SOME CONTRIBUTIONS

TO

THE DESIGN AND ANALYSIS OF EXPERIMENTS

WITH SPECIAL REFERENCE

TO

(WEIGHING DESIGNS, PARTIALLY BALANCED DESIGNS AND DESIGNS FOR TWO-WAY ELIMINATION OF HETEROGENEITY)

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CONTENTS

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NOTATIONS

All capital letters denote matrices and the underlined letters denote vectors. I_n denotes the n^{th} order identity matrix, E_{mn} denotes an $m \times n$ matrix with positive unit elements everywhere and O_{mn} denotes an $m \times n$ null matrix. The transpose of X is denoted by X'. The inverse of X is denoted by X^{-1}. The Kronecker product and ordinary product of two matrices A and B are denoted respectively by $A \times B$ and AB. The determinant of A is denoted by A or det A. Diag $(a_1, a_2, \ldots, a_n)$ stands for the diagonal matrix with elements $a_1, a_2, \ldots, a_n$.

The Kronecker delta, δ_{ij}, is used to denote the value 1 or 0 according as $i = j$ or $i \neq j$.

The symbols used in one chapter have no bearing on the symbols used in another chapter, unless specifically mentioned.

The numbers in square brackets like [ ] refer to Bibliography.

Notations like "BIBD" are not to be confused with the product of the matrices.

E(y) means expectation of y; y may be a matrix, a vector or a scalar. V(y) means variance of y.

GENERAL INTRODUCTION

This dissertation contains the author's various contributions to the design and analysis of experiments. The main topics covered in this thesis are weighing designs, partially balanced designs and designs for two-way elimination of heterogeneity. Some of the material of the chapters 2, 3, 4, 5 (marked with asterisks in the thesis) has been compiled from the author's published papers [9], [10], [11], [12], [13], [14].

Chapters I and II deal with chemical balance weighing designs. Yates [71] and Hotelling [36] observed that the weights of the objects can be more accurately determined by weighing them in groups. If n weighing operations are made to weigh p objects, the minimum variance that each estimated weight may have, will be $\frac{\sigma^2}{n}$ where σ^2 is the variance of each weighing. Since we an interested in the weights of the objects and not in the estimate of σ^2, the minimum number of weighing operations to weigh n objects is n. Let $X = ((x_{ij}))$ be the weighing design matrix of order n for weighing n objects in n weighings with a chemical balance having no bias, where $x_{ij} = +1$ or -1 if the j^{th} object is included in the i^{th} weighing by being placed in the left or right pan and $x_{ij} = 0$ if the j^{th} object is not included in the i^{th} weighing. It can be seen that $(X'X)^{-1}\sigma^2$ is the variance covariance matrix of the estimated weights. Hotelling has shown that the minimum variance for the estimated weight is obtained if X is a matrix consisting of ± 1 such that the columns are orthogonal. Hence, we call a weighing design to be optimum if $X'X = nI_n$; i.e. X is a Hadamard matrix (H_n). It may be remarked that a necessary condition for the existence of Hadamard matrices is $n \equiv 0 \bmod 4$ with the possible exception of $n = 2$. A complete summary of the status of the existence of H_n is given by Bose and Shrikhande [22] and conjectured that for every order of $n \equiv 0 \bmod 4$, H_n exists. In section 1.2 of the chapter I, we give a method of construction of H_n where $n = 2\,[(2p^k + 1)^2 + 1]$, $p^k \equiv 3 \bmod 4$.

In the absense of optimum weighing designs, the problem is to find the best weighing design. There are three well known criteria to decide this problem. They are due to (i) Kishen [42] (ii) Ehrenfeld [30] and (iii) Mood [44]. Banerjee [7] has given an expository article reviewing the work done in weighing designs till the year 1950. Raghavarao [52] found best weighing designs subject to some restrictions. No general solution for finding best weighing designs when $n \not\equiv 0 \bmod 4$ was obtained in the previous literature in weighing designs. In sections 1.3, 1.4, 1.5 we show that when n is odd, P_n matrix (cf. Raghavarao [50]) if it exists, is the best weighing design under the three efficiency criteria. Also, it is shown that when $n \equiv 2 \bmod 4$, T_n (cf. Raghavarao [52]) if it exists is the best weighing design with Ehrenfeld's definition of efficiency; and U_n (cf. Ehilich [29]) if it exists, is the best one in Mood's definition of efficiency.

Chapter II deals with the designs with some restrictions. Sections 2.2, 2.4, 2.6, 2.7 and 2.8 are confined to the class of designs, where these give

(i) equal variances for the estimated weights

(ii) equal correlations for the estimated weights.

The second condition is the same as that of Banerjee [6]. Raghavarao [50], [51] found best weighing designs under the two conditions mentioned above for some $n \equiv 1 \mod 4$ and $n \equiv 2 \mod 4$ with the three efficiency definitions. No best designs when $n \equiv 3 \mod 4$ were found under the above class of designs in the previous literature. In section 2.8, best weighing designs for some $n \equiv 3 \mod 4$ and also for some $n \equiv 1 \mod 4$, when P_n does not exist are obtained under the conditions (i) and (ii).

In section 2.10, we show that, when $n \equiv 2 \mod 4$, U_n and T_n are the best designs subject to the condition, viz. the variances of the estimated weights are equal, with Kishen's, Mood's and Kishen's , Ehrenfeld's definitions of efficiency respectively. The best weighing designs (except P_n) under the conditions (i) and (ii) mentioned above may be improved in some cases by relaxing the second condition, viz. the estimated weights are equally correlated. Designs subject to the condition (i) may be obtained with the help of symmetrical partially balanced incomplete block designs ("SPBIBD"). Two designs for n = 9 and 21 are obtained with the help of "SPBIB" designs. These designs are more efficient than R_9 and R_{21} respectively.

The problem in section 2.11, first mentioned by Mood, is to find rough estimates of the weights of some objects and accurate estimates of others. Designs for some odd n are obtained in the cases (a) n-1 objects are weighed accurately with equal precision (b) n-2 objects are weighed accurately with equal precision.

"PBIB" designs are very useful in design of experiments. They were first introduced by Bose and Nair [20] and later generalised by Nair and Rao [45]. The analysis, combinatorial properties and constructions of these designs have been extensively studied by many authors. In order to equip an experimenter with a wide class of designs, there is a necessity to develop the scope of partially balanced designs. Chapters 3,4,5 deal with some partially balanced designs. Group divisible designs were introduced by Bose and Connor [17] and were extended to m-associate classes by Roy [56]. Further development was done by Raghavarao [53]. In chapter 3 we define a new association scheme for a "PBIB" design which belongs to group divisible family and we name it as "GPBIB" design. "BIB" design, group divisible design, group divisible m-associate design come under this family as particular cases. Some general constructions for obtaining "GPBIB" designs are given in section 3.5. Bose and Shimamoto [21] classified two associate "PBIB" designs as group divisible, 'L_i' ($i = 2,3$), triangular('T'), cyclic ('C') and simple ('S_1') type designs. In section 3.6, "GPBIB", 3-associate designs are classified as 'GD' 3-associate, 'GL_i' ($i = 1,2$), 'GT', 'GC' and 'GS_1' type designs. Raghavarao have studied 'GD' 3-associate designs in detail [53]. In the present work, some combinatorial properties and non-existence of 'GL_2' and 'GT' designs have been established.

In chapter 3, we restrict our attention to Shah's [58] intra-inter group partially balanced designs having two groups of treatments with replicate numbers r_1 and r_2 respectively. A detailed study of these designs is given in this chapter. These designs are to achieve partial balance with in the groups and balance (i.e. treatment differences are estimated with the same variance) between the groups and they are named as partially balanced block designs with two different number of replicates. Our work also gives some combinatorial properties and some methods of construction.

In incomplete block designs like "BIB", "PBIB" including Lattice designs the block sizes are constant. In agronomic experiments it is some times not agriculturally feasible to lay out blocks of equal sizes. Therefore, Kishen [41] introduced the Symmetrically Unequal Block ("SUB") arrangements which share the property of complete balance (in the sense of constant λ , i.e. any two treatments occur together λ times in the blocks), but involve blocks of different sizes. The analysis of these designs is obtained on the assumption of equal intra block error variances for blocks of different sizes. This assumption may be reasonable when the block sizes do not vary widely. In case the block sizes differ in "SUB" arrangements widely, we attempt to meet the above assumption by using a different incomplete block designs, having unequal block sizes (where the block sizes do not vary much), which are more general than "SUB" arrangements. These are called, in chapter 5, partially balanced designs - an extension of the definition of "PBIB" designs. Using the association matrices [19], we arrive at a result which gives the necessary and sufficient condition of a connected design to be partially balanced. Some constructions of these designs are given with the help of existing incomplete block designs. Lastly, in section 5.4, we give some constructions of binary equi-replicate partially balanced designs, pairwise balanced (which are also partially balanced) designs and binary equi-replicate balanced designs - all having two unequal block sizes-by using 2-associate "PBIB" designs.

Designs for two-way elimination of heterogeneity were fully studied by many authors, where the column - row incidence matrix is complete .

Pottoff [49] gave the analysis of the designs for two-way elimination of heterogeneity in general and classified some designs where row-column incidence matrix is incomplete. In chapter 6, using the concepts of orthogonality and balancing designs for two-way elimination of heterogeneity are classified into 64 classes. (Here row-column, treatment-row, treatment-column incidence matrices may not be necessarily complete). We restrict our attention to the designs where the three incidence matrices are binary and they belong to 32 classes where the property of balancing is attributed to treatments. Our main aim in this chapter is to get more designs under different classes where the row-column incidence matrices are incomplete. It is proved in this work, that the classes 17, 18, 19, 21, 22, 25 and 27 are impossible. The designs obtained in the classes 5, 8, 23, 31, are all new. The designs of Pottoff [49] where the row-column incidence matrices are incomplete, come under the class 29. For the class 29, we arrive at a result that the design should have the same number of treatments, columns and rows. A measure of non-orthogonality for two-way design, similar to Shah's [57] measure of non-orthogonality in the case of incomplete block designs, is given in section 6.6. Finally some constructed designs (for $v \leq 23$, $r \leq 12$) are given as an appendix at the end of the chapter.

The results given in this thesis are believed to be new and will improve substantially the existing knowledge of the design and analysis of experiments.

The following are, in short, the new results discussed in the thesis :

1. (a) Construction of H_n , when $n = 2 [(2p^k + 1)^2 + 1]$ where $p^k \equiv 3 \bmod 4$.

(b) When n is odd, P_n matrix (cf. Raghavarao [50]), if it exists, is the best weighing design under the three efficiency criteria.

(c) For $n \equiv 2 \bmod 4$, T_n matrix (cf. Raghavarao [52]), if it exists, is the best weighing design with Ehrenfeld's definition of efficiency and U_n (cf. Ehlich [29]*), if it exists, is the best weighing design under Mood's definition of efficiency.

2. (a) Best weighing designs are obtained for some $n \equiv 3 \bmod 4$ and also for some $n \equiv 1 \bmod 4$, when P_n does not exist under the 3 efficiency definitions subject to the conditions

(i) the variances of the estimated weights are equal

(ii) the estimated weights are equally correlated.

(b) When $n \equiv 2 \bmod 4$, T_n, if it exists, is the best weighing design under Kishen's and Ehrenfeld's definitions of efficiency; U_n, if it exists, is the best weighing design under Kishen's and Mood's definitions of efficiency, subject to the condition, viz., the variances of the estimated weights are equal.

(c) When $n \equiv 1 \bmod 4$, if the corresponding H_{n-1} exists, best weighing designs can be obtained for the cases (i) n-1 objects are weighed accurately with equal precision, (ii) n-2 objects are weighed accurately with equal precision of Mood's [44] problem, viz., finding the designs which give equal variances for some estimated weights where these should be weighed accurately.

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* We call the matrices, constructed by Ehlich for $n \equiv 2 \bmod 4$, as U_n matrices.

* 3. Introduction of group divisible family of "PBIB" designs and their study.

4. A detailed study of partially balanced block designs with two different number of replications which are particular cases of Shah's intra - inter group partially balanced designs.

5. Partially balanced designs with unequal block sizes and their use.

** 6. Using the concepts of orthogonality and balancing, classification of the designs for two-way elimination of heterogeneity is made. Designs are obtained in the classes 5, 8, 23, 31 where the row-column incidence matrices are incomplete and binary.

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* Adhikari [1] had done a similar work. The author's contribution is an independent one.

** Similar work was done by Agrawal [2], [3], [4]. But the author has done this study independently.

CHAPTER 1

ON OPTIMUM WEIGHING DESIGNS

1.1 INTRODUCTION

Suppose we are given n objects, whose weights are to be found in n weighings with a chemical balance having no bias. Let

$$x_{ij} = 1 \text{ if the } j^{th} \text{ object is placed in the left pan in the } i^{th} \text{ weighing}$$
$$= -1 \text{ if the } j^{th} \text{ object is placed in the right pan in the } i^{th} \text{ weighing}$$
$$= 0 \text{ if the } j^{th} \text{ object is not weighed in the } i^{th} \text{ weighing.}$$

The n^{th} order matrix $X = ((x_{ij}))$ is known as the design matrix. Also, let y_i be the result recorded in the i^{th} weighing, ϵ_i is the error in the result, w_j the true weight of the j^{th} object, so that we have n equations

(1.1.1) $$x_{i1} w_1 + x_{i2} w_2 + \cdots + x_{in} w_n + \epsilon_i = y_i \qquad i=1,2,\ldots,n.$$

We assume X to be non-singular matrix. The method of least squares or the theory of linear estimation gives the estimated weights $\hat{\underline{w}}$ by the equation

(1.1.2) $$\hat{\underline{w}} = (X'X)^{-1} X' \underline{y}$$

where $\hat{\underline{w}}$ and $\underline{y}$ are the column vectors of the estimated weights and the observations respectively. If σ^2 is the variance of each weighing, then

(1.1.3) $$V(\hat{\underline{w}}) = (X'X)^{-1}\sigma^2 = S^{-1}\sigma^2 = C\sigma^2 = ((c_{ij}))\sigma^2$$

where $$S = X'X \text{ and } C = ((c_{ij})) = S^{-1}.$$

From Hotelling's results, we know that the minimum variance of the each estimated weight may have, is σ^2/n and in this case $X'X = nI_n$. We call such weighing design X to be optimum. Mood [44] has pointed out that

Hadamard matrices, H_n, when used as weighing designs, satisfy this optimality condition, viz. $H_n' H_n = nI_n$. It may be remarked here that a necessary condition for the existence of Hadamard matrices is $n \equiv 0 \bmod 4$ with the possible exception of $n = 2$. A complete summary of the status of the existence H_n is given by Bose and Shrikhande [22] and it is conjectured that, for every $n \equiv 0 \bmod 4$, H_n exists. Williamson [69] gave some methods of construction of Hadamard matrices. In one method, he gave the construction of H_n where $n = 2(p^k+1)$ and p is odd prime, k is positive integer. In section 1.2 of this chapter, we give a method of construction of H_n where $n = 2[(2p^k+1)^2+1]$, $p^k \equiv 3 \bmod 4$.

In the absence of above type optimum weighing designs, best weighing designs are determined from the efficiency definitions of Kishen [42], Mood [44] and Ehrenfeld [30].

Kishen treats the reciprocal of the increase in variance resulting from the adoption of any design other than the optimum design, as the efficiency of the design. This efficiency can be measured by

$$(1.1.4) \qquad 1 \Big/ \sum_{i=1}^{n} c_{ii}$$

This criterion in Kiefer's [39], [40] notation is termed as A-optimum.

Mood considers as best that weighing design which gives the smallest corresponding joint confidence region for the estimated weights. Here we use the term, "smallest confidence region" as defined by Neyman [46]. Hence a design will be called best in the sense of Mood, if the determinant of the matrix C is minimum and this is the case when det. S is maximum. Thus the efficiency of a weighing design X can be measured in the sense of Mood, by

(1.1.5) $$\det.S \,/\, \max \det.S$$

If λ_{min} is the minimum distinct characteristic root of S, then the efficiency of a weighing design X, can be measured, in the sense of Ehrenfeld, by

(1.1.6) $$\lambda_{min} \,/\, \max \lambda_{min}$$

The last two definitions of efficiency were first introduced by Wald [68] for statistical designs in general. In Kiefer's notations, these two are denoted as D-optimum, E-optimum respectively.

Best weighing designs, when n is odd in the sense of Kishen's definition of efficiency, are obtained in section 1.3. Sections 1.4 and 1.5 give the best weighing designs for $n \equiv 1 \bmod 4$, $n \equiv 2 \bmod 4$ under Ehrenfeld's and Mood's definitions of efficiency.

1.2 CONSTRUCTION OF SOME NEW HADAMARD MATRICES

DEFINITION 1.2.1 : An $n+1^{th}$ order matrix which contains ± 1 or 0 is called T_{n+1} matrix, if

(1.2.1) $$T'_{n+1} T_{n+1} = nI_{n+1}$$

(here, we consider T_{n+1} matrices whose diagonal elements are zeros).

Raghavarao [52] showed that these matrices are optimum weighing designs with $n+1 \equiv 2 \bmod 4$ in Kishen's and Ehrenfeld's definitions of efficiency subject to the conditions (i) the variances of the estimated weights are equal (ii) the estimated weights are equally correlated.

Some of them are obtained with the help of Williamson's [69] S_n matrices.

DEFINITION 1.2.2 : An n^{th} order square matrix having elements ± 1 in non-diagonal and zeros in diagonal is called S_n matrix if

$$S_n'S_n = S_nS_n' = nI_n - E_{nn} \tag{1.2.2}$$

We can easily show that, if S_n exists, then it is symmetrical one. Hence

$$T_{n+1} = \begin{bmatrix} 0 & E_{1n} \\ E_{n1} & S_n \end{bmatrix} \tag{1.2.3}$$

When $n = p^h$ where p is odd prime and h a positive integer such that $p^h \equiv 1 \bmod 4$, then there always exists S_n matrix. The problem is unsolved when n is not a prime or a power of prime. Williamson used T_{n+1} for the construction of Hadamard matrices of order 2n+2.

$$H_{2n+2} = \begin{bmatrix} T_{n+1} + I_{n+1} & T_{n+1} - I_{n+1} \\ -T_{n+1} + I_{n+1} & T_{n+1} + I_{n+1} \end{bmatrix} \tag{1.2.4}$$

We give construction of H_{2n+2} with the help of S_n where n is a perfect square and need not be a power of prime, but $n = (2t+1)^2$ where $t = p^k \equiv 3 \bmod 4$, p is a prime and k is positive integer.

DEFINITION 1.2.3 : A skew symmetric matrix of order n having zeros as diagonal elements and ± 1 as non-diagonal elements is called Σ_n if

$$\Sigma_n'\Sigma_n = \Sigma_n\Sigma_n' = nI_n - E_{nn} \tag{1.2.5}$$

If $n = p^h$ where p is odd prime and h a positive integer, such that

$p^h \equiv 3 \bmod 4$, there always exists Σ_n matrix (cf. Williamson)

THEOREM 1.2.1 : If Σ_t exists, then Σ_{2t+1} also exists and it is given by

$$\Sigma_{2t+1} = \begin{bmatrix} \Sigma_t & \Sigma_t + I_t & -E_{t1} \\ \Sigma_t - I_t & -\Sigma_t & E_{t1} \\ E_{1t} & -E_{1t} & 0 \end{bmatrix}$$

The proof of this theorem is simple and hence it is omitted. Here 2t+1 need not be power of prime.

Example:

$$\Sigma_7 = \begin{bmatrix} 0 & + & + & - & + & - & - \\ - & 0 & + & + & - & + & - \\ - & - & 0 & + & + & - & + \\ + & - & - & 0 & + & + & - \\ - & + & - & - & 0 & + & + \\ + & - & + & - & - & 0 & + \\ + & + & - & + & - & - & 0 \end{bmatrix}$$

+ means +1

− means −1

$$\Sigma_{15} = \begin{bmatrix} 0 & + & + & - & + & - & - & + & + & + & - & + & - & - & - \\ - & 0 & + & + & - & + & - & - & + & + & + & - & + & - & - \\ - & - & 0 & + & + & - & + & - & - & + & + & + & - & + & - \\ + & - & - & 0 & + & + & - & + & - & - & + & + & + & - & - \\ - & + & - & - & 0 & + & + & - & + & - & - & + & + & + & - \\ + & - & + & - & - & 0 & + & + & - & + & - & - & + & + & - \\ + & + & - & + & - & - & 0 & + & + & - & + & - & - & + & - \\ - & + & + & - & + & - & - & 0 & - & - & + & - & + & + & + \\ - & - & + & + & - & + & - & + & 0 & - & - & + & - & + & + \\ - & - & - & + & + & - & + & + & + & 0 & - & - & + & - & + \\ + & - & - & - & + & + & - & - & + & + & 0 & - & - & + & + \\ - & + & - & - & - & + & + & + & - & + & + & 0 & - & - & + \\ + & - & + & - & - & - & + & - & + & - & + & + & 0 & - & + \\ + & + & - & + & - & - & - & - & - & + & - & + & + & 0 & + \\ + & + & + & + & + & + & + & - & - & - & - & - & - & - & 0 \end{bmatrix}$$

DEFINITION 1.2.4 : The Kronecker product A x B of matrices A and B is given by

$$(1.2.7) \qquad A \times B = \begin{bmatrix} a_{11}B & a_{12}B & \cdot\cdot & a_{1n}B \\ a_{21}B & a_{22}B & \cdot\cdot & a_{2n}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \cdot\cdot & a_{mn}B \end{bmatrix}$$

where $A = ((a_{ij}))$, $B = ((b_{kl}))$ are respectively $m \times n$, $p \times q$ matrices and $a_{ij}\ B$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$ is itself a $p \times q$ matrix. We shall use the symbol 'x' in a product of matrices to denote the Kronecker product.

We also know that

(a) The Kronecker product is associative and distributive with respect to addition of matrices, i.e.

$$(1.2.8) \qquad \begin{aligned} (A_1 \times A_2) \times C &= A_1 \times (A_2 \times C) \\ (A + B) \times C &= A \times C + B \times C \\ C \times (A + B) &= C \times A + C \times B \end{aligned}$$

where A and B are conformable for addition

(b) If A is $m \times k$ matrix, B is $k \times n$ matrix, C is $p \times l$ matrix and D is $l \times q$ matrix, then

$$(1.2.9) \qquad AB \times CD = A \times C \cdot B \times D$$

both sides being $mp \times nq$ matrices.

When S_n or Σ_n exists, using the concept of Kronecker product of matrices we can always construct S_{n^2}.

THEOREM 1.2.2 : If either S_n or Σ_n exists, then S_{n^2} always exists and it is given by

$$(1.2.10) \qquad S_{n^2} \begin{array}{l} = S_n \times S_n + E_{nn} \times I_n - I_n \times E_{nn} \quad \text{(or)} \\ = \Sigma_n \times \Sigma_n + E_{nn} \times I_n - I_n \times E_{nn} \end{array}$$

PROOF : Let X be the matrix equivalent to the right hand side of (1.2.10). We know that $S_n'S_n (\text{or } \Sigma_n' \Sigma_n) = nI_n - E_{nn}$ consider $X'X$. Using the definition 1.2.4 and the properties of Kronecker product of matrices, we can easily show that

$$(1.2.11) \qquad X'X = n^2 I_{n^2} - E_{n^2 n^2} \quad \text{and} \quad X E_{n^2 1} = 0_{n^2 1}$$

The diagonal elements of X are zeros and also it is symmetric with ± 1 as non-diagonal elements. Hence X is our required S_{n^2}

Hence, if Σ_t exists, by the theorem 1.2.1 there exists Σ_{2t+1}. And by the theorem 1.2.2, $S_{(2t+1)^2}$ exists. This ensures the existence of $T_{(2t+1)^2+1}$ and $H_2 [(2t+1)^2 + 1]$.

Let $t = 7, 19, 31$. Then we have the following designs which are new to the existing literature.

$$T_{226}, \; T_{1522}, \; T_{3970}$$
$$H_{452}, \; H_{3044}, \; H_{7940}$$

1.3 BEST WEIGHING DESIGNS WHEN n IS ODD WITH KISHEN'S EFFICIENCY DEFINITION

LEMMA 1.3.1 : Let $S = ((s_{ij}))$ be a p^{th} order positive definite matrix each of whose diagonal elements $= n \geq p$. A necessary condition that $\text{tr } S^{-1}$ (tr means trace) $\leq \dfrac{p(n+p-2)}{(n-1)(n+p-1)}$ is that all the characteristic roots of S should lie in the closed interval ($\frac{n(n-1)}{n+p-2}$, $n+p-1$) .

PROOF : Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the characteristic roots of S.

(1.3.1) $$\operatorname{tr} S^{-1} - \frac{p(n+p-2)}{(n-1)(n+p-1)} = \sum_{i=1}^{p} \frac{1}{\lambda_i} - \frac{p(n+p-2)}{(n-1)(n+p-1)}$$

$$\geq \frac{1}{\lambda_i} + \frac{(p-1)^2}{np-\lambda_i} - \frac{p(n+p-2)}{(n-1)(n+p-1)}$$

(1.3.2) $$= \frac{p(n+p-2)}{(n-1)(n+p-1)} \frac{\{\lambda_i-(n+p-1)\}\{\lambda_i - \frac{n(n-1)}{n+p-2}\}}{\lambda_i \quad (np-\lambda_i)}$$

$$\geq 0 \text{ for } \lambda_i \geq n+p-1 \text{ or } \lambda_i \leq \frac{n(n-1)}{n+p-2}$$

$\therefore$ a sufficient condition for $\sum_{i=1}^{p} \frac{1}{\lambda_i} \not\geq \frac{p(n+p-2)}{(n-1)(n+p-1)}$ is that

$\lambda_i \geq n+p-1$ or $\lambda_i \leq \frac{n(n-1)}{n+p-2}$ for some i

$\therefore$ a necessary condition for $\sum_{i=1}^{p} \frac{1}{\lambda_i} \leq \frac{p(n+p-2)}{(n-1)(n+p-1)}$ is that

$\frac{n(n-1)}{n+p-2} \leq \lambda_i \leq n+p-1$ for all i

LEMMA 1.3.2 : A necessary condition for $\operatorname{tr} S^{-1} \leq \frac{p(n+p-2)}{(n-1)(n+p-1)}$ is that $\lambda_{max}(S) \leq n+p-1$. (i.e. the maximum characteristic root of S $\leq$ $n+p-1$).

Proof of this lemma follows from the lemma 1.3.1.

THEOREM 1.3.1 : In the lemma 1.3.1, when S is symmetric and $s_{ij} \equiv 1 \bmod 2$ $(i,j = 1,2,\ldots, p)$, tr $S^{-1} \geq \dfrac{p(n+p-2)}{(n-1)(n+p-1)}$.

PROOF : Suppose $\lambda_{max}(S) \leq n+p-1$. Let

(1.3.3) $$S = \begin{pmatrix} nI_1 & \underline{s}_1' \\ \underline{s}_1 & S_1 \end{pmatrix} \quad \text{and} \quad C = ((c_{ij})) = S^{-1}$$

Let $\underline{\xi}$ be the normalised characteristic vector of S_1 corresponding to $\lambda_{max}(S_1)$

(1.3.4) $$\lambda_{max}(S_1) = (o \ \underline{\xi}') \begin{pmatrix} nI_1 & \underline{s}_1' \\ \underline{s}_1 & S_1 \end{pmatrix} \begin{pmatrix} o \\ \underline{\xi} \end{pmatrix}$$

(1.3.5) $$\leq \lambda_{max}(S) \leq n+p-1$$

Let $\lambda_{max}(S_1) = n+p-1-x$, $x \geq o$ and $\lambda_1^*, \lambda_2^*, \ldots, \lambda_{p-1}^*$ be the characteristic roots of S_1 .

$$\text{Tr } C = \text{tr } (S_1 - \tfrac{1}{n}\underline{s}_1 \underline{s}_1')^{-1} + \frac{1}{n - \underline{s}_1' S_1^{-1} \underline{s}_1}$$

$$= \text{tr } \left(S_1^{-1} + \frac{S_1^{-1}\underline{s}_1\underline{s}_1' S_1^{-1}}{n - \underline{s}_1' S_1^{-1} \underline{s}_1}\right) + \frac{1}{n - \underline{s}_1' S_1^{-1} \underline{s}_1}$$

(1.3.6) $$= \text{tr } S_1^{-1} + \frac{1 + \underline{s}_1' S_1^{-2} \underline{s}_1}{n - \underline{s}_1' S_1^{-1} \underline{s}_1}$$

But, $$\text{tr } C - \frac{p(n+p-2)}{(n-1)(n+p-1)} = \text{tr } S_1^{-1} + \frac{1 + \underline{s}_1' S_1^{-2} \underline{s}_1}{n - \underline{s}_1' S_1^{-1} \underline{s}_1} - \frac{p(n+p-2)}{(n-1)(n+p-1)}$$

$$\geq \frac{1}{\lambda_1^*} + \frac{(p-2)^2}{n(p-1)-\lambda_1^*} + \frac{1+\underline{s}_1' S_1^{-2} \underline{s}_1}{n-\underline{s}_1' S_1^{-1}\underline{s}_1} - \frac{p(n+p-2)}{(n-1)(n+p-1)}$$

$$\geq \frac{1}{n+p-1-x} + \frac{(p-2)^2}{n(p-1)-(n+p-1-x)} + \frac{(n+p-1-x)^2+p-1}{(n+p-1-x)\{n(n+p-1-x)-(p-1)\}}$$

$$- \frac{p(n+p-2)}{(n-1)(n+p-1)}$$

$$\geq \quad 0 \quad \text{for} \quad n \geq 3,\ p \geq 2 \text{ and } 0 \leq x \leq 1$$

Equality sign, it takes when $x = 1$. Hence by lemma 1.3.2, a necessary condition for $\text{tr } C \leq \frac{p(n+p-2)}{(n-1)(n+p-1)}$ is that $\lambda_{max}(S_1) \leq n+p-2$ and

(1.3.7) $$\text{tr } C = \sum_{i=1}^{p} c_{ii} \geq p \min_i c_{ii}$$

(1.3.8) $$c_{11} = \frac{1}{n-\underline{s}_1' S_1^{-1}\underline{s}_1} \geq \frac{1}{n - \frac{\underline{s}_1'\underline{s}_1}{\lambda_{max}(S_1)}}$$

(1.3.9) $$\geq \frac{1}{n-\frac{p-1}{n+p-2}} = \frac{n+p-2}{(n-1)(n+p-1)}$$

$$\therefore \quad \text{tr } C \geq \frac{p(n+p-2)}{(n-1)(n+p-1)}.$$ Hence the theorem.

COROLLARY 1.3.1.1 : Let X be a $n \times p$ order matrix, each of whose elements is ± 1. Let $S = X'X$ which is non-singular. Then

$$\text{tr } S^{-1} \geq \frac{p(n+p-2)}{(n-1)(n+p-1)}, \quad \text{when } n \text{ is odd}$$

The proof of this corollary follows from the theorem 1.3.1 and a lemma from Ehilich [29] viz. when n is odd, all the elements of $S (= X'X)$ are odd.

COROLLARY 1.3.1.2 : Let X be a n^{th} order matrix whose elements are ± 1 or o and $X'X (= S)$ be a non-singular matrix. Then $\operatorname{tr} S^{-1} \geq \frac{2n}{2n-1}$, when n is odd.

For proving this corollary, the following lemma is useful.

LEMMA 1.3.3 : Let $S = \begin{pmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{pmatrix}$ be a symmetric positive definite matrix where the orders of S, S_{11}, S_{22} be $n \times n$, $n-p \times n-p$, $p \times p$ respectively. Let λ_1, λ_2 be the maximum characteristic roots of S_{11}, S_{22} respectively. Then

$$\operatorname{tr} S^{-1} \geq \operatorname{tr} S_{11}^{-1} + \operatorname{tr} S_{22}^{-1} + \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} \operatorname{tr} \frac{S_{12} S'_{12}}{\lambda_1 \lambda_2} \left(I_{n-p} - \frac{S_{12} S'_{12}}{\lambda_1 \lambda_2} \right)^{-1} \tag{1.3.10}$$

PROOF : $\operatorname{tr} S^{-1} = \operatorname{tr} \begin{pmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{pmatrix}^{-1}$

$$= \operatorname{tr} (S_{11} - S_{12} S_{22}^{-1} S'_{12})^{-1} + \operatorname{tr} (S_{22} - S'_{12} S_{11}^{-1} S_{12})^{-1} \tag{1.3.11}$$

$$= \operatorname{tr} S_{11}^{-1} + \operatorname{tr} S_{11}^{-2} S_{12} (S_{22} - S'_{12} S_{11}^{-1} S_{12})^{-1} S'_{12} + \operatorname{tr} (S_{22} - S'_{12} S_{11}^{-1} S_{12})^{-1}$$

$$\geq \operatorname{tr} S_{11}^{-1} + \frac{1}{\lambda_1^2} \operatorname{tr} S'_{12} S_{12} (S_{22} - S'_{12} S_{11}^{-1} S_{12})^{-1} + \operatorname{tr} (S_{22} - S'_{12} S_{11}^{-1} S_{12})^{-1} \quad (\text{cf. } [5])$$

$$\geq \operatorname{tr} S_{11}^{-1} + \frac{1}{\lambda_1} \operatorname{tr} \frac{S'_{12} S_{12}}{\lambda_1 \lambda_2} + \frac{1}{\lambda_1^2 \lambda_2^2} \operatorname{tr} (S_{12} S'_{12})^2 (S_{11} - S_{12} S_{22}^{-1} S'_{12})^{-1}$$

$$+ \operatorname{tr} (S_{22} - S'_{12} S_{11}^{-1} S_{12})^{-1}$$

On expanding all the terms in the above manner, we get

$$\text{tr } S^{-1} \geqslant \text{tr } S_{11}^{-1} + \text{tr } S_{22}^{-1} + \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \text{tr } \frac{S_{12}S_{12}'}{\lambda_1 \lambda_2} + \text{tr } \left(\frac{S_{12}S_{12}'}{\lambda_1 \lambda_2}\right)^2 + \ldots\ldots$$

$$= \text{tr } S_{11}^{-1} + \text{tr } S_{22}^{-1} + \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} \text{ tr } \frac{S_{12}S_{12}'}{\lambda_1 \lambda_2} \left(I_{n-p} - \frac{S_{12}S_{12}'}{\lambda_1 \lambda_2} \right)^{-1}$$

It can be easily seen that $\text{tr } \frac{S_{12}S_{12}'}{\lambda_1 \lambda_2} \left(I_{n-p} - \frac{S_{12}S_{12}'}{\lambda_1 \lambda_2} \right)^{-1} \geqslant 0$

Hence

(1.3.12) $$\text{tr } S^{-1} \geqslant \text{tr } S_{11}^{-1} + \text{tr } S_{22}^{-1}$$

PROOF OF THE COROLLARY 1.3.1.2 : Let $X = n \left[\underset{n-p}{X_1} \vdots \underset{p}{X_2} \right]$ where the order of X_1 be $n \times n-p$. Let the each column of X_1 contain atleast one zero and the elements of X_2 be ± 1. Let

(1.3.13) $$S = X'X = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix} \begin{matrix} n-p \\ p \end{matrix}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n-p \qquad p$$

Evidently we know that $\text{tr } S_{11}^{-1} \geqslant \frac{n-p}{n-1}$ and by the corollary 1.3.1.1 ,

$\text{tr } S_{22}^{-1} \geqslant \frac{p(n+p-2)}{(n-1)(n+p-1)}$. Hence by (1.3.12) we get

$$\text{tr } S^{-1} \geqslant \text{tr } S_{11}^{-1} + \text{tr } S_{22}^{-1}$$

$$\geqslant \frac{n-p}{n-1} + \frac{p(n+p-2)}{(n-1)(n+p-1)} = 1 + \frac{1}{n+p-1}$$

$$\geqslant 1 + \frac{1}{2n-1}$$

Hence the corollary.

The efficiency of a weighing design in Kishen's definition of efficiency is given by

$$(1.3.14) \qquad 1/\sum_{i=1}^{n} c_{ii} = \frac{1}{\text{tr } S^{-1}}$$

When n is odd, by the corollary 1.3.1.2 we have that $\text{tr } S^{-1} \geq \frac{2n}{2n-1}$

Hence the efficiency of a weighing design $\leq \frac{2n-1}{2n}$. When we use P_n matrix [50] as weighing design, we get the efficiency of the design to be $\frac{2n-1}{2n}$ which is maximum possible. Hence

THEOREM 1.3.2 : P_n matrix, if it exists, is the best weighing design when n is odd, in Kishen's definition of efficiency.

1.4 BEST WEIGHING DESIGNS WITH EHRENFELD'S DEFINITION OF EFFICIENCY

LEMMA 1.4.1 : Let $S = ((s_{ij}))$ be a n^{th} order positive definite matrix. Then the minimum characteristic root of S is less than or equal to $\frac{1}{2}\left(s_{ii} + s_{jj} - \sqrt{(s_{ii} - s_{jj})^2 + 4s_{ij}s_{ji}}\right)$ for all $i,j\ (i \neq j) = 1,2,\ldots, n$.

PROOF : Let $\begin{pmatrix} \xi_i \\ \xi_j \end{pmatrix}$ be the normalised characteristic vector of $\begin{pmatrix} s_{ii} & s_{ij} \\ s_{ji} & s_{jj} \end{pmatrix}$ corresponding to the minimum characteristic root of $\begin{pmatrix} s_{ii} & s_{ij} \\ s_{ji} & s_{jj} \end{pmatrix}$

Then

$$(1.4.1) \qquad \text{minimum characteristic root of } \begin{pmatrix} s_{ii} & s_{ij} \\ s_{ji} & s_{jj} \end{pmatrix}$$

$$= (0_{1\,i-1}\ \xi_i\ 0_{1\,j-i-1}\ \xi_j\ 0_{1\,n-j})\ S \begin{bmatrix} 0_{i-1\,1} \\ \xi_i \\ 0_{j-i-1\,1} \\ \xi_j \\ 0_{n-j\,1} \end{bmatrix}$$

$\geqslant$ minimum characteristic root of S

But minimum characteristic root of $\begin{pmatrix} s_{ii} & s_{ij} \\ s_{ji} & s_{jj} \end{pmatrix}$

$$= \tfrac{1}{2} \left[s_{ii} + s_{jj} - \sqrt{(s_{ii} - s_{jj})^2 + 4s_{ij}s_{ji}} \right]$$

Hence the result.

Let X be a n^{th} order matrix (weighing design) with the elements ± 1 or o. Let $X'X (= S)$ be a non-singular matrix. Easily we can observe that S is symmetric and $s_{ii} \leqslant n$ for all i. The efficiency of the weighing design with Ehrenfeld's definition of efficiency is $\dfrac{\lambda_{min} \text{ of } S}{\max \lambda_{min} \text{ of } S}$; λ_{min} is the minimum characteristic root of S.

Case (i) : When n is odd.

(a) The weighing design does not contain zeros. Hence by lemma of Ehilich [29], viz., all s_{ij}'s are odd and by lemma 1.4.1, we have

$$(1.4.2) \qquad \lambda_{min}(S) \leq n - |s_{ij}| \leq n-1$$

(b) At least one column of the weighing design contains at least one zero. By the lemma 1.4.1, we get

$$(1.4.3) \qquad \lambda_{min}(S) \leq \tfrac{1}{2} \left[n-1 + n - \sqrt{(-1)^2 + 4s_{ij}^2} \right]$$

$$\leq \tfrac{1}{2} (2n-2) = n-1 .$$

Combining (a) and (b), we get $\lambda_{min}(S) \leq n-1$; which shows that the efficiency of the weighing design is maximum possible when $\lambda_{min}(S) = n-1$. We know that, if P_n [50] exists, it gives $\lambda_{min}(P_n' P_n) = n-1$. Hence

THEOREM 1.4.1 : In the sense of Ehrenfeld's definition of efficiency for odd n, P_n, if it exists, is the best weighing design.

Case (ii): When $n \equiv 2 \bmod 4$. ($n \neq 2$)

(a) the weighing design does not contain zeros in any column. The following lemmas of Ehilich [29] are useful.

LEMMA 1.4.2 : All the elements in S are even.

LEMMA 1.4.3 : There exists no 3 columns of X which possess pairwise inner product $\equiv 0 \bmod 4$.

By the lemmas 1.4.1, 1.4.2 and 1.4.3 we have

$$(1.4.4) \qquad \lambda_{min}(S) \leq n-2 \leq n-1 .$$

(b) at least one column of the design contains at least one zero. Suppose i^{th} column of the design contains at least one zero. Hence $s_{ii} \leq n-1$. j^{th} column does not contain zeros. If $s_{ii} = n-1$, then we can easily see that s_{ij} is odd. By the lemma 1.4.1, we have

$$(1.4.5) \qquad \lambda_{min}(S) \leq \tfrac{1}{2}(n-1 + n - \sqrt{1 + 4}\ ) \leq n-1$$

If $s_{ii} = n-2$, then we get

$$(1.4.6) \qquad \lambda_{min}(S) \leq \tfrac{1}{2}(n-2 + n) \leq n-1$$

On combining (a) and (b), we see that $\lambda_{min}(S) \leq n-1$, which shows that the efficiency of the design is maximum possible when $\lambda_{min}(S) = n-1$. We know that, if T_n [52] exists, it gives $\lambda_{min}(T_n'T_n) = n-1$. Hence

THEOREM 1.4.2 : When $n(\neq 2) \equiv 2 \bmod 4$, T_n is the best weighing design in the sense of Ehrenfeld's definition of efficiency, if it exists.

1.5 BEST WEIGHING DESIGNS WITH MOOD'S DEFINITION OF EFFICIENCY

Let X be n^{th} order matrix with the elements ± 1 or o. Let $X'X = S$ be a non-singular matrix. The following two results are due to Wojtas [70].

RESULT 1.5.1 : When n is odd, the maximum det. $S \leq (2n-1)(n-1)^{n-1}$

RESULT 1.5.2 : When $n(>2) \equiv 2 \bmod 4$, the maximum det. $S \leq 4(n-1)^2(n-2)^{n-2}$

The efficiency of a weighing design in Mood's definition is given by det. S / max. det. S .

Case (i) n is odd. The efficiency of the design is maximum possible if det S. $= (2n-1)(n-1)^{n-1}$. We know that, if P_n exists, it gives the maximum determinant. Hence

THEOREM 1.5.1 : When n is odd, P_n, if it exists, is the best weighing design in the sense of Mood's definition of efficiency.

Case (ii) $n(>2) \equiv 2 \bmod 4$. The efficiency of the design is maximum possible if det $S = 4(n-1)^2(n-2)^{n-2}$, Ehilich [29] constructed matrices when $n \equiv 2 \bmod 4$, n upto 38, which we call them U_n matrices; give maximum determinant. Hence

THEOREM 1.5.2 : When $n(>2) \equiv 2 \bmod 4$, U_n, if it exists, is the best weighing design in the sense of Mood's definition of efficiency.

(1.5.1) $$U_n = \begin{bmatrix} A_1 & A_2 \\ -A_2' & A_1' \end{bmatrix}$$ where A_1, A_2 are cyclic matrices

of order n/2 with the elements ± 1. This method of construction was given by Ehilich [29] and he constructed U_n matrices for n upto 38 using electronic computor. The following table gives the designs for n upto and including 38, adopted from [29]. Here, only the first row of A_1 and A_2 are given. In the table + means + 1 and - means - 1 .

TABLE 1.5.1

n	A_1	A_2
2	+	+
6	+ + +	+ + -
10	+ + + + -	+ + + + -
14	+ + + + + + -	- + + - + - -
18	+ + + + - + + + -	+ + + + + - + - -
26	+ + + + + - + + + - - + -	+ + + + + - + + + - - + -
30	- - - + + + - + + + - + + - +	- - + - + + + + + + - + + + +
38	- - - + + + - - + + - + + + + + + - +	- - + - + + - + - + + + - + + + + + +

The construction U_n is easy when $P_{n/2}$ exists.

(1.5.2) $$U_n = \begin{bmatrix} P_{n/2} & P_{n/2} \\ -P_{n/2} & P_{n/2} \end{bmatrix} .$$ Here $P_{n/2}$ need not be cyclic.

We know that P_{25} , P_{41} exist. Hence we have two more new U_n designs for $n = 50$ and 82 .

CHAPTER 2

WEIGHING DESIGNS UNDER RESTRICTED CONDITIONS

2.1 INTRODUCTION

This chapter is continuation to the first chapter on weighing designs. Here also we suppose that n objects are given to be weighed in n weighings with a chemical balance having no bias. In sections 2.8, 2.8_a, 2.8_b, 2.8_c of this chapter, we obtain best weighing designs subject to the conditions; (i) the variances of the estimated weights are equal (ii) the estimated weights are equally correlated; with the three efficiency criteria, when $n \equiv 3 \bmod 4$ and $n \equiv 1 \bmod 4$ where P_n does not exist for some favourable cases. The results, in sections 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7, are useful for obtaining the above said designs. Section 2.9 provides some designs subject to the conditions, viz. (i) the variances of the estimated weights are equal $(ii)_a$ the estimated weights are uncorrelated. Designs are obtained under the restriction, viz. (i) the variances of the estimated weights are equal, in section 2.10. Some of these designs are better (in the sense of efficiency) than the designs obtained with the conditions (i) and (ii). Finally, we get designs in section 2.11 for the problem in which the experimenter might be interested in finding rough estimates of the weights of some objects and accurate estimates of others. We restrict the designs to the cases (a) n-1 objects are weighed accurately (b) n-2 objects are weighed accurately; the variances of the estimated weights of these objects are assumed to be equal. The sections marked with asterisks are from author's published paper [14].

2.2* DESIGNS $[n, s, \lambda]$ AND THE FUNDAMENTAL NECESSARY CONDITION

We shall confine ourselves to the case when (i) the variances of the estimated weights are equal (ii) the estimated weights are equally correlated. In this case, we get design X with the parameters n, s, λ where n its size, s the number of zeros in any column and

$$\lambda = \sum_{i=1}^{n} x_{ij} x_{ij'} \qquad j, j' = 1, 2, \ldots, n \quad j \neq j' \tag{2.2.1}$$

Thus we get

$$X'X = (n-s-\lambda)\, I_n + \lambda E_{nn} \tag{2.2.2}$$

We denote this weighing design X by $[n, s, \lambda]$

$$\det X = \pm (\det.\ X'X)^{\frac{1}{2}} \tag{2.2.3}$$

$$= \pm (n-s+\overline{n-1}\,\lambda)^{\frac{1}{2}} (n-s-\lambda)^{n-1/2}$$

Case (i) n is even. Since (2.2.3) is a real integral value $(n-s+\overline{n-1}\,\lambda)(n-s-\lambda)$ must be a perfect square.

THEOREM 2.2.1 : A necessary condition for the existence of X, when n is even is that $(n-s+\overline{n-1}\,\lambda)$ is a perfect square.

Let

$$(n-s+\overline{n-1}\,\lambda)(n-s-\lambda) = f^2, \text{ then} \tag{2.2.4}$$

$$n = \frac{s'(\lambda+2) + [s'^2 \lambda^2 + 4(1+\lambda) f^2]^{\frac{1}{2}}}{2(1+\lambda)} \quad \text{where } s' = s+ \tag{2.2.5}$$

Case (ii) n is odd. Since (2.2.3) is real integral value, $n-s+\overline{n-1}\,\lambda$ is a perfect square.

THEOREM 2.2.2 : A necessary condition for the existence of X when n is odd is that $n-s+\overline{n-1}\lambda$ is a perfect square.

Let

(2.2.6) $$n-s+\overline{n-1}\lambda = d^2$$ when d is some integer.

Since we are considering odd n, it takes values either 1 mod 4 or 3 mod 4. Let

(2.2.7) $$n = 4t+1 \quad \text{if } n \equiv 1 \bmod 4$$
$$= 4t+3 \quad \text{if } n \equiv 3 \bmod 4$$

where t is non-negative integer. Let s take one of the values $4t'$, $4t'+1$, $4t'+2$, $4t'+3$ where $t'(<t)$ is also some non-negative integer.

Case (ii_a) $n = 4t+1$. For $n-s+\overline{n-1}\lambda$ to be a perfect square, s should be o, 1 mod 4. Hence

REMARK $2.2.2_a$: The weighing designs $[4t+1, 4t'+2, \lambda]$, $[4t+1, 4t'+3, \lambda]$ do not exist.

Case (ii_b) $n = 4t+3$. When $s-2\lambda \equiv$ o, 1 mod 4, $n-s+\overline{n-1}\lambda$ is not a perfect square. Hence

REMARK $2.2.2_b$: The weighing designs $[4t+3, 4t', \lambda]$, $[4t+3, 4t'+1, \lambda]$ when λ is even, and the designs $[4t+3, 4t'+2, \lambda]$, $[4t+3, 4t'+3, \lambda]$ when λ is odd do not exist.

2.3 THE LEGENDRE SYMBOL, THE HILBERT NORM RESIDUE SYMBOL AND THE HASSE-MINKOWSKI INVARIANT

The Hasse-Minkowski invariant was first used by Shrikhande [59] in the design of experiments to show the impossibility of symmetrical balanced incomple block designs when v is odd. It is well recognised by now that the Hasse-Minkowski invariant is an important tool to show the impossibility of various designs. Since we will be using this invariant in the chapters 2 and 3, we give, in this section, a brief resume of the important properties of the Legendre symbol, the Hilbert norm residue symbol and the Hasse-Minkowski invariant.

The Legendre symbol is defined as

$$(2.3.1) \qquad (a/p) = \begin{cases} +1 & \text{if } a \text{ is quadratic residue of } p \\ -1 & \text{if } a \text{ is non-quadratic residue of } p. \end{cases}$$

A slight generalisation of the Legendre symbol is the Hilbert norm residue symbol $(a,b)_p$. If a and b are any non-zero rational numbers, we define $(a,b)_p$ to have the value + 1 or -1 according as the congruence

$$(2.3.2) \qquad ax^2 + by^2 \equiv 1 \bmod(p^r)$$

has or has not for every value of r, rational solutions x_r, y_r. Here p is any prime including the conventional prime $p_\infty = \infty$.

Many properties of $(a,b)_p$ are given by Jones [37], Pall [48]. We mention the following the known properties of $(a,b)_p$ taken from the above references for ready reference .

(a) If m and m' are integers not divisible by odd prime

$$(1.3.3) \qquad (m,m')_p = +1$$

$$(2.3.4) \qquad (m,p)_p = (m/p)$$

Moreover, if $m \equiv m' \not\equiv o \bmod p$

$$(2.3.5) \qquad (m,p)_p = (m', p)_p$$

(b) For arbitrary non-zero integers m, m', n, n' and s; and for every prime p

$$(2.3.6) \qquad (-m, m)_p = +1$$

$$(2.3.7) \qquad (mm', n)_p = (m, n)_p \, (m', n)_p$$

(2.3.8) $$(mm', m-m')_p = (m, -m')_p$$

(2.3.9) $$(m, n)_p = (n, m)_p$$

(2.3.10) $$\prod_{j=1}^{m} (j, j+1)_p = ((m+1)!, -1)_p$$

and

(2.3.11) $$(as^2, b)_p = (a, b)_p .$$

Now, let A, B be two rational, symmetric and non-singular matrices of the same order n such that $A = CBC'$ where C is a rational non-singular matrix. Then A and B are said to be rationally congruent. The rational congruence relation between A and B is denoted, symbolically, by $A \sim B$. Let d_i ($i = 1, 2, \ldots, n$) be the leading principal minor determinant of order i and suppose $d_i \neq 0$ for all i . Define $d_0 = 1$. Then the Hasse-Minkowski invariant of A is given by

(2.3.12) $$c_p(A) = (-1, -1)_p \prod_{i=0}^{n-1} (d_{i+1}, -d_i)_p \quad \text{for each prime } p .$$

The following two lemmas regarding the Hasse-Minkowski invariant will be useful.

LEMMA 2.3.1 : For a n x n diagonal matrix Δ_n with each diagonal element d,

(2.3.13) $$c_p(\Delta_n) = (-1, -1)_p (-1, d)_p^{n(n+1)/2}$$

LEMMA 2.3.2 : If $A = eI_n + fE_{nn}$ where e and f are non-zero rationals, then

(2.3.14) $$c_p(A) = (-1,-1)_p (-1, e)_p^{n(n-1)/2} (-1, g)_p (n, g)_p (n, e)_p (g, e)_p^{n-1}$$

where $g = e + nf$.

2.4[a] ON THE IMPOSSIBILITIES OF $[n, s, \lambda]$

Case (i) $\lambda = o$. Since we are considering non-singular weighing designs its inverse exists and it is also a matrix with rational elements. Thus $I_n = (X^{-1})'(X'X)(X^{-1})$. We have that I_n and $X'X$ are rationally congruent and they can be written $X'X \sim I_n$. Hence

(2.4.1) $$c_p(X'X) = c_p(I_n) = (-1,-1)_p$$

But $X'X = (n-s)I_n$. From the lemma 2.3.1, we see that

(2.4.2) $$c_p(X'X) = (-1, -1)_p \, (-1, n-s)_p^{n(n+1)/2}$$

On equating the right hand sides of (2.4.1) and (2.4.2), we get for all primes that

(2.4.3) $$(-1, n-s)_p^{n(n+1)/2} = +1$$

(2.4.3) always holds when $n \equiv o \bmod 4$ or $\equiv 1 \bmod 2$; and when $n \equiv 2 \bmod 4$, it becomes

(2.4.4) $$(-1, n-s) = +1$$

This result can be stated in the form of the following theorem.

THEOREM 2.4.1 : A necessary condition for the existence of $[n, s, o]$ where $n \equiv 2 \bmod 4$, is that $(-1, n-s)_p = +1$ for all primes p.

Examples for some non-existing designs :

n	6	14	18	34	58	78	94
s	3	2	4	4	2	2	2

Case (ii) $\lambda \neq o$. Here

(2.4.5) $$X'X = (n-s-\lambda)\, I_n + \lambda E_{nn} .$$

From the lemma 2.3.2, we see that

$$(2.4.6) \qquad c_p(X'X) = (-1,-1)_p \, (-1,\ n-s-\lambda)_p^{n(n-1)/2} \, (-1,g)_p (n,g)_p (n,n-s-\lambda)_p (n-s-\lambda,g)_p^{n-1}$$

where $g = n-s + \overline{n-1}\lambda$

Case(ii_a) : n is even; we have $g\,(n-s-\lambda) = f^2$ (cf. Theorem 2.2.1).

On equating right hand sides of (2.4.6) and (2.4.1), we get

$$(2.4.7) \qquad (-1,\ n-s-\lambda)_p^{n(n-1)/2} \, (-(n-s-\lambda),\ g)_p = +1$$

It follows from (2.4.7) that

$$(2.4.8) \qquad (-\overline{n-s-\lambda}, g)_p = +1 \qquad \text{if } n \equiv 0 \bmod 4$$

$$(2.4.9) \qquad (n-s-\lambda, g)_p = +1 \qquad \text{if } n \equiv 2 \bmod 4$$

Case (ii_b) n is odd; we have $g = d^2$ (this is considered after considering the theorem 2.2.2). On equating the right hand sides of (2.4.6) and (2.4.1) we get

$$(2.4.10) \qquad (-1,\ n-s-\lambda)_p^{n(n-1)/2} \, (n,\ n-s-\lambda)_p = +1$$

It follows from (2.4.10) that

$$(2.4.11) \qquad (n,\ n-s-\lambda)_p = +1 \qquad \text{for } n \equiv 1 \bmod 4$$

$$(2.4.12) \qquad (-n,\ n-s-\lambda)_p = +1 \qquad \text{for } n \equiv 3 \bmod 4$$

The results (2.4.8), (2.4.9), (2.4.11), (2.4.12) can be stated in the form of following theorem.

THEOREM 2.4.2 : If the theorems 2.2.1 and 2.2.2 satisfy, then a necessary condition, for the existence of $[n, s, \lambda]$ with $\lambda \neq 0$, is that $(-\overline{n-s-\lambda},\ n-s + \overline{n-1}\lambda)_p = +1$ or $(n-s-\lambda,\ n-s + \overline{n-1}\lambda)_p = +1$ or $(n, n-s-\lambda)_p = +1$ or $(-n,\ n-s-\lambda)_p = +1$ according as $n \equiv 0 \bmod 4$ or $2 \bmod 4$ or $1 \bmod 4$ or $3 \bmod 4$ for all primes p.

Examples for some non-existing designs.

n	19	27	31	43
s	12	15	10	o
λ	1	2	2	3

2.5* STRUCTURE OF THE DESIGN $[n, s, \lambda]$ WITH $\lambda \neq o$

The distribution of the elements + 1, o, -1 in the matrix is particular interest. Let the first row of this matrix contain r positive units and t zeros. We bring them in the columns 1,2,...,r and in the last t columns respectively. Then we construct the matrix X_1 of n x n-1 in which the first row vanishes and it is given as

(2.5.1) $$X_1 = (\underline{x}_i - x_{1i}\,\underline{x}_1)_{i=2,3,\dots,n}$$

where $\underline{x}_i$ is the i^{th} column vector of X. After deleting the first row in X_1, there remains a matrix X_2 which gives that

(2.5.2) $$\det.\, X_2'X_2 = \det.\, X'X$$

and

(2.5.3) $$X_2'X_2 = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}' & X_{22} & X_{23} \\ X_{13}' & X_{23}' & X_{33} \end{bmatrix}$$

where

(2.5.4)
$$X_{11} = (n-s-\lambda)\left(I_{r-1} + E_{r-1\ r-1}\right), \quad X_{23} = 2\lambda\ E_{n-t-r\ t}$$
$$X_{12} = -(n-s-\lambda)\ E_{r-1\ n-t-r}, \quad X_{33} = (n-s-\lambda)I_t + \lambda E_{tt}$$
$$X_{22} = (n-s-\lambda)\ I_{n-t-r} + (n-s+3\lambda)\ E_{n-t-r\ n-t-r}, \quad X_{13} = o_{r-1\ t}$$

Hence from (2.5.2), (2.5.3), (2.5.4) we get that

$$(2.5.5) \qquad r = \frac{n-t}{2} \pm \left(\frac{d}{2\lambda}\right) \left\{ \lambda(s+\lambda-t) \right\}^{\frac{1}{2}}$$

where $d^2 = n-s+\overline{n-1}\,\lambda$. Since the design is non-singular, we know that can not be -1 for $s \geq 1$. Thus from (2.5.5) we see that, because r takes integral values, t is of the form $s-(i^2-1)\lambda \geq o$. Further i takes different values only when it takes the values including zero. This is due to $\sum_{j=1}^{n} t_j = ns$, where t_j is the number of zeros in the j^{th} row of X. When i does not take the value zero, then all t_j^s are equal and each is equivalent to s. Same is the case with $s < 3\lambda$. Hence

LEMMA 2.5.1 : A necessary condition for the existence of $[n,s,\lambda]$ having rows with different number of zeros, where $s > o$ and $\lambda \neq o$, is that $n-s-\lambda$ is even or $s \geq 3\lambda$.

Let $X_o = [n,s,o]$ which gives

$$(2.5.6) \qquad X_o' X_o = (n-s)I_n = X_o X_o' .$$

Hence

LEMMA 2.5.2 : The designs $[n,s,o]$ having rows with different number of zeros are non-existent.

Now consider the designs having every row s zeros and $\lambda \neq o$. Then (2.5.5) becomes

$$(2.5.7) \qquad r = (n-s \pm d)/2$$

and (2.5.7) shows that every row of X contains either $\frac{n-s+d}{2}$ or $\frac{n-s-d}{2}$ positive units. Let n_1 be the number of rows of X where each row contains $\frac{n-s+d}{2}$ positive units and let n_2 be the number of rows of X where each row contains $\frac{n-s-d}{2}$ positive units and $n_1 + n_2 = n$. Write

$$(2.5.8) \qquad X = \begin{bmatrix} X_{n_1 n} \\ X_{n_2 n} \end{bmatrix}$$

where $X_{n_i n}$ is $n_i \times n$ matrix $(i = 1,2)$ such that

$$(2.5.9)\qquad \begin{cases} X_{n_1 n}\; E_{n1} = d\, E_{n_1 1} \\ X_{n_2 n}\; E_{n1} = -d\, E_{n_2 1} \end{cases}$$

Let

$$(2.5.10)\qquad X^* = \begin{bmatrix} X_{n_1 n} \\ -X_{n_2 n} \end{bmatrix}$$

Consequently

$$(2.5.11)\qquad X'X = X^{*'}X^* \text{ and } X^* E_{n1} = d\, E_{n1}$$

Hence, we get

$$(2.5.12)\qquad X^{*'}X^* = X^* X^{*'} = (n-s-\lambda)I_n + \lambda E_{nn}$$

and also that every row and column of X^* has the same number of positive units. Since X with $\lambda \neq o$ implies X^*, we use X^* for $[n, s, \lambda]$ and X_o for $[n, s, o]$ here afterwards when X has every row with s zeros.

2.6* SOME NON-EXISTING DESIGNS $[n, s, \lambda]$

Let N be a matrix obtained from X^*(or X_o) by changing the negative units to zeros. Let M be a matrix obtained from X^*(or X_o) by changing positive units and negative units to zeros and zeros of X^*(or X_o) to positive units. Hence

$$(2.6.1)\qquad X^* \text{ (or } X_o) = 2N + M - E_{nn}$$

Let

$$(2.6.2)\qquad NN' = ((\lambda_{ij})),\; MM' = ((\mu_{ij})),\; MN' + NM' = ((\gamma_{ij}))$$

We can deduce from (2.5.12), (2.6.1) and (2.6.2) that

$$(2.6.3)\qquad 4((\lambda_{ij})) + 2((\gamma_{ij})) + ((\mu_{ij})) = (n-s-\lambda)I_n + (2s+\lambda-n)E_{nn} + 2((r_i + r_j))$$

where r_i is the number of positive units in the i^{th} row of X^*(or X_o). All r_i^{s} are equal in the case of X^*.

Let s = o, then $M = O_{nn}$. Hence (2.6.3) gives

LEMMA 2.6.1 : A necessary condition for the existence of X^* with $s = o$ or for the existence of [n, o, o] except for $n = 2$, is that $n \equiv \lambda \mod 4$.

Again on considering (2.6.3), when n is even and λ is odd or when n is odd and λ is even, we have that μ_{ij} is odd for $i,j\,(i \neq j) = 1,2,\ldots,n$. From the matrix MM' we get that

(2.6.4) $$ns^2 = ns + \sum_{i}\sum_{\substack{j \\ i \neq j}}^{n} \mu_{ij}$$

When all μ_{ij}s are odd, it follows from (2.6.4) that

(2.6.5) $$s(s-1)+1 \geq n$$

LEMMA 2.6.2 : A necessary condition for the existence of X^* (or X_o) when n is even and λ is odd or when n is odd and λ is even is that $n \leq s(s-1) + 1$.

(This lemma is the generalisation of Raghavarao's lemma 2.1 of [50]).

Let

(2.6.6) $$((\eta_{ij})) = (E_{nn} - M)\,(E_{nn} - M)' = MM' + (n-2s)E_{nn}$$

From (2.6.3) and (2.6.6) we have

(2.6.7) $$4((\lambda_{ij})) + 2((\gamma_{ij})) + ((\eta_{ij})) = (n-s-\lambda)I_n + \lambda E_{nn} + 2((r_i + r_j))$$

It follows from (2.6.7) that η_{ij}s are odd when λ is odd. Hence

LEMMA 2.6.3 : When λ is odd, a necessary condition for the existence of X^* or X_o is that

(2.6.8) $$n \leq (n-s)(n-s-1) + 1$$

Some examples of non-existing designs.

n	s	λ	reference	n	s	λ	reference
11	2	0	Lemmas 2.5.2, 2.6.2	21	17	3	Lemmas 2.5.1, 2.6.3
11	2	4	Lemmas 2.5.1, 2.6.2	22	18	1	Lemmas 2.5.1, 2.6.3
12	3	1	Lemmas 2.5.1, 2.6.2	23	3	2	Lemmas 2.5.1, 2.6.2
19	3	0	Lemmas 2.5.2, 2.6.2	27	2	0	Lemmas 2.5.2, 2.6.2
19	3	10	Lemmas 2.5.1, 2.6.2	29	4	2	Lemmas 2.5.1, 2.6.2

Now we can show easily that the existence of $[n, o, \lambda]$ with $\lambda \neq o$ implies the existence of symmetrical balanced incomplete block ("SBIB") design with the parameters $v^* = b^* = n$, $r^* = k^* = \frac{n + d}{2}$, $\lambda^* = \frac{n + 2d + \lambda}{4}$; and consequently if a "SBIB" design exists with the above parameters we get $[n, o, \lambda]$

2.7* THE NON-EXISTENCE OF THE DESIGNS $[n,1,1]$ and $[n,1,3]$.

By the lemma 2.5.1, we see that each row (and each column) of these designs contains one zero. By the lemma 2.6.2 these designs never exist when n is even. Hence, in this section n means odd n. Thus on transforming X to X^*, we get that every row (and every column) contains the same number of positive units. Let this number be r. Hence

$$(2.7.1) \qquad r = \frac{n-1+d}{2}$$

Consider the design $[n,1,1]$. Here $M = I_n$ and

$$(2.7.2) \qquad X^* = 2N + I_n - E_{nn} \qquad \text{also}$$

$$(2.7.3) \qquad X^*X^{*'} = 4NN' + 2(N + N') + (n - 4r - 2)\, E_{nn} + I_n$$

$$= (n-2)\, I_n + E_{nn}$$

Hence

$$(2.7.4) \qquad 2NN' + (N+N') = \frac{n-3}{2}\, I_n + \left(2r - \frac{n-3}{2}\right) E_{nn}$$

Let $N = ((n_{ij}))$ where $n_{ij} = 1$ or o for all $i,j = 1,2,\ldots, n$. (2.7.4) gives

(2.7.5) $$2\lambda_{ij} + n_{ij} + n_{ji} = 2r - \frac{n-3}{2} \quad i,j\ (i \neq j) = 1,2,\ldots,n.$$

Let $2r - \frac{n-3}{2}$ be odd. Hence X^* should be skew symmetric which is impossible due to the fact that every row and column of X^* contains $\frac{n-1+d}{2}$ positive units and $\frac{n-1-d}{2}$ negative units. Thus it follows that N is symmetric and $n \equiv 3 \bmod 4$. Let $n = 4t + 3$.

(2.7.6) $$NN' = rI_n + (r-t-1)\,N + (r-t)\,(E_{nn} - I_n - N)$$

Easily we can show, by using association matrices of Bose and Mesner [19], that N is symmetrical partially balanced incomplete block ("SPBIB") design with the following parameters

$$v = b = n, \quad r = k = \frac{n-1+d}{2}, \; n_1 = r, \; n_2 = n-r-1, \; \lambda_1 = r-t-1, \quad \lambda_2 = r-t$$

and

(2.7.7) $$P_1 = \begin{bmatrix} r-t-1 & t \\ & n-r-t-1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} r-t & t \\ & n-r-t-2 \end{bmatrix}$$

For existence of this design we must have Δ to be a perfect square and η be an integer where

$$\Delta = \gamma^2 + 2\beta + 1, \quad \gamma = p_{12}^2 - p_{12}^1, \quad \beta = p_{12}^2 + p_{12}^1 \quad \text{and}$$

(2.7.8) $$\eta = \frac{(v-1)(1-\gamma) - 2n_1}{2\,\Delta^{\frac{1}{2}}}$$

This result is due to Connor and Clatworthy [27]. For the parameters given in (2.7.7), $\Delta = 4t + 1$, $\eta = (n-1-2r)/2(4t+1)^{\frac{1}{2}}$. On substituting the value of r we get

(2.7.9) $$\eta = -\left\{\frac{2t+1}{4t+1}\right\}^{\frac{1}{2}}$$ which is not an integer except for $t = o$.

Hence it follows that "SPBIB" design with the parameters given in (2.7.7) does not exist; this shows that the design $[n,1,1]$ is impossible for $n > 3$. Similarly we can prove that the design $[n,1,3]$ is also non-existent for $n > 5$.

2.8* BEST WEIGHING DESIGNS WITH KISHEN'S DEFINITION OF EFFICIENCY WHEN $n \equiv 3$ MOD 4 AND $n \equiv 1$ MOD 4 WHERE P_n DOES NOT EXIST

Let the matrices $[n,o,-1]$, $[n,o,3]$, $[n,o,5]$ be denoted by Σ_n^*, Q_n R_n respectively. By the lemma 2.6.1 we have that $n \equiv 3 \bmod 4$ and $n \equiv 1 \bmod 4$ are necessary conditions for the existence of Σ_n^*, Q_n and R_n respectively.

When the matrix $X'X$ is of the form $(n-s-\lambda)\, I_n + \lambda E_{nn}$, the variance of each estimated weight, is

$$\frac{n-s+\overline{n-2}\,\lambda}{(n-s-\lambda)(n-s+\overline{n-1}\,\lambda)}\,\sigma^2 \tag{2.8.1}$$

Therefore, the efficiency can be measured, in Kishen's definition, by

$$\frac{(n-s-\lambda)\ (n-s+\overline{n-1}\,\lambda)}{n\ \ (n-s+\overline{n-2}\,\lambda)} = f(n-s,\ \lambda), \tag{2.8.2}$$

in Raghavarao's notation.

Case (i) $n \equiv 3 \bmod 4$. With the help of the sections 2.2, 2.4, 2.5, 2.6 and 2.7, we can show that the design sets $[n,o,o]$, $[n,o,1]$, $[n,o,2]$, $[n,1,o]$, $[n,2,o]$ and $[n,1,1]$ do not exist.

Now consider

$$f(n,3) - f(n-s,\lambda) = \frac{(n-3)(4n-3)}{n(4n-6)} - \frac{(n-s-\lambda)\ (n-s+\overline{n-1}\,\lambda)}{n(n-s+\overline{n-2}\,\lambda)} \tag{2.8.3}$$

$n-s$, λ are positive, $n > s+\lambda$ or $s = o$, $\lambda = -1$.

$$= \frac{(n-s')\{n(4s'-9)-6s'+9\} + \lambda\{(4s'-13)n^2+(4-s')6n-9\}}{n(4n-6)(n-s+\overline{n-2}\,\lambda)} \tag{2.8.4}$$

where $s' = s+\lambda$.

For Q_n to be efficient, (2.8.4) should be positive. Evidently this is positive when $n > 3$ and $\lambda = -1$. Also it is positive when $s' > 3$. When $s' = 3$, (2.8.4) becomes

$$\frac{(n-3)^2(3-\lambda)}{n(4n-6)(n-3+\overline{n-1}\,\lambda)} \tag{2.8.5}$$

which is non-negative since $\lambda \leq 3$. Hence (2.8.4) is positive for $s' \geq 3$. Also we know that $[n, s, \lambda]$ does not exist for $0 \leq s < 3$. Thus

THEOREM 2.8.1 : For $n > 3$ and $n \equiv 3 \bmod 4$, Q_n is the best weighing design in Kishen's definition of efficiency.

Case (ii) $n \equiv 1 \bmod 4$ where P_n does not exist. The matrix P_n is $[n, 0, 1]$. The sections 2.2, 2.4, 2.5, 2.6 and 2.7 enable us to show that the design sets $[n,0,\lambda]$ $\lambda = -1, 0, 2, 3, 4$; $[n,1,\lambda]$ $\lambda = 0, 1, 2, 3, 4$, $[n,2,\lambda]$, $[n,3,\lambda]$ and $[n, 4, 0]_{n > 13}$ do not exist. Also $[9, 4, 0]$ does not exist. For $n = 13$, P_n exists and it is the best weighing design.

Consider the difference

$$f(n,5) - f(n-s,\lambda) = \frac{(n-5)(6n-5)}{n(6n-10)} - \frac{(n-s-\lambda)(n-s+\overline{n-1}\,\lambda)}{n(n-s+\overline{n-2}\,\lambda)} \tag{2.8.6}$$

$$= \frac{(n-s')\left[(6n-10)s'-25(n-1)\right] + \lambda\left[n(6n-10)s'-(31n^2-60n+25)\right]}{n(6n-10)(n-s+\overline{n-2}\,\lambda)} \tag{2.8.7}$$

where $s' = s + \lambda$.

As in the case (i), (2.8.7) can be shown positive for $s' \geq 5$. Hence we have the following theorem

THEOREM 2.8.2 : For $n > 5$ and $n \equiv 1 \bmod 4$ also when P_n does not exist, R_n is the best weighing design in the sense of Kishen's definition of efficiency.

2.8_a^* BEST WEIGHING DESIGNS WITH EHRENFELD'S DEFINITION OF EFFICIENCY WHEN $n \equiv 3$ MOD 4 AND $n \equiv 1$ MOD 4, WHERE P_n DOES NOT EXIST.

We know that $(n-s-\lambda)$ and $(n-s+\overline{n-1}\,\lambda)$ are the distinct characteristic roots of $X'X$ with the multiplicities n-1 and 1 respectively, when $\lambda \neq o$. If $\lambda = o$, n-s is the only distinct characteristic root and it has multiplicity n. In either case, among the distinct roots $n-s-\lambda$ is always minimum except when $\lambda = -1$ and s=o, in which case the minimum root is one. Hence from (1.1.6) we measure the efficiency of the design $[n, s, \lambda]$ in Ehrenfeld's definition, by

$$(2.8.8) \qquad f_1(n-s, \lambda) = \begin{cases} \dfrac{n-s-\lambda}{\max. \lambda_{min}} & n-s > \lambda > o \\ \dfrac{1}{\max. \lambda_{min}} & s=o, \lambda = -1 \end{cases}$$

The following two theorems provide best weighing designs when $n \equiv 3 \bmod 4$ and $n \equiv 1 \bmod 4$ where P_n does not exist respectively. The proofs are omitted, because they are similar to the proofs of the theorems 2.8.1 and 2.8.2.

THEOREM 2.8.3 : In Ehrenfeld's definition of efficiency, Q_n is the best weighing design for $n > 3$ and $n \equiv 3 \bmod 4$.

THEOREM 2.8.4 : For $n > 5$ and $n \equiv 1 \bmod 4$, also when P_n does not exist, R_n is best design in Ehrenfeld's definition of efficiency.

2.8_b^* BEST WEIGHING DESIGNS IN THE SENSE OF MOOD'S DEFINITION OF EFFICIENCY WHEN $n \equiv 3$ MOD 4 AND $n = 1$ MOD 4, WHERE P_n DOES NOT EXIST

Consider the difference of the determinants $Q_n'Q_n$ and $X'X$ where X is any $[n, s, \lambda]$; $n \equiv 3 \bmod 4$.

$$(2.8.9) \qquad \det. Q_n' Q_n - \det. X'X = (4n-3)(n-3)^{n-1} - (n-s+\overline{n-1}\lambda)(n-s-\lambda)^{n-1}$$

$$(2.8.10) \qquad = n(n-3)^{n-1}\left\{ (4 - \frac{3}{n}) - (\lambda + 1 - \frac{s'}{n})(1 - \frac{s'-3}{n-3})^{n-1} \right\}$$

where $s' = s + \lambda$. For larger values of n the expression in the braces of

(2.8.10) tends to $4 - (\lambda+1)\, e^{-s'+3}$. Thus we have

$$(2.8.11) \qquad 4 - (\lambda+1)\, e^{-s'+3} \geq 4 - (s'+1)\, e^{-s'+3} \geq 0 \quad \text{for } s' \geq 3$$

It follows from (2.8.10) and (2.8.11) that the difference of the determinants is positive for $n > 3$, $s' \geq 3$ and $\lambda \geq 0$. We know that $[n, s, \lambda]$ does not exist for $s' < 3$ except for $s = 0$, $\lambda = -1$. From (2.8.10)

$$(2.8.12) \qquad \det. Q_n' Q_n - \det. \Sigma_n^{*'} \Sigma_n^{*} = (n-3)^{n-1} \left\{ 4n-3 - \left(1 + \frac{4}{n-3}\right)^{n-1} \right\}$$

and

$$(2.8.13) \qquad 4n-3-\left(1 + \frac{4}{n-3}\right)^{n-1} \text{ tends to } 4n-3-e^{4} \text{ for large values of } n.$$

Hence we have the difference (2.8.12) of the determinants is positive for $n > 14$ and Q_{15} does not exist. Thus

THEOREM 2.8.5 : For $n > 15$, Q_n is best weighing design in the sense of Mood's definition of efficiency and Σ_n^{*} is the best one for $n \leq 15$.

$n \equiv 1 \bmod 4$. Consider the difference of the determinants $R_n' R_n$ and $X'X$ where X is any $[n, s, \lambda]$.

$$(2.8.14) \qquad \det R_n' R_n - \det X'X = (6n-5)(n-5)^{n-1} - (n-s+\overline{n-1}\,\lambda)(n-s-\lambda)^{n-1}$$

$$(2.8.15) \qquad = n(n-5)^{n-1} \left\{ \left(6 - \frac{5}{n}\right) - \left(\lambda + 1 - \frac{s'}{n}\right)\left(1 - \frac{s'-5}{n-5}\right)^{n-1} \right\}$$

For large values of n the expression in the braces of (2.8.15) tends to $6-(\lambda+1)\, e^{-s'+5}$ which is greater than $6-(s'+1)\, e^{-s'+5}$ and

$$(2.8.16) \qquad 6-(s'+1)\, e^{-s'+5} \geq 0 \quad \text{for } s' \geq 5$$

It follows from (2.8.15) and (2.8.16) that for $n > 5$ and $s' \geq 5$ the difference of the determinants is positive. Also we know that $[n, s, \lambda]$ does not exist for $s' < 5$ except for $s = 0$, $\lambda = 1$. Hence

THEOREM 2.8.6 : For $n > 5$ and $n \equiv 1 \bmod 4$, also when P_n does not exist R_n is best weighing design in Mood's definition of efficiency.

2.8$_c^*$ CONSTRUCTION OF Q_n, Σ_n^* AND R_n MATRICES

We know from the section 2.6 that the existence of "SBIB" design with the parameters $v^* = b^* = n$, $r^* = k^* = \frac{n-d}{2}$, $\lambda^* = \frac{n-2d+\lambda}{4}$ where $\lambda \neq 0$, implies the existence of $[n, 0, \lambda]$. Let N denote the incidence matrix of "SBIB" design with the above parameters. Then the design X, $[n, 0, \lambda]$, is obtained by

(2.8.17) $$X = 2N - E_{nn}$$

Let $d_1^2 = 4n-3$ and $d_2^2 = 6n-5$ where d_1 and d_2 are integral values. Thus we have the following table

TABLE 2.8.1

v^*	"SBIBD" r^*	λ^*	Corresponding weighing design
n	$\frac{n-d_1}{2}$	$\frac{n-2d_1+3}{4}$	Q_n
n	$\frac{n-1}{2}$	$\frac{n-3}{4}$	Σ_n^*
n	$\frac{n-d_2}{2}$	$\frac{n-2d_2+5}{4}$	R_n

Now we give some numerical examples from the table 11.3 and pages 469-70 of Cochran and Cox [26].

TABLE 2.8.2*

v	"SBIBD" r	λ	Weighing design
3	2	1	Σ_3^*
7	3	1	Σ_7^*
7	6	5	Q_7
9	8	7	R_9
11	5	2	Σ_{11}^*
15	7	3	Σ_{15}^*
21	5	1	R_{21}
31	10	3	Q_{31}

* See these constructions at the end of this chapter.

2.9 WEIGHING DESIGNS $[n, s, o]$

In this section, we give some designs $[n,s,o]$. These may be used when the experimenter wants to weigh a large number of objects and when the pans do not allow more objects at a time, subject to the conditions (i) the variances of the estimated weights are equal (ii) the correlation of any two estimated weights is zero. For the construction of some of these designs, we may use the concept of Kronecker product of two matrices.

THEOREM 2.9.1 : The Kronecker product of $[n_1, s_1, o]$ and $[n_2, s_2, o]$ is $[n_1n_2, n_1s_2 + n_2s_1 - s_1s_2, o]$.

The proof of this theorem is simple and hence it is omitted.

Example : Let $X_1 = I_3 = [3,2,0]$, $X_2 = [4,0,0] = H_4$

$$X = I_3 \times H_4 = [12,8,0].$$

Some of the designs are given at the end of this chapter.

2.10 DESIGNS SUBJECT TO THE CONDITION, VIZ. THE VARIANCES OF THE ESTIMATED WEIGHTS ARE EQUAL

Case (i) $n \equiv 2 \bmod 4$. Consider the class of designs which give equal variances for the estimated weights. Let X be n^{th} order design matrix under this class. From the section 1.1 we have

(2.10.1) $$v(\hat{\underline{w}}) = (X'X)^{-1}\sigma^2 = S^{-1}\sigma^2 = C\,\sigma^2 = (\!(c_{ij})\!)\,\sigma^2$$

Here all c_{ii}s are equal.

THEOREM 2.10.1 : For the class of designs under section 2.10, $c_{ii} \geq \frac{1}{n-1} (i = 1,2,\ldots, n)$.

PROOF : Let

(2.10.2) $$S = \begin{bmatrix} s_{11}\ I_1 & \underline{s}_1' \\ \underline{s}_1 & S_1 \end{bmatrix}$$

Suppose X contains atleast one zero; say it is in first column of X. Then we have $s_{11} \leq n-1$. Also

(2.10.3) $$c_{11} = \frac{1}{s_{11} - \underline{s}_1' S_1^{-1} \underline{s}_1} \geq \frac{1}{s_{11}} \geq \frac{1}{n-1}$$

But by the definition of X, $c_{11} = c_{ii}$ $(i = 2,3,\ldots,n)$. Hence all $c_{ii} \geq \frac{1}{n-1}$.

Suppose X contains no zeros. First we prove the following lemma.

LEMMA 2.10.1 : Let X be a n x n matrix where $n \equiv 2 \bmod 4$ and its elements be $\pm$ 1. Let $S = (\!(s_{ij})\!) = X'X$, non-singular matrix. Then $\operatorname{tr} S^{-1} \geq \frac{n}{n-1}$.

We have the following three lemmas of Ehilich [29] are evident.

(See the lemmas 1.4.2 and 1.4.3)

LEMMA 2.10.1_a : The number of s_{ij}'s where each $s_{ij} \equiv o \bmod 4$ in $S \le \frac{n^2}{2}$. If equality sign exists, then X can be so arranged in the form as $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix}$ where the orders of S_{11} and S_{22} are equal; every s_{ij} in S_{11} and $S_{22} \equiv 2 \bmod 4$ and every s_{ij} in $S_{12} \equiv o \bmod 4$. (cf. [29]).

PROOF OF THE LEMMA 2.10.1 : By the lemma 2.10.1_a, we can always arrange any S as $\begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix}$ where the elements of S_{11} and $S_{22} \equiv 2 \bmod 4$ and their orders be $\frac{n}{2}$ and $\frac{n}{2}$ respectively. Hence

$$\operatorname{tr} S^{-1} = \operatorname{tr} \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix}^{-1}$$

$$\geqslant \operatorname{tr} S_{11}^{-1} + \operatorname{tr} S_{22}^{-1} \quad \text{[cf. 1st chapter, (1.3.12)]}$$

(2.10.4) $$= \tfrac{1}{2} (\operatorname{tr} S_{11}^{*-1} + \operatorname{tr} S_{22}^{*-1})$$

where the elements of S_{11}^* and S_{22}^* are odd; and by the theorem 1.3.1

$\operatorname{tr} S_{11}^{*-1} \geqslant \frac{n}{n-1}$ and $\operatorname{tr} S_{22}^{*-1} \geqslant \frac{n}{n-1}$. Hence

(2.10.5) $$\operatorname{tr} S^{-1} \geqslant \frac{n}{n-1}.$$

But for the class of designs under section 2.10, $\operatorname{tr} S^{-1} = n\, c_{ii}$ which gives $c_{ii} \geqslant \frac{1}{n-1}$ $(i = 1,2,\ldots, n)$. Hence the theorem 2.10.1.

We know for T_n matrix all c_{ii}'s are equal and each is equivalent to $\frac{1}{n-1}$. Also for U_n matrix (cf. 1st chapter),

$$(2.10.6) \quad U_n' U_n = \begin{bmatrix} (n-2)I_{n/2} + 2E_{n/2\ n/2} & 0_{n/2\ n/2} \\ 0_{n/2\ n/2} & (n-2)I_{n/2} + 2E_{n/2\ n/2} \end{bmatrix}$$

$$(2.10.7) \quad (U_n' U_n)^{-1} = \begin{bmatrix} \frac{1}{n-2}(I_{n/2} - \frac{1}{n-1} E_{n/2\ n/2}) & 0_{n/2\ n/2} \\ 0_{n/2\ n/2} & \frac{1}{n-2}(I_{n/2} - \frac{1}{n-1} E_{n/2\ n/2}) \end{bmatrix}$$

Evidently, it is seen from (2.10.7) that the diagonal terms of $(U_n' U_n)^{-1}$ are equal and each $= \frac{1}{n-1}$. From the sections 1.4 and 1.5, we see that T_n and U_n are the best designs in Ehrenfeld's and Mood's definitions of efficiency respectively. Thus

THEOREM 2.10.2 : Under the class of designs of section 2.10, T_n is the best design in Kishen's and Ehrenfeld's definitions of efficiency; U_n is the best design in Kishen's and Mood's definitions of efficiency.

Case (ii): n is odd. If P_n exists, we know that it is the best weighing design in the sense of the three efficiency criteria. When P_n does not exist, we have Q_n, Σ_n^* and R_n matrices. But they may not be best under the class of designs of section 2.10. For example, consider R_9 and R_{21}. The relative efficiency of the designs R_9 and R_{21} to the designs 14 and 15 (given at the end of the chapter) is less than one under all the three efficiency criteria. "SPBIB" designs may be used for the construction of some weighing designs under the class of section 2.10. These designs, for some n (as in the case of n = 9 and 21), may be more efficient than the designs R_n and Σ_n^* .

Let N' be a m-associate "SPBIB" design whose parameters are

(2.10.8) $v = b = n,\ r,\ n_1, n_2, \ldots, n_m ; \lambda_1, \lambda_2, \ldots \lambda_m$ and $((p^i_{jk}))$

Let $B_1, B_2, \ldots, B_m$ be the association matrices [19]. Let

(2.10.9) $$X = 2N - E_{nn}$$ which is non-singular and

(2.10.10) $$X'X = 4N'N + (n-4r)E_{nn}$$

$$= nB_o + \sum_{i=1}^{m} \lambda_i^* B_i \quad \text{where } B_o = I_n \text{ and}$$

$$\lambda_i^* = n - 4(r - \lambda_i) \quad i = 1, 2, \ldots, m$$

From (2.10.10) and by the properties of association matrices [58], we can show that X gives equal variances for the estimated weights. Designs 14,15; given at the end of the chapter, were constructed with the help of 2 - associate "SPBIB" designs.

When P_n does not exist, where $n = n_1 n_2 \equiv 1 \bmod 4$ and if P_{n_1} and P_{n_2} exist, then we can construct design X with the help of Kronecker product of matrices as

(2.10.11) $$X = P_{n_1} \times P_{n_2}$$

The matrix X gives equal variances for the estimated weights and each is given by

(2.10.12) $$\frac{4}{(2n_1 - 1)(2n_2 - 1)} \sigma^2$$

In Kishen's definition its efficiency is

(2.10.13) $$\frac{(2n_1 - 1)(2n_2 - 1)}{4 n_1 n_2}$$

TABLE 2.10.1

n	n_1	n_2	λ_{min}	Efficiency with Kishen's definition
65	5	13	48	.865
125	5	25	96	.882
169	13	13	144	.914
205	5	41	160	.889
325	13	25	288	.942

2.11 DESIGNS WHICH GIVE EQUAL VARIANCES FOR THE PART OF ESTIMATED WEIGHTS OF THE OBJECTS WHERE THESE SHOULD BE WEIGHED ACCURATELY

In this section we give some designs which give equal variances of some estimated weights of the objects where these should be weighed accurately. The problem was first mentioned by Mood in [44]. We restrict our attention to the cases that (i) n-1 objects are weighed accurately, (ii) n-2 objects are weighed accurately. We can show easily that $c_{ii} \geq \frac{1}{n}$ for $i = 1,2,\ldots, n$. Our aim is to obtain designs which give smaller c_{ii}'s (corresponding to the objects where these should be weighed accurately) as compared to the c_{ii}'s of the best designs of the sizes n-1 and n-2. For $n \equiv 0 \bmod 4$, the problem is solved if H_n exists. When $n \equiv 2 \bmod 4$, we may use T_n matrices if they exist. In this section we give designs for odd n under Kishen's definition of efficiency.

Case (i) $n \equiv 1 \bmod 4$ (a) n-1 objects are weighed with equal precision.

Let $S = \begin{bmatrix} S_1 & \underline{s}_1 \\ \underline{s}_1' & s_{nn} \end{bmatrix}$. Let $c_{11}, c_{22} \ldots\ldots c_{n-1\ n-1}$ corresponding to the required c_{ii}'s in which we are interested.

$$(2.11.1) \qquad \sum_{i=1}^{n-1} c_{ii} = (n-1)\, c_{ii} = tr\Big(S_1 - \frac{\underline{s}_1\, \underline{s}_1'}{s_{nn}}\Big)^{-1} \geq tr\, S_1^{-1}$$

$$\geq \frac{2n-3}{2(n-1)} \quad \text{(cf. theorem 1.3.1)}$$

$$(2.11.2) \therefore \qquad c_{ii} \geq \frac{2n-3}{2(n-1)^2} \qquad \text{for } i = 1,2,\ldots, n-1$$

Let H_{n-1} be the Hadamard matrix of order n-1, which is arranged, so that first row of it $= E_{1\ n-1}$. Let $H_{n-1}' = (E_{n-1\,1}\ \underline{h}_2 \ldots \underline{h}_{n-1})$ and

$$(2.11.3) \qquad X = \begin{bmatrix} E_{1\ n-1} & 1 \\ h_2' & 0 \\ h_{n-1}' & 0 \\ E_{1\ n-1} & -1 \end{bmatrix}$$

which gives

$$(2.11.4) \qquad X'X = \begin{bmatrix} (n-1)I_{n-1} + E_{n-1\ n-1} & 0_{n-1\ 1} \\ 0_{1\ n-1} & 2 \end{bmatrix}$$

and

$$c_{ii} = \frac{2n-3}{2(n-1)^2} \qquad i = 1,2,\ldots, n-1$$

$$c_{nn} = \tfrac{1}{2}\ , \quad \text{Hence}$$

THEOREM 2.11.1 : The design X defined in (2.11.3) is the best for the case $(i)_a$ under the class of designs in section 2.11, (b) n-2 objects are weighed accurately. Let

$$(2.11.5) \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix} \text{ where } \quad S_{22} = \begin{pmatrix} s_{n-1\ n-1} & s_{n-1\ n} \\ s_n & s_{nn} \end{pmatrix}$$

Let c_{ii} ($i = 1,2,\ldots, n-2$) correspond to the required c_{ii}^s in which we are interested.

$$(2.11.6) \qquad \sum_{i=1}^{n-2} c_{ii} = (n-2)\, c_{ii} = tr(S_{11} - S_{12}S_{22}^{-1}S_{12}')^{-1}$$

$$\geqslant tr\, S_{11}^{-1} \geqslant \frac{2(n-2)^2}{(n-1)(2n-3)} \qquad \text{(cf. theorem 1.3.1)}$$

$$(2.11.7) \qquad c_{ii} \geqslant \frac{2(n-2)}{(n-1)(2n-3)}$$

Consider Σ_{n-2}^* where $\Sigma_{n-2}^*\ E_{n-2\ 1} = -\,E_{n-2\ 1}$ and let

$$(2.11.8) \qquad X = \begin{pmatrix} \Sigma_{n-2}^* & E_{n-2\ 2} \\ E_{2\ n-2} & I_2 \end{pmatrix}$$

which gives

(2.11.9)
$$X'X = \begin{bmatrix} (n-1)I_{n-2} + E_{n-2\ n-2} & O_{n-2\ 2} \\ O_{2\ n-2} & I_2 + (n-2)E_{2\ 2} \end{bmatrix}$$

and

$$c_{ii} = \frac{2(n-2)}{(n-1)(2n-3)} \text{ for } i = 1,2,\ldots,n-2 \text{ and } c_{n-1\ n-1} = c_{nn} = \frac{n-1}{2n-3}$$

Hence

THEOREM 2.11.2 : The design X defined in (2.11.8) is the best one for the case $(i)_b$ under the class of designs of section 2.11.

The c_{nn} for X in (2.11.3) is $\frac{1}{2}$ and $c_{n-1\ n-1} = c_{nn}$ for X in (2.11.8) is $\sim \frac{1}{2}$. If the experimenter wants to improve them, he has to use different type of designs. For the case $(i)_a$ the following design is useful, if it exists.

Let

(2.11.10)
$$X = \begin{bmatrix} H_{n-1} & E_{n-1\ 1} \\ E_{1\ n-1} & -1 \end{bmatrix} \quad \text{where}$$

(2.11.11)
$$H'_{n-1} E_{n-1\ 1} = d\, E_{n-1\ 1} \quad \text{and } d \text{ is positive integer.}$$

Hence we get

(2.11.12)
$$X'X = \begin{pmatrix} (n-1)I_{n-1} + E_{n-1\ n-1} & (d-1)E_{n-1\ 1} \\ (d-1)E_{1\ n-1} & nI_1 \end{pmatrix}$$

For inverting (2.11.12), we use the following method of inversion for partitioned matrix.

GENERAL PARTITIONED MATRIX AND ITS INVERSE *1

Greenberg and Sarhan [34] utilize the special form of a partitioned matrix in order to invert it. In the general form of their matrix the diagonal sub-matrices are themselves diagonal matrices. We here consider

*1 See the reference [11].

diagonal sub-matrices of the form $aI_m + bE_{mm}$.

Consider a non-singular (h x h) partitioned matrices L and M of order (v x v) defined as follows.

$$L = ((L_{ij})) \text{ where}$$

(2.11.13) $$L_{ij} = b_{ij}E_{v_iv_j}, \quad L_{ii} = a_i I_{v_i} + b_{ii} E_{v_iv_i}, \quad b_{ij} = b_{ji} \text{ and } a_i \neq o \qquad i,j = 1,2,\ldots, h$$

$$M = ((M_{ij})) \text{ where}$$

(2.11.14) $$M_{ij} = y_{ij} E_{v_iv_j}, \quad M_{ii} = x_i I_{v_i} + y_{ii} E_{v_iv_j}, \quad y_{ij} = y_{ji} \text{ and } x_i \neq o \qquad i,j = 1,2,\ldots, h$$

(2.11.15) $$\sum_{i=1}^{h} v_i = v$$

Let

(2.11.16) $$B = ((b_{ij})), \quad Y = ((y_{ij})), \quad D_1 = \text{diag}(v_1, v_2, \ldots, v_h),$$

$$D_2 = \text{diag}(a_1, a_2, \ldots, a_h)$$

THEOREM 2.11.3 : If M is the inverse of L , then

(2.11.17) $$Y = -(BD_1 + D_2)^{-1} BD_2^{-1}$$

PROOF : We have $LM = I_v$

(2.11.18) $$\therefore \quad \sum_{k=1}^{h} L_{ik} M_{ki} = I_{v_i}, \quad \sum_{k=1}^{h} L_{ik} M_{kj} = 0_{v_iv_j}$$

The equations (2.11.8), on left hand side, show that the coefficients of $E_{v_iv_i}$ and $E_{v_iv_j}$ are zeros and the coefficients of I_{v_i} are one. Hence

(2.11.19)
$$x_i = \frac{1}{a_i}$$

$$\sum_{k=1}^{h} v_k b_{ik} y_{kj} + a_i y_{ij} = -\frac{b_{ij}}{a_j} \qquad i,j = 1,2,\ldots, h$$

Then (2.11.19) can be written as

$$(BD_1 + D_2)\, Y = - BD_2^{-1} \qquad \text{so that}$$

$$Y = - (BD_1 + D_2)^{-1}\, BD_2^{-1} \qquad \text{when B is non-singular (2.11.17)}$$

can also be written as

(2.11.20) $$Y = - (D_2D_1 + D_2B^{-1}D_2)^{-1}$$

Hence from (2.11.12) and (2.11.17) we have

(2.11.21)
$$c_{ii} = \frac{1}{n-1}\left[1 - \frac{n-(d-1)^2}{(n-1)(2n-d-1^2)}\right] \qquad i = 1,2,\ldots, n-1$$
$$c_{nn} = \frac{2}{2n-(d-1)^2}$$

For the construction of X in (2.11.10) we use the condition 2.11.11. Hence $n-1 = d^2 (= 4t^2 \text{ say})$. To satisfy this, "SBIB" design of family 'A' [63] is used. Let N be a "SBIB" design of family 'A' where v = n-1. Hence we get H_{n-1} as

(2.11.22) $$H_{n-1} = 2N - E_{n-1\ n-1}$$

The parameters of N are $v = b = n-1 = 4t^2$, $r = k = 2t^2 \pm t$, $\quad = t^2 \pm t$.
We know that "SBIB" design of family 'A' for n-1 = 16, 36, 64, 100 exist

Case $(i)_b$: We have that Σ^*_{n-2} always exists if H_{n-1} exists, where

(2.11.23) $$\Sigma^*_{n-2}\, E_{n-2\ 1} = \Sigma^{*\prime}_{n-2}\, E_{n-2\ 1} = E_{n-2\ 1} \quad \text{and}$$
$$\Sigma^{*\prime}_{n-2}\, \Sigma^*_{n-2} = (n-1)\, I_{n-2} - E_{n-2\ n-2}$$

Let

(2.11.24) $$X = \begin{bmatrix} \Sigma^*_{n-2} & E_{n-2\ 2} \\ E_{2n-2} & \begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix} \end{bmatrix}$$

which gives

$$(2.11.25) \qquad X'X = \begin{bmatrix} (n-1)I_{n-2} + E_{n-2\ n-2} & E_{n-2\ 2} \\ E_{2\ n-2} & 4I_2 + (n-4)E_{22} \end{bmatrix}$$

From (2.11.17) and (2.11.25) we get that

$$(2.11.26) \qquad c_{ii} = \frac{1}{n-1}\left(1 - \frac{n-3}{2(n-2)^2}\right) \qquad i=1,2,\ldots,n-2$$

$$c_{n-1\ n-1} = c_{nn} = \frac{1}{8}\left(\frac{n-1}{n-2}\right)^2$$

Case (ii) $n \equiv 3 \bmod 4$, (a) $n-1$ objects are weighed accurately with equal precision.

If we use T_{n-1} matrix for the objects which should be weighed accurately, c_{ii} of T_{n-1} is $\frac{1}{n-2}$. Our aim is to get designs where its c_{ii} $(i = 1,2,\ldots, n-1)$ are smaller than $\frac{1}{n-2}$. Let

$$(2.11.27) \qquad X = \begin{pmatrix} T_{n-1} & E_{n-1\ 1} \\ E_{1\ n-1} & -1 \end{pmatrix} \qquad \text{where}$$

$$(2.11.28) \qquad T'_{n-1} E_{n-1\ 1} = dE_{n-1\ 1} \quad \text{and } d \text{ is any positive integer}$$

$$(2.11.29) \qquad X'X = \begin{bmatrix} (n-2)I_{n-1} + E_{n-1\ n-1} & (d-1)E_{n-1\ 1} \\ (d-1)E_{1\ n-1} & nI_1 \end{bmatrix}$$

From (2.11.17) and (2.11.29) we get that

$$(2.11.30) \qquad c_{ii} = \frac{1}{n-2}\left[1 - \frac{\overline{n-d-1}^2}{(n-1)(2n-\overline{d-1}^2)-n}\right] \qquad i=1,2,\ldots, n-1$$

$$c_{nn} = \frac{2n-3}{n(2n-3)-(n-1)\overline{d-1}^2}$$

Two designs (Nos. 16, 17) defined in (2.11.27) for n = 11 and 27 are given at the end of the chapter.

(b) n-2 objects are weighed accurately with equal precision.

If S_{n-2} exists, we write

(2.11.31) $$X = \begin{bmatrix} S_{n-2} & E_{n-2\ 2} \\ E_{2\ n-2} & \begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix} \end{bmatrix}$$ where

$$S'_{n-2} S_{n-2} = (n-2)I_{n-2} - E_{n-2\ n-2} \quad \text{and} \quad S_{n-2} E_{n-2\ 1} = 0_{n-2\ 1}$$

and

(2.11.32) $$X'X = \begin{bmatrix} (n-2)I_{n-2} + E_{n-2\ n-2} & 0_{n-2\ 2} \\ 0_{2\ n-2} & 4I_2 + (n-4)E_{22} \end{bmatrix}$$

from (2.11.32) we see that

(2.11.33) $$c_{ii} = \frac{1}{n-2}\left[1 - \frac{1}{2(n-2)}\right] \qquad i=1,2,\ldots, n-2$$

$$c_{n-1\ n-1} = c_{nn} = \frac{n}{8(n-2)}$$

We know that S_{n-2} does not exist for n = 23, 35, 59, 71, 79, 95 when we consider n upto 100, and the existence of S_{n-2} is unknown for n = 47, 67, 87.

APPENDIX 2.1

In appendix 2.1, we give some weighing designs connected to the chapter 2. In the following designs + means +1 , - means -1.

1.

$$\Sigma_3^* = \begin{bmatrix} + & + & - \\ - & + & + \\ + & - & + \end{bmatrix}$$

2.

$$\Sigma_7^* = \begin{bmatrix} + & + & - & - & - & + & + \\ - & + & - & + & + & - & + \\ + & + & + & + & - & - & - \\ + & - & - & + & + & + & - \\ + & - & + & - & + & - & + \\ - & + & + & - & + & + & - \\ - & - & + & + & - & + & + \end{bmatrix}$$

3.

$$Q_7 = \begin{bmatrix} - & + & + & + & + & + & + \\ + & - & + & + & + & + & + \\ + & + & - & + & + & + & + \\ + & + & + & - & + & + & + \\ + & + & + & + & - & + & + \\ + & + & + & + & + & - & + \\ + & + & + & + & + & + & - \end{bmatrix}$$

4.

$$R_9 = \begin{bmatrix} - & + & + & + & + & + & + & + & + \\ + & - & + & + & + & + & + & + & + \\ + & + & - & + & + & + & + & + & + \\ + & + & + & - & + & + & + & + & + \\ + & + & + & + & - & + & + & + & + \\ + & + & + & + & + & - & + & + & + \\ + & + & + & + & + & + & - & + & + \\ + & + & + & + & + & + & + & - & + \\ + & + & + & + & + & + & + & + & - \end{bmatrix}$$

5.

$$\Sigma_{11}^* = \begin{bmatrix} + & + & - & + & - & + & - & + & - & + & - \\ - & + & + & + & - & + & + & - & - & - & + \\ + & - & + & - & - & + & - & - & + & + & + \\ - & - & + & + & + & + & - & + & + & - & - \\ + & + & + & - & + & - & - & + & - & - & + \\ - & - & - & - & + & + & + & + & - & + & + \\ + & - & + & + & + & - & + & - & - & + & - \\ - & + & + & - & - & - & + & + & + & + & - \\ + & + & - & - & + & + & + & - & + & - & - \\ - & + & - & + & + & - & - & - & + & + & + \\ + & - & - & + & - & - & + & + & + & - & + \end{bmatrix}$$

6.

$$\Sigma^{s}_{15} = \begin{bmatrix}
- & + & + & + & + & - & + & - & - & + & - & - & + & + & - \\
+ & - & + & + & + & + & - & + & - & - & - & - & - & + & + \\
+ & + & - & + & + & - & + & - & + & - & + & - & - & - & + \\
+ & + & + & - & + & - & - & + & - & + & + & + & - & - & - \\
+ & + & + & + & - & + & - & - & + & - & - & + & + & - & - \\
- & - & + & + & - & - & + & + & + & + & - & + & - & - & + \\
- & - & - & + & + & + & - & + & + & + & + & - & + & - & - \\
+ & - & - & - & + & + & + & - & + & + & - & + & - & + & - \\
+ & + & - & - & - & + & + & + & - & + & - & - & + & - & + \\
- & + & + & - & - & + & + & + & + & - & + & - & - & + & - \\
- & + & - & - & + & - & - & + & + & - & - & + & + & + & + \\
+ & - & + & - & - & - & - & - & + & + & + & - & + & + & + \\
- & + & - & + & - & + & - & - & - & + & + & + & - & + & + \\
- & - & + & - & + & + & + & - & - & - & + & + & + & - & + \\
+ & - & - & + & - & - & + & + & - & - & + & + & + & + & -
\end{bmatrix}$$

7.

$$R_{21} = \begin{bmatrix}
+ & + & - & - & - & - & + & - & + & - & - & - & - & - & - & - & - & - & + & - & - \\
- & + & + & - & - & - & - & + & - & + & - & - & - & - & - & - & - & - & - & + & - \\
- & - & + & + & - & - & - & - & + & - & + & - & - & - & - & - & - & - & - & - & + \\
+ & - & - & + & + & - & - & - & - & + & - & + & - & - & - & - & - & - & - & - & - \\
- & + & - & - & + & + & - & - & - & - & + & - & + & - & - & - & - & - & - & - & - \\
- & - & + & - & - & + & + & - & - & - & - & + & - & + & - & - & - & - & - & - & - \\
- & - & - & + & - & - & + & + & - & - & - & - & + & - & + & - & - & - & - & - & - \\
- & - & - & - & + & - & - & + & + & - & - & - & - & + & - & + & - & - & - & - & - \\
- & - & - & - & - & + & - & - & + & + & - & - & - & - & + & - & + & - & - & - & - \\
- & - & - & - & - & - & + & - & - & + & + & - & - & - & - & + & - & + & - & - & - \\
- & - & - & - & - & - & - & + & - & - & + & + & - & - & - & - & + & - & + & - & - \\
- & - & - & - & - & - & - & - & + & - & - & + & + & - & - & - & - & + & - & + & - \\
- & - & - & - & - & - & - & - & - & + & - & - & + & + & - & - & - & - & + & - & + \\
+ & - & - & - & - & - & - & - & - & - & + & - & - & + & + & - & - & - & - & + & - \\
- & + & - & - & - & - & - & - & - & - & - & + & - & - & + & + & - & - & - & - & + \\
+ & - & + & - & - & - & - & - & - & - & - & - & + & - & - & + & + & - & - & - & - \\
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- & - & + & - & + & - & - & - & - & - & - & - & - & - & + & - & - & + & + & - & - \\
- & - & - & + & - & + & - & - & - & - & - & - & - & - & - & + & - & - & + & + & - \\
- & - & - & - & + & - & + & - & - & - & - & - & - & - & - & - & + & - & - & + & + \\
+ & - & - & - & - & + & - & + & - & - & - & - & - & - & - & - & - & + & - & - & +
\end{bmatrix}$$

8.

$$
Q_{31} = \left[\begin{array}{ccccccccccccccccccccccccccccccc}
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-&+&+&+&+&+&+&+&+&+&+
\end{array}\right]
$$

9.

$$
[6,2,0] = \left[\begin{array}{cccccc}
0&+&+&0&+&- \\
+&0&+&-&0&+ \\
+&+&0&+&-&0 \\
0&+&-&0&+&+ \\
-&0&+&+&0&+ \\
+&-&0&+&+&0
\end{array}\right]
$$

10.

$$
[7,3,0] = \left[\begin{array}{ccccccc}
-&+&+&0&+&0&0 \\
0&-&+&+&0&+&0 \\
0&0&-&+&+&0&+ \\
+&0&0&-&+&+&0 \\
0&+&0&0&-&+&+ \\
+&0&+&0&0&-&+ \\
+&+&0&+&0&0&-
\end{array}\right]
$$

11.

$$[10,2,0] = \begin{bmatrix} 0 & + & + & - & + & 0 & + & + & + & - \\ + & 0 & + & + & - & - & 0 & + & + & + \\ - & + & 0 & + & + & + & - & 0 & + & + \\ + & - & + & 0 & + & + & + & - & 0 & + \\ + & + & - & + & 0 & + & + & + & - & 0 \\ 0 & + & - & - & - & 0 & + & - & + & + \\ - & 0 & + & - & - & + & 0 & + & - & + \\ - & - & 0 & + & - & + & + & 0 & + & - \\ - & - & - & 0 & + & - & + & + & 0 & + \\ + & - & - & - & 0 & + & - & + & + & 0 \end{bmatrix}$$

12.

$$[13,4,0] = \begin{bmatrix} 0 & - & + & 0 & + & - & + & 0 & + & - & + & 0 & + \\ 0 & 0 & - & - & + & + & + & + & 0 & + & 0 & - & + \\ + & 0 & 0 & + & + & - & 0 & 0 & - & + & - & + & + \\ - & 0 & + & 0 & 0 & 0 & + & + & + & + & - & + & - \\ + & + & + & - & 0 & + & + & 0 & - & - & 0 & + & 0 \\ 0 & + & 0 & - & + & 0 & - & - & + & + & + & + & 0 \\ + & + & - & + & + & 0 & 0 & + & + & - & 0 & 0 & - \\ - & + & - & + & - & 0 & + & 0 & 0 & 0 & + & + & + \\ + & - & 0 & + & 0 & + & + & - & 0 & + & + & 0 & - \\ 0 & + & + & + & 0 & + & 0 & - & + & 0 & - & - & + \\ + & - & 0 & 0 & - & + & - & + & + & 0 & 0 & + & + \\ - & 0 & + & + & + & + & - & + & - & 0 & + & 0 & 0 \\ + & + & + & 0 & - & - & 0 & + & 0 & + & + & - & 0 \end{bmatrix}$$

13.

$$[14,5,0] = \begin{bmatrix} 0 & + & + & - & + & - & - & 0 & + & + & 0 & + & 0 & 0 \\ - & 0 & + & + & - & + & - & 0 & 0 & + & + & 0 & + & 0 \\ - & - & 0 & + & + & - & + & 0 & 0 & 0 & + & + & 0 & + \\ + & - & - & 0 & + & + & - & + & 0 & 0 & 0 & + & + & 0 \\ - & + & - & - & 0 & + & + & 0 & + & 0 & 0 & 0 & + & + \\ + & - & + & - & - & 0 & + & + & 0 & + & 0 & 0 & 0 & + \\ + & + & - & + & - & - & 0 & + & + & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & + & + & 0 & + & + & - & + & - & - \\ + & 0 & 0 & 0 & + & 0 & + & - & 0 & + & + & - & + & - \\ + & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & + & - & + \\ 0 & + & + & 0 & 0 & 0 & + & + & - & - & 0 & + & + & - \\ + & 0 & + & + & 0 & 0 & 0 & - & + & - & - & 0 & + & + \\ 0 & + & 0 & + & + & 0 & 0 & + & - & + & - & - & 0 & + \\ 0 & 0 & + & 0 & + & + & 0 & + & + & - & + & - & - & 0 \end{bmatrix}$$

14. Design under section 2.10 for n = 9.

$$
\begin{bmatrix}
+ & - & - & + & + & + & + & + & + \\
- & + & - & + & + & + & + & + & + \\
- & - & + & + & + & + & + & + & + \\
+ & + & + & + & - & - & + & + & + \\
+ & + & + & - & + & - & + & + & + \\
+ & + & + & - & - & + & + & + & + \\
+ & + & + & + & + & + & + & - & - \\
+ & + & + & + & + & + & - & + & - \\
+ & + & + & + & + & + & - & - & +
\end{bmatrix}
$$

Variance of each estimated weight = $.185\,\sigma^2$

15. Design under section 2.10 for n = 21.

$$
\begin{bmatrix}
+ & - & - & - & - & - & - & - & - & - & + & + & - & - & - & - & + & + & - & + & + \\
- & + & - & - & - & - & - & - & - & + & - & + & - & - & - & + & - & + & + & - & + \\
- & - & + & - & - & - & - & - & - & + & + & - & - & - & - & + & + & - & + & + & - \\
- & + & + & + & - & - & - & - & - & - & - & - & - & + & + & - & - & - & - & + & + \\
+ & - & + & - & + & - & - & - & - & - & - & - & + & - & + & - & - & - & + & - & + \\
+ & + & - & - & - & + & - & - & - & - & - & - & + & + & - & - & - & - & + & + & - \\
- & + & + & - & + & + & + & - & - & - & - & - & - & - & - & - & + & + & - & - & - \\
+ & - & + & + & - & + & - & + & - & - & - & - & - & - & - & + & - & + & - & - & - \\
+ & + & - & + & + & - & - & - & + & - & - & - & - & - & - & + & + & - & - & - & - \\
- & - & - & - & + & + & - & + & + & + & - & - & - & - & - & - & - & - & - & + & + \\
- & - & - & + & - & + & + & - & + & - & + & - & - & - & - & - & - & - & + & - & + \\
- & - & - & + & + & - & + & + & - & - & - & + & - & - & - & - & - & - & + & + & - \\
- & + & + & - & - & - & - & + & + & - & + & + & + & - & - & - & - & - & - & - & - \\
+ & - & + & - & - & - & + & - & + & + & - & + & - & + & - & - & - & - & - & - & - \\
+ & + & - & - & - & - & + & + & - & + & + & - & - & - & + & - & - & - & - & - & - \\
- & - & - & - & + & + & - & - & - & - & + & + & - & + & + & + & - & - & - & - & - \\
- & - & - & + & - & + & - & - & - & + & - & + & + & - & + & - & + & - & - & - & - \\
- & - & - & + & + & - & - & - & - & + & + & - & + & + & - & - & - & + & - & - & - \\
- & - & - & - & - & - & - & + & + & - & - & - & - & + & + & - & + & + & + & - & - \\
- & - & - & - & - & - & + & - & + & - & - & - & + & - & + & + & - & + & - & + & - \\
- & - & - & - & - & - & + & + & - & - & - & - & + & + & - & + & + & - & - & - & +
\end{bmatrix}
$$

Variance of the each estimated weight = $.053\,\sigma^2$

16. Design under the section 2.11 for n = 11.

$$\begin{bmatrix}
0 & + & + & + & + & + & + & + & + & + & + \\
+ & 0 & + & + & + & + & + & + & - & - & - \\
+ & + & 0 & + & + & + & - & - & + & + & - \\
+ & + & + & 0 & + & - & + & - & + & - & + \\
+ & + & + & + & 0 & - & - & + & - & + & + \\
+ & + & + & - & - & 0 & + & + & + & + & - \\
+ & + & - & + & - & + & 0 & + & + & - & + \\
+ & + & - & - & + & + & + & 0 & - & + & + \\
+ & - & + & + & - & + & + & - & 0 & + & + \\
+ & - & + & - & + & + & - & + & + & 0 & + \\
+ & - & - & + & + & - & + & + & + & + & 0
\end{bmatrix}$$

$c_{11} = \frac{19}{169}$, $c_{ii} = \frac{18}{169}$

17. Design under the section 2.11 for n = 27.

$$\begin{bmatrix}
0 & + \\
+ & 0 & - & + & - & - & + & + & + & + & - & - & + & - & - & + & - & + & + & - & - & + & + & + & + & + & + \\
+ & - & 0 & - & + & - & - & + & + & + & + & - & - & + & + & - & + & - & + & + & - & - & + & + & + & + & + \\
+ & + & - & 0 & - & + & - & - & + & + & + & + & - & - & + & + & - & + & - & + & + & - & - & + & + & + & + \\
+ & - & + & - & 0 & - & + & - & - & + & + & + & + & - & + & + & + & - & + & - & + & + & - & - & + & + & + \\
+ & - & - & + & - & 0 & - & + & - & - & + & + & + & + & + & + & + & + & - & + & - & + & + & - & - & + & + \\
+ & + & - & - & + & - & 0 & - & + & - & - & + & + & + & + & + & + & + & + & - & + & - & + & + & - & - & + \\
+ & + & + & - & - & + & - & 0 & - & + & - & - & + & + & + & + & + & + & + & + & - & + & - & + & + & - & - \\
+ & + & + & + & - & - & + & - & 0 & - & + & - & - & + & - & + & + & + & + & + & + & - & + & - & + & + & - \\
+ & + & + & + & + & - & - & + & - & 0 & - & + & - & - & - & - & + & + & + & + & + & + & - & + & - & + & + \\
+ & - & + & + & + & + & - & - & + & - & 0 & - & + & - & + & - & - & + & + & + & + & + & + & - & + & - & + \\
+ & - & - & + & + & + & + & - & - & + & - & 0 & - & + & + & + & - & - & + & + & + & + & + & + & - & + & - \\
+ & + & - & - & + & + & + & + & - & - & + & - & 0 & - & - & + & + & - & - & + & + & + & + & + & + & - & + \\
+ & - & + & - & - & + & + & + & + & - & - & + & - & 0 & + & - & + & + & - & - & + & + & + & + & + & + & - \\
+ & - & + & + & + & + & + & + & - & - & + & + & - & + & 0 & + & - & + & + & - & - & - & - & + & + & - & + \\
+ & + & - & + & + & + & + & + & + & - & - & + & + & - & + & 0 & + & - & + & + & - & - & - & - & + & + & - \\
+ & - & + & - & + & + & + & + & + & + & - & - & + & + & - & + & 0 & + & - & + & + & - & - & - & - & + & + \\
+ & + & - & + & - & + & + & + & + & + & + & - & - & + & + & - & + & 0 & + & - & + & + & - & - & - & - & + \\
+ & + & + & - & + & - & + & + & + & + & + & + & - & - & + & + & - & + & 0 & + & - & + & + & - & - & - & - \\
+ & - & + & + & - & + & - & + & + & + & + & + & + & - & - & + & + & - & + & 0 & + & - & + & + & - & - & - \\
+ & - & - & + & + & - & + & - & + & + & + & + & + & + & - & - & + & + & - & + & 0 & + & - & + & + & - & - \\
+ & + & - & - & + & + & - & + & - & + & + & + & + & + & - & - & - & + & + & - & + & 0 & + & - & + & + & - \\
+ & + & + & - & - & + & + & - & + & - & + & + & + & + & - & - & - & - & + & + & - & + & 0 & + & - & + & + \\
+ & + & + & + & - & - & + & + & - & + & - & + & + & + & + & - & - & - & - & + & + & - & + & 0 & + & - & + \\
+ & + & + & + & + & - & - & + & + & - & + & - & + & + & + & + & - & - & - & - & + & + & - & + & 0 & + & - \\
+ & + & + & + & + & + & - & - & + & + & - & + & - & + & + & + & + & - & - & - & - & + & + & - & + & 0 & + \\
+ & + & + & + & + & + & + & - & - & + & + & - & + & - & + & - & + & + & - & - & - & - & + & + & - & + & 0
\end{bmatrix}$$

$c_{11} = \frac{51}{961}$, $c_{ii} = \frac{38}{961}$

CHAPTER 3.

GROUP DIVISIBLE FAMILY OF "PBIB" DESIGNS*

3.1 INTRODUCTION.

In this chapter we make systematic study of some partially balanced incomplete block designs which belong to group divisible family. For the sake of brevity we denote them as "GPBIB" designs. The definition of "GPBIB" design is given in section 3.2. In section 3.3, We give the characterisation of (m+1) associate "GPBIB" design. The analysis of these designs is given in section 3.4. Section 3.5 deals with some methods of construction of "GPBIB" designs. Some combinatorial properties of these designs and necessary conditions for the existence of a class of these designs are studied in section 3.6, where specialisation is also given to "GL_2" and "GT" designs.

3.2. DEFINITION OF (m+1) ASSOCIATE "GPBIB" DESIGN.

An (m+1) associate PBIB design belongs to group divisible family, if it satisfies the following conditions:

(i) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

(ii) There are v(= ut) treatments and these can be grouped into t groups of u each, such that any two treatments of the same group are either 1st, 2nd, ... or m^{th} associates while two treatments from different groups are $(m+1)^{th}$ associates. The association scheme is same for all the t groups.

(iii) Each treatment is replicated r times.

---

* See the reference [12]

(iv) Every pair of treatments which are i^{th} associates $(i = 1,2,\ldots, m+1)$ occur together in λ_i blocks.

We suppose that the association scheme of these groups is known. Let the secondary parameters of the groups be

(3.2.1) $$n_i (i=1,2,\ldots,m), \quad \sum_{i=1}^{m} n_i = u-1$$

$$P_l^* = ((p_{ij}^l)) \qquad (l,i,j = 1,2,\ldots,m)$$

Hence we get the secondary parameters of the above "GPBIB" design as

$$n_i (i=1,2,\ldots,m) \text{ and } n_{m+1} = u(t-1)$$

$$P_l = \begin{bmatrix} P_l^* & 0_{m\,1} \\ 0_{1\,m} & u(t-1) \end{bmatrix} \qquad l = 1,2,\ldots, m$$

(3.2.2)

$$P_{m+1} = \left[\begin{array}{c|c} 0_{mm} & \begin{matrix} n_1 \\ n_2 \\ \cdot \\ \cdot \\ n_m \end{matrix} \\ \hline n_1 n_2 \ldots n_m & u(t-2) \end{array}\right]$$

3.3 CHARACTERIZATION OF (m+1) ASSOCIATE "GPBIB" DESIGN.

Let $N = ((n_{ij}))$ be the incidence matrix of a "GPBIB" design where $n_{ij} = 1$ or 0 according as the i^{th} treatment occurs in the j^{th} block or not $(i = 1,2,\ldots,v$ and $j = 1,2,\ldots,b)$. Let

(3.3.1) $$N = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_t \end{bmatrix}$$

where N_i $(i = 1,2,\ldots,t)$ is the part of the incidence matrix N corresponding to i^{th} group of treatments in b blocks. Also let the association matrices, first introduced by Bose and Mesner [19], of the groups be $B_o, B_1,\ldots,B_m$. Hence we can write

(3.3.2) $$NN^1 = I_t \times (P-Q) + E_{tt} \times Q$$

where $$P = \sum_{i=o}^{m} \lambda_i B_i \ , \quad Q = \lambda_{m+1} E_{uu}$$

Let the eigen values of P be $\sum_{i=o}^{m} n_i \lambda_i$, $\theta_1, \theta_2, \ldots, \theta_m$ with the multiplicities $1, \alpha_1, \alpha_2, \ldots, \alpha_m$ respectively. Here $n_o=1$, $\lambda_o= r$

The det(NN') can be written as

(3.3.3) $$\det(NN') = \det[P+(t-1)\,Q]\,[\det(P-Q)]^{t-1}$$

Further we know that det(P+aQ), where a is any real number, as

(3.3.4) $$\det(P+aQ) = \left[\sum_{i=o}^{m} n_i \lambda_i + au\,\lambda_{m+1}\right] \theta_1^{\alpha_1} \theta_2^{\alpha_2} \cdots \theta_m^{\alpha_m}$$

Let $$\theta_{m+1} = \sum_{i=o}^{m} n_i \lambda_i - u\,\lambda_{m+1} = rk - v\lambda_{m+1} \quad \text{and}$$

$$\theta_o = \sum_{i=o}^{m} n_i \lambda_i + u(t-1)\,\lambda_{m+1} = rk$$

Hence

(3.3.5) $$\det(NN') = \theta_o \ \theta_1^{\alpha_1 t} \theta_2^{\alpha_2 t} \cdots \theta_m^{\alpha_m t} \ \theta_{m+1}^{t-1}$$

We know from the result of Connor and Clatworthy [27] that the characteristic roots of NN' can not be negative for an existing design. Thus we have the following theorem :

THEOREM 3.3.1 : A necessary condition for the existence of (m+1) associate "GPBIB" design is that $\theta_i \geq 0$ ($i = 1,2,\ldots, m+1$).

Some examples of the non-existing designs, using the theorem 3.3.1, will be given in section 3.6.

From (3.3.5), we can classify existing (m+1) associate "GPBIB" designs into different classes. For example the following are the two cases.

(i) Group regular designs characterised by $\theta_{m+1} = 0$ and $\theta_i > 0$ ($i=1,2,\ldots,m$).

(ii) Regular designs characterised by $\theta_i > 0$ ($i=1,2,\ldots,m+1$).

3.4. ANALYSIS OF (m+1) ASSOCIATE "GPBIB" DESIGNS

The analysis of (m+1) associate "GPBIB" design can be obtained with the help of association matrices; this being the particular case of Shah's generalised analysis [58]. With the usual intra block model, the normal equations giving the column vector of the intra block estimates of the treatment effects $\underset{\sim}{\tau}$ are

(3.4.1) $$\underset{\sim}{Q} = C\,\hat{\underset{\sim}{\tau}}$$

where $\underset{\sim}{Q} = \underset{\sim}{T} - \frac{1}{k} N\underset{\sim}{B}$ and $C = rI_v - \frac{1}{k} NN'$

$\underset{\sim}{T}$ and $\underset{\sim}{B}$ being the column vectors of the treatment totals and block totals respectively.

(3.4.2) $$C = I_t \times \left(\sum_{i=0}^{m} c_i B_i \right) - \frac{\lambda_{m+1}}{k} E_{vv}$$

(3.4.3) $$c_o = \frac{\lambda_{m+1} + r(k-1)}{k}, \quad c_i = \frac{\lambda_{m+1} - \lambda_i}{k}, \quad i=1,2,\ldots,m.$$

Hence

$$(3.4.4) \qquad C + \frac{\lambda_{m+1}}{K} E_{vv} = I_t \times \left(\sum_{i=0}^{m} c_i B_i \right)$$

By the theorem 3.2 of Shah [58], we can show that the inverse of $\sum_{i=0}^{m} c_i B_i$ is of the form $\sum_{i=0}^{m} d_i B_i$. The solutions of d_i^s are obtained from the independent equations, namely,

$$(3.4.5) \qquad \sum_{i=0}^{m} \sum_{j=0}^{m} p_{ij}^{l} c_i d_j = 1 \quad \text{if } l = 0$$
$$= 0 \quad \text{if } l = 1,2,..,m.$$

Hence we have that

$$(3.4.6) \qquad \hat{\underline{\tau}} = \left[I_t \times \sum_{i=0}^{m} d_i B_i \right] \underline{Q}$$

$$(3.4.7) \qquad V(\hat{\tau}_i - \hat{\tau}_j) = 2(d_o - d_s)\sigma^2 \quad \text{or}$$

$$= 2d_o \sigma^2$$

according as the i^{th} and j^{th} treatments are s^{th} associates (s=1,2,..,m) or $(m+1)^{th}$ associates, where σ^2 is the intrablock error variance. The average variance of the design is

$$(3.4.8) \qquad \frac{2}{v-1} \left[(v-1)d_o - \sum_{i=0}^{m} n_i d_i \right] \sigma^2$$

and its efficiency is

$$(3.4.9) \qquad \left(\frac{v-1}{r} \right) \left[(v-1)d_o - \sum_{i=0}^{m} n_i d_i \right]^{-1}$$

3.5 CONSTRUCTION OF "GPBIB" DESIGNS.

In this section we give some methods of construction of "GPBIB" designs by using balanced incomplete block ("BIB") designs of family "A" (Cf. Shrikhande [63]), and some particular row-orthogonal matrices having elements ± 1 and 0 with m associate "PBIB" designs. A "BIB" design with the parameters v, b, r, k, λ belongs to the family 'A' if $b = 4(r-\lambda)$. Two series of "BIB" designs of family 'A' were given by Shrikhande and Raghavarao in [64]. A matrix X of order $n \times m$, having the elements $+1, -1, 0$ and every row containing s zeros, is row-orthogonal if $XX' = (m-s)I_n$. When $m=n$, X becomes $[n, s, o]$ (Cf. Chapter 2). For the construction of "GPBIB" design we use X, the row-orthogonal matrix, which satisfies the property that when we change -1^s, $+1^s$ of X to zeros and zeros of X to $+1^s$ the resulting design is a "BIB" design. We denote this row-orthogonal matrix as $X(n,m,s)$.

$$\text{eg.} \qquad X(3,6,2) = \begin{pmatrix} 0 & 1 & -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

If N is any incidence matrix of a binary design of order $v \times b$, then its complement design is given by $N^*(=E_{vb}-N)$.

THEOREM 3.5.1 : Let N_1 be a m associate "PBIB" design with the parameters

(3.5.1) $$v' = u,\ b',\ r',\ k',\ n_1',\ n_2',\ \ldots,\ n_m', \lambda_1', \lambda_2', \ldots, \lambda_m',\ P_i' = ((p^{i\,'}_{jk}))$$
$$(i,j,k=1,2,\ \ldots\ m)$$

and let N_2 be a "BIB" design of family 'A' with the parameters $v''=t, b'', r'', k'', \lambda''$; then

(3.5.2) $$N = [N_2 \times N_1 + N_2^* \times N_1^*]$$

is $(m+1)$ associate "GPBIB" design. The parameters of the design are

(3.5.3) $$v = ut,\ b = b'b'',\ r = r'r'' + (b'-r')(b''-r''),\ k = k'k'' + (v'-k')(v''-k'')$$
$$n_i = n_i',\ n_{m+1} = u(t-1),\ \lambda_i = b''\lambda_i' + f \quad (i=1,2,\ldots,m),\ \lambda_{m+1} = f-g$$
where $f = (b''-r'')(b'-2r')$, $g = (r''-\lambda'')(b'-4r')$

$$P_i = \begin{bmatrix} P_i' & 0_{m1} \\ 0_{1m} & u(t-1) \end{bmatrix}, \quad P_{m+1} = \left[\begin{array}{c|c} 0_{mm} & \begin{matrix} n_1 \\ n_2 \\ \cdot \\ \cdot \\ n_m \end{matrix} \\ \hline n_1 n_2 \cdots n_m & u(t-2) \end{array}\right]$$

$i = 1,2, \ldots , m.$

PROOF :

(3.5.4) $$NN' = [N_2 \times N_1 + N_2^{*} \times N_1^{*}][N_2 \times N_1 + N_2^{*} \times N_1^{*}]'$$

On simplification (3.5.4) becomes

(3.5.5) $$NN' = I_t \times [b'' N_1 N_1' + g\, E_{v'v'}] + (f-g)\, E_{vv}$$

Since N_1 is a "PBIB" design, we can show easily from (3.5.5) that N is "GPBIB" design with the parameters given in (3.5.3).

COROLLARY 3.5.1.1 : N is group divisible design if N_1 is a "BIB" design.

COROLLARY 3.5.1.2 : N is group divisible 3 associate design if N_1 is group divisible design.

THEOREM 3.5.2 : Let N_1 be a m associate "PBIB" design with the parameters given in (3.5.1). Further let $u = 2k'$. Let $X(t,n,s)$ be a row-orthogonal matrix with the elements ± 1 and 0 . Let M be a "BIB" design with the parameters t,n,s,k'', λ'' obtained from $X(t,n,s)$ on changing zeros to $+1^s$ and $\pm 1^s$ of $X(t,n,s)$ to zeros. Let L be a design obtained from $X(t,n,s)$ on changing -1^s to zeros. Then

(3.5.6) $$N = L \times N_1 + (E_{tn} - L - M) \times N_1^{*}$$

is (m+1) associate "GPBIB" design with the parameters

$v = ut,\ b = nb',\ r = (n-s)r',\ k = (t-k'')k',\ n_i = n_i' (i=1,2,\ldots,m)$

(3.5.7) $n_{m+1} = u(t-1)$, $\lambda_i = (n-s)\lambda_i'$ $(i=1,2,\ldots,m)$, $\lambda_{m+1} = \dfrac{r'(n-2s+\lambda'')}{2}$

and P_i $(i=1,2,\ldots,m+1)$ Cf. (3.5.3)

The proof of this theorem is similar to the proof of theorem 3.5.1 Since $u = 2k'$, we get $\theta_{m+1} = o$. We may obtain regular designs by relaxing the condition $u=2k'$ and using particular $X(t,n,s)$ which gives both L and M as "BIB" designs.

eg. $$X(7,7,3) = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix}$$

THEOREM 3.5.3 : Let N_1 be a m associate "PBIB" design with the parameters given in (3.5.1) . Then

(3.5.8) $$N = I_t \times N_1 + (E_{tt} - I_t) \times E_{ub'}$$

is (m+1) associate "GPBIB" design with the parameters

$v = ut$, $b = tb'$, $r = r' + (t-1)b'$, $k = k' + (t-1)u$, $n_i = n_i'$, $n_{m+1} = u(t-1)$

(3.5.9) $\lambda_i = \lambda_i' + (t-1)b'$, $\lambda_{m+1} = 2r' + (t-2)b'$ and P_i $(i=1,2,\ldots,m+1)$ Cf. (3.5.3)
$(i=1,2,\ldots,m)$

THEOREM 3.5.4 : Let N_1 be a (m+1) associate "GPBIB" design with the parameters

(3.5.10) $v_1 = u_1 t_1$, b_1, r_1, k_1, n_i', λ_i', P_i' $(i=1,2,\ldots,m+1)$

Let $$N_1 = \begin{bmatrix} N_{11} \\ N_{12} \\ \cdot \\ \cdot \\ N_{1t_1} \end{bmatrix} \text{ where } N_{1i} N_{1j}' = \lambda_{m+1}' E_{u_1 u_1}, \quad i,j = 1,2,\ldots,t_1, \; i \neq j$$

Let N_2 be a "BIB" design with the parameters u_2, b_2, r_2, $k_2 = t_1$, λ_2''. Substitute t_1 distinct N_{1i}^{s} in place of t_1 distinct units and $O_{u_1b_1}$ in place of $v_2 - t_1$ zeros in every block of N_2. The resulting matrix is (m+1) associate "GPBIB" with the parameters

$$v = u_1u_2,\ b = b_1b_2,\ r = r_1r_2,\ k = k_1,\ n_i = n_i',\ n_{m+1} = u_1(v_2-1) \quad (i=1,2,\ldots,m)$$

$$\lambda_i = r_2\,\lambda_i'\ (i=1,2,\ldots,m),\ \lambda_{m+1} = \lambda_2''\,\lambda_{m+1}'$$

(3.5.11)

$$P_i = \left[\begin{array}{c|c} ((p_{jk}^{i'})) & o_{m1} \\ \hline o_{1m} & u_1(v_2-1) \end{array}\right],\quad P_{m+1} = \left[\begin{array}{c|c} o_{mm} & \begin{array}{c} n_1 \\ n_2 \\ \cdot \\ \cdot \\ n_m \end{array} \\ \hline n_1n_2\cdots n_m & u_1(v_2-2) \end{array}\right]$$

$(i=1,2,\ldots,m)$

The proofs of the theorems 3.5.3 and 3.5.4 are evident and hence they are omitted.

COROLLARY 3.5.4.1 : Let N_1 be a "BIB" design with the parameters $u_1t_1 = v_1$, b_1, r_1, k_1, λ_1' and N_2 be another "BIB" design with the parameters v_2, b_2, r_2, $k_2 = t_1$, λ_2'. Write

$$N_1 = \begin{bmatrix} N_{11} \\ N_{12} \\ \vdots \\ N_{1t_1} \end{bmatrix} \qquad \text{where} \quad N_{1i}\,N_{1j}' = \lambda_1'\,E_{u_1u_1}.$$

By the above theorem 3.5.4, we get a group divisible design with the parameters

$$v = mn,\ b = b_1b_2,\ r = r_1r_2,\ k = k_1,\ m = v_2,\ n = u_1, \lambda_1 = r_2\lambda_1', \lambda_2 = \lambda_1'\lambda_2'.$$

COROLLARY 3.5.4.2 : Let $u_1 = 1$ in corollary 3.6.4.1. Then we get "BIB" design with the parameters $v = v_2$, $b = b_1b_2$, $r = r_1r_2$, $k = k_1$, $\lambda = \lambda_1'\lambda_2'$.

3.6 3 ASSOCIATE "GPBIB" DESIGNS

Bose and Shimamoto [21] classified 2 associate "PBIB" designs as group divisible ("GD") type, "L_i" (i=2,3) type, triangular ("T") type, cyclic ("C") type and simple ("S_1") type "PBIB" designs. Here, we classify 3 associate "GPBIB" designs as "GD" 3 associate, "GL_i" (i=2,3), "GT", "GC" and "GS_1" type designs. Roy [56] and Raghavarao [53] studied "GD" 3 associate designs in detail. In this section we give some combinatorial properties and non-existence of certain "GL_2" and "GT" designs. For the details of "L_2" and "T" type designs we refer [62], [61], [28].

THEOREM 3.6.1 : In a (m+1) associate "GPBIB" design with $\theta_{m+1}=o$, k is divisible by t and further every block of the design contains k/t treatments from each group.

PROOF : Let e_j^i be the number of treatments from i^{th} group in the j^{th} block. Then

(3.6.1) $$\sum_{j=1}^{b} e_j^i = ur \; , \quad \sum_{j=1}^{b} e_j^i \,(e_j^i - 1) = u\,(\sum_{l=1}^{m} n_l \lambda_l)$$

Define $e_{\cdot}^i = b^{-1} \sum_{j=1}^{b} e_j^i = \frac{ur}{b} = \frac{k}{t}$. Then

(3.6.2) $$\sum_{j=1}^{b} (e_j^i - e_{\cdot}^i)^2 = u(r + \sum_{l=1}^{m} n_l \lambda_l) - bk^2/t^2 = o$$

since $r + \sum_{l=1}^{m} n_l \lambda_l = rk - v\lambda_{m+1} + u\lambda_{m+1} = \theta_{m+1} + u\lambda_{m+1} = u\lambda_{m+1} = \frac{urk}{v}$

(3.6.2) shows that all e_j^i (i=1,2,...,b) are equal; $e_j^i = e_1^i = \frac{k}{t}$ an integer for all i = 1,2,...,t.

3.6$_a$ "GL_2" DESIGNS

Here, u ($=n^2$) is a perfect square .

$$\det(NN') = \theta_o \, \theta_1^{2(n-1)t} \, \theta_2^{(n-1)^2 t} \, \theta_3^{t-1} \qquad (3.6.3)$$

where $\theta_o = rk$, $\theta_1 = r + (n-2)(\lambda_1 - \lambda_2) - \lambda_2$, $\theta_2 = r + \lambda_2 - 2\lambda_1$ and $\theta_3 = rk - v\lambda_3$.

It can be observed that θ_i^s(i=o,1,2,3) are the distinct eigen roots of NN' with the respective multiplicities $1, 2(n-1)t, (n-1)^2t, t-1$. The designs with the following parameters violate the necessary condition; viz. theorem 3.3.1, and hence are impossible.

v	b	r	k	n	t	λ_1	λ_2	λ_3	
18	21	7	6	3	2	0	2	3	($\theta_3 < o$)
27	27	8	8	3	3	5	0	2	($\theta_2 < o$)
64	48	9	12	4	4	1	5	1	($\theta_1 < o$)
70	50	12	18	5	3	9	2	2	($\theta_2 < o$)

COROLLARY 3.6.1.1 : A necessary condition for the existence of "GL_2" design with θ_3=o is that k is divisible by t.

THEOREM 3.6.2 : In "GL_2" design with θ_1=o and θ_3=o , k is divisible by nt and further every group in every block of the design contains k/nt treatments from each of the n rows (or columns) of the "L_2" association scheme.

Proof follows from the corollary 3.6.1.1 and theorem 2.1 in [54].

COROLLARY 3.6.2.1 : A necessary condition for the existence of "GL_2" design with $\theta_1=0$, $\theta_3=0$ and $n\neq 4$ is that k is divisible by nt.

We use the condition $n \neq 4$ in the above corollary because "L_2" association scheme is not unique for n=4. (cf. Shrikhande [61]).

The designs with the following parameters violate the conditions of the corollaries 3.6.1.1 and 3.6.2.1 and hence are impossible.

u	b	r	k	n	t	λ_1	λ_2	λ_3	
18	16	8	9	3	2	5	2	4	(cor. 3.6.1.1)
18	81	36	8	3	2	6	21	16	(cor. 3.6.2.1)
27	81	36	12	3	3	6	21	16	(cor. 3.6.2.1)
36	64	16	9	3	4	2	3	4	(cor. 3.6.1.1)
75	36	12	25	5	3	5	3	4	(cor 3.6.1.1)

3.6_b NON-EXISTENCE OF CERTAIN SYMMETRICAL "GL_2" DESIGNS

From Shrikhande's [59] and Connor and Clatworthy's [27] results it follows that

THEOREM 3.6.3 : A necessary condition for the existence of symmetrical regular "GL_2" design is that $\theta_2^{(n-1)^2t} \quad \theta_3^{t-1}$ should be a perfect square.

The following corollary is obvious.

COROLLARY 3.6.3.1: For a regular symmetrical "GL_2" design, θ_2 is perfect square if t is odd and n is even; θ_3 is perfect square if t is even.

The following designs violate the condition of the theorem 3.6.3 and hence are non-existing.

$v = b$	$r = k$	n	t	λ_1	λ_2	λ_3
18	7	3	2	2	1	2
36	10	3	4	4	5	2
108	12	6	3	1	2	1
180	24	6	5	2	4	3

Further necessary conditions for the existence of regular "GL_2" designs can be obtained with the help of the Hasse - Minkowshi invariant (cf. chapter 2). From Singh and Shukla [65] we have the following theorem.

THEOREM 3.6.4 : If M is an irreducible positive definite, rational symmetric and generalised stochastic matrix of order v, with the rational eigen values θ_o, θ_1, θ_2 and θ_3 with respective multiplicities $\alpha_o=1$, α_1, α_2 and α_3 and Q_1, Q_2 are the gramians of the rational vectors generating the eigen spaces corresponding to θ_1, θ_2 respectively then the Hasse - Minkowski p invariant of M is given by

$$(3.6.4) \quad c_p(M) = (-1,-1)_p \left(\theta_o, -v\, \theta_1^{\alpha_1} \theta_2^{\alpha_2} \theta_3^{\alpha_3} \right)_p (\theta_1,\theta_2)_p^{\alpha_1\alpha_2} (\theta_1,\theta_3)_p^{\alpha_1\alpha_3} (\theta_2,\theta_3)_p^{\alpha_2\alpha_3}$$

$$\prod_{i=1}^{3} (-1,\theta_i)_p^{\frac{\alpha_i(\alpha_i+1)}{2}} (\theta_1, |Q_1|)_p (\theta_2, |Q_2|)_p (\theta_3, v|Q_1|.|Q_2|)_p$$

Let $M = NN'$. The c_p (NN') can be calculated in the usual way [47] and by using (3.6.4), we get on further simplification

$$(3.6.5) \quad c_p(NN') = (-1,-1)_p(-1,\theta_1)_p^{(n-1)t}(-1,\theta_2)_p^{\frac{\alpha_2(\alpha_2+1)}{2}}(-1,\theta_3)_p^{\frac{t(t-1)}{2}}(t,\theta_3)_p$$

where $\alpha_2 = (n-1)^2 t$.

Since $NN' \sim I_v$, we should have $c_p(NN') = (-1,-1)_p$ for all primes. Thus

THEOREM 3.6.5 : A necessary condition for the existence of a regular symmetric "GL_2" design is that

$$(-1,\theta_1)_p^{(n-1)t}(-1,\theta_2)_p^{\frac{\alpha_2(\alpha_2+1)}{2}}(-1,\theta_3)_p^{\frac{t(t-1)}{2}}(t,\theta_3)_p = 1 \quad \text{for all}$$

primes p.

The following corollary can be deduced easily

COROLLARY 3.6.5.1 : Necessary conditions for the existence of regular symmetric "GL_2" design are

(i) $t \equiv 2 \bmod 4$, n is even; $(-1,\theta_2)_p = 1$

(ii) $t \equiv 1 \bmod 4$, n is odd; $(t,\theta_3)_p = 1$

(3.6.6) (iii) $t \equiv 1 \bmod 4$, n is even; $(-1,\theta_1)_p\,(t,\theta_3)_p = 1$

(iv) $t \equiv 3 \bmod 4$, n is even; $(-1,\theta_1)_p\,(-t,\theta_3)_p = 1$

(v) $t \equiv 3 \bmod 4$, n is odd; $(-t,\theta_3)_p = 1$

The designs with the following parameters do not satisfy the above corollary and hence are non-existent:

v = b	r = k	n	t	λ_1	λ_2	λ_3
27	11	3	3	6	8	3
45	13	3	5	5	7	3
72	36	6	2	25	26	10
108	28	6	3	14	16	3

3.6$_c$ "GT" DESIGNS

Here $u = n(n-1)/2$

(3.6.7) $$NN' = \theta_o\, \theta_1^{\alpha_1}\, \theta_2^{\alpha_2}\, \theta_3^{\alpha_3}$$

where $\theta_o = rk$, $\theta_1 = r + (n-4)\lambda_1 - (n-3)\lambda_2$, $\theta_2 = (r - 2\lambda_1 + \lambda_2)$ and $\theta_3 = rk - v\lambda_3$.

It can be observed that θ_i's $(i = o, 1, 2, 3)$ are the distinct characeristic roots of NN' with the respective multiplicities $1, \alpha_1 = (n-1)t, \alpha_2 = \frac{n(n-3)\ t}{2}$, $\alpha_3 = t-1$. The designs with the following parameters violate the necessary condition (cf. theorem 3.3.1) and hence are impossible.

v	b	r	k	n	t	λ_1	λ_2	λ_3	
12	9	6	8	4	2	2	4	5	$(\theta_3 < o)$
30	60	20	10	5	3	18	4	3	$(\theta_2 < o)$
30	20	10	15	6	2	4	8	4	$(\theta_1 < o)$

COROLLARY 3.6.1.2 : A necessary conditon for the existence of "GT" design with $\theta_3 = o$ is that k is divisible by t.

THEOREM 3.6.6 : In a "GT" design with $\theta_1 = o$, $\theta_3 = o$, 2k is divisible by nt, further every group in every block of the design contains 2k/nt treatments from each of the n rows of the triangular association scheme.

The proof of it follows from the corollary 3.6.1.2 and the theorem 1.1 of [54].

COROLLARY 3.6.6.1 : A necessary condition for the existence of "GT" design with $\theta_1 = o$, $\theta_3 = o$ and $n \neq 8$ is that 2k is divisible by nt.

The designs with the following parameters violate the conditions of the corollaries 3.6.1.2 and 3.6.6.1 and hence are non-existing.

v	b	r	k	n	t	λ_1	λ_2	λ_3	
12	8	4	6	4	2	1	4	2	(Cor. 3.6.6.1)
12	24	8	3	4	2	1	0	2	(Cor. 3.6.1.2)
18	81	18	4	4	3	1	2	4	(Cor. 3.6.1.2)
30	25	10	12	5	3	2	6	4	(Cor. 3.6.6.1)

We use the condition, $n \neq 8$, in the corollary 3.6.6.1, because triangular association scheme is not unique for $n = 8$ [35].

3.6_d NON-EXISTENCE OF CERTAIN SYMMETRICAL "GT" DESIGNS

From Shrikhande's [59], Connor and Clatworthy's [27] results we have the following theorem.

THEOREM 3.6.7 : A necessary condition for the existence of symmetrical regular "GT" designs is that $\theta_1^{\alpha_1} \theta_2^{\alpha_2} \theta_3^{\alpha_3}$ should be perfect square.

The following corollary is obvious.

COROLLARY 3.6.7.1 :

(i) when t is even; θ_3 should be a perfect square

(ii) t is odd, $n \equiv 1 \bmod 4$; θ_2 should be a perfect square

(iii) t is odd, $n \equiv 0 \bmod 4$; θ_1 should be perfect square

(iv) t is odd, $n \equiv 2 \bmod 4$; $\theta_1\theta_2$ should be perfect square.

The following designs violate the necessary condition of the above theorem 3.6.7 and hence are non-existing.

v = b	r = k	n	t	λ_1	λ_2	λ_3
12	7	4	2	5	4	3
18	9	4	3	5	4	4
30	10	5	3	3	4	3
45	12	6	3	6	4	2

Further necessary conditions for the existence of symmetrical regular "GT" designs may be obtained with the help of the Hasse - Minkowski invariant. Using Ogawas results [47] and (3.6.4) the $c_p(NN')$ can be evaluated; and on further simplification we get

$$(3.6.8) \quad c_p(NN') = (-1,-1)_p \prod_{i=1}^{3} (-1,\theta_i)_p^{\frac{\alpha_i(\alpha_i+1)}{2}} (\theta_1,\theta_2)_p^{\alpha_1\alpha_2} (\theta_1\theta_2, n-2)_p^{(n-1)t} (\theta_1\theta_2, n)_p^{t}$$

$$(t,\theta_1)_p^{v(n-1)} (t,\theta_2)_p^{\frac{vn(n-3)}{2}} (t,\theta_3)_p \; (\theta_2,u)_p^{t}$$

Hence

THEOREM 3.6.8: A necessary condition for the existence of a regular "GT" design is that

$$\prod_{i=1}^{3} (-1,\theta_i)_p^{\frac{\alpha_i(\alpha_i+1)}{2}} (\theta_1,\theta_2)_p^{\alpha_1\alpha_2} (\theta_1\theta_2, n-2)_p^{\alpha_1'} (\theta_1\theta_2,n)_p^{t} (t,\theta_1)_p^{u\alpha_1} (t,\theta_2)_p^{u\alpha_2} (t,\theta_3)_p (\theta_2,u)_p^{t} = 1$$

The following corollary can be deduced easily.

COROLLARY 3.6.8.1: Necessary conditions for the existence of regular symmetric "GT" designs are

(i) $t \equiv 2 \bmod 4$, $n \equiv 2 \bmod 4$; $(-1,\theta_1\theta_2)_p = 1$

(ii) $t \equiv 2 \bmod 4$, $n \equiv 0 \bmod 4$; $(-1,\theta_1)_p = 1$

(iii) t is odd, $n \equiv 1 \bmod 4$; $(t,\theta_3)_p \, (n,\theta_1)_p = 1$

(iv) t is odd, $n \equiv 3 \bmod 4$; $(-n,\theta_1)_p(\theta_2,un)_p(t,\theta_3)_p = 1$

(v) t is odd, $n \equiv 2 \bmod 4$; $(u,\theta_2)_p(\theta_1,-\theta_2)_p(t,\theta_3)_p = 1$

(vi) t is odd, $n \equiv 0 \bmod 4$; $(\theta_2, \frac{(n-2)(n-3)}{2})_p \, (t,\theta_3)_p = 1$

The following designs violate the above necessary condition and hence are non-existing.

v = b	r = k	n	t	λ_1	λ_2	λ_3
17	7	4	2	5	4	3
30	8	6	2	1	3	2
30	10	5	3	3	4	3

CHAPTER 4.

PARTIALLY BALANCED BLOCK DESIGNS WITH TWO DIFFERENT NUMBER OF REPLICATIONS*

4.1 INTRODUCTION

Shah [58] defined intra-inter group partially balanced designs and he gave the intra block analysis for the generalised designs. In this chapter we restrict our attention to designs of this type having two groups to achieve partial balance (as in the definition of partially balanced designs) with in the groups and balance (i.e. treatment differences are estimated with the same variance) between the groups. We shall call these designs as partially balanced block designs ("PBBD"). In section 4.2, we give the definitions and the relations of the parameters of the design. Analysis part is dealt in section 4.3. Some methods of construction are given in the last section.

4.2. DEFINITIONS AND RELATIONS

DEFINITION 4.2.1 : An incomplete block design with two different replicates is said to be "PBB" design if it satifies the following conditions.

(i) The experimental material is devided into b blocks of k plots each; different treatments being applied to the units in the same block.

(ii) There are v treatments divided into two groups of v_1 and v_2 treatments respectively; the treatment of i^{th} group occur in exactly $r_i (i=1,2)$ blocks.

(iii) There can be established relations of association between any two treatments in the $i^{th} (i=1,2)$ group satisfying the following requirements

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* See the reference [13]

(a) Two treatments are either 1st, 2nd, .., or m_i th associates

(b) Each treatment has exactly n_{ij} j^{th} associates $(j=1,2,..,m_i)$

(c) Given any two treatments which are q^{th} associates, the number of treatments common to the s^{th} associates of the first and t^{th} associates of the second is $p^q_{i.st}$ and is independent of the pair of treatments with which we start. Also

$$p^q_{i.st} = p^q_{i.ts}$$

(d) Two treatments which are j^{th} associates occur together in exactly λ_{ij} blocks.

(iv) Two treatments which are from different groups occur together λ times in blocks.

Now further define each treatment in the i^{th} group to be its own o^{th} associate and o^{th} associate of no other treatment. We may thus consistently write

(4.2.1) $\lambda_{io} = r_i$, $n_{io} = 1$, $p^o_{i.st} = \delta_{st}\, n_{is}$, $p^t_{i.os} = p^t_{i.so} = \delta_{st}$

Then the relations between the parameters are

$$bk = v_1 r_1 + v_2 r_2 \, , \qquad \sum_{j=o}^{m_i} n_{ij} = v_i \, ,$$

(4.2.2) $$\sum_{j=o}^{m_i} n_{ij}\,\lambda_{ij} = r_i k - v_{i'}\lambda \, , \qquad \sum_{t=o}^{m_i} p^q_{i.st} = n_{is} \quad \left(\begin{smallmatrix} i,i' = 1,2 \\ i \neq i' \end{smallmatrix}\right)$$

$$n_{is}\, p^s_{i.qt} = n_{iq}\, p^q_{i.st} = n_{it}\, p^t_{i.qs} \qquad (q,s,t = o,1,2,..,m_i)$$

Let $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ be the incidence matrix of the design where $N_i\,(i=1,2)$ is the incidence matrix of the i^{th} group treatments in b blocks satisfying the above conditions. Let B_{ij} be the association matrices (cf. Bose and Mesner [19], Shah [58]) of the design N_i ($j=1,2,\ldots,m_i$). Hence we have

(4.2.3) $$N_i N_i' = \sum_{j=o}^{m_i} \lambda_{ij} B_{ij} \quad \text{and}$$

(4.2.4) $$N\,N' = \begin{bmatrix} N_1N_1' & N_1N_2' \\ N_2N_1' & N_2N_2' \end{bmatrix}$$

(4.2.5) $$= \begin{bmatrix} \sum_{j=o}^{m_1} \lambda_{1j} B_{1j} & \lambda E_{v_1v_2} \\ \lambda E_{v_2v_1} & \sum_{j'=o}^{m_2} {}_{2j'} B_{2j'} \end{bmatrix}$$

When $\lambda = o$, the design is disconnected and hence we give the restriction $\lambda > o$. Obviously $\lambda_{ij} \leq r_i$. With little algebra and following from [19] we get

(4.2.6) $$\det.\ NN' = k^2(r_1r_2-b\lambda) \prod_{j=1}^{m_1} \theta_{1j}^{\alpha_{1j}} \prod_{j'=1}^{m_2} \theta_{2j'}^{\alpha_{2j'}}$$

where $\theta_{iu} = \sum_{j=o}^{m_i} \lambda_{ij}\ s_{i.uj}$ ($u=o,1,\ldots,m_i$) is the characteristic root NN' with the multiplicity α_{iu}, $\theta_{io} = r_ik - v_{i'}\lambda$ ($i,i'=1,2$ and $i \neq i'$)

and $s_{i.uj}^{\cdot s}$ are the distinct characteristic roots of B_{ij}. Since NN' is positive (or atleast semi-) definite, from the results of Connor and Clatworthy [27] $\theta_{iu}{}^s$ must not be negative; also $r_1r_2-b\lambda \geq o$. Hence

THEOREM 4.2.1 : A necessary condition for the existence of "PBB" design is that $\theta_{ij} \geq o$ and $r_1r_2-b\lambda \geq o$. ($j=1,2,\ldots,m_i$ and $i=1,2$).

THEOREM 4.2.2 : In a "PBB" design with $r_1r_2=b\lambda$, $N_i{}^s$ ($i=1,2$) are "PBIB" designs.

PROOF : Let e_{ij} be the number of treatments from the i^{th} group occuring in the j^{th} block of the design. Then

$$\sum_{j=1}^{b} e_{ij} = v_i r_i , \quad \sum_{j=1}^{b} e_{ij}(e_{ij}-1) = v_i \left[r_i(k-1) - v_{i'}\lambda \right] \quad (i,i'=1,2) \; i \neq i'$$

Define $$e_{i\cdot} = \sum_{j=1}^{b} e_{ij} / b = \frac{v_i r_i}{b}$$

Then $$\sum_{j=1}^{b} (e_{ij} - e_{i\cdot})^2 = \frac{v_1 v_2 (r_1 r_2 - b\lambda)}{b}$$
$$= 0 , \quad \text{since } b\lambda = r_1 r_2$$

Hence $e_{i1} = e_{i2} = \ldots = e_{ib} = \frac{v_i r_i}{b} = k_i$ (say) which is an integer. Thus it follows that N_i is a "PBIB" design with the block size k_i .

4.3 ANALYSIS

The analysis of these designs be obtained with the help of the association matrices [58]. We use the notations given in chapter II of Chakrabarti [25]. Assuming the usual model and denoting by $\underline{\tau}^*$ and $\underline{\tau}$ the vectors of the treatment effects with and without recovery of interblock information, we get the solutions of the normal equations as

(4.3.1) $$\hat{\underline{\tau}} = (C + \frac{\lambda}{K} E_{vv})^{-1} \underline{Q}$$

$$\hat{\underline{\tau}}^* = (wC + \frac{w'}{K} NN' + \frac{w-w'}{K} \lambda E_{vv})^{-1} \underline{P}$$

(4.3.2) $$C + \frac{\lambda}{K} E_{vv} = \begin{bmatrix} \sum_{j=0}^{m_1} c_{1j} B_{1j} & 0_{v_1 v_2} \\ 0_{v_2 v_1} & \sum_{j'=0}^{m_2} c_{2j'} B_{2j'} \end{bmatrix}$$

where $B_{io} = I_{v_i}$, $c_{io} = \frac{r_i(k-1)+\lambda}{k}$, $c_{ij} = \frac{\lambda-\lambda_{ij}}{k}$ $(j=1,2,\ldots,m_i;\ i=1,2)$

and

$$(4.3.3) \quad wC + \frac{w'}{k}NN' + \frac{w-w'}{k}\lambda E_{vv} = \begin{bmatrix} \sum_{j=o}^{m_1} c^*_{1j}B_{1j} & 0_{v_1v_2} \\ 0_{v_2v_1} & \sum_{j'=o}^{m_2} c^*_{2j'}B_{2j'} \end{bmatrix}$$

where $c^*_{io} = wr_i - \frac{w-w'}{k}(r_i-\lambda)$, $c^*_{ij} = \frac{w-w'}{k}(\lambda-\lambda_{ij})$ $(j=1,2,\ldots,m_i;\ i=1,2)$

Hence by the theorem 3.2 of [58] we have

$$(4.3.4) \quad \begin{bmatrix} \sum_{j=o}^{m_1} c_{1j}B_{1j} & 0_{v_1v_2} \\ 0_{v_2v_1} & \sum_{j'=o}^{m_2} c_{2j'}B_{2j'} \end{bmatrix}^{-1} = \begin{bmatrix} \sum_{j=o}^{m_1} d_{1j}B_{1j} & 0_{v_1v_2} \\ 0_{v_2v_1} & \sum_{j'=o}^{m_2} d_{2j'}B_{2j'} \end{bmatrix}$$

and

$$(4.3.5) \quad \begin{bmatrix} \sum_{j=o}^{m_1} c^*_{1j}B_{1j} & 0_{v_1v_2} \\ 0_{v_2v_1} & \sum_{j'=o}^{m_2} c^*_{2j'}B_{2j'} \end{bmatrix}^{-1} = \begin{bmatrix} \sum_{j=o}^{m_1} d^*_{1j}B_{1j} & 0_{v_1v_2} \\ 0_{v_2v_1} & \sum_{j'=o}^{m_2} d^*_{2j'}B_{2j'} \end{bmatrix}$$

where d_{ij}, d^*_{ij} s are obtained from their respective m_i+1 independent equations, namely,

$$(4.3.6) \quad \sum_{j=o}^{m_1} \sum_{j'=o}^{m_2} p^l_{i.jj'} \; c_{ij}d_{ij'} \begin{cases} = 1 & \text{if } l = o \\ = o & \text{if } l = 1,2,\ldots,m_i \end{cases}$$

and

$$(4.3.7) \quad \sum_{j=o}^{m_1} \sum_{j'=o}^{m_2} p^l_{i.jj'} \; c^*_{ij}d^*_{ij'} \begin{cases} = 1 & \text{if } l = o \\ = o & \text{if } l = 1,2,\ldots,m_i \end{cases}$$

(4.3.8) $wV(\hat{\tau}_s - \hat{\tau}_t) = 2(d_{io}-d_{ij})$ if the treatments s and t are from the i^{th} group and further if they are j^{th} associates $(j=1,2,\ldots,m_i;\ i=1,2)$

$= 2(d_{1o} + d_{2o})$ if they are from different groups.

(4.3.9) $V(\hat{\tau}_s^* - \hat{\tau}_t^*) = 2(d_{io}^* - d_{ij}^*)$ if the treatments s and t are from the i^{th} group and further if they are j^{th} associates $(j=1,2,..,m_i)$ $i=1,2$

$= 2(d_{1o}^* + d_{2o}^*)$ if they are from different groups.

4.4 CONSTRUCTION

TYPE A DESIGNS : We achieve partial balance in the first group and balance in the second group.

CONSTRUCTION 4.4.1 : Let N_1^* be the incidence matrix of m associate "PBIB" design with the parameters

(4.4.1) $v_1^*, b^*, r_1^*, k_1^*, n_1^*, n_2^*, ..., n_m^*, \lambda_1^*, \lambda_2^*, .., \lambda_m^*$ and p_{jk}^{i*} $(i,j,k=1,2,..,m)$

Let N_2^* be the incidence matrix of a "BIB" design with the parameters v_2^*, b^*, r_2^*, k_2^* and λ^*. Further let

(4.4.2) $$N_2^* = \begin{pmatrix} \underline{n}_1 \\ \underline{n}_2 \\ \vdots \\ \underline{n}_{v_2^*} \end{pmatrix}$$ where $\underline{n}_l$ be the l^{th} row vector of N_2^* $l=1,2,..,v_2^*$

Let $$N_1 = \underbrace{N_1^* \; N_1^* \cdots N_1^*}_{v_2^* \text{ times}}$$ and

(4.4.3) $$N_2 = \begin{bmatrix} \underline{n}_1 & \underline{n}_2 & \cdot & \cdot & \underline{n}_{v_2^*} \\ \underline{n}_2 & \underline{n}_3 & \cdot & \cdot & \underline{n}_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \underline{n}_{v_2^*} & \underline{n}_1 & \cdot & \cdot & \underline{n}_{v_2^*-1} \end{bmatrix},$$ then $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$

is partially balanced block design with the parameters

(4.4.4) $v = v_1^* + v_2^*$, $b = v_2^* b^*$, $r_1 = v_2^* r_1^*$, $r_2 = v_2^* r_2^*$, $k = k_1^* + k_2^*$

$n_{1j} = n_j^*$, $\lambda_{1j} = \lambda_j^*$, $p_{1.jl}^{i} = p_{jl}^{i*}$ $(i,j,l=1,2,..,m)$; $\mu = v_2^* \lambda^*$ (be the number of times two treatments in the second group, together occur in the blocks), and $\lambda = r_1^* k_2^*$

PROOF : Obviously, we know that N_1 is a "PBIB" design having the association scheme as N_1^* and N_2 is a "BIB" design. Further

$$(4.4.5) \qquad N_2 N_1' = \begin{bmatrix} n_1 & n_2 & \cdot & \cdot & n_{v_2^*} \\ n_2 & n_3 & \cdot & \cdot & n_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n_{v_2^*} & n_1 & \cdot & \cdot & n_{v_2^*-1} \end{bmatrix} \begin{bmatrix} N_1^{*'} \\ N_1^{*'} \\ \cdot \\ N_1^{*'} \end{bmatrix}$$

$$(4.4.6) \qquad = k_2^* E_{v_2^* b^*} N_1^{*'} = r_1^* k_2^* E_{v_2^* v_1^*} = \lambda E_{v_2 v_1}$$

Example :

$$N_1^* = \begin{bmatrix} 0&0&1&1 \\ 0&1&0&1 \\ 1&0&1&0 \\ 1&1&0&0 \end{bmatrix}, \qquad N_2^* = \begin{bmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0 \\ 0&1&1&1&1&0&1&1&1&1&0&1&1&1&1&0 \\ 1&0&1&1&1&1&0&1&1&1&1&0&0&1&1&1 \\ 1&1&0&1&1&1&1&0&0&1&1&1&1&0&1&1 \\ 1&1&1&0&0&1&1&1&1&0&1&1&1&1&0&1 \end{bmatrix}$$

is "PBB" design of type A with the parameters $v = 8$, $b = 16$, $r_1 = 8$, $r_2 = 12$, $k = 5$, $n_{11} = 2$, $n_{12} = 1$, $\lambda_{11} = 4$, $\lambda_{12} = 0$, $p^1_{1.11} = 0$, $\mu = 8$, $\lambda = 6$.

COROLLARY 4.4.1.1 : When N_2^* is a randomised block design i.e. $N_2^* = E_{v_2^* b^*}$, $N = \begin{pmatrix} N_1^* \\ N_2^* \end{pmatrix}$ is "PBB" design with the parameters

$$(4.4.7) \qquad v = v_1^* + v_2^*,\ b = b^*,\ r_1 = r_1^*,\ r_2 = b^*,\ k = v_2^* + k_1^*,\ n_{1i} = n_i^*,\ \lambda_{1i} = \lambda_i^*$$

$$p^i_{1.jl} = p^{i*}_{jl} \ (i,j,l = 1,2,\ldots,m), \quad \mu = b^*, \ \lambda = r_1^*$$

TYPE B DESIGNS : We get partial balance in both the groups.

CONSTRUCTION 4.4.2 : Let N_1 and N_2 be the incidence matrices of two "PBIB" designs with the parameters

(4.4.8) $v_1', b_1', r_1', k_1', n_{1i}', \lambda_{1i}', p_{1.jl}^{i'}$ $(i,j,l = 1,2,\ldots, m_1)$

$v_2', b_2', r_2', k_2', n_{2i}', \lambda_{2i}', p_{2.j'l'}^{i''}$ $(i',j',l' = 1,2,\ldots, m_2)$

with the condition $v_1' + k_2' = v_2' + k_1'$. Then

$$N = \begin{bmatrix} N_1 & E_{v_1' b_2'} \\ E_{v_2' b_1'} & N_2 \end{bmatrix}$$

is "PBB" design with the parameters

(4.4.9) $v = v_1' + v_2'$, $b = b_1' + b_2'$, $r_1 = r_1' + b_2'$, $r_2 = r_2' + b_1'$, $k = v_1' + k_2'$

$n_{ij} = n_{ij}'$, $\lambda_{ij} = \lambda_{ij}' + b_{i'}'$, $p_{i.st}^{j} = p_{i.st}^{j'}$ $(j,s,t = 1,2,\ldots,m_i$ and $i,i' = 1,2)$, $\lambda = r_1' + r_2'$.
$i \neq i'$

The proof of this construction is obvious.

Example:

$$N_1 = \begin{bmatrix} 0&0&1&1&0 \\ 0&0&0&1&1 \\ 1&0&0&0&1 \\ 1&1&0&0&0 \\ 0&1&1&0&0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0&0&1&1 \\ 1&1&0&0 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&0&1 \\ 0&1&1&0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0&0&1&1&0&1&1&1&1 \\ 0&0&0&1&1&1&1&1&1 \\ 1&0&0&0&1&1&1&1&1 \\ 1&1&0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1&1&1 \\ 1&1&1&1&1&0&0&1&1 \\ 1&1&1&1&1&1&1&0&0 \\ 1&1&1&1&1&1&0&1&0 \\ 1&1&1&1&1&0&1&0&1 \\ 1&1&1&1&1&1&0&0&1 \\ 1&1&1&1&1&0&1&1&0 \end{bmatrix}$$

the parameters are

$v = 11$, $b = 9$, $r_1 = 6$, $r_2 = 7$, $k = 8$, $n_{11} = 2$

$n_{12} = 2$, $\lambda_{11} = 5$, $\lambda_{12} = 4$, $p_{1.11}^{1} = 0$, $n_{21} = 1$,

$n_{22} = 4$, $\lambda_{21} = 5$, $\lambda_{22} = 6$, $p_{2.11}^{1} = 0$, $\lambda = 4$.

COROLLARY 4.4.2.1 : If either N_1 or N_2 is "BIB" design, then N is "PBB" design of type A.

CONSTRUCTION 4.4.3 : Let N_1 and N_2 be the incidence matrices of two "PBIB" designs with the parameters as in (4.4.8) with the condition $v_1' = 2k_1'$ or $v_2' = 2k_2'$ and $b_1' = b_2' = b'$. Then

$$(4.4.10) \qquad N = \begin{bmatrix} N_1 & E_{v_1'b'} - N_1 \\ N_2 & N_2 \end{bmatrix} \quad \text{or} \quad N = \begin{bmatrix} N_1 & N_1 \\ N_2 & E_{v_2'b'} - N_2 \end{bmatrix}$$

is "PBB" design with the parameters

$$(4.4.11) \quad v = v_1' + v_2',\ b = 2b',\ r_1 = b' \text{ or } 2r_1,\ r_2 = 2r_2' \text{ or } b',\ k = k_1' + k_2'$$

$$n_{ij} = n_{ij}',\ \lambda_{ij} = 2\lambda_{ij}',\ p^j_{i.st} = p'^j_{i.st}\ (j,s,t = 1,2,\ldots,m_i \text{ and } i = 1,2),\ \lambda = r_2' \text{ or } r_1'$$

example :

$$N_1 = N_2 = \begin{bmatrix} 0&0&1&1 \\ 0&1&0&1 \\ 1&0&1&0 \\ 1&1&0&0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0&0&1&1&0&0&1&1 \\ 0&1&0&1&0&1&0&1 \\ 1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&1&0&0 \\ 0&0&1&1&1&1&0&0 \\ 0&1&0&1&1&0&1&0 \\ 1&0&1&0&0&1&0&1 \\ 1&1&0&0&0&0&1&1 \end{bmatrix}$$

the parameters are

$v = 8 = b$, $r_1 = r_2 = 4$, $k = 4$, $n_{11} = n_{21} = 2$,
$n_{12} = n_{22} = 1$, $\lambda_{11} = \lambda_{21} = 2$, $\lambda_{12} = \lambda_{22} = 0$,
$p^1_{1.11} = p^1_{2.11} = 0$, $\lambda = 2$.

COROLLARY 4.4.3.1 : If one of the N_i's $(i=1,2)$ is a "BIB" design in the construction 4.4.3, then N is "PBB" design of type A.

CONSTRUCTION 4.4.4 : If N is "PBB" design, then $E_{vb} - N$ is also "PBB" design.

CONSTRUCTION 4.4.5 : Let $N_1 = \begin{pmatrix} N_{11} \\ N_{12} \end{pmatrix}$ and $N_2 = \begin{pmatrix} N_{21} \\ N_{22} \end{pmatrix}$ be two

"PBB" designs, such that either all N_{ij}'s (i,j=1,2) are "PBIB" designs or only N_{21} and N_{22} are "PBIB" designs and the block sizes of N_{21}, N_{22} are the same. Then

$$N = \begin{bmatrix} N_{11} \times N_{21} \\ N_{12} \times N_{22} \end{bmatrix} \quad \text{is "PBB" design .}$$

PROOF : Let λ_i be the number of times that any two treatments from different groups of N_i (i=1,2) occur together in the blocks of the design N_i . Consider

$$(4.4.12) \qquad NN' = \begin{bmatrix} N_{11} \times N_{21} \\ N_{12} \times N_{22} \end{bmatrix} \left(N_{11}' \times N_{21}' \quad N_{12}' \times N_{22}' \right)$$

$$(4.4.13) \qquad = \begin{bmatrix} N_{11}N_{11}' \times N_{21}N_{21}' & \lambda_1 \lambda_2 \; E_{v_1 v_2} \\ \lambda_1 \lambda_2 \; E_{v_2 v_1} & N_{12}N_{12}' \times N_{22}N_{22}' \end{bmatrix}$$

where v_i (i=1,2) is the number of treatments of the i^{th} group of N. Because of the properties of partially balanced block design and the Kronecker product of matrices [67] (see also chapter 4), we can have the diagonal matrices of (4.4.13) are of the form as in (4.2.8). The purpose of the condition that all N_{ij} (i,j=1,2) are "PBIB" designs or N_{21} and N_{22} having the same block sizes are "PBIB" designs is to attain constant block sizes for N.

COROLLARY 4.4.5.1 : Let $N_1 = \begin{pmatrix} N_{11} \\ N_{12} \end{pmatrix}$ is a "PBB" design and $N_2 = E_{v_2 b_2}$, then $N = N_1 \times E_{v_2 b_2}$ is also "PBB" design .

TYPE C DESIGNS : We restrict that one of the groups is having 2 associate classes.

CONSTRUCTION 4.4.6 : Let $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ be a "PBB" design such that N_2 is "BIB" design. Then $N = \begin{pmatrix} N_1 \\ N_2 \times E_{n1} \end{pmatrix}$ is "PBB" design of type c .

The proof of this is similar to the proof of the construction 4.4.5.

CONSTRUCTION 4.4.7 : Let N_1^* be a t associate "PBIB" design and N_2^* be a semi - regular group divisible design [17] with the parameters

$$(4.4.14) \qquad \begin{array}{l} v_1^*,\ b^*,\ r_1^*,\ k_1^*,\ n_i^*,\ \lambda_i^*,\ p_{jl}^{i*}\ (i,j,l=1,2,\dots,t) \\ v_2^*,\ b^*,\ r_2^*,\ k_2^*,\ m^*,\ n^*,\ \lambda_1^{**},\ \lambda_2^{**} \end{array}$$

respectively.

Let $N_1 = \underbrace{N_1^*\ N_1^* \cdot N_1^*}_{n^* \text{ times}}$ and let

$$N_2^* = \begin{bmatrix} N_{21}^* \\ N_{22}^* \\ \cdot \\ \cdot \\ N_{2m^*}^* \end{bmatrix} \quad \text{further} \quad N_{2i}^* = \begin{bmatrix} \underline{n}_{1i} \\ \underline{n}_{2i} \\ \cdot \\ \cdot \\ \underline{n}_{n^*i} \end{bmatrix}, \quad \begin{array}{l} i = 1,2,\dots, m^* \\ j = 1,2,\dots, n^* \end{array}$$

where $\underline{n}_{ji}$ is the j^{th} row vector of i^{th} group of N_2^*. Let

$$N_{2i} = \begin{bmatrix} \underline{n}_{1i} & \underline{n}_{2i} & \cdot\ \cdot & \underline{n}_{n^*i} \\ \underline{n}_{2i} & \underline{n}_{3i} & \cdot\ \cdot & \underline{n}_{1i} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \underline{n}_{n^*i} & \underline{n}_{1i} & \cdot\ \cdot & \underline{n}_{n^*-1\,i} \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} N_{21} \\ N_{22} \\ \cdot \\ \cdot \\ N_{2m^*} \end{bmatrix}$$

Then $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ is "PBB" design with the parameters

$$(4.4.15) \quad v = v_1^* + v_2^*,\ b = n^*b^*,\ r_1 = n^*r_1^*,\ r_2 = n^*r_2^*,\ k = k_1^* + k_2^*,\ n_{1j} = n_j^*,$$

$$\lambda_{1j} = n^*\lambda_j^*,\ p_{1.sl}^{j} = p_{1.sl}^{j*}\ (j,s,l = 1,2,\dots,t);\ n_{21} = n^* - 1,$$

$$n_{22} = (m^*-1)n^*,\ \lambda_{21} = n^*\lambda_1^{**},\ \lambda_{22} = n^*\lambda_2^{**},\ p_{2.11}^{1} = n^*-2,\ \lambda = c^*r_1^*$$

where $c^* = k_2^* / m^*$

PROOF : We know that N_1 is t associate "PBIB" design having the association scheme as N_1^* and N_2 is semi-regular group divisible design.

Further

$$(4.4.16) \qquad N_2 N_1' = \begin{bmatrix} N_{21} N_1' \\ N_{22} N_1' \\ \cdot \\ \cdot \\ N_{2m^*} N_1' \end{bmatrix} \qquad \text{where}$$

$$(4.4.17) \qquad N_{2i} N_1' = (\underline{n}_{1i} + \underline{n}_{2i} + \cdots + \underline{n}_{n^*i}) N_1' \qquad (i=1,2,\ldots,m^*)$$

Since N_2^* is semi-regular group divisible design, from [17] we have that $k_2^* = c^* m^*$ and every block of it contains c^* treatments from each group.

Hence $\qquad N_{2i} N_1' = c^* r_1^* E_{n^* v_1^*}$ $\qquad$ which gives

$$N_2 N_1' = c^* r_1^* E_{v_2^* v_1^*}$$

CONSTRUCTION 4.4.8 : Let N_1^* be a t associate "PBIB" design with the parameters given as in (4.4.14). Let N_2^* be a 'L_2' type "PBIB" design with the parameters

(4.4.18) $v_2^* = s^{*2}$, b_2^*, r_2^*, k_2^*, $n_{21}^* = 2(s^*-1)$, $n_{22}^* = (s^*-1)^2$, $\lambda_{21}^*, \lambda_{22}^*$, $P_{2.11}^{1*} = s^*-2$

such that $r_2^* k_2^* - v_2^* \lambda_{21}^* = s^*(r_2^* - \lambda_{21}^*)$. Let $N_2^* = \begin{bmatrix} N_{21}^* \\ N_{22}^* \\ \cdot \\ N_{2s^*}^* \end{bmatrix}$ where

$N_{2i}^* N_{2i}^{*'} = (r_2^* - \lambda_{21}^*) I_{s^*} + \lambda_{21}^* E_{s^* s^*}$ and $N_{2i}^* N_{2j}^{*'} = (\lambda_{21}^* - \lambda_{22}^*) I_{s^*} + \lambda_{22}^* E_{s^* s^*}$

$i,j = 1,2,\ldots,s^*$ and $i \neq j$.

Let $\qquad N_{2i}^* = \begin{bmatrix} \underline{n}_{1i} \\ \underline{n}_{2i} \\ \cdot \\ \cdot \\ \underline{n}_{s^*i} \end{bmatrix}$ where $\underline{n}_{1i}$ is the 1^{th} row vector of N_{2i}^*

Let

$$N_{2i} = \begin{bmatrix} n_{1i} & n_{2i} & \cdot & \cdot & n_{s^*i} \\ n_{2i} & n_{3i} & \cdot & \cdot & n_{1i} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n_{s^*i} & n_{1i} & \cdot & \cdot & n_{s^*-1i} \end{bmatrix}$$

and

$$N_1 = \underbrace{(N_1^* \, N_1^* \cdot \cdot \, N_1^*)}_{s^* \text{times}}, \quad N_2 = \begin{bmatrix} N_{21} \\ N_{22} \\ \cdot \\ \cdot \\ N_{2s^*} \end{bmatrix} ; \text{ then } N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

is "PBB" design of type "C" with the parameters

(4.4.19) $v = v_1^* + v_2^*$, $b = s^* b^*$, $r_1 = s^* r_1^*$, $r_2 = s^* r_2^*$, $k = k_1^* + k_2^*$, $n_{1j} = n_j^*$,

$\lambda_{1j} = \lambda_j^*$, $p_{1.ql}^j = p_{1.ql}^{j*}$ $(j,q,l=1,2,\ldots,t)$, $n_{21} = 2(s^*-1)$, $n_{22} = (s^*-1)^2$,

$\lambda_{21} = s\lambda_{21}^*$, $\lambda_{22} = s^*\lambda_{22}^*$, $p_{2.11}^1 = s^*-2$ and $\lambda = c^* r_1^*$ where $c^* = k_2^*/s^*$.

PROOF : Since N_2^* is "L_2" type "PBIB" design with the condition $r_2^* k_2^* - v_2^* \lambda_{21}^* = s^*(r_2^* - \lambda_{21}^*)$, we have from [54] that $k_2^* = c^* s^*$ and the number of ones in every column of N_{2i}^* $(i=1,2,\ldots,s^*)$ is c^*.

Hence $N_2 N_1' = c^* r_1^* E_{v_2 v_1}$. Also we know that N_1 is t associate "PBIB" design having the association scheme as N_1^* and it is seen that N_2 is "L_2" type "PBIB" design. N is "PBB" design with the parameters given in (4.4.19).

CONSTRUCTION 4.4.9 : Let N_1^* be t associate "PBIB" design with the parameters as in (4.4.14). Let N_2^* be a 2 associate cyclic "PBIB" design with the parameters

(4.4.20) v_2^*, b^*, r_2^* , k_2^* , n_{21}^* , n_{22}^* , λ_{21}^* λ_{22}^* $p_{2.11}^{1*}$

Let

$$N_1 = \underbrace{N_1^* \, N_1^* \ldots \, N_1^*}_{v_2^* \text{ times}} \quad \text{and} \quad N_2^* = \begin{bmatrix} n_1 \\ n_2 \\ \cdot \\ \cdot \\ n_{v_2^*} \end{bmatrix}$$

where n_l is the l^{th} row vector of N_2^* ($l=1,2,\ldots,v_2^*$), also let

$$N_2 = \begin{bmatrix} n_1 & n_2 & \cdot & \cdot & n_{v_2^*} \\ n_2 & n_3 & \cdot & \cdot & n_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n_{v_2^*} & n_1 & \cdot & \cdot & n_{v_2^*-1} \end{bmatrix}, \text{ then } N = \binom{N_1}{N_2} \text{ is}$$

"PBB" design of type C with the parameters

(4.4.21) $v = v_1^* + v_2^*$, $b = v_2^* b^*$, $r_1 = v_2^* r_1^*$, $r_2 = v_2^* r_2^*$, $k = k_1^* + k_2^*$,

$n_{1j} = n_j^*$, $\lambda_{1j} = v_2^* \lambda_{1j}^*$, $p_{1.ql}^j = p_{1.ql}^{j*}$ $(j,q,l = 1,2,\ldots,t)$, $n_{21} = n_{21}^*$, $n_{22} = n_{22}^*$

$\lambda_{21} = v_2^* \lambda_{21}^*$, $\lambda_{22} = v_2^* \lambda_{22}^*$, $p_{2.11}^1 = p_{2.11}^{1*}$, $\lambda = r_1^* k_2^*$.

PROOF : Easily we can see that N_1 is "PBIB" design having the association scheme as N_1^* . N_2 is "PBIB" design having the association scheme as N_2^* . The second one is due to the property of cyclic "PBIB" designs [21]. Also

$$\begin{aligned} N_2 N_1' &= (n_1 + n_2 + \cdots + n_{v_2^*}) N_1' \\ &= r_1^* k_2^* E_{v_2^* v_1^*} \end{aligned}$$

CHAPTER 5

PARTIALLY BALANCED DESIGNS WITH UNEQUAL BLOCK SIGES

5.1. INTRODUCTION

This chapter deals with the analysis and constructions of partially balanced designs with unequal block sizes. Section 5.2 gives the necessary and sufficient condition for a connected design to be partially balanced and also the analysis of the design. The analysis of the design is obtained on the assumption of equal intra block error variance for the blocks of different sizes. In section 5.3, we give some methods of construction of these designs with the known incomplete block designs. In the last section we study equi-replicate binary two associate partially balanced designs with two unequal block sizes and some of their applications.

5.2 PARTIALLY BALANCED DESIGNS AND THEIR ANALYSIS

Let there be v treatments and b blocks of $k_1, k_2, \ldots, k_b$ plots respectively and let the i^{th} treatment be replicated r_i times $(i = 1,2,\ldots,v)$. Following Chakrabarti [25] we have that the normal equations for estimating the vector of treatment constants $\underline{\tau}$ can be written as

(5.2.1) $$\underline{Q} = C\,\hat{\underline{\tau}}$$

where

(5.2.2) $$\underline{Q} = \underline{T} - N\,\mathrm{diag}(k_1^{-1}, k_2^{-1}, \ldots, k_b^{-1})\,\underline{B}$$
$$C = \mathrm{diag}(r_1, r_2, \ldots, r_v) - N\,\mathrm{diag}(k_1^{-1}, k_2^{-1}, \ldots, k_b^{-1})\,N'$$

The matrix C defined above is called C matrix of the design. If the design is connected, $C + aE_{vv}$ is non-singular where a is some non-zero real number.

(5.2.3) $$\hat{\underline{\tau}} = (C + aE_{vv})^{-1}\,\underline{Q}$$

is a solution of the equation $\underline{Q} = C\,\hat{\underline{\tau}}$ [58].

DEFINITION 5.2.1 : A connected design is said to be partially balanced, if it satisfies the following conditions

(i) There can be established relations of association between any two treatments satisfying the following requirements

(a) Two treatments are either 1st, 2nd, .. , or m^{th} associates

(b) Each treatment has exactly n_i i^{th} associates $(i = 1,2,..,m)$

(c) Given any two treatments which are i^{th} associates, the number of treatments which are common to the j^{th} associates of the first and the k^{th} associates of the second is p^i_{jk} and is independent of the pair of treatments with which we start.

Also $p^i_{jk} = p^i_{kj}$ $(i,j,k = 1,2,..., m)$

(ii) All treatments are estimated with the same variance, say $u_o\sigma^2$, and with m different covariances, say $u_1\sigma^2$, $u_2\sigma^2$, ..., $u_m\sigma^2$. The covariance of i^{th} and j^{th} estimated treatments is $u_k\sigma^2$, if they are k^{th} associates $(i,j = 1,2,..., v$ and $k = 1,2,..., m)$.

Now define each treatment to be its own o^{th} associate and the o^{th} associate of no other treatment. We may thus consistently write

$$(5.2.4) \quad n_o = 1, \quad p^o_{st} = \delta_{st}\, n_s, \quad p^t_{os} = p^t_{so} = \delta_{st} \qquad (s,t = o,1,2,...,m)$$

Then the relations between the parameters are

$$(5.2.5) \qquad \sum_{i=o}^{m} n_i = v \quad , \quad \sum_{k=o}^{m} p^i_{jk} = n_i$$

$$n_i\, p^i_{jk} = n_j\, p^j_{ik} = n_k\, p^k_{ij} \qquad (i,j,k = o,1,2,..., m)$$

Further we define the association matrices B_o, B_1, ..., B_m [19].

$$(5.2.6) \qquad B_i = ((b^{\beta}_{\alpha i})) = \begin{bmatrix} b^1_{1i} & b^2_{1i} & \cdot & \cdot & b^v_{1i} \\ b^1_{2i} & b^2_{2i} & \cdot & \cdot & b^v_{2i} \\ \cdot & \cdot & & & \cdot \\ \vdots & \vdots & & & \vdots \\ b^1_{vi} & b^2_{vi} & & & b^v_{vi} \end{bmatrix}$$

where $b^{\beta}_{\alpha i} = 1$ if the treatments α and β are i^{th} associates

$= o$ otherwise.

B_i is the symmetric matrix in which each row total and column total is n_i $(i = o,1,2,\ldots, m)$. Evidently $B_o = I_v$. Among the numbers $b^{\beta}_{\alpha o}, b^{\beta}_{\alpha 1}, \ldots, b^{\beta}_{\alpha m}$ only one is unity i.e. $b^{\beta}_{\alpha i} = 1$, if α and β are i^{th} associates. Hence

$$(5.2.7) \qquad \sum_{i=o}^{m} B_i = E_{vv}$$

and also we can deduce that

$$(5.2.8) \qquad B_j B_k = B_k B_j = \sum_{l=o}^{m} p^l_{jk} B_l \quad \text{for all } j,k = o,1,\ldots, m.$$

From (5.2.3) we get

$$(5.2.9) \qquad V(\hat{\tau}) = \left[(C + aE_{vv})^{-1} - \frac{1}{av^2} E_{vv} \right] \sigma^2$$

Since the design is partially balanced, from the definition 5.2.1 we have that

$$(5.2.10) \qquad V(\hat{\tau}) = \left[\sum_{i=o}^{m} u_i B_i \right] \sigma^2$$

On equating right hand side of (5.2.9) and (5.2.10), we get

$$(5.2.11) \qquad (C + aE_{vv})^{-1} = \sum_{i=o}^{m} d_i B_i$$

where $d_k = u_k + \frac{1}{av^2}$ $(k = o,1,2,\ldots, m)$. Hence from the theorem 3.2 of [58], we can deduce that

$$(5.2.12) \qquad C = \sum_{i=o}^{m} c_i B_i$$

where c_i's $(i = o,1,\ldots, m)$ are real numbers obtained from (5.2.11) and satisfy the relation $\sum_{i=o}^{m} n_i c_i = o$. Thus, we have the following theorem.

THEOREM 5.2.1 : The necessary and sufficient condition for a connected design to be partially balanced is $C = \sum_{i=o}^{m} c_i B_i$ where B_i's are association matrices defined in (5.2.6), (5.2.7) and (5.2.8).

Hence, for a partially balanced design

$$(5.2.13) \qquad \hat{\underline{\tau}} = \left(\sum_{i=o}^{m} d_i B_i \right) \underline{Q}$$

With the little algebra it can be shown that the d_i's are the solutions of the equations

$$(5.2.14) \qquad \sum_{i=o}^{m} \sum_{j=o}^{m} p^{l}_{ij}\, c_i\, d_j = 1 - \frac{1}{v} \quad \text{if } l = o$$
$$= -\frac{1}{v} \quad \text{if } l = 1,2,\ldots, m .$$

where $c_o = r_l - \sum_{j=1}^{b} \frac{n^2_{lj}}{k_j}$ and it is same for all $l = 1,2,\ldots,v$

$$c_i = - \sum_{j=1}^{b} \frac{n_{lj}\, n_{l'j}}{k_j} \quad \text{where the treatments } l,l'$$

are i^{th} associates $(l,l' = 1,2,\ldots, v)$, $l \neq l'$, and $i = 1,2,\ldots, m$).

Since the (m+1) equations in (5.2.14) are not independent, any m of them can be taken with an additional convenient restriction like

$$\Sigma d_j = o \quad \text{or for some } j,\ d_j = o$$

Finally we have

(5.2.15) $\quad V(\hat{\tau}_s - \hat{\tau}_t) = 2(d_o - d_i)\sigma^2$ if the treatments s and t are i^{th} associates ($s,t = 1,2,\ldots,v$ and $i = 1,2,\ldots,m$)

The average variance is given by

(5.2.16) $$\frac{2}{v-1}\left[(v-1)d_o - \sum_{i=1}^{m} n_i d_i\right]\sigma^2$$

5.3 CONSTRUCTION OF PARTIALLY BALANCED DESIGNS WITH UNEQUAL BLOCK SIZES

In the previous literature of partially balanced designs with unequal block sizes, we have limitted number of designs which are confined to "SUB" arrangements (cf. Kishen [41] and Raghavarao [55]). Here we give some methods of construction with the known incomplete block designs.

CONSTRUCTION 5.3.1 : Let $N_1, N_2, \ldots, N_l$ be l incidence matrices of m associate partially balanced designs having the same association scheme. Then

(5.3.1) $$N = (N_1 N_2 \ldots N_l)$$

is also m associate partially balanced design with the same association scheme.

PROOF : Let $C_1, C_2, \ldots, C_l, C$ be the C - matrices of the designs $N_1, N_2, \ldots, N_l$ and N respectively. Let

(5.3.2) $$C_j = \sum_{i=0}^{m} c_{ij} B_i \qquad (j = 1,2,\ldots, l)$$

where B_i's are the association matrices of the designs $N_1, N_2, \ldots, N_l$.

Obviously

(5.3.3) $$C = C_1 + C_2 + \ldots + C_l$$

$$(5.3.3_a) \qquad C = \sum_{j=1}^{l} \sum_{i=o}^{m} c_{ij} B_i$$

$$= \sum_{i=o}^{m} \left(\sum_{j=1}^{l} c_{ij} \right) B_i$$

Let $\sum_{j=1}^{l} c_{ij} = c_i$. Hence

$$(5.3.4) \qquad C = \sum_{i=o}^{m} c_i B_i$$

and by the theorem 5.2.1, N is partially balanced design with the association scheme same as N_j $(j = 1,2,\ldots, l)$.

COROLLARY 5.3.1.1 : Let $N_1, N_2, \ldots, N_l$ be the incidence matrices of l , m-associate "PBIB" designs having the same association scheme Let N_j be having the parameters

$$(5.3.5) \qquad v, b_j, r_j, k_j, \lambda_{1j}, \lambda_{2j} \ldots, \lambda_{mj}, n_1, n_2, \ldots, n_m, ((p^i_{uw}))$$

$$(j = 1,2,\ldots, l \text{ and } i,u,w = 1,2,\ldots, m)$$

Then N is binary equi-replicate m associate partially balanced design having l different block sizes with the parameters

$$(5.3.6) \qquad v, b = \sum_{j=1}^{l} b_j, \quad r = \sum_{j=1}^{l} r_j, \; k_1, k_2, \ldots, k_l \; ; \; \lambda_1, \lambda_2, \ldots, \lambda_m \; ;$$

$n_1, n_2, \ldots, n_m$; $((p^i_{uw}))$ $(i,u,w = 1,2,\ldots, m)$ where $\lambda_i = \sum_{j=1}^{b} \lambda_{ij}$ $(i = 1,2,\ldots,m)$ be the number of times two treatments, which are i^{th} associates, occur together in the blocks.

COROLLARY 5.3.1.2 : Let N_1 be a balanced design and N_2 be a partially balanced where the number of treatments in both the designs is same. Then $N = (N_1 N_2)$ is a partially balanced design whose association scheme is same as N_2 .

The following are 3 examples based on the corollary 5.3.1.2 where the designs are non-equi-replicated partially balanced designs.

(i)

$$N = \begin{bmatrix} 1&1&1&1&1&1&1&1&0&0&0&0&0&0&1&1&0 \\ 1&1&0&0&0&0&0&0&1&1&1&1&0&0&0&1&1 \\ 0&0&1&1&0&0&0&0&1&1&1&1&1&0&0&0&1 \\ 0&0&0&0&1&1&0&0&1&1&1&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&1&0&1&1&0&0 \end{bmatrix}$$

In the above example 13 blocks are having sizes 2 each and the remaining 4 blocks have sizes 4 each. The first treatment is replicated 10 times and the remaining treatments are replicated 8 times each.

(ii)

$$N = \begin{bmatrix} 2&2&2&0&0&0&0&1&1 \\ 1&0&0&1&1&0&1&0&1 \\ 0&1&0&1&1&1&0&1&0 \\ 0&0&1&1&1&1&1&0&0 \end{bmatrix}$$

(iii)

$$N = \begin{bmatrix} 2&2&2&2&0&0&0&0&1&0&1&1&0 \\ 1&0&0&0&0&1&1&1&0&1&0&1&1 \\ 0&1&0&0&1&0&1&1&1&0&1&0&1 \\ 0&0&1&0&1&1&0&1&1&1&0&1&0 \\ 0&0&0&1&1&1&1&0&0&1&1&0&1 \end{bmatrix}$$

Examples (ii) and (iii) are ternary partially balanced designs. In (ii), the first 5 blocks have sizes, 3 each and the remaining 4 blocks have 2 plots each. The first treatment is replicated 8 times and for the remaining treatments, each is replicated 5 times. In (iii) the design has equal block sizes. The first treatment is replicated 11 times and the rest are having replications, 7 each.

When the design has constant block sizes (k), we know that

$$c_o = r_l - \sum_{j=1}^{b} n_{lj}^2 / k \qquad (l = 1,2,\ldots, v)$$

$$= \frac{kr_l - \sum_{j=1}^{b} n_{lj}^2}{k}$$

Further, when the design is binary

$$c_o = \frac{r_l(k-1)}{k} \qquad \text{i.e. } r_l \text{ is constant.}$$

Thus, in a binary partially balanced design with equal block sizes, all treatments are replicated with the same number of times .

CONSTRUCTION 5.3.2 : The Kronecker product of two equi-replicate partially balanced designs N_1 and N_2 ($N = N_1 \times N_2$) with s and t associate classes is an equi-replicate partially balanced design with atmost t+s+st associate classes.

(This construction is similar to the construction $4\cdot 2_a$ of [67]. The proof of this is given with the help of association matrices. For the properties of Kronecker product of matrices, see the section 2 of the 1st chapter)

PROOF : N_1 and N_2 are two equi-replicate partially balanced designs. Let their parameters be

$$(5.3.7) \qquad v_1,\ b_1,\ r_1,\ n_{1i_1},\ {}_1p^{i_1}_{j_1k_1} \qquad (i_1,\ j_1,\ k_1 = 1,2,\ldots,s)$$

$$v_2,\ b_2,\ r_2,\ n_{2i_2},\ {}_2p^{i_2}_{j_2k_2} \qquad (i_2,\ j_2,\ k_2 = 1,2,\ldots,t)$$

Let B_{1i_1} $(i_1 = 0,1,\ldots,s)$, B_{2i_2} $(i_2 = 0,1,2,\ldots,t)$ be the corresponding association matrices. Let

Let B_{1i_1} $(i_1 = o,1,\ldots,s)$, B_{2i_2} $(i_2 = o,1,2,\ldots,t)$ be the corresponding association matrices. Let

$$(5.3.8) \qquad C_1 = \sum_{i_1=o}^{s} c_{1i_1} B_{1i_1} \quad , \quad C_2 = \sum_{i_2=o}^{t} c_{2i_2} B_{2i_2}$$

be the respective C - matrices of N_1 and N_2. Let C be the C - matrix of N and it is given by

$$(5.3.9) \qquad C = r_2 C_1 \times I_{v_2} + r_1 I_{v_1} \times C_2 - C_1 \times C_2$$

$$= r_2 \Big(\sum_{i_1=o}^{s} c_{1i_1} B_{1i_1} \Big) \times I_{v_2} + r_1 I_{v_1} \times \Big(\sum_{i_2=o}^{t} c_{2i_2} B_{2i_2} \Big)$$

$$- \Big(\sum_{i_1=o}^{s} c_{1i_1} B_{1i_1} \Big) \times \Big(\sum_{i_2=o}^{t} c_{2i_2} B_{2i_2} \Big)$$

$$(5.3.10) \qquad = r_2 \sum_{i_1=o}^{s} c_{1i_1} (B_{1i_1} \times B_{2o}) + r_1 \sum_{i_2=o}^{t} c_{2i_2} (B_{1o} \times B_{2i_2})$$

$$- \sum_{i_1=o}^{s} \sum_{i_2=o}^{t} c_{1i_1} c_{2i_2} (B_{1i_1} \times B_{2i_2})$$

Hence

$$(5.3.11) \qquad C = \sum_{i_1=o}^{s} \sum_{i_2=o}^{t} c(i_1, i_2) B(i_1, i_2)$$

where

$$(5.3.12) \qquad B(i_1, i_2) = B_{1i_1} \times B_{2i_2} \quad (i_1 = o,1,\ldots,s;\ i_2 = o,1,\ldots,t)$$

$$c(o,o) = r_2 c_{1o} + r_1 c_{2o} - c_{1o} c_{2o}$$

$$c(i_1,o) = (r_2 - c_{2o}) c_{1i_1} \quad (i_1 = 1,2,\ldots,s)$$

$$c(o,i_2) \quad = \quad (r_1 - c_{1o})\, c_{2i_2} \qquad (i_2 = 1,2,\ldots,t)$$

$$c(i_1,i_2) \quad = \quad - c_{1i_1}\, c_{2i_2} \qquad (i_1=1,2,\ldots,s;\ i_2=1,2,\ldots,t)$$

Further we have

(5.3.13) $$B(o,o) \quad = \quad I_{v_1} \times I_{v_2} = I_v \qquad \text{where } v = v_1 v_2$$

$$\sum_{i_1=o}^{s} \sum_{i_2=o}^{t} B(i_1,i_2) \quad = \quad E_{vv}$$

(5.3.14) $$B(i_1,i_2)B(j_1,j_2) = B_{1i_1} \times B_{2i_2} \cdot\cdot B_{1j_1} \times B_{2j_2}$$

$$= B_{1i_1} B_{1j_1} \times B_{2i_2} B_{2j_2}$$

$$= B_{1j_1} B_{1i_1} \times B_{2j_2} B_{2i_2}$$

(5.3.15) $$B(i_1,i_2)B(j_1,j_2) = B_{1j_1} \times B_{2j_2} \cdot B_{1i_1} \times B_{2i_2}$$

(5.3.16) $$B(i_1,i_2)B(j_1,j_2) = B(j_1,j_2)\cdot B(i_1,\ i_2)$$

Also from (5.3.14)

(5.3.17) $$B(i_1,i_2)B(j_1,j_2) = \Big(\sum_{l_1=o}^{s} {}_1p^{l_1}_{i_1j_1}\ B_{1l_1}\Big) \times \Big(\sum_{l_2=o}^{t} {}_2p^{l_2}_{i_2j_2}\ B_{2l_2}\Big)$$

$$= \sum_{l_1=o}^{s} \sum_{l_2=o}^{t} {}_1p^{l_1}_{i_1j_1}\ {}_2p^{l_2}_{i_2j_2}\ (B_{1l_1} \times B_{2l_2})$$

Let

(5.3.18) $$p^{(l_1,\ l_2)}_{(i_1,i_2)(j_1,j_2)} = {}_1p^{l_1}_{i_1j_1} \cdot {}_2p^{l_2}_{i_2j_2}$$

Hence

(5.3.19) $$B(i_1,i_2)B(j_1,j_2) = \sum_{l_1=o}^{s} \sum_{l_2=o}^{t} p^{(l_1,l_2)}_{(i_1,i_2)(j_1,j_2)}\ B(l_1,l_2)$$

Thus, we get N as equi-replicate partially balanced design with atmost s+t+st associate classes having the parameters

(5.3.20) $v = v_1 v_2,\ b = b_1 b_2,\ r = r_1 r_2,\ n(i_1,i_2) = n_{1i_1}\, n_{2i_2}$ where

$n_{1o} = n_{2o} = 1$; $p^{(i_1,i_2)}_{(j_1,j_2)(k_1,k_2)}$ $\quad i_1,j_1,k_1 = o,1,2,\ldots,s$; $\quad i_2,j_2,k_2 = o,1,2,\ldots,t$

and $B(i_1,i_2)^s$ are the association matrices of the scheme .

5.4 EQUI-REPLICATE BINARY TWO ASSOCIATE PARTIALLY BALANCED DESIGNS WITH TWO UNEQUAL BLOCK SIZES.

Let N_1 and N_2 be two 2 associate "PBIB" designs having the same association scheme. Let the parameters of these designs be

(5.4.1) $v,\ b_1,\ r_1,\ k_1,\ \lambda_{11},\ \lambda_{21},\ n_1,\ n_2,\ p^1_{11}$

$v,\ b_2,\ r_2,\ k_2,\ \lambda_{12},\ \lambda_{22},\ n_1,\ n_2,\ p^1_{11}$

respectively. Then by the corollary 5.3.1.1 we have

(5.4.2) $N = (N_1\ N_2)$

is an equi-replicate binary two associate partially balanced with two unequal block sizes having the parameters

(5.4.3) $v,\ b = b_1 + b_2,\ r = r_1 + r_2,\ k_1,\ k_2,\ \lambda_1 = \lambda_{11} + \lambda_{12},\ \lambda_2 = \lambda_{21} + \lambda_{22}$

$n_1,\ n_2,\ p^1_{11}$.

Table No. 5.4.1 gives a list of these designs which can be constructed with the help of 2 associate "PBIB" design tables of Bose, Shrikhande and Clatworthy [24].

In agronomic experiments it is sometimes not agriculturally feasible to layout blocks of equal sizes. It was, therefore, Kishen [41] introduced symmetrically unequal block arrangements which share the property of complete balance (in the sense of constant λ i.e. any two treatments together occur λ times in the blocks), but which involve blocks of different sizes. The analysis of these designs is obtained on the assumption of equal intrablock error variance for blocks of different sizes. This assumption may be reasonable when the block sizes do not differ much. We hesitate to use these designs when the block sizes vary too much. In such situations we may prefer partially balanced designs, whose block sizes are not varying much, to "SUB" arrangements, however "SUB" arrangements give better efficiency. For example the design 129 of the table No. 5.4.1 with the parameters

$v = 15,\ b = 33,\ r = 10,\ k_1 = 10,\ k_2 = 4,\ n_1 = 4,\ n_2 = 10, \lambda = 3,\ p^1_{11} = 3$ has the

efficiency 0.84 and the design 135 of the table 5.4.1 with the parameters

$v = 15,\ b = 33,\ r = 10,\ k_1 = 5,\ k_2 = 4,\ n_1 = 2,\ n_2 = 12,\ \lambda_1 = 0,\ \lambda_2 = 3,\ p^1_{11} = 1$

has the efficiency 0.83. Here we prefer the second design to the first for the analysis purpose. Moreover, "SUB" arrangements are particular cases of partially balanced designs with unequal block sizes [55].

Also we can construct pairwise balanced (which are also partially balanced under section 5.4) designs and balanced designs with the help of "PBIB" designs. An arrangement v objects (called treatments) in b sets (called blocks) will be defined a pairwise balanced design of index λ and of type $(v, k_1, \ldots, k_l)$, if each block contains either $k_1, k_2, \ldots, k_l$ treatments which are all distinct and every pair of distinct treatments occurs in blocks of the design (cf. Bose and Shrikhande [23]). It is obvious that N defined in (5.4.2) is also pairwise balanced if $\lambda_1 = \lambda_2 (= \lambda)$ in (5.4.3) ; such a pairwise balanced design is "SUB" arrangement if one of the N_i^* (i=1,2) is simple "PBIB" design (cf. [21] and cf. Raghavarao [55]. Let us denote N which is partially balanced and pairwise balanced as "PBPB" design.

CONSTRUCTION 5.4.1 : Let N_1, N_2 be 2-associate "PBIB" designs having the same association scheme with the parameters given in (5.4.1). Let

$$\frac{\lambda_{11} - \lambda_{21}}{\lambda_{22} - \lambda_{12}} = \frac{l}{m} \tag{5.4.4}$$

where l and m are positive integers. Then

$$N = (N_1^* \; N_2^*) \tag{5.4.5}$$

where $N_1^* = (\underbrace{N_1 \; N_1 .. \; N_1}_{m \text{ times}})$, $N_2^* = (\underbrace{N_2 \; N_2 .. \; N_2}_{l \text{ times}})$

is a "PBPB" design of index $\lambda \; (= m\lambda_{11} + l\lambda_{12})$ and of type (v, k_1, k_2)

PROOF : Evidently we know that N_1^*, N_2^* are 2-associate "PBIB" designs with the same association scheme. Then N is equi-replicate binary partially balanced design with two unequal block sizes having the parameters.

$$v,\; b = mb_1 + lb_2,\; r = mr_1 + lr_2,\; k_1,\; k_2,\; \lambda_1 = m\lambda_{11} + l\lambda_{12}, \tag{5.4.6}$$

$$\lambda_2 = m\lambda_{21} + l\lambda_{22},\; n_1,\; n_2,\; p_{11}^1$$

By (5.4.4) we have that $\lambda_1 = \lambda_2$. Hence N is "PBPB" design. (When we want that r to be small, then we take l and m in the construction in such a way that they are prime to each other).

COROLLARY 5.4.1.1 : The design $E_{vb} - N$ where N is "PBPB" design with index λ and of the type (v, k_1, k_2) is again "PBPB" design with the index $\lambda^* = b - 2r + \lambda$ and of the type $(v, v-k_1, v-k_2)$.

DEFINITION 5.4.1 : The partially balanced design (N) defined in (5.4.2) is regular if NN' is non-singular.

*LEMMA 5.4.1 : If A is non-singular matrix of order v such that (i) $AE_{v1} = cE_{v1}$, (ii) $AA'E_{v1} = dE_{v1}$ where c and d are scalars, then then $A'E_{v1} = \frac{d}{c} E_{v1}$

The proof is obvions and is omitted.

THEOREM 5.4.1 : In a regular binary equi-replicate partially balanced design with different block sizes, $b > v$.

PROOF : We know that in a regular partially balanced design defined in corollary 5.3.1.1 with the parameters given (5.3.6) with $b = v$,

$$NE_{v1} = rE_{v1}$$

$$NN'E_{v1} = (r + \sum_{i=1}^{m} n_i \lambda_i) E_{v1}$$

Hence by the above lemma we get

$$N'E_{v1} = \frac{r + \sum_{i=1}^{m} n_i \lambda_i}{r} E_{v1}$$

which shows that all the block sizes are equal. Thus we have, in a regular binary equi-replicate partially balanced design with different block sizes, $b > v$.

COROLLARY 5.4.1.1 : In a "PBPB" design with different block sizes $b > v$.

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* See reference [9]

$(5.4)_a$ BALANCED DESIGNS WITH TWO UNEQUAL BLOCK SIZES

Let the C-matrix of the design N defined in (5.4.2) be

(5.4.7) $$C = c_0 I_v + c_1 B_1 + c_2 B_2$$

where

(5.4.8) $$c_0 = \sum_{i=1}^{2} \frac{r_i(k_i-1)}{k_i}, \quad c_1 = -\left(\frac{\lambda_{12}}{k_1} + \frac{\lambda_{12}}{k_2}\right), \quad c_2 = -\left(\frac{\lambda_{21}}{k_1} + \frac{\lambda_{22}}{k_2}\right)$$

For the design N to be balanced, we must have

$$\frac{\lambda_{11}}{k_1} + \frac{\lambda_{12}}{k_2} = \frac{\lambda_{21}}{k_1} + \frac{\lambda_{22}}{k_2}$$

(5.4.9) i.e. $$\frac{k_1}{k_2} = \frac{\lambda_{11} - \lambda_{21}}{\lambda_{22} - \lambda_{12}}$$

CONSTRUCTION 5.4.2 : Let N_1, N_2 be two 2-associate "PBIB" designs having the same association scheme with the parameters given in (5.4.1). Let

(5.4.10) $$\frac{k_1}{k_2} = \frac{p}{q} \quad \text{and} \quad \frac{\lambda_{11} - \lambda_{21}}{\lambda_{22} - \lambda_{12}} = \frac{l}{m}$$

where l and m, p and q are positive integers. Then

(5.4.11) $$N = (N_1^* \; N_2^*)$$

where $$N_1^* = \underbrace{(N_1 \; N_1 \, .. \, N_1)}_{pm \text{ times}}, \quad N_2^* = \underbrace{(N_2 \; N_2 .. \; N_2)}_{ql \text{ times}}$$

is an equi-replicate binary balanced design.

PROOF : It is evident that N_1^*, N_2^* are 2-associate "PBIB" designs having the same association scheme with the parameters

$$(5.4.12) \qquad v^* = v,\ b_1^* = pm\, b_1,\ r_1^* = pmr_1,\ k_1^* = k_1,\ \lambda_{11}^* = pm\,\lambda_{11}\ ,$$

$$\lambda_{21}^* = pm\,\lambda_{21},\ n_1^* = n_1,\ n_2^* = n_2,\ p_{11}^{1*} = p_{11}^1$$

$$v^* = v,\ b_2^* = ql\ b_2,\ r_2^* = ql\ r_2,\ k_2^* = k_2,\ \lambda_{12}^* = ql\,\lambda_{12}\ ,$$

$$\lambda_{22}^* = ql\,\lambda_{22},\ n_1^* = n_1,\ n_2^* = n_2,\ p_{11}^{1*} = p_{11}^1$$

Consequently

$$(5.4.13) \qquad \frac{\lambda_{11}^* - \lambda_{21}^*}{\lambda_{22}^* - \lambda_{12}^*} = \frac{pm\,(\lambda_{11} - \lambda_{21})}{ql\,(\lambda_{22} - \lambda_{12})} = \frac{p}{q} = \frac{k_1}{k_2} = \frac{k_1^*}{k_2^*}$$

Hence N is equi-replicate balanced design.

(When we want the number of replications to be small, then we take

$$N_1^* = (\underbrace{N_1N_1 \cdots N_1}_{u\ \text{times}}),\quad N_2^* = (\underbrace{N_2N_2 \cdots N_2}_{w\ \text{times}})$$

where $\frac{u}{w} = \frac{pm}{ql}$, u and w are positive integers prime to each other.)

COROLLARY 5.4.2.1 : The design $E_{vb} - N$, where N is defined in (5.4.11), is never a balanced design unless $k_1 = k_2$ in which case N is a "BIB" design.

*1 THEOREM 5.4.3 : When b = v, there will not exist an equi-replicate binary balanced design with different block sizes.

PROOF : When the connected design is balanced, C-matrix of the design can be written as

$$(5.4.14) \qquad C = \frac{vr-b}{v-1}\left(I_v - \frac{1}{v} E_{vv}\right)$$

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*1 See the reference [10]

(5.4.15) i.e. $N\,\mathrm{diag}(k_1^{-1}, k_2^{-1}, \ldots, k_b^{-1})\,N' = \frac{1}{v-1}\,[(v-r)I_v+(r-1)E_{vv}]$

Since N is non-singular, we have that

(5.4.16) $(N')^{-1}\,\mathrm{diag}(k_1, k_2,\ldots,k_b)\,N^{-1} = \frac{v-1}{v-r}\,[I_v - \frac{r-1}{(v-1)r}E_{vv}]$

Hence

(5.4.17) $\mathrm{Diag}(k_1, k_2,\ldots, k_b) = \frac{v-1}{v-r}\,[N'N - \frac{r-1}{(v-1)r}N'E_{vv}N]$

From (5.4.17) we get

(5.4.18) $k_j = \frac{v-1}{v-r}k_j - \frac{r-1}{(v-r)r}k_j^2 \quad (j=1,2,\ldots,v)$

(5.4.19) $0 = \frac{v-1}{v-r}\lambda_{ij} - \frac{r-1}{(v-r)r}k_ik_j \quad (i,j=1,2,\ldots,v;\ i\neq j)$

Equations (5.4.18) and (5.4.19) give that N is "BIB" design. Hence we have that in an equi-replicate binary balanced design with different block sizes $b > v$.

We give some equi-replicate ($r \leq 15$) binary balanced designs with 2 different block sizes using 2-associate "PBIB" design tables [24] in table No. 5.4.2.

TABLE 5.4.1

($r \leq 10$)

S.No.	Reference	v	b	r	k_1	k_2	λ_1	λ_2	n_1	n_2	P_{11}^1
1	$S_1 + SR_2$	6	11	6	4	3	2	3	1	4	0
2	$S_1 + SR_3$	6	15	8	4	3	2	4	1	4	0
3	$S_1 + SR_5$	6	19	10	4	3	2	5	1	4	0
4	$S_1 + R_1$	6	9	5	4	3	4	2	1	4	0
5	$S_2 + SR_1$	6	10	6	4	3	4	3	1	4	0
6	$S_2 + SR_3$	6	18	10	4	3	4	5	1	4	0

S.No.	Reference	v	b	r	k_1	k_2	λ_1	λ_2	n_1	n_2	p_{11}^1
7	S2 + R1	6	12	7	4	3	6	3	1	4	0
8	R2 + R3	6	18	10	4	3	6	4	2	3	1
9	S3 + SR1	6	13	8	4	3	6	4	1	4	0
10	S3 + SR2	6	17	10	4	3	6	5	1	4	0
11	S3 + R1	6	15	9	4	3	8	4	1	4	0
12	S4 + SR1	6	16	10	4	3	8	5	1	4	0
13	SR1+ R4	6	16	10	3	4	4	6	1	4	0
14	S6 + S7	8	10	6	4	6	6	3	1	6	0
15	S6 + S9	8	14	9	4	6	9	5	1	6	0
16	S7 + S8	8	16	9	6	4	9	4	1	6	0
17	S7 + SR7	8	12	7	6	4	3	4	1	6	0
18	S6 + R5	8	14	6	4	3	3	2	1	6	0
19	S7 + SR8	8	16	9	6	4	3	5	1	6	0
20	S9 + SR7	8	20	10	6	4	4	3	1	6	0
21	S9 + R5	8	16	9	6	3	6	5	1	6	0
22	SR7+ R5	8	16	7	4	3	0	3	1	6	0
23	SR8+ R5	8	20	9	4	3	0	4	1	6	0
24	S12+ SR13	9	21	8	6	3	2	3	2	6	1
25	S12+ R8	9	12	6	6	4	5	2	2	6	1
26	S12+ R9	9	18	7	6	3	4	2	2	6	1
27	S12+ R10	9	21	8	6	3	5	2	2	6	1
28	S13+ R8	9	15	8	6	4	7	3	2	6	1
29	S13+ R9	9	21	9	6	3	6	3	2	6	1
30	S14+ SR12	9	18	9	6	3	6	4	2	6	1
31	S13+ SR12	9	15	7	6	3	4	3	2	6	1

S.No.	Reference	v	b	r	k_1	k_2	λ_1	λ_2	n_1	n_2	p^1_{11}
32	S14+ R8	9	18	10	6	4	9	4	2	6	1
33	SR12+SR14	9	18	9	3	6	3	5	2	6	1
34	SR12+R8	9	18	7	3	4	3	2	2	6	1
35	SR14+R8	9	18	10	6	4	6	5	2	6	1
36	R8 +R9	9	24	9	4	3	5	2	2	6	1
37	R8 +R10	9	27	10	4	3	6	2	2	6	1
38	LS1 +LS3	9	27	10	4	3	2	4	4	4	1
39	LS1 +LS6	9	21	8	4	3	3	2	4	4	1
40	LS1 +LS8	9	15	8	4	6	4	4	4	4	1
41	LS1 +LS10	9	18	9	4	5	3	5	4	4	1
42	LS3 +LS8	9	24	10	3	6	4	4	4	4	1
43	LS6 +LS8	9	18	8	3	6	5	2	4	4	1
44	LS6 +LS10	9	21	9	3	5	4	3	4	4	1
45	LS8 +LS10	9	15	9	6	5	5	5	4	4	1
46	S17 +S18	10	15	8	4	8	8	4	1	8	0
47	S17 +S19	10	20	10	4	6	10	4	1	8	0
48	S18 +S19	10	15	10	8	6	10	6	1	8	0
49	S17 +SR16	10	18	8	4	5	4	3	1	8	0
50	S18 +SR16	10	13	8	8	5	4	5	1	8	0
51	S18 +SR17	10	17	10	8	5	4	6	1	8	0
52	S19 +SR16	10	18	10	6	5	6	5	1	8	0
53	T1 +T6	10	15	5	4	3	2	0	6	3	3
54	T1 +T7	10	25	8	4	3	3	0	6	3	3
55	T1 +T10	10	17	8	4	5	3	4	6	3	3
56	T1 +T15	10	10	5	4	6	3	1	6	3	3

S.No.	Reference	v	b	r	k_1	k_2	λ_1	λ_2	n_1	n_2	p^1_{11}
57	T1 + T16	10	15	8	4	6	5	2	6	3	3
58	T1 + T19	10	15	9	4	7	6	4	6	3	3
59	T2 + T6	10	20	7	4	3	2	0	6	3	3
60	T2 + T7	10	30	10	4	3	3	0	6	3	3
61	T2 + T9	10	16	7	4	5	3	2	6	3	3
62	T2 + T14	10	30	10	4	3	3	2	6	3	3
63	T2 + T15	10	15	7	4	6	4	1	6	3	3
64	T2 + T16	10	20	10	4	6	6	2	6	3	3
65	T2 + T18	10	20	10	4	6	5	4	6	3	3
66	T3 + T6	10	25	9	4	3	4	0	6	3	3
67	T3 + T9	10	21	9	4	5	4	2	6	3	3
68	T3 + T15	10	20	9	4	6	5	1	6	3	3
69	T6 + T10	10	22	9	3	5	3	4	6	3	3
70	T6 + T15	10	15	6	3	6	3	1	6	3	3
71	T6 + T16	10	26	9	3	6	5	2	6	3	3
72	T6 + T19	10	20	10	3	7	6	4	6	3	3
73	T7 + T9	10	26	9	3	5	3	2	6	3	3
74	T7 + T12	10	30	10	3	4	3	2	6	3	3
75	T7 + T15	10	25	9	3	6	4	1	6	3	3
76	T9 + T12	10	16	7	5	4	2	4	6	3	3
77	T9 + T14	10	26	9	5	3	2	4	6	3	3
78	T9 + T15	10	11	6	5	6	3	3	6	3	3
79	T9 + T16	10	16	9	5	6	5	4	6	3	3
80	T9 + T18	10	16	9	5	6	4	6	6	3	3
81	T9 + T19	10	16	10	5	7	6	6	6	3	3

S.No.	Reference	v	b	r	k_1	k_2	λ_1	λ_2	n_1	n_2	p^1_{11}
82	T10 + T12	10	22	10	5	4	3	6	6	3	3
83	T10 + T15	10	17	9	5	6	4	5	6	3	3
84	T12 + T14	10	30	10	4	3	2	4	6	3	3
85	T12 + T15	10	15	7	4	6	3	3	6	3	3
86	T12 + T16	10	20	10	4	6	5	4	6	3	3
87	T12 + T18	10	20	10	4	6	4	6	6	3	3
88	T14 + T15	10	25	9	3	6	3	3	6	3	3
89	T15 + T19	10	15	10	6	7	7	5	6	3	3
90	S23 + S24	12	10	6	6	9	6	3	2	9	1
91	S24 + S29	12	16	9	9	6	9	4	2	9	1
92	S23 + S31	12	14	9	6	9	9	5	2	9	1
93	S26 + S27	12	25	10	4	6	10	3	1	10	0
94	S26 + S28	12	21	10	4	10	10	5	1	10	0
95	S27 + S28	12	16	10	6	10	10	6	1	10	0
96	SR20+ SR26	12	18	9	4	8	3	5	1	10	0
97	SR21+SR25	12	28	10	3	6	2	4	3	8	2
98	R15 + R17	12	36	10	4	3	4	2	1	10	0
99	S22 +SR25	12	15	8	8	6	4	4	3	8	2
100	S22 +SR27	12	35	10	8	3	2	3	3	8	2
101	S22 +R18	12	31	9	8	3	4	2	3	8	2
102	S23 +SR20	12	15	6	6	4	3	2	1	10	0
103	S23 +SR26	12	15	9	6	8	6	5	1	10	0
104	S25 +SR25	12	18	10	8	6	6	5	3	8	2
105	S26 +SR22	12	23	9	4	6	5	3	1	10	0
106	S26 + R16	12	35	10	4	3	5	2	1	10	0

S.No.	Reference	v	b	r	k_1	k_2	λ_1	λ_2	n_1	n_2	p^1_{11}
107	S27 +SR22	12	22	9	6	4	7	3	1	10	0
108	S27 + R16	12	30	10	6	3	5	3	1	10	0
109	S28 + R15	12	18	9	10	4	7	5	1	10	0
110	S28 +SR22	12	14	9	10	6	5	6	1	10	0
111	SR22+ R15	12	20	8	6	4	2	3	1	10	0
112	SR22+ R16	12	28	9	6	3	0	3	1	10	0
113	SR24+ R15	12	24	10	6	4	2	4	1	10	0
114	S40 + S41	14	14	7	6	8	7	3	1	12	0
115	S40 + S42	14	28	9	6	4	9	2	1	12	0
116	S41 + S42	14	28	10	8	4	10	3	1	12	0
117	R24 + R25	14	42	10	4	3	0	2	1	12	0
118	S40 + SR33	14	19	9	6	7	3	4	1	12	0
119	S40 + R24	14	21	7	6	4	3	2	1	12	0
120	S40 + R25	14	35	9	6	3	3	2	1	12	0
121	S41 +SR33	14	19	10	8	7	4	5	1	12	0
122	S41 + R24	14	21	8	8	4	4	3	1	12	0
123	S41 + R25	14	35	10	8	3	4	3	1	12	0
124	S42 +SR32	14	29	10	4	7	6	3	1	12	0
125	SR32+ R24	14	22	8	7	4	0	3	1	12	0
126	SR33+ R24	14	26	10	7	4	0	4	1	12	0
127	S47 + S49	15	20	10	6	9	10	4	2	12	1
128	R27 + R28	15	45	10	4	3	0	2	2	12	1
129	S46 + R31	15	33	10	10	4	3	3	4	10	3
130	S47 +SR37	15	28	10	6	5	4	3	2	12	1
131	S47 + R27	15	25	8	6	4	4	2	2	12	1

S.No.	Reference	v	b	r	k_1	k_2	λ_1	λ_2	n_1	n_2	p^1_{11}
132	S47 + R28	15	40	10	6	3	4	2	2	12	1
133	S48 +SR36	15	31	9	10	3	4	3	4	10	3
134	S49 + R27	15	25	10	9	4	6	4	2	12	1
135	SR37+ R27	15	33	10	5	4	0	3	2	12	1
136	T20 + T23	15	26	6	5	3	2	0	8	6	4
137	T20 + T24	15	46	10	5	3	3	0	8	6	4
138	T20 + T25	15	12	6	5	10	4	2	8	6	4
139	T20 + T26	15	18	10	5	10	7	4	8	6	4
140	T20 + T29	15	36	8	5	3	1	2	8	6	4
141	T20 + T22	15	22	8	5	6	3	2	8	6	4
142	T21 + T23	15	32	8	5	3	3	0	8	6	4
143	T21 + T25	15	18	8	5	10	5	2	8	6	4
144	T21 + T28	15	27	7	5	3	2	1	8	6	4
145	T22 + T25	15	16	8	6	10	4	4	8	6	4
146	T22 + T27	15	20	10	6	9	4	6	8	6	4
147	T22 + T28	15	25	7	6	3	1	3	8	6	4
148	T22 + T29	15	40	10	6	3	1	4	8	6	4
149	T23 + T25	15	26	8	3	10	4	2	8	6	4
150	T25 + T27	15	16	10	10	9	6	6	8	6	4
151	T25 + T29	15	36	10	10	3	3	4	8	6	4
152	T27 + T28	15	25	9	9	3	3	5	8	6	4
153	S54 +SR40	16	22	7	8	4	3	2	3	12	2
154	S54 + R35	16	38	9	8	3	3	2	3	12	2
155	S54 + R36	16	30	9	8	4	5	2	3	12	2
156	S55 +SR40	16	28	10	8	4	6	3	3	12	2

S.No.	Reference	v	b	r	k_1	k_2	λ_1	λ_2	n_1	n_2	p_{11}^1
157	SR40+ R35	16	48	10	4	3	0	2	3	12	2
158	S60 + S61	18	25	10	6	9	10	3	2	15	1
159	S68 +SR52	20	31	8	10	4	3	2	4	15	3
160	S69 +SR51	20	26	8	8	5	4	2	3	16	2
161	S77 + S78	21	28	9	9	6	9	2	2	18	1
162	S84 + R45	24	39	10	8	5	5	2	3	20	2
163	S87 +SR64	25	35	9	10	5	4	2	4	20	3
164	S96 +SR70	30	40	10	10	6	5	2	4	25	3

TABLE 5.4.2

S.No.	Reference	v	b	r	k_1	k_2	e
1	S*2 + SR3	6	18	8	2	3	.75
2	R*4 + S2	6	18	8	2	4	.75
3	SR3 + S4	6	24	14	3	4	.86
4	S*7 + SR7	8	12	5	2	4	.80
5	R5 + S9	8	16	9	3	6	.89
6	S*9 + SR10	8	24	10	2	4	.80
7	S*9 + R6	8	32	11	2	3	.73
8	SR7 + S11	8	20	13	4	6	.92
9	R6 + S8	8	36	15	3	4	.80
10	SR12+ S13	9	15	7	3	6	.86
11	R11 + S13	9	27	11	3	6	.82
12	S*12+SR*13	9	21	13	3	6	.92

S.No.	Reference	v	b	r	k_1	k_2	e
13	LS9 + LS3	9	30	14	3	6	.86
14	S*21+SR19	10	30	12	2	5	.83
15	T14 + T16	10	30	12	3	6	.83
16	S*18+R*14	10	25	13	2	6	.89
17	S*28+SR24	12	18	7	2	6	.86
18	R17 +SR22	12	32	10	3	6	.80
19	R22 + S23	12	46	13	3	6	.77
20	R23 + S25	12	36	14	4	8	.86
21	R24 + S41	14	21	8	4	8	.88
22	R25 + S40	14	35	9	3	6	.77
23	R28 + S49	15	40	12	3	9	.83
24	SR40+ S54	16	22	7	4	8	.86
25	R39 + S54	16	42	12	4	8	.83
26	SR42+ S55	16	44	14	4	8	.86
27	SR46+ S61	18	28	11	6	9	.91
28	R41 +SR47	18	66	15	3	9	.80

In the table e is the efficiency of the design and N^* implies the complement of N.

CHAPTER 6

DESIGNS FOR TWO-WAY ELIMINATION OF HETEROGENEITY

6.1 INTRODUCTION

Some times in a design the position within the block is important as a source of variation, and the experiment gains in efficiency by eliminating the positional effect. The classical example is due to Youden in his studies on the tobacco mosaic virus. Different types of designs for two-way elimination of heterogeneity, i.e. Latin squares, Youden squares and other extended designs, have been studied. For these designs, the row-column incidence matrices are complete; i.e. information is available in all the row-column cells. Pottoff [49] studied some designs where the row-column incidence matrices are incomplete. In this chapter we shall study some designs which possess the properties of orthogonality and balancing. The row-column, treatment-row and treatment-column incidence matrices of our designs need not be complete. In section 6.2 we give the preliminaries and the analysis of general two-way design. In section 6.3, using the concepts of balancing and orthogonality we give the classification of two-way designs. Section 4 presents the study of some of these classes in detail where the incidence matrices (i.e. row-column, treatment-row and treatment-column incidence matrices) obtained are binary designs. Section 6.5 deals with some constructions for some particular classes. A measure of non-orthogonality is given in section 6.6. Finally at the end of the chapter, an appendix of 60 designs, which give incomple row-column incidence matrices, is added.

6.2 PRELIMINARIES AND THE ANALYSIS OF TWO-WAY DESIGN

Consider a two-way design (i.e. a design for two-way elemination of heterogeneity) with u rows and w columns. Let there be v treatments; the i^{th} treatment being replicated r_i times. Let l_{ij} denote the number of times the i^{th} treatment occurs in the j^{th} row, $m_{ij'}$ denote the number of times i^{th} treatment occurs in the j'^{th} column and $n_{jj'}$ be the number of plots in the $(j,j')^{th}$ cell. $i = 1,2,\ldots,v$; $j = 1,2,\ldots,u$; $j' = 1,2,\ldots, w$. Let

$$(6.2.1) \qquad L = ((l_{ij})), \quad M = ((m_{ij'})), \quad N = ((n_{jj'})) \quad \text{and}$$

$$\sum_{i=1}^{v} l_{ij} = s_j, \quad \sum_{i=1}^{v} m_{ij'} = t_{j'}, \quad \sum_{j=1}^{u} l_{ij} = r_i \quad \text{and} \quad \sum_{i}^{v}\sum_{j}^{u} l_{ij} = n$$

We call L,M,N as treatment-rwo, treatment-column,row-column incidence matrices respectively. Let $\underline{R} = \{R_1, R_2, \ldots, R_u\}$, $\underline{C} = \{C_1, C_2, \ldots, C_w\}$, $\underline{T} = \{T_1, T_2, \ldots, T_v\}$ denote the column vectors of row totals, column totals, treatment totals. Let $G = \underline{R}' E_{u1} = \underline{C}' E_{w1} = \underline{T}' E_{v1}$ be the grand total. Let the expectation of the yield of the plot in the j^{th} row and j'^{th} column having treatment i be $\mu + \alpha_j + \beta_{j'} + \tau_i$ where α_j = the effect of j^{th} row, $\beta_{j'}$ = the effect of j'^{th} column and τ_i = the effect of i^{th} treatment. Let $\underline{\alpha}, \underline{\beta}, \underline{\tau}$ be the column vectors of α_j s, $\beta_{j'}$ s, τ_i s respectively. The normal equations are

$$(6.2.2) \quad \begin{bmatrix} G \\ \underline{R} \\ \underline{C} \\ \underline{T} \end{bmatrix} = \begin{bmatrix} nI_1 & E_{1u}\mathrm{diag}(s_1,s_2,\ldots,s_u) & E_{1w}\mathrm{diag}(t_1,t_2,\ldots,t_w) & E_{1v}\mathrm{diag}(r_1,r_2,\ldots,r_v) \\ \mathrm{diag}(s_1,s_2,\ldots,s_u)E_{u1} & \mathrm{diag}(s_1,s_2,\ldots,s_u) & N & L' \\ \mathrm{diag}(t_1,t_2,\ldots,t_w)E_{w1} & N' & \mathrm{diag}(t_1,t_2,\ldots,t_w) & M' \\ \mathrm{diag}(r_1,r_2,\ldots,r_v)E_{v1} & L & M & \mathrm{diag}(r_1,r_2,\ldots,r_v) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \\ \hat{\underline{\tau}} \end{bmatrix}$$

On premultiplying (6.2.2) by $[O_{v1} - L\,\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1}) \;\; O_{vw} \;\; I_v]$ on the both the sides we get

(6.2.3) $$\underline{P}_1 = F\hat{\underline{\beta}} + D_1\hat{\underline{\tau}}$$

where $$\underline{P}_1 = \underline{T} - L\,\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1})\,\underline{R}$$

(6.2.4) $$F = M - L\,\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1})\,N$$

$$D_1 = \mathrm{diag}(r_1, r_2, \ldots, r_v) - L\,\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1})\,L'$$

and again on premultiplying (6.2.2) by $[O_{w1} - N'\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1}) \;\; I_w \;\; O_{wv}]$ on both the sides we get

(6.2.5) $$\underline{P}_2 = D_2\hat{\underline{\beta}} + F'\hat{\underline{\tau}}$$

where $$\underline{P}_2 = \underline{C} - N'\,\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1})\,\underline{R}$$

(6.2.6) $$D_2 = (\mathrm{diag}(t_1, t_2, \ldots, t_w) - N'\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1})\,N$$

Let D_2^* be the pseudo inverse of D_2 i.e. $D_2^* D_2 D_2^* = D_2^*$ and $D_2 D_2^* D_2 = D_2$.

DEFINITION 6.2.1 : A design is said to be singly connected if all the elementary column or row (treatment or row; treatment or column) contrasts are estimable when treatment (column; row) effects are neglected.

DEFINITION 6.2.2 : A singly connected design is said to be doubly connected if all the elementary treatment (column; row) contrasts are estimable.

We assume here that the design is singly connected in the column-row sense. Hence the rank of $D_2 = w-1$. Also, we have

$$D_1 E_{v1} = O_{v1},\quad D_2 E_{w1} = O_{w1},\quad F E_{w1} = O_{v1},\quad F' E_{v1} = O_{w1},\quad \hat{\underline{\tau}}' E_{v1} = 0,\; \hat{\underline{\beta}}' E_{w1} = 0 .$$

From (6.2.3) and (6.2.5) we get that

(6.2.7) $$\underline{P} - F D_2^* \underline{P}_2 = (D_1 - F D_2^* F')\,\hat{\underline{\tau}}$$

Let

(6.2.8) $\underline{Q} = \underline{P}_1 - FD_2^{*}\underline{P}_2$ Hence

$\Delta = D_1 - FD_2^{*}F'$

(6.2.9) $\underline{Q} = \Delta\,\hat{\underline{\tau}}$

Evidently

(6.2.10) $E_x(\underline{Q}) = \Delta\,\underline{\tau}$ (E_x means expectation)

$V(\underline{Q}) = \Delta\sigma^2$

where σ^2 is the error variance. Q_i^s $(i = 1,2,\ldots,v)$ are adjusted treatment yields. Δ is similar to C-matrix of the treatment block design. When $N = E_{uw}$, we have

(6.2.11) $$\underline{Q} = \underline{T} - \frac{L\underline{R}}{w} - \frac{M\underline{C}}{u} + \frac{L\,E_{u1}G}{uw}$$

$$\Delta = \mathrm{diag}(r_1,r_2,\ldots,r_v) - \frac{LL'}{w} - \frac{MM'}{u} + \frac{L\,E_{uu}L'}{uw}$$

The following table 6.2.1 represets the analysis of variance table for a two-way design. Let v-1 be the rank of Δ.

TABLE 6.2.1

Source	Degrees of freedom	Sum of Squares	Sum of Squares	Degrees of freedom	Source
Rows (ignoring columns and treatments)	u-1	$\sum_{j=1}^{u}\frac{R_j^2}{s_j} - \frac{G^2}{n}$	$\sum_{j=1}^{u}\frac{R_j^2}{s_j} - \frac{G^2}{n}$	u-1	Rows (ignoring columns and treatments)
Columns (eliminating rows and ignoring treatments)	w-1	$\underline{P}_2'\,D_2^{*}\,\underline{P}_2$	+	w-1	Columns (eliminating rows and treatments)
Treatments (eliminating rows and columns)	v-1	$\sum_{i=1}^{v}\hat{\tau}_i Q_i$	$\underline{P}_1'\,D_1^{*}\,P_1$	v-1	Treatments (eliminating rows and ignoring columns)
Error	n-u-w-v+2	+	$\rightarrow$	n-u-w-v+2	Error
Total	n-1	$\Sigma y^2 - \frac{G^2}{n}$	$\Sigma y^2 - \frac{G^2}{n}$	n-1	Total

+ Obtained by subtraction. We assume that the rank of $D_1 = v-1$.

6.3 CLASSIFICATION OF TWO-WAY DESIGNS

Orthogonality and balancing are the desirable properties, because they increase efficiency and also simplify the analysis.

DEFINITION 6.3.1 : A design for obtaining information on several sets of parameters is said to be an orthogonal design if the estimates of estimable parameters of the different groups are uncorrelated.

In the design for two-way elimination of heterogeneity we have 3 sets of parameters in which we are interested. They are the row effects, column effects and treatment effects.

DEFINITION 6.3.2 : A doubly connected design is said to be balanced if all the elementary treatment (row; column) contrasts are estimable with the same variance.

i.e. Δ can be written as $\theta(I_v - \frac{1}{v}E_{vv})$ where θ is the non-zero characteristic root of Δ with the multiplicity v-1.

From the normal equations (6.2.2), we get the estimated treatment effects, estimated column effects and the estimated row effects and they are given by (we assume here the design is doubly connected in treatment, row and column senses)

(6.3.1) $\underline{Q}_1 = \Delta_1 \hat{\underline{\tau}}$, $\underline{Q}_4 = \Delta_4 \hat{\underline{\tau}}$

(6.3.2) $\underline{Q}_2 = \Delta_2 \hat{\underline{\beta}}$, $\underline{Q}_5 = \Delta_5 \hat{\underline{\beta}}$

(6.3.3) $\underline{Q}_3 = \Delta_3 \hat{\underline{\alpha}}$, $\underline{Q}_6 = \Delta_6 \hat{\underline{\alpha}}$

where

(6.3.4)
$$\begin{cases} \underline{Q}_1 = \underline{P}_1 - F D_2^* \underline{P}_2 , & \Delta_1 = D_1 - F D_2^* F' \\ \underline{Q}_2 = \underline{P}_2 - F' D_1^* \underline{P}_1 , & \Delta_2 = D_2 - F' D_1^* F \\ \underline{Q}_3 = \underline{P}_4 - H' D_3^* \underline{P}_3 , & \Delta_3 = D_4 - H' D_3^* H \\ \underline{Q}_4 = \underline{P}_3 - H D_4^* H' , & \Delta_4 = D_3 - H D_4^* H' \end{cases}$$

$$(6.3.4)\qquad \begin{cases} \underline{Q}_5 = \underline{P}_6 - E'D_5^{*}\underline{P}_5 \ , & \Delta_5 = D_6 - E'D_5^{*}E \\ \underline{Q}_6 = \underline{P}_5 - E\,D_6^{*}P_6 \ , & \Delta_6 = D_5 - E\,D_6^{*}E' \end{cases}$$

and

$$(6.3.5)\qquad \begin{cases} \underline{P}_3 = \underline{T} - M\,\mathrm{diag}(t_1^{-1},t_2^{-1},\ldots,t_w^{-1})\,\underline{C} \\ \underline{P}_4 = \underline{R} - N\,\mathrm{diag}(t_1^{-1},t_2^{-1},\ldots,t_w^{-1})\,\underline{C} \\ \underline{P}_5 = \underline{R} - L'\mathrm{diag}(r_1^{-1},r_2^{-1},\ldots,r_v^{-1})\,\underline{T} \\ \underline{P}_6 = \underline{C} - M'\mathrm{diag}(r_1^{-1},r_2^{-1},\ldots,r_v^{-1})\,\underline{T} \\ D_3 = \mathrm{diag}(r_1,r_2,\ldots,r_v) - M\,\mathrm{diag}(t_1^{-1},t_2^{-1},\ldots,t_w^{-1})\,M' \\ D_4 = \mathrm{diag}(s_1,s_2,\ldots,s_u) - N\,\mathrm{diag}(t_1^{-1},t_2^{-1},\ldots,t_w^{-1})\,N' \\ D_5 = \mathrm{diag}(s_1,s_2,\ldots,s_u) - L'\mathrm{diag}(r_1^{-1},r_2^{-1},\ldots,r_v^{-1})\,L \\ D_6 = \mathrm{diag}(t_1,t_2,\ldots,t_w) - M'\mathrm{diag}(r_1^{-1},r_2^{-1},\ldots,r_v^{-1})\,M \\ H = L - M\,\mathrm{diag}(t_1^{-1},t_2^{-1},\ldots,t_w^{-1})\,N' \\ E = N - L'\mathrm{diag}(r_1^{-1},r_2^{-1},\ldots,r_v)\,M \end{cases}$$

We can easily show that $\underline{Q}_1 = \underline{Q}_4$, $\underline{Q}_2 = \underline{Q}_5$ and $\underline{Q}_3 = \underline{Q}_6$ also $\Delta_1 = \Delta_4$, $\Delta_2 = \Delta_5$ and $\Delta_3 = \Delta_6$; $\underline{Q}_1$, $\underline{Q}_2$, $\underline{Q}_3$ are the column vectors of adjusted treatment yields, adjusted column yields, adjusted row yields respectively.

In order that the adjusted treatment yields are orthogonal to the adjusted column yields, we must have Expectation of $\underline{E}(\underline{Q}_1\,\underline{Q}_2') = 0_{vw}$ and

$$(6.3.6)\qquad \begin{aligned} E_x(\underline{Q}_1\underline{Q}_2') &= E_x(\,\underline{P}_1 - F\,D_2^{*}\underline{P}_2)(\underline{P}_2' - P_1'D_1^{*}F) \\ &= E_x(\underline{P}_1\,\underline{P}_2') - F\,D_2^{*}E_x(P_2P_2') - E_x(\underline{P}_1\underline{P}_1')\,D_1^{*}F + \\ &\qquad F\,D_2^{*}\,E_x(\underline{P}_2\underline{P}_1')\,D_1^{*}F \end{aligned}$$

$$\begin{aligned} E_x(\underline{P}_1\underline{P}_1') &= E_x\,[\underline{T} - L\,\mathrm{diag}(s_1^{-1},s_2^{-1},\ldots,s_u^{-1})\underline{R}\,] \\ &\qquad [\underline{T}' - \underline{R}'\,\mathrm{diag}(s_1^{-1},s_2^{-1},\ldots,s_u^{-1})L'\,] \\ &= D_1\,\sigma^2 \end{aligned}$$

Similarly we have

$$E_x(\underline{P}_2\ \underline{P}_2') = D_2\,\sigma^2$$

$$E_x(\underline{P}_1\ \underline{P}_2') = F\,\sigma^2$$

Hence

(6.3.7) $$E_x(\underline{Q}_1\ \underline{Q}_2') = -\Delta_1 D_1^{*} F\,\sigma^2$$

When (6.3.6) holds, we have $\Delta_1 D_1^{*} F = O_{vw}$ i.e.

$$D_1^{*} F = [a_1 E_{v1}, a_2 E_{v1}, \ldots, a_w E_{v1}]$$

where $a_1, a_2, \ldots, a_w$ are real numbers. This equation gives that

(6.3.8) $$F = O_{vw} \text{ i.e. } M = L\,\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1})\,N$$

Thus, inorder that the adjusted treatment yields may be orthogonal to the adjusted column yields, the design should satisfy the condition (6.3.8). In a similar way we can show that the adjusted treatment yields may be orthogonal to the adjusted row yields if

(6.3.9) $$L = M\,\mathrm{diag}(t_1^{-1}, t_2^{-1}, \ldots, t_w^{-1})\,N'$$

and the adjusted row yields are orthogonal to the adjusted column yields if

(6.3.10) $$N = L'\,\mathrm{diag}(r_1^{-1}, r_2^{-1}, \ldots, r_v^{-1})\,M$$

If the adjusted treatment yields are orthogonal to both the adjusted row yields and adjusted column yields, we get from the equations (6.3.8) and (6.3.9) that

(6.3.11)
$$M\,\mathrm{diag}(t_1^{-1}, t_2^{-1}, \ldots, t_w^{-1})\,D_2 = O_{vw}$$
$$L\,\mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_u^{-1})\,D_4 = O_{vu}$$

and (6.3.11) gives

(6.3.12)
$$\frac{m_{ij'}}{t_{j'}} \text{ is constant for all } j'\ (j' = 1,2,\ldots,w) \text{ i.e. } \frac{m_{ij'}}{t_{j'}} = b_i \text{ (say)}$$
$$\frac{l_{ij}}{s_j} \text{ is constant for all } j\ (j=1,2,\ldots,u) \text{ i.e. } \frac{l_{ij}}{s_j} = c_i \text{ (say)}$$

(6.3.12) can be written as

$$
(6.3.13) \qquad b_i = \frac{m_{i1}}{t_1} = \frac{m_{i2}}{t_2} = \ldots = \frac{m_{iw}}{t_w} = \frac{\sum_{j=1}^{w} m_{ij'}}{\sum_{j=1}^{w} t_{j'}} = \frac{r_i}{n}
$$

$$
c_i = \frac{l_{i1}}{s_1} = \frac{l_{i2}}{s_2} = \ldots = \frac{l_{iu}}{s_u} = \frac{\sum_{j=1}^{u} l_{ij}}{\sum_{j=1}^{u} s_j} = \frac{r_i}{n}
$$

Thus from (6.3.13) we get

$$
(6.3.14) \qquad M = ((m_{ij'})) = \left(\left(\frac{r_i t_{j'}}{n}\right)\right); \quad L = ((l_{ij})) = \left(\left(\frac{r_i s_j}{n}\right)\right)
$$

$$
i = 1,2,\ldots,v; \; j = 1,2,\ldots,u; \; j' = 1,2,\ldots,w.
$$

Similary if the adjusted column yields are orthogonal to both the adjusted treatment yields and the adjusted row yields, then

$$
(6.3.15) \qquad M = \left(\left(\frac{r_i t_{j'}}{n}\right)\right), \quad N = \left(\left(\frac{s_j t_{j'}}{n}\right)\right)
$$

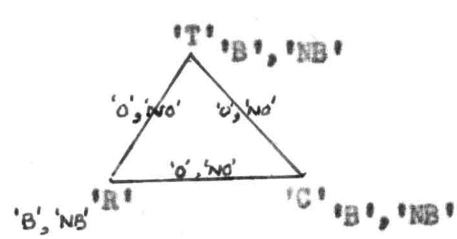
and if the adjusted row yields are orthogonal to both the adjusted treatment yields and adjusted column yields, then

$$
(6.3.16) \qquad L = \left(\left(\frac{r_i s_j}{n}\right)\right), \quad N = \left(\left(\frac{s_j t_{j'}}{n}\right)\right)
$$

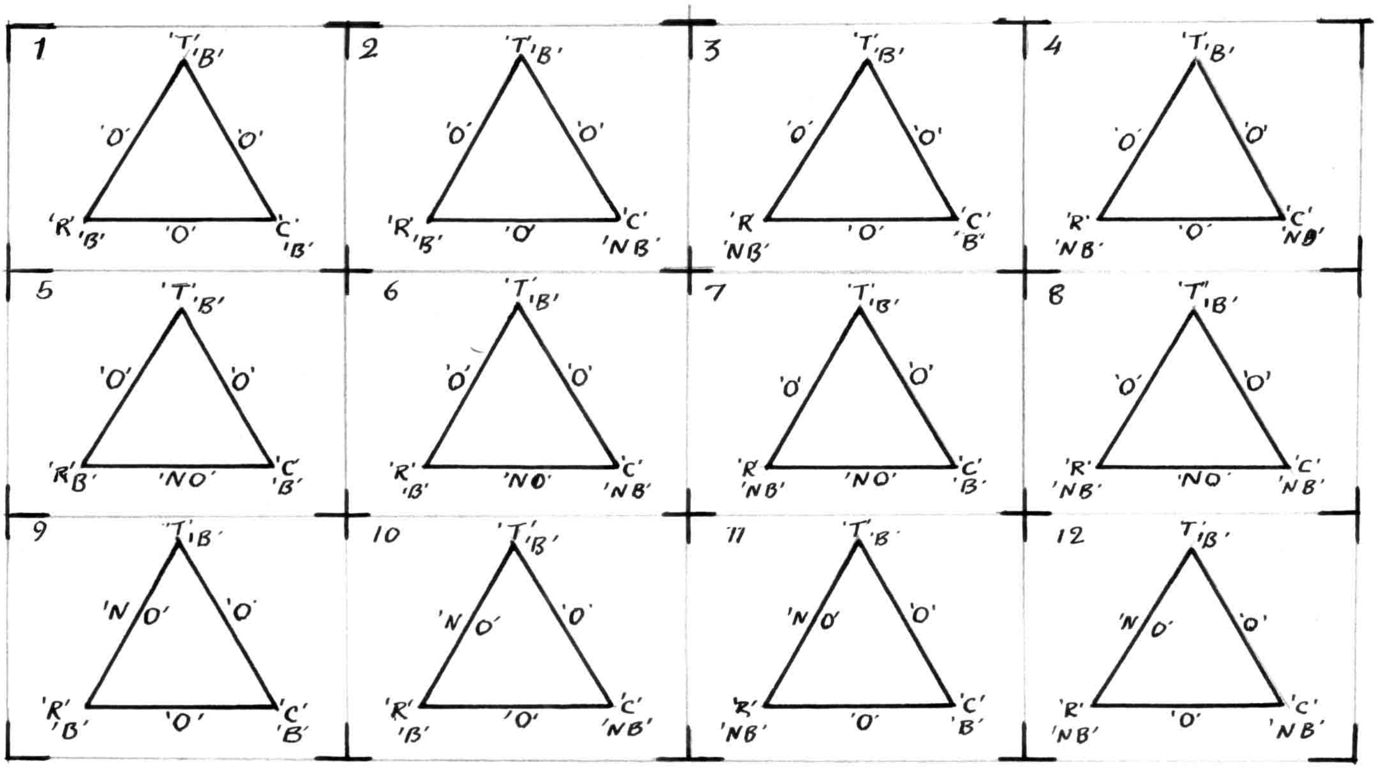
Finally, if the adjusted treatment yields, the adjusted column yields and the adjusted row yields are mutually orthogonal, then

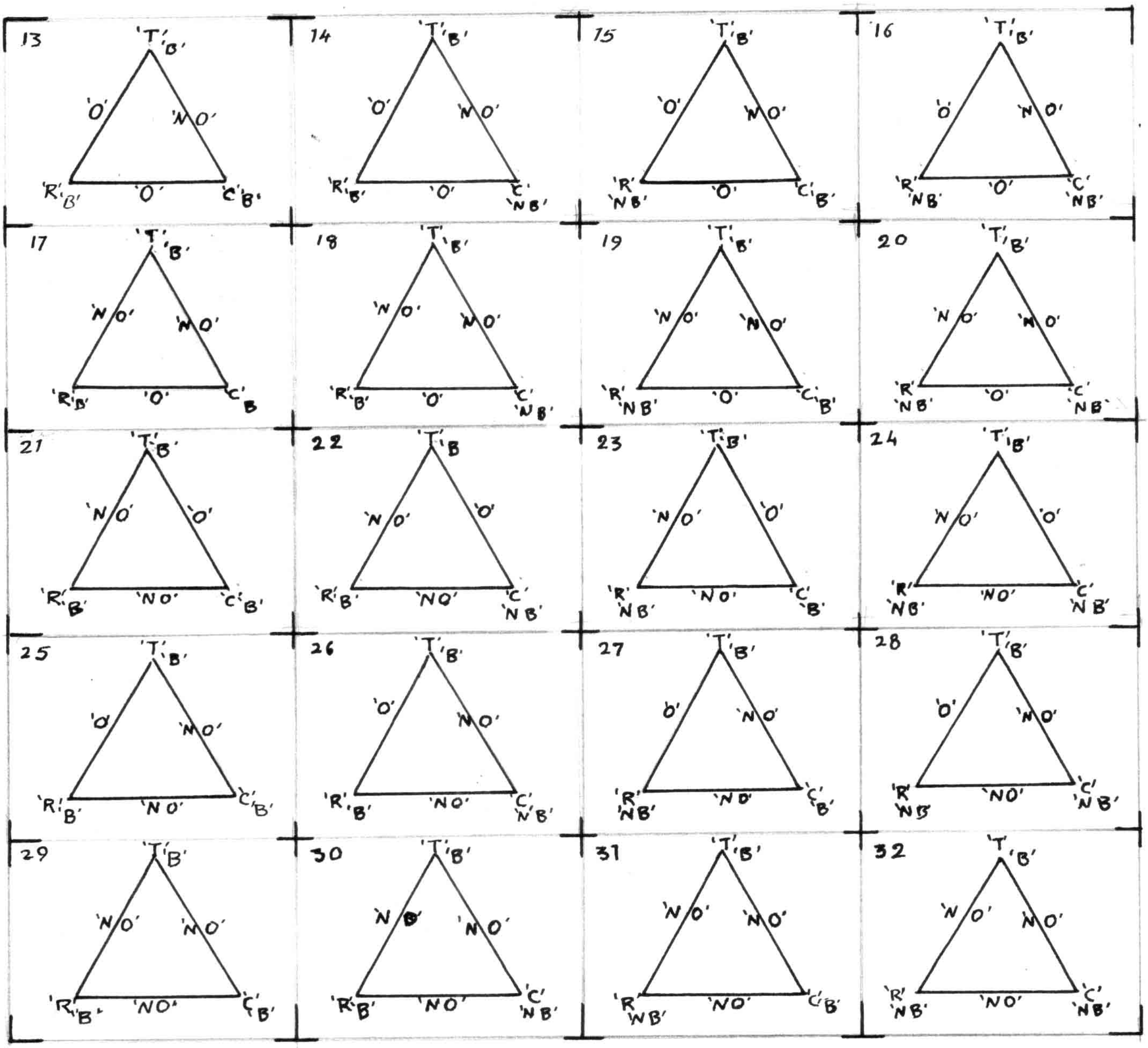
$$
(6.3.17) \qquad L = \left(\left(\frac{r_i s_j}{n}\right)\right), \quad M = \left(\left(\frac{r_i t_{j'}}{n}\right)\right), \quad N = \left(\left(\frac{s_j t_{j'}}{n}\right)\right)
$$

Now we classify the design on the basis of the following two properties (i) orthogonality or non-orthogonality (ii) balancing or non-balancing. Let 'B', 'NB', 'O', 'NO' denote for balancing, non-balancing, orthogonality, non-orthogonality respectively. Let



be a triangle whose edges represent treatments ('T'), rows ('R') and columns ('C') with the second property mentioned above. The sides of the triangle represent the first property between ('T', 'R') ; ('R', 'C')and ('C','T'). Hence we get 64 classes of designs. Out of them we study 32 classes where treatments 'T' takes the property 'B'. See the following 32 classes.




13 'T' 'B' 'O' 'NO' 'R' 'B' 'O' 'C' 'B'
14 'T' 'B' 'O' 'NO' 'R' 'B' 'O' 'C' 'NB'
15 'T' 'B' 'O' 'NO' 'R' 'NB' 'O' 'C' 'B'
16 'T' 'B' 'O' 'NO' 'R' 'NB' 'O' 'C' 'NB'
17 'T' 'B' 'NO' 'NO' 'R' 'B' 'O' 'C' 'B'
18 'T' 'B' 'NO' 'NO' 'R' 'B' 'O' 'C' 'NB'
19 'T' 'B' 'NO' 'NO' 'R' 'NB' 'O' 'C' 'B'
20 'T' 'B' 'NO' 'NO' 'R' 'NB' 'O' 'C' 'NB'
21 'T' 'B' 'NO' 'O' 'R' 'B' 'NO' 'C' 'B'
22 'T' 'B' 'NO' 'O' 'R' 'B' 'NO' 'C' 'NB'
23 'T' 'B' 'NO' 'O' 'R' 'NB' 'NO' 'C' 'B'
24 'T' 'B' 'NO' 'O' 'R' 'NB' 'NO' 'C' 'NB'
25 'T' 'B' 'O' 'NO' 'R' 'B' 'NO' 'C' 'B'
26 'T' 'B' 'O' 'NO' 'R' 'B' 'NO' 'C' 'NB'
27 'T' 'B' 'O' 'NO' 'R' 'NB' 'NO' 'C' 'B'
28 'T' 'B' 'O' 'NO' 'R' 'NB' 'NO' 'C' 'NB'
29 'T' 'B' 'NO' 'NO' 'R' 'B' 'NO' 'C' 'B'
30 'T' 'B' 'NO' 'NO' 'R' 'B' 'NO' 'C' 'NB'
31 'T' 'B' 'NO' 'NO' 'R' 'NB' 'NO' 'C' 'B'
32 'T' 'B' 'NO' 'NO' 'R' 'NB' 'NO' 'C' 'NB'

6.4 STUDY OF THE CLASSES WHERE TREATMENTS TAKES THE PROPERTY, BALANCING

In this section we study the designs, where the treatments are attributed with the property balancing. We restrict our attention to the designs for which (i) $r_i = r$, $t_{j'} = t$, $s_j = s$ ($i = 1,2,\ldots,v$; $j' = 1,2,\ldots,v$; $j = 1,2,\ldots,u$) (ii) the incidence matrices L, M and N are binary.

CLASSES 1-4 : The sides of the triangle TRC, i.e. 'TR', 'RC' and 'CT' take the property 'O'. Also $n = vr = us = wt$. Hence by (6.3.17), we have

$$L = \frac{s}{v} E_{vu} = \frac{r}{u} E_{vu}$$

(6.4.1)
$$M = \frac{t}{v} E_{vw} = \frac{r}{w} E_{vw}$$

$$N = \frac{t}{u} E_{uw} = \frac{s}{w} E_{uw}$$

Since L, M, N are binary, we get $v = u = w = r = s = t$ i.e. two-way design is latin square for the classes 1-4.

CLASSES 5-8 : The sides 'TR', 'TC' take the property 'O'. By (6.3.14) we get

(6.4.2) $$L = E_{vu}, \quad M = E_{vu} \quad \because s = t = v \text{ and } r = u = w .$$

CLASS 5 : The edges 'T', 'R', 'C' take 'B'. Then we must have

(6.4.3) $$\Delta_1 = D_1 - F D_2^{*} F' = \theta_1 (I_v - \frac{1}{v} E_{vv})$$

(6.4.4) $$\Delta_2 = D_2 - F' D_1^{*} F = \theta_2 (I_w - \frac{1}{w} E_{ww})$$

(6.4.5) $$\Delta_3 = D_4 - H' D_3^{*} H = \theta_3 (I_u - \frac{1}{u} E_{uu})$$

where θ_1, θ_2, θ_3 are the non-zero characteristic roots of Δ_1, Δ_2, Δ_3 respectively.

From the property of orthogonality we have that $F = O_{vu} = H$. Hence

(6.4.6) $$\Delta_1 = D_1 = u(I_v - \frac{1}{v}E_{vv}) \quad \text{i.e.} \quad \theta_1 = u$$

(6.4.7) $$\Delta_2 = D_2 = vI_u - \frac{1}{v}NN'$$

(6.4.8) $$\Delta_3 = D_4 = vI_u - \frac{1}{v}N'N$$

Hence from (6.4.4), (6.4.5) and (6.4.7), (6.4.8) we have

$$\theta_2 = \theta_3 = \frac{u(v-1)}{u-1}$$ and N is E_{uu} if $v = u$ or a symmetrical "BIB" design if $v < u$.

CLASS 6 : The edge 'R' takes 'B'. By (6.4.2), (6.4.5), (6.4.8) we get $N = E_{uu}$ if $v = u$ or N is symmetrical "BIB" design if $v < u$. Thus, class 6 reduces to the class 5. Similarly we can show that class 7 reduces to the class 5.

CLASS 8 : The edges 'R' and 'C' both take 'NB'. In this case we get N as incomplete design with u rows, u columns which is not "BIB" design. In the next section, in the construction of this class of designs we get N as symmetrical "PBIB" design.

CLASSES 9-12 : The sides 'RC', 'CT' take 'O'. By (6.3.15) we have

(6.4.9) $$M = E_{vw}\ , \quad N = E_{uw} \quad \text{and } v = u = t, \quad w = r = s$$

which shows that $$\Delta_1 = D_1 = rI_v - \frac{LL'}{r} = \theta_1 (I_v - \frac{1}{v}E_{vv})$$

and hence L is either symmetrical "BIB" design or E_{vv}. The classes 10,11,12 reduce to the class 9.

Similarly we can show that the classes 14,15,16 reduce to the class 13 and here M is symmetrical "BIB" design or E_{vv} .

CLASSES 17-20 : The side 'RC' takes 'O'. Hence by (6.3.10) we have

(6.4.10) $$rN = L'M \quad \text{and} \quad E = O_{uw}$$

CLASS 17,18,19 : All edges 'T', 'R', 'C' take 'B'. Hence from (6.4.3), (6.4.4), (6.4.5) and since $\Delta_1 = \Delta_4$ we get

(6.4.11) $$\theta_1 F D_2 = \theta_2 D_1 F$$

(6.4.12) $$\theta_1 H D_4 = \theta_3 D_3 H$$

(6.4.13) $$FF' = \frac{\theta_2}{\theta_1}(D_1^2 - \theta_1 D_1)$$

(6.4.14) $$F'F = \frac{\theta_1}{\theta_2}(D_2^2 - \theta_2 D_2)$$

(6.4.15) $$HH' = \frac{\theta_3}{\theta_1}(D_3^2 - \theta_1 D_3)$$

(6.4.16) $$H'H = \frac{\theta_1}{\theta_3}(D_4^2 - \theta_3 D_4)$$

Also by (6.4.10), (6.3.4) and since $\Delta_2 = \Delta_5$, $\Delta_3 = \Delta_6$, we get

(6.4.17) $$\Delta_2 = tI_w - \frac{1}{r} M'M = \theta_2(I_w - \frac{1}{w}E_{ww})$$

(6.4.18) $$\Delta_3 = sI_u - \frac{1}{r} L'L = \theta_3(I_u - \frac{1}{u}E_{uu})$$

which show that L' and M' are "BIB" designs. Again from (6.4.11) and (6.4.12) we get respectively

(6.4.19) $$\theta_1 M D_2 = \theta_2 D_1 M$$

(6.4.20) $$\theta_1 L D_4 = \theta_3 D_3 L$$

and on premultiplying (6.4.19) by M' on both the sides, we deduce that N' is a "BIB" design. Then $D_2 = \theta(I_w - \frac{1}{w}E_{ww})$ where $\theta = \frac{u(s-1)}{w-1}$ also $\theta_2 = \frac{v(r-1)}{w-1}$.

The non-zero characteristic roots FF' and F'F are the same. Hence from (6.4.13) and (6.4.14) we see that the non-zerocharacteristic roots of D_1 are θ_1 and $\frac{\theta\theta_1}{\theta_2}$

with the multiplicities v-w and w-1 respectively. And from (6.4.19) we get that

$$D_2 = \frac{r\theta_2^2}{r\theta_2 + \theta_1(\theta_2 - t)} \left(I_w - \frac{1}{w}E_{ww}\right)$$

Hence

$$\frac{r\theta_2^2}{r\theta_2 + \theta_1(\theta_2 - t)} = \theta \quad \text{which gives} \quad \theta_1 = \frac{v(s-r)(r-1)}{(v-t)(s-1)}$$

Since $\theta_1 > 0$ and $v > t$, we must have $s > r$ i.e. $v > u$. Then there will be atleast v-u non-zero characteristic roots for D_1 equivalent to r. i.e. $\theta_1 = r$ or $\frac{\theta\theta_1}{\theta_2} = r$. Since $\theta_1 = \frac{v(s-r)(r-1)}{(v-t)(s-1)}$, $\frac{\theta\theta_1}{\theta_2}$ does not take the values r.

Hence $\theta_1 = r$, and

(6.4.21) $$\text{tr}\,\Delta_1 = (v-1)\theta_1 = (v-1)r$$

Also

(6.4.22) $$\text{tr}\,\Delta_1 = vr - u - \frac{1}{\theta}\,\text{tr}FF' = vr - u - \frac{r(w-1)}{v(r-1)}(v-u) .$$

On equating the right hand sides of (6.4.21) and (6.4.22), we get

$(w-1)(s-r) = (r-1)(s-v)$ i.e. $s > v$ which is impossible . Hence class 19 is impossible. Similarly we can show, on considering (6.4.12), (6.4.15), (6.4.16) and (6.4.18), that class 18 is impossible. Since class 17 is a particular case of class 18 or class 19, we get class 17 is also impossible.

CLASSES 21-24 : The side 'TC' takes '0' . Hence by (6.3.8)

(6.4.23) $$sM = LN$$

(6.4.24) $$\begin{cases} \Delta_1 = D_1 = rI_v - \frac{LL'}{s} \\ \Delta_2 = D_2 = tI_w - \frac{N'N}{s} \\ \Delta_3 = D_4 - H'D_3^*H \quad \text{and} \quad H = \frac{LD_4}{s} \end{cases}$$

From (6.4.15), (6.4.16) and (6.4.24), we can show in a similar way as in the cases of classes 17,18 that classes 21,22 are not possible.

CLASS 23 : 'T' the treatments take 'B' and 'C' the columns take 'B'. By (6.4.24) we see that L and N' are "BIB" designs. Some constructions for the class 23 are given in the next section.

CLASS 24 : 'T' takes the property 'B'. Hence L must be a "BIB" design.

CLASSES 25-28 : As in the case of classes 21,22, we have here that classes 25, 27 do not exist. Classes 26, 28 are similar to the classes 23, 24 respectively. Here the properties of L, N' are attributed to M,N respectively.

CLASSES 29-32 : All sides of the triangle take the property 'NO'.

CLASS 29 : All edges take the property 'B'. Hence addition to the equations, namely (6.4.13), (6.4.14), (6.4.15) and (6.4.16), we have

$$EE' = \frac{\theta_2}{\theta_3}(D_5^2 - \theta_3 D_5) \tag{6.4.25}$$

$$E'E = \frac{\theta_3}{\theta_2}(D_6^2 - \theta_2 D_6) \tag{6.4.26}$$

Let

(6.4.27)

$$\left[\begin{array}{l}
\lambda_i \ (i = 1,2,\ldots,v-1) \text{ be the non-zero characteristic roots of } D_1 \\
\delta_j \ (j = 1,2,\ldots,w-1) \text{ be the non-zero characteristic roots of } D_2 \\
\lambda'_{i'} \ (i' = 1,2,\ldots,v-1) \text{ be the non-zero characteristic roots of } D_3 \\
\delta'_{j'} \ (j' = 1,2,\ldots,u-1) \text{ be the non-zero characteristic roots of } D_4 \\
\rho_k \ (k = 1,2,\ldots,u-1) \text{ be the non-zero characteristic roots of } D_5 \\
\rho'_{k'} \ (k' = 1,2,\ldots,w-1) \text{ be the non-zero characteristic roots of } D_6
\end{array}\right.$$

Thus the characteristic roots of

$$(6.4.28)\quad \begin{cases} FF' & \text{are given by} \quad \frac{\theta_2}{\theta_1}(\lambda_i^2 - \theta_1\lambda_i) \\ F'F & \text{are given by} \quad \frac{\theta_1}{\theta_2}(\delta_j^2 - \theta_2\delta_j) \\ HH' & \text{are given by} \quad \frac{\theta_3}{\theta_1}(\lambda_{i'}'^2 - \theta_1\lambda_{i'}') \\ H'H & \text{are given by} \quad \frac{\theta_1}{\theta_3}(\delta_{j'}'^2 - \theta_3\delta_{j'}') \\ EE' & \text{are given by} \quad \frac{\theta_2}{\theta_3}(\rho_k^2 - \theta_3\rho_k) \\ E'E & \text{are given by} \quad \frac{\theta_3}{\theta_2}(\rho_{k'}'^2 - \theta_2\rho_{k'}') \end{cases}$$

Also we know that the non-zero characteristic roots of FF' and $F'F$ are the same;

the non-zero characteristic roots of HH' and $H'H$ are the same;

the non-zero characteristic roots of EE' and $E'E$ are the same.

Let $v \le w \le u$. From (6.4.28) we get

$$(6.4.29)\quad \begin{cases} \delta_{j_i} = \frac{\theta_2}{\theta_1}\lambda_i \ (i=1,2,\ldots,v-1); \text{ for the remaining } w-v\ \delta_j\text{s, each } \delta_j = \theta_2 \\ \delta'_{j'_{i'}} = \frac{\theta_3}{\theta_1}\lambda'_{i'} \ (i'=1,2,\ldots,v-1); \text{ for the remaining } u-v\ \delta'_{j'}\text{s, each } \delta'_{j'} = \theta_3 \\ \rho_{k_{k'}} = \frac{\theta_3}{\theta_2}\rho'_{k'} \ (k'=1,2,\ldots,w-1); \text{ for the remaining } u-w\ \rho_k\text{s, each } \rho_k = \theta_3 \end{cases}$$

Again from (6.4.27) we obtain

$$(6.4.30)\quad \begin{cases} \rho_{k_i} = \frac{s}{r}\lambda_i \quad (i=1,2,\ldots,v-1); \text{ for the remaining } u-v\ \rho_k\text{s, each } \rho_k = s \\ \delta'_{j'_j} = \frac{s}{t}\delta_j \quad (j=1,2,\ldots,w-1); \text{ for the remaining } u-w\ \delta'_{j'}\text{s each } \delta'_{j'} = s \\ \rho'_{k'_{i'}} = \frac{t}{r}\lambda'_{i'} \quad (i'=1,2,\ldots,v-1); \text{ for the remaining } w-v\ \rho'_{k'}\text{s, each } \rho'_{k'} = t \end{cases}$$

From (6.4.29) and (6.4.30) we see

$$(6.4.31)\quad \begin{cases} \rho_k = \frac{s}{r}\lambda_i \ (i=1,2,\ldots,v-1), = s \ \text{(the eigen values are u-v in number) and} \\ \rho_k = \frac{s}{r}\lambda_i \ (i=1,2,\ldots,v-1), = \frac{\theta_3 t}{\theta_2} \ \text{(the eigen values are w-v in number),} \\ \qquad = \theta_3 \ \text{(the eigen values are u-w in number).} \end{cases}$$

Hence $\theta_3 = s$ and $\theta_2 = t$. Further

$tr\ H'D_3^* H = w(t-1) - s(u-1) = s-w$ which is impossible unless $s=w$, in which case $H = O_{vu}$. Since we are considering non-orthogonal design, H can not be O_{vu}. Thus the above (6.4.31) holds only when $u = w = v$ and hence $\theta_1 = \theta_2 = \theta_3$ and $r = s = t$. Evidently if N is symmetrical "BIB" design, L and M are also symmetrical "BIB" designs with the same parameters.

6.5 CONSTRUCTION OF DESIGNS IN SOME PARTICULAR CASES

The constructions, which we give in this section, deal with the designs for which the row-column incidence matrices are incomplete.

DESIGNS FOR THE CLASSES 5 AND 8 : Let there exists an Youden square which corresponds to a symmetrical "BIB" design; the parameters of the "BIB" design are $v^*(= u = w)$, $r^*(= v)$, λ^*. Now write the above symmetrical "BIB" design and keep the integers i (i=1,2,.., v) in the place of units of the design in the following way. Denote the unit by i, where l^{th} treatment occurs in the k^{th} block, if l^{th} treatment occurs in k^{th} column, i^{th} row of Youden square ($l = 1,2,\ldots, u$; $k = 1,2,\ldots, u$). The numbered matrix thus obtained is our required design for the class - 5.

Similarly, we can construct two-way design for the class - 8 using extended Youden square [18] of a symmetrical "PBIB" design. When the number of treatments is small, it is some times useful to have design of class-5 or class-8, where the number of replicates exceeds the number of treatments.

CONSTRUCTION OF TWO-WAY DESIGN FOR THE CLASS 29 IN A PARTICULAR CASE : Let $L = M$ or $E_{vv} - M \pm I_v$, $L = N$ which are symmetrical "BIB" designs, such that $L = M = N = E_{vv} - I_v$ or $L + L' = E_{vv} \pm I_v$, $M + M' = E_{vv} \pm I_v$.

THEOREM 6.5.1 : There always exists a two-way design for which $L = M = N = E_{vv} - I_v$ where $v \neq 2,6$.

PROOF : We know that there exist at least 2 mutual orthogonal latin squares of order v where $v \neq 2,6$. Let 'L_1' and 'L_2' be two mutual orthogonal latin squares with the numbers 1,2,..., v . Now perform the operations of interchanging rows and columns in 'L_1' to get 'L_1^*' in such a way that the diagonal numbers in 'L_1^*' being the same. Let 'L_2^*' be the resulting latin square of 'L_2' after performing the same operations as in the case of 'L_1' to 'L_1^*' . Hence by the property of orthogonal latin squares, we get the diagonal elements of 'L_2^*' are 1,2,..., v. On eliminating the diagonal elements in 'L_2^*' we obtain our required design which satisfies the condition that $L = M = N = E_{vv} - I_v$.

THEOREM 6.5.2 : When $v \equiv 3 \bmod 4 = p^n$ where p is prime and n is any positive integer, we can always construct two-way design for which $L = N = M$ (or $= E_{vv} \pm I_v - M$) as symmetrical "BIB" designs such that $N + N' = E_{vv} \pm I_v$.

Let the elements of $GF(p^n)$ be $o, x^o, x^1, \ldots, x^{p^n-2}$. Let us denote them by 1, 2, 3, .., v respectively. Write the matrix X where

(6.5.1) $$X = ((x_{ij})) = ((x^{j'} - x^{i'}))$$

$$x^{j'} - x^{i'} = x^{k'} \quad \text{if } k' \text{ is even}$$
$$= o \quad \text{otherwise}$$

$$i, j = 1 ; \quad i, j = 2 ; \quad i, j = 3 ; \ldots\ldots ; i, j = v$$
$$x^{i'}, x^{j'} = o; \quad i', j' = o ; \quad i', j' = 1 ; \ldots\ldots ; i', j' = v-2$$

Then rewrite the matrix X with the numbers in such a way that $x^{k'}$ in i^{th} row takes the corresponding row number of $x^{k'}$ in i^{th} column. The matrix thus obtained is our required design.

PROOF : Let i be the row number of $x^{k'}$ in the j^{th} column where k' is even. Let $(x^{k'}, l)_j = i$ i.e. the element of $x^{k'}$ in the j^{th} row, l^{th} column takes the number i. Suppose

$$(x^{k'}, l)_j = (x^{k''}, l)_q = i$$

We know that the elements $x^{k'}$; $x^{k''}$ are obtained on subtraction of x^{j-2}, x^{q-2} from x^{l-2}. Hence

(6.5.2)
$$x^l - x^j = x^{k'+2}$$
$$x^l - x^q = x^{k''+2}$$

Also

(6.5.3)
$$x^j - x^i = x^{k'+2}$$
$$x^q - x^i = x^{k''+2}$$

From (6.5.2) and (6.5.3), we get

$$x^l + x^i = 2x^j$$
$$= 2x^q$$

which is impossible unless $j = q$. Hence

(6.5.4)
$$(x^{k'}, l)_j \neq (x^{k''}, l)_q$$

Let $x^{k'}$ be even if k' is even. In order to get even $(x^{k'})^s$ in the l^{th} column, j takes a set of $\frac{p^n-1}{2}$ values. Now we show that i is one of them or i does not belong to this set.

$$(x^{k'}, l)_j = i$$

$$\therefore \quad x^{k'+2} = x^l - x^j = x^j - x^i$$

$$\text{i.e.} \quad x^l - x^i = 2(x^j - x^i).$$

Let $2 = x^q$. For $x^j - x^i$ to be even, x^q should be even. Thus, when x^q is even, i belongs to the set of $\frac{p^n-1}{2}$ values mentioned above and when x^q is odd, i does not belong to this set.

We know that N is symmetrical "BIB" design (cf. Bose [16]) such that $N + N'$ is $E_{vv} - I_v$. Also we can show easily that L = N and L = M if $2 = x^q$ is even or L = N and $L = E_{vv} - I_v - M$ if $2 = x^q$ is odd.

If we introduce the elements 1,2,..., v in the diagonal of two-way design obtained above in such a way that no two same integers are repeated in any row or column, then we get a two-way design for which L = M = N are symmetrical "BIB" designs such that L = N = M (or $= E_{vv} + I_v - M$) and $N + N' = E_{vv} + I_v$.

CONSTRUCTION OF THE DESIGN IN SOME CASES FOR THE CLASS-23 : I. Let there exists a symmetrical "BIB" design with the parameters (v, r, λ). Then two "BIB" designs with the following parameters are possible.

(6.5.5)

	v^*	b^*	r^*	k^*	λ^*
(i)	r	$v-1$	$v-r$	$r-\lambda$	$v-2r+\lambda$
(ii)	$v-r$	$v-1$	r	$r-\lambda$	λ

We construct two-way design in some particular cases for which L takes one of the above two designs, N' takes the other and $M = E_{v-r\ r}$ (or $E_{r\ v-r}$).

THEOREM 6.5.3 : For symmetrical "BIB" design ($v = 2n + 1$, $r = n \pm 1, \lambda$) where $n = p^m$, p is prime and m is positive integer, if the premitive root x of $GF(p^m)$ has the property $2 = x^{s'}$; $1 + x^{2t+1} = x^{s''}$; s', s'' are either both even or both odd positive integers and $t (\neq \frac{p^m-3}{4})$ is any positive integer, then there exists a two-way design for the class - 23.

Write

$$(6.5.6) \quad \sum^{*}_{2n+1} = \begin{bmatrix} -S_n - I_n & S_n - I_n & E_{n1} \\ S_n - I_n & S_n + I_n & -E_{n1} \\ E_{1n} & -E_{1n} & -1 \end{bmatrix} \text{ or } \begin{bmatrix} -\Sigma_n - I_n & \Sigma_n - I_n & E_{n1} \\ \Sigma_n + I_n & \Sigma_n - I_n & -E_{n1} \\ -E_{1n} & E_{1n} & -1 \end{bmatrix}$$

The meanings of S_n and Σ_n are the same as defined in chapter 1. We can show easily that $\sum^{*}_{2n+1}$ consists a "SBIB" design, when we change negative units to zeros in it and also we know that S_n does not exist for $n \equiv 3 \bmod 4$, Σ_n does not exist for $n \equiv 1 \bmod 4$. Let

(6.5.7)

$$\text{(i)} \quad N' \to \begin{array}{c|c} \text{part 1} & \text{part 2} \\ (S_n + I_n & -S_n + I_n) \end{array} \text{ or } \begin{array}{c|c} \text{part 1} & \text{part 2} \\ (\Sigma_n + I_n & \Sigma'_n + I_n) \end{array}$$

$$\text{(ii)} \quad L \to \left[\begin{array}{c|c} S_n - I_n & S_n + I_n \\ E_{1n} & -E_{1n} \end{array}\right] \text{ or } \left[\begin{array}{c|c} \Sigma_n + I_n & \Sigma_n - I_n \\ -E_{1n} & E_{1n} \end{array}\right]$$

be two matrices obtained from $\sum^{*}_{2n+1}$. (i) and (ii) consist two "BIB" designs if we eliminate negative units in them. We construct two-way design for which N' corresponds to the first "BIB" design, L corresponds to second "BIB" design.

Case (i) $n \equiv 1 \bmod 4$. Write S_n, $-S_n$ in terms of Galois Field elements as

(6.5.8)

$$\left[\begin{array}{lll} S_n = ((x_{ij})) = ((x^{j'} - x^{i'})) & \text{where} & \\ x^{j'} - x^{i'} = x^k & \text{if } k \text{ is even} & \\ \phantom{x^{j'} - x^{i'}} = o & \text{other-wise} & \text{and} \\ -S_n = ((x_{ij})) = ((x^{j'} - x^{i'})) & \text{where} & \\ x^{j'} - x^{i'} = x^k & \text{if } k \text{ is odd} & \\ \phantom{x^{j'} - x^{i'}} = o & \text{otherwise.} & \\ i, j = 1;\ i,j = 2;\ i,j = 3;\ \ldots;\ i,j = n & & \\ x^{i'}, x^{j'} = o;\ i',j' = o;\ i',j' = 1;\ \ldots;\ i',j' = n-2 & & \end{array}\right.$$

Also we know S_n is symmetric matrix.

(a) Let $2 = x^{s'}$ where s' is even. Now place the numbers (i) in the place of Galois Field elements of S_n as $(x^k, 1)_j = i$. Since $2 = x^{s'}$ is even, i takes the same set of values as j takes, but in different order. The last treatment of L, namely n + 1 is kept in the diagonal units of the first part of N'; and the treatments arranged in S_n are transformed identically on S_n of the first part of N'. Now remains the transforming of the treatments in the l^{th} column in the 2nd part of L ($l = n+1, n+2, .., 2n$) to the second part of N' such that every row of N' contains all the treatments. Take $-S_n$. Let x^{2k+1} is the element in the j^{th} row, l^{th} column of $-S_n$. Let $(x^{2(k+t+1)}, 1)_j = i$ in $-S_n$, if $x^{2(k+t+1)}$ occurs in the j^{th} column, i^{th} row of S_n. Hence

$$x^{2k+1} = x^{l-2} - x^{j-2} \quad \text{and}$$

$$x^{2t+1}(x^{l-2} - x^{j-2}) = x^{j-2} - x^{i-2}$$

Suppose $l = i$ i.e. $(1 + x^{2t+1})(x^{l-2} - x^{j-2}) = o$ which shows that either $1 + x^{2t+1} = o$ or $j = l$. If $j = l$, the Galois Field element is zero. Hence no question of transforming element for the number i. $x^{2t+1} = -1$, then $t = \frac{p^m-3}{4}$. We eliminate this value, because we do not want the i^{th} treatment in the i^{th} column except in diagonal places. Consider

$$x^{2t+1}(x^{l-2} - x^{i-2}) = (1 + x^{2t+1})(x^{j-2} - x^{i-2}).$$

Since $x^{j-2} - x^{i-2}$ is even power of x, for left hand side to be even, $1 + x^{2t+1}$ should be even power of x. Thus, for i belongs to the set of values which j takes, $1 + x^{2t+1} = x^{s''}$ where s'' is even positive integer.

Let $1 + x^{2t+1} = x^{s''}$ be even. Now place the numbers i in the place of Galois Field elements of $-S_n$. Hence l^{th} column ($l=n+1, n+2,..,2n$) treatments of the second part (corresponds to L) are transformed to the

i^{th} row $(i = 1,2,\ldots, n)$ of $-S_n$. Since $-S_n$ is symmetric, we transpose the numbered $-S_n$ and place in the second part of (i) which corresponds to N'. We transpose the numbered $-S_n$ for obtaining all the treatments in every row of N'. Different treatments $1,2,\ldots, n$ are placed in the diagonal units of the second part of (i).

(b) $2 = x^{s'}$, $1 + x^{2t+1} = x^{s''}$ where s', s'' are odd. The integers of i^{th} $(i = 1,2,\ldots, n)$ column of S_n of the first part of (ii) are transformed to the i^{th} row of S_n $(i = 1,2,\ldots, n)$ of the first part of (i) and transpose the numbered S_n in place of S_n of the first part of (i). For $-S_n$ of (i), keep the numbers 'i' as

$(x^{2(k+t+1)}, 1)_j = i$, if $x^{2(k+t+1)}$ occurs in the j^{th} column, i^{th} row of S_n, where x^{2k+1} is the element of $-S_n$ in the j^{th} row, i^{th} column.

Case (ii) $n \equiv 3 \bmod 4$. In order to get a two-way design, we use similar procedures as in the case of $n \equiv 1 \bmod 4$. Here we use skew symmetric Σ_n instead of symmetric S_n.

II. Construction of design for the class-23 where $M = E_{vv} - I_v$, L, N' are "BIB" designs.

We have that (for $n = p^m$) where p is prime and m is positive integer).

$$(6.5.9)\quad \begin{aligned} &(i) \quad [S_n \mid -S_n] \quad \text{or} \quad [\Sigma_n \mid -\Sigma_n] \\ &(ii) \quad [-S_n \mid S_n] \quad \text{or} \quad [-\Sigma_n \mid \Sigma_n] \end{aligned}$$

Correspond to two "BIB" designs when we eliminate negative units in (i) and (ii). Let N' corresponds to one of them and L corresponds to another. Then we can construct two-way design in a similar way as in the theorem 6.5.3. Here $M = E_{nn} - I_n$.

Construction of the design for the class-31 : We consider a design for which $L = N'$ and $M = E_{vv} - I_v$. Consider

$$(6.5.10) \quad \begin{array}{llll} \text{(i)} & [S_n \mid -S_n] & \text{or} & [\Sigma_n \mid -\Sigma_n] \\ \text{(ii)} & [S_n \mid -S_n] & \text{or} & [\Sigma_n \mid -\Sigma_n] \end{array}$$

which contain the same "BIB" designs when we eliminate negative units in them. We can construct our required design, when $n = p^m$, by using the method given in the theorem 6.5.3.

6.6 A MEASURE OF NON-ORTHOGONALITY OF A TWO-WAY DESIGN

Let us now consider the extreme values of the multiple correlation between the estimates of estimable contrasts $l_1' \Delta_1 \tau$ and the regression function of $l_2' \Delta_2 \beta$, $l_3' \Delta_3 \alpha$ (where l_1, l_2, l_3 are v - column, w - column u - column vectors respectively) is given by

$$(6.6.1) \qquad R^2 = \frac{\sigma_{12}^2 \sigma_{33} + \sigma_{13}^2 \sigma_{22} - 2\sigma_{12} \sigma_{13} \sigma_{23}}{\sigma_{11} (\sigma_{22} \sigma_{33} - \sigma_{23}^2)}$$

where σ_{ij} is the covariance of i,j (i,j = 1,2,3) functions, viz. $l_1' \Delta_1 \hat{\tau}$, $l_2' \Delta_2 \hat{\beta}$, $l_3 \Delta_3 \hat{\alpha}$. On maximising R^2 subject to the conditions $l_1' \Delta_1 l_1 = 1$, $l_2' \Delta_2 l_2 = 1$, $l_3' \Delta_3 l_3 = 1$, the following equations are obtained.

$$(6.6.2) \qquad \det. \begin{bmatrix} -R^2 I_v & F D_2^* & H D_4^* \\ F' D_1^* & I_w & -E D_5^* \\ H' D_3^* & -E' D_6^* & I_u \end{bmatrix} = 0$$

Since the rank of $\Delta_3 = u-1$ and $\frac{1}{\sqrt{u}} E_{u1}$ is the normalised eigen vector for the zero root of Δ_3, then

$$\det. \begin{bmatrix} I_v & -ED_5^* \\ -E'D_6^* & I_u \end{bmatrix} \neq o$$

Hence (6.6.2) can be written as

(6.6.3) $$\det. \left[R^2 I_v - (FD_2^* \quad HD_4^*) \begin{pmatrix} I_v & -ED_5^* \\ -E'D_6^* & I_u \end{pmatrix}^{-1} \begin{pmatrix} F'D_1^* \\ H'D_3^* \end{pmatrix} \right] = o .$$

By a similar argument as Shah [57] made in his article, we have a measure of non-orthogonality as

(6.6.4) $$\delta = \pi (1 - R_i^2)$$

where R_i^2 are the non-zero eigen values of

$$(FD_2^* \quad HD_4^*) \begin{pmatrix} I_v & -ED_5^* \\ -E'D_6^* & I_u \end{pmatrix}^{-1} \begin{pmatrix} F'D_1^* \\ H'D_3^* \end{pmatrix}$$

which are different multiple correlation coefficients; and δ lies between o and 1 .

Example: Let us find the measure of non-orthogonality (given above) for a two-way design whose incidence matrices are $L = M = N = E_{vv} - I_v$.

The eigen values R_i^2 are found to be $\frac{2}{(v-1)(v-2)}$ with the multiplicity v-1. Hence

$$\delta = \left[\frac{v(v-3)}{(v-1)(v-2)} \right]^{v-1} .$$

APPENDIX 6.1

Appendix 6.1 consists of 60 two-way designs whose row-column incidence matrices are binary and incomplete. The integers (1,2,..., v) in the matrix N (or N') represent the treatments. If we replace them by units, then N becomes row-column incidence matrix.

Design number	v	w	u	r	t	s	Class number
1	4	4	4	3	3	3	29
2	4	5	5	5	4	4	5
3	4	6	6	6	4	4	8
4	4	7	7	7	4	4	5
5	4	8	8	8	4	4	8
6	4	9	9	9	4	4	8
7	4	9	12	9	4	3	23
8	4	10	10	10	4	4	8
9	4	12	12	12	4	4	8
10*	4	12	16	12	4	3	24
11	5	5	5	4	4	4	29
12	5	6	6	6	5	5	5
13	5	5	10	4	4	2	23
14	5	6	10	6	5	3	23
15	5	9	9	9	5	5	8
16	5	11	11	11	5	5	5
17	6	5	10	5	6	3	29
18	6	6	10	5	5	3	23
19	6	7	7	7	6	6	5
20	6	8	8	8	6	6	8
21	6	9	9	9	6	6	8
22	6	10	10	10	6	6	8
23*	6	10	15	10	6	4	23
24	6	11	11	11	6	6	5
25	6	12	12	12	6	6	8

Design number	v	w	u	r	t	s	Class number
26	7	7	7	3	3	3	29
27	7	7	7	4	4	4	29
28	7	7	7	6	6	6	29
29	7	7	14	6	6	3	31
30	7	7	14	6	6	3	23
31	7	8	8	8	7	7	5
32	7	8	14	8	7	4	23
33	7	10	10	10	7	7	8
34	8	8	8	7	7	7	29
35	8	7	14	7	8	4	23
36	8	8	14	7	7	4	23
37	8	9	9	9	8	8	5
38	8	10	10	10	8	8	8
39	8	12	12	12	8	8	8
40	9	9	18	8	8	4	31
41	9	9	18	8	8	4	23
42	9	10	18	10	9	5	23
43	9	4	12	4	9	3	23
44*	9	9	12	8	8	6	31
45	9	10	10	10	9	9	5
46	9	9	9	8	8	8	29
47	9	12	12	12	9	9	8
48	10	9	18	9	10	5	23
49	10	6	15	6	10	4	23
50	10	10	10	9	9	9	29

Design number	v	w	u	r	t	s	Class number
51	10	10	18	9	9	5	29
52	10	12	12	12	10	10	8
53	11	11	11	5	5	5	29
54	11	11	11	6	6	6	29
55	11	11	22	10	10	5	23
56	11	12	22	12	11	6	23
57	12	11	22	11	12	6	23
58	13	13	26	12	12	6	23
59	19	19	19	9	9	9	29
60	23	23	23	11	11	11	29

* The construction of the design is arbitrary.

1.

$N =$

	3	4	2
4		1	3
2	4		1
3	1	2	

2.

$N =$

	1	2	3	4
4		1	2	3
3	4		1	2
2	3	4		1
1	2	3	4	

3.

$N =$

	1	2		3	4
1		3	4		2
2	3		1	4	
	4	1		2	3
3		4	2		1
4	2		3	1	

4.

N =

		1		2	3	4
4			1		2	3
3	4			1		2
2	3	4			1	
	2	3	4			1
1		2	3	4		
	1		2	3	4	

5.

N =

1	2	4	3				
				1	2	3	4
2	3			4	1		
		1	2			4	3
3			4	2			1
	1	3			4	2	
4		2		3		1	
	4		1		3		2

6.

N =

1	4		2			3		
	1	4		2			3	
		1	4		2			3
3			1	4		2		
	3			1	4		2	
		3			1	4		2
2			3			1	4	
	2			3			1	4
4		2			3			1

7.

N'=

1			4			2			3		
2				4			3			1	
3					4			1			2
	2				1	4				3	
	3		2				4				1
	1			3				4	2		
		3			2		1		4		
		1	3					2		4	
		2		1		3					4

8.

N =

	1				3	4			2
2	3			4					1
	4	2				1	3		
	2		4		1			3	
		3	2	1					4
4			3			2	1		
		1		2		3		4	
3		4			2			1	
			1	3	4		2		
1							4	2	3

9.

N =

1	2			3		4					
	1	2			3		4				
		1	2			3		4			
			1	2			3		4		
				1	2			3		4	
					1	2			3		4
4						1	2			3	
	4						1	2			3
3		4						1	2		
	3		4						1	2	
		3		4						1	2
2			3		4						1

10.

N'=

2				3				4				1			
	1				4				3				2		
		1				4				2				3	
			1				4				3				2
4				1				2				3			
	4				3				2				1		
		4				1				3				2	
			2				1				4				3
3				2				1				4			
	3				2				1				4		
		2				3				4				1	
			3				2				1				4

11.

N =

	4	3	2	1
4		2	1	5
3	2		5	4
2	1	5		3
1	5	4	3	

12.

N =

	5	4	3	2	1
1		5	4	3	2
2	1		5	4	3
3	2	1		5	4
4	3	2	1		5
5	4	3	2	1	

13.

N'=

	4		2				5		3
3		1						5	4
	5			2	4			1	
5				1		3	2		
		4	3		2	1			

14.

N'=

	3		5		1	4		2	
4		5			3	2	1		
	1			4		5	3		2
2				3	5			4	1
		2	1				4	3	5
1	2	3	4	5					

15.

N =

		1	3	2		4	5	
1		3		4		2		5
	1	4		3	2	5		
3		5	1		4		2	
2				1	5		3	4
5	2				1	3	4	
	4	2	5				1	3
4	3		2	5				1
	5		4		3	1		2

16.

N =

1	2	3	4	5						
2		5			3	1			4	
3	5				4		2	1		
4			5			3		2		1
5				3			4		1	2
	3	4				2	1			5
	1		3		2				5	4
		2		4	1			5		3
		1	2				5	4	3	
	4			1		5		3	2	
			1	2	5	4	3			

17.

N'=

6	3		5		1		2		4
4	6	5				2		1	3
	1	6		4	2		3	5	
2			6	3		1	5	4	
		2	1	6	4	3			5

18.

N'=

	4		2		6	5		3	
3		1			5	6	4		
	5			2		4	6		1
5				1	3			6	2
		4	3				1	2	6
1	2	3	4	5					

19.

N =

4	1	5	2	6	3	
1	5	2	6	3		4
5	2	6	3		4	1
2	6	3		4	1	5
6	3		4	1	5	2
3		4	1	5	2	6
	4	1	5	2	6	3

20.

N =

1	3	5		2	6	4	
3	1		5	4	2		6
5		1	3	6		2	4
	5	3	1		4	6	2
2	4	6		1	5	3	
4	2		6	3	1		5
6		2	4	5		1	3
	6	4	2		3	5	1

21.

N =

1	4		2	5		3	6	
	1	4		2	5		3	6
4		1	5		2	6		3
3	6		1	4		2	5	
	3	6		1	4		2	5
6		3	4		1	5		2
2	5		3	6		1	4	
	2	5		3	6		1	4
5		2	6		3	4		1

22.

N =

1	3	5			2	4	6		
	1	3	5			2	4	6	
		1	3	5			2	4	6
5			1	3	6			2	4
3	5			1	4	6			2
2	4		6		1	3		5	
	2	4		6		1	3		5
6		2	4		5		1	3	
	6		2	4		5		1	3
4		6		2	3		5		1

23.

N'=

1	2	3	6	5	4									
4	6					5	1	3	2					
5		2				4				1	3	6		
6			1				2			4			3	5
	5			1				4		3		2		6
	3				1				6	2	5		4	
		6		3			5		4		1			2
		4			2		3	6				1	5	
			2	4		3			1			5	6	
			3		5	2		1			6			4

24.

N =

					1	2	3	4	5	6
	5		3	1			2	6		4
		4	1	2		3			6	5
	3	1		5	4		6		2	
	1	2	5		6	4		3		
1			4	6	2			5	3	
2		3		4		6	5	1		
3	2		6			5	1		4	
4	6			3	5	1				2
5		6	2		3		4			1
6	4	5						2	1	3

25.

N =

1	3	5	2	4	6						
6			1				2	4		5	3
	1			2		5		3	4		6
		2			3	4	6		5	1	
5	2	4				1	3	6			
4				1	5	6				3	2
	6		3		2		4		1		5
		6	5	3				1	2	4	
2	5	1							3	6	4
3				5	4		1	2	6		
	4		6		1	3		5		2	
		3	4	6		2	5				1

26.

N =

		6		2		4
4				3	7	
	7		6	1		
6	3					5
			3		2	1
2		7	5			
	4	1			5	

27.

N =

1		6		2		4
4	2			3	7	
	7	3	6	1		
6	3		4			5
			3	5	2	1
2		7	5		6	
	4	1			5	7

28.

N =

	2	3	4	5	6	7
2		4	5	6	7	1
3	4		6	7	1	2
4	5	6		1	2	3
5	6	7	1		3	4
6	7	1	2	3		5
7	1	2	3	4	5	

29.

N'=

		6		2		4			7		3		5
4				3	7		6				1	5	
	7		6	1				4		5	2		
6	3					5	2	7					1
			3		2	1				6		7	4
2		7	5				4		1	3			
	4	1			5			3	6			2	

30.

N'=

	7		3		5				4		6		2
		6	5			1	7				4	3	
4					2	5		6		1	7		
		1		2	7		3	5					6
6	4	7								2		1	3
	3			1		4	5		2	7			
2			6	3				1	5			4	

31.

N =

	1	2	3	4	6	5	7
7		1	2	3	4	6	5
5	7		1	2	3	4	6
6	5	7		1	2	3	4
4	6	5	7		1	2	3
3	4	6	5	7		1	2
2	3	4	6	5	7		1
1	2	3	4	6	5	7	

32.

N'=

1		6		2		4			7		3		5
4	2			3	7		6				1	5	
	7	3	6	1				4		5	2		
6	3		4			5	2	7					1
			3	5	2	1				6		7	4
2		7	5		6		4		1	3			
	4	1			5	7		3	6			2	
							1	2	3	4	5	6	7

33.

N =

		1	2		3	4	5	6	7
1	2	3	4	5	6	7			
7			1	3	4	6		5	2
2	3	4	5		1		7		6
3		2		1	7	5	6		4
4	1	5	6			2		7	3
6	7			2	5	1	3	4	
5	6	7	3	4			1	2	
	5	6		7	2		4	3	1
	4		7	6		3	2	1	5

34.

N =

	2	3	4	5	6	7	8
3		1	8	6	5	2	4
4	5		1	2	7	6	3
5	4	6		1	3	8	7
6	8	5	7		1	4	2
7	3	2	6	8		1	5
8	6	4	3	7	2		1
2	1	7	5	4	8	3	

35.

N'=

1		6		2		4	8	7		3		5	
4	2			3	7			8	6	5			1
	7	3	6	1			4		8			2	5
6	3		4			5			1	8	2	7	
			3	5	2	1	6	4	7		8		
2		7	5		6			3			1	8	4
	4	1			5	7	2			6	3		8

36.

N'=

	7		3		5		8		4		6		2
		6	5			1	7	8			4	3	
4					2	5		6	8	1	7		
		1		2	7		3	5		8			6
6	4	7								2	8	1	3
	3			1		4	5		2	7		8	
2			6	3				1	5			4	8
1	2	3	4	5	6	7							

37.

N =

	2	3	4	5	6	7	8	1
1		2	3	4	5	6	7	8
8	1		2	3	4	5	6	7
7	8	1		2	3	4	5	6
6	7	8	1		2	3	4	5
5	6	7	8	1		2	3	4
4	5	6	7	8	1		2	3
3	4	5	6	7	8	1		2
2	3	4	5	6	7	8	1	

38.

N =

1	4	5	7		2	3	6	8	
	1	4	5	7		2	3	6	8
7		1	4	5	8		2	3	6
5	7		1	4	6	8		2	3
4	5	7		1	3	6	8		2
2	3	6	8		1	4	5	7	
	2	3	6	8		1	4	5	7
8		2	3	6	7		1	4	5
6	8		2	3	5	7		1	4
3	6	8		2	4	5	7		1

39.

N =

1	5		2	6		3	7		4	8	
	1	5		2	6		3	7		4	8
5		1	6		2	7		3	8		4
4	8		1	5		2	6		3	7	
	4	8		1	5		2	6		3	7
8		4	5		1	6		2	7		3
3	7		4	8		1	5		2	6	
	3	7		4	8		1	5		2	6
7		3	8		4	5		1	6		2
2	6		3	7		4	8		1	5	
	2	6		3	7		4	8		1	5
6		2	7		3	8		4	5		1

40.

N'=

	6		8		2		4				7		9		3		5
6		5		3	1							9			8	7	4
	5			2			9	8	7			6		4	1		
8				7		5	1			9	6			3			2
	3	2	7			4			9					8		6	1
2	1					9		7			4	3	8			5	
			5	4	9			6	3	8	1					2	
4		9	1					3		7			6	5	2		
		8			7	6	3		5	4		2	1				

41.

N'=

	7		9		3		5				8		2		4		6
9		7		8	4							5			6	1	3
	4			6			7	1	2			8		9	5		
3				9		2	6			1	5			7			8
	9	1	6			8			4					2		3	7
5	8					3		4			2	1	7			9	
			3	1	8			2	6	5	9					4	
7		6	2					5		3			4	1	9		
		4			5	1	2		8	6		7	3				

42.

N'=

	6		8		2		4		1	7		9		3		5	
6		5		3	1				9	2	7		8	4			
	5			2			9	8		4	3		6			7	1
8				7		5	1		3			4	9		2	6	
	3	2	7			4				9	1	6	5		8		
2	1					9		7	5	8				6	3		4
			5	4	9			6				3	1	8	7		2
4		9	1					3	7		6	2				8	5
		8			7	6	3				4			5	1	2	9
1	2	3	4	5	6	7	8	9									

43.

N'=

1	6	8		9	4		7	3		2	5
2	4	9	5		7	1		8	6		3
3	5	7	8	6		9	2		1	4	
			1	2	3	4	5	6	7	8	9

44.

N'=

	3	4	2		9	8		6	5	7	
9		1	5	8		6	7		4	3	
5	1			2	8		4	7	6	9	
	9	8		6	1	2	5		3		7
2		7	6		3		9	8	1		4
3	7		1	4		9		5	2		8
	4	5	9	3			2	1		8	6
6		2		7	5	4		3		1	9
8	6		7		4	1	3			2	5

45.

N =

	1	2	3	4	5	6	7	8	9
9		1	2	3	4	5	6	7	8
8	9		1	2	3	4	5	6	7
7	8	9		1	2	3	4	5	6
6	7	8	9		1	2	3	4	5
5	6	7	8	9		1	2	3	4
4	5	6	7	8	9		1	2	3
3	4	5	6	7	8	9		1	2
2	3	4	5	6	7	8	9		1
1	2	3	4	5	6	7	8	9	

46.

N =

	2	3	4	5	6	7	8	9
2		4	5	6	7	8	9	1
3	4		6	7	8	9	1	2
4	5	6		8	9	1	2	3
5	6	7	8		1	2	3	4
6	7	8	9	1		3	4	5
7	8	9	1	2	3		5	6
8	9	1	2	3	4	5		7
9	1	2	3	4	5	6	7	

47.

N =

1	4	7		2	5	8		3	6	9	
	1	4	7		2	5	8		3	6	9
7		1	4	8		2	5	9		3	6
4	7		1	5	8		2	6	9		3
3	6	9		1	4	7		2	5	8	
	3	6	9		1	4	7		2	5	8
9		3	6	7		1	4	8		2	5
6	9		3	4	7		1	5	8		2
2	5	8		3	6	9		1	4	7	
	2	5	8		3	6	9		1	4	7
8		2	5	9		3	6	7		1	4
5	8		2	6	9		3	4	7		1

48.

N'=

10	6		8		2		4		1		9		3		5		7
6	10	5		3	1					2		7			9	4	8
	5	10		2			9	8	6		3	1		7	4		
8			10	7		5	1			6	2	4		9			3
	3	2	7	10		4			8				5	1		9	6
2	1				10	9		7			5	8	4	6		3	
			5	4	9	10		6	2	3	8				7	1	
4		9	1				10	3		5			7	2	6	8	
		8			7	6	3	10	4	1		5	2				9

49.

N'=

1			4	5	6		2	10	9	3	7	8		
	1	3	9		8	10	4		2	6		5		7
	6	1	10	7		9	8	2		5	3		4	
2		8		9	1	3		5	7	4			6	10
3	5			1	10	2	7				6	9	8	4
4	2	7	1					8	6		10	3	9	5

50.

N =

	7	6	5	8	9	10	2	3	4
10		1	7	6	8	9	3	4	5
9	10		2	1	7	8	4	5	6
8	9	10		3	2	1	5	6	7
2	8	9	10		4	3	6	7	1
4	3	8	9	10		5	7	1	2
6	5	4	8	9	10		1	2	3
3	4	5	6	7	1	2		10	9
5	6	7	1	2	3	4	10		8
7	1	2	3	4	5	6	9	8	

51.

N'=

		8		2		4		6	10		9		3		5		7
			5			6	1	3		10		7			9	4	8
2			8		9	5			6		10	1		7	4		
	1	5			7			8		6	2	10		9			3
4					2		3	7	8				10	1		9	6
		2	1	7			9				5	8	4	10		3	
6	5	9					4		2	3	8				10	1	
	3			4	1	9				5			7	2	6	10	
8	6		7	3					4	1		5	2				10
1	2	3	4	5	6	7	8	9									

52.

N =

1	3	5	7	9		2	4	6	8	10	
	1	3	5	7	9		2	4	6	8	10
9		1	3	5	7	10		2	4	6	8
7	9		1	3	5	8	10		2	4	6
5	7	9		1	3	6	8	10		2	4
3	5	7	9		1	4	6	8	10		2
2	4	6	8	10		1	3	5	7	9	
	2	4	6	8	10		1	3	5	7	9
10		2	4	6	8	9		1	3	5	7
8	10		2	4	6	7	9		1	3	5
6	8	10		2	4	5	7	9		1	3
4	6	8	10		2	3	5	7	9		1

53.

N =

		8		10		2		4		6
3				6			4	11	7	
	10		1	9		6	11			
5	9					8			6	3
			2		1	11		8	3	
7	8	5	11					10		
	5				4		1	3		10
9			10	7	3					2
		2	7				6		1	5
11		4			2	9	5			
	1	7		4	9				8	

54.

N =

1		8		10		2		4		6
3	2			6			4	11	7	
	10	3	1	9		6	11			
5	9		4			8			6	3
			2	5	1	11		8	3	
7	8	5	11		6			10		
	5				4	7	1	3		10
9			10	7	3		8			2
		2	7				6	9	1	5
11		4			2	9	5		10	
	1	7		4	9				8	11

55.

			8		10		2		4		6		3		5		7		9		11	
	3				6			4	11	7				10	9		8	5				1
		10		1	9		6	11				8					5			2	4	7
	5	9					8			6	3			1		2	11		10	7		
				2		1	11		8	3		10	6	9					7			4
N'=	7	8	5	11					10							1		4	3		2	9
		5				4		1	3		10	2		6	8	11					9	
	9			10	7	3					2		4	11				1		6	5	
			2	7				6		1	5	4	11			8	10	3				
	11		4			2	9	5					7		6	3				1		8
		1	7		4	9				8		6			3			10	2	5		

56.

	1		10		2		4		6		8			7		9		11		3		5
	6	2			11			3	7	4		9				10			5	8	1	
		11	3	6	1		10	9					5		2	7		8	4			
	8	6		4			3			5	9	11	1					2			7	10
				3	5	8	1		2	11					7		4	9		10	6	
N'=	10	7	11	8		6			5			3	9	2	1					4		
		3				5	7	10	1		4		8				9		6	11		2
	2			9	3	10		8			7	5			11	4	1					6
			6	5				7	9	2	1			4	10				11		8	3
	4		9			11	5	2		10		7		8			3	6	1			
		4	1			7				9	11		10	5		6	2				3	
												1	2	3	4	5	6	7	8	9	10	11

57.

	1		10		2		4		6		8	12	3		5		7		9		11	
	6	2			11			3	7	4			12	10	9		8	5				1
		11	3	6	1		10	9				8		12			5			2	4	7
	8	6		4			3			5	9			1	12	2	11		10	7		
				3	5	8	1		2	11		10	6	9		12			7			4
N'=	10	7	11	8		6			5							1	12	4	3		2	9
		3				5	7	10	1		4	2		6	8	11		12			9	
	2			9	3	10		8			7		4	11				1	12	6	5	
			6	5				7	9	2	1	4	11			8	10	3		12		
	4		9			11	5	2		10			7		6	3				1	12	8
		4	1		8	7				9	11	6			3			10	2	5		12

58.

N =

	3		5		7		9		11		13	
8		6	13					5		10	7	
	1				4	12	10	13		5		
10	9			8	3					7		12
			1				6	2	12	3		7
12		2	11			10	5					9
		9			1				8	4	2	5
2		11		4	13			12	7			
	4	7		11			1				10	6
4				13		6	3			2	9	
	12	8	6	9		13			1			
6	11					3		8	5			4
			2	10	8	11		3			1	
		4		6		8		10		12		2
				3	11	9	12		4			1
9			7	2					6		11	8
		1				5	13	11	2		6	
11	13	10			9	4					8	
	8			1				7	3	13	4	
13	10		3	12			11	6				
	6		10			1				9	5	3
3			12		5	2			13	8		
	7	5	8		12			1				11
5					2		7	4			3	10
		13	9	7	10		2			1		
7	5	12					4		9	6		

59.

N =

		12		14		16		18		2		4		6		8		10
3			12		18	5		8	17			16	19	11				
	15		1		8			10			16	5				14	11	7
5					14		2	7		10	19			18	3	13		
	13	9	17		1		10			12			18	7				16
7							16		4	9		12	3			2	5	15
		18	15	11	19		1		12			14			2	9		
9	7	17							18		6	11		14	5			4
				2	17	13	3		1		14			16			4	11
11		6	9	19							2		8	13		16	7	
	6	13				4	19	15	5		1		16			18		
13	9			8	6	3							4		10	15		18
			8	15				6	3	17	7		1		18			2
15		2	11			10	13	5							2		12	17
		4			10	17				8	5	19	9		1		2	
17	14	19		4	13			12	15	7							8	
	4			6			12	19				10	7	3	11		1	
1	10		16	3		6	15			14	17	9						
	1		6			8			14	3				12	9	5	13	

60.

N =

		14		16		18		20		22		2		4		6		8		10		12
4						3	9		12	23		18	21	6				5	17		16	
	12		8	1	20	16			9			11				23	21	14			15	
6	18							5	11		14	3		20	23	8				7	19	
	17		14		10	1	22	18			11			13				3	23	16		
8	21		20							7	13		16	5		22	3	10				9
			19		16		12	1	2	20			13			15				5	3	18
10		11	23		22							9	15		18	7		2	5	12		
	5	20			21		18		14	1	4	22			15			17				7
12				13	3		2							11	17		20	9		4	7	14
		9	7	22			23		20		16	1	6	2			17			19		
14	9	16				15	5		4							13	19		22	11		5
				11	9	2			3		22		18	1	8	4			19			21
16		8	11	18				17	7		6							15	21		2	13
		23				13	11	4			5		2		20	1	10	6			21	
18	4	15		10	13	20				19	9		8							17	23	
	23			3				15	13	6			7		4		22	1	12	8		
20	3		6	17		12	15	22				21	11		10							19
			3			5				17	15	8			9		6		2	1	14	10
22		21	5		8	19		14	17	2				23	13		12					
	16	12			5			7				19	17	10			11		8			
2				23	7		10	21		16	19	4				3	15		14			
	6	1	18	14			7			9				21	19	12			13		10	

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