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HERMITE AND SPECIAL HERMITE EXPANSIONS REVISITED

S. THANGAVELU

1. Introduction. On a compact Riemannian manifold M consider a first-order pseudodifferential operator $p(X,D)$ that is positive and selfadjoint. Let λ_j, $j=0,1,2,\ldots,$ be the sequence of its eigenvalues and $e_j(x)$ the corresponding eigenfunctions. The family $\{e_j\}$ then forms an orthonormal basis for $L^2(M)$. Let E_jf be the projection of f onto the jth eigenspace so that we have

$$f=\sum_{j=0}^{\infty}E_jf, \tag{1.1}$$

where the series converges in the L^2-norm. For functions f in $L^p(M)$, where p is other than 2, the above series may not converge to f in the L^p-norm, and one is led to consider the Bochner-Riesz means $S_\lambda^\delta f$.

The Bochner-Riesz means are defined by the equation

$$S_\lambda^\delta f(x)=\sum_{\lambda_j\leqslant\lambda}\left(1-\frac{\lambda_j}{\lambda}\right)^\delta E_jf(x), \tag{1.2}$$

and we want to know if $S_\lambda^\delta f$ converges to f in the L^p-norm as $\lambda\to\infty$. Let $\delta(p)=\max\{n|1/p-1/2|-1/2,0\}$ be the critical index, where n is the dimension of the manifold. Then a necessary condition for the convergence of $S_\lambda^\delta f$ to f in the L^p-norm is that $\delta>\delta(p)$. In [6] Sogge proved that this condition is also sufficient as long as $1\leqslant p\leqslant 2(n+1)/(n+3)$ or $p\geqslant 2(n+1)/(n-1)$. This result includes previously known results for the multiple Fourier series ($M=T^n$, the n-torus) and spherical harmonic expansions ($M=S^n$, the $(n+1)$-sphere).

Let us leave the premises of compact manifolds and proceed to noncompact situations. The simplest example is the case of the standard Laplacian $-\Delta$ on $\mathbb{R}^n$, and in this case the operator does not have point spectrum. The spectral decomposition is given by the Fourier transform, and one is led to consider the Bochner-Riesz means

$$S_R^\delta f(x)=(2\pi)^{-n/2}\int_{|\xi|\leqslant R}\left(1-\frac{|\xi|^2}{R^2}\right)^\delta \hat{f}(\xi)e^{ix\cdot\xi}d\xi, \tag{1.3}$$

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where $\hat{f}$ is the Fourier transform of f given by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} f(x)\,dx.$$

As before, one has the conjecture that S_R^δ are uniformly bounded on $L^p(\mathbb{R}^n)$ if and only if $\delta > \delta(p)$. Again this has been proved only in the case when $1 \leqslant p \leqslant 2(n+1)/(n+3)$ or $p \geqslant 2(n+1)/(n-1)$. However, when $n = 2$, the conjecture has been proved for all p; this is the celebrated theorem of Carleson and Sjölin.

In this paper we are mainly concerned with the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ and the special Hermite operator

$$L = -\Delta + \frac{1}{4}|z|^2 - i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

on $\mathbb{C}^n$. For the operator H the eigenfunctions are the normalized Hermite functions $\Phi_\alpha(x)$, $\alpha \in \mathbb{N}^n$, with eigenvalues $(2|\alpha| + n)$ where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Thus one has the Hermite expansion

$$f = \sum_\alpha (f, \Phi_\alpha)\Phi_\alpha, \tag{1.4}$$

where the sum is extended over all multi-indices $\alpha \in \mathbb{N}^n$. In the case of the operator L, the eigenfunctions are given by the special Hermite functions $\Phi_{\alpha\beta}$, α, $\beta \in \mathbb{N}^n$, and one has

$$L(\Phi_{\alpha\beta}) = (2|\beta| + n)\Phi_{\alpha\beta}.$$

Notice that the eigenvalues depend only on β, which means that the eigenspaces are infinite-dimensional. The special Hermite expansion then takes the form

$$f(z) = \sum_\alpha \sum_\beta (f, \Phi_{\alpha\beta})\Phi_{\alpha\beta}, \tag{1.5}$$

where both α, $\beta \in \mathbb{N}^n$. The Bochner-Riesz means for the expansions (1.4) and (1.5) were studied in [10].

First consider the case of the Hermite operator H. The Bochner-Riesz means are defined by

$$S_R^\delta f = \sum \left(1 - \frac{2k+n}{R}\right)_+^\delta P_k f, \tag{1.6}$$

where $P_k f$ are the projections

$$P_k f = \sum_{|\alpha|=k} (f, \Phi_\alpha)\Phi_\alpha.$$

In the 1-dimensional case it is known that S_R^δ are uniformly bounded on $L^p(\mathbb{R})$ if and only if $\delta > (2/3)((1/p) - (1/2)) - (1/6)$. Thus the Hermite series of an $L^1(\mathbb{R})$-function converges in the norm if and only if $\delta > 1/6$. This comes as a surprise, because in the higher-dimensional case the situation is different. When $n \geqslant 2$, the conjecture is S_R^δ are uniformly bounded on $L^p(\mathbb{R}^n)$ if and only if $\delta > \delta(p)$, where $\delta(p)$ is the same critical index defined in the case of the standard Laplacian on $\mathbb{R}^n$.

As the 1-dimensional case has been settled, let us concentrate on the higher-dimensional case. In [10] we proved that the conjecture is true when $p = 1 : S_R^\delta f$ converges to f in $L^1(\mathbb{R}^n)$ if and only if $\delta > (n-1)/2$. We also proved that the conjecture is true if we consider only radial functions. Thus the condition $\delta > \delta(p)$ is necessary. In 1994, Karadzhov [3] proved the conjecture in the range $1 \leqslant p \leqslant 2n/(n+2)$. It still remains open to see if the conjecture is true in the range $2n/(n+2) < p \leqslant 2(n+1)/(n+3)$. Our investigations indicate that the conjecture may be false in the above range. Though we are not able to prove this, we have strong reasons to believe that it may be the case. First of all, unlike the case of the standard Laplacian, the Bochner-Riesz means in our case are not translation invariant. The kernels can be expressed as oscillatory integrals whose asymptotic behavior is not well understood. As we have shown in [9], there is a critical region in which these integrals behave like Airy functions. Instead of considering global estimates, in this paper we prove the following local estimate.

THEOREM 1. *Let B be a fixed compact subset of $\mathbb{R}^n$. Let $2(n+1)/(n-1) \leqslant p \leqslant \infty$ and $\delta > \delta(p)$. Then the uniform estimates*

$$\int_B |S_R^\delta f(x)|^p dx \leqslant C_B \int_{\mathbb{R}^n} |f(x)|^p dx$$

hold where C_B depends only on B.

COROLLARY 1. *Let B, δ and $p < \infty$ be as above. Then for any $f \in L^p(\mathbb{R}^n)$*

$$\lim_{R\to\infty} \int_B |S_R^\delta f(x) - f(x)|^p dx = 0.$$

It is likely that the method used in [6] can be applied to prove the uniform boundedness of the operators $\chi_B S_R^\delta \chi_B$. But Theorem 1 is stronger than this result. Moreover, the proof of Theorem 1 is elementary, and we do not need the sophisticated theory of Fourier integral operators.

We remark that Theorem 1 remains true even when $n = 1$. We believe that this behavior of the Riesz means is mainly due to the noncompactness of the underlying manifold. The case of the standard Laplacian seems to be special as in that case one can make use of dilation and translation invariance. In fact, appealing to a transplantation theorem of Kenig-Stanton-Tomas [4] we can prove the Bochner-Riesz conjecture for the standard Laplacian from the above local estimates. To be precise, let B be any compact subset of $\mathbb{R}^n$ containing zero as a point of density, and set $\chi_B f(x) = \chi_B(x) f(x)$. Then their transplantation theorem says that the uniform boundedness of the operators $\chi_B S_R^\delta \chi_B$ on $L^p(\mathbb{R}^n)$ implies the uniform boundedness of the Bochner-Riesz means (with same δ) associated to the standard Laplacian. Thus we get a new proof of the Bochner-Riesz conjecture for Δ.

Thus we are convinced that it is not only reasonable but also natural to study local estimates for the Bochner-Riesz means associated to Hermite and special Hermite expansions. To strengthen our point of view, let us compare the behavior of the eigenvalues and eigenfunctions in the compact and noncompact situations. In the compact case each eigenspace is finite-dimensional, but in the noncompact case this need not be true. As we have already mentioned, in the case of the special Hermite operator each eigenspace is infinite-dimensional. In the case of the Hermite operator each eigenspace is finite-dimensional, but still the behavior of the eigenvalues is different. Let $N(\lambda)$ stand for the number of eigenvalues $\lambda_j \leqslant \lambda$. In the case of a compact Riemannian manifold, when P is a second-order elliptic differential operator, the Weyl formula says that (see Sogge [6])

$$N(\lambda) = c\lambda^{n/2} + O\left(\lambda^{(n-1)/2}\right). \tag{1.7}$$

On the other hand, in the case of the Hermite operator, the dimension of the eigenspace corresponding to the eigenvalue $(2k + n)$ is $(k + n - 1)!/k!(n - 1)!$, and consequently

$$N(\lambda) = \sum_{(2k+n) \leqslant \lambda} \frac{(k + n - 1)!}{k!(n - 1)!} = O(\lambda^n). \tag{1.8}$$

The Weyl formula for $N(\lambda)$ is proved by observing that $N(\lambda)$ is the trace of the partial-sum operator. That is, if

$$S_\lambda f = \sum_{\lambda_j \leqslant \lambda} E_j f,$$

and if $S_\lambda(x, y)$ is the kernel of this operator, then

$$N(\lambda) = \int_M S_\lambda(x, x)\, dx.$$

In the case of the Hermite operator,

$$S_\lambda(x,y) = \sum_{(2k+n)\leqslant\lambda} \Phi_k(x,y), \tag{1.9}$$

where $\Phi_k(x,y)$ are the kernels of the projections P_k given by

$$\Phi_k(x,y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y). \tag{1.10}$$

If we integrate $S_\lambda(x,x)$ over $\mathbb{R}^n$, we only get

$$\int_{\mathbb{R}^n} S_\lambda(x,x)\,dx = \sum_{(2k+n)\leqslant\lambda}\sum_{|\alpha|=k} 1 = O(\lambda^n).$$

On the other hand, if B is any compact subset of $\mathbb{R}^n$, then using the estimate

$$\sup_{x\in\mathbb{R}^n} |\Phi_k(x,x)| \leqslant ck^{(n/2)-1}, \tag{1.11}$$

which was proved in [10] (see Lemma 3.2.2), we get

$$\int_B S_\lambda(x,x)\,dx = \sum_{(2k+n)\leqslant\lambda}\int_B \Phi_k(x,x)\,dx \leqslant c\lambda^{n/2}.$$

Thus, when restricted to compact subsets, the Hermite functions seem to behave like the eigenfunctions in the compact case.

The main ingredient in the study of the Bochner-Riesz means is the so-called restriction theorems for the spectral projection operators. In the case of compact manifolds, let χ_λ be the projections defined by

$$\chi_\lambda f = \sum_{\lambda\leqslant\lambda_j\leqslant\lambda+1} E_j f.$$

It has been established in [6] that

$$\|\chi_\lambda f\|_2 \leqslant c\lambda^{\delta(p)}\|f\|_p\,, \quad 1\leqslant p\leqslant \frac{2(n+1)}{n+3}. \tag{1.12}$$

In the case of the Hermite expansions, the relevant restriction theorem takes the form

$$\|P_k f\|_2 \leqslant ck^{(1/2)\delta(p)-(1/4)}\|f\|_p. \tag{1.13}$$

This estimate already fails in the 1-dimensional case. When $n = 1$, $P_k f = (f, h_k)h_k$, where h_k are the 1-dimensional Hermite functions, and hence

$$\|P_k f\|_2 = |(f, h_k)| \leqslant \|f\|_1 \|h_k\|_\infty.$$

As $\|h_k\|_\infty = O(k^{-1/12})$ we see that the above restriction theorem cannot hold when $n = 1$, $p = 1$.

On the other hand, it has been proved in [10] that when $n \geqslant 2$, one has the estimate

$$\|P_k f\|_2 \leqslant c k^{(n-2)/4} \|f\|_1.$$

Further, when f is radial, the estimates (1.13) hold in the range $1 \leqslant p < 2n/(n+1)$. In [3] Karadzhov proved the estimates (1.13) in the range $1 \leqslant p \leqslant 2n/(n+2)$, thus proving the Bochner-Riesz conjecture in the same range of p. It is natural to expect that the estimates are valid in the range $1 \leqslant p \leqslant 2(n+1)/(n+3)$ but, unfortunately, it is still open if it is true or not.

If we are only interested in proving Theorem 1, the local version of the Bochner-Riesz conjecture, then we don't need the full force of the estimates (1.13). What we need is the following local version of the estimates (1.13). In the next section we establish the following result.

THEOREM 2. *Let B be any compact subset of $\mathbb{R}^n$, and let $1 \leqslant p \leqslant 2(n+1)/(n+3)$. Then for any $f \in L^p(\mathbb{R}^n)$, we have*

$$\|P_k \chi_B f\|_2 \leqslant C_B k^{(1/2)\delta(p) - 1/4} \|f\|_p,$$

where C_B depends only on B.

This is the main result of this paper. It may appear as if the techniques developed in [6] for the case of compact Riemannian manifolds can be adapted to prove Theorem 2 by considering functions supported in a fixed compact set. Unfortunately, this is not the case. The proof of the restriction theorems depends heavily on the orthogonality of the eigenfunctions, and once we restrict the Hermite functions to a fixed compact set, their orthogonality is spoiled. We use the generating function identity satisfied by the Hermite functions to prove Theorem 2.

We remark that this theorem is true even when $n = 1$. As we have already noted,

$$\|P_k \chi_B f\|_2 = \left| \int_B f(x) h_k(x)\, dx \right|,$$

and the asymptotic estimates of h_k, which can be found in Szego [8], show that

$$\sup_{x \in B} |h_k(x)| \leqslant C_B k^{-1/4},$$

and this proves the above theorem for $n = 1$, $p = 1$. Theorem 2 is proved in Section 2. Once the restriction theorem is proved, Theorem 1 can be established. In fact, in order to prove the theorem it is enough to show that $S_R^\delta \chi_B$ are uniformly bounded on $L^{p'}(\mathbb{R}^n)$, and this follows from the fact that S_R^δ and χ_B are selfadjoint. Using some pointwise estimates for the kernel of S_R^δ and the boundedness properties of $P_k \chi_B$ one can establish the uniform boundedness of $S_R^\delta \chi_B$. For details we refer to [10].

Having said so much about Hermite expansions, we now turn our attention to the case of special Hermite expansions. Introduced and studied in [10], they can be put in the compact form

$$f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k, \tag{1.14}$$

where φ_k are the Laguerre functions

$$\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{(1/4)|z|^2},$$

and $f \times \varphi_k$ stands for the twisted convolution

$$f \times \varphi_k(z) = \int_{\mathbb{C}^n} f(z-w)\varphi_k(w) e^{(i/2)\Im z \cdot \bar{w}} dw.$$

As $f \times \varphi_k$ is the projection of f onto the kth eigenspace spanned by $\{\Phi_{\alpha\beta} : \alpha, \beta \in \mathbb{N}^n,\ |\beta| = k\}$, the Bochner-Riesz means take the form

$$S_R^\delta f = (2\pi)^{-n} \sum \left(1 - \frac{2k+n}{R}\right)_+^\delta f \times \varphi_k, \tag{1.15}$$

and the natural conjecture is that the S_R^δ are uniformly bounded on $L^p(\mathbb{C}^n)$ if and only if $\delta > \delta(p) = 2n((1/p) - (1/2)) - (1/2)$.

This conjecture has been settled completely in the case of radial functions (see [10]). We also proved in [10] that the conjecture is true for $1 \leqslant p \leqslant 2n/(n+1)$. The main ingredient in the proof of the conjecture is again a restriction theorem, namely, the estimates

$$\|f \times \varphi_k\|_2 \leqslant c k^{n((1/p)-(1/2))-(1/2)} \|f\|_p. \tag{1.16}$$

Recently we have shown that (1.16) is valid in a slightly bigger range $1 \leqslant p \leqslant 2(3n+1)/(3n+4)$, and consequently the Bochner-Riesz means are uniformly bounded on $L^p(\mathbb{C}^n)$ for $\delta > \delta(p)$ whenever $1 \leqslant p \leqslant 2(3n+1)/(3n+4)$ [5].

As in the case of Hermite expansions, our efforts to extend the above range up to $2(2n+1)/(2n+3)$ have yielded no fruit, and we have to be content with the following local restriction theorem.

THEOREM 3. *Let $B \subset \mathbb{C}^n$ be any compact subset, and let $1 \leqslant p \leqslant 2(2n+1)/(2n+3)$. Then for all $f \in L^p(\mathbb{C}^n)$,*

$$||\chi_B f \times \varphi_k||_2 \leqslant C_B k^{n((1/p)-(1/2))-(1/2)} ||f||_p,$$

where C_B depends only on B.

Once we have the above restriction theorem it is routine to prove the following theorem and its corollary.

THEOREM 4. *Let B be as above, $2(2n+1)/(2n-1) \leqslant p \leqslant \infty$, and let $\delta > \delta(p)$. Then for all $f \in L^p(\mathbb{C}^n)$*

$$\int_B |S_R^\delta f(z)|^p dz \leqslant C_B \int_{\mathbb{C}^n} |f(z)|^p dz.$$

COROLLARY 2. *Let B and δ be as above. Then*

$$\lim_{R\to\infty} \int_B |S_R^\delta f(z) - f(z)|^p dz = 0$$

for all $f \in L^p(\mathbb{C}^n)$ provided $2(2n+1)/(2n-1) \leqslant p < \infty$.

The Bochner-Riesz means for the special Hermite expansions is a twisted convolution operator whose kernel can be explicitly calculated. This is good news, and we can expect something better in the case of special Hermite expansions. We have already mentioned that the Bochner-Riesz conjecture was proved in the range $1 \leqslant p \leqslant 2(3n+1)/(3n+4)$. It would be interesting to see if the same is true in the Hermite case, that is, if the conjecture is true for $2n/(n+2) \leqslant p \leqslant 2(3n+2)/(3n+8)$. We see in the next section that this follows if we can get good estimates of certain oscillatory integrals. Unfortunately, the estimates we get are good enough only for a smaller range of p.

There is other evidence to strengthen our belief that special Hermite expansions behave better than the Hermite expansions. We can use the Carleson-Sjölin theorem for the Bochner-Riesz means on $\mathbb{R}^2$ of the standard Laplacian to deduce the following local version for the special Hermite expansions on $\mathbb{C}$.

THEOREM 5. *Let $n = 1$, $1 \leqslant p < 4/3$ and B be any compact subset of $\mathbb{C}$. Then the operators $\chi_B S_R^\delta \chi_B$ are uniformly bounded on $L^p(\mathbb{C})$ if and only if $\delta > \delta(p) = 2((1/p) - (1/2)) - (1/2)$.*

The results concerning special Hermite expansions are proved in Section 3.

2. Hermite expansions. We now proceed to prove the local restriction theorem for the Hermite projection operators P_k. Let B be a compact subset of $\mathbb{R}^n$, and let $f \in L^p(\mathbb{R}^n)$. As P_k are projections,

$$\begin{aligned}||P_k\chi_B f||_2^2 &= (P_k\chi_B f, P_k\chi_B f)\\ &= \int_{\mathbb{R}^n} P_k(\chi_B f)(x)\chi_B f(x)\,dx.\end{aligned}$$

Applying Hölder's inequality we get

$$||P_k\chi_B f||_2^2 \leqslant ||P_k\chi_B f||_{L^{p'}(B)}||f||_p.$$

Theorem 2 follows, once we show that

$$||P_k\chi_B f||_{L^{p'}(B)} \leqslant C_B k^{n((1/p)-(1/2))-1}||f||_{L^p(B)} \tag{2.1}$$

for $1 \leqslant p \leqslant 2(n+1)/(n+3)$. To prove this we imbed P_k in an analytic family G_k^α of operators and then use Stein's analytic interpolation theorem.

Recall that the projection operators P_k are integral operators with kernel $\Phi_k(x, y)$ given by (1.10). From Mehler's formula for the Hermite functions it follows that $\Phi_k(x, y)$ verify the generating function identity

$$\sum_{k=0}^{\infty} r^k\Phi_k(x, y) = \pi^{-n/2}(1-r^2)^{-n/2}e^{-(1/2)((1+r^2)/(1-r^2))(|x|^2+|y|^2)+(2r/1-r^2)x\cdot y} \tag{2.2}$$

(see Lemma 1.1.36 in [10]), which is valid for all r with $|r| < 1$. The right-hand side of (2.2) can be written as the product

$$(1-r)^{-n/2}e^{-(1/4)((1+r)/(1-r))|x-r|^2}(1+r)^{-n/2}e^{-(1/4)((1-r)/(1+r))|x+y|^2}.$$

Now, the Laguerre functions of type $\alpha > -1$ are given by the generating function

$$\sum_{k=0}^{\infty} r^k L_k^\alpha\left(\frac{1}{2}t^2\right)e^{-(1/4)t^2} = (1-r)^{-\alpha-1}e^{-(1/4)((1+r)/(1-r))t^2}. \tag{2.3}$$

In view of (2.3), we can write (2.2) as

$$\begin{aligned}\sum_{k=0}^{\infty} r^k\Phi_k(x, y) = \pi^{-n/2}&\left(\sum_{j=0}^{\infty} r^j L_j^{(n/2)-1}\left(\frac{1}{2}|x-y|^2\right)\right)e^{-(1/4)|x-y|^2}\\ &\times\left(\sum_{i=0}^{\infty}(-r)^i L_i^{(n/2)-1}\left(\frac{1}{2}|x+y|^2\right)\right)e^{-(1/4)|x+y|^2}.\end{aligned} \tag{2.4}$$

This yields the formula

$$\Phi_k(x,y) = \pi^{-n/2}\sum_{j=0}^{k}(-1)^j L_j^{(n/2)-1}\left(\frac{1}{2}|x+y|^2\right)e^{-(1/4)|x+y|^2} \times L_{k-j}^{(n/2)-1}\left(\frac{1}{2}|x-y|^2\right)e^{-(1/4)|x-y|^2}. \tag{2.5}$$

We now define for each α the functions Φ_k^α by means of the generating function

$$\sum_{k=0}^{\infty} r^k\Phi_k^\alpha(x,y) = \pi^{-n/2}(1-r^2)^{-\alpha+(n/2)-1}e^{-(1/2)((1+r^2)/(1-r^2))(|x|^2+|y|^2)+(2r/(1-r^2))x\cdot y}. \tag{2.6}$$

Then it is clear that

$$\Phi_k^\alpha(x,y) = \pi^{-n/2}\sum_{j=0}^{k}(-1)^j L_j^{\alpha-(n/2)}\left(\frac{1}{2}|x+y|^2\right)e^{-(1/4)|x+y|^2} \times L_{k-j}^{\alpha-(n/2)}\left(\frac{1}{2}|x-y|^2\right)e^{-(1/4)|x-y|^2} \tag{2.7}$$

and that $\Phi_k^{n-1}(x,y) = \Phi_k(x,y)$. As the Laguerre polynomials L_k^α are defined even for α complex, $\operatorname{Re}\alpha > -1$ we can define $\Phi_k^\alpha(x,y)$ for all α with $\operatorname{Re}\alpha > (n/2)-1$.

Regarding the kernels Φ_k^α, we prove the following pointwise estimates.

PROPOSITION 1 (i) $\sup_{\mathbb{R}^n\times\mathbb{R}^n}|\Phi_k^{n/2}(x,y)| \leqslant C$.

(ii) $\sup_{B\times B}|\Phi_k^{((n-1)/2)+i\tau}(x,y)| \leqslant C_B e^{a|\tau|}k^{-1/2}$ *for any compact* $B\subset\mathbb{R}^n$, $\tau\in\mathbb{R}$.

(iii) $\Phi_{2k}^{n+i\tau}(x,y) = \sum_{j=0}^{k}A_j^{i\tau}\Phi_{2k-2j}^{n-1}(x,y)$, *and* $\Phi_{2k+1}^{n+i\tau}(x,y) = \sum_{j=0}^{k}A_j^{i\tau}\Phi_{2k+1-2j}^{n-1}(x,y)$, *where* A_j^β *are the binomial coefficients defined by* $A_j^\beta = \Gamma(j+\beta+1)/\Gamma(j+1)\Gamma(\beta+1)$.

Assuming the proposition for a moment we show how we get the $L^p - L^{p'}$ estimates for the projections. We consider the analytic family of operators

$$G_k^\alpha f(x) = \int_{\mathbb{R}^n} f(y)\Phi_k^{((n-1)/2)+((n+1)/2)\alpha}(x,y)\,dy, \tag{2.8}$$

for $0\leqslant \operatorname{Re}\alpha\leqslant 1$. One can check that this is an admissible analytic family of operators (in the sense of Stein). When $\alpha = 1+i\tau$, part (iii) of the proposition gives

$$G_{2k}^{1+i\tau} f = \sum_{j=0}^{k} A^{(n+1)/2i\tau} P_{2k-2j} f,$$

$$G_{2k+1}^{1+i\tau} f = \sum_{j=0}^{k} A^{((n+1)/2)i\tau} P_{2k+1-2j} f.$$

Using asymptotic properties of the Γ function and the orthogonality of the projections one can show that

$$||G_k^{1+i\tau} f||_2 \leqslant Ce^{a|\tau|}||f||_2, \tag{2.9}$$

where $a > 0$. When $\alpha = i\tau$ and $f \in L^1(\mathbb{R}^n)$,

$$G_k^{i\tau}\chi_B f(x) = \int_B \Phi_k^{((n-1)/2)+((n+1)/2)i\tau}(x, y) f(y)\, dy, \tag{2.10}$$

and the estimate (ii) of the proposition shows that

$$||G_k^{i\tau}\chi_B f||_{L^\infty(B)} \leqslant Ce^{a|\tau|} k^{-1/2} ||f||_{L^1(B)}. \tag{2.11}$$

Now we interpolate (2.9) and (2.11) using the analytic interpolation theorem of Stein. The result is

$$||G_k^{\alpha}\chi_B f||_{L^{p'}(B)} \leqslant C_B k^{-(1/2)(1-\alpha)} ||f||_{L^p(B)}, \tag{2.12}$$

where $\alpha = (n-1)/(n+1)$ and $p = (1 - (\alpha/2))^{-1}$. For this choice of α, $G_k^\alpha \chi_B f = P_k \chi_B f$ and $p = 2(n+1)/(n+3)$ and we have

$$||P_k\chi_B f||_{L^{p'}(B)} \leqslant C_B k^{-1/(n+1)} ||f||_{L^p(B)}. \tag{2.13}$$

Note that when $p = 2(n+1)/(n+3)$, $n((1/p) - (1/2)) - 1 = -1/(n+1)$, and therefore

$$||P_k\chi_B f||_{L^{p'}(B)} \leqslant C_B k^{\delta(p)-(1/2)} ||f||_{L^p(B)} \tag{2.14}$$

is valid for $p = 2(n+1)/(n+3)$. We already know that (2.14) is valid when $p = 1$ also. Now the Riesz-Thorin interpolation shows that (2.14) is valid for $1 \leqslant p \leqslant 2(n+1)/(n+3)$, proving Theorem 2.

We now proceed to prove the proposition. As parts (i) and (iii) are easy we consider them first. From (2.7) we have

$$\Phi_k^{n/2}(x,y) = \sum_{j=0}^{k}(-1)^j L_j^0\left(\frac{1}{2}|x+y|^2\right)e^{-(1/4)|x+y|^2} \times L_{k-j}^0\left(\frac{1}{2}|x-y|^2\right)e^{-(1/4)|x-y|^2}. \tag{2.15}$$

If we take $x, y \in \mathbb{R}^2$, then with $n = 2$ the kernel $\Phi_k^{n-1}(x,y)$ of P_k on $L^2(\mathbb{R}^2)$ is given by

$$\Phi_k^1(x,y) = \pi^{-1}\sum_{j=0}^{k}(-1)^j L_j^0\left(\frac{1}{2}|x+y|^2\right)L_{k-j}^0\left(\frac{1}{2}|x-y|^2\right)e^{-(1/4)|x+y|^2}e^{-(1/4)|x-y|^2}. \tag{2.16}$$

The estimate (1.11) with $n = 2$ shows that $|\Phi_k^1(x,y)| \leqslant C$ for all x and y in $\mathbb{R}^2$, and now a comparison of (2.15) and (2.16) shows that estimate (i) of the proposition is true.

We remark in passing that instead of using (ii) and (iii), if we use (i) and (iii) in the analytic interpolation, we get the estimate

$$||P_k f||_{p'} \leqslant c k^{\delta(p)-1/2}||f||_p \tag{2.17}$$

for $1 \leqslant p \leqslant 2n/(n+2)$. This gives another proof of Karadzhov's theorem.

In order to prove (iii) we observe that

$$\sum_{k=0}^{\infty} r^k \Phi_k^{n+i\tau}(x,y) = \pi^{-n/2}(1-r^2)^{-(n/2)-i\tau-1}e^{-(1/2)((1+r^2)/(1-r^2))(|x|^2+|y|^2)+(2r/(1-r^2))x\cdot y}.$$

As the binomial coefficients $A_k^{i\tau}$ verify

$$\sum_{k=0}^{\infty} r^{2k}A_k^{i\tau} = (1-r^2)^{-i\tau-1},$$

we get the identity

$$\sum_{k=0}^{\infty} r^k \Phi_k^{n+i\tau}(x,y) = \pi^{-n/2}\left(\sum_{j=0}^{\infty} r^j \Phi_j^{n-1}(x,y)\right)\left(\sum_{\ell=0}^{\infty} A_\ell^{i\tau} r^{2\ell}\right). \tag{2.18}$$

Equating the coefficients of r^k on both sides of (2.18) we prove (iii) of the proposition.

Finally we turn our attention to the proof of (ii). To this end we express $\Phi_k^{((n-1)/2)+i\tau}$ as an oscillatory integral.

LEMMA 1. *Let τ be real and $\alpha = ((n-1)/2) + i\tau$. Then*

$$\Phi_k^\alpha(x, y) = ce^{(\pi/2)\tau} \int_{-\pi/2}^{\pi/2} (\sin 2t)^{-(1/2)-i\tau} e^{-2t\tau} e^{(2k+1)it} e^{i\varphi(t,x,y)} \, dt,$$

where c is a constant and

$$\varphi(t, x, y) = -x \cdot y \operatorname{cosec} 2t + \frac{1}{2}(|x|^2 + |y|^2) \cot 2t.$$

Proof. In the generating function for $\Phi_k^\alpha(x, y)$, we replace r by re^{-2it} getting

$$\sum_{k=0}^{\infty} r^k e^{-2kit} \Phi_k^\alpha(x, y) = \pi^{-n/2}(1 - r^2 e^{-4it})^{-\alpha+(n/2)-1} e^{B(r,t,x,y)},$$

where

$$B(r, t, x, y) = -\frac{1}{2}\left(\frac{1 + r^2 e^{-4it}}{1 - r^2 e^{-4it}}\right)(|x|^2 + |y|^2) + \frac{2re^{-2it}}{1 - r^2 e^{-4it}} x \cdot y.$$

From the above identity we get

$$r^k \Phi_k^\alpha(x, y) = \pi^{-(n/2)-1} \int_{-\pi/2}^{\pi/2} (1 - r^2 e^{-4it})^{-\alpha+(n/2)-1} e^{2kit} e^{B(r,t,x,y)} \, dt.$$

Letting $r \to 1$ and noting that $B(r, t, x, y) \to \varphi(t, x, y)$, we obtain the lemma. Taking the limit under the integral sign can be justified. For details we refer to Proposition 5.2.1 of [10].

We use the above representation in establishing estimate (ii) of the proposition. For the sake of simplicity we assume $\tau = 0$, the general case being completely similar. Setting $R = (2k + 1)$ we consider the integral

$$J_0 = \int_{-\pi/2}^{\pi/2} (\sin 2t)^{-1/2} e^{iRt} e^{i\varphi(t,x,y)} \, dt.$$

Replacing x and y by $R^{1/2}x$ and $R^{1/2}y$, respectively, and writing $\psi(t, x, y) = t + \varphi(t, x, y)$, we look at

$$J = \int_{-\pi/2}^{\pi/2} (\sin 2t)^{-1/2} e^{iR\psi(t,x,y)} \, dt. \tag{2.19}$$

We show that for $|x|^2+|y|^2 \leqslant 1/2$, we have the estimate $|J| \leqslant CR^{-1/2}$, from which part (ii) of the proposition follows immediately.

We write J as a sum of two integrals

$$J = \int_{-\pi/4}^{\pi/4} + \int_{\pi/4 \leqslant |t| \leqslant \pi/2} (\sin 2t)^{-1/2} e^{iR\psi(t)} dt.$$

In the second integral we can make a change of variable to bring it to the form

$$\int_{-\pi/4}^{\pi/4} (\sin 2t)^{-1/2} e^{iR\psi(t,x,-y)} dt.$$

Therefore, we only need to estimate the integral

$$I = \int_0^{\pi/4} (\sin 2t)^{-1/2} e^{iR\psi(t,x,y)} dt. \tag{2.20}$$

We estimate this integral using the method of stationary phase.

Let us write $a^2 = |x|^2 + |y|^2$, $b = x \cdot y$, and $\lambda = \cos 2t$. Then a simple calculation shows that $-\psi'(t)\sin^2 2t = a^2 - 2b\lambda + \lambda^2 - 1$ and $\psi''(t)\sin^3 2t = 4(a^2\lambda - b\lambda^2 - b)$. The stationary points are given by $\lambda = b \pm m$, where $m^2 = 1 - a^2 + b^2$. If we assume $a^2 \leqslant 1/2$, then in the interval $0 \leqslant t \leqslant \pi/4$ there is exactly one stationary point given by $\cos 2t_1 = b + m$, and at the stationary point

$$\psi''(t_1) = 4m \operatorname{cosec} 2t_1 \geqslant 2 \operatorname{cosec} 2t_1 \tag{2.21}$$

as we are assuming $a^2 \leqslant 1/2$.

Now the method of stationary phase says that the main term in the asymptotic expansion of the oscillatory integral I is

$$\left(\frac{\pi}{2}\right)^{1/2} R^{-1/2} (\psi''(t_1))^{-1/2} (\sin 2t_1)^{-1/2} e^{iR\psi(t_1) + i(\pi/4)},$$

which is bounded by a constant times $R^{-1/2}$ in view of (2.21). This heuristic argument can be made rigorous. We use the method of stationary phase in the following form known as the Van der Corput lemma. (A proof can be found in [10].)

LEMMA 2. *Suppose ψ is real valued and smooth on $[a,b]$. Assume that $|\psi^{(k)}(t)| \geqslant 1$, and when $k=1$, $\psi'(t)$ is monotonic. Then*

$$\left| \int_a^b \varphi(t) e^{iR\psi(t)} dt \right| \leqslant CR^{-1/k} \left\{ |\varphi(b)| + \int_a^b |\varphi'(t)| \, dt \right\}.$$

In order to apply this lemma we need to get lower bounds for the first and second derivatives of $\psi(t)$. The following lemmas give the required bounds. These lemmas were proved in [10] (see Lemmas 5.3.2 and 5.3.3).

LEMMA 3. *Assuming that $a^2 \leqslant 1/2$, the lower bound $|\psi''(t)| \geqslant \operatorname{cosec} 2t_1$ is valid under any of the following two conditions:*
(i) $(1/2)|x-y| \leqslant t \leqslant t_1 + (1/20)\sin 2t_1,\ b \leqslant 0$;
(ii) $t_1 - (1/20)\sin 2t_1 \leqslant t \leqslant t_1 + (1/20)\sin 2t_1,\ b \geqslant 0$.

LEMMA 4. *Assume again $a^2 \leqslant 1/2$. The lower bound $|\psi'(t)| \geqslant 1/20$ is valid under any of the following two conditions:*
(i) $t \geqslant t_1 + (1/20)\sin 2t_1,\ b \geqslant 0$ *or* $b \leqslant 0$;
(ii) $t \leqslant t_1 - (1/20)\sin 2t_1,\ b \geqslant 0$.

We now proceed to estimate the integral I using these two lemmas. Let $\beta = (1/2)|x-y|$ and consider first the case $\beta \geqslant R^{-1}$. We write $I = I_1 + I_2$, where

$$I_1 = \int_0^{\beta/2} (\sin 2t)^{-(1/2)} e^{iR\psi(t)}\,dt.$$

We first estimate I_1; to do this we claim

$$\begin{aligned} -\psi'(t)\sin^2 2t &\geqslant \frac{1}{2}|x-y|^2, \\ |\psi''(t)\sin^3 2t| &\leqslant 8|x-y|^2 \end{aligned} \tag{2.22}$$

for $0 \leqslant t \leqslant \beta/2$. These estimates can be easily proved. For example, in [10, p. 118] we have shown that

$$|\varphi''(t)\sin^3 2t| \leqslant 4|\varphi'(t)|\sin^2 2t$$

and $-\varphi'\sin^2 2t = 4b\sin^2 t + |x-y|^2$, which gives the estimate

$$|\varphi'(t)|\sin^2 t \leqslant \sin^2\beta/2 + |x-y|^2 \leqslant 2|x-y|^2$$

as $2b \leqslant a^2 \leqslant 1/2$ and $t \leqslant \beta/2$. As $\psi''(t) = \varphi''(t)$, we finally get

$$|\psi''(t)\sin^3 2t| \leqslant 8|x-y|^2.$$

Similarly, one can show that $-\psi'(t)\sin^2 2t \geqslant (1/2)|x-y|^2$.

Once we have the estimates (2.22), I_1 can be easily estimated. We write

$$I_1 = \frac{-i}{R}\int_0^{\beta/2} \frac{(\sin 2t)^{3/2}}{\psi'(t)\sin^2 2t}\, d\left(e^{iR\psi(t)}\right).$$

Using the estimates (2.22) and integrating by parts, we get

$$|I_1| \leqslant CR^{-1}|x-y|^2 \int_0^{\beta/2} t^{1/2}\, dt \leqslant CR^{-1}|x-y|^{-1/2}.$$

As $|x-y| \geqslant 2R^{-1}$ we get the right estimate, namely, $|I_1| \leqslant CR^{-1/2}$.

In the estimation of I_2 we consider two cases. First assume that $b \leqslant 0$. The critical point t_1 is given by $\sin^2 2t_1 = a^2 - 2b\cos 2t_1$, and as $b \leqslant 0$ we have $(1/2)|x-y|^2 \leqslant a^2 \leqslant \sin^2 2t_1 \leqslant a^2 - 2b = |x-y|^2$. Thus $\sin 2t_1 \sim |x-y|$. We split the integral I_2 into two parts $I_2 = I_3 + I_4$, where

$$I_3 = \int_{\beta/2}^{t_1+(1/20)\sin 2t_1} (\sin 2t)^{-1/2} e^{iR\psi(t)}\, dt.$$

Using the estimates of Lemma 3, this integral is bounded by

$$|I_3| \leqslant CR^{-1/2}(\sin 2t_1)^{1/2}\{(\sin 2t_1)^{-1/2} + |x-y|^{-1/2}\},$$

which is bounded by $CR^{-1/2}$. Similarly, the second integral is bounded by $CR^{-1/2}$.

Next assume that $b \geqslant 0$ (still we are assuming that $\beta \geqslant R^{-1/2}$). Let t_0 be the point at which $\psi''(t) = 0$. As $b \geqslant 0$ we observe that $\sin^2 2t_1 \geqslant |x-y|^2$. We split the integral into four parts corresponding to the intervals $\beta/2 \leqslant t \leqslant t_1 - (1/20)\sin 2t_1$, $t_1 - (1/20)\sin 2t_1 \leqslant t \leqslant t_1 + (1/20)\sin 2t_1$, $t_1 + (1/20)\sin 2t_1 \leqslant t \leqslant t_0$, and $t_0 \leqslant t \leqslant \pi/4$. As above using the lower bounds given in Lemmas 3 and 4 we get the estimate $|I_2| \leqslant CR^{-1/2}$.

Thus we have estimated I when $\beta \geqslant R^{-1}$. So, assume now $\beta \leqslant R^{-1}$ and again consider two cases. When $b \leqslant 0$, without loss of generality let us assume $\beta \leqslant R^{-1} < t_1 + (1/20)\sin 2t_1$ and write

$$I = \int_0^{\beta} + \int_{\beta}^{t_1+(1/20)\sin 2t_1} + \int_{t_1+(1/20)\sin 2t_1}^{\pi/4}.$$

As $\beta \leqslant R^{-1}$, a mere integration gives the required estimate for the first integral. For the second and third integrals, we use the lower bounds given in the lemmas. Noting that $\sin 2t_1 \sim |x-y|$ we get the estimate $|I| \leqslant CR^{-1/2}$.

When $b \geqslant 0$ and $\beta \leqslant R^{-1}$, we assume, without loss of generality, that $\beta \leqslant R^{-1} < t_1 - (1/20)\sin 2t_1$. Note that in this case $\sin^2 2t_1 \geqslant |x-y|^2$. We split the integral into four parts, corresponding to $0 < t \leqslant R^{-1}$ $R^{-1} < t \leqslant t_1 - (1/20)\sin 2t_1$, $t_1 - (1/20)\sin 2t_1 \leqslant t \leqslant t_1 + (1/20)\sin 2t_1$, and $t_1 + (1/20)\sin 2t_1 \leqslant t \leqslant \pi/4$. In each case we get the estimate $CR^{-1/2}$.

Thus we have got the estimate $|J| \leqslant CR^{-1/2}$, which in turn completes the proof of Proposition 1.

We conclude this section with the following remark. For $a^2 \geqslant 1/2$ there are two stationary points of $\psi(t)$ that approach each other as $a^2 \to 1$. So, the behavior of the oscillatory integral is more complicated when $1/2 \leqslant a^2 \leqslant 1$, and we do not get good estimates in that region.

3. Special Hermite expansions. In this section we prove Theorems 4 and 5. For technical convenience we would consider the Cesaro means σ_N^δ rather than the Riesz means S_R^δ. The behavior of the Cesaro means that are defined by

$$\sigma_N^\delta f = (2\pi)^{-n} \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta f \times \varphi_k \tag{3.1}$$

is completely analogous to that of the Riesz means, and so it is enough to prove Theorems 4 and 5 when S_R^δ is replaced by σ_N^δ. The convenience of using σ_N^δ in place of S_R^δ arises from the fact that the kernel of σ_N^δ is explicitly known. In fact, we have $\sigma_N^\delta f = (2\pi)^{-n} f \times s_N^\delta$, where

$$s_N^\delta(z) = \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta \varphi_k(z). \tag{3.2}$$

As the Laguerre polynomials $L_k^\alpha(t)$ satisfy the identity

$$\sum_{k=0}^{N} A_{N-k}^\delta L_k^\alpha(t) = L_N^{\alpha+\delta+1}(t), \tag{3.3}$$

and as $\varphi_k(z) = L_k^{n-1}((1/2)|z|^2) e^{-(1/4)|z|^2}$, we have

$$s_N^\delta(z) = \frac{1}{A_N^\delta} L_N^{\delta+n}\left(\frac{1}{2}|z|^2\right) e^{-(1/4)|z|^2}. \tag{3.4}$$

One can then use asymptotic properties of the Laguerre polynomials in the study of Cesaro means.

We first take up the proof of Theorem 4. As in the case of Hermite expansions, it is enough to prove Theorem 3. Once we establish the local restriction theorem we can proceed as in [10] to prove Theorem 4. In order to prove Theorem 3, it is enough to establish the estimates

$$\|f \times \varphi_k\|_{L^{p'}(B)} \leqslant C k^{2n((1/p)-(1/2))-1} \|f\|_{L^p(B)}, \tag{3.5}$$

for $1 \leqslant p \leqslant 2(2n+1)/(2n+3), f \in L^p(B)$. To this end we imbed $f \to f \times \varphi_k$ into an analytic family of operators and then apply Stein's interpolation theorem.

To define the analytic family, consider

$$\psi_k^\alpha(z) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha\left(\frac{1}{2}|z|^2\right) e^{-(1/4)|z|^2}, \tag{3.6}$$

and observe that we can define ψ_k^α even for complex values of α, provided $\operatorname{Re}\alpha > -1$. We then set

$$G_k^\alpha f = f \times \psi_k^{-(1/2)+(n+(1/2))\alpha},$$

and for this family we establish the following proposition.

PROPOSITION 2. *We have*
(i) $\|G_k^{i\tau} f\|_{L^\infty(B)} \leqslant C_B(1+|\tau|)^{1/2}\|f\|_{L^1(B)}$;
(ii) $\|G_k^{1+i\tau} f\|_{L^2(B)} \leqslant C_B(1+|\tau|)^n k^{-n}\|f\|_{L^2(B)}$,
where B is a compact subset of $\mathbb{C}^n$ and $\tau \in \mathbb{R}$.

Assume the proposition for a moment. When $\alpha = (2n-1)/(2n+1)$, $G_k^\alpha f$ reduces to

$$f \times \psi_k^{n-1}(z) = \frac{\Gamma(k+1)\Gamma(n)}{\Gamma(k+n)} f \times \varphi_k(z).$$

Hence interpolating between (i) and (ii) we get (3.5) when $p = 2(2n+1)/(2n+3)$. As (3.5) is known to be valid for $p = 1$, an application of Riesz-Thorin proves that (3.5) is valid for $1 \leqslant p \leqslant 2(2n+1)/(2n+3)$.

Estimate (ii) of the proposition follows from the better estimate

$$\|G_k^{1+i\tau} f\|_2 \leqslant C(1+|\tau|)^n k^{-n}\|f\|_2$$

proved in [10] (see Proposition 2.6.2). In order to prove (i) we only need to get the estimate

$$\sup_{z\in B} |\psi_k^{-(1/2)+i\tau}(z)| \leqslant C_B(1+|\tau|)^{(1/2)}, \tag{3.7}$$

where B is a compact subset of $\mathbb{C}^n$.

Now the Laguerre functions can be expressed in terms of the Bessel functions. Indeed, we have the formula

$$e^{-x}x^{\alpha/2}L_k^\alpha(x) = \frac{1}{\Gamma(k+1)}\int_0^\infty e^{-t}t^{k+\alpha/2}J_\alpha(2\sqrt{tx})\,dt$$

for $\alpha > -1$ (see Theorem 5.4 in Szego [8]). So we can write

$$\psi_k^\alpha(z) = \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} \int_0^\infty e^{-t} t^{k+\alpha} \frac{J_\alpha(\sqrt{2t}|z|)}{(\sqrt{2t}|z|)^\alpha} e^{(1/4)|z|^2} dt. \tag{3.8}$$

From this expression it is clear that (3.7) follows once we show that the estimate

$$|J_{-(1/2)+i\tau}(z)| \leqslant C(\tau)|z^{-(1/2)+i\tau}|$$

holds for the Bessel function. The Bessel functions satisfy the relation

$$J_{\alpha-1}(z) + J_{\alpha+1}(z) = 2\alpha z^{-1} J_\alpha(z),$$

and therefore we are done if we show that $|J_\alpha(z)| \leqslant C(\alpha)|z^\alpha|$ for all $\operatorname{Re}\alpha \geqslant 1/2$. But in this case we can use the integral representation

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha + (1/2))\Gamma(1/2)} \int_{-1}^1 (1-t^2)^{\alpha-1/2} e^{itz}\, dt,$$

which is valid for $\operatorname{Re}\alpha > -1/2$. From this it is clear that $|J_\alpha(z)| \leqslant C(\alpha)|z^\alpha|$ for $\operatorname{Re}\alpha > -1/2$.

This concludes the proof of the proposition, which implies the local restriction theorem from which Theorem 4 follows in a routine fashion.

We now proceed to the proof of Theorem 5. Without loss of generality we assume that B is the unit ball $|z| \leqslant 1$, and we prove the theorem for the Cesaro means. As we have already noted, the operators σ_N^δ are twisted convolution operators with kernel s_N^δ given by (3.4). In view of (3.8) we see that when $n = 1$

$$s_N^\delta(z) = \frac{2^{\delta+1}}{A_N^\delta \Gamma(N+1)} \int_0^\infty e^{-t} t^{N+\delta+1} \frac{J_{\delta+1}(\sqrt{2t}|z|)}{(\sqrt{2t}|z|)^{\delta+1}} e^{(1/4)|z|^2} dt. \tag{3.9}$$

Since we want to prove the uniform boundedness of $\chi_B \sigma_N^\delta \chi_B$, we choose a cutoff function $\varphi \in C_0^\infty(|t| \leqslant 3)$ such that $\varphi(t) = 1$ for $|t| \leqslant 2$, and we write

$$s_N^\delta(z) = s_N^\delta(z)\varphi(|z|) + s_N^\delta(z)(1 - \varphi(|z|)).$$

When $z, w \in B$, $|z - w| \leqslant 2$, so the second kernel vanishes identically. Hence

$$\chi_B \sigma_N^\delta \chi_B f(z) = (2\pi)^{-n} \chi_B(z)(\chi_B f) \times (s_N^\delta \varphi)(z).$$

So we have to prove the uniform boundedness of the operator

$$T_N^\delta f(z) = \int_{\mathbb{C}} s_N^\delta(z-w)\varphi(|z-w|)f(w)e^{(i/2)\operatorname{Im} z\cdot\bar{w}}\,dw$$

on the prescribed L^p-spaces.

Let us define the kernels $k_t(z)$ by

$$k_t(z) = t\frac{J_{\delta+1}(\sqrt{2t}|z|)}{(\sqrt{2t}|z|)^{\delta+1}}\varphi(|z|)e^{(1/4)|z|^2} \tag{3.10}$$

for $t > 0$. In view of (3.9) we have

$$T_N^\delta f(z) = \frac{2^{\delta+1}}{A_N^\delta\Gamma(N+1)}\int_0^\infty e^{-t}t^{N+\delta}T_t f(z)\,dt,$$

where $T_t f(z) = f \times k_t(z)$. By Minkowski's integral inequality, the uniform boundedness of T_N^δ follows from the uniform boundedness of the operators T_t.

To study the operators T_t we use the result of Carleson-Sjölin, together with the following result of Cowling [2].

THEOREM 6. *Let k be a distribution with compact support on $\mathbb{C}^n$, and suppose that $1 \leqslant p \leqslant \infty$. Then the twisted convolution operator $f \to f \times k$ is bounded on $L^p(\mathbb{C}^n)$ if and only if the convolution operator $f \to f * k$ is bounded on $L^p(\mathbb{C}^n)$.*

In view of this theorem of Cowling, we only need to show that the $f \to f * k_t$ are uniformly bounded on $L^p(\mathbb{C})$. But these are closely related to the Bochner-Riesz means associated to the standard Laplacian on $\mathbb{R}^2$. In fact, the Bochner-Riesz means are given by

$$S_t^\delta f(z) = t^2\int_{\mathbb{C}}\frac{J_{\delta+1}(t|z-w|)}{(t|z-w|)^{\delta+1}}f(w)\,dw.$$

In [1], Carleson-Sjölin proved that these are uniformly bounded on $L^p(\mathbb{C})$ provided $\delta > 2((1/p)-(1/2)) - 1/2$ and $1 \leqslant p < 4/3$. A close examination of their proof shows that the same is true of the truncated operators $f \to f * k_t$. This concludes the proof of Theorem 5.

4. Concluding remarks. We would like to conclude this paper with the following questions that merit further investigation. In what follows let S_R^δ, S_N^δ, and S_t^δ stand for the Bochner-Riesz means for the Hermite, special Hermite, and Fourier expansions, respectively.

The transplantation theorem of Kenig-Stanton-Tomas [4] says that if B is the unit ball, then the uniform boundedness of $\chi_B S_R^\delta \chi_B$ or $\chi_B S_N^\delta \chi_B$ on L^p implies the same for S_t^δ. One wonders if the converse is true. In the case of special Hermite expansions this means that the truncated operators

$$T_t f(z) = t^{2n} \int_{|w| \leqslant 1} \frac{J_{\delta+n}(t|w|)}{(t|w|)^{\delta+n}} f(z-w)\, dw \tag{4.1}$$

are uniformly bounded whenever the nontruncated operators are uniformly bounded. In the previous section we just saw that this is indeed the case when $n = 1$. It would be interesting to see if the same is true when $n \geqslant 2$. But in the case of Hermite expansions we do not even have a clue how to go about this.

Our second question is the following. Suppose that the uniform boundedness of S_t^δ and $\chi_B S_R^\delta \chi_B$ are equivalent. Then what happens to the Bochner-Riesz conjecture for S_R^δ? Is it true, or do we have to be content with the local version? We already know that this is the case when $n = 1$. So we wonder if there is a δ_c such that the full conjecture fails for $0 \leqslant \delta \leqslant \delta_c$.

Finally, it is reasonable to consider mixed-norm versions of the Bochner-Riesz conjecture. This has been done successfully in the case of S_t^δ, and similar questions for S_R^δ and S_N^δ are interesting problems worth investigating. We hope to settle some of these problems in the future.

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Note added in proof. The global version of Theorem 3 was recently proved by K. Stempack and J. Zienkiewicz in "Twisted convolution and Riesz means" (to appear in J. Analyse Math.).

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STATISTICS AND MATHEMATICS DIVISION, INDIAN STATISTICAL INSTITUTE, 8TH MILE MYSORE ROAD, BANGALORE 560059, INDIA; veluma@isibang.ernet.in