SEQUENTIAL ANALYSIS
Vol. 23, No. 4, pp. 523–525, 2004

Discussion on "Likelihood Ratio Identities and Their Applications to Sequential Analysis" by Tze L. Lai

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This relatively brief, yet detailed survey of a collection of beautiful, important results is a fitting tribute both to Wald and the vitality of Sequential Analysis by one of its most creative practitioners. Many of the results are due to Professor Lai himself. As Professor Lai points out again and again, all these diverse results originate in a sense in Wald's use of an appropriate likelihood ratio test to handle questions relating to

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boundary crossing of a random walk. In fact many of the results have their origin in the Wald inequalities or Wald approximations, obtained by neglecting excess over boundaries. Second order results arise from estimating the excess over boundaries. What is equally striking is that these provide good approximations to solutions of very hard optimization problems. Incidentally, some of the methods of classical sequential analysis, as presented in Wald's book, were also independently developed in Britain by Barnard and his group but the method based on likelihood ratio was Wald's.

It is clear that many of the results have wide applications. It would be wonderful if the editor can persuade Professor Lai to write another review of actual or potential applications. That too would be a fitting tribute to Wald, whose results on sequential analysis were considered so important that they remained classified material till after the end of the second world war.

May I add a couple of more historical comments? I missed a reference to Bahadur's[1] elegant proof of Wald's Fundamental Identity using Wald's method of likelihood ratio and analytic continuation. Bahadur was probably attracted to this because of his interest in the so called method of exponential tilting used in large deviations. Who by the way first used this term? Could it be Bahadur? Or was it Cramer or Chernoff?

Similar ideas also appear in Barndorff–Nielsen's magic (saddle point) formula for the maximum likelihood estimate, see my monograph on Higher Order Asymptotics.[3] There was a beautiful presentation of likelihood ratio methods in Varadhan's IMS Presidential talk on ''The Art and Science of Large Deviations''.

Finally, may I point out that most of Professor Ghosh's[2] chapter on sequential tests of composite hypotheses (to which Professor Lai refers) is based on my 1962 thesis submitted to Calcutta University and a joint paper with Professor Hall and Wijsman[4]? This is just to set the record straight. Once again we are all indebted to Professor Lai for this wonderful survey.

REFERENCES

1. Bahadur, R.R. A note on the fundamental identity of sequential analysis. Ann. Math. Statist. **1958**, *29*, 534–543.
2. Ghosh, B.K. *Sequential Tests of Statistical Hypotheses*; Addison–Wesley Publishing Co.: Reading, MA, 1970.
3. Ghosh, J.K. *Higher Order Asymptotics*; NSF–CBMS Regional Conference Series in Probability and Statistics Institute for Mathematical Statistics; 1994; Vol. 4.

4. Hall, W.J.; Wijsman, R.A.; Ghosh, J.K. The relationship between sufficiency and invariance with applications in sequential analysis. Ann. Math. Statist. **1965**, *36*, 575–614. Correction in Ann. Statist. **1995**, *23*, 705.