Second-Order Pitman Admissibility and Pitman Closeness: The Multiparameter Case and Stein-Rule Estimators

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In a multiparameter estimation problem, for first-order efficient estimators, second-order Pitman admissibility, and Pitman closeness properties are studied. Bearing in mind the dominant role of Stein-rule estimators in multiparameter estimation theory, such second-order properties are also studied for shrinkage maximum likelihood estimators.

1. INTRODUCTION

The classical *maximum likelihood estimators* (MLE) are generally *best asymptotically normal* (BAN) and are known to be asymptotically *first-order efficient* (FOE) in the light of conventional *quadratic risk functions* as well as the *generalized Pitman closeness criterion* (GPCC). In this characterization, an asymptotic representation for BAN estimators in terms of an average of independent summands plus a remainder term converging to zero at a faster rate plays the basic role; we may refer to Keating, Mason, and Sen [7, Chap. 6] for some systematic exposition of this feature, mostly,

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dealing with the single parameter situation. This FOE-isomorphism of quadratic risk and GPCC remains in tact in the general multiparameter case as well (viz., Sen [13]).

The past three decades have witnessed a phenomenal growth of research literature on *higher order asymptotic efficiency* of FOE estimators. In this context, quadratic and other conventional *loss functions* and *concentration probabilities* have been used extensively. The recent monograph of Ghosh [4] provides an up to date account of the developments in this broad and active area of research wherein the multiparameter estimation problems have also been treated adequately.

Recently, Rao [9] has revived interest in comparing estimates through PCC, which shows that the marginal distribution of estimators or their risk functions in their usual sense do not capture all that is relevant in comparing them. The present authors [5] studied the *second-order Pitman admissibility* and *second-order Pitman closeness* of BAN (FOE) estimators in the single parameter case. There remains a natural need to comprehend the general multiparameter case with respect to both of these second-order efficiency criteria, and the current study is primarily geared towards this basic objective.

In the multiparameter case, the classical MLE may not be generally admissible (relative to a chosen quadratic risk), and some alternative versions, known as the *Stein-rule* (SR) or *shrinkage* estimators, dominate the MLE, often, in a finite sample setup, and more generally, in a well-defined asymptotic setup. Stein [15] initiated this line of research for the simple multinormal mean vector estimation problem when the covariance matrix is specified. During the past 40 years, the dominance of Stein-rule versions over the classical MLE and other conventional estimators in the multiparameter case has been studied extensively, covering some finite sample results for suitable *exponential families of distributions* and extending the findings to suitable asymptotic setups for a much wider class; we may refer to Sen [11] for some relevant first-order asymptotics for SRMLEs. Dominance of the SR-estimators in the light of the GPCC has been studied in a finite sample setup by Sen, Kubokawa, and Saleh [14], where the asymptotic case has also been treated briefly. However, all these studies relate to the (asymptotic) dominance with respect to suitable FOE criteria. From an asymptotic perspective (as relevant to FOE/BAN estimators), there is a basic feature of multiparameter estimation problems with special emphasis on Stein-rule estimators which merits a critical appraisal. The SR-estimators are generally adapted to a chosen pivot (say, θ_0). For every $\theta \neq \theta_0$, a MLE $\hat{\theta}_n$ and its plausible Stein-rule versions are asymptotically PC-equivalent in the sense that they share the FOE property in the conventional asymptotic setups; the asymptotic PC-dominance of SRMLE over the classical MLE studied by Sen, Kubokawa, and Saleh [14] pertains only to a *Pitman-neighborhood* of the pivot θ_0, beyond which the dominance becomes asymptotically imperceptible. Therefore, it is of natural interest to compare a MLE and its SR versions in the light of second-order efficiency properties. This is one of the basic objectives of the current study.

A general second-order PC result in the multiparameter case is presented in Section 2. In this context, special attention is paid to the multivariate location model. This provides a natural motivation for SR estimators which are then treated in Section 3, covering *location*, *scale* as well as *location-scale* models. An extension of the main theorem (in Section 2), having some interest on its own, is presented in the concluding section.

2. A General Result

Consider a sequence $\{X_i;\ i \geqslant 1\}$ of independent and identically distributed (i.i.d.) random variables or vectors (r.v.) with a common density $f(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, ..., \theta_p)'$ is an unknown vector parameter belonging to a *parametric space* $\boldsymbol{\Theta}$ which $\mathscr{R}^p$ or some open subset thereof, and p is a positive integer. We adopt the same regularity assumptions as in Bhattacharya and Ghosh [2, p. 439] with $s = 3$ (in their notation), and $f(\cdot; \boldsymbol{\theta})$ and $g(\cdot; \boldsymbol{\theta})$ in their notation interpreted respectively as $\ln f(\cdot; \boldsymbol{\theta})$ and $f(\cdot; \boldsymbol{\theta})$ in our notation. Let $\mathscr{I} = ((\mathscr{I}_{ij}))$ be the $p \times p$ per observation *Fisher information* matrix which is assumed to be positive definite (p.d.) at each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Let $\mathscr{I}^{-1} = ((\mathscr{I}^{ij}))$, and for each $1 \leqslant i, j, u \leqslant p$, let

$$S_{i \cdot j \cdot u} = E_{\boldsymbol{\theta}}\{ D_i \ln f(X_1; \boldsymbol{\theta}))(D_j \ln f(X_1; \boldsymbol{\theta}))(D_u \ln f(X_1; \boldsymbol{\theta}))\}; \tag{2.1}$$

$$S_{i \cdot ju} = E_{\boldsymbol{\theta}}\{D_i \ln f(X_1; \boldsymbol{\theta}))(D_j D_j \ln f(X_1; \boldsymbol{\theta}))\}; \tag{2.2}$$

$$S_{iju} = E_{\boldsymbol{\theta}}\{ D_i D_j D_u \ln f(X_1; \boldsymbol{\theta})\}; \tag{2.3}$$

$$\bar{S}_{iju} = S_{i \cdot ju} - S_{i \cdot j \cdot u}, \tag{2.4}$$

where D_i stands for the partial differentiation operator with respect to θ_i $(1 \leqslant i \leqslant p)$. Note that for each i, j, u, $\mathscr{I}_{ij}$, $\mathscr{I}^{ij}$, $S_{i \cdot j \cdot u}$, $S_{i \cdot ju}$, and $\bar{S}_{iju}$ are generally functions of $\boldsymbol{\theta}$; for notational simplicity, this dependence is, however, suppressed.

Base on a sample $X_1, ..., X_n$ of size n, let $\hat{\boldsymbol{\theta}}$ $(= \hat{\boldsymbol{\theta}}_n) = (\hat{\theta}_1, ..., \hat{\theta}_p)'$ be MLE of $\boldsymbol{\theta}$, defined in the sense of Theorem 3 of Bhattacharya and Ghosh [2]; again for notational simplicity, the subscript n is dropped. Along the lines of Ghosh and Sinha [6] and Pfanzagl and Wefelmeyer [8], who studied in detail the second-order efficiency properties of adjusted MLEs under conventional quadratic and other convex risk functions, we consider here a class $\mathscr{C}$ of estimators of $\boldsymbol{\theta}$ of the form

$$\hat{\boldsymbol{\theta}} + n^{-1}\mathbf{d}(\hat{\boldsymbol{\theta}}), \tag{2.5}$$

where the components of $\mathbf{d}(\hat{\boldsymbol{\theta}})$ are sufficiently smooth and have functional forms free from n (see Theorem 2.1 below). As seen later, consideration of such estimators will enable us to improve upon $\hat{\boldsymbol{\theta}}$ in many models of interest with regard to second-order Pitman closeness. The following theorem plays a crucial role in this context.

THEOREM 2.1. *Let* $\mathbf{T}_n^* = \hat{\boldsymbol{\theta}} + n^{-1}\mathbf{d}^*(\hat{\boldsymbol{\theta}})$ *and* $\mathbf{T}_n = \hat{\boldsymbol{\theta}} + n^{-1}\mathbf{d}(\hat{\boldsymbol{\theta}})$ *be estimators of* $\boldsymbol{\theta}$, *where* $\mathbf{d}^*(\boldsymbol{\theta}) = (d_1^*(\boldsymbol{\theta}), ..., d_p^*(\boldsymbol{\theta}))'$, $\mathbf{d}(\boldsymbol{\theta}) = (d_1(\boldsymbol{\theta}), ..., d_p(\boldsymbol{\theta}))'$, *and for each i*, $d_i^*(\boldsymbol{\theta})$ *and* $d_i(\boldsymbol{\theta})$ *are continuously differentiable over* $\boldsymbol{\Theta}$, *with functional forms free from n, such that their partial derivatives fulfil the local Lipschitz conditions. Then for each* $\boldsymbol{\theta}$ *at which* $\boldsymbol{\phi}(\boldsymbol{\theta}) = (\phi_1(\boldsymbol{\theta}), ..., \phi_p(\boldsymbol{\theta}))' = \mathbf{d}(\boldsymbol{\theta}) - \mathbf{d}^*(\boldsymbol{\theta}) \neq \mathbf{0}$,

$$\begin{aligned}
\Delta_n(\boldsymbol{\theta}) &= P_{\boldsymbol{\theta}}\{(\mathbf{T}_n^* - \boldsymbol{\theta})'\,\mathscr{I}(\mathbf{T}_n^* - \boldsymbol{\theta}) < (\mathbf{T}_n - \boldsymbol{\theta})'\,\mathscr{I}(\mathbf{T}_n - \boldsymbol{\theta})\} \\
&= \tfrac{1}{2} + (2\pi n)^{-1/2}\,\{\boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\boldsymbol{\phi}(\boldsymbol{\theta})\}^{-3/2}\Big[\tfrac{1}{2}\{\boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\boldsymbol{\phi}(\boldsymbol{\theta})\}^2 \\
&\quad + \boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\boldsymbol{\phi}(\boldsymbol{\theta})\Big[\boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\mathbf{d}^*(\boldsymbol{\theta}) + \operatorname{tr}\{\mathbf{B}(\boldsymbol{\theta})\} \\
&\quad + \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{u=1}^{p}\phi_i(\boldsymbol{\theta})\,\mathscr{I}^{ju}(S_{u\cdot ij} + \tfrac{1}{2}S_{iju})\Big] \\
&\quad - \tfrac{1}{6}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{u=1}^{p}\phi_i(\boldsymbol{\theta})\,\phi_j(\boldsymbol{\theta})\,\phi_u(\boldsymbol{\theta})\,\bar{S}_{iju} - \boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\mathbf{B}(\boldsymbol{\theta})\,\boldsymbol{\phi}(\boldsymbol{\theta})\Big] \\
&\quad + o(n^{-1/2}),
\end{aligned} \tag{2.6}$$

where $\mathbf{B}(\boldsymbol{\theta})$ *is a* $p \times p$ *matrix with* (i, j)*th element* $D_j\phi_i(\boldsymbol{\theta})$, *for* $i, j = 1, ..., p$.

Proof. Observe that for each $\boldsymbol{\theta}$ with $\boldsymbol{\phi}(\boldsymbol{\theta}) \neq \mathbf{0}$, we may write equivalently

$$\Delta_n(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}(V_n > 0), \tag{2.7}$$

where

$$V_n = \{\boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\boldsymbol{\phi}(\boldsymbol{\theta})\}^{-1/2}[\boldsymbol{\phi}(\hat{\boldsymbol{\theta}})'\,\mathscr{I}\{n^{1/2}(\mathbf{T}_n^* - \boldsymbol{\theta})\} + \tfrac{1}{2}n^{-1/2}\boldsymbol{\phi}(\hat{\boldsymbol{\theta}})'\,\mathscr{I}\boldsymbol{\phi}(\hat{\boldsymbol{\theta}})]. \tag{2.8}$$

Let $\mathbf{R} = (R_1, ..., R_p)' = n^{1/2}\mathscr{I}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$. Note that for each i $(=1, ..., p)$,

$$R_i = H_{1i} + n^{-1/2}\left[\sum_{j=1}^{p}\sum_{u=1}^{p}\mathscr{I}^{ju}H_{2ij}H_{1u} + \frac{1}{2}\sum_{j=1}^{p}\sum_{u=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}S_{ijs}\mathscr{I}^{ju}\mathscr{I}^{st}H_{1u}H_{1t}\right] + o(n^{-1/2}), \tag{2.9}$$

where

$$H_{1i} = n^{-1/2}\sum_{u=1}^{p} D_i \ln f(X_u; \boldsymbol{\theta}), \qquad i = 1, ..., p; \tag{2.10}$$

$$H_{2ij} = n^{-1/2}\sum_{u=1}^{p} (D_i D_j \ln f(X_u; \boldsymbol{\theta}) + \mathscr{I}_{ij}), \qquad i, j = 1, ..., p. \tag{2.11}$$

Therefore, the approximate cumulants of $\mathbf{R}$ under $\boldsymbol{\theta}$ are given by

$$\begin{aligned}\kappa_{1n}(R_i) &= n^{-1/2}\sum_{j=1}^{p}\sum_{u=1}^{p}\mathscr{I}^{ju}(S_{u\cdot ju} + \tfrac{1}{2}S_{uji}) + o(n^{-1/2}),\\ \kappa_{2n}(R_i, R_j) &= \mathscr{I}_{ij} + o(n^{-1/2}),\\ \kappa_{3n}(R_i, R_j, R_u) &= n^{-1/2}\bar{S}_{iju} + o(n^{-1/2}), \qquad i, j, u = 1, ..., p.\end{aligned} \tag{2.12}$$

The fourth and higher order cumulants of $\mathbf{R}$ under $\boldsymbol{\theta}$ are $o(n^{-1/2})$. Since,

$$\mathscr{I}\{n^{1/2}(\mathbf{T}_n^* - \boldsymbol{\theta})\} = \mathbf{R} + n^{-1/2}\mathscr{I}\mathbf{d}^*(\boldsymbol{\theta}) + o(n^{-1/2}); \tag{2.13}$$

$$\boldsymbol{\phi}(\hat{\boldsymbol{\theta}}) = \boldsymbol{\phi}(\boldsymbol{\theta}) + n^{-1/2}\mathbf{B}(\boldsymbol{\theta})\,\mathscr{I}^{-1}\mathbf{R} + o(n^{-1/2}), \tag{2.14}$$

it follows from (2.8) and (2.12) that for each $\boldsymbol{\theta}$ with $\boldsymbol{\phi}(\boldsymbol{\theta}) \neq \mathbf{0}$, the approximate cumulants of V_n under $\boldsymbol{\theta}$ are given by

$$\begin{aligned}\kappa_{1n}^*(V_n) &= n^{-1/2}\{\boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\,\boldsymbol{\phi}(\boldsymbol{\theta})\}^{-1/2}\Bigg[\tfrac{1}{2}\boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\,\boldsymbol{\phi}(\boldsymbol{\theta}) + \boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\,\mathbf{d}^*(\boldsymbol{\theta})\\ &\quad + \operatorname{tr}[\mathbf{B}(\boldsymbol{\theta})] + \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{u=1}^{p}\phi_i(\boldsymbol{\theta})\,\mathscr{I}^{ju}(S_{u\cdot ij} + \tfrac{1}{2}S_{uij})\Bigg] + o(n^{-1/2}),\\ \kappa_{2n}^*(V_n) &= 1 + o(n^{-1/2}),\\ \kappa_{3n}^*(V_n) &= n^{-1/2}\{\boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\,\boldsymbol{\phi}(\boldsymbol{\theta})\}^{-3/2}\Bigg[\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{u=1}^{p}\phi_i(\boldsymbol{\theta})\,\phi_j(\boldsymbol{\theta})\,\phi_u(\boldsymbol{\theta})\,\bar{S}_{iju}\\ &\quad + 6\boldsymbol{\phi}(\boldsymbol{\theta})'\,\mathscr{I}\,\mathbf{B}(\boldsymbol{\theta})\,\boldsymbol{\phi}(\boldsymbol{\theta})\Bigg] + o(n^{-1/2}).\end{aligned} \tag{2.15}$$

The fourth and higher order cumulants of V_n under $\boldsymbol{\theta}$ are of order $o(n^{-1/2})$. The proof can now be completed by using (2.7) and an Edgeworth expansion for the distribution of V_n under $\boldsymbol{\theta}$. ▮

The stochastic expansions used in the above proof are over a set with $P_{\boldsymbol{\theta}}$-probability $1-o(n^{-1/2})$. Note that in Theorem 2.1, $\mathscr{I}=\mathscr{I}(\boldsymbol{\theta})$ has been used as a Riemannian metric in the fashion of Amari [1] and Sen [10], among others. Since we have essentially used a quadratic norm (reducible to the conventional Euclidean distance by suitable linear transformation), this sophistication could have been avoided by an appeal to simpler Euclidean distances. However, the transformation leading to such a Euclidean norm may generally depend on the unknown $\boldsymbol{\theta}$, except in the particular case of some location-scale models. As such, we prefer to proceed in the manner outlined before. Along the line of Ghosh, Sen, and Mukerjee [5] we present here the notion of second-order Pitman closeness in a multiparameter setting, which will be helpful in exploiting the implications of this theorem, and we intend to discuss them as well.

With estimators $\mathbf{T}_n$ and $\mathbf{T}_n^*$ of $\boldsymbol{\theta}$, let

$$\begin{aligned}\alpha_{n1}(\boldsymbol{\theta})&=P_{\boldsymbol{\theta}}\{\mathbf{T}_n^*-\boldsymbol{\theta})'\,\mathscr{I}(\mathbf{T}_n^*-\boldsymbol{\theta})<(\mathbf{T}_n-\boldsymbol{\theta})'\,\mathscr{I}(\mathbf{T}_n-\boldsymbol{\theta})\}-\tfrac{1}{2};\\ \alpha_{n2}(\boldsymbol{\theta})&=n^{1/2}\alpha_{n1}(\boldsymbol{\theta}).\end{aligned}\tag{2.16}$$

Then, $\mathbf{T}_n^*$ will be superior to $\mathbf{T}_n$ with regard to second-order Pitman closeness if

(a) $\lim_{n\to\infty}\alpha_{n2}(\boldsymbol{\theta})\geqslant 0$, for each $\boldsymbol{\theta}$ for which a finite limit exists,

(b) $\lim_{n\to\infty}\alpha_{n1}(\boldsymbol{\theta})$ exists and $\lim_{n\to\infty}\alpha_{n1}(\boldsymbol{\theta})\geqslant 0$, for each $\boldsymbol{\theta}$ for which $\lim_{n\to\infty}\alpha_{n2}(\boldsymbol{\theta})$ does not exist finitely,

the inequality being strict for some $\boldsymbol{\theta}$ $(\in\boldsymbol{\Theta})$ either in (a) or (b).

With reference to a class, $\mathscr{C}$, of estimators of $\boldsymbol{\theta}$, $\mathbf{T}_n$ $(\in\mathscr{C})$ will be called *second-order Pitman-inadmissible* in $\mathscr{C}$ if there exists some other estimator $\mathbf{T}_n^*$ $(\in\mathscr{C})$, such that $\mathbf{T}_n$ and $\mathbf{T}_n^*$ are not one-to-one functionally related to each other, and $\mathbf{T}_n^*$ is superior to $\mathbf{T}_n$ with regard to the second-order Pitman closeness definition given above. Otherwise, $\mathbf{T}_n$ will be called *second-order Pitman admissible* in $\mathscr{C}$.

In the light of the above definitions and Theorem 2.1, we consider the following important class of multiparameter estimation problems.

Location Models. Here $f(\mathbf{x};\boldsymbol{\theta})$ is assumed to be of the form

$$f(\mathbf{x};\boldsymbol{\theta})=f^*(x^{(1)}-\theta_1,\ldots,x^{(p)}-\theta_p),\tag{2.17}$$

where $\mathbf{x}=(x^{(1)},\ldots,x^{(p)})'\in\mathscr{R}^p$, $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_p)'\in\boldsymbol{\Theta}\subset\mathscr{R}^p$, and the functional form of the density f^* is free from $\boldsymbol{\theta}$ and is assumed to be given. Then, it can be seen that $\mathcal{I} = ((\mathcal{I}_{ij}))$ and $S_{i \cdot ju}$, $S_{i \cdot j \cdot u}$, $\bar{S}_{iju}$ $(1 \leqslant i, j, u \leqslant p)$ are all free from $\boldsymbol{\theta}$. Hence, under the setup of Theorem 2.1, we have

$$\begin{aligned}
\boldsymbol{\phi}(\boldsymbol{\theta})' \mathcal{I} \boldsymbol{\phi}(\boldsymbol{\theta}) & \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \phi_i(\boldsymbol{\theta}) \mathcal{I}^{ju}(S_{u \cdot ij} + \tfrac{1}{2} S_{iju}) \\
& - \tfrac{1}{6} \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \phi_i(\boldsymbol{\theta}) \phi_j(\boldsymbol{\theta}) \phi_u(\boldsymbol{\theta}) \bar{S}_{iju} \\
& = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \lambda_{iju} \phi_i(\boldsymbol{\theta}) \phi_j(\boldsymbol{\theta}) \phi_u(\boldsymbol{\theta}) \\
& = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \bar{\lambda}_{iju} \phi_i(\boldsymbol{\theta}) \phi_j(\boldsymbol{\theta}) \phi_u(\boldsymbol{\theta}), \qquad (2.18)
\end{aligned}$$

where

$$\lambda_{iju} = \mathcal{I}^{ij} \sum_{v=1}^{p} \sum_{w=1}^{p} \mathcal{I}^{vw}(S_{w \cdot uv} + \tfrac{1}{2} S_{uvw}) - \tfrac{1}{6} \bar{S}_{iju} \qquad (2.19)$$

and

$$\bar{\lambda}_{iju} = \tfrac{1}{6}(\lambda_{iju} + \lambda_{iuj} + \lambda_{uij} + \lambda_{uji} + \lambda_{jiu} + \lambda_{jui}), \qquad 1 \leqslant i, j, u \leqslant p, \qquad (2.20)$$

are constants, free from $\boldsymbol{\theta}$. Having these simplifications at hand, we consider first the special case:

$$\bar{\lambda}_{iju} = 0 \qquad \forall\, i, j, u = 1, ..., p. \qquad (2.21)$$

Note that if the location model exhibits sufficient symmetry so as to ensure $S_{i \cdot j \cdot u} = S_{i \cdot ju} = S_{iju} = 0$ $\forall i, j, u = 1, ..., p$, then (2.21) follows from (2.19) and (2.20); we refer to the multivariate normal and multivariate Cauchy location models for some easy verification of these symmetry conditions. In this setup, we denote the MLE of $\boldsymbol{\theta}$ by $\hat{\boldsymbol{\theta}}$ and take

$$\mathbf{T}_n = \hat{\boldsymbol{\theta}}; \qquad \mathbf{T}_n^* = \hat{\boldsymbol{\theta}}\{1 - n^{-1}(1 + \hat{\boldsymbol{\theta}}' \mathcal{I} \hat{\boldsymbol{\theta}})^{-1} h\}, \qquad (2.22)$$

where h $(\neq 0)$ is a constant (free from n). Then applying Theorem 2.1 with

$$\begin{aligned}
-\mathbf{d}^*(\boldsymbol{\theta}) &= \boldsymbol{\phi}(\boldsymbol{\theta}) = \{1 + \boldsymbol{\theta}' \mathcal{I} \boldsymbol{\theta}\}^{-1} h \boldsymbol{\theta}, \\
\mathbf{B}(\boldsymbol{\theta}) &= \{1 + \boldsymbol{\theta}' \mathcal{I} \boldsymbol{\theta}\}^{-1} h \{\mathbf{I}_p - 2\{1 + \boldsymbol{\theta}' \mathcal{I} \boldsymbol{\theta}\}^{-1} \boldsymbol{\theta}\boldsymbol{\theta}' \mathcal{I}\},
\end{aligned} \qquad (2.23)$$

where $\mathbf{I}_p$ is the $p \times p$ identity matrix, and using (2.18) and (2.21), we arrive at the following result for $\boldsymbol{\theta} \neq \mathbf{0}$.

$$P_{\boldsymbol{\theta}}\{(\mathbf{T}_n^* - \boldsymbol{\theta})' \mathscr{I} (\mathbf{T}_n^* - \boldsymbol{\theta}) < (\mathbf{T}_n - \boldsymbol{\theta})' \mathscr{I} (\mathbf{T}_n - \boldsymbol{\theta})\}$$
$$= \tfrac{1}{2} + (2\pi n \boldsymbol{\theta}' \mathscr{I} \boldsymbol{\theta})^{-1/2} \operatorname{sgn}(h)[(p-1) - (h/2)(1 + \boldsymbol{\theta}' \mathscr{I} \boldsymbol{\theta})^{-1} \boldsymbol{\theta}' \mathscr{I} \boldsymbol{\theta}]$$
$$+ o(n^{-1/2}). \tag{2.24}$$

Also, for $\boldsymbol{\theta} = \mathbf{0}$, by (2.22),

$$P_{\mathbf{0}}\{\mathbf{T}_n^{*\prime} \mathscr{I} \mathbf{T}_n^* < \mathbf{T}_n' \mathscr{I} \mathbf{T}_n\} = P_{\mathbf{0}}\{[h^2 - 2hn(1 + \hat{\boldsymbol{\theta}}' \mathscr{I} \hat{\boldsymbol{\theta}})] \hat{\boldsymbol{\theta}}' \mathscr{I} \hat{\boldsymbol{\theta}} < 0\}, \tag{2.25}$$

so that

$$\lim_{n \to \infty} P_{\mathbf{0}}\{\mathbf{T}_n^{*\prime} \mathscr{I} \mathbf{T}_n^* < \mathbf{T}_n' \mathscr{I} \mathbf{T}_n\} = \{1 + \operatorname{sgn}(h)\}/2. \tag{2.26}$$

For $p > 2$, by (2.24) and (2.26), $\mathbf{T}_n^*$ will be superior to $\mathbf{T}_n = \hat{\boldsymbol{\theta}}$ with regard to second-order Pitman closeness if and only if

$$0 < h \leqslant 2(p-1). \tag{2.27}$$

It is easy to see that with h as in (2.27), $\mathbf{T}_n^*$ will dominate $\mathbf{T}_n$ not only at $\boldsymbol{\theta} = \mathbf{0}$ but also for every $\boldsymbol{\theta} \in \boldsymbol{\Theta} = \mathscr{R}^p$. Thus with location models satisfying (2.21) and for $p \geqslant 2$, it is possible to improve upon the MLE under the criterion of second-order Pitman closeness.

Continuing with the setup given by (2.21) and for $p \geqslant 2$, it may be of interest to compare various choices of h wihtin the range given by (2.27). This, in turn, calls for a study of second-order Pitman admissibility with reference to the class, $\mathscr{C}_1$, of estimators $\mathbf{T}_n^*$ given by (2.22) wherein h is chosen to be free from n, satisfying (2.27). To that effect, consider two estimators

$$\mathbf{T}_{nj} = \hat{\boldsymbol{\theta}}\{1 - n^{-1} h_j (1 + \hat{\boldsymbol{\theta}}' \mathscr{I} \hat{\boldsymbol{\theta}})^{-1}\}, \qquad j = 1, 2, \tag{2.28}$$

where $h_1 (\neq) h_2$ are constants, free from n, satisfying (2.27). Analogously to (2.24) and (2.26), it can be seen that for $\boldsymbol{\theta} \neq \mathbf{0}$,

$$P_{\boldsymbol{\theta}}\{(\mathbf{T}_{n2} - \boldsymbol{\theta})' \mathscr{I} (\mathbf{T}_{n2} - \boldsymbol{\theta}) < (\mathbf{T}_{n1} - \boldsymbol{\theta})' \mathscr{I} (\mathbf{T}_{n1} - \boldsymbol{\theta})\}$$
$$= \tfrac{1}{2} + (2\pi n \boldsymbol{\theta}' \mathscr{I} \boldsymbol{\theta})^{-1/2} \operatorname{sgn}(h_2 - h_1)$$
$$\times [(p-1) - \tfrac{1}{2}(h_1 + h_2)\{1 + \boldsymbol{\theta}' \mathscr{I} \boldsymbol{\theta}\}^{-1} \boldsymbol{\theta}' \mathscr{I} \boldsymbol{\theta}] + o(n^{-1/2}), \tag{2.29}$$

and at $\boldsymbol{\theta} = \mathbf{0}$,

$$\lim_{n \to \infty} P_{\mathbf{0}}\{\mathbf{T}_{n2}' \mathscr{I} \mathbf{T}_{n2} < \mathbf{T}_{n1}' \mathscr{I} \mathbf{T}_{n1}\} = [1 + \operatorname{sgn}(h_2 - h_1)]/2. \tag{2.30}$$

From (2.29) and (2.30), one can check easily that an estimator $\mathbf{T}_n^* = \hat{\boldsymbol{\theta}}\{1 - n^{-1}h\{1 + \hat{\boldsymbol{\theta}}'\mathcal{I}\hat{\boldsymbol{\theta}}\}^{-1}$ ($\in \mathcal{C}_1$) is second-order Pitman admissible in the class $\mathcal{C}_1$ if

$$p - 1 \leqslant h \leqslant 2(p-1), \qquad p \geqslant 2. \tag{2.31}$$

On the other hand, if $0 < h < p-1$, then $\mathbf{T}_n^*$ can be shown to be dominated by another estimator $\mathbf{T}_n^{**} = \hat{\boldsymbol{\theta}}\{1 - n^{-1}h^o(1 + \hat{\boldsymbol{\theta}}'\mathcal{I}\hat{\boldsymbol{\theta}})^{-1}\}$ ($\in \mathcal{C}_1$) whenever h^0, free from n, is so chosen that $h < h^o < 2p - 2 - h$, and, hence, $\mathbf{T}_n^*$ is second-order Pitman inadmissible in $\mathcal{C}_1$.

Next, we consider location models for which (2.21) may not hold, i.e.,

$$\bar{\lambda}_{iju} \neq 0 \qquad \text{for some} \quad i, j, u\ (=1, ..., p). \tag{2.32}$$

Although it is difficult to find natural examples of such location models, for the sake of completeness, we briefly discuss this case as well. With $\mathbf{T}_n = \hat{\boldsymbol{\theta}}$ and $\mathbf{T}_n^* = \hat{\boldsymbol{\theta}} - n^{-1}\boldsymbol{\xi}$, where $\boldsymbol{\xi} = (\xi_1, ..., \xi_p)'$, $\xi_1, ..., \xi_p$ are constants (free from n), and $\boldsymbol{\xi} \neq \mathbf{0}$, it is easily seen from Theorem 2.1 and (2.18) that for every $\boldsymbol{\theta} \in \mathcal{R}^p$,

$$\begin{aligned} &P_{\boldsymbol{\theta}}\{(\mathbf{T}_n^* - \boldsymbol{\theta})' \mathcal{I}(\mathbf{T}_n^* - \boldsymbol{\theta}) < (\mathbf{T}_n - \boldsymbol{\theta})' \mathcal{I}(\mathbf{T}_n - \boldsymbol{\theta})\} - \tfrac{1}{2} \\ &\quad = (2\pi n)^{-1/2} (\boldsymbol{\xi}'\mathcal{I}\boldsymbol{\xi})^{-3/2} \left[\sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \bar{\lambda}_{iju}\xi_i\xi_j\xi_u - \tfrac{1}{2}(\boldsymbol{\xi}'\mathcal{I}\boldsymbol{\xi})^2 \right] \\ &\qquad + o(n^{-1/2}). \end{aligned} \tag{2.33}$$

Hence, $\mathbf{T}_n^*$ is superior to $\mathbf{T}_n$ with regard to second-order Pitman closeness if $\boldsymbol{\xi}$ is so chosen that

$$\sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \bar{\lambda}_{iju}\xi_i\xi_j\xi_u > \tfrac{1}{2}(\bar{\lambda}'\mathcal{I}\bar{\lambda})^2. \tag{2.34}$$

Under (2.32), it can be shown that such a choice of $\boldsymbol{\xi}$ is always possible. For example, if $\bar{\lambda}_{111} \neq 0$ then $\boldsymbol{\xi}' = (\mathcal{I}_{11}^{-2}\bar{\lambda}_{111}, \mathbf{0}')$ satisfies (2.34). Estimators corresponding to different choices of $\boldsymbol{\xi}$ satisfying (2.34) can be compared, under the criterion of second-order Pitman closeness, in a straightforward manner using Theorem 2.1.

3. Stein-Rule Estimators

To motivate our general results, we look back into the location models satisfying (2.21) and note that $\mathbf{T}_n^*$, as given by (2.22), is quite similar in form to a conventional Stein-Rule estimator which in an asymptotic setup has captured a much wider domain of estimation problems including the location models as special cases; for such asymptotics, we may refer to Sen [11], where the details for SRMLE are given in a broader setup. For allied second-order efficiency properties in a conventional risk formulation we may refer to Ghosh [4]. As such, for such models too, one may as well be interested in examining the behavior of SR-estimators vis-à-vis the MLE $\mathbf{T}_n = \hat{\boldsymbol{\theta}}$. Hence, we consider an estimator of the form

$$\tilde{\mathbf{T}}_n = \hat{\boldsymbol{\theta}}\{1 - n^{-1}(\hat{\boldsymbol{\theta}}'\mathscr{I}\hat{\boldsymbol{\theta}})^{-1} h\}, \tag{3.1}$$

which is well defined, provided $P_{\boldsymbol{\theta}}\{\hat{\boldsymbol{\theta}} = \mathbf{0}\}$ $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$. Here also, the constant h is assumed to be free from n. In this formulation, (3.1) resembles the usual Stein-rule estimators, albeit the latter may be of more general form than in (3.1). Although the pair $(\tilde{\mathbf{T}}_n, \mathbf{T}_n)$ does not exactly satisfy the conditions of Theorem 2.1, virtually repeating the same line of attack, it can be shown that for $\boldsymbol{\theta} \neq \mathbf{0}$, the proof and conclusion of Theorem 2.1 remain valid with this pair. Towards this formulation, we denote the median of a central chi square distribution with *p degrees of freedom* (DF) by ψ_p. Then, it will be proved in the Appendix that

$$p - 1 < \psi_p < 2(p-1) \qquad \forall p \geqslant 2. \tag{3.2}$$

With the help of this inequality, analogously to (2.27), it can be shown that $\tilde{\mathbf{T}}_n$ is superior to $\mathbf{T}_n = \hat{\boldsymbol{\theta}}$ with regard to the second-order Pitman closeness if and only if

$$0 < h \leqslant 2(p-1). \tag{3.3}$$

Also, denoting by $\mathscr{C}_2$ the class of estimators $\tilde{\mathbf{T}}_n$ given by (3.1) with h, free from n, satisfying (3.3), it can be shown, analogously to (2.31), that $\tilde{\mathbf{T}}_n$ $(\in \mathscr{C}_2)$ is second-order Pitman admissible if and only if

$$p - 1 \leqslant h \leqslant \psi_p. \tag{3.4}$$

Therefore, the first-order Pitman closeness properties of SRMLEs studied earlier by a host of researchers can be extended to the second-order case by imposing additional regularity conditions, as needed to justify (3.1)–(3.3).

Let us next consider the multiparameter *scale models*. Here $f(\mathbf{x}; \boldsymbol{\theta})$ is of the form

$$f(\mathbf{x}; \boldsymbol{\theta}) = (\theta_1 \cdots \theta_p)^{-1} f^*(x^{(1)}/\theta_1, ..., x^{(p)}/\theta_p), \tag{3.5}$$

where $\mathbf{x}' = (x^{(1)}, ..., x^{(p)})$, $\boldsymbol{\theta}' = (\theta_1, ..., \theta_p)$, the θ_j are all positive, i.e., $\boldsymbol{\theta} > \mathbf{0}$, and the functional form of the density f^* is assumed to be known and free from the unknown parameter vector $\boldsymbol{\theta}$. Then

$$\begin{aligned} &\mathscr{I}_{ij} = (\theta_i \theta_j)^{-1} q_{ij}, \qquad \mathscr{I}^{ij} = \theta_i \theta_j q^{ij}, \qquad i, j = 1, ..., p; \\ &S_{i \cdot j \cdot u} = (\theta_i \theta_j \theta_u)^{-1} q_{i \cdot j \cdot u}, \qquad S_{i \cdot ju} = (\theta_i \theta_j \theta_u)^{-1} q_{i \cdot ju}; \\ &S_{iju} = (\theta_i \theta_j \theta_u)^{-1} q_{iju}, \qquad \bar{S}_{iju} = (\theta_i \theta_j \theta_u)^{-1} \bar{q}_{iju}, \qquad i, j, u = 1, ..., p, \end{aligned} \tag{3.6}$$

where q_{ij}, q^{ij}, $q_{i \cdot j \cdot u}$, $q_{i \cdot ju}$, q_{iju}, and $\bar{q}_{iju}$ are constants free from $\boldsymbol{\theta}$, and $\mathbf{Q} = ((q_{ij}))$ is p.d.

Let $\mathbf{T}_n = \hat{\boldsymbol{\theta}} = (\hat{\theta}_1, ..., \hat{\theta}_p)'$ be the MLE of $\boldsymbol{\theta}$, and as a rival estimator, we consider

$$\mathbf{T}_n^* = \hat{\boldsymbol{\theta}} - n^{-1}(\xi_1 \hat{\theta}_1, ..., \xi_p \hat{\theta}_p)', \tag{3.7}$$

where the ξ_j are constants free from n, and $\boldsymbol{\xi} = (\xi_1, ..., \xi_p)' \neq \mathbf{0}$. We apply Theorem 2.1 to the pair $(\mathbf{T}_n^*, \mathbf{T}_n)$ with

$$\boldsymbol{\phi}(\boldsymbol{\theta}) = -\mathbf{d}^*(\boldsymbol{\theta}) = (\xi_1 \theta_1, ..., \xi_p \theta_p)'; \qquad \mathbf{B}(\boldsymbol{\theta}) = \operatorname{diag}(\xi_1, ..., \xi_p) \tag{3.8}$$

and use (3.6) to obtain, for each $\boldsymbol{\theta}$ $(> \mathbf{0})$,

$$\begin{aligned} &P_{\boldsymbol{\theta}}\{(\mathbf{T}_n^* - \boldsymbol{\theta})' \mathscr{I} (\mathbf{T}_n^* - \boldsymbol{\theta}) < (\mathbf{T}_n - \boldsymbol{\theta})' \mathscr{I} (\mathbf{T}_n - \boldsymbol{\theta})\} \\ &\quad = \tfrac{1}{2} + (2\pi n)^{-1/2} (\boldsymbol{\xi}' \mathbf{Q} \boldsymbol{\xi})^{-3/2} \left[\sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \bar{\beta}_{iju} \xi_i \xi_j \xi_u - \tfrac{1}{2} (\boldsymbol{\xi}' \mathbf{Q} \boldsymbol{\xi})^2 \right] \\ &\qquad + o(n^{-1/2}), \end{aligned} \tag{3.9}$$

where

$$\bar{\beta}_{iju} = [\beta_{iju} + \beta_{iuj} + \beta_{uij} + \beta_{uji} + \beta_{jiu} + \beta_{jui}]/6, \qquad i, j, u = 1, ..., p, \tag{3.10}$$

with

$$\beta_{iju} = q_{ij}(1 - \delta_{ju} + l_u) - \tfrac{1}{6} \bar{q}_{iju}, \qquad i, j, u = 1, ..., p, \tag{3.11}$$

where δ_{ju} is the usual Kronecker delta, and

$$l_u = \sum_{w=1}^{p} \sum_{v=1}^{p} q^{vw} (q_{w \cdot uv} + \tfrac{1}{2} q_{uvw}), \qquad u = 1, ..., p. \tag{3.12}$$

By (3.9), $\mathbf{T}_n^*$ will be superior to $\mathbf{T}_n = \hat{\boldsymbol{\theta}}$ with regards to the second-order Pitman closeness criterion if and only if $\boldsymbol{\xi}$ is such that

$$\sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \bar{\beta}_{iju} \xi_i \xi_j \xi_u > \tfrac{1}{2}(\boldsymbol{\xi}' \mathbf{Q} \boldsymbol{\xi})^2. \tag{3.13}$$

It may be noted that such a choice of $\boldsymbol{\xi}$ is possible whenever

$$\bar{\beta}_{iju} \neq 0 \qquad \text{for some} \quad i, j, u = 1, ..., p; \tag{3.14}$$

and this condition holds in many models of interest. Example 1 cited below is an illustration. Under (3.14), estimators corresponding to different choices of $\boldsymbol{\xi}$ satisfying (3.13) can then be compared in a straightforward way using Theorem 2.1.

EXAMPLE 1. Let $f(\mathbf{x}; \boldsymbol{\theta})$ be given by a product of p *gamma* densities with known shape parameters r_i (>0) and unknown scale parameters θ_i (>0), $i = 1, ..., p$. Then we have the following simplifications:

$$\begin{aligned} &q_{ii} = r_i, \qquad q^{ii} = r_i^{-1}, \qquad q_{iii} = 4r_i; \\ &q_{i \cdot i \cdot i} = -q_{i \cdot ii} = \bar{q}_{iii} = 2r_i, \qquad i = 1, ..., p; \\ &q_{ij} = q^{ij} = 0 \qquad \forall i \neq j = 1, ..., p; \\ &q_{iju} = q_{i \cdot j \cdot u} = q_{i \cdot ju} = \bar{q}_{iju} = 0, \qquad \text{unless } i, j, u \text{ are all equal.} \end{aligned} \tag{3.15}$$

Hence, by (3.10) through (3.12), we obtain that

$$\begin{aligned} &\bar{\beta}_{iii} = -\tfrac{1}{3} r_i, \qquad i = 1, ..., p, \\ &\bar{\beta}_{iij} = \bar{\beta}_{iji} = \bar{\beta}_{jii} = \tfrac{1}{3} r_i \qquad \forall i \neq j = 1, ..., p; \\ &\bar{\beta}_{iju} = 0 \qquad \text{whenever } i, j, u \text{ are all distinct.} \end{aligned} \tag{3.16}$$

Therefore (3.14) holds, and it may be seen that (3.13) is satisfied if, in particular, $\boldsymbol{\xi} = (-\tfrac{1}{3} r_1^{-1}, \mathbf{0}')'$.

Remark. Motivated by Dasgupta [3], one may also wish to compare, in a scale model, $\mathbf{T}_n = \hat{\boldsymbol{\theta}}$ with

$$\tilde{\mathbf{T}}_n = \hat{\boldsymbol{\theta}} - n^{-1} \left[\prod_{i=1}^{p} \hat{\theta}_i \right]^{1/p} \boldsymbol{\xi}, \tag{3.17}$$

where $\boldsymbol{\xi} \neq \mathbf{0}$, and the coordinate elements ξ_i are constants, free from n. By (3.6) and Theorem 2.1, after some simplifications, it can be shown that for each $\boldsymbol{\theta}$ ($>\mathbf{0}$),

$$P_{\boldsymbol{\theta}}\{(\tilde{\mathbf{T}}_n-\boldsymbol{\theta})\,\mathscr{I}(\tilde{\mathbf{T}}_n-\boldsymbol{\theta})<(\mathbf{T}_n-\boldsymbol{\theta})\,\mathscr{I}(\mathbf{T}_n-\boldsymbol{\theta})\}-\tfrac{1}{2}$$
$$=(2\pi n)^{-1/2}\left[\sum_{i=1}^{p}\sum_{j=1}^{p}(\theta_i\theta_j)^{-1}\xi_i\xi_j q_{ij}\right]^{-3/2}$$
$$\times\left[\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{u=1}^{p}(\theta_i\theta_j\theta_u)^{-1}\gamma_{iju}\xi_i\xi_j\xi_u\right.$$
$$\left.-(1/2)\left[\prod_{i=1}^{p}\theta_i\right]^{1/p}\left\{\sum_{i=1}^{p}\sum_{j=1}^{p}(\theta_i\theta_j)^{-1}\xi_i\xi_j q_{ij}\right\}^2\right]+o(n^{-1/2}), \tag{3.18}$$

where $\gamma_{iju}=q_{ij}l_u-\frac{1}{6}\bar{q}_{iju}$ $(i, j, u=1, ..., p)$ with l_u given by (3.12). Hence if $\tilde{\mathbf{T}}_n$ is superior to $\mathbf{T}_n$ with regard to second-order Pitman closeness, then we must have $\forall\boldsymbol{\theta}>\mathbf{0}$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{u=1}^{p}(\theta_i\theta_j\theta_u)^{-1}\gamma_{iju}\xi_i\xi_j\xi_u$$
$$\geqslant\tfrac{1}{2}\left\{\prod_{i=1}^{p}\theta_i\right\}^{1/p}\left[\sum_{i=1}^{p}\sum_{j=1}^{p}(\theta_i\theta_j)^{-1}\xi_i\xi_j q_{ij}\right]^2. \tag{3.19}$$

The case of $p=1$ has been treated in [5], so we let $p\geqslant 2$. Then multiplying both sides of (3.19) by $\theta_1^3\{\prod_{i=1}^{p}(\theta_1/\theta_i)\}^{1/p}$ and keeping θ_1 (>0) fixed while allowing $\theta_2, ..., \theta_p\to\infty$, we get that $0\geqslant\frac{1}{2}(q_{11}\xi_1^2)^2$. Since $\mathbf{Q}$ is p.d., this yields $\xi_1=0$. Similarly, $\xi_2=\cdots=\xi_p=0$, i.e., $\boldsymbol{\xi}=\mathbf{0}$, which is a contradiction. Hence, for $p\geqslant 2$, no choice of $\boldsymbol{\xi}\neq\mathbf{0}$ can ensure the superiority of $\tilde{\mathbf{T}}_n$ to $\mathbf{T}_n=\hat{\boldsymbol{\theta}}$ under the criterion of second-order Pitman closeness. This may be contrasted with the findings in Dasgupta [3] who worked with quadratic type loss.

Finally, we consider the *location-scale* models in a univariate setup where $p=2$, $\boldsymbol{\theta}=(\theta_1, \theta_2)'$, $\theta_1\in\mathscr{R}$, $\theta_2>0$, and the density $f(x;\boldsymbol{\theta})$ is of the form

$$f(x;\boldsymbol{\theta})=\theta_2^{-1}f^*((x-\theta_1)/\theta_2), \qquad x\in\mathscr{R}, \tag{3.20}$$

where the form of f^* is free from $\boldsymbol{\theta}$. Then we have

$$\mathscr{I}_{ij}=\theta_2^{-2}q_{ij}, \qquad \mathscr{I}^{ij}=\theta_2^2 q^{ij} \qquad \forall i, j=1, 2;$$
$$S_{i\cdot j\cdot u}=\theta_2^{-3}q_{i\cdot j\cdot u}, \qquad S_{i\cdot ju}=\theta_2^{-3}q_{i\cdot ju}; \tag{3.21}$$
$$S_{iju}=\theta_2^{-3}q_{iju}, \qquad \bar{S}_{iju}=\theta_2^{-3}\bar{q}_{iju} \qquad \forall i, j, u=1, 2,$$

where the q_{ij}, q^{ij}, $q_{i\cdot j\cdot u}$, $q_{i\cdot ju}$, q_{iju}, and $\bar{q}_{iju}$ are constants free from $\boldsymbol{\theta}$ and $\mathbf{Q}=((q_{ij}))$ is p.d. We apply Theorem 2.1 to the pair: $(\mathbf{T}_n^*, \mathbf{T}_n)$, where $\mathbf{T}_n=\hat{\boldsymbol{\theta}}$ is the MLE and

$$\mathbf{T}_n^*=\hat{\boldsymbol{\theta}}-n^{-1}\hat{\theta}_2\boldsymbol{\xi}, \qquad \boldsymbol{\xi}=(\xi_1, \xi_2)'\neq\mathbf{0}, \tag{3.22}$$

the constants ξ_1, ξ_2 being free from n. Then analogously to (3.13), it can be shown that $\mathbf{T}_n^*$ is superior to $\mathbf{T}_n$ with regard to second-order Pitman closeness if and only if $\boldsymbol{\xi}$ is such that

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{u=1}^{2} \bar{\gamma}_{iju} \xi_i \xi_j \xi_u > \tfrac{1}{2}(\boldsymbol{\xi}' \mathbf{Q} \boldsymbol{\xi})^2, \tag{3.23}$$

where

$$\bar{\gamma}_{iju} = \{\gamma_{iju} + \gamma_{iuj} + \gamma_{uij} + \gamma_{uji} + \gamma_{jiu} + \gamma_{jui}\}/6; \qquad \gamma_{iju} = q_{ij} l_u - \tfrac{1}{6} \bar{q}_{iju}, \tag{3.24}$$

and l_u is defined as in (3.12) with $p = 2$. A choice of $\boldsymbol{\xi}$ satisfying (3.23) is possible whenever $\bar{\gamma}_{iju} \neq 0$, for some $i, j, u\ (=1, 2)$—a condition which holds in many models arising in practice. For example, under the univariate normal or Cauchy location-scale models, it can be shown that $\bar{\gamma}_{222} \neq 0$, so that (3.23) holds, in particular, if ξ_2 equals $\bar{\gamma}_{222}/q_{22}^2$ and ξ_1 is sufficiently close to 0. Extensions to multivariate location-scale models can be treated in a similar but admittedly more complex manner.

4. An Extension of Theorem 2.1

For our study of second-order Pitman closeness, instead of using the per observation Fisher information matrix $\mathscr{I}(\boldsymbol{\theta})$ $(\equiv \mathscr{I})$ as a Riemannian metric, we may as well use a $p \times p$ matrix $\mathscr{M}(\boldsymbol{\theta})$ $(\equiv \mathscr{M})$ which is p.d. for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Then, under the setup of Theorem 2.1 and with the same notational system, one can show that for each $\boldsymbol{\theta}$ with $\boldsymbol{\phi}(\boldsymbol{\theta}) \neq \mathbf{0}$,

$$\begin{aligned}
&P_{\boldsymbol{\theta}}\{(\mathbf{T}_n^* - \boldsymbol{\theta})' \mathscr{M} (\mathbf{T}_n^* - \boldsymbol{\theta}) < (\mathbf{T}_n - \boldsymbol{\theta})' \mathscr{M} (\mathbf{T}_n - \boldsymbol{\theta})\} \\
&\quad = \tfrac{1}{2} + (2\pi n)^{-1/2} \{\tilde{\boldsymbol{\phi}}(\boldsymbol{\theta})' \mathscr{I} \tilde{\boldsymbol{\phi}}(\boldsymbol{\theta})\}^{-3/2} \Big[\tfrac{1}{2} \{\tilde{\boldsymbol{\phi}}(\boldsymbol{\theta})' \mathscr{I} \tilde{\boldsymbol{\phi}}(\boldsymbol{\theta})\} \{\boldsymbol{\phi}(\boldsymbol{\theta})' \mathscr{M} \boldsymbol{\phi}(\boldsymbol{\theta})\} \\
&\qquad + \tilde{\boldsymbol{\phi}}(\boldsymbol{\theta})' \mathscr{I} \boldsymbol{\phi}(\boldsymbol{\theta}) \{ \boldsymbol{\phi}(\boldsymbol{\theta})' \mathscr{M} \mathbf{d}^*(\boldsymbol{\theta}) + \operatorname{tr}(\mathscr{I}^{-1} \mathscr{M} \mathbf{B}(\boldsymbol{\theta})) \\
&\qquad + \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \tilde{\boldsymbol{\phi}}_i(\boldsymbol{\theta}) \mathscr{I}^{ju} (S_{u \cdot ij} + (1/2) S_{iju}) \} \\
&\qquad - \tfrac{1}{6} \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{u=1}^{p} \tilde{\phi}_i(\boldsymbol{\theta}) \tilde{\phi}_j(\boldsymbol{\theta}) \tilde{\phi}_u(\boldsymbol{\theta}) \bar{S}_{iju} - \tilde{\boldsymbol{\phi}}(\boldsymbol{\theta})' \mathscr{M} \mathbf{B}(\boldsymbol{\theta}) \tilde{\boldsymbol{\phi}}(\boldsymbol{\theta}) \Big] \\
&\qquad + o(n^{-1/2}),
\end{aligned} \tag{4.1}$$

where $\tilde{\boldsymbol{\phi}}(\boldsymbol{\theta}) = (\tilde{\phi}_1(\boldsymbol{\theta}), \ldots, \tilde{\phi}_p(\boldsymbol{\theta}))' = \mathscr{I}^{-1} \mathscr{M} \boldsymbol{\phi}(\boldsymbol{\theta})$.

As before, under the formulated criterion of second-order Pitman closeness and for a given $\mathcal{M}$, this generalized version of Theorem 2.1 can be useful in comparing estimators as well as in finding estimators superior to the MLE $\hat{\boldsymbol{\theta}}$, which is, generally, only first-order efficient with reference to specific models. Thus, under a symmetric location model (e.g., the multivariate normal or Cauchy location models), where $\mathcal{I}$ is free from the location parameter $\boldsymbol{\theta}$ and

$$S_{i \cdot j \cdot u}(\boldsymbol{\theta}) \equiv S_{i \cdot ju}(\boldsymbol{\theta}) \equiv S_{iju}(\boldsymbol{\theta}) \equiv 0 \qquad \forall i, j, u = 1, ..., p, \tag{4.2}$$

if $\mathcal{M}$ is taken as the $p \times p$ identity matrix I, then proceeding as in the derivation of (2.24) and (2.26), it can be shown from the above generalized version that $\mathbf{T}_n^* = \hat{\boldsymbol{\theta}} - n^{-1} h\{1 + \hat{\boldsymbol{\theta}}' \mathcal{I} \mathcal{I} \hat{\boldsymbol{\theta}}\}^{-1} \mathcal{I} \hat{\boldsymbol{\theta}}$ will be superior to $\mathbf{T}_n = \hat{\boldsymbol{\theta}}$ with regard to second-order Pitman closeness provided $p \geqslant 2$ and the constant h, free from n, satisfies: $0 < h \leqslant 2(p-1)$. Of course, with reference to the first order Pitman closeness, this dominance holds even for a bigger class of SR estimators; we may refer to Sen [13] for some details.

Appendix: Bounds for the Chi Square Median

Our main interest centers around the inequalities in (3.2), and we establish these bounds by invoking some simple properties of the chi square density. First, by an appeal to the *mean-median-mode* (MMM)-inequality for (noncentral) chi square distributions (viz., Sen [12] it follows in particular that

$$\psi_p < E(\chi_p^2) = p \qquad \text{for every} \quad p \geqslant 2. \tag{5.1}$$

Let $G_p(x) = P\{\chi_p^2 \leqslant x\} = \int_0^x g_p(y)\, dy$, $x \geqslant 0$, where $g_p(\cdot)$ stands for the density function of χ_p^2. Then, by partial integration, we have

$$G_p(x) = G_{p+2}(x) + 2g_{p+2}(x) \qquad \text{for every} \quad p \geqslant 1, x \geqslant 0. \tag{5.2}$$

Therefore,

$$\begin{aligned} G_{p+2}(p+1) &= [G_{p+2}(p+1) - G_{p+2}(p-1)] + G_p(p-1) - 2g_{p+2}(p-1) \\ &= G_p(p-1) + g_{p+2}(p-1) \\ &\quad \times \left\{ \int_{p-1}^{p+1} [g_{p+2}(x)/g_{p+2}(p-1) - 1]\, dx \right\}. \end{aligned} \tag{5.3}$$

Note that $g_{p+2}(\cdot)$, for $p \geqslant 1$, is *strongly unimodal* with *mode* equal to p, and hence, writing $x = p + u$, $u \in (-1, +1)$, we have

$$g_{p+2}(p+u)/g_{p+2}(p-u) = \exp\left\{-u - \frac{p}{2}\ln((p-u)/(p+u))\right.$$
$$= \exp\{A_p(u)\}, \qquad \text{say.} \tag{5.4}$$

Then $A_p(0) = 0$, and for every $u \in (-1, +1)$, $(d/du)\, A_p(u) = u^2(p^2 - u^2)^{-1} \geqslant 0$. Therefore, $A_p(u)$ is positive for every $u \in (0, 1)$, and hence, for every $x \in (p-1, p+1)$,

$$g_{p+2}(x)/g_{p+2}(p-1) = g_{p+2}(p+(x-p))/g_{p+2}(p-1) > 1, \tag{5.5}$$

so that from (5.3) and (5.5), we have

$$G_{p+2}(p+1) > G_p(p-1) \qquad \text{for every} \quad p \geqslant 1. \tag{5.6}$$

On the other hand, by virtue of the reproductive property of χ_p^2 (in terms of independent χ_1^2 variables), we have by the central limit theorem,

$$\lim_{p \to \infty} G_{p+2}(p+1) = \tfrac{1}{2}. \tag{5.7}$$

Therefore, the monotonicity in (5.6) and the limit in (5.7) imply that

$$G_p(p-1) \leqslant \tfrac{1}{2} \qquad \text{for every} \quad p \geqslant 1. \tag{5.8}$$

Note that for $p = 1$, $G_p(p-1) = G_1(0) = 0$, so that we have a strict inequality in (5.8). Since for every $p \geqslant 2$, $p \leqslant 2(p-1)$ and $G_p(\cdot)$ is absolutely continuous, we conclude from (5.8) and (5.1) that (3.2) holds. This completes the proof of (3.2). In passing, we may note that if $\psi_p(\lambda)$ stands for the median of a noncentral chi square distribution with p DF and noncentrality parameter $\lambda \geqslant 0$, then by the subadditive property of $\psi_p(\lambda)$ (viz., Sen [12]), we have

$$\psi_p(\lambda) \leqslant \psi_p + \lambda \qquad \forall \lambda \geqslant 0, p \geqslant 1. \tag{5.9}$$

Thus, the upper bound in (3.2) extends directly to the noncentral case. However, it remains open to resolve whether

$$\psi_p(\lambda) \text{ is } \geqslant p - 1 + \lambda \qquad \forall \lambda \geqslant 0, p \geqslant 1; \tag{5.10}$$

the bounds given in Sen [12] may not suffice for the last inequality.

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