CONTRIBUTIONS TO ERGODIC THEORY

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CONTENTS

INTRODUCTION

Ergodic theory is chiefly concerned with the study of transformations on a measure space which preserve the measure. Interesting classes of such transformations are the classes of ergodic, weakly mixing and mixing transformations. The bulk of this thesis is devoted to the study of a closely related family of transformations called the weakly stable transformations; these are more general than the weakly mixing transformations.

This thesis is divided into three parts. Part I contains preliminary ideas, notations and some results on the invertibility and continuity properties of transition functions (Chapters 1 and 2). In Chapter 3 we collect known results on the splitting theorem and the ergodic theorem in general Banach spaces for later reference.

Part II begins with motivating the introduction and study of weakly stable transformations (Chapter 4). In Chapter 5, some simple properties of weakly stable transformations are exhibited. Chapter 6 gives a generalization of the classical mixing theorem for an invertible measure-preserving transformation and relates the weak stability of such a transformation to the equality of the invariant σ-field in the product space with the product of the invariant σ-fields.

In Chapter 7, we look at weak stability from a different angle. Here we introduce weak stability for semigroups of contractions on an arbitrary Hilbert space using reversible vectors. This coincides with the previous definition if we consider the group generated by the induced unitary operator in the L_2-space. Generalizations of the results of Chapter 6 are proved for semigroups of contractions in a Hilbert space. As corollaries, we get some known generalizations of the mixing theorem to semigroups of transformations. Chapter 8 gives another result on the weak stability of semigroups of measure-preserving transformations relating it to the symmetric invariant sets in the product space.

Chapter 9 is concerned with a family of transformations on a probability space endowed with a probability distribution. Associated with the family are a skew product transformation and a transition function. The weak stability properties of these are related to similar properties of the family of transformations.

We study automorphisms of compact groups in Chapter 10. Here the weak stability of an automorphism enables the subspace of invariant functions in the L_2-space to be spanned by the invariant characters. Part II ends with Chapter 11 in which the ergodic decomposition of a transformation is considered. It is proved that if almost all ergodic components are weakly mixing, then the transformation is weakly stable.

The problem of existence of invariant measures for families of transformations is considered in Part III. The procedure of using Banach limits for the invariant measure problem for a single transformation is by now well established. The corresponding invariant means for amenable semigroups are utilized to get invariant measures for families of transformations. It is hoped that this will stimulate further research in this direction.

PART I

P R E L I M I N A R I E S

CHAPTER 1

BASIC IDEAS AND DEFINITIONS

Let X be an abstract set of points and $\underline{A}$ a σ-field of subsets of X. The pair $(X, \underline{A})$ is called a measurable space; elements of $\underline{A}$ are measurable sets. If X is a topological space, we take $\underline{A}$ to be the σ-field generated by the family of all open sets and call it the Borel σ-field.

A transformation, or a function T from a measurable space $(X, \underline{A})$ to a measurable space $(Y, \underline{B})$ is called measurable if the inverse image of every measurable set in Y is measurable in X. The identity transformation I on a measurable space is trivially measurable. A measurable transformation T on a measurable space is invertible, if there exists another measurable transformation T^{-1} such that $TT^{-1} = T^{-1}T = I$. We shall, throughout most of this work, consider only measurable sets and measurable transformations.

A set A is strictly invariant for a transformation T (or strictly T-invariant) if $T^{-1}A = A$. A real - or complex-valued Borel measurable function f on X is strictly invariant (or strictly T-invariant) if $f(Tx) = f(x)$ for all $x \in X$. For a given transformation T, the class of all strictly invariant sets is a σ-field. This we denote by $\underline{I}^o = \underline{I}^o(T)$ and call the strictly invariant σ-field. A measurable function is strictly invariant if and only if it is measurable with respect to $\underline{I}^o$.

Let m be a probability measure on (X, $\underline{A}$), i.e., a non-negative and countably additive set function defined on $\underline{A}$ with m(X) = 1. (Occasionally we consider infinite measures m, but then we shall assume that X is the disjoint union of a countable number of sets A_i with $m(A_i) < \infty$ and call such an m σ-finite.) One of the basic principles of measure theory is that sets of zero measure are negligible. Thus we do not distinguish between two functions or transformations which are equal almost everywhere (i.e., outside a set of measure zero); such functions are considered equivalent. Two sets A and B are equivalent if their characteristic functions 1_A and 1_B are equivalent, i.e., $m(A + B) = 0$. (+ stands for the symmetric difference operation.) Two sub-σ-fields $\underline{A}_1$ and $\underline{A}_2$ of $\underline{A}$ are equivalent, if for every set in one of them, there is a set in the other equivalent to the first one; in such a case, we write just $\underline{A}_1 = \underline{A}_2$. A sub-$\sigma$-field $\underline{A}_1$ of $\underline{A}$ is called trivial if it is equivalent to the trivial σ-field consisting of the empty set and the whole space only.

The family of all equivalence classes of measurable sets is a Boolean σ-algebra with the natural operations of union and intersection. The measure on X taken over to the Boolean σ-algebra is positive - any non-zero element has positive measure. This Boolean σ-algebra with this measure is called the measure algebra of the measure space $(X, \underline{A}, m)$.

Given a measure-preserving transformation T on X, we can define a mapping of the measure algebra as follows: the image of an equivalence class of sets is defined by taking one representative A and forming the equivalence class containing $T^{-1} A$. This mapping, denoted by T^{-1}, is well-defined and measure-preserving. If T is invertible, then T^{-1} on the measure algebra is an automorphism.

We now introduce the L_p-spaces. The real $L_p(X) = L_p(X, \underline{A}, m)$ (sometimes written as L_p) for $1 \leq p < \infty$ is the space of equivalence classes of real-valued functions f with $|f|^p$ integrable. The complex L_p-spaces are similarly defined. L_p, for each p, is a Banach space with norm $\|f\|_p = [\int |f|^p \, dm]^{1/p}$, while L_2 is a Hilbert space with inner product $(f_1, f_2) = \int f_1 \bar{f}_2 \, dm$. ($\bar{f}$ denotes the complex conjugate of the function f.) We denote by $L_\infty(X) = L_\infty$ the space of all equivalence classes of (real/

complex- valued) essentially bounded measurable functions on X. L_∞ is a Banach space with essential supremum as norm.

Given a sub-σ-field $\underline{A}_1$ of $\underline{A}$, the conditional expectation of a function f is a function $E(f|\underline{A}_1)$ measurable with respect to $\underline{A}_1$ and having the property

$$\int_A f\,dm = \int_A E(f|\underline{A}_1)\,dm$$

for all $A \in \underline{A}_1$. The conditional probability $P(A|\underline{A}_1)$ of a set A is defined as the conditional expectation of 1_A:

$$P(A|\underline{A}_1) = E(1_A|\underline{A}_1).$$

A set A is invariant for a transformation T (or T-invariant) if $m(A + T^{-1}A) = 0$. A function f is invariant for T (or T-invariant) if $f(Tx) = f(x)$ a.e. The σ-field of all invariant sets is denoted by $\underline{I} = \underline{I}(T)$ and called the invariant σ-field. A transformation T is called measure-preserving if $m(T^{-1}A) = m(A)$ for all $A \in \underline{A}$. For a given transformation T, a measure m' is invariant if T is measure-preserving in the space $(X, \underline{A}, m')$. A transformation T is non-singular if $m(A) = 0$ implies $m(T^{-1}A) = 0$. Any measure-preserving transformation is non-singular.

Measurable transformations are only a particular case of a more general class of functions called transition functions. A transition function $P = P(\cdot,\cdot)$ on X is a function from $X \times \underline{A}$ to the unit interval $[0, 1]$ (with the usual Borel σ-field) such that (i) for fixed $x \in X$, $P(x, \cdot)$ is a probability measure on $(X, \underline{A})$ and (ii) for fixed $A \in \underline{A}$, $P(\cdot, A)$ is a measurable function on $(X, \underline{A})$. A measurable transformation T induces a transition function P by the formula $P(x, A) = 1_A(Tx)$ for all $x \in X$, $A \in \underline{A}$. We call such transition functions induced or degenerate. The transition function $P(x, A) = 1_A(x)$ corresponds to the identity transformation and may itself be denoted by I.

The product of two transition functions P_1 and P_2 is defined as the transition function P_1P_2 given by

$$(P_1P_2)(x, A) = \int P_1(y, A)\, P_2(x, dy).$$

If P_1 and P_2 are both degenerate, then P_1P_2 is also so and corresponds to the product of the transformations concerned. The above multiplication is associative in the set of all transition functions on $(X, \underline{A})$ and $I(x, A) = 1_A(x)$ acts as the identity for this multiplication.

We shall call a transition function P invertible if there exists another transition function P^{-1} such that $PP^{-1} = I$. Then $P^{-1}P = I$ and P^{-1} is unique. We call P^{-1} the inverse of P. It is easy to show that an induced transition function $P(x, A) = 1_A(Tx)$ is invertible if and only if the inducing transformation T is invertible. The following theorem goes further and shows that any invertible transition function is induced. In proving this, we assume that single point sets are measurable.

<u>Theorem 1.1.</u> A transition function P is invertible if and only if there exists an invertible measurable transformation T such that $P(x, A) = 1_A(Tx)$.

<u>Proof.</u> We need to prove only the 'only if' part. Let P be invertible. For a fixed $x \in X$, we have

$$\int P^{-1}(y, \{x\})\, P(x, dy) = 1.$$

The integrand lies between 0 and 1 and hence $P^{-1}(y, \{x\}) = 1$ a.e. $P(x, \cdot)$. Choose and fix a y_0 such that $P^{-1}(y_0, \{x\}) = 1$. Considering the equation

$$\int P(z, \{y_0\})\, P^{-1}(y_0, dz) = 1,$$

we see that $P(z, \{y_o\}) = 1$ a.e. $P^{-1}(y_o, \cdot)$. But the measure $P^{-1}(y_o, \cdot)$ is concentrated at x. Hence $P(x, \{y_o\}) = 1$. Defining T by the equation $y_o = Tx$, we see that T is well-defined. Besides $P(x, A) = 1_A(Tx)$ for all $x \in X$, $A \in \underline{A}$ and hence T is measurable and invertible.

A transition function P is non-singular with respect to the measure m if $m(A) = 0$ implies that $P(x, A) = 0$ a.e. This coincides with the non-singularity of a transformation when we consider the induced transition function. A measure m is invariant for a transition function P if $m(A) = \int P(x, A) m(dx)$ for all $A \in \underline{A}$. It is clear that invariant measures for a transformation correspond to those for the induced transition function.

Transformations and transition functions induce certain operators in L_p-spaces which are of great interest in our studies. Let T, respectively P, preserve the measure m. In L_2 or in L_∞, we consider the operator U given by

$$(Uf)(x) = f(Tx),$$
$$(Uf)(x) = \int f(y) P(x, dy).$$

The operator U is always a contraction; on L_2, the operator

U induced by T is an isometry. But on L_1, we introduce operators in a different way. We assume only that the transition function P is non-singular. Then, for $f \in L_1$, the measure $\bar{m}'$ given by $\bar{m}'(A) = \int P(x, A) f(x) m(dx)$ is absolutely continuous with respect to m and so there is a $g \in L_1$ with $\bar{m}'(A) = \int_A g\,dm$. Defining a transformation V by $Vf = g$, we see that V is a contraction. Moreover, the adjoint of V is the operator U on L_∞ defined above. For the case of a non-singular transformation T, the operator V is given by $\int_A Vf\,dm = \int_{T^{-1}A} f\,dm$. The importance of V becomes evident when we consider the problem of the existence of a finite invariant measure equivalent to the given measure m. (See Part III). This problem is the same as that of the existence of a strictly positive invariant function for the induced contraction V on L_1.

CHAPTER 2

CONTINUITY PROPERTIES OF TRANSITION FUNCTIONS

We discuss, in this chapter, the continuity, equicontinuity and quasi-equicontinuity properties of transition functions. The motivation for this discussion comes from the recent work of Rosenblatt [49] on ergodic decompositions. Some of the results are useful in Part III.

Throughout this chapter, X denotes a compact Hausdorff space and $\underline{A}$ the Borel σ-field of X. The transition functions P we study are assumed to have the property that for each $x \in X$, $P(x, \cdot)$ is a regular probability measure. Let $C(X)$ be the Banach space of all real-valued continuous functions on X with supremum norm. We consider the operator U.

$$(Uf)(x) = \int f(y)\, P(x, dy)$$

induced by P on the space of all real-valued bounded functions on X. Rosenblatt [49] makes assumptions of the following kind on the operator U.

(a) For each $f \in C(X)$, $Uf \in C(X)$.

(b) For each $f \in C(X)$, the family $\{U^n f\}$, $n = 0,1,2,\ldots$ is equicontinuous.

(c) For each $f \in C(X)$, the family $\{U^n f\}$ $n = 0,1,2,\ldots$ is quasi-equicontinuous.

(For definitions of equicontinuity and quasi-equicontinuity of a subset of $C(X)$, see Dunford and Schwartz [13].) Our aim, in this chapter, is to show that these are equivalent to certain natural conditions on the transition function itself.

Let $M(X)$ denote the Banach space of regular finite signed measures on X with total variation as norm. By Riesz's theorem, $C^*(X)$, the conjugate space of $C(X)$ is isometrically isomorphic with $M(X)$. Throughout what follows, we shall consider $M(X)$ with the $C(X)$-topology of $C^*(X)$, called the weak topology of $M(X)$. A base for the neighbourhood system at $m \in M(X)$ is given by

$$N(m; f_1,\ldots,f_n; \varepsilon) = \{m' : |\int f_i dm - \int f_i dm'| < \varepsilon,\ 1 \leq i \leq n\}$$

where $\varepsilon > 0$, n a positive integer and $f_1,\ldots, f_n \in C(X)$ are arbitrary. $M(X)$ with the weak topology is a locally convex linear topological space. $P(X)$, the set of probability measures in $M(X)$ is a compact convex subset of $M(X)$.

The space X, being compact Hausdorff, has a unique uniformity $\underline{X}$ inducing the given topology. $M(X)$ has a natural uniformity $\underline{M}$, a typical element of a base for which is the set $\{(m_1, m_2): m_1 - m_2 \in A\}$ where A is a neighbourhood of the origin in $M(X)$. We consider X and $M(X)$ as uniform spaces with uniformities $\underline{X}$ and $\underline{M}$ respectively.

<u>Definition 2.1</u> A transition function P on X is <u>continuous</u> if the map $x \to P(x, \cdot)$ from X to $M(X)$ is continuous.

It is immediate that the operator U takes $C(X)$ into $C(X)$ if and only if P is continuous. Besides the following theorem holds, so that the continuity of a transition function is a natural generalization of that of a transformation.

<u>Theorem 2.1</u> The transition function $P(x, A) = 1_A(Tx)$ is continuous if and only if the transformation T is a continuous map of X into itself.

<u>Proof</u>. Let D be the set of degenerate measures in $P(X)$. X is homeomorphic with D (in the relative weak topology) under the map $h : x \to \mu_x(A) = 1_A(x)$. The truth of the theorem is now immediate.

We now recall the well-known definitions of equicontinuity and quasi-equicontinuity in a general set up.

Definition 2.2 (Kelley [36]). Let F be a family of maps from a topological space X to a uniform space $(Y, \underline{\bar{Y}})$. F is equicontinuous at $x \in X$ if, for each $A^* \in \underline{\bar{Y}}$, there is a neighbourhood N_x of x such that $(f(x), f(y)) \in A^*$ for all $y \in N_x$ and all $f \in F$. F is equicontinuous if it is equicontinuous at each $x \in X$.

Definition 2.3 (Bartle [2]). A family F of maps from a compact Hausdorff space X to a uniform space $(Y, \underline{\bar{Y}})$ is quasi-equicontinuous if, given a net x_α converging to x, for every $A^* \in \underline{\bar{Y}}$ and every α_o, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \geq \alpha_o$ such that, for every $f \in F$, there is an i, $1 \leq i \leq n$ with the property that $(f(x_{\alpha_i}), f(x)) \in A^*$.

Definition 2.4. We call a family $\{ P_s \}$ of transition functions on X equicontinuous (quasi-equicontinuous) if the family qua a family of maps from X to the uniform space $(M(X), \underline{\bar{M}})$ is equicontinuous (quasi-equicontinuous).

In a similar way, we consider equicontinuous (quasi-equicontinuous) family of maps of X into itself.

Theorem 2.2 A family $\{T_s\}$ of measurable maps of X into itself is equicontinuous (quasi-equicontinuous) if and only if the family $\{P_s\}$ of induced transition functions is equicontinuous (quasi-equicontinuous).

Proof. As in the proof of Theorem 2.1, we consider D which is homeomorphic to X. With respect to the relative uniformity on D, the homeomorphism h (as well as its inverse) is uniformly continuous. Using this, it is easy to verify the assertion.

The following result connects our definitions with those of Rosenblatt [49].

Theorem 2.3 A family $\{P_s\}$ of transition functions on X is equicontinuous (quasi-equicontinuous) if and only if, for every $f \in C(X)$, the family of functions $\{U_s f\}$ is equicontinuous (quasi-equicontinuous).

Proof. We give the proof for the case of quasi-equicontinuity. The proof for the equicontinuity case is similar.

Let $\{P_s\}$ be quasi-equicontinuous. Fix an $f \in C(X)$. Let a net x_α converging to x be given. Then for a given $\varepsilon > 0$ and α_0, we have to find $\alpha_1, \ldots, \alpha_n \geq \alpha_0$ such that for every s, there is an i with the property that

$$|(U_s f)(x_{\alpha_i}) - (U_s f)(x)| < \epsilon.$$

Let A be the neighbourhood of the origin in $M(X)$ defined by

$$A = \{ m : | \int f\,dm | < \epsilon \}.$$

Let $A^* = \{ (m_1, m_2) : m_1 - m_2 \in A \}$. Since $\{ P_s \}$ is quasi-equicontinuous, given A^* and α_o, there exist $\alpha_1, \ldots, \alpha_n \geq \alpha_o$ such that, for every s, there is an i with the property that $(P_s(x_{\alpha_i}, \cdot), P_s(x, \cdot)) \in A^*$. Hence

$$|(U_s f)(x_{\alpha_i}) - (U_s f)(x)| =$$

$$|\int f(y)\, P_s(x_{\alpha_i}, dy) - \int f(y)\, P_s(x, dy)|$$

$$< \epsilon.$$

The other part is proved by (essentially) retracing the steps.

CHAPTER 3

SEMIGROUPS OF OPERATORS

For semigroups of operators on a Banach space, there are two important results - the ergodic theorem and the splitting theorem. Increasingly more general versions of the splitting theorem were proved by Jacobs [30 , 31] and deLeeuw and Glicksberg [10]. Ergodic theorems have been proved, among others, by Alaoglu and Birkhoff [1], Eberlein [16] and Day [8]. We give here an account of these results in a form more general than what we need in Parts II and III; we also mention a connection between these two.

We first present the splitting theorem following deLeeuw and Glicksberg [10].

Let H be a Banach space. Elements of H will be denoted by f, g, h etc. An operator semigroup (or a semigroup of operators) on H is a subsemigroup of the multiplicative semigroup of bounded operators on H containing the identity operator I. For an operator semigroup $\underline{U}$, the orbit $(\underline{U} f)$ of an element $f \in H$ is the set $\{ Uf: U \in \underline{U} \}$. We call $\underline{U}$ weakly almost periodic if each orbit has compact closure in the weak topology of H. Any such semigroup is bounded : there exists a constant C such that $\|U\| \leq C$

for all $U \in \underline{U}$. If H is a Hilbert space, then the converse is also true: any bounded semigroup is weakly almost periodic.

For any operator semigroup $\underline{U}$, we denote by $w(\underline{U})$ the weak operator closure of $\underline{U}$. $w(\underline{U})$ is again an operator semigroup. The weak orbit closure of an element f is denoted by $w(\underline{U}\,f)$. We then have $w(\underline{U}\,f) = (w(\underline{U})f)$.

An element f is invariant (for or under a semigroup $\underline{U}$) if $Uf = f$ for all $U \in \underline{U}$. A subspace L of H is invariant if $UL \subseteq L$ for all $U \in \underline{U}$. It is easy to see that the weak closure $w(\underline{U})$ has the same invariant elements and invariant subspaces as $\underline{U}$.

We now introduce the reversible, flight and almost periodic elements for a given operator semigroup $\underline{U}$ on H.

<u>Definition 3.1</u> An element $f \in H$ is <u>reversible</u> if for each $U \in w(\underline{U})$, there is a $U_o \in w(\underline{U})$ such that $U_o U f = f$.

The set $R = R(\underline{U})$ of reversible elements is an invariant subset of H but need not be a linear subspace.

<u>Definition 3.2</u> An element $f \in H$ is <u>flight</u> if $0 \in w(\underline{U}\,f)$.

The set $F = F(\underline{U})$ of flight elements is in general neither invariant nor a linear subspace.

We call a finite dimensional invariant subspace L a unitary subspace if $\underline{U}$ restricted to L is contained in a bounded group of operators on L. Any unitary subspace is contained in R.

<u>Definition 3.3</u> The set $A(\underline{U})$ of <u>almost periodic</u> elements is the closed linear span of the unitary subspaces.

We shall say that the splitting theorem holds for a weakly almost periodic operator semigroup $\underline{U}$ on H if $F(\underline{U})$ is a closed invariant linear subspace of H, $A(\underline{U}) = R(\underline{U})$ and $H = R(\underline{U}) \oplus F(\underline{U})$.

<u>Theorem 3.1</u> The splitting theorem holds for any semigroup of contractions on a Hilbert space. If H is an ~~reflexive~~ arbitrary Banach space and $\underline{U}$ is an abelian weakly almost periodic semigroup, then the splitting theorem holds for $\underline{U}$.

We now turn to ergodic theorems. As before, $\underline{U}$ is a weakly almost periodic operator semigroup on the Banach space H. Besides $\underline{U}$, we consider the convex hull $[\underline{U}]$ of $\underline{U}$. We say that the ergodic theorem holds for the semigroup $\underline{U}$ if for each $f \in H$, there exists exactly one invariant element in $w([\underline{U}]f)$. (This is also the strong closure of the orbit $([\underline{U}]f)$.)

Theorem 3.2 (Jacobs [32], Theorem 1.2.1). The ergodic theorem holds for an abelian weakly almost periodic semigroup $\underline{U}$ on a Banach space.

Suppose we start with a weakly almost periodic semigroup $\underline{U}$ which is also convex, i.e., closed under convex linear combinations. An interesting result (Theorem 7.4) of deLeeuw and Glicksberg [10] for such semigroups asserts that the ergodic theorem holds if and only if the splitting theorem holds.

For our purposes we need a more general ergodic theorem than the above one. To this end, we first introduce the notion of amenable semigroups (Day [8]).

Let S be a topological semigroup, i.e., a semigroup as well as a Hausdorff topological space such that the map $(s, s') \to ss'$ from $S \times S$ to S is continuous. We denote by $C(S)$ the Banach space of bounded real-valued continuous functions on S with sup. norm. $C^*(S)$, as usual, will denote the conjugate space of $C(S)$. A mean M on $C(S)$ is an element of $C^*(S)$ such that, for each $f \in C(S)$, we have $\inf_s f(s) \leq M(f) \leq \sup_s f(s)$. It is clear that an element M of $C^*(S)$ is a mean if and only if (i) $f \geq 0$ implies $M(f) \geq 0$ and (ii) $M(1) = 1$. An element M of $C^*(S)$ is called right invariant if $M(R_s f) = M(f)$ for all $f \in C(S)$ and all $s \in S$, where $(R_s f)(s') = f(s's)$. Left invariance of elements of $C^*(S)$ is similarly defined. A semigroup S is called right (left) amenable if there exists a right (left) invariant mean. S is amenable if there exists a mean which is both left and right invariant. It is known (Day [8]) that all compact and solvable topological groups as well as abelian topological semigroups are amenable.

We now give an ergodic theorem due to Day [8] and Eberlein [16]. Let $\underline{U}$ denote a bounded operator semigroup on H. An element in the convex hull $[\underline{U}]$ is called an average of $\underline{U}$. $\underline{U}$ is said to be strongly ergodic under a net U_α of averages of $\underline{U}$ if, for each $U \in \underline{U}$, we have $\|U_\alpha(U - I)f\| \to 0$ as well as $\|(U - I)U_\alpha f\| \to 0$ for every $f \in H$.

Let S be an amenable topological semigroup and for each $s \in S$, U_s be a bounded operator on H such that $U_{s_1 s_2} = U_{s_1} U_{s_2}$ for all $s_1, s_2 \in S$ (i.e., $s \to U_s$ is a homomorphism onto an operator semigroup) and such that the map $s \to \phi(U_s f)$ for fixed $f \in H$ and $\phi \in H^*$, the conjugate space of H, is continuous. Let further $\underline{U} = \{U_s : s \in S\}$ be a bounded semigroup. We call $\underline{U}$ a continuous bounded representation of S. Theorem 8.1 (in conjunction with the remarks in Section 10) of Day [8] and Theorem 3.1 of Eberlein [16] together yield the following result.

<u>Theorem 3.3</u> A continuous bounded representation $\underline{U}$ of an amenable topological semigroup S is strongly ergodic under a net U_α of averages of $\underline{U}$. Moreover the following four conditions on an element f are equivalent:

i) $g \in w([\underline{U}]f)$ and g is invariant

ii) $g = \lim U_\alpha f$

iii) $g = \lim U_\alpha f$ weakly

iv) g is a weak cluster point of the set $\{U_\alpha f\}$.

Let us call an element f satisfying any one of the above four conditions ergodic. The following result is due to Eberlein [16].

<u>Theorem 3.4</u> The ergodic elements form a closed linear invariant subspace H_o. The transformation $\overline{U}$ defined on H_o by $\overline{U}f = g$ is a bounded operator on H_o with $\overline{U} = \overline{U}^2 = \overline{U}U = U\overline{U}$ on H_o for all $U \in \underline{U}$.

In case $\underline{U}$ is an abelian weakly almost periodic semigroup, every element of H is ergodic and so Theorem 3.2 follows from the above results.

CHAPTER 4

INTRODUCING PART II

Let T be a measure-preserving transformation on a probability space $(X, \underline{\bar{A}}, m)$. It is well-known (and can be deduced from the individual ergodic theorem) that the sequence $m(T^{-n} A \cap B)$, $n = 0,1,2,\ldots$ is Cesaro convergent for every pair A and B of measurable sets. This property of the sequences $m(T^{-n} A \cap B)$ characterises, in a sense, all measure-preserving transformations. For, a result of Dowker [12] implies that, under certain mild conditions, T is essentially measure-preserving, in the sense that there is an invariant probability measure equivalent to m, if and only if, the sequences $m(T^{-n} A \cap B)$ are Cesaro convergent.

Because of this property, we may try to study a transformation T through the associated sequences $m(T^{-n} A \cap B)$, $A, B \in \underline{\bar{A}}$. Indeed this approach is not new in ergodic theory; we need to cite only the classical concepts of ergodicity, weak mixing and mixing for transformations. T is ergodic, weakly mixing or mixing according as, for every $A, B \in \underline{\bar{A}}$, the sequence $m(T^{-n} A \cap B)$ converges in the Cesaro sense, in the strong Cesaro sense or in the ordinary sense, to the limit $m(A)m(B)$. Another interesting class of transformations is that for each of which the sequences $m(T^{-n} A \cap B)$ converge to a limit. These are the stable transformations studied recently by Maitra [41].

Motivated by this observation, we introduce and study transformations T for which $m(T^{-n}A \cap B)$, for $A, B \in \underline{\bar{A}}$, is strong Cesaro convergent to some limit. We call these the weakly stable transformations. It is not difficult to see that weak stability is just weak mixing minus ergodicity - a measure-preserving transformation is weakly mixing if and only if it is weakly stable and ergodic. (A similar relation holds between stable and mixing transformations too.) This makes it interesting and desirable to ask which properties of weakly mixing transformations follow from the hypothesis of ergodicity and which from that of weak stability. We shall see that many of the well-known theorems on weak mixing have their analogues in terms of weak stability, which simplify, in the presence of ergodicity, to the corresponding theorems on weak mixing. We shall also find some characteristic properties of weakly stable transformations.

Since transformations are only a particular case of transition functions, which themselves are a particular case of operators in Hilbert spaces, it is of interest to know if the properties we are studying of transformations are not essentially operator-theoretic. Indeed, we shall see that weak stability makes sense and has interesting consequences for contractions on a Hilbert space. (Note that we cannot talk of ergodicity or weak mixing in an arbitrary Hilbert space.) More generally, we study the weak stability properties of families of contractions. Here it is natural to impose some condition on the family as an abstract set. The simplest one is to ask for the semigroup property; one could go further and ask for the amenability of the semigroup. In a different direction, one could ask for a measure to be given on a σ-field of subsets of the family. We consider all these situations and obtain generalizations of known results for weakly mixing transformations.

CHAPTER 5

DEFINITIONS AND SIMPLEST PROPERTIES

In this chapter, we introduce the weakly stable and stable transformations on a probability space and study some of their elementary properties. On the unit interval, we notice the existence of non-trivial weakly stable non-weakly mixing transformations. From the well-known category theorems, we derive some corollaries on the category of sets of weakly stable and stable transformations on the unit interval. We end the discussion with the weak stability and stability properties of powers and roots of transformations.

Let T be a measure-preserving transformation on a probability space $(X, \underline{A}, m)$. T is said to be ergodic if, for all $A, B \in \underline{A}$, we have

$$\lim_n \frac{1}{n} \sum_0^{n-1} m(T^{-j} A \cap B) = m(A)m(B).$$

Equivalently, T is ergodic if and only if every invariant set has measure zero or one (i.e., the invariant σ-field is trivial), or if and only if every invariant function is a constant. We introduce the following definition.

<u>Definition 5.1</u> A measure-preserving transformation T is <u>weakly stable</u> if, for $A, B \in \underline{A}$, the sequence $m(T^{-n}A \cap B)$ is strong Cesaro convergent, i.e., if there exists a constant $C(A,B)$ such that

$$\lim \frac{1}{n} \sum_{0}^{n-1} |m(T^{-j}A \cap B) - C(A,B)| = 0.$$

Maitra [41] calls T <u>stable</u> if the sequence $m(T^{-n}A \cap B)$ is convergent to a limit $C(A, B)$ for $A, B \in \underline{A}$. A transformation T is (weakly) mixing if it is (weakly) stable with the constant $C(A, B) = m(A)m(B)$ for all $A, B \in \underline{A}$.

Every stable transformation is clearly weakly stable. Since strong Cesaro convergence implies Cesaro convergence, it follows from the individual ergodic theorem that, if T is a weakly stable transformation, then $C(A,B) = \int_B P(A|\underline{I})dm$. It is immediate from this that (weakly) mixing transformations are precisely those which are (weakly) stable and ergodic.

The simplest example of a weakly stable transformation is the identity. This is indeed stable. If X is a finite or a countably infinite set, $\underline{A}$ the class of all subsets of X and m a probability measure giving positive mass to singletons, then the identity is the only invertible weakly stable transformation on X. On the unit interval however, we have non-trivial examples of invertible weakly stable transformations. In fact, we can find non-trivial examples of weakly stable transformations which are not weakly mixing as the ensuing discussion shows.

Let X be the closed unit interval, $\underline{A}$ the σ-field of Borel sets and m the Lebesgue measure. Let T be an invertible weakly stable transformation on X. We shall show that X is essentially the union of two disjoint sets on one of which T is the identity and on the other, antiperiodic. In other words, we shall prove that, for every integer $n \geq 2$, the set A_n of all periodic points of period n has measure zero. Clearly A_n is measurable and strictly invariant. If $m(A_n) > 0$, then by an argument similar to that of Halmos ([24], p. 70), we may find a measurable subset B of A_n such that $B, TB, \ldots, T^{n-1}B$ are disjoint, $T^n B = B$ and $m(B) = \frac{1}{n} m(A_n)$. But then the sequence $m(T^n B \cap B)$ is not strong Cesaro convergent - a contradiction. (This argument shows, incidentally, that no periodic transformation of period greater than one, on the unit interval, is weakly stable. This is true more generally -see Corollary 6.5.)

This result may justifiably make one wonder whether the weak stability of a transformation T which is not weakly mixing is due only to the occurrence of a set of positive measure on which T is the identity. This is not so. In Chapter 10, we give examples of families of antiperiodic weakly stable (in fact, stable) but not weakly mixing transformations on every k-dimensional torus. Since the normalised measure space of a torus is point isomorphic to that of the unit interval, we may conclude that there exist such transformations even on the latter.

We now give some results regarding the category of sets of invertible stable and weakly stable transformations on the unit interval $(X, \bar{\underline{A}}, m)$. Let $(\bar{\underline{A}}^*, m^*)$ be the measure algebra induced by $(X, \underline{A}, m)$ and $\underline{T}$ the group of all automorphisms of $(\underline{A}^*, m^*)$ equipped with the weak topology (Halmos [24], p. 61). Then with the obvious definitions of weak stability etc., for elements of $\bar{\underline{T}}$, we have the following observations:

(i) The set of all weakly stable automorphisms contains the set of all weakly mixing automorphisms and hence is a dense set of the second category in $\bar{\underline{T}}$. It is not the whole of $\bar{\underline{T}}$ however. In fact, any ergodic but not weakly mixing automorphism is an example of an automorphism which is not weakly stable.

(ii) The set of all weakly stable but not weakly mixing automorphisms, being a subset of the complement of a dense G_δ, is of the first category. It is dense. To see this, let T be the automorphism induced by an antiperiodic, weakly stable but not weakly mixing transformation on $(X, \underline{A}, m)$ (such an one exists, as we noted earlier), observe that any automorphism conjugate to T is also weakly stable but not weakly mixing and apply the Conjugacy Lemma of Halmos ([24], p. 77).

(iii) The set of all stable automorphisms is a dense set of the first category. This is because, the set of mixing automorphisms is a dense set of the first category and the set of stable, but not mixing automorphisms is, as an argument similar to that in (ii) shows, also a dense set of the first category.

(iv) The set of all weakly stable automorphisms which are not stable is a set of the second category. It is dense since it contains the conjugacy class of every weakly mixing non-mixing automorphism.

We now return to the study of stable and weakly stable transformations on an arbitrary probability space. A few properties of these may be deduced from the definition. In proving these and later too, we shall make use of the following two facts about strong Cesaro convergence without explicit mention.

(a) A bounded sequence a_n of complex numbers is strong Cesaro convergent to a if and only if there exists a set D of natural numbers of density one such that a_n converges to a on D.

(b) If a bounded sequence a_n of complex numbers is strong Cesaro convergent to a and D is any set of natural numbers of positive density, then a_n is strong Cesaro convergent to a on D.

<u>Theorem 5.1</u> If T is weakly stable (and invertible) then so is T^k for every non-negative (and negative) integer k.

<u>Proof</u>. Let T be weakly stable. If $k = o$, $T^o = I$ is weakly stable. Since T^{-1} (when T is invertible) is easily seen to be weakly stable, to complete the proof, it is enough to show that T^k is weakly stable for every positive integer k. But this is true because the set of positive multiples of k has density $\frac{1}{k}$. The theorem is proved.

<u>Remark</u> The analogue of Theorem 5.1 is true and trivial for stable transformations.

<u>Theorem 5.2</u> If T is weakly stable (and invertible), then for every positive (and negative) integer k we have
$\underline{I}(T) = \underline{I}(T^k)$.

Proof. It is enough to prove that $\bar{\underline{I}}(T^k) \subseteq \bar{\underline{I}}(T)$ for every positive integer k. Let $A \in \bar{\underline{I}}(T^k)$ and let, for arbitrary $B \in \bar{\underline{A}}$, $C(A,B)$ denote the strong Cesaro limit of the sequence $m(T^{-n} A \cap B)$. Then $C(A,B)$ is also the strong Cesaro limit of the sequence $m(T^{-nk} A \cap B)$. But, for every n, $m(T^{-nk} A \cap B) = m(A \cap B)$ and therefore $C(A,B) = m(A \cap B)$. On the other hand, $C(T^{-1}A, B) = C(A,B)$. It follows that $A \in \bar{\underline{I}}(T)$. The theorem is proved.

Corollary 5.1 No weakly stable automorphism of a measure algebra can have finite orbits of order greater than one.

While it is true that every power of a weakly stable transformation is weakly stable, a similar statement does not hold for roots.*) For example, if T is a periodic transformation of period $n > 1$ on the unit interval, then $T^n = I$ is weakly stable, but T is not weakly stable (as we have noted earlier). We can however give a necessary and sufficient condition.

Theorem 5.3 If T is invertible and weakly stable and if T_o is a root of T, then T_o is weakly stable if and only if $\bar{\underline{I}}(T_o) = \bar{\underline{I}}(T)$.

*) We consider only measure-preserving roots.

<u>Proof</u>. If T_o is weakly stable, then Theorem 5.2 implies that $\underline{I}(T_o) = \underline{I}(T)$. Suppose, conversely, that $\underline{I}(T_o) = \bar{\underline{I}}(T)$ and that $T_o^k = T$, k a positive integer. First, note that for all $A \in \bar{\underline{A}}$, the function $P(A|\bar{\underline{I}}(T)) = P(A|\underline{I}(T_o))$ is invariant for T_o. Now, for every fixed r, $0 \leq r < k$ and arbitrary measurable sets A and B, $m(T_o^{-r-kn} A \cap B) = m(T^{-n} A \cap T_o^r B)$ and hence is strong Cesaro convergent to the limit $\int_{T_o^r B} P(A|\underline{I}(T_o))\,dm = \int_B P(A|\underline{I}(T_o))\,dm$. Since this limit is independent of r, it follows that the sequence $m(T_o^{-n} A \cap B)$ is itself strong Cesaro convergent for $A, B \in \bar{\underline{A}}$. T_o is therefore weakly stable.

<u>Corollary 5.2</u> (Blum and Friedman [4]). Any root T_o of an invertible weakly mixing transformation T is weakly mixing.

<u>Proof.</u> Since T_o is a root of T, $\bar{\underline{I}}(T_o) \subseteq \bar{\underline{I}}(T)$. Since $\underline{I}(T)$ is trivial, $\underline{I}(T) \subseteq \underline{I}(T_o)$ and hence $\underline{I}(T_o) = \bar{\underline{I}}(T)$ and is trivial. By Theorem 5.3, T_o is weakly stable and hence weakly mixing.

Remark. The analogue of Theorem 5.3 is true for invertible stable transformations and the proof is, if anything, easier in this case. We therefore have the following corollary.

Corollary 5.3 A weakly stable root of an invertible stable transformation is stable.

CHAPTER 6

THE STABILITY THEOREM-I

The ergodic and weak mixing properties of an invertible measure-preserving transformation T on a probability space $(X, \underline{A}, m)$ are well-known to influence strongly the spectral structure of the induced unitary operator U on $L_2(X)$. T is ergodic if and only if the number 1 is a simple eigen value of U; T is weakly mixing if and only if 1 is simple as well as the only eigen value of U. The latter statement is part of a result known as the 'Mixing Theorem' (Halmos [24]). To introduce the other part of this theorem, we consider the product measure space $(X^{(2)}, \underline{A}^{(2)}, m^{(2)})$ where $X^{(2)} = X \times X$, $\underline{A}^{(2)} = \underline{A} \times \underline{A}$ and $m^{(2)} = m \times m$. Given any measure-preserving transformation T on X, we consider the measure-preserving transformation $T^{(2)} = T \times T$ on $X^{(2)}$ defined by $T^{(2)}(x_1, x_2) = (Tx_1, Tx_2)$. $T^{(2)}$ is called the Cartesian square of T. The mixing theorem states that an invertible T is weakly mixing if and only if $T^{(2)}$ is ergodic. Indeed, a look at the proof shows that $T^{(2)}$ is weakly mixing if T is weakly mixing and T is weakly mixing if $T^{(2)}$ is ergodic. This kind of splitting the mixing theorem is very helpful in our analysis of weakly stable transformations.

In this chapter, we shall prove a generalization of the mixing theorem to the case of weakly stable transformations. We can, and shall in the next chapter, prove this result for certain semigroups of contractions in Hilbert spaces, but that needs more refined operator-theoretical techniques. Here, we shall prove the following Stability Theorem by elementary methods.

Theorem 6.1 The following conditions on an invertible measure-preserving transformation T on a probability space $(X, \underline{A}, m)$ are equivalent.

1) T is weakly stable.
2) The number 1 is the only eigen value of U.
3) $T^{(2)}$ is weakly stable.
4) $\underline{I}\,(T^{(2)}) = (\underline{I}(T))^{(2)}$.

We shall need the following lemma for the proof of the theorem. We denote the subspace, and the projection into the subspace, of invariant functions for U in $L_2(X)$ by $K = K(U)$.

Lemma 6.1 T is weakly stable if and only if for every $f, g \in L_2(X)$, there is a constant $C_{f,g}$ such that

$$\lim_n \frac{1}{n} \sum_{j=o}^{n-1} |(U^j f, g) - C_{f,g}| = 0.$$

If T is weakly stable, then $C_{f,g} = (Kf, g)$ for all $f, g \in L_2$.

The proof of the first part is straightforward. The second part is a consequence of the mean ergodic theorem.

Proof of Theorem 6.1 We shall first show that (1) <==> (3).

(3) => (1). If $T^{(2)}$ is weakly stable, then for $A, B \in \bar{\underline{A}}$, $m(T^{-n} A \cap B) = m^{(2)}(T^{(2)-n} (A \times X) \cap (B \times X))$ is strong Cesaro convergent and hence T is weakly stable.

(1) => (3). Let T be weakly stable. For rectangles $A \times B$ and $C \times D$ in $\bar{\underline{A}}^{(2)}$, we have that the sequence

$$m^{(2)}(T^{(2)-n}(A \times B) \cap (C \times D)) = m(T^{-n} A \cap C) m(T^{-n} B \cap D)$$

is strong Cesaro convergent. It follows that for every pair A^*, B^* of sets in $\underline{A}_o^{(2)}$, the field of finite disjoint unions of rectangles in $\underline{A}^{(2)}$, $m^{(2)}(T^{(2)-n} A^* \cap B^*)$ is strong Cesaro convergent. Approximating now sets $A^*, B^* \in \bar{\underline{A}}^{(2)}$ by sets from the field $\bar{\underline{A}}_o^{(2)}$, it is not difficult to show that the sequence $m^{(2)}(T^{(2)-n} A^* \cap B^*)$ is strong Cesaro convergent to the limit $(K^* 1_{A^*}, 1_{B^*})$ where K^* is the projection on the subspace of $L_2(X^{(2)})$ consisting of invariant functions for $T^{(2)}$. Hence $T^{(2)}$ is weakly stable.

We shall next prove the implications (1) => (4) => (2) => (1). For notational convenience, we let $\bar{\underline{I}} = \underline{I}(T)$ in what follows.

(1) => (4)+) By the above part of the proof, the weak stability of T implies that of $T^{(2)}$. Hence, for $A, B, C, D \in \bar{\underline{A}}$, the sequence $m^{(2)}(T^{(2)-n}(A \times B) \cap (C \times D))$ converges in the strong Cesaro sense to $\int_{C \times D} P(A \times B \mid \underline{I}(T^{(2)}))\, dm^{(2)}$. But

$$m^{(2)}(T^{(2)-n}(A \times B) \cap (C \times D)) = m(T^{-n}A \cap C)\; m(T^{-n}B \cap D)$$

and so the sequence strong Cesaro converges also to $\int_C P(A|\underline{I})dm \int_D P(B|\bar{\underline{I}})dm.$ Thus

$$\int_{C \times D} \left\{ P(A \times B|\underline{I}(T^{(2)})) - P(A|\bar{\underline{I}})\, P(B|\underline{I}) \right\} dm^{(2)} = 0.$$

Fixing A and B, the integral vanishes over finite disjoint unions of measurable rectangles and hence over all $\bar{\underline{A}}^{(2)}$-measurable sets. Thus

+) The author is indebted to Dr. J. K. Ghosh for the proof of this implication.

$$P(A \times B \mid \underline{I}\,(T^{(2)})) = P(A|\underline{I})\, P(B|\underline{I}) \quad \text{a.e.}$$

The right side is an $\bar{\underline{I}}^{(2)}$-measurable function. Moreover, for any $E^* \in \underline{I}^{(2)}$, since $\bar{\underline{I}}^{(2)} \subseteq \underline{I}\,(T^{(2)})$,

$$\int_{E^*} P(A|\bar{\underline{I}})\, P(B|\bar{\underline{I}})\, d\, m^{(2)} = \int_{E^*} P(A \times B \mid \bar{\underline{I}}\,(T^{(2)}))\, dm^{(2)}$$
$$= m^{(2)}\,((A \times B) \cap E^*)$$

and so $P(A \times B|\bar{\underline{I}}\,(T^{(2)})) = P(A \times B|\bar{\underline{I}}^{(2)})$ a.e.
From this it follows that $\bar{\underline{I}}\,(T^{(2)}) = \underline{I}^{(2)}$.

(4) => (2): Let $\underline{I}(T^{(2)}) = \bar{\underline{I}}^{(2)}$. If λ is an eigen value of U with eigen function f, consider $F(x,y) = f(x)\,\overline{f(y)}$. Then $U^{(2)} F = F$ and hence F is $\underline{I}(T^{(2)})$ - measurable, i.e., $\underline{I}^{(2)}$-measurable. Thus a.e. section of F is $\bar{\underline{I}}$-measurable, i.e., invariant for T. Taking a suitable section, we see that f is T-invariant and $\lambda = 1$.

(2) => (1). If 1 is the only eigen value of U and if $E(\cdot)$ denotes the spectral measure associated with U, i.e.,

$$U = \int \lambda\, d\, E,$$

then for any fixed $f \perp K$, the measure $\mu(A) = (E(A)f,f)$ defined on the unit circle is non-atomic. Then

$$\frac{1}{n} \sum_{j=0}^{n-1} |(U^j f, f)|^2 = \frac{1}{n} \sum_{j=0}^{n-1} | \int \lambda^j \mu(d\lambda)|^2$$

tends to zero as $n \to \infty$ (Halmos [24], p. 40). It follows that $(U^n f, f)$ is strong Cesaro convergent to zero. This extends, by standard arguments, to $(U^n f, g)$ for any $g \in L_2$ and $f \perp K$. Since for $f \in K$, it is trivial that $(U^n f, g)$ is strong Cesaro convergent for any $g \in L_2$, it follows that T is weakly stable. The proof of the theorem is complete.

We now have a number of corollaries.

Corollary 6.1 T is weakly mixing if and only if 1 is the only eigen value of U and is simple.

Corollary 6.2 If T is weakly mixing, then $T^{(2)}$ is weakly mixing; if $T^{(2)}$ is ergodic, then T is weakly mixing.

Proof. If T is weakly mixing, then $\underline{I}(T^{(2)}) = \underline{\bar{I}}^{(2)}$ as well as $\underline{I}$ is trivial. Thus $\underline{I}(T^{(2)})$ is trivial and so $T^{(2)}$ is weakly stable and ergodic, i.e., weakly mixing. If $T^{(2)}$ is ergodic, it is clear that T is ergodic and $\underline{\bar{I}}(T^{(2)}) = \underline{\bar{I}}^{(2)}$ both being trivial. Thus T is weakly mixing.

Corollary 6.3 T is weakly stable if and only if every finite dimensional U-invariant subspace of $L_2(X)$ consists of invariant functions only.

Proof. If the condition is satisfied, then 1 is the only eigen value of U and T is weakly stable. Conversely, if L is any finite dimensional subspace of L_2 invariant for U, then U restricted to L is unitary and $U \neq I$ on L would imply the existence of eigen values for U other than 1.

Corollary 6.4 If T is weakly stable, then U on L_2 has no finite orbits of order greater than one.

Proof. If there is a finite orbit of order $k > 1$ for U, then there will be a finite dimensional invariant subspace on which U is not the identity.

The converse of Corollary 6.4 is not true in general. Any ergodic rotation on the circle group will serve as a counter-example. But if X is a compact abelian group, m the normalised Haar measure on the Borel σ-field of X and T a continuous automorphism of X, then the converse does hold. See Corollary 10.1.

A sequence A_n of measurable sets in X is called a separating sequence, if, for every pair of points $x \neq y$, there is an integer n such that $x \in A_n$ and $y \in X - A_n$. A lemma of Halmos and von Neumann [25] says that on a measure space with a separating sequence of sets, two invertible transformations which induce the same automorphism of the measure algebra, differ on at most a set of measure zero. This leads to the following corollary.

Corollary 6.5 If there is a separating sequence of sets in X, then no periodic transformation of period greater than one can be weakly stable.

Proof. If $T^n = I$, $n > 1$, then $U^n = I$. So U will have finite orbits of order greater than one, since $U = I$ would mean, by the above remarks, that $T = I$ a.e.

The conclusion of Corollary 6.5 may not hold if no separating sequence of sets exist. For example, if $\underline{A}$ is the trivial σ-field, all transformations, including the periodic ones, are weakly stable.

CHAPTER 7

THE STABILITY THEOREM-II

In this chapter, we shall show that the Stability Theorem-I (Theorem 6.1) can be proved in a fairly general set-up - for certain semigroups of contractions on a Hilbert space H. We start with the splitting theorem (see Chapter 3) and define weak stability for a semigroup of contractions in terms of its reversible functions. For the semigroup generated by a single contraction U, the weak stability is equivalent to the strong Cesaro convergence of the sequences $(U^n f, g)$ where $f, g \in H$, so that the weak stability introduced here coincides with the earlier definition in the case of a measure-preserving transformation. The main results (Theorem 7.2 and Theorem 7.3) are about the tensor product of operators acting on the tensor product Hilbert space. Applying these to semigroups of contractions induced by measure-preserving transformations and transition functions, we get generalizations of the mixing theorem.

Let $\underline{U}$ be a semigroup of contractions on the Hilbert space H. The splitting theorem (Theorem 3.1) then gives H as the direct sum of the subspaces $R = R(\underline{U})$ of reversible elements and $F = F(\underline{U})$ of flight elements for $\underline{U}$. The subspace R is spanned by a family of mutually orthogonal finite dimensional invariant subspaces and the restriction of the semigroup $\underline{U}$ to R is a group of unitary operators. It is clear from the definition of a reversible element that any invariant element is reversible: $K(\underline{U}) \subseteq R(\underline{U})$. The semigroups of our interest are precisely those for which these two subspaces coincide.

<u>Definition 7.1</u> A semigroup $\underline{U}$ of contractions on H is <u>weakly stable</u> if every reversible element is invariant.

Examples of weakly stable semigroups are easy to find. Indeed, it follows from Lemma 7.3 of deLeeuw and Glicksberg [10] that any convex semigroup of contractions, i.e., any semigroup of contractions closed under the formation of convex linear combinations is weakly stable.

Let us first consider the semigroup $\underline{U}$ to be $\{U^n: n \geq 0\}$, generated by a single contraction U. It can be seen that R in this case is spanned by the eigen vectors of U with eigen values of modulus one. Hence U is weakly stable if and only if every eigen vector of U corresponding to an eigen value of modulus one is invariant. Moreover the weak stability of U can be characterised in terms of the sequences $(U^n f, g)$ where $f, g \in H$, as the following theorem shows. We need a lemma due to Foguel [17] for its proof.

<u>Lemma 7.1</u> For a contraction U on a Hilbert space H, let

$$L = \{ f : \|U^n f\| = \|f\|, \ \|U^{*n} f\| = \|f\| \text{ for } n = 0,1,2,\ldots \}$$

where U^* is the adjoint of U. Then L is a subspace of H invariant under U and U^*. The restriction of U to L is unitary. If $g \perp L$, then $U^n g$ tends to 0 weakly.

<u>Theorem 7.1</u> A contraction U on H is weakly stable if and only if for every $f, g \in H$, the sequence $(U^n f, g)$ is strong Cesaro convergent.

<u>Proof.</u> Let U be weakly stable. By the splitting theorem, $H = K(U) \oplus F(U)$. We shall apply Lemma 7.1 to $F = F(U)$ and write $F = F_1 \oplus F_2$ where F_1 and F_2 are subspaces invariant under U, U on F_1 is unitary and $U^n f$ tends to 0 weakly for every $f \in F_2$. It is immediate that the sequence $(U^n f, g)$ is strong Cesaro convergent for all $f \in K \oplus F_2$ and all $g \in H$. Let U_0 be the restriction of U to F_1. Considering the spectral representation of U_0, we can prove, as in Theorem 6.1, that for all $f, g \in F_1$, the sequence $(U_0^n f, g)$ is strong Cesaro convergent to zero. Putting these together, we see that for all $f, g \in H$, the sequence $(U^n f, g)$ is strong Cesaro convergent.

Let now $(U^n f, g)$ be strong Cesaro convergent for every $f, g \in H$. If U is not weakly stable, then there exists an element f such that $\|f\| = 1$, $Uf = \lambda f$, $|\lambda| = 1$ and $\lambda \neq 1$. Then $(U^n f, f) = \lambda^n$ which is not strong Cesaro convergent - a contradiction. Thus U is weakly stable and the theorem is proved.

A few interesting results on the weak stability for powers and roots*) of a contraction on a Hilbert space can be proved as in Chapter 5. Indeed analogues of Theorems 5.1, 5.2 and 5.3 as well as Corollaries 5.2 (for an L_2-space) and 5.3 hold good.

We now introduce the tensor product of two Hilbert spaces H_1 and H_2 with inner products $(\;,\;)_1$ and $(\;,\;)_2$ respectively. Consider the set H^* of all formal finite linear combinations of formal products $f \cdot g$ with $f \in H_1$ and $g \in H_2$. The product is assumed to satisfy the relations $(f_1 + f_2) \cdot g = f_1 \cdot g + f_2 \cdot g$, $f \cdot (g_1 + g_2) = f \cdot g_1 + f \cdot g_2$, $(\alpha f) \cdot g = f \cdot (\alpha g) = \alpha (f \cdot g)$ for all complex numbers α and $f, f_1, f_2 \in H_1$, $g, g_1, g_2 \in H_2$. The set H^* is linear. The tensor product $H_1 \times H_2$ is defined to be the completion of H^* with respect to the inner product:

$$(f_1 \cdot g_1, f_2 \cdot g_2) = (f_1, f_2)_1 \, (g_1, g_2)_2 .$$

*) We consider only roots which are contractions.

If $H_1 = H_2 = H$, the tensor product $H \times H$ is denoted by $H^{(2)}$ and called the tensor square of H. It is clear that if H is the L_2-space of a probability space X, then $H^{(2)}$ is the L_2-space of the Cartesian square $X^{(2)}$.

Given a contraction U on H, we define its tensor square $U^{(2)} = U \times U$ on $H^{(2)}$ by writing $U^{(2)}(f.g) = Uf.Ug$ for all $f, g \in H$. $U^{(2)}$ is a contraction on $H^{(2)}$. When U is the isometry on $L_2(X)$ induced by a measure-preserving transformation T on X, $U^{(2)}$ is the isometry on $L_2(X^{(2)})$ induced by $T^{(2)}$. For a semigroup $\underline{\bar{U}}$ of contractions on H, the set $\underline{\bar{U}}^{(2)} = \{ U^{(2)} : U \in \underline{\bar{U}} \}$ is again a semigroup of contractions on $H^{(2)}$. We shall be using the splitting theorem both for $\underline{\bar{U}}$ and for $\underline{\bar{U}}^{(2)}$. Obvious notations will be used for the subspaces of reversible, flight and invariant elements for $\underline{U}^{(2)}$. The tensor square of $w(\underline{\bar{U}})$ is denoted by $w(\underline{\bar{U}})^{(2)}$; so also that of $R(\underline{\bar{U}})$ etc. To prove our main results, we need the following two lemmas.

<u>Lemma 7.2</u> $w(\underline{\bar{U}}^{(2)}) = w(\underline{\bar{U}})^{(2)}$.

<u>Proof</u>. Let $W^{(2)} \in w(\underline{\bar{U}})^{(2)}$ with $W \in w(\underline{\bar{U}})$. There is a net U_α in $\underline{\bar{U}}$ converging to W in the weak operator topology. Then $U_\alpha^{(2)}$ converges to $W^{(2)}$ in the weak operator topology in $H^{(2)}$ and hence $W^{(2)} \in w(\underline{\bar{U}}^{(2)})$. Thus $w(\underline{\bar{U}})^{(2)} \subseteq w(\underline{\bar{U}}^{(2)})$.

If now $W \in w(\bar{\underline{U}}^{(2)})$, then there is a net $U_\alpha^{(2)}$ in $\bar{\underline{U}}^{(2)}$ converging to W in the weak operator topology in $H^{(2)}$. Since $U_\alpha \in \underline{U} \subseteq w(\bar{\underline{U}})$ and $w(\bar{\underline{U}})$ is weakly compact, there is a weakly convergent subnet U_β with limit $U_o \in w(\underline{U})$. It follows that $W = U_o^{(2)} \in w(\underline{U})^{(2)}$. Thus $w(\bar{\underline{U}}^{(2)}) \subseteq w(\underline{U})^{(2)}$ and the lemma is proved.

<u>Lemma 7.3</u> $R(\bar{\underline{U}}^{(2)}) = R(\underline{U})^{(2)}$.

<u>Proof</u>. We shall first show that $R(\underline{U})^{(2)} \subseteq R(\bar{\underline{U}}^{(2)})$. Let f and g be reversible elements for $\underline{U}$ and consider $f \cdot g$. Since $w(\bar{\underline{U}}^{(2)}) = w(\bar{\underline{U}})^{(2)}$, any element of $w(\bar{\underline{U}}^{(2)})$ is of the form $W^{(2)}$ for some $W \in w(\bar{\underline{U}})$. Since f and g are reversible and since $w(\bar{\underline{U}})$ acts as a group of unitary operators on $R(\underline{U})$, given $W \in w(\bar{\underline{U}})$, there exists a $W_o \in w(\bar{\underline{U}})$ such that, for all reversible elements h, $W_o W h = h$ and hence $W_o^{(2)} W^{(2)}(f \cdot g) = f \cdot g$ with $W_o^{(2)} \in w(\bar{\underline{U}})^{(2)} = w(\bar{\underline{U}}^{(2)})$. This shows that $f \cdot g \in R(\bar{\underline{U}}^{(2)})$.

Besides, it is easy to see that $f \in F(\underline{U})$, $g \in H$ imply that $f \cdot g$ and $g \cdot f$ are in $F(\bar{\underline{U}}^{(2)})$. Thus $F(\bar{\underline{U}}^{(2)})$ contains the subspaces $F \times R$, $R \times F$ and $F \times F$. Since
$H^{(2)} = R(\bar{\underline{U}}^{(2)}) \oplus F(\bar{\underline{U}}^{(2)})$ as well as

$= R^{(2)} \oplus F \times R \oplus R \times F \oplus F \times F$, we see that
$R(\bar{\underline{U}}^{(2)}) = R(\underline{U})^{(2)}$. The lemma is proved.

Our first main result shows that weak stability is preserved under passage to tensor squares if there exists at least one non-zero invariant element.

Theorem 7.2 A semigroup $\underline{U}$ of contractions on H with $K(\underline{U}) \neq 0$ is weakly stable if and only if its tensor square $\underline{U}^{(2)}$ is weakly stable.

Proof. Let $\underline{U}$ be weakly stable. Then $R(\underline{U}) = K(\underline{U})$ and so $R(\underline{U}^{(2)}) = R(\underline{U})^{(2)} = K(\underline{U})^{(2)} \subseteq K(\underline{U}^{(2)})$. Since the opposite inclusion holds always, we see that $R(\underline{U}^{(2)}) = K(\underline{U}^{(2)}) = K(\underline{U})^{(2)}$. This shows that $\underline{U}^{(2)}$ is weakly stable.

Conversely, suppose that $\underline{U}^{(2)}$ is weakly stable. Then $R(\underline{U}^{(2)}) = K(\underline{U}^{(2)})$. If $\underline{U}$ is not weakly stable, then there will exist a non-invariant vector f in $R(\underline{U})$. Let g be a non-zero invariant vector. Then $f.g \in R(\underline{U}) \times R(\underline{U}) = R(\underline{U}^{(2)})$ but is not invariant under $\underline{U}^{(2)}$ - a contradiction. The theorem is proved.

The condition $K(\underline{U}) \neq 0$ is used only in proving that if $\underline{U}^{(2)}$ is weakly stable, then $\underline{U}$ is weakly stable. That this result may not hold if $K(\underline{U}) = 0$ is seen by considering the single contraction $U = -I$. In this case, U is not weakly stable but $U^{(2)}$ is the identity on $H^{(2)}$ and so is weakly stable.

We need to introduce some terminology before presenting the next theorem. By a <u>conjugation</u> J on a Hilbert space H, is meant a one-one conjugate linear map of H onto H such that $J^2 = I$ and $(Jf, Jg) = (g,f)$ for all $f, g \in H$. An operator U on H is said to be <u>real</u> with respect to a conjugation J (Stone [50]) if $UJ = JU$. Let us call a family $\underline{U}$ of operators <u>real</u> if there exists a conjugation J with respect to which every $U \in \underline{U}$ is real. Our result below is for real semigroups of contractions.

<u>Theorem 7.3</u> A real semigroup $\underline{U}$ of contractions is weakly stable if and only if $K(\underline{U}^{(2)}) = K(\underline{U})^{(2)}$.

<u>Proof.</u> The proof of the necessity part (for not necessarily real semigroups) is contained in that of the preceding theorem.

Suppose now that $\underline{U}$ is not weakly stable. Then, by the splitting theorem, there exists a minimal finite dimensional subspace R_1 of R which is invariant under $\underline{U}$ but does not consist of invariant elements alone. Without loss of generality, we can assume that $R_1 \perp K$. Since $\underline{U}$ is real, $J(K) = K$ where J is the associated conjugation. Hence $J(R_1) \perp K$. If $\{ f_1, \ldots, f_n \}$ is an orthonormal basis for R_1, then $\{ Jf_1, \ldots, Jf_n \}$ is an orthonormal basis for $J(R_1)$. It can be shown that the non-zero element $\sum_{i=1}^{n} (f_i \cdot Jf_i)$ is invariant for

$\underline{U}^{(2)}$ (using the fact that the restriction of any $U \in \underline{U}$ to R_1 is unitary), but is orthogonal to $K \times K = K(\underline{U}^{(2)})$. This completes the proof.

We shall now turn to applications of the above results. Let X be a probability space and $X^{(2)}$ its Cartesian square. As we have already noted, $L_2(X^{(2)})$ is the tensor square of $L_2(X)$. We shall consider the natural conjugation J in $L_2(X)$ which takes any element f to its complex conjugate $\bar{f}$. The condition of reality of a family $\underline{U}$ of operators then means that $U\bar{f} = \overline{Uf}$ for every f and every $U \in \underline{U}$. Let us take a semigroup $\underline{U}$ of contractions on $L_2(X)$. Along with the weak stability of $\underline{U}$, we can consider the following two properties $\underline{U}$ may possess.

Definition 7.2 A semigroup $\underline{U}$ is ergodic if every invariant function is a constant.

Definition 7.3 A semigroup $\underline{U}$ is weakly mixing if it is weakly stable and ergodic, i.e., if every reversible function is a constant.

From Theorems 7.2 and 7.3, we get the following results.

Corollary 7.1 A semigroup $\underline{U}$ with $K(\underline{U}) \neq 0$ is weakly mixing if and only if its tensor square $\underline{U}^{(2)}$ is weakly mixing.

Corollary 7.2 A real semigroup $\underline{U}$ with $K(\underline{U}) \neq 0$ is weakly mixing if and only if $\underline{U}^{(2)}$ is ergodic.

When $\underline{U}$ is a semigroup of isometries, the weak stability of $\underline{U}$ can be defined without any reference to reversible vectors, by virtue of the following lemma.

Lemma 7.4 A semigroup $\underline{U}$ of isometries on H is weakly stable if and only if every finite dimensional invariant subspace of H consists only of invariant vectors.

The proof of this lemma is done by showing that any finite dimensional invariant subspace of H is contained in $R(\underline{U})$.

Let now $\underline{U}_o$ be an arbitrary family of isometries on $L_2(X)$. With the usual definitions of invariant functions and subspaces for $\underline{U}_o$, we call $\underline{U}_o$ ergodic if every invariant function is a constant and weakly stable (weakly mixing) if every finite dimensional subspace of $L_2(X)$ invariant for $\underline{U}_o$, consists only of invariant functions (constants, respectively). It follows that the ergodicity, weak stability, weak mixing and reality of $\underline{U}_o$ and $\underline{U}$, the semigroup generated by $\underline{U}_o$ are respectively equivalent. Thus Theorem 7.2 and Corollary 7.1 hold for an arbitrary family $\underline{U}_o$ of isometries for which there exists at least one non-zero invariant vector. If this condition is satisfied and if $\bar{\underline{U}}_o$ is real, then Theorem 7.3 and Corollary 7.2 also hold for $\bar{\underline{U}}_o$.

We can apply these results to an arbitrary family $\bar{\underline{T}}_o$ of measure-preserving transformations on X. Let $\bar{\underline{U}}_o$ be the family of induced isometries on $L_2(X)$. Let $\bar{\underline{T}}_o^{(2)}$ be the Cartesian square of the family $\bar{\underline{T}}_o$, i.e., $\bar{\underline{T}}_o^{(2)} = \{T^{(2)} : T \in \bar{\underline{T}}_o\}$. Then the tensor square $\bar{\underline{U}}_o^{(2)}$ of $\bar{\underline{U}}_o$ is the family of isometries on $L_2(X^{(2)})$ induced by $\bar{\underline{T}}_o^{(2)}$. Let $\bar{\underline{I}}_o$ be the σ-field of sets invariant under $\bar{\underline{T}}_o$. $\bar{\underline{T}}_o$ is ergodic if $\bar{\underline{I}}_o$ is trivial. Clearly, $\bar{\underline{T}}_o$ is ergodic if and only if $\bar{\underline{U}}_o$ is ergodic. Let us call $\bar{\underline{T}}_o$ weakly stable or weakly mixing according as $\bar{\underline{U}}_o$ is weakly stable or weakly mixing. The family $\bar{\underline{U}}_o$ is obviously real and constants are non-zero invariant functions. So the preceding results hold for $\bar{\underline{U}}_o$, i.e., $\bar{\underline{T}}_o$ is weakly stable if and only if $\bar{\underline{T}}_o^{(2)}$ is so and also if and only if $\underline{I}(\bar{\underline{T}}_o^{(2)}) = \underline{I}(\bar{\underline{T}}_o)^{(2)}$ (which is equivalent to the relation $K(\bar{\underline{U}}_o^{(2)}) = K(\bar{\underline{U}}_o)^{(2)}$). Besides $\bar{\underline{T}}_o$ is weakly mixing if and only if $\bar{\underline{T}}_o^{(2)}$ is ergodic. This generalizes the Cartesian square part of the classical mixing theorem. This result has been obtained by Moore [42] for Borel transformation groups and by Dye [15] for amenable topological semigroups of transformations. Dye [15] has also obtained, for such semigroups of transformations, another interesting equivalent condition. To this we shall return in a few moments.

The generality of our results enables us to apply them to semigroups of transition functions also. Let $\underline{P}$ be a semigroup of transition functions with invariant measure m and let $\underline{U}$ be the semigroup of contractions induced on $L_2(X)$. $\underline{U}$ then is real and constants are invariant for $\underline{U}$. Thus Theorems 7.2 and 7.3 as well as Corollaries 7.1 and 7.2 hold for $\underline{U}$.

It is clear that our results in this chapter generalize the equivalence of conditions (2), (3) and (4) of Theorem 6.1. For a single contraction U, Theorem 7.1 connects these up with condition (1) of Theorem 6.1. It is interesting to note that Dye [15] has suitably generalised condition (1) to the case of amenable topological semigroups of transformations. We now describe this part of his results along with a corollary on the weak stability of such semigroups.

Let S be an amenable topological semigroup (see Chapter 3) and C(S) the Banach space of complex-valued continuous functions on S with sup.norm. (For the following results, Dye requires only the existence of both a right mean and a left mean on C(S).) We say that a function $f \in C(S)$ is almost convergent with limit M_o if $M(f) = M_o$ for each right mean and each left mean M on C(S).

Let $\underline{U} = \{U_s\}$ be a (weakly) continuous representation of S onto a semigroup of isometries on a Hilbert space H. As we have already noted, any finite dimensional subspace of H invariant under $\underline{U}$ is contained in R. This together with Lemma 3.4 of Dye [15] allows us to identify the subspace $F = F(\underline{U})$ of flight vectors as:

$$F(\underline{U}) = \{ f : |(U_s f, g)| \text{ is almost convergent to } 0 \text{ for every } g \in H \}.$$

The proof of the following result (part of Theorem 1 of Dye [15]) is clear.

<u>Theorem 7.4</u> Let $\underline{U}$ be a continuous isometric representation of an amenable topological semigroup S on H. Then $\underline{U}$ has no finite dimensional invariant subspace if and only if for every $f, g \in H$, the function $|(U_s f, g)|$ is almost convergent to zero.

From this, we get the following corollary on the weak stability of $\underline{U}$.

<u>Corollary 7.3</u> Let $\underline{U}$ be as in Theorem 7.4. Then $\underline{U}$ is weakly stable if and only if for every $f, g \in H$, $|(U_s f, g) - (K(\underline{U})f, g)|$ is almost convergent to zero.

<u>Proof.</u> One applies Theorem 7.4 to the restriction of $\underline{U}$ to $K(\underline{U})^{\perp}$.

To apply this to the case of measure-preserving transformations on X, let $s \to T_s$ be an anti-homomorphism (i.e., $T_{s_1 s_2} = T_{s_2} T_{s_1}$) into the set of measure-preserving transformations on X such that the map $s \to m(T_s^{-1} A \cap B)$ is continuous for all $A, B \in \underline{\mathbf{A}}$. Then we have the following result.

<u>Corollary 7.4</u> For a semigroup $\{ T_s \}$ of measure-preserving transformations on X with the above-mentioned properties, the weak stability is equivalent to the almost convergence to zero of the functions $|m(T_s^{-1} A \cap B) - \int_B P(A | \underline{I}) dm|$ where $A, B \in \underline{\mathbf{A}}$.

CHAPTER 8

THE STABILITY THEOREM-III

There is another interesting mixing theorem due to Hopf [28], concerning the weak mixing of a measure-preserving transformation T on a probability space $(X, \bar{\underline{A}}, m)$. We consider again the Cartesian square $(X^{(2)}, \bar{\underline{A}}^{(2)}, m^{(2)})$ of $(X, \bar{\underline{A}}, m)$ and the Cartesian square $T^{(2)}$ of the transformation T. A set $A^* \in \bar{\underline{A}}^{(2)}$ is called <u>symmetric</u> if $m^{(2)}(A^* + \bar{A}^*) = 0$ where $\bar{A}^* = \{(x,y): (y, x) \in A^*\}$. Let $\underline{A}_s^{(2)}$ be the σ-field of symmetric sets in $\bar{\underline{A}}^{(2)}$ and $m_s^{(2)}$ the restriction of $m^{(2)}$ to $\underline{A}_s^{(2)}$. Since the inverse image of a symmetric set under $T^{(2)}$ is symmetric, $T^{(2)}$ may be restricted to a transformation $T_s^{(2)}$ on the measure space $(X^{(2)}, \underline{A}_s^{(2)}, m_s^{(2)})$. Hopf [28] has proved that T is weakly mixing if and only if $T_s^{(2)}$ is ergodic. Since this means that if every symmetric $T^{(2)}$-invariant set has measure 0 or 1 then T is weakly mixing, this result, in a way, is stronger than the mixing theorem we have referred to earlier.

In this chapter, we shall prove the following generalization of the above symmetric mixing theorem. Recall the definitions of weak stability, weak mixing and ergodicity for a semigroup $\underline{T}$ of measure-preserving transformations.

Theorem 8.1 A countable abelian semigroup $\underline{T}$ of measure-preserving transformations is weakly stable if and only if a.e. section of every symmetric $\bar{\underline{T}}^{(2)}$-invariant set is $\bar{\underline{T}}$-invariant.

Proof. The necessity part is true for the non-abelian case also. Let $\bar{\underline{T}}$ be weakly stable and $\bar{\underline{U}}$ the semigroup of induced isometric operators. Let $\underline{A}^*$ be a symmetric $\underline{T}^{(2)}$-invariant set. It is sufficient to show that almost every y-section of the real symmetric function $1_{A^*}(x, y)$ is $\bar{\underline{T}}$-invariant.

Consider the compact Hermitian operator V on $L_2(X)$ defined by

$$(Vf)(x) = \int 1_{A^*}(x, y)\, f(y)\, m(dy).$$

Let $\lambda_o = 0, \lambda_1, \lambda_2, \ldots$ be the eigen values of V and K_n, $n = 0, 1, 2, \ldots$ be the corresponding eigen subspaces. Since, as is easily verified, V commutes with each $U \in \bar{\underline{U}}$, every K_n is invariant under $\underline{U}$. But K_n, for $n \geq 1$, is finite dimensional and so, by the weak stability of $\bar{\underline{U}}$, consists only of invariant functions. It follows that $UL = L$ for each $U \in \bar{\underline{U}}$. i.e., for each $T \in \underline{T}$ and $f \in L_2(X)$,

$$\int 1_{A*} (Tx,y)\, f(y)\, m(dy) = \int 1_{A*}(x,y)\, f(y)\, m(dy).$$

From this, it is easy to deduce that $1_{A*}(Tx,y) = 1_{A*}(x,y)$ a.e. on $X^{(2)}$ for all $T \in \underline{\bar{T}}$ and hence a.e. y-section of $1_{A*}(x,y)$ is $\underline{\bar{T}}$-invariant.

To prove the sufficiency of the condition in case $\underline{\bar{T}}$ is abelian, we first note that the splitting theorem (Theorem 3.1) for the abelian case gives $R(\underline{\bar{U}})$, the subspace of reversible functions for $\underline{\bar{U}}$, as the direct sum of one dimensional invariant subspaces; each one dimensional subspace being characterised by a function λ_U on $\underline{\bar{U}}$ of modulus one with the property that $Uf = \lambda_U f$ for each $U \in \underline{\bar{U}}$ and f a non-zero function in the subspace considered. Thus to show that $\underline{\bar{U}}$ is weakly stable, it is enough to show that every such λ_U is identically equal to 1. Assume the contrary and let $\lambda_U \not\equiv 1$ and f be such that $Uf = \lambda_U f$ for all $U \in \underline{\bar{U}}$. Then $f \perp K(\underline{\bar{U}})$. Consider the non-zero $\underline{\bar{U}}^{(2)}$-invariant function $F(x,y) = f(x)\,\overline{f(y)}$. If $f(x) = f_1(x) + if_2(x)$, then $\mathrm{Re}\, F(x,y) = f_1(x)f_1(y) + f_2(x)\, f_2(y)$ is a real symmetric invariant function for $\underline{\bar{U}}^{(2)}$ and so the assumed condition implies that a.e. y-section of $\mathrm{Re}\, F(x,y)$ is $\underline{\bar{U}}$-invariant. Since $f \perp K(\underline{\bar{U}})$, f_1 as well as f_2 are orthogonal to $K(\underline{\bar{U}})$. For a fixed y, $\mathrm{Re}\, F(x,y)$ is a linear combination of $f_1(x)$ and $f_2(x)$ and so orthogonal to $K(\bar{U})$. Thus we get a contradiction if we can find a y_o such that Re $F(x,y_o)$ is non-zero and invariant under $\bar{U}$.

Since $f \not\equiv 0$, the set $\{ x: f(x) \neq 0 \} = B$ is of positive measure. We have $B = B_1 \bar{\cup} B_2 \bar{\cup} B_3$ where $B_1 = \{ x: f_1(x) = 0, f_2(x) \neq 0 \}$, $B_2 = \{x: f_1(x) \neq 0, f_2(x) = 0 \}$ and $B_3 = \{x: f_1(x) \neq 0, f_2(x) \neq 0 \}$. If B_1 has positive measure, a suitable y_o can be taken so that Re $F(x,y_o)$ is a non-zero $\bar{U}$-invariant function. Similar is the case if B_2 has positive measure. In case B_1 as well as B_2 are of zero measure, B_3 must have positive measure. If B_3 is the singleton set $\{ y_o \}$, then Re $F(x, y_o) \not\equiv 0$. If there exist two points y_1 and y_2 in B_3 such that

$$\frac{f_1(y_1)}{f_2(y_1)} \neq \frac{f_1(y_2)}{f_2(y_2)},$$

then Re $F(x,y) = 0$ over $B_3 \times B_3$ would imply that $f_1(x) = f_2(x) = 0$ over B_3 - a contradiction. So Re $F(x,y) \neq 0$ over $B_3 \times B_3$ and a suitable y_o-section can be taken for which Re $F(x,y_o)$ is non-zero invariant. If finally

$$\frac{f_1(y)}{f_2(y)} = \frac{f_1(z)}{f_2(z)} = k$$

for all y and z in B_3, then Re $F(x,y) = (k^2+1)[f_2(x)f_2(y)] \neq 0$

over $B_3 \times B_3$ and hence again a suitable y_0-section can be taken. Thus in all cases, there is a y_0-section such that $\operatorname{Re} F(x,y_0)$ is non-zero and invariant. The proof of the theorem is complete.

Corollary 8.1 A countable abelian semigroup $\bar{T}$ is weakly mixing if and only if every symmetric $\bar{T}^{(2)}$-invariant set has measure zero or one.

Proof. Let the condition be satisfied. Clearly $\bar{T}$ is ergodic. If A^* is any symmetric $\bar{T}^{(2)}$-invariant set, then a.e. section of A^* has measure one if A^* has measure one and measure zero if A^* has measure zero. In either case, a.e. section of A^* is $\bar{T}$-invariant and so $\bar{T}$ is weakly stable. Thus $\bar{T}$ is weakly mixing.

If $\bar{T}$ is weakly mixing, then for any symmetric invariant set A^*, a.e. y-section of A^* is $\bar{T}$-invariant and hence has measure zero or one. The set $\{$ y: y-section of A^* has measure one $\}$ is $\bar{T}$-invariant and has measure zero or one. Then A^* itself has measure zero or one.

Corollary 8.2 $\bar{I}(\bar{T}_s^{(2)}) \subseteq \bar{I}(\bar{T})^{(2)}$ if and only if $\bar{I}(\bar{T}^{(2)}) = \bar{I}(\bar{T})^{(2)}$ for a countable abelian semigroup $\bar{T}$.

The proof is done by using Theorem 8.1 and the results of Chapter 7. The point of interest here is that weak stability is used in proving a general assertion about certain σ-fields in the product space.

CHAPTER 9

SKEW PRODUCTS AND TRANSITION FUNCTIONS

. In this chapter, we continue our study of families of measure-preserving transformations on a probability space $(X, \underline{A}, m)$ but now with a different set up. Let $\underline{T} = \{ T_y : y \in Y \}$ be a family of measure-preserving transformations on X. We assume that Y itself is a probability space $(Y, \underline{B}, \mu)$. This set up has been considered by various authors (see Gládysz [19] for references) and random ergodic theorems have been proved. Kakutani [34] discussed the ergodicity of the family $\underline{T}$ and Gladysz [19] followed up with a discussion of the weak mixing properties of $\underline{T}$. The interesting results relate such properties of the family $\underline{T}$ with those of a certain 'skew product' transformation and of a transition function - which we proceed to introduce.

We shall assume throughout this chapter that the family $\underline{T}$ satisfies the following property: $A \in \underline{A}$ implies that the set $\{ (x,y): T_y x \in A \}$ is a measurable subset of $X \times Y$. The definitions of invariant sets, functions and subspaces for $\underline{T}$, used earlier, have to be modified slightly in accordance with the principle that sets of zero measure are negligible; e.g., a subspace L of $L_2(X)$ is invariant if $U_y L \subseteq L$ for almost all $y \in Y$, where U_y is the isometry induced by T_y. This has to be kept in mind when we talk of ergodicity, weak stability etc., of the family $\underline{T}$.

The family $\underline{T}$ induces a transition function $P(x, A)$ on $X \times \underline{A}$ as follows. (See Kakutani [34].)

$$P(x, A) = \mu \{ y: T_y x \in A \} \qquad x \in X,\ A \in \underline{A}.$$

This transition function leaves the measure m invariant:

$$\int P(x, A)\, m(dx) = m(A).$$

The contraction U on $L_2(X)$ induced by $P(x,A)$ satisfies the relation:

$$(Uf)(x) = \int_Y f(T_y x)\mu(dy).$$

Hence we also have

$$(U^k f)(x) = \int_Y \cdots \int_Y f(T_{y_1} T_{y_2} \cdots T_{y_k} x)\mu(dy_1)\mu(dy_2)\cdots \mu(dy_k)$$

for $k = 2,3,\ldots$

Let $(Y^*, \underline{B}^*, \mu^*)$ be the one-sided infinite product of $(Y, \underline{B}, \mu)$ with itself. The nth coordinate of a point $y^* \in Y^*$

is denoted by $y_n(y^*)$ for $n = 0,1,2,\ldots$ The shift χ in Y^* defined by $y_n(\chi y^*) = y_{n+1}(y^*)$ is a measure-preserving transformation on Y^*. The skew product transformation φ is defined on $(X \times Y^*, \underline{A} \times \underline{B}^*, m \times \mu^*)$ by the equation

$$\varphi(x,y^*) = (T_{y_o(y^*)}\, x, \chi y^*).$$

The transformation φ preserves the measure $m \times \mu^*$. It is easily seen that

$$\varphi^n(x,y^*) = (T_{y_{n-1}(y^*)} \cdots T_{y_o(y^*)} x, \chi^n y^*)$$

for $n = 1,2,\ldots$.

Kakutani [34] proved that the ergodicities of $\underline{T}$, U and φ are equivalent. He posed the interesting problem of discussing the weak mixing properties of $\underline{T}$, U and φ. In this chapter we shall discuss the weak stability properties of $\underline{T}$, U and φ and obtain, as a corollary, a result on their weak mixing properties. More precisely, we prove that the weak stability of φ and U are equivalent and equivalent to the following property of the family $\underline{T}$ which, as we shall see below, is weaker than the weak stability of $\underline{T}$.

<u>Definition 9.1</u> The family $\underline{T}$ is called <u>weakly</u> <u>G-stable</u> if $|\lambda| = 1$, $f \in L_2(X)$ and $U_y f = \lambda f$ for almost all y imply that $\lambda = 1$.

This definition corresponds to the definition of weak mixing (hereinafter referred to as weak G-mixing) of the family $\underline{T}$ according to Gladysz [19]. Gladysz has proved the equivalence of the weak G-mixing of $\underline{T}$ and the weak mixing of φ. Here we go a step further and bring the operator U into the picture. Besides we do not assume the invertibility of the transformations in $\underline{T}$. First of all, let us establish that the weak G-stability of $\underline{T}$ is weaker than its weak stability.

If $\underline{T}$ is weakly stable and if there is an $f \in L_2(X)$ and a complex number λ of modulus one such that $U_y f = \lambda f$ for almost all y, then the subspace spanned by f is an one dimensional invariant subspace and so consists of invariant functions, i.e., f is invariant. Hence $\underline{T}$ is weakly G-stable. But the converse is not true in general. Consider $X = Y =$ the circle group with Legesgue measure and $T_y x = xy$ for all $x \in X$, $y \in Y$. The family $\underline{T}$ is then weakly G-mixing and hence weakly G-stable. For, if $f \in L_2(X)$ with $U_y f = \lambda f$ for almost all y, $|\lambda| = 1$, let $f(x) = \Sigma\, c_n x^n$. (We know that the functions $f_n(x) = x^n$, $n = 0, \pm 1, \pm 2, \ldots$ form a complete orthonormal basis for $L_2(X)$.) Then $(U_y f)(x) = \Sigma\, c_n x^n y^n = \Sigma\, c_n \lambda x^n$ and hence $c_n y^n = \lambda\, c_n$ for all n. If f is not a constant, some c_n with $n \neq 0$ is non-zero, say, $c_{n_0} \neq 0$.

Then $y^{n_o} = \lambda$, i.e., y is a root of λ. Thus $U_y f = \lambda f$ can hold only for at most a countable number of values of y (those y which are roots of λ) - a contradiction. Hence f must be a constant. However, the family $\underline{\bar{T}}$ is not weakly stable, since the one dimensional subspace of $L_2(X)$ generated by the function $f(x) = x^n$, $n \neq 0$ is invariant, but contains the non-invariant function $f(x) = x^n$.

For proving our main result on the weak G-stability of the family $\underline{\bar{T}}$, we need the following special case of a lemma of Gladysz [19].

Lemma 9.1 If $F(x, y^*) \in L_2(X \times Y^*)$ is such that

$$F(\varphi(x, y^*)) = aF(x, y^*) \quad \text{a.e.}$$

for some constant a of modulus one, then there exists a measurable function $g(x)$ such that

$$F(x, y^*) = g(x) \quad \text{a.e.}$$

Theorem 9.1 The following statements are equivalent.

(1) $\underline{\bar{T}}$ is weakly G-stable

(2) U is weakly stable

(3) φ is weakly stable.

<u>Proof</u>. We shall prove that (3) => (2) => (1) => (3).

If φ is weakly stable, then for $F, G \in L_2(X \times Y^*)$, the sequence $(F(\varphi^k), G)$ is strong Cesaro convergent. Taking now $f, g \in L_2(X)$ and putting $F(x, y^*) = f(x)$ and $G(x, y^*) = g(x)$, we have

$$F(\varphi^k(x, y^*)) = f(T_{y_{k-1}(y^*)} \cdots T_{y_o(y^*)}\, x)$$

and so

$$(F(\varphi^k), G) =$$

$$= \int_{X \times Y^*} f(T_{y_{k-1}(y^*)} \cdots T_{y_o(y^*)} x)\overline{g(x)}\; m(dx)\mu^*(dy^*)$$

$$= \int_X \int_Y \cdots \int_Y f(T_{y_{k-1}} \cdots T_{y_o} x)\overline{g(x)}\; m(dx)\; \mu(dy_{k-1}) \cdots \mu(dy_o)$$

$$= (U^k f, g)$$

and hence the sequence $(U^k f, g)$ is strong Cesaro convergent. By Theorem 7.1, U is weakly stable. Thus (3) => (2).

Let U be weakly stable. If λ is a complex number of modulus one such that, for some $f \in L_2(X)$, $(U_y f)(x) = \lambda f(x)$ a.e. (y), then

$$(Uf)(x) = \int_Y (U_y f)(x)\; \mu(dy)$$
$$= \lambda\, f\, (x).$$

The weak stability of U implies that $\lambda = 1$ and hence (2) => (1).

Let now $\underline{T}$ be weakly G-stable. If φ is not weakly stable, then there is a $\lambda \neq 1$, $|\lambda| = 1$ and an $F \in L_2(X \times Y^*)$ such that

$$F(\varphi(x, y^*)) = \lambda F(x,y^*) \quad \text{a.e.}$$

By Lemma 9.1, there is a function $g \in L_2(X)$ such that

$$F(x,y^*) = g(x) \quad \text{a.e.}$$

Hence we have

$$g(T_{y_o(y^*)} x) = \lambda g(x) \quad \text{a.e.}$$

It is now easy to see that for almost all y,

$$g(T_y x) = \lambda g(x) \quad \text{a.e.}(x)$$

Since $\underline{T}$ is weakly G-stable, $\lambda = 1$ - a contradiction. Thus (1) => (3).

<u>Corollary 9.1</u> The following statements are equivalent.

(1) $\underline{T}$ is weakly G-mixing

(2) U is weakly mixing

(3) φ is weakly mixing.

It would be interesting to extend the result of this chapter to families of contractions on an arbitrary Hilbert space. Specifically, let $(Y, \underline{B}, \mu)$ be a probability space and let, for each $y \in Y$, U_y be a contraction on a Hilbert space H. The weak G-stability of $\underline{U} = \{ U_y \}$ is defined in the obvious way. The equation

$$(Uf, g) = \int (U_y f, g) \, \mu(dy)$$

for $f, g \in H$, yields a new contraction U (under suitable measurability assumptions). Is the weak stability of U equivalent to the weak G-stability of the family $\underline{U}$? It is easy to see that if U is weakly stable, then $\underline{U}$ is weakly G-stable. We are unable to answer the other part of the question.

CHAPTER 10

AUTOMORPHISMS OF COMPACT GROUPS

Throughout this chapter, X is a compact topological group with $\underline{A}$, the σ-field of Borel sets and m the normalised Haar measure on X. Our aim is to study the weak stability properties of continuous automorphisms of X. For this, we shall need the following facts from the representation theory for compact groups.

A representation $\underline{V} = \{ V(x) \}$ of X in a Hilbert space H is a strongly continuous homomorphism $x \to V(x)$ of X into the group of bounded invertible operators on H. If H is finite dimensional, we call $\underline{V}$ finite dimensional. If V(x) is unitary for each $x \in X$, then $\underline{V}$ is called a unitary representation.

Two representations $\underline{V}_1$ and $\underline{V}_2$ of X in Hilbert spaces H_1 and H_2 respectively are called equivalent if there exists an isomorphism W from H_2 to H_1 such that $V_1(x)W = WV_2(x)$ for all $x \in X$.

A representation $\underline{V}$ of X in H is irreducible if no proper subspace of H is invariant under all V(x), $x \in X$.

An important result for a compact group X is that every representation of X in a Hilbert space is equivalent to a unitary representation. Moreover, every irreducible unitary representation of X is finite dimensional.

Let $\{ V(x) \}$ be an irreducible representation of X acting on H. With respect to some orthonormal basis for H, we may consider the representation as a set of matrices $\{ v_{ij}(x) \}$. For fixed i and j, the function $v_{ij}(x)$ on X is continuous; the functions $\{ v_{ij} \}$ are called the matrix functions of the representation. The trace of the matrix $\{ v_{ij}(x) \}$ is called the character of the (irreducible) representation; it is independent of the orthonormal basis chosen. Two irreducible representations are equivalent if and only if they have the same character. We denote the set of all characters by $Ch(X)$.

Let the family of all equivalence classes of irreducible representations of X be indexed by α. With the equivalence class with index α, we can associate uniquely a finite dimensional subspace S_α of $L_2(X)$, spanned by the matrix functions of any representation in that class. The celebrated Peter-Weyl theorem then says that the S_α's are mutually orthogonal and $L_2(X) = \oplus S_\alpha$.

If X is a compact abelian group, the foregoing theory simplifies very much. Here every irreducible representation is one dimensional and so an irreducible unitary representation of X is only a continuous homomorphism of X into the circle group. This is also the character of the representation. The set of all characters is a group and forms an orthonormal basis for $L_2(X)$.

We now come to the study of continuous automorphisms of the compact group X. These are invertible measure-preserving transformations on X. For convenience in presentation and for motivating the results in the general case, we shall consider abelian groups first.

Let T be a continuous automorphism of the compact abelian group X. The induced unitary operator takes characters to characters and indeed is an automorphism of the character group $Ch(X)$. It is well known that T is ergodic if and only if U on $Ch(X)$ has no finite orbits on the set of non-constant characters in $Ch(X)$. Besides, the ergodicity of T is equivalent to its mixing (Halmos [24]). Motivated by this result, Maitra [41] has shown that T is stable if and only if U has no finite orbits of order greater than one on $Ch(X)$. The following theorem and its corollary on the weak stability of T are immediate consequences of this result and Corollary 6.4.

<u>Theorem 10.1</u> T is weakly stable if and only if U on $Ch(X)$ has no finite orbits of order greater than one.

<u>Corollary 10.1</u> T is weakly stable if and only if U on $L_2(X)$ has no finite orbits of order greater than one.

Note that Theorem 10.1 and Maitra's result quoted above show that every weakly stable automorphism of X is stable. An example of an automorphism X which is not weakly stable is the automorphism which takes every element of X to its inverse.

We shall now consider examples of weakly stable automorphisms. If X is the circle group, then the only weakly stable automorphism of X is the identity. On the tori, however, we can find examples of non-trivial weakly stable automorphisms. To see this, let, for any integer $k \geq 2$, $X^{(k)}$ be the k-dimensional torus. We know (see Jacobs [33]) that there exists a one-one correspondence between automorphisms of $X^{(k)}$ and matrices of order k with integral entries and determinant ± 1 such that if T is an automorphism and (t_{ij}) the corresponding matrix, then for every

$(x_1, x_2, \ldots, x_k) \in X^{(k)}$, $T(x_1, x_2, \ldots, x_k) = (x_1', x_2', \ldots, x_k')$ where $x_j' = x_1^{t_{1j}} x_2^{t_{2j}} \ldots x_k^{t_{kj}}$ for $1 \leq j \leq k$. Moreover, the character group $\mathfrak{Ch}(X^k)$ of $X^{(k)}$ may be identified with $Z^{(k)}$, the k-fold direct product of the group of integers Z with itself such that the action of U on $\mathfrak{Ch}(X^{(k)})$ coincides with the action of (t_{ij}) on $Z^{(k)}$ defined by $(t_{ij})(n_1, n_2, \ldots, n_k) = (n_1', n_2', \ldots, n_k')$ where $n_i' = \sum_{j=1}^{k} n_j t_{ij}$.

T is therefore weakly stable as soon as (t_{ij}) does not have orbits of order $p > 1$ on $Z^{(k)}$. For this, a sufficient condition is that no eigen value $\lambda \neq 1$ of (t_{ij}) should be a root of unity. (For, if T were not weakly stable, then there would exist an element $n \in Z^{(k)}$ of order $p > 1$ for (t_{ij}). It can then be seen that (t_{ij}) must admit an eigen value different from unity, which is however a pth root of unity.) This condition is mild enough to enable us to construct many weakly stable automorphisms of $X^{(k)}$. In fact, we can have families of weakly stable automorphisms none of which is weakly mixing. E.g., if (t_{ij}) is an integral-entried matrix of order k such that $t_{ii} = 1$ for $1 \leq i \leq k$, $t_{ij} = 0$ for $i > j$, $1 \leq i, j \leq k$ and $t_{12} \neq 0$, then the corresponding automorphism on $X^{(k)}$ is antiperiodic and weakly stable but (see Jacobs [33]) not weakly mixing.

An interesting property of weakly stable automorphisms of a compact abelian group is given by the following theorem.

<u>Theorem 10.2</u> An automorphism T is weakly stable if and only if the subspace of invariant functions in $L_2(X)$ is spanned by the invariant characters in $Ch(X)$.

<u>Proof</u>. Let T be weakly stable. $K(U)$ denotes the subspace of invariant functions. Let $\{ f_\alpha \}$ be the family of non-invariant characters in $Ch(X)$. To show that $K(U)$ is spanned by the invariant characters, it is enough to prove that if $f = \Sigma\, c_\alpha f_\alpha$ with $\Sigma |c_\alpha|^2 < \infty$ is any invariant function, then $f = 0$. But this follows easily from the fact that U has only infinite orbits on the set $\{ f_\alpha \}$.

If T is not weakly stable and g is any non-invariant character in $Ch(X)$ such that for some integer $p > 1$, $U^p g = g$, $Ug, \ldots, U^{p-1} g$ are all distinct, then the function $h = g + Ug + \ldots + U^{p-1} g$ is a non-zero invariant function. h is therefore in $K(U)$ but is not in the span of the invariant characters.

We shall now consider T to be a continuous automorphism of a general compact group X.

Given an irreducible representation $\underline{V} = \{ V(x) \}$ of X, let us write $\varrho \underline{V} = \varrho \{V(x)\} = \{V(Tx)\}$ which is another irreducible representation of X acting on the same Hilbert space as $\underline{V}$. It is trivial that ϱ takes equivalent representations to equivalent representations and so induces a map of the set $\underline{F}$ of all S'_α s onto itself, which also can be denoted by ϱ. Indeed $\varrho(S_\alpha)$, for any α, is the image of S_α under U. It is easy to check that the action of ϱ on $\underline{F}$ may be identified with the action of U on $Ch(X)$ and hence that U maps $Ch(X)$ onto itself.

Kaplansky [35] has observed that even for the non-abelian case, the conditions of ergodicity and mixing are equivalent for automorphisms T of X and that T is ergodic if and only if U has no finite orbits on the set of non-constant characters in $Ch(X)$. Looking at this result and Theorem 10.1, one is tempted to conclude that, even here, T is weakly stable if and only if U has no finite orbits of order greater than one on $Ch(X)$. But this would be rash. For, if T is any inner automorphism of X, i.e., $Tx = yxy^{-1}$ for some $y \in X$, then U is the identity on $Ch(X)$, but no inner automorphism, except the identity, can be weakly stable; in particular, no inner automorphism can be ergodic. For, if T is an inner automorphism of X, then $\rho(S_\alpha) = S_\alpha$ for every α and hence every S_α is a finite dimensional invariant subspace of U. When T is weakly stable, U is the identity on every S_α and hence on $L_2(X)$. The following result explains the situation.

Theorem 10.3 T is weakly stable if and only if for every irreducible representation $\underline{V}$ of X, $\rho\,\underline{V}$ is equivalent to $\underline{V}$ implies that $\rho\underline{V} = \underline{V}$ and U on $\mathbf{Ch}(X)$ has no finite orbits of order greater than one.

Proof. Let T be weakly stable. Suppose that $\underline{V}$ is an irreducible representation of X of type α. If $\rho\,\underline{V}$ is equivalent to $\underline{V}$ then S_α is invariant under U and hence U is the identity on S_α. If v_{ij} are the matrix functions of $\underline{V}$, then $v_{ij}(x) = v_{ij}(Tx)$ a.e. for each fixed i and j. Since v_{ij} is continuous, $v_{ij}(x) = v_{ij}(Tx)$ for all $x \in X$. Thus $\rho\,\underline{V} = \underline{V}$. The second part is a consequence of Corollary 6.4 and the fact that U leaves $\mathbf{Ch}(X)$ invariant.

Conversely, let the automorphism T satisfy the conditions of the theorem. Let $\underline{F}_o$ be the subset of $\underline{F}$ corresponding to invariant characters in $\mathbf{Ch}(X)$. The first condition shows that U is the identity on every $S_\alpha \in \underline{\bar{F}}_o$. The second condition implies that ρ has only infinite orbits on $\underline{F} - \underline{F}_o$. It follows that for every function $f \in S_\alpha \in \underline{F} - \underline{F}_o$, the functions $f, Uf, U^2f, \ldots$ are mutually orthogonal. A proof similar to the abelian case yields the result that T is, in fact, stable.

<u>Corollary 10.2</u> T is weakly stable if and only if, for every irreducible representation $\underline{\bar{V}}$ of X, $\rho\,\underline{\bar{V}}$ is equivalent to $\underline{V}$ implies that $\rho\underline{V} = \underline{V}$ and U on $L_2(X)$ has no finite orbits of order greater than one.

<u>Corollary 10.3</u> Every weakly stable automorphism of X is stable.

The following theorem generalizes Theorem 10.2.

<u>Theorem 10.4</u> An automorphism T is weakly stable if and only if the subspace $K(U)$ of invariant functions in $L_2(X)$ is the direct sum of the S_α's corresponding to the invariant characters of X.

<u>Proof.</u> As before, let $\underline{F}_o$ be the set of S_α's corresponding to invariant characters and let $S = \oplus \{ S_\alpha : S_\alpha \in \underline{\bar{F}}_o \}$. We

have to show that T is weakly stable if and only if $S = K(U)$.

Let T be weakly stable. If $S_\alpha \in \underline{F}_0$, then U leaves S_α invariant and hence is the identity on S_α. Therefore $S \subseteq K(U)$. Let $f \in S^\perp$ and invariant. f can then be written (uniquely) in the form Σf_α with $f_\alpha \in S_\alpha \in \bar{\underline{F}} - \underline{F}_0$. Since f is invariant, $\Sigma f_\alpha = \Sigma U^n f_\alpha$ for $n = 1,2,\ldots$. The fact that ρ has only infinite orbits on $\bar{\underline{F}} - \underline{F}_0$ now implies that, for every f_α, there exists a sequence α_k of distinct α's such that $\|f_\alpha\| = \|f_{\alpha_k}\|$ for all k. Since $\Sigma \|f_\alpha\|^2 = \|f\|^2 < \infty$, it follows that $f = 0$.

Conversely let $S = K(U)$. If $\bar{\underline{V}}$ is any irreducible representation of X, of type α, say, such that $\rho \underline{V}$ is equivalent to $\underline{V}$, then $\rho(S_\alpha) = S_\alpha \subseteq S = K(U)$ and so U is the identity on S_α. This in turn implies that $\rho \bar{\underline{V}} = \bar{\underline{V}}$. Since we can show that U has no finite orbits on the non-invariant characters just as in the abelian case, an application of Theorem 10.3 yields that T is weakly stable.

It would be interesting to discuss, more generally, what subspaces associated with T are spanned by the S_α's corresponding to the characters lying in the subspace (in the abelian case, by the characters lying in that subspace). For instance, is this true for the subspace of reversible functions? We conjecture that the subspace of reversible functions is always spanned by the S_α's corresponding to the reversible characters. This is trivially true for weakly stable automorphisms. When T is not weakly stable, we are unable to say anything definite.

Before closing this chapter, we wish to remark that the necessity parts in all the theorems and corollaries of this chapter (excepting, of course, Corollary 10.3) hold for groups of automorphisms on X also.

CHAPTER 11

ERGODIC DECOMPOSITIONS AND WEAK STABILITY

It is a well-known result in ergodic theory that, under suitable conditions, a measure-preserving transformation on a probability space may be expressed as a direct integral of ergodic transformations in an essentially unique fashion. Since, in a manner of speaking, ergodicity is to measure-preservingness what (weak) mixing is to (weak) stability, it is reasonable to ask whether (weakly) stable transformations are expressible as direct integrals of (weakly) mixing transformations. Maitra [41] has considered this question and answered it in the negative. He has an example of a stable transformation none of whose (ergodic) components is even weakly mixing.

In this chapter, we shall show that while the components of a weakly stable transformation need not be weakly mixing, it is nevertheless true that if T is a transformation almost all of whose components are weakly mixing, then T is weakly stable. Indeed our result of this chapter shows that T is weakly stable under even a weaker condition on collections of components of $T_\bullet$. We are unable to say anything similar for stable transformations, since we use spectral methods for proving our result.

Let us first give a brief sketch of the decomposition theory. Let $(X, \underline{A}, m)$ be a probability space for which the following two conditions hold :

(a) $\underline{A}$ is countably generated.

(b) For every countably generated sub-σ-field $\underline{B}$ of $\underline{A}$, there exists a real-valued function $P(A|\underline{B})(x)$, $x \in X$, $A \in \underline{A}$ such that

i) for each $x \in X$, $P(A|\underline{B})(x)$ is a probability measure on $(X, \underline{A})$

ii) for each $A \in \underline{A}$, $P(A|\underline{B})(x)$ is a $\underline{B}$-measurable function and

iii) for each $A \in \underline{A}$ and $B \in \underline{B}$, $\int_B P(A|\underline{B})(x)\, m(dx) = m(A \cap B)$.

The simplest example of a space with these properties is a Borel set in an Euclidean space. Other examples are Lusin spaces; more generally perfect probability spaces (see Blackwell [3]). In what follows, we assume that conditions (a) and (b) hold for the space X.

Let T be an invertible measure-preserving transformation on X with the property that $\underline{I}^o$, the σ-field of strictly invariant sets is countably generated. One can then show that there exists a set $N \in \bar{\underline{I}}^o$ with $m(N) = 0$ such that for every $A \in \bar{\underline{I}}^o$, $P(A|\bar{\underline{I}}^o)(x) = 1_A(x)$ for all $x \notin N$.

Since $\bar{\underline{I}}^o$ is countably generated, it has atoms and every set in $\bar{\underline{I}}^o$ is a union of atoms. Let the atoms of $\underline{I}^o$ which are disjoint with N be indexed by a set Y and let for $y \in Y$, X_y denote the corresponding atom. Each X_y is made into a probability space by requiring the measurable subsets $\bar{\underline{A}}_y$ of X_y to be of the form $X_y \cap A$ with $A \in \underline{A}$ and defining a measure m_y on $(X_y, \bar{\underline{A}}_y)$ by

$m_y(X_y \cap A) = P(A|\bar{\underline{I}}^o)(x)$ where $x \in X_y$ is arbitrary. (m_y is well-defined because the function $P(A|\bar{\underline{I}}^o)(x)$ is constant on the X_y's for every $A \in \bar{\underline{A}}$.)

We convert Y itself into a probability space $(Y, \underline{C}, \mu)$ by declaring a subset of Y to be measurable if and only if the union of the corresponding atoms of $\bar{\underline{I}}^o$ is in $\underline{I}^o$ and defining the measure μ of this subset to be the m-measure of the corresponding set in $\bar{\underline{I}}^o$.

For each $y \in Y$, X_y is a strictly invariant set and so the transformation T may be restricted to an invertible map T_y of X_y. The decomposition theorem of Halmos [22] then says that for almost all $y \in Y$, T_y is measure-preserving on $(X_y, \underline{A}_y, m_y)$ and, indeed, is ergodic. The transformations T_y are called the components of T and T is said to be the direct integral of the T_y's over the measure space $(Y, \underline{C}, \mu)$.

We are now in a position to present our theorem. Let $Y_o \subseteq Y$ be the set (of measure one) of all y such that T_y is an ergodic measure-preserving transformation on $(X_y, \underline{A}_y, m_y)$. For each $y \in Y_o$, let U_y denote the induced unitary operator on $L_2(X_y)$.

<u>Theorem 11.1</u> Let, for any complex number λ of modulus one, C_λ denote the set of all $y \in Y_o$ such that λ is an eigen value of U_y. If the inner measure of C_λ is zero for every $\lambda \neq 1$, then T is weakly stable.

<u>Proof</u> Suppose T to be not weakly stable and let $\lambda \neq 1$ be an eigen value of U. Choose and fix a bounded measurable function f defined everywhere on X and a strictly invariant set $A \subseteq X - N - \bigcup_{y \notin Y_o} X_y$ of positive

measure with the property that, for every $x \in A$, $f(Tx) = \lambda f(x) \neq 0$. Then, for every $X_y \subseteq A$, the function f restricted to X_y, say f_y, is a non-zero bounded measurable function on $(X_y, \underline{A}_y, m_y)$ and $f_y(T_y x) = \lambda f_y(x)$ for all $x \in X_y$, so that λ is an eigen value of U_y. If now C is the set of all $y \in Y_o$ for which $X_y \subseteq A$, then $C \subseteq C_\lambda$ and $\mu(C) = m(A) > 0$. Hence the inner measure of C_λ is positive.

<u>Corollary 11.1</u> If T_y is weakly mixing for almost all $y \in Y$, then T is weakly stable.

<u>Remarks</u> (1) The above corollary can be generalised to locally compact groups of transformations acting measurably on sufficiently smooth Borel spaces, using the decomposition theory for such cases, given by Varadarajan [52].

(2) Let T be an invertible measurable transformation on a measurable space $(X, \underline{A})$. Then the set $P_o(X)$ of all invariant probability measures is a convex set of which the set of extreme points is the set $P_e(X)$ of ergodic measures. (A measure m is ergodic, etc., if T is ergodic, etc., in the space $(X, \bar{\underline{A}}, m)$.) Under the assumption

(A1) If all ergodic measures vanish over a strictly invariant set A, then all invariant measures vanish over A

Blum and Hanson [6] have proved that any invariant measure m has the following integral representation with respect to a unique measure μ_m on a certain σ-field π of subsets of $P_e(X)$.

$$(*) \qquad m(A) = \int_{P_e(X)} \sigma(A)\mu_m(d\sigma) \qquad A \in \underline{A} .$$

The set $P_{ws}(X)$ of weakly stable measures forms a convex set of which the set $P_{wm}(X)$ of weakly mixing measures is the set of extreme points. We can now ask if every weakly stable measure can be represented as an integral over the set $P_{wm}(X)$. Let us make the following parallel assumption:

(A2) If all weakly mixing measures vanish over a strictly invariant set A, then all weakly stable measures vanish over A.

Under (A2), we can show that the set $P_{wm}(X)$ is a thick subset of the measure space $(P_e(X), \pi, \mu_m)$ whenever m is weakly stable. In such a case we can restrict μ_m and π to the set $P_{wm}(X)$. Thus under assumptions (A1) and (A2), the equation (*) holds, where now m is weakly stable and the integral is over the set $P_{wm}(X)$ only.

Let X be a compact Hausdorff space and T a homeomorphism of X. Then $P_o(X)$ is non-empty and the assumption (A1) holds by Lemma 4.5 of Varadarajan [52]. If further the set $P_{ws}(X)$ is non-empty and closed (in the weak topology) then the same lemma implies that (A2) also holds. We are unable to decide under what additional conditions $P_{ws}(X)$ is a non-empty closed set.

PART III

I N V A R I A N T   M E A S U R E S

CHAPTER 12

INTRODUCING PART III

There are two kinds of questions regarding the existence of invariant measures for a given family of measurable transformations. One may take just a measurable space and ask for a measure invariant under each one of the transformations. Or one may start with a measure space and ask for an invariant measure which is stronger than the given measure. The former is important, e.g., in the context of ergodic decompositions - representing an invariant measure in terms of ergodic measures. Then it is necessary to know that there exist non-trivial invariant measures. Kryloff and Bogoliouboff [39] (see Oxtoby [47]) discussed this problem for a homeomorphism of a compact metric space and Fomin [18] for groups of homeomorphisms. The latter problem comes up when we want to see if the individual ergodic theorem is valid for a given measurable transformation in a measure space; it is valid, for instance, if there exists an invariant measure which is stronger than the given measure. This problem has been fruitfully investigated by numerous authors and has a long history. We shall content ourselves with mentioning the works of

Hopf [27], Halmos [23], Dowker [12], Hajian and Kakutani [21] and Sucheston [51]. There have been fewer investigations on (stronger) invariant measures for families of transformations. Cotlar and Ricabarra [7] and Rechard [48] and most recently Blüm and Friedman [5] have made interesting contributions in this direction.

One of the problems of ergodic theory is to investigate what properties possessed by transformations allow themselves to be carried over to transition functions and more generally to contractions in L_1-spaces. The problem of existence of an invariant measure for a transition function is of independent interest and has been studied, among others, by Doeblin [11], Kryloff and Bogoliouboff [37 ; 38] and Harris [26]. Ito [29] obtained a number of interesting necessary and sufficient conditions generalizing from the transformation case. Some of the results of Ito [29] were generalized to a positive contraction on an L_1-space by Dean and Sucheston [9], Neveu [46] and Hajian and Ito [20].

Our work, in the following two chapters, is concerned with both the questions mentioned in the beginning. In Chapter 13, we consider the existence of an invariant measure for families - more precisely, semigroups - of continuous mappings on a compact Hausdorff space and obtain a unified generalization of the results of Fomin [18]. In the last chapter, we consider the existence of a finite invariant (equivalent) measure for families of transformations, transition functions and contractions on L_1-spaces and obtain generalizations of the results of Cotlar and Ricabarra [7], Rechard [48] and Blum and Friedman [5]. The main idea in the proofs is to use invariant means for amenable semigroups.

CHAPTER 13

INVARIANT MEASURES ON COMPACT HAUSDORFF SPACES

In this chapter, we prove two results on the existence of an invariant probability measure for a family of transition functions or transformations on a compact Hausdorff space. The first result is for amenable topological semigroups of transition functions, from which the results of Fomin [18] follow as corollaries. The second one is for an equicontinuous group of homeomorphisms.

Let X be a compact Hausdorff space. We shall use the notations and definitions introduced in Chapter 2. We note that if P_1 and P_2 are regular continuous transition functions, then P_1P_2 is also a regular continuous transition function. This can be proved as in Lemma 2 of Rosenblatt [49]. If S is an amenable topological semigroup and if $s \to P_s$ is a homomorphism into the set of regular continuous transition functions on X such that for each fixed $x \in X$, the map $s \to P_s(x, \cdot)$ from S to $\underline{M}(X)$ (with the weak topology) is continuous, then we call $\underline{P} = \{P_s : s \in S\}$ a continuous representation of S by regular continuous transition functions on X. Similar is the definition of a continuous representation of S by continuous transformations on X; continuity of the representation means that the map $s \to T_s x$ for fixed $x \in X$ is continuous. We note that if $T_{s_1 s_2} = T_{s_1} T_{s_2}$ then the induced transition functions satisfy the relation $P_{s_1 s_2} = P_{s_1 s_2}$.

<u>Theorem 13.1</u> There exists a regular Borel probability measure invariant for any continuous representation of an amenable semigroup by regular continuous transition functions on a compact Hausdorff space.

<u>Proof</u>. Let $\underline{P}$ be a continuous representation of an amenable semigroup S by regular continuous transition functions on the compact Hausdorff space X. Consider the transformations U_s defined for bounded real-valued functions on X by the equation

$$(U_s f)(x) = \int f(y)\, P_s(x,dy).$$

By the continuity of P_s, the transformation U_s is an operator on $C(X)$, the Banach space of bounded real-valued continuous functions on X. Let us fix an $x \in X$ and an

$f \in C(X)$. Since the maps $s \to P_s(x, \cdot)$ and $P_s(x, \cdot) \to \int f(y)\, P_s(x, dy)$ are continuous, it follows that the bounded real-valued function $(U_s f)(x)$ on S is continuous.

Let M be any invariant mean on $C(S)$ and put

$$\lambda(f) = M((U_s f)(x)) \qquad f \in C(X) .$$

Then λ is a non-negative linear functional on $C(X)$ with the property $\lambda(1) = M(1) = 1$. Moreover, it follows from the (left) invariance of M that $\lambda(f) = \lambda(U_s f)$ for all $s \in S$. The measure m which corresponds to λ by Riesz's theorem, has the required properties.

Remarks (1) The above theorem remains true if S is only left amenable.

(2) Instead of fixing a point $x \in X$, we may fix a probability measure μ on X and assuming that the map $s \to \int P_s(x, \cdot\,)\, \mu(dx)$ is continuous, get an invariant measure from the linear functional

$$\lambda(f) = M(\int (U_s f)(x)\, \mu(dx)).$$

It is clear that all the invariant measures are obtained in this way.

(3) The above result was obtained independently of the work of Lloyd [40] where many interesting properties of the set of invariant measures are proved. The author is indebted to Mr. K. Viswanath for this reference.

<u>Corollary 13.1</u> There exists a regular Borel probability measure invariant for any continuous representation of an amenable semigroup by continuous maps of a compact Hausdorff space.

Since any abelian, solvable, or compact topological group is amenable, the existence of an invariant measure for such groups $\underline{T}$ of homeomorphisms of a compact metric space X such that the map $(s, x) \rightarrow T_s x$ is continuous, is immediate (Fomin [18]).

The next result is for groups of mappings on a compact Hausdorff space X. By virtue of Theorem 1.1, we restrict ourselves to homeomorphisms of X. The space X has a unique uniformity $\underline{\underline{X}}$ inducing the given topology of X. Recall Definition 2.2 of the equicontinuity of a family of maps from a topological space to a uniform space. We also need the following definition (Definition V.10.7 in Dunford and Schwartz [13]).

Definition 13.1 A family F of linear maps of a linear topological space X is equicontinuous on a subset K of X, if for every neighbourhood A of the origin in X, there is a neighbourhood B of the origin such that if m_1, $m_2 \in K$ and $m_1 - m_2 \in B$, then $f(m_1 - m_2) \in A$ for all $f \in F$.

We now have the following theorem for groups of mappings of X.

Theorem 13.2 There exists a regular invariant Borel probability measure for any equicontinuous group of homeomorphisms on a compact Hausdorff space.

Proof. Let T be the given equicontinuous group of homeomorphisms on X. Let, for each $s \in S$, $\eta_s(m) = m\, T_s^{-1}$ for $m \in M(X)$. Then $\{ \eta_s \}$ is a group of continuous linear maps on M(X) leaving P(X) invariant. We shall show that $\{ \eta_s \}$ is equicontinuous on P(X).

Let A be the neighbourhood of the origin in $M(X)$, given by

$$A = \{m : |\int f_i \, dm| < \epsilon, \quad 1 \leq i \leq n\}$$

where $\epsilon > 0$ and $f_1, \ldots, f_n \in C(X)$. Since each f_i is uniformly continuous, given $\partial > 0$, there exists a $C_i^* \in \overline{\underline{X}}$ such that $(x, y) \in C_i^*$ implies that $|f_i(x) - f_i(y)| < \partial$. Putting $C^* = \bigcap C_i^*$, we see that C^* is non-empty. Since $\underline{T}$ is equicontinuous and X compact, $\overline{\underline{T}}$ is uniformly equicontinuous, i.e., given $C^* \in \overline{\underline{X}}$, there is a $D^* \in \overline{\underline{X}}$ such that $(x, y) \in D^*$ implies $(T_s x, T_s y) \in C^*$ for all s and hence $|f_i(T_s x) - f_i(T_s y)| < \partial$ for all s and all $i = 1, 2, \ldots, n$. This shows that the family of functions $\{f_i T_s : 1 \leq i \leq n, \ s \in S\}$ is equicontinuous. Being bounded, this family is conditionally compact by Arzela's theorem and so totally bounded. For the $\epsilon > 0$ associated with A above, there exist functions $g_1, \ldots, g_k \in C(X)$ such that for any i and any s, there is a j with $\| f_i T_s - g_j \| < \frac{\epsilon}{4}$. Define a neighbourhood B of the origin in $M(X)$ by

$$B = \{m : |\int g_i \, dm| < \frac{\epsilon}{2}, \quad 1 \leq i \leq k\}.$$

It can be seen that for $m_1, m_2 \in P(X)$ such that $m_1 - m_2 \in B$, we have $\eta_s(m_1 - m_2) \in A$ for all s, i.e., $\{\eta_s\}$ is equicontinuous on $P(X)$. Hence, by Kakutani's fixed point theorem (Theorem V. 10.8 in Dunford and Schwartz [13]), there exists an $m \in P(X)$ such that $\eta_s m = m$ for each $s \in S$. This is the required invariant measure.

The above theorem and its proof generalize the method of Dunford and Schwartz [14] for getting the Haar measure on a compact group. As a corollary, we get the following result of Fomin [18].

<u>Corollary 13.2</u> Let $\{T_s\}$ be a compact group of homeomorphisms of a compact Hausdorff space X such that the map $(s, x) \to T_s x$ is continuous. Then there exists a regular invariant Borel probability measure for $\{T_s\}$.

<u>Proof</u>. Under the assumptions, the group is equicontinuous, by Theorem 7.16 in Kelley [36].

CHAPTER 14

FINITE INVARIANT EQUIVALENT MEASURES ON MEASURE SPACES

In this chapter, $(X, \underline{A}, m)$ will denote a fixed probability space. We consider the question of existence of a finite measure equivalent to m and invariant for a family of transition functions on X. We assume that the transition functions under consideration are all non-singular with respect to m. We also consider, more generally, the existence of a strictly positive invariant function for a family of contractions on the real $L_1(X)$. Our results are for (weakly) continuous representations of amenable semigroups by contractions and generalize a number of known results.

We could start with a σ-finite measure space and ask for a finite invariant measure stronger than the given measure and then, by standard procedures, reduce this to the above problem. We do not give the details here.

Recall how a non-singular transition function P induces a contraction V on $L_1(X)$ (Chapter 1). This V is given by

$$\int_A Vf\,dm = \int P(x,A)f(x)m(dx) \qquad A\in \underline{A}.$$

The existence of a finite invariant equivalent measure μ for P is equivalent to the existence of a strictly positive invariant function f_0 for V, via the equation

$$\mu(A) = \int_A f_0\,dm \qquad A\in \bar{\underline{A}}.$$

We denote by L_1^+ and L_∞^+ respectively the cones of non-negative functions in L_1 and L_∞. The adjoint U of an operator V on L_1 is defined on L_∞ by the equation

$$\langle Vf, g \rangle = \langle f, Ug \rangle$$

where, and in what follows, $\langle f, g\rangle = \int fg\,dm$ with $f\in L_1$ and $g\in L_\infty$. In case V is induced by a transition function P, U is the familiar operator

$$(Uf)(x) = \int f(y)\, P(x,dy)$$

acting on L_∞.

If S is an amenable topological semigroup and $s \to V_s$ a homomorphism of S into the set of positive contractions on L_1 such that the map $s \to \langle V_s f, g \rangle$ for each

$f \in L_1$ and $g \in L_\infty$ is continuous, then we call $\underline{V} = \{ V_s \}$ a continuous representation of S by positive contractions on L_1. The continuity condition, for the case of transition functions, is that the map $s \to \int P_s(x, A)f(x)\, m(dx)$ for every fixed $A \in \underline{A}$ and $f \in L_1$ is continuous; for the case of transformations, that the map $s \to m(T_s^{-1} A \cap B)$ for every pair A and B of measurable sets is continuous.

We use the methods of Neveu [46] and invariant means for amenable semigroups in proving our first result.

<u>Theorem 14.1</u> For a continuous representation $\underline{V}$ of an amenable semigroup S by positive contractions on $L_1(X)$, the following conditions are equivalent.

(A) There exists a strictly positive invariant function f in L_1.

(B) $g \in L_\infty^+$ and $\inf_s \langle V_s 1, g \rangle = 0$ imply that $g = 0$ a.e.

<u>Proof</u>. Let (A) hold. Fix $g \in L_\infty^+$ and let $f_0 \in L_1^+$. The general inequality $f_0 \leq a + (f_0 - a)^+$ implies that

$$\langle V_s f_0, g \rangle \leq a \langle V_s 1, g \rangle + \| (f_0 - a)^+ \|_1 \, \| g \|_\infty$$

for any real number a and for all s, since V_s is a contraction. Hence $\inf_s \langle V_s 1, g \rangle = 0$ implies that $\inf_s \langle V_s f_o, g \rangle = 0$. Taking the given invariant function f for f_o, we see that $\langle f, g \rangle = 0$ and hence that $g = 0$ a.e., i.e., (B) holds.

Let now condition (B) be satisfied. For any fixed $g \in L_\infty$, the real-valued function $\langle V_s 1, g \rangle$ of s is bounded and continuous. Let M be any invariant mean on $C(G)$ and put $\lambda(g) = M(\langle V_s 1, g \rangle)$. λ is then a positive linear functional on L_∞. Besides, $\lambda(U_s g) = \lambda(g)$ for every $s \in S$, as can be seen by using the invariance of M. We can now use Lemma 1 of Neveu [46] in the same way as he does, to get a strictly positive invariant function f in L_1.

<u>Corollary 14.1</u> There exists a finite invariant measure equivalent to m for a continuous representation $\underline{\bar{P}}$ of an amenable semigroup S by transition functions on X if and only if the following condition holds.

(B) $m(A) > 0$ implies that $\inf_s Q_s(A) > 0$

where $Q_s(A) = \int P_s(x, A)m(dx)$.

For the case of transformations, condition (B) says that if $m(A) > 0$, then $\inf_s m(T_s^{-1} A) > 0$. Applying Corollary 14.1 to an abelian semigroup of transformations, we get Theorem 4 of Blum and Friedman [5].

We shall now study the following condition for a family $\{ V_s \}$ of positive contractions on L_1.

Condition (E). Given $\epsilon > 0$, there exists a $\partial > 0$ such that $m(A) < \partial$ implies that $\sup_s \int_A V_s 1\, dm < \epsilon$.

Dean and Sucheston [9] have proved the necessity of condition (E) for the existence of a strictly positive invariant function in L_1 for the semigroup generated by a positive contraction on L_1. The same proof goes through for the case of a family of positive contractions also. We include the proof below for the sake of completeness.

<u>Theorem 14.2</u> For any family $\{ V_s \}$ of positive contractions on L_1, condition (E) is necessary for the existence of a strictly positive invariant element in L_1.

<u>Proof</u>. We remark first that $\int_A V_s 1\, dm = \int U_s 1_A\, dm$. Let f be a strictly positive invariant function in L_1 and let

$\varepsilon > 0$ be given. Considering the sets $F_\alpha = \{ x: f(x) > \alpha \}$, we can find an $\alpha > 0$ such that $m(F_\alpha^c) < \frac{\varepsilon}{2}$ and so $\int_{F_\alpha^c} U_s 1_A \, dm < \frac{\varepsilon}{2}$. On the other hand, as is easily seen,

$\int_{F_\alpha} U_s 1_A \, dm < \frac{1}{\alpha} \int_A f \, dm$. We can choose a $\partial > 0$ such that if $m(A) < \partial$ then $\int_A f \, dm < \frac{\varepsilon \alpha}{2}$ and hence, $m(A) < \partial$ implies that $\int U_s 1_A \, dm < \varepsilon$ for all s. This proves the necessity of condition (E).

For the group generated by an invertible transformation, condition (E) is sufficient for the existence of a finite invariant equivalent measure, but it is no longer sufficient if the transformation is not one-one. Rechard [48] has proved interesting results in this direction, using mean ergodic theorems. Ito [29] has generalized Rechard's methods for the case of the semigroup generated by a transition function. Our results below, for a continuous representation of an amenable semigroup by positive contractions are based on this method and we use Eberlein's ergodic theorem. For transformations however, we give a simpler proof, similar to the one by Hajian and Kakutani [21].

Following Rechard [48], we introduce the notion of a measure being asymptotically stronger than a given measure with respect to a family of contractions on L_1.

Definition 14.1 A measure m^* is asymptotically stronger than m for a family $\{ V_s \}$ of contractions on L_1 if $m^*(A) = 0$ implies that $\inf_s \int_A V_s 1 dm = 0$.

The definition may be adapted to families of transformations and transition functions by considering the induced contractions.

We first consider the case of transformations. Recall that a (non-singular) transformation T is called conservative if any measurable set A such that the sets $T^{-n} A$, $n \geq 0$ are disjoint, has measure zero. This is equivalent to incompressibility: $m(T^{-1}A - A) = 0$ for a measurable set A implies that $m(A - T^{-1}A) = 0$. If T is conservative, then T^n is conservative for every n. We may call a semigroup $\{ T_s \}$ of transformations conservative if each T_s is conservative.

Theorem 14.3. Let $\underline{\overline{T}}$ be a continuous representation of an amenable semigroup S by transformations on X. If condition (E) holds, then there exists a finite invariant measure m^* weaker and asymptotically stronger than m. If S is a group or if $\{T_s\}$ is conservative, then m^* is equivalent to m.

Proof. Fix some invariant mean M on $C(S)$ and let $m^*(A) = M(m(T_s^{-1} A))$ for $A \in \underline{A}$. m^* is a non-negative, finitely additive, invariant set function with $m^*(X) = 1$. Given $\epsilon > 0$, by condition (E), there is a $\partial > 0$ such that $m(A) < \partial$ implies that $m(T_s^{-1} A) < \epsilon$ for all s. Hence $m^*(A) = M(m(T_s^{-1} A)) \leq \epsilon$. This implies that m^* is continuous and so countably additive. Besides $m^* \ll m$. If $m^*(A) = 0$, then $\inf_s m(T_s^{-1} A) \leq M(m(T_s^{-1} A)) = 0$ and so m^* is asymptotically stronger than m.

If S is a group, i.e., $\underline{T}$ is a group of invertible transformations, it follows from Theorem 1 of Rechard [48] that m^* is stronger than m. Let now $\underline{T}$ be conservative. Write $m^*(A) = \int_A f\,dm$ and $B = \{ x: f(x) = 0 \}$. We claim that B is invariant under each T_s. For, consider the sets $T_s^{-1} B - B$. If $m(T_s^{-1} B - B) > 0$ for some s, then

$$0 < \int_{T_s^{-1}B-B} f\,dm = \int_{T_s^{-1}B} f\,dm = \int_B f\,dm = 0$$

a contradiction. Hence $m(T_s^{-1}B - B) = 0$ for each s. The conservativeness of T_s implies that $m(B - T_s^{-1} B) = 0$ for each s and hence B is invariant. Since

$$0 = \int_B f\,dm = m^*(B) = M(m(T_s^{-1}B)) = m(B),$$ we see that $f > 0$ a.e. Hence $m << m^*$ and the proof is complete.

Theorem 14.4 Let $\underline{T} = \{ T_s : s \in S \}$ be an arbitrary semi-group of transformations on X. If there exists a finite measure m^* invariant under $\underline{T}$ which is weaker than m and asymptotically stronger than m, then, given $\epsilon > 0$, there exists a $\partial > 0$ and a s_0 such that $m(A) < \partial$ implies that $m(T_{ss_0}^{-1} A) < \epsilon$ for all s. If S is a group or if $m << m^*$, then condition (E) holds.

Proof. The first part is essentially contained in Theorem 3 of Rechard [48]. If S is a group, $S = \{ s\, s_0 : s \in S \}$ and so the second assertion follows. The third assertion follows from Theorem 14.2.

A positive contraction V on L_1 is called conservative if for some strictly positive $f \in L_1$, we have $\sum_0^\infty V^n f = \infty$ a.e. If this is true for some such f, then this is true for every such f, so that the definition of conservativeness is independent of the function chosen. If V is conservative, then V^n is conservative for every positive integer n. One way to see this is to use Lemma 6 of Hajian and Ito [20] which states that V is conservative if and only if every U-subinvariant function (i.e., $g \in L_\infty$ such that $Ug \leq g$) is invariant. We shall call a semi-group $\{V_s\}$ of contractions conservative if each V_s is conservative. A transition function may be called conservative if the induced contraction is conservative.

<u>Theorem 14.5</u> For a continuous representation $\underline{V}$ of an amenable semigroup S by positive contractions on L_1, condition (E) is sufficient for the existence of a non-negative invariant function in L_1. If $\underline{V}$ is conservative, then the function obtained is strictly positive.

<u>Proof</u>. We use the results mentioned in Chapter 3. By Theorem 3.3 the semigroup $\underline{V}$ is strongly ergodic under a net V_α of averages of $\{V_s\}$. We first show that for each $f = 1_A$, $A \in \underline{A}$, the orbit $\{V_\alpha f\}$ is conditionally weakly compact. It is sufficient to show the conditional weak sequential compactness of $\{V_s f\}$. For this, we need only to show by Theorem IV.8.9 in Dunford and Schwartz [13],

that the set $\{ V_s f \}$ is bounded (obvious) and that given $\epsilon > 0$, there exists a $\delta > 0$ such that for any set B with $m(B) < \delta$, we have $\sup_s \int_B V_s f dm < \epsilon$. Since V_s is positive and $f = 1_A \leq 1$, $\int_B V_s f \, dm \leq \int_B V_s 1 dm$ and so, by condition (E), this is fulfilled. Hence, for each $f = 1_A$, the orbit $\{ V_\alpha f \}$ contains a weak cluster point and so the element f is ergodic. Since, by Theorem 3.4, the set of ergodic elements forms a closed linear subspace, it follows that all the elements in L_1 are ergodic and there is an operator $\overline{V}$ on L_1 such that $\overline{V} f = \lim_\alpha V_\alpha f$ in norm for all $f \in L_1$ and $\overline{V} V_s = V_s \overline{V} = \overline{V}^2 = \overline{V}$ for all s. The function $\overline{V} 1$ is a non-negative invariant function. The measure $m^*(A) = \int_A \overline{V} 1 dm$ is weaker and asymptotically stronger than m. If the semigroup $\underline{\overline{V}}$ is conservative, it can be shown, as in Hajian and Ito [20] that $\overline{V}1$ is strictly positive. m^* would then be equivalent to m. The theorem is proved.

A similar result holds for the case of a continuous representation of an amenable semigroup by transition functions.

We finally come to two simple necessary and sufficient conditions for families of transformations. The first one is a generalization of the notion of weakly wandering set due to Hajian and Kakutani [21].

Definition 14.2 A measurable set A is weakly wandering for a family $\{ T_\alpha \}$ of transformations if there exists a sequence α_n of indices such that $A, T^{-1}_{\alpha_1} A, T^{-1}_{\alpha_2} A, \ldots$ are all pairwise disjoint.

Let us call a transformation T bothways measurable if $A \in \underline{A}$ implies that $T^{-1} A \in \underline{\bar{A}}$ and $TA \in \underline{\bar{A}}$. T is bothways non-singular if $m(A) = 0$ implies that $m(TA) = 0 = m(T^{-1} A)$.

Theorem 14.6 There exists a finite invariant equivalent measure for a continuous representation of an amenable semi-group by bothways measurable and bothways non-singular transformations if and only if every weakly wandering set has measure zero.

The proof is done by showing that if every weakly wandering set has measure zero, then condition (B) is satisfied. Since a proof of this has already appeared in print, in Blum and Friedman [5], we refrain from giving the proof*.

* The author proved this result independently of Blum and Friedman.

The concept of boundedness for a group of transformations has been introduced by Cotlar and Ricabarra [7]. Let $\underline{T} = \{ T_s \}$ be a group of (invertible) transformations on X. Two measurable sets E and F are called equivalent if $E = \bigcup_1^\infty E_i$, $F = \bigcup_1^\infty F_i$, $E_i \cap E_j = \emptyset$, $F_i \cap F_j = \emptyset$ for all $i \neq j$ and there is a sequence s_i of elements in S such that $F_i = T_{s_i}^{-1} E_i$ for each i. A set E is bounded if it is not equivalent to any proper subset of itself. The group $\underline{T}$ of transformations is bounded if the whole space is bounded.

<u>Theorem 14.7</u> There exists a finite invariant equivalent measure for a continuous representation $\underline{T}$ of an amenable group by transformations on X if and only if $\underline{T}$ is bounded.

<u>Proof.</u> It is easily shown that the condition is necessary. We assert that if $\underline{T}$ is bounded, then every weakly wandering set has measure zero. If not, let A be a weakly wandering set of positive measure and s_i the associated sequence of elements of S. Then the sets $E = A \cup \bigcup_1^\infty T_{s_i}^{-1} A$ and $F = \bigcup_1^\infty T_{s_i}^{-1} A$ are seen to be equivalent and $m(E - F) > 0$. Hence $\underline{T}$ is not bounded and the proof is complete.

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