j=1}^{k_n} [1 - F_{nj}(y)] \\ &= \sum_{j=1}^{k_n} \int [1 - \cos(x, y)] dF_{nj}(x) \qquad (4.8) \\ &= \int [1 - \cos(x, y)] dM_n(x) \\ &= f_n(y) \quad \text{say.} \end{aligned}$$

We also have $\lambda_n(y) = \exp[-f_n(y)]$ and hence for any given ε there exists a δ depending only on ε such that

$$(4.9) \qquad f_n(y) \leq \varepsilon \quad \text{if} \quad 1 - \lambda_n(y) \leq \delta$$

Since λ_n is compact we have from theorem 4.2.3 for any given $\delta > 0$ a compact sequence S_n of S-operators depending on δ only such that

$$(4.10) \quad 1 - \lambda_n(y) \leq (S_n y, y) + \frac{\delta}{2}$$

From (4.9) and (4.10) it follows that whenever $(S_n y, y) \leq \delta/2$, $f_n(y) \leq \varepsilon$ and hence from lemma 4.4.1 we have

$$(4.11) \qquad f_n(y) \leq (S_n' y, y) + \varepsilon$$

where

$$S_n' = 8 \varepsilon \delta^{-1} S_n$$

Since S_n is compact so is S_n' and theorem 4.2.3 and (4.11) imply that μ_n is weakly conditionally compact. The proof of the theorem is now complete.

We will denote by $K(x, y)$ the following function:

$$K(x, y) = e^{i(x,y)} - 1 - \frac{i(x, y)}{1+||x||^2}$$

<u>Theorem 4.4.7.</u> Let μ_n for each n be of the form $e(F_n)$ where F_n is a finite measure. Let $\mu_n * x_n \Rightarrow \mu$ for some suitably chosen elements x_n in X. We further assume that F_n is increasing. Then F_n increases to a measure F which may be σ-finite but gives finite mass outside every neighbourhood of the origin and for which

$$\int_{||x|| \leq 1} ||x||^2 \, dF < \infty$$

In addition

$$\mu(y) = \exp \left[i(x_o, y) + \int K(x,y) dF(x) \right]$$

where x_o is a fixed element of X.

<u>Proof:</u> Let $\lambda_n(y)$ be defined as

$$\lambda_n(y) = \exp \left[\int K(x,y) d\, F_n \right]$$

Then $\lambda_n(y)$ is the characteristic function of λ_n which is the shift of μ_n by the element

$$Z_n = - \int \frac{x}{1 + ||x||^2} \, dF_n$$

We will now show that $\lambda_n(y)$ converges uniformly in y over bounded spheres. For this purpose we write

$$\text{(4.12)} \qquad \int K(x,y) dF_n = \int_{||x|| \leq 1} K(x,y) dF_n + \int_{||x|| > 1} K(x,y) dF_n$$

Let F be the limit of F_n. From theorems 4.4.3 and 4.4.6 it follows that F is finite outside every neighbourhood of the origin and

$$\int_{||x|| \leq 1} ||x||^2 \, dF < \infty$$

Since

$$| K(x, y) | \leq 2 + \frac{||X|| \cdot ||Y||}{1+||x||^2}$$
$$\leq 2 + || Y ||$$

it follows that

$$(4.13) \qquad \sup_{y \in S} \left| \int_{||X|| > 1} K(x,y) dF_n - \int_{||x|| > 1} K(x,y) dF \right| \to 0 \quad \text{as} \quad n \to \infty$$

for every bounded sphere S. On the other hand if $||x|| \leq 1$

$$| K(x,y) | \leq | e^{i(x,y)} - 1 - i(x, y) | + \frac{|(x,y)| \, ||x||^2}{1 + ||x||^2}$$
$$\leq \tfrac{1}{2} (x,y)^2 + |(x,y)| \, ||x||^2$$
$$< \tfrac{1}{2} ||x||^2 \, ||y||^2 + ||y|| \, ||x||^2$$

Since $\int ||x||^2 \, dF$ is finite it follows that

$$(4.14) \qquad \sup_{y \in S} \left| \int_{||x|| \leq 1} K(x,y) dF_n - \int_{||x|| \leq 1} K(x,y) dF \right| \to 0 \quad \text{as} \quad n \to \infty$$

for every bounded sphere S. (4.12), (4.13) and (4.14) imply that

$$\sup_{y \in S} | \lambda_n(y) - \lambda(y) | \to 0 \quad \text{as} \quad n \to \infty$$

for every bounded sphere S where

$$\lambda(y) = \exp\left[\int K(x,y)\,dF \right]$$

Since λ_n is shift compact from theorem 4.4.5 it follows that $\lambda_n \Rightarrow \lambda$ and λ has to be a shift of μ. Hence the theorem follows.

Theorem 4.4.8. Let $\mu(y)$ be a function of the form

$$\mu(y) = \exp\left[\int K(x, y)\,dF(x) \right]$$

where M is a σ-finite measure giving finite mass out side every neighbourhood of the identity and for which

$$\int_{||x|| \leq 1} ||x||^2 \, dF < \infty$$

Then $\mu(y)$ is the characteristic function of an infinitely divisible distribution.

Proof: Let N_n denote the sphere of radius $\frac{1}{n}$ around the origin and F_n the restriction of F to N_n'. Then F_n increases to F. Let

$$\mu_n(y) = \exp\left[\int K(x, y)\,dF_n \right]$$

From the proof of theorem 4.4.7 it follows that

$$\sup_{y \in V} | \mu_n(y) - \mu(y) | \to 0 \quad \text{as} \quad n \to \infty$$

for every bounded sphere V. In view of theorems 4.4.1 and 4.4.5 it is enough to show that μ_n is shift compact. We will now show that $\lambda_n = | \mu_n |^2$ is compact.

$$\lambda_n = |\mu_n|^2 = |e(F_n)|^2 = e(M_n) \quad \text{where} \quad M_n = F_n + \bar{F}_n .$$

Since F_n increases to F it follows that M_n increases to M where $M = F + \bar{F}$. Without any loss of generality we can assume that F and hence M vanishes outside the sphere $||x|| \leq 1$. We further have

$$\begin{aligned}
1 - \lambda_n(y) &= 1 - \exp\left[\int [\cos(x,y)-1] dM_n \right] \\
&= 1 - \exp\left[- \int [1 - \cos(x,y)] dM_n \right] \\
&\leq \int [1 - \cos(x,y)] dM_n \\
&\leq \int [1 - \cos(x,y)] dM \\
&\leq \tfrac{1}{2} \int_{||x|| \leq 1} (x,y)^2 d\,M \\
&= \tfrac{1}{2} (Sy, y)
\end{aligned}$$

Since

$$\int_{||x|| \leq 1} ||x||^2 dM = 2 \int_{||x|| \leq 1} ||x||^2 \, dF < \infty$$

it follows that S is an S-operator. Since S is a fixed S-operator independent of n it follows from theorem 4.2.3 that λ_n is compact. Consequently μ_n is shift compact and the theorem is proved.

Gaussian distributions are defined in the Hilbert Space in exactly the same manner as in definition 3.6.1. We will now prove

<u>Theorem 4.4.9.</u> A distribution μ on X is Gaussian if and only if $\mu(y)$ is of the form

$$\mu(y) = \exp\left\{ i(x_0, y) - (Sy, y) \right\}$$

where x_0 is a fixed element and S an S-operator.

<u>Proof:</u> Let us take a countable dense subset $y_1, y_2, \ldots y_n, \ldots$ in X and consider the map τ from X in to Z^∞, the countable product of the circle groups, defined as follows

$$x \to [e^{i(x,y_1)}, \ldots e^{i(x,y_n)}, \ldots]$$

Let H be the image of X under τ in Z^∞. Then τ is a both ways measurable isomorphism of the two groups X and H. If μ is Gaussian on X then $\mu\tau^{-1}$ is Gaussian in H and hence in Z^∞. From theorem 3.6.1 we have

$$\hat{\mu\tau^{-1}}(\theta) = \theta(Z) \exp[-\phi(\theta)] \tag{4.15}$$

where θ is a character on Z^∞, Z is a fixed element of Z^∞ and ϕ a function with properties specified in theorem 3.6.1. $\theta(Z)$ denotes the value of the character θ at the point Z. Further any character θ on Z^∞ is of the form

$$\theta(Z) = Z_1^{n_1} \ldots Z_k^{n_k} \tag{4.16}$$

where $n_1, \ldots n_k$ are integers and $Z_1, \ldots Z_k$ the first k co-ordinates of Z in Z^∞. Therefore

$$\hat{\mu\tau^{-1}}(\theta) = \hat{\mu}(n_1 y_1 + \ldots n_k y_k)$$

where $\hat{\mu}(y)$ is the characteristic function on X and $n_1, \ldots n_k$ are related to θ by the relation (4.16). Hence

$$|\mu \tau^{-1}(\theta)| = |\mu| (n_1 y_1 + \dots n_k y_k) = e^{-\phi(\theta)}$$

Since for any θ and θ'

$$\phi(\theta + \theta') + \phi(\theta - \theta') = 2 [\phi(\theta) + \phi(\theta')]$$

it follows that

(4.17) $$h(y + y') + h(y - y') = 2 [h(y') + h(y)]$$

whenever y, y' are of the form $n_1 y_1 + \dots n_k y_k$, where

$$h(y) = -\log |\mu(y)|$$

Since $y_1, y_2, \dots$ are dense and h is continuous in the norm topology of X it follows that the relation (4.17) is valid for any pair y, y'. This implies that $h(y)$ can be put in the form

$$h(y) = (Sy, y)$$

where S is a positive semidefinite Hermitian operator. From the continuity of $\mu(y)$ in the S- topology it follows that S is an S-operator. If we now consider the distribution λ on X defined by the equation

$$\lambda(y) = \exp [-(Sy, y)]$$

we have

$$\lambda \tau^{-1}(\theta) = e^{-\phi(\theta)}$$

Hence $\lambda\tau^{-1}$ is a shift of $\mu\tau^{-1}$. Since both $\lambda\tau^{-1}$ and $\mu\tau^{-1}$ give unit mass for the subgroup H of Z^{∞} the element Z by which $\lambda\tau^{-1}$ is shifted to obtain $\mu\tau^{-1}$ belongs to H and hence $Z = \tau(x_0)$ for some $x_0 \in X$. Consequently

$$\mu = \lambda * x_0$$

or

$$\mu(y) = \exp\left[i\,(x_0, y) - (Sy, y)\right]$$

conversely if $\mu(y)$ is of the form

$$\mu(y) = \exp\left[i(x_0, y) - (Sy, y)\right]$$

the distribution $\mu\tau^{-1}$ in Z^{∞} is Gaussian and hence so is μ.

We now prove the representation theorem for infinitely divisible distributions.

<u>Theorem 4.4.10</u>. A function $\mu(y)$ is the characteristic function of an infinitely divisible distribution μ on X if and only if it is of the form

$$\mu(y) = \exp\left[i(x_0, y) - (Sy, y) + \int K(x, y)\,dM \quad\right] \tag{4.18}$$

where x_0 is a fixed element of X, S an S-operator and F a σ-finite measure giving finite mass outside every neighbourhood of the origin and for which

$$\int_{\|x\| \leq 1} \|x\|^2 dM < \infty$$

Here $K(x, y)$ is the function

$$K(x, y) = \left[e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|x\|^2} \right]$$

The representation (4.18) of $\mu(y)$ is unique.

<u>Proof:</u> Let $\mu(y)$ be the characteristic function of an infinitely divisible distribution μ. Then in the same manner as in the proof of theorem 3.7.1 we can construct a sequence of distributions λ_n such that

i) $\lambda_n = e(M_n)$

ii) M_n increases to M

iii) $\lambda_n * x_n \Rightarrow \lambda$

iv) $\mu = \lambda * \mu_0$ where μ_0 is Gaussian

Now using theorems 4.4.7 and 4.4.9 we can complete the proof. Sufficiency is immediate from theorems 4.4.8 and 4.4.9. Uniqueness is proved in an exactly same manner as theorem 3.8.1 but keeping in mind that the space X playing the role of the character group is connected.

4.5. Compactness criteria

In the present section we will find out the necessary and sufficient conditions inorder that a sequence μ_n of infinitely divisible distributions may be weakly conditionally compact.

If μ is any infinitely divisible distribution by $\mu = [x, S, M]$ we will mean that the three quantities occurring in the representation of μ according to Theorem 4.4.10 are respectively x, S and M. We will associate with any such $\mu = [x, S, M]$ another S-operator which we will denote by T.

$$(5.1) \qquad (Ty, y) = 2(Sy, y) + \int_{\|x\| \leq 1} (x,y)^2 dM(x)$$

T is an S-operator since

$$\int_{\|x\| \leq 1} (x,y)^2 \, dM(x)$$

<u>Lemma 4.5.1.</u> In order that a sequence μ_n of Gaussian distributions with covariance operators S_n be shift compact it is necessary and sufficient that S_n should be compact. [If μ_n is Gaussian with covariance operator S its characteristic function is $\exp[i(x_0,y) - \frac{1}{2}(Sy,y)]$]

<u>Proof</u>: Sufficiency is immediate from theorem 4.2.4. We will prove necessity. If μ_n is shift compact then $|\mu_n|^2$ is compact. But $|\mu_n|^2$ is Gaussian with mean zero and covariance operator $2S_n$

$$|\mu_n|^2(y) = \exp[-(S_n y, y)]$$

From theorem 4.2.3 it follows that there exists a compact sequence U_n of S-operators such that

$$1 - |\mu_n|^2 (y) \leq (U_n y, y) + \epsilon$$

Hence there is a δ such that for any n $(U_n y, y) \leq \delta$ implies $(S_n y, y) \leq 1$. From this we can deduce that

$$(S_n y, y) \leq \delta^{-1} (U_n y, y)$$

Since δ is independent of n and U_n is compact, S_n is also compact.

<u>Lemma 4.5.2.</u> Let μ be a symmetric infinitely divisible distribution with $\mu = [0, 0, M]$. Further let M be concentrated in the unit sphere. Then

$$\int ||x||^4 d\mu \leq \int ||x||^4 dM + 3 \left[\int ||x||^2 dM \right]^2 < \infty.$$

<u>Proof:</u> It is enough to prove the theorem when M is finite since the other case can be obtained by limit ~~transposition~~ transition. Let $M(X) = t$ and F be the distribution such that $F(X) = 1$ and $M = tF$. Then

$$\mu = e^{-t} \sum \frac{t^r F^r}{r!}$$

$$\begin{aligned} \int ||x||^4 dF^r &= E \, || X_1 + \dots X_r ||^4 \\ &= rE \, ||X_1||^4 + r(r-1)[E \, ||X_1||^2]^2 + 2r(r-1)E(X_1, X_2)^2 \\ &\leq r \, E \, ||X_1||^4 + 3r(r-1) \, (E \, ||X_1||^2)^2. \end{aligned}$$

Expectation is with respect to F and $X_1, \dots X_r$ are independent random variables in X with the same distribution F. Terms with zero expectation have been omitted.

$$\int ||x||^4 d\mu \leq e^{-t} \sum \frac{rt^r E||X_1||^4}{r!} + 3e^{-t} \sum \frac{r(r-1)t^r (E||X_1||^2)^2}{r!}$$

$$= t\, E\, ||X_1||^4 + 3t^2\, (E\, ||X_1||^2)^2$$

$$= t \int ||x||^4\, dF + 3t^2\, [\int ||x||^2\, dF]^2$$

$$= \int ||x||^4\, dM + 3\, [\int ||x||^2\, dM]^2$$

<u>Lemma 4.5.3.</u> Let μ_n be symmetric infinitely divisible distributions such that

$$\mu_n = [0, 0, M_n]$$

with M_n vanishing outside the sphere $||x|| \leq 1$ for all n. If μ_n is compact then

$$\sup_n \int ||x||^4\, d\mu_n < \infty$$

<u>Proof:</u> Since M_n is concentrated in the unit sphere

$$\int ||x||^4\, dM_n \leq \int ||x||^2\, dM_n .$$

Theorem 4.4.6 implies that

$$\sup_n \int ||x||^2\, d\, M_n < \infty$$

and hence an application of lemma 4.5.2 will complete the proof.

<u>Remark:</u> In the same manner as lemma 4.5.2 it can be shown that if, M is symmetric and vanishes outside the sphere $||x|| \leq 1$ for

$$\mu = [0, 0, M]$$

we have

$$\int (x, y)^2 \, dM = \int (x, y)^2 \, d\mu \quad \text{for all } y$$

Lemma 4.5.4. Let μ_n be a weakly conditionally compact sequence of symmetric distribution such that

$$\sup_n \int ||x||^4 \, d\mu_n < \infty$$

Then if S_n is the covariance operator of μ_n, S_n is compact.

Proof: If θ_n is a sequence of distributions on the real line such that $\theta_n \Rightarrow \theta$ and $\int x^2 \, d\theta_n$ is uniformly bounded then

$$\int x d\theta_n \Rightarrow \int x d\theta \quad \text{as} \quad n \Rightarrow \infty.$$

Theorem 4.2.5 can be applied now and the lemma follows at once.

Theorem 4.5.1. Let μ_n be symmetric distributions that are infinitely divisible with representations

$$\mu_n = [0, S_n, M_n]$$

Then inorder that μ_n be compact it is necessary and sufficient that

1) M_n restricted outside any neighbourhood of the identity is weakly conditionally compact.

2) T_n as defined in (5.1) is compact.

Proof: We will first prove sufficiency. Let unit sphere be chosen as the neighbourhood and let us write $M_n = M_n^{(1)} + M_n^{(2)}$ where $M_n^{(1)}$ and $M_n^{(2)}$ are respectively the restrictions of M_n inside the unit sphere and outside the unit sphere. Since $M_n^{(2)}$ is weakly conditionally compact and $F_n \Rightarrow F$ implies $e(F_n) \Rightarrow e(F)$ it is enough to shew that the distributions

$$\lambda_n = [0, S_n, M_n^{(1)}]$$

form a compact sequence.

$$\int (x,y)^2 d\lambda_n(x) = 2(S_n y,y) + \int (x,y)^2 dM_n^{(1)}(x)$$

$$= 2(S_n y, y) + \int_{||x|| \leq 1} (x,y)^2 \, dM_n(x)$$

$$= (T_n y, y)$$

Since T_n is compact sufficiency follows from theorem 4.2.4. Necessity of 1) is a consequence of theorem 4.4.3 and 2) follows from lemmas 4.5.1, 4.5.2 and 4.5.4.

<u>Theorem 4.5.2</u>. In order that a sequence μ_n of infinitely divisible distributions with representations

$$\mu_n = [x_n, S_n, M_n]$$

be shift compact it necessary and sufficient that

i) M_n restricted to the complement of any neighbourhood N of the origin is weakly conditionally compact.

ii) T_n as defined in (5.1) is compact.

<u>Proof</u>: Since μ_n is shift compact if and only $|\mu_n|^2$ is compact what we need are conditions for the compactness of

$$|\mu_n|^2 = [0, 2S_n, M_n + \tilde{M}_n].$$

In addition we have

$$\int_{||x||\leq 1} (x, y)^2 \, d(M_n + \tilde{M}_n) = 2 \int_{||x||\leq 1} (x,y)^2 \, dM_n .$$

Hence the theorem follows from 4.5.1

<u>Theorem 4.5.3.</u> In order that μ_n with representations

$$\mu_n = [x_n, S_n, M_n]$$

be weakly conditionally compact it necessary and sufficient that inaddition to the conditions of theorem 4.5.2 x_n should be compact in X.

<u>Proof:</u> In order to prove the theorem it suffices to show that whenever

$$\mu_n = [x_n, S_n, M_n]$$

is shift compact

$$\lambda_n = [0, S_n, M_n]$$

is compact. Let $F_n \Rightarrow F$. Then the following convergence takes place in norm.

$$\int \frac{x}{1 + ||x||^2} \, d F_n \Rightarrow \int \frac{x}{1 + ||x||^2} \, dF.$$

Hence we can assume that M_n vanishes for all n outside the sphere $||x|| \leq 1$. Let us now consider

$$f_n(y) = \int K(x, y)dM_n(x)$$

$$|f_n(y)| \leq \tfrac{1}{2} \int_{||x|| \leq 1} (x,y)^2 dM_n(x) + \int_{||x|| \leq 1} \frac{|(x,y)|}{1+||x||^2} ||x||^2 \, dM_n(x)$$

$$\leq \tfrac{1}{2} (T_n y, y) + \int |(x, y)| \, ||x||^2 \, dM_n(x)$$

$$\leq \tfrac{1}{2} (T_n y, y) + \left[\int (x,y)^2 ||x||^2 \, dM_n \int ||x||^2 dM_n \right]^{\frac{1}{2}}$$

$$\leq \tfrac{1}{2} (T_n y, y) + c(T_n y, y)^{\frac{1}{2}}$$

Hence given any $\rho > 0$ there is a $\delta > 0$ such that $| f_n(y)| \leq \rho$ whenever $(T_n y, y) \leq \delta$. But since $\lambda_n(y) \mathrel{\overset{+}{=}} \exp [f_n(y)]$, for any $\varepsilon > 0$ there exists a $\rho > 0$ such that $1 - R\lambda_n(y) \leq \varepsilon$ whenever $| f_n(y)| \leq \rho$. Consequently for any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $(T_n y, y) \leq \delta$ it follows that $1 - R\lambda_n(y) \leq \varepsilon$. Here R denotes the real part. Now lemma 4.4.1 and theorem 4.2.3. imply the validity of the theorem since T_n is compact.

<u>Remark.</u> In defining the operator T we could have taken any bounded sphere around the origin instead of the unit sphere. When M_n restricted outside any neighbourhood of the origin is known to be weakly conditionally compact, the compactness of T_n when it is based on some sphere implies the compactness of T_n when it is based on any finite sphere.

Since the representation is unique one can give conditions for the convergence of

$$\mu_n = [x_n, S_n, M_n]$$

to the distribution

$$\mu = [x, S, M]$$

interms of $[x_n, S_n, M_n]$ and $[x, S, M]$

However we will need later in the chapter only the following:

<u>Theorem 4.5.4.</u> Let μ_n be a sequence such that μ_n has the representation

$$\mu_n = [x_n, S_n, M_n]$$

If $\mu_n \to \mu$, μ is Gaussian if and only if

$$M_n(N') \to 0 \text{ as } n \to \infty$$

for every neighbourhood N of the origin.

<u>Proof</u>: Let μ be Gaussian. Since μ cannot be written as $e(F) * \lambda$ with λ infinitely divisible it follows that $M_n(N') \to 0$ as $n \to \infty$ for every neighbourhood N of the origin. Conversely if $M_n(N') \to 0$ for every neighbourhood N in exactly the same manner as in the proof of theorem 3.6.1 it can be shown that $\mu(y)$ is of the form

$$\mu(y) = \exp[i(x_0, y) - (Sy, y)],$$

which shows that μ is Gaussian.

4.6. Accompanying laws.

Definition 4.6.1. A sequence α_{nj} of distributions $j = 1, 2, \ldots k_n$, $n = 1, 2, \ldots$ is said to be uniformly infinitesimal if for any neighbourhood N of the origin

$$\lim_{n \to \infty} \inf_{1 \leq j \leq k_n} \alpha_{nj}(N) = 1$$

Theorem 4.6.1. In order that α_{nj} be uniformly infinitesimal it is necessary that

$$\lim_{n \to \infty} \sup_{1 \leq j \leq k_n} \sup_{\|y\| \leq K} | \alpha_{nj}(y) - 1 | = 0$$

for every constant K.

Proof: This is immediate from theorem 4.4.4.

Theorem 4.6.2. Let α_{nj} be uniformly infinitesimal distributions with non-negative characteristic functions. Let

$$\mu_n = \prod_{j=1}^{k_n} \alpha_{nj}$$

and λ_n be defined as

$$\lambda_n = \prod_{j=1}^{k_n} e(\alpha_{nj})$$

In order that $\mu_n \Rightarrow \mu$ it is necessary and sufficient that $\lambda_n \Rightarrow \mu$.

Proof: Let μ_n be compact. Since the inequality $e^{x-1} \geq x$ is valid for all x we have

$$e(\alpha_{nj})(y) \geq \alpha_{nj}(y) \qquad \text{for } j = 1,2,\ldots k_n, \quad n = 1,2,\ldots$$

Since $\alpha_{nj}(y)$ is non-negative

$$\lambda_n(y) \geq \mu_n(y).$$

Or

$$1 - \lambda_n(y) \leq 1 - \mu_n(y).$$

From the compactness of μ_n and theorem 4.2.3 it follows that λ_n is compact. Now let λ_n be compact. It follows from theorem 4.5.2 and the remark made after theorem 4.5.3 that

1) F_n restricted to N' is weakly conditionally compact for every neighbourhood N.

2) The sequence S_n of operators defined by

$$(S_n y, y) = \int_{\|x\| \leq \ell} (x,y)^2 dF_n(x)$$

is compact for every ℓ

Here F_n denotes the sum $\alpha_{n1} + \alpha_{n2} + \ldots \alpha_{nk_n}$. We will now show that μ_n is compact. We have

$$\begin{aligned} 1 - \mu_n(y) &\leq \sum_{j=1}^{k_n} [1 - \alpha_{nj}(y)] \\ &= \int [1 - \cos(x,y)]\, dF_n(x) \\ &= \int_{\|x\| \leq \ell} [1-\cos(x,y)] dF_n(x) + \int_{\|x\| > \ell} [1-\cos(x,y)] dF_n(x) \\ &\leq \tfrac{1}{2} \int_{\|x\| \leq \ell} (x,y)^2 dF_n(x) + 2\, F_n[\|x\| > \ell\,] \\ &= \tfrac{1}{2} (S_n\, y,y) + 2F_n\, [\|x\| > \ell\,] \end{aligned}$$

Since F_n is weakly compact out side any neighbourhood we can choose such that for all n

$$F_n [\, ||x|| > \ell \,] \leq \epsilon/2$$

Since for that fixed ℓ S_n is compact theorem 4.2.3. shows that μ_n is compact.

We will now complete the proof by showing that whenever λ_n is compact, for every constant K

$$\sup_{||y|| \leq K} | \lambda_n(y) - \mu_n(y) | \to 0$$

To this end it is enough to show that

$$\sup_{||y|| \leq K} \sup_n \sum_{j=1}^{k_n} [1 - \alpha_{nj}(y)] < \infty.$$

But the expression is equal to

$$\sup_{||y|| \leq K} \sup_n \int [1 - \cos(x,y)] dF_n$$

$$\leq \sup_{||y|| \leq K} \sup_n \int_{||x|| \leq 1} [1 - \cos(x,y)] dF_n + \sup_n 2F_n(||x|| > 1)$$

$$\leq \tfrac{1}{2} K^2 \sup_n \int_{||x|| \leq 1} ||x||^2 dF_n + 2 \sup_n F_n(||x|| > 1)$$

$$< \infty.$$

The last step follows since $\lambda_n = e(F_n)$ is compact.

Lemma 4.6.1. Let α_{nj} be uniformly infinitesimal. Then if x_{nj} is defined by the relation

$$x_{nj} = \int_{||x|| \leq 1} d\, \alpha_{nj}$$

then

$$\sup_{1 \leq j \leq k_n} ||x_{nj}|| \to 0 \quad \text{as} \quad n \to \infty.$$

Proof: Let ε be arbitrary and V the sphere $||x|| \leq \varepsilon$. Then

$$||x_{nj}|| \leq || \int_V x\, d\, \alpha_{nj} || + || \int_{\varepsilon < ||x|| \leq 1} x\, d\, \alpha_{nj} ||$$

$$\leq \varepsilon + \alpha_{nj}(V')$$

Hence

$$\limsup_{n \to \infty} \sup_{1 \leq j \leq k_n} ||x_{nj}|| \leq \varepsilon .$$

Since ε is arbitrary the lemma follows.

Lemma 4.6.2. Let α_{nj} be uniformly infinitesimal. Let x_{nj} be defined as in lemma 4.6.1. Then if $\theta_{nj} = \alpha_{nj} * (-x_{nj})$ there exists a n_0 such that for all $1 \leq j \leq k_n$ and $n \geq n_0$ we have

$$|| \int_{||x|| \leq 1} x d\theta_{nj} || \leq 2\, \theta_{nj}\, [\,||x|| > 1]$$

Proof: Let n_0 be so chosen that

$$\sup_{1 \leq j \leq k_n} ||x_{nj}|| \leq 1/4 \quad \text{for all } n \geq n_o.$$

Then

$$\int_{||x|| \leq 1} x \, d\, \theta_{nj} = \int_{||x-x_{nj}|| \leq 1} (x-x_{nj}) d\, \alpha_{nj}$$

$$= \int_{||x-x_{nj}|| \leq 1} x d\, \alpha_{nj} - \int_{||x|| \leq 1} x d\, \alpha_{nj} + x_{nj}\, \alpha_{nj}\, [\, ||x-x_{nj}|| > 1].$$

Therefore for $n \geq n_o$ and $1 \leq j \leq k_n$,

$$|| \int_{||x|| \leq 1} x \, d\, \theta_{nj} || \leq \int_{[||x-x_{nj}|| \leq 1 \,\Delta\, ||x|| \leq 1]} ||x|| d\, \alpha_{nj} + ||x_{nj}||\, \theta_{nj}\, [\, ||x|| > 1]$$

$$\leq \int_{\frac{3}{4} \leq ||x|| \leq \frac{5}{4}} ||x||\, d\, \alpha_{nj} + \frac{1}{4}\, \theta_{nj}\, [\, ||x|| > 1]$$

$$\leq \frac{5}{4}\, \alpha_{nj}\, [\, ||x|| > \tfrac{3}{4}] + \frac{1}{4}\, \theta_{nj}\, [\, ||x|| > 1]$$

$$\leq \frac{5}{4}\, \theta_{nj}\, [\, ||x|| > 1] + \frac{1}{4}\, \theta_{nj}\, [\, ||x|| > 1]$$

$$\leq 2\, \theta_{nj}\, [\, ||x|| > 1]\, .$$

<u>Lemma 4.6.3</u>. Let F_n be a sequence of σ-finite measures such that F_n restricted to N' is finite and weakly conditionally compact for every neighbourhood N of the origin. Then for any $\varepsilon > 0$ there exists a compact set K such that

$$F_n(K') \leq \varepsilon \quad \text{for } n = 1, 2, \ldots$$

<u>Proof:</u> Let ε be any positive number. Choose a sequence N_r of neighbourhoods of the origin decreasing to the origin. Let A_r be defined as $N_{r-1} - N_r$, N_0 being taken as the whole space. From the conditions of the lemma it is possible to find a compact set K_r in A_r such that for all n

$$F_n(A_r - K_r) \leq \frac{\varepsilon}{2^r}$$

Let K be defined as

$$K = \bigcup_{r=1}^{\infty} K_r \cup \{0\}$$

Since

$$K \cap N_r' = K_1 \cup K_2 \ldots \cup K_r$$

and N_r decreases to the origin it follows that K is compact and

$$F_n(K') \leq \sum_{r=1}^{\infty} F_n(A_r - K_r) \leq \varepsilon.$$

We now proceed to prove the main theorem of the section.

Let α_{nj} be uniformly infinitesimal sequence of distributions on X. Let μ_n, x_{nj}, θ_{nj}, λ_n be defined as follows:

$$\mu_n = \prod_{j=1}^{k_n} \alpha_{nj}$$

$$x_{nj} = \int_{\|x\| \leq 1} x \, d\, \alpha_{nj}$$

$$\theta_{nj} = \alpha_{nj} * (-x_{nj})$$

$$\lambda_n = \prod_{j=1}^{k_n} e(\theta_{nj}) * \left(\sum_{j=1}^{k_n} x_{nj} \right)$$

In what follows we will adopt the above notation.

Theorem 4.6.3. If μ_n is shift compact then so is λ_n.

Proof: Since μ_n is shift compact $|\mu_n|^2$ is compact. But

$$|\mu_n|^2 = \prod_{j=1}^{k_n} |\alpha_{nj}|^2$$

It follows now from theorem 4.6.2 that if one defines

$$F_n = \sum_{j=1}^{k_n} |\alpha_{nj}|^2 = \sum_{j=1}^{k_n} |\theta_{nj}|^2 \tag{6.1}$$

then $e(F_n)$ is compact. We can now apply theorem 4.5.2 and lemma 4.6.3 and deduce that for any $\varepsilon > 0$ there exists a compact set K such that

$$F_n(K') \leq \varepsilon \qquad \text{for } n = 1, 2, \ldots \tag{6.2}$$

Let us now define the S-operators T_n by the formula

$$(T_n y, y) = \int_{||x|| \leq \ell} (x,y)^2 \, dF_n(x).$$

Then for any finite ℓ the sequence T_n is compact. Let us now define G_n as

$$G_n = \sum_{j=1}^{k_n} G_{nj} \tag{6.3}$$

In order to complete the theorem we have to show that $e(G_n)$ is shift compact or

a) G_n is weakly conditionally compact when restricted outside any neighbourhood of the identity.

b) If the S-operators S_n are defined as

$$(S_n y, y) = \int_{||x|| \leq 1} (x,y)^2 dG_n$$

Then S_n is compact.

Since α_{nj} is uniformly infinitesimal by lemma 4.6.1. G_{nj} are also uniformly infinitesimal. Hence for any $\varepsilon > 0$ there exists a compact set C such that

$$G_{nj}(C) \geq 1 - \varepsilon \quad \text{for all } n \text{ and } 1 \leq j \leq k_n.$$

In the same manner as in the proof of theorem 3.5.1 we have for all n and $1 \leq j \leq k_n$

$$
\begin{aligned}
|\theta_{nj}|^2(K') &= \int \theta_{nj}(K' + x)\,d\theta_{nj}(x) \\
&\geq \int_C \theta_{nj}(K' + x)\,d\theta_{nj}(x) \\
&\geq (1 - \varepsilon) \inf_{x \in C} \theta_{nj}(K' + x) \\
&= (1 - \varepsilon)[1 - \sup_{x \in C} \theta_{nj}(K + x)] \\
&\geq (1 - \varepsilon)\, \theta_{nj}(\overline{K + C}') \\
&= (1 - \varepsilon)\, \theta_{nj}(K_1')
\end{aligned} \tag{6.4}
$$

Where K_1 is another compact set.

In a similar manner it can be shown that if V and N are two neighbourhoods of the origin such that $V + V \subset N$ and

(6.5) $\qquad \theta_{nj}(V) \geq 1 - \varepsilon \quad$ for all $n \geq n_0$, $1 \leq j \leq k_n$

then for $n \geq n_0$ and $1 \leq j \leq k_n$

(6.6) $\qquad |\theta_{nj}|^2(V') \geq (1 - \varepsilon)\, \theta_{nj}(N')$

(6.2), (6.3) and (6.4) imply that for any $\varepsilon > 0$ there exists a compact set K_1 such that

(6.7) $\qquad \theta_n(K_1') \leq \varepsilon \quad$ for all n.

On the other hand (6.1), (6.3), (6.6) and the weak compactness of F_n

when restricted to the complement of any neighbourhood of the origin imply that for any neighbourhood N of the origin,

$$(6.8) \qquad \sup_n G_n(N') < \infty$$

(6.7) and (6.8) prove that G_n is weakly conditionally compact when restricted outside any neighbourhood of the origin. To prove b) let us consider

$$\int_{||x||\leq 2} (x,y)^2 d|\theta_{nj}|^2(x) = \iint_{||x_1-x_2||\leq 2} (x_1-x_2,y)^2 d\theta_{nj}(x_1)d\theta_{nj}(x_2)$$

$$\geq \int_{||x_1||\leq 1} \int_{||x_2||\leq 1} (x_1-x_2,y)^2 d\theta_{nj}(x_1)d\theta_{nj}(x_2)$$

$$= 2\,\theta_{nj}(||x||\leq 1) \int_{||x||\leq 1} (x,y)^2 d\theta_{nj}(x) - 2\Big[\int_{||x||\leq 1}(x,y)d\theta_{nj}\Big]^2$$

Since θ_{nj} is uniformly infinitesimal we can assume that $\theta_{nj}\,(||x|| \leq 1) \geq \frac{1}{2}$ for all suitably large n and for all $1 \leq j \leq k_n$. Hence

$$(S_n y,\, y) \leq (T_n y, y) + 2(U_n y,\, y)$$

where

$$(U_n y,\, y) = \sum_{j=1}^{k_n} \Big[\int(x,y)d\theta_{nj}(x)\Big]^2$$

Since we know that T_n is compact in order to show that S_n is compact it is enough to prove that U_n is compact. We will show now that trace of U_n tends to zero as $n \to \infty$, and this would complete the proof.

Let us put

$$y_{nj} = \int_{||x|| \leq 1} x \, d\Theta_{nj}$$

Then for the trace of U_n we have

$$\text{Trace of } U_n = \sum_{j=1}^{k_n} ||y_{nj}||^2$$

$$\leq \left(\sup_{1 \leq j \leq k_n} || y_{nj} || \right) \left(\sum_{j=1}^{k_n} || y_{nj} || \right)$$

From lemmas 4.6.1, 4.6.2 and (6.8) it follows that Trace of $U_n \to 0$ as $n \to \infty$.

<u>Theorem 4.6.4</u>. If λ_n is shift compact then so is μ_n.

<u>Proof:</u>

P.T.O.

$$1 - |\mu_n|^2(y) = 1 - \prod_{j=1}^{k_n} |\theta_{nj}|^2(y)$$

$$\leq \sum_{j=1}^{k_n} [1 - |\theta_{nj}|^2(y)]$$

$$\leq 2 \sum_{j=1}^{k_n} [1 - \text{Real } \theta_{nj}(y)]$$

$$= 2 \sum_{j=1}^{k_n} \int [1 - \cos(x, y)] d\, \theta_{nj}$$

$$= 2 \int [1 - \cos(x,y)] \; d\, G_n(x)$$

$$= 2 \int_{||x|| \leq \ell} [1 - \cos(x,y)] dG_n(x) + 2G_n[||x|| > \ell \;]$$

$$\leq \int_{||x|| \leq} (x,y)^2 dG_n(x) + 2\, G_n[||x|| > \ell \;]$$

$$(6.9) \qquad = (S_n y, y) + 2\, G_n[||x|| > \ell \;]$$

From the shift compactness of λ_n for any $\varepsilon > 0$ we can choose ℓ such that

$$(6.10) \qquad G_n\, [||x|| > \ell \;] \leq \frac{\varepsilon}{2} \quad \text{for all } n$$

and the sequence of operators S_n defined by

$$(S_n y, y) = \int_{||x|| \leq \ell} (x,y)^2 d\, G_n(x)$$

is compact. From (6.9), (6.10) the compactness of S_n and theorem 4.2.3 the present theorem follows.

<u>Theorem 4.6.5.</u> Let λ_n be shift compact. Then for any finite number ℓ we have

$$\lim_{n\to\infty} \sup_{||y||\leq \ell} | \lambda_n(y) - \mu_n(y) | = 0.$$

<u>Proof:</u> Since θ_{nj} is uniformly infinitesimal it follows from theorem 4.6.1 that

$$\lim_{n\to\infty} \sup_{||y||\leq \ell} \sup_{1\leq j\leq k_n} | \theta_{nj}(y) - 1 | = 0$$

Hence it is enough to show that

$$(6.11) \qquad \sup_{n} \sup_{||y||\leq \ell} \sum_{j=1}^{k_n} | \theta_{nj}(y) - 1 | < \infty$$

To this end we have

$$\theta_{nj}(y) - 1 = \int [e^{i(x,y)} - 1] d\,\theta_{nj}$$

$$= \int_{||x||\leq 1} [e^{i(x,y)} - 1 - i(x,y)] d\theta_{nj} + i \int_{||x||\leq 1} (x,y) d\theta_{nj} + \int_{||x||>1} [e^{i(x,y)} - 1] d\theta_{nj}$$

$$|\theta_{nj}(y) - 1| \leq \frac{1}{2} \int_{||x||\leq 1} (x,y)^2 d\theta_{nj} + ||y|| \, \Big|\Big| \int_{||x||\leq 1} x\, d\theta_{nj} \Big|\Big| + 2\, \theta_{nj}[||x||>1]$$

(6.11) follows at once from the compactness of S_n, lemma 4.6.2 and (6.8).

Theorem 4.6.6. In order that $\mu_n * x_n \Rightarrow \mu$, where x_n are arbitrary points in X, it is necessary and sufficient that $\lambda_n * x_n \Rightarrow \mu$.

Proof: This is an immediate consequence of theorems 4.6.3, 4.6.4, 4.6.5 and 4.4.5.

Theorem 4.6.7. Limit distribution of sums of independent uniformly infinitesimal random variables in X is infinitely divisible.

Proof: Theorems 4.6.6 and 4.4.1 imply the present theorem since λ_n is infinitely divisible for each n.

Theorem 4.6.8. Let $\mu_n * x_n \Rightarrow \mu$. In order that μ may be Gaussian it is necessary and sufficient that for each neighbourhood N

$$\lim_{n \to \infty} G_n(N') = 0,$$

where

$$G_n = \sum_{j=1}^{k_n} e_{nj}$$

Proof: This follows immediately from theorems 4.6.6 and 4.5.4.

Chapter V

SOME PROPERTIES OF THE SEMIGROUP OF DISTRIBUTIONS

5.1. A factorization theorem

According to a theorem of Khintchine [9] any distribution on the real line can be written as the convolution of two distributions one of which is the convolution of a finite or countable number of indecomposable distributions and the other is a distribution without any indecomposable factor and hence infinitely divisible. We will extend this result with a slight modification when X is a locally compact abelian separable metric group. The modification is necessary due to the existence of idempotent distributions.

Theorem 5.1.1. Let μ be any distribution on X. It can then be written as $\lambda_G * \lambda$ where λ is a distribution on X without any idempotent factors and λ_G is the maximal idempotent factor of μ.

Proof: Let H be the closed subgroup generated by the set of all characters y such that $\hat{\mu}(y) \neq 0$. H has interior and is hence open. Its annihilator G in X is therefore compact and the normalised Haar measure λ_G of G is the maximal idempotent factor of μ. If we denote by τ the canonical homomorphism from X onto X/G, then the distribution $\mu\tau^{-1}$ on X/G has no idempotent factors. We now choose a Borel set A contained in X such that τ restricted to A maps A onto X/G in a one-to-one manner. The existence of such a Borel set follows from a result due to Mackey ([14], lemma 1.1 page 102)

By a result of Kuratowski ([11], page 252) it follows that the inverse of τ from X / G onto A is a measurable map. Hence $\mu\tau^{-1}$ induces a measure λ on A. This λ satisfies the requirements of the theorem.

We now introduce a function $\theta(\alpha)$ defined for all factors α of μ, similar to a function introduced by Khintchine and to serve the same purpose. Let μ be a distribution on X without any idempotent factors. Then there is a sequence $y_1, y_2, \ldots$ of characters in Y such that $\mu(y_i) \neq 0$ for $i = 1, 2, \ldots$ and the smallest closed subgroup containing $y_1, y_2, \ldots$ is Y. Since $-\log|\mu(y_i)|$ is well defined we can choose a sequence ε_n of positive numbers such that

$$-\sum_{n=1}^{\infty} \varepsilon_n \log |\mu(y_n)| < \infty .$$

This implies at once that for every factor α of μ the function

$$\theta(\alpha) = -\sum_{n=1}^{\infty} \varepsilon_n \log |\alpha(y_n)|$$

is well defined and has the following obvious properties.

(1.1) i) $\theta(\alpha) \geq 0$

ii) $\theta(\alpha) = 0$ if and only if α is degenerate

iii) $\theta(\alpha_1 * \alpha_2) = \theta(\alpha_1) + \theta(\alpha_2)$

iv) If $\alpha_n \Rightarrow \alpha$ then $\theta(\alpha_n) \to \theta(\alpha)$

v) $\theta(\alpha) = \theta(\beta)$ if α and β are translates of each other.

Theorem 5.1.2. Let μ be a distribution without any idempotent or indecomposable factors. Then μ is infinitely divisible.

Proof: In view of the remarks made after theorem 3.5.2 it is enough to factorize μ in the form

(1.2) $$\mu = \alpha_{n1} * \dots \alpha_{n2^n}$$

in such a way that any limit of shifts of α_{nj} is degenerate. From the properties (i) – (v) of the Θ - function it is clear that it is sufficient to factorize μ in the form (1.2) with $\Theta(\alpha_{nj})$ equal to $2^{-n}\,\Theta(\mu)$ for all n and $1 \leq j \leq 2^n$. If μ satisfies the conditions of the theorem so does any factor of μ. Thus it is sufficient to prove that μ can be written as $\mu_1 * \mu_2$ with $\Theta(\mu_1) = \Theta(\mu_2) = \frac{1}{2}\,\Theta(\mu)$. A repetition of this will then complete the proof. In order to prove this we first observe that

$$\inf_{\alpha \in F(\mu),\, \Theta(\alpha) \neq 0} \Theta(\alpha) = 0,$$

where $F(\mu)$ denotes the set of all factors of μ. If this were not so then the class $F(\mu)$ being shift compact the infimum would be attained at a nondegenerate distribution which has to be indecomposable. Thus there are factors of μ with arbitrarily small Θ - values. We now take two distributions μ_1, μ_2 such that $\mu = \mu_1 * \mu_2$ and $|\,\Theta(\mu_1) - \Theta(\mu_2)|$ is a minimum. From the shift compactness of $F(\mu)$ it follows that the minimum is attained. The minimum is zero for otherwise by transferring a factor of μ_1 or μ_2 with an arbitrarily small Θ - value to

μ_2 or μ_1 we can make $|\theta(\mu_1) - \theta(\mu_2)|$ smaller. This completes the proof.

Theorem 5.1.3. Any distribution μ on X can be written as $\lambda_H * \lambda_1 * \lambda_2$ where λ_H is the maximal idempotent factor of μ, λ_1 is the product of a finite or countable number of indecomposable distributions and λ_2 is an infinitely divisible distribution without any indecomposable or idempotent factors.

Proof: An application of theorem 5.1.1. shows that μ can be written as $\lambda_H * \lambda$ where λ_H is the maximal idempotent factor and λ has no idempotent factors. Thus we can define a function θ on $F(\lambda)$ satisfying properties (i) - (v) of (1.1). Let now δ_1 be the maximum of $\theta(\alpha)$ as α varies over the indecomposable factors of λ. If δ_1 is greater than zero we write $\lambda = \alpha_1 * P_1$ where α_1 is indecomposable and $\theta(\alpha_1) \geq \delta_1/2$. We now denote by δ_2 ~~the δ_2~~ the maximum of $\theta(\alpha)$ as α varies over the indecomposable factors of P_1. If $\delta_2 > 0$ then we write $P_1 = \alpha_2 * P_2$ where α_2 is indecomposable and $\theta(\alpha_2) \geq \delta_2/2$. We repeat this procedure successively. If the process terminates at the n^{th} stage then $\lambda = \alpha_1 * \dots \alpha_n * P_n$ and P_n has no idempotent or indecomposable factors. Otherwise the process continues ad-infinitum. Since

$$\sum_{j=1}^{\infty} \theta(\alpha_j) < \infty, \quad \theta(\alpha_n) \to 0 \text{ as } n \to \infty.$$

The sequence $\alpha_1 * \alpha_2 \dots * \alpha_n$ being increasing in the sense of convolution by theorem 3.2.1, will converge after a suitable shift. Absorbing these shifts in α_n we can suppose that the sequence $\alpha_1 * \dots \alpha_n$ converges. Since P_n will now be compact we take any limit λ_2 and write $\lambda = \lambda_1 * \lambda_2$ where $\lambda_1 = \prod_{j=1}^{\infty} \alpha_j$. We will now show that λ_2 has no idempotent or indecomposable factors inorder to complete the proof. That it has no idempotent factors is obvious. Since every factor of λ_2 is also a factor of P_n for each n if α is any indecomposable factors of λ_2 we have

$$\theta(\alpha) \leq \delta_n \leq 2\,\theta(\alpha_n)$$

and since $\theta(\alpha_n) \to 0$ as $n \to \infty$ it follows that $\theta(\alpha) = 0$ or α is degenerate. Consequently λ_2 has no indecomposable factors. An application of theorem 5.1.2 is all we need to complete the proof.

5.2. Nonatomic nature of infinitely divisible distributions

Let X be a locally compact group and μ an infinitely divisible distribution on X with a representation, according to theorem 3.7.1, interms of the Gaussian component ϕ and the measure M. What are the conditions on ϕ and M in order that μ may be non-atomic, purely atomic or a mixture ? This problem was considered by J. R. Blum and M. Rosenblatt [25] when X is the real line.

Let μ be any measure. We will define a function $A(\mu)$ as follows. If the measure is non-atomic then $A(\mu)$ is defined to be zero. If there are atoms then $A(\mu)$ is the mass corresponding to the atom with maximum mass.

(2.1) $$A(\mu) = \operatorname*{Max}_{x} \mu[\{x\}]$$

where $\{x\}$ is the single point set consisting of x only. For any two distributions μ_1, μ_2 we have at once

(2.2) $$A(\mu_1 * \mu_2) \leq \text{Min } [A(\mu_1), A(\mu_2)].$$

For any two measures F_1 and F_2 we have

(2.3) $$A(F_1 * F_2) \leq \text{Min } [A(F_1)F_2(X), A(F_2)F_1(X)].$$

Lemma 5.2.1. Any non-degenerate Gaussian distribution is non-atomic.

Proof: Since $\phi(y)$ is not identically zero there is a y_0 such that $\phi(y_0) > 0$. Let us now consider the induced distribution on the circle group by the map $x \Rightarrow (x, y_0)$. The induced distribution has a characteristic function

$$\alpha(n) = e^{-n^2 \phi(y_0)}$$

Since $\phi(y_0) > 0$, this distribution is non-atomic and hence the original distribution on X is also non-atomic.

Lemma 5.2.2. Let F be a totally finite measure and $\mu = e(F)$. Then

$$A(\mu) \leq e^{-F(X)} \frac{A(F)}{F(X)}.$$

Proof:

$$\mu = e^{-F(X)}\left[1 + F + \frac{F^2}{2!} + \dots\right]$$

where 1 is the distribution degenerate at the identity.

$$A(\mu) \leq e^{-F(X)}\left[1 + A(F) + \frac{A(F^2)}{2!} + \dots\right]$$

$$\leq e^{-F(X)}\left[1 + A(F) + \frac{F(X)\,A(F)}{2\,!} \dots + \frac{[F(X)]^{n-1}}{n\,!} A(F) + \dots\right]$$

$$= e^{-F(X)} + e^{-F(X)} A(F)\left[1 + \frac{F(X)}{2!} + \dots \frac{[F(X)]^{n-1}}{n!} + \dots\right]$$

$$= e^{-F(X)} + e^{-F(X)} A(F)\left[\frac{e^{F(X)} - 1}{F(X)}\right]$$

$$= e^{-F(X)} + A(F)\left[\frac{1 - e^{-F(X)}}{F(X)}\right]$$

$$\leq e^{-F(X)} + \frac{A(F)}{F(X)} .$$

Lemma 5.2.3. Let μ be infinitely divisible with a representation interms of the Gaussian component ϕ and the measure M. Let ϕ be identically zero and M non-atomic and totally infinite. Then μ is non-atomic.

Proof: Let F_n be the restriction of M to the complement N_n' of a neighbourhood N_n. Let N_n decrease to the identity. Since $e(F_n)$ is a factor of μ we have from lemma 5.2.2.

$$A(\mu) \leq A[e(F_n)] \leq e^{-F_n(X)} + \frac{R(F_n)}{F_n(X)}$$

$$= e^{-F_n(X)}$$

Since $F_n(X)$ increases to ∞ as n increases to ∞ as n increases it follows that $A(\mu) = 0$ or μ is non-atomic.

Lemma 5.2.4. Let μ be infinitely divisible and have a representation interms of the Gaussian component ϕ and the measure M. Let ϕ be identically zero and M purely atomic. Let the masses of M be concentrated at the points $g_1, g_2, \ldots, g_n, \ldots$ of X and the corresponding masses be $p_1, p_2, \ldots p_n, \ldots$ If $\sum_{n=1}^{\infty} p_n = \infty$, then μ is non-atomic.

Proof: Let ε be any positive number. Let q_n be defined as $\mathrm{Min}\ (\varepsilon, p_n)$. Then $\sum q_n$ also diverges. Let for any N, k be so chosen that

$$q_1 + q_2 + \ldots q_k \geq N.$$

If by F one denotes the measure with masses $q_1, q_2, \ldots q_k$ at the points $g_1, g_2, \ldots g_k$ respectively we have $F \leq M$ and $A(F) \leq \varepsilon$. Since $e(F)$ is factor of μ we have

$$A(\mu) \leq A[e(F)] \leq e^{-N} + \frac{\varepsilon}{N}$$

Since ε, N are arbitrary it follows that $A(\mu)$ is equal to zero or μ is non-atomic.

Theorem 5.2.1. Let μ be infinitely divisible with a representation interms of the Gaussian component ϕ and the measure M. In order that μ have an atom of positive mass it is necessary and sufficient that

1) $\phi \equiv 0$

2) $M(X) < \infty$

In that case μ is purely atomic if and only if M is purely atomic.

Proof: The necessity is contained in lemmas 5.2.1, 5.2.2 and 5.2.3. Sufficiency is obvious for $e(F)$ gives a mass of $e^{-F(X)}$ to the identity. The second part of the theorem follows from the expansion for $\mu = e(F)$ interms of the convolution powers of F.

5.3. Gaussian Distributions.

It is well known that in an n-dimensional vector space any symmetric Gaussian distribution has for its spectrum the whole space or a proper subspace of lower dimension. We shall show now that a similar situation prevails in a locally compact group.

Theorem 5.3.1. Let μ be any symmetric Gaussian distribution on X. Then the spectrum of μ is a connected subgroup of X.

Proof: Let μ have the characteristic function $\hat{\mu}(y) = \exp[-\phi(y)]$. Let C be the spectrum of μ. Since μ is symmetric it follows that $-C = C$. Hence to show that C is a subgroup it is enough to show that $C + C \subset C$. To this end we consider the two distribution α and β which are also Gaussian and are defined by the relation

$$\alpha(y) = \exp\left[- \frac{\phi(y)}{4}\right]$$
$$\beta(y) = \exp\left[- \frac{\phi(y)}{2}\right].$$

We have $\alpha^2 = \beta$ and $\beta^2 = \mu$. From the relation $\phi(2y) = 4\,\phi(y)$ it follows that if we denote by τ the homomorphism $x \to 2x$ in X

$$\alpha\tau^{-1} = \mu.$$

Hence if A and B denote the spectra of α and β respectively

(3.1) $$\overline{A + A} = B$$

(3.2) $$\overline{B + B} = C$$

(3.3) $$\overline{2A} = C$$

Since $2A \subset A + A$ it follows that $C \subset B$ and hence

$$C + C \subset C$$

In order to show that C is connected let us consider the character group of G. If H is the subgroup of Y defined by

$$H = [\, y : \phi(y) = 0 \,]$$

then Y/H is the character group of C and $\phi(y)$ which can be considered as a function on Y/H has the following properties in Y/H

1) $\phi(y) \neq 0$ for $y \neq e$

2) $\phi(ny) = n^2 \phi(y)$

3) $\phi(y)$ is continuous.

Hence Y/H cannot have any compact subgroups. Therefore C is connected.

5.4. Processes with independent increments.

Let X be a finite dimensional vector space or a Hilbert Space. Let $x(t)$ be an X valued stochastic process with independent increments. The increments need not be homogeneous in time. Let us suppose that the process is uniformly stochastically continuous in any bounded t-interval. Then every marginal will be infinitely divisible. From the uniqueness of the representation it will follow that if μ_t is the marginal distribution of $x(t)$ then the representation

$$\mu_t = [x_t, S_t, M_t]$$

has the following properties

1) x_t is continuous

2) $(S_t y, y)$ is a non-decreasing continuous function of t for each fixed y

3) $M_t(A)$ is a non-decreasing continuous function of t for each fixed closed set A not containing the origin.

However when X is a locally compact group in general the representation is not unique and if we choose for each t the representation of μ_t without any consideration the properties 1), 2), 3) mentioned above may not be satisfied. We will show presently that there is exactly one choice of the representation which ensures these properties.

The non-uniqueness of a single representation and the uniqueness of the representation when a family μ_t is involved perhaps requires a bit of explanation. Let us take some infinitely divisible distribution μ. Suppose there are two infinitely divisible distributions which are not even shifts of one another such that $\alpha^2 = \beta^2 = \mu$. Then naturally one cannot expect a unique representation for μ. On the other hand when we consider a semigroup μ_t, with $\mu_1 = \mu$ then $\mu_{\frac{1}{2}}$ specifies the root of μ which is taken. Hence among the many representations of μ we need consider only those which correspond to the root $\mu_{\frac{1}{2}}$ of μ. It is this type of internal consistency which makes possible a unique representation.

<u>Theorem 5.4.1.</u> Let α_{st} be a family of distributions on X defined for $0 \leq s \leq t$ such that

a) $\alpha_{st} * \alpha_{tu} = \alpha_{su}$ for $s \leq t \leq u$

b) α_{st} converges to the distribution degenerate at the origin whenever $t - s$ converges to zero in such a manner that t and s remain bounded. Then there exists a family of representations $[x_t, \phi_t, M_t]$ such that

1) x_t is continuous in t

2) $\phi_t(y)$ is a non-decreasing continuous function of t for each fixed y.

3) $M_t(A)$ is a non-decreasing continuous function of t for each fixed closed set A not containing the identity.

4) For every $s \leq t$, α_{st} has the representation

$$\alpha_{st} = [\, x_t - x_s, \phi_t - \phi_s, M_t - M_s]$$

x_t, ϕ_t, M_t with the above properties is unique.

<u>Proof</u>: From conditions a) and b) we conclude that each α_{st} is infinitely divisible. Since the Gaussian component is unique just as in the proof of theorem 3.9.1. We can have it removed and assume without any loss of generality that α_{st} has no Gaussian component for any pair $s \leq t$. Let us consider for any integer n the set of real numbers $r/n!$ as r varies over the integers and represent $\alpha_{\frac{r}{n!}, \frac{r+1}{n!}}$ as

$$\alpha_{\frac{r}{n!}\ \frac{r+1}{n!}} = [\, Z_{rn}, 0, F_{rn}]$$

We define M_{tn} for any t and n by the relation

$$M_{tn} = \sum_{r=0}^{k-1} F_{rn}$$

where k is so chosen that $\frac{k}{n!} \leq t < \frac{k+1}{n!}$. The distribution with representation $[0, 0, M_{tn}]$ is a factor of α_{ot} for all n. Hence we can choose a subsequence n_j such that M_{tn_j} converges weakly for each rational t to M_t in the suitable sense (when restricted outside each neighbourhood of the identity). Since M_{tn} is monotonic in t so will be M_t over the rationals. M_t for each rational t will be such that $[0, 0, M_t]$ represents a translate of α_{ot}. We can show easily that M_t can be extended for all t and then $[0, 0, M_t]$ would represent a shift of α_{ot} for all t. Introducing now the Gaussian part also we have

$$\alpha_{ot} = [\, x_t,\ \phi_t,\ M_t \,]$$

or

$$\alpha_{st} = [\, x_t - x_s,\ \phi_t - \phi_s,\ M_t - M_s \,]$$

As for uniqueness, since we already know that ϕ is unique, if M_t and M_t' are two representations, we can obtain in the same way as in theorem 3.8.1 for all $s \leq t$, and any pair y_1, y_2 of characters

$$\exp [\ \int (x, y_1)\ [\ 1 - R(x, y_2)]\ d\ G_{st}\ ] = 1$$

where $G_{st} = (M_t - M_s) - (M_t' - M_s')$. From this we conclude

$$\int (x, y_1)\ [1 - R(x, y_2)]\ d\ G_{st} = 2\pi i\, n(s, t)$$

where $n(s, t)$ is an integer valued continuous function of s and t and since $n(s, s) = 0$, $n(s, t) = 0$. Therefore $\theta_{st} = 0$.

5.5. The case of a complete separable metric group.

Let X be any abelian complete separable metric group with a sequence $y_1, y_2, \dots y_n, \dots$ of characters separating points of X. Let us consider the map τ from X into Z^{∞}, the countable product of the circle groups, defined by

$$x \to [\, y_1(x), \dots y_n(x), \dots \,]$$

τ is a continuous one to one map of X on to H, a subgroup of Z^{∞} which is the image of X under τ. From a theorem of Kuratowski H is a measurable subgroup of Z^{∞} and the inverse τ^{-1} is a measurable homomorphism of H onto X. For the study of the semigroup of distributions on X to some extent the map τ can be exploited.

Lemma 5.5.1. Let μ be a distribution on $H \subset Z^{\infty}$. Let λ_1, λ_2 be two distributions on Z^{∞} such that $\lambda_1 * \lambda_2 = \mu$. Then there are translates μ_1, μ_2 of λ_1, λ_2 respectively such that $\mu_1(H) = \mu_2(H) =$ and $\mu_1 * \mu_2 = \mu$.

Proof: $\mu(H) = \int \lambda_1(H - x) d\, \lambda_2(x) = 1$

Hence $\lambda_1(H - x) = 1$ almost everywhere with respect to λ_2. Hence there is a translate μ_1 of λ_1 such that $\mu_1(H) = 1$. Similarly there is a translate μ_2 of λ_2 such that $\mu_2(H) = 1$. Since $\mu_1 * \mu_2 = \mu'$ is a translate of μ and $\mu'(H) = 1$, it follows that μ' is a translate of μ by an element of H. Absorbing this shift in μ_1 or μ_2 we have at once

$$\mu_1 * \mu_2 = \mu$$

and $\quad \mu_1(H) = \mu_2(H) = 1.$

Hence as far as the factorisation properties are concerned factorisation within the subgroup H is equivalent to factorisation in the whole group Z^{∞}. Therefore the map between the distributions on X and the distributions on H induced by τ is an isomorphism of the two semigroups which takes Gaussian distribution to Gaussian distributions, infinitely divisible distributions to infinitely divisible distributions and indecomposable distributions to indecomposable distributions. Using these we will prove a few results.

<u>Theorem 5.5.1</u>. Let α_{nj} be uniformly infinitesimal distributions on X. Let $\mu_n = \prod_{j=1} \alpha_{nj}$. If $\mu_n * x_n \Rightarrow \mu$ then μ is infinitely divisible in X.

<u>Proof:</u> Let $\beta_{nj} = \alpha_{nj}\tau^{-1}$. Since τ is continuous β_{nj} is uniformly infinitesimal. If $\lambda_n = \prod_{j=1} \beta_{nj}$ then $\lambda_n = \mu_n\tau^{-1}$ and $\lambda_n * \tau(x_n) \Rightarrow \mu\tau^{-1}$. Therefore $\mu\tau^{-1}$ is an infinitely divisible distribution in Z^{∞}. By lemma 5.5.1, $\mu\tau^{-1}$ is infinitely divisible in H and hence μ is infinitely divisible in X.

<u>Theorem 5.5.2</u>. Let μ be any distribution on X. Then $\mu = \lambda_B * \lambda_1 * \lambda_2$, where λ_B is the maximal idempotent factor, λ_1 is the product of a finite or countable number of indecomposable

factors and λ_2 is infinitely divisible without indecomposable or idempotent factors.

<u>Proof:</u> Let us consider $\mu\tau^{-1} = \alpha$ on H and factorise α as $\alpha_0 * \alpha_1 * \alpha_2$ according to theorem 5.1.3. It is clear that we can pass over to X and we have to show only convergence in X. This follows at once from theorems 2.3.1 and 3.2.1.

In an exactly similar manner we can prove

<u>Theorem 5.5.3.</u> Let $\mu(y)$ be the characteristic function of a distribution μ. Then μ is Gaussian if and only if

$$\mu(y) = y(x) \exp [-\phi(y)]$$

where x is a fixed point, $y(x)$ is the value of the character y at the point x and $\phi(y)$ is a continuous non-negative function satisfying

$$\phi(y_1 + y_2) + \phi(y_1 - y_2) = 2[\phi(y_1) + \phi(y_2)]$$

for every pair y_1, y_2 of characters.

We have worded the theorem 5.5.3 differently from the locally compact case. The difference is in the assumption that $\mu(y)$ is a characteristic function. In the locally compact case this would follow from positive definiteness and continuity. In a Hilbert Space this would follow from positive definiteness and continuity in the S-topology. A similar condition is as yet unknown for the case of a separable and complete metric group. Even when X is a Banach space no general conditions are known under which a positive definite functions corresponds to a measure. Before we can make a detailed study of the semigroup $\mathfrak{M}$ of distributions on X, with characteristic functions as an important tool, we have to answer the above question. We are still very far away from the answer.

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