On NBUE property of one component system supported by an inactive standby and a repair facility

A.D. Dharmadhikari[a,*], J.K. Ghosh[b,c], M.M. Kuber[a]

[a] Department of Statistics, University of Poona, Pune 411 007, India
[b] Stat-Math Division, Indian Statistical Institute, Calcutta 700 035, India
[c] Department of Statistics, Purdue University, West Lafayette, IN 47907, USA

Received 12 October 1994; accepted 1 November 1996

Abstract

The paper deals with the aging property of a one-component system supported by an identical, inactive standby and a perfect repair facility. We assume that all the lifetimes and repair times induced by the operating and under-repair components are mutually independent and repair times are arbitrary. It is shown that the lifetime of a system that begins with one (both) operative component (s) having NWUE (NBUE) lifetimes is NWUE (NBUE).

Keywords: Aging classes; NBUE; NWUE; Repairable systems; Inactive standby; Perfect repair facility

1. Introduction

Barlow and Proschan (1976) studied the aging properties of the coherent systems via the aging properties of components under the assumption that the components may have repair facilities. They grouped the components of the system, say ϕ into two mutually exclusive groups. Components subjected to repair upon failure were put in one group, say, E_1 and rest of them in other group, say E_2. The main result of Barlow and Proschan is, if

(i) lifetimes of the components in group E_1 are exponential,
(ii) lifetimes of the components in group E_2 are IFR,
(iii) repair times of components in group E_1 are DFR, repairs restore the components at age zero,
(iv) lifetimes and repair times of all the components are mutually independent.

then the lifetime distribution of the coherent system is NBU.

One component system supported by one active standby and a repair facility is a special case of the above model when ϕ is a 'parallel' system, group E_1 has two components and group E_2 is empty.

Hence, using the above result, one understands that when a component having exponential life is supported by an active standby having exponential life and a DFR repair facility the lifetime of the system would be NBU.

A natural question is, can one depart from 'exponential' life? Barlow and Proschan felt so. They conjectured that the coherent system ϕ would have NBU life even when all the components in group E_1 have IFR lifetimes and rest of the assumptions are unchanged.

However, in 1977 Miller gave a counter example and showed that it is not possible to depart from component having 'exponential' life to component having arbitrary IFR life.

In this paper we study the effect of inactive standby and a repair facility. Precisely, we study the aging properties of a one component system supported by an inactive standby and a repair facility, (Barlow and Proschan, 1975, Model 3, p. 203) with two initial conditions. In the first case, we start the system with a new component and a new standby. In the second case, we start observing the process when the first component has failed so that one component (the standby) starts functioning and repair of the other (the failed component) has just began. Let F and G be the common distribution functions of the life-times and repair times of the components, respectively. We prove that in the first case if F is NBUE, the system's life is NBUE (no matter what is G) and in the second case, if F is NWUE, the system's life is NWUE. More can be said when F and G are both exponentials.

In the following discussion, $\{X_n, n \geqslant 0\}$ and $\{Y_n, n \geqslant 1\}$ indicate successive life-times and repair times of the components, respectively. We assume that $\{X_n, n \geqslant 0\}$ and $\{Y_n, n \geqslant 1\}$ are mutually independent sequences of non-negative independent random variables (r.v.'s). Further, let X_n's have common distribution function F and survival function $\bar{F}$, Y_n's have common distribution function G and survival function $\bar{G}$. Let $F(0) = 0 = G(0)$ and

$$\int_0^\infty \bar{F}(x)\,\mathrm{d}x = \mu_F < \infty, \qquad \int_0^\infty \bar{G}(x)\,\mathrm{d}x = \mu_G < \infty. \tag{1.1}$$

Let Z_i, $i \geqslant 1$ denote the independent and identically distributed r.v.'s whose common distribution is the conditional distribution of X_i given that $X_i > Y_i$. Let H and $\bar{H}$ denote the common distribution function and survival function of the r.v.'s Z_i, respectively. Then

$$\bar{H}(t) = p^{-1} \int_t^\infty G(x)\,\mathrm{d}F(x) \quad \text{where } p = \int_0^\infty G(x)\,\mathrm{d}F(x). \tag{1.2}$$

Define

$$K = \inf\{i \geq 1: X_i < Y_i\}, \quad T_1 = \sum_{i=1}^{k} X_i, \tag{1.3}$$

$$T_0 = X_0 + T_1, \qquad S_m = \sum_{i=1}^{m} Z_i, \quad m \geq 1.$$

Note that T_1 is the time of first failure of the above model given that, initially, one component is operative, other is under repair and age of the operative component as well as the elapsed repair time are zero. Similarly, T_0 is the time of first system failure when, initially, both the components are operative and ages of the components are zero. Then

$$\bar{R}_1(t) = P[T_1 > t] = \bar{F}(t) + \sum_{m=1}^{\infty} p^m \int_0^t \bar{F}(t-u)\,\mathrm{d}H^{(m)}(u), \tag{1.4}$$

$$\bar{R}_0(t) = P[T_0 > t] = \bar{F}(t) + \int_0^t \bar{R}_1(t-u)\,\mathrm{d}F(u), \tag{1.5}$$

where $H^{(m)}(\cdot)$ is the m-fold convolution of H with itself.

Since these are integral equations it is difficult to calculate exact reliability at time t even if F and G are completely known. Barlow and Proschan (1975) and Birolini (1985) have obtained the Laplace transforms of reliability and availability while Bhattacharjee and Kandar (1983) have given bounds for the steady-state availability of this model. As stated earlier, we study aging properties of R_0 and R_1 in terms of aging properties of F. This would be helpful to provide bounds for the reliability at time t.

In Section 2 we state and prove the results we require and the results dealing with the system under study are given in Section 3. In a special case, when component lifetimes are exponential, we derive explicit expressions for $R_i(t)$ leading to stronger aging properties.

2. Technical preliminaries

The necessary results are summarized in Lemmas 1, 2 and Theorem 1.

Lemma 1. *Let $\bar{F}$, $\bar{H}$ and p be as given in* (1.1) *and* (1.2), *respectively; then*

$$\bar{F}(t) \leq \bar{H}(t) \leq p^{-1}\bar{F}(t) \quad \textit{for all } t \geq 0.$$

Proof. Note that the Radon Nikodym derivative of $H(t)$ with respect to $F(t)$ is $p^{-1}G(t)$ which is increasing in t. So the pair (H, F) possesses the generalized monotone-likelihood ratio property. Hence, we have $\bar{F}(t) \leqslant \bar{H}(t)$ for all $t \geqslant 0$ (see Lehmann, 1986, pp. 78, 85).

Also, $p\bar{H}(t) = \int_t^\infty G(x)\,\mathrm{d}F(x) \leqslant \int_t^\infty \mathrm{d}F(x) = \bar{F}(t)$ for all $t \geqslant 0$. □

Lemma 2. *Let $\bar{F}$, μ_F, p and $\bar{R}_1$ be as in* (1.1), (1.2) *and* (1.4), *respectively; then*

$$\mu_{R_1} = \int_0^\infty \bar{R}_1(x)\,\mathrm{d}x = \frac{\mu_F}{1-p}.$$

Proof. Using Wald's equation (cf. Barlow and Proschan, 1975, p. 169) in the definition of T_1 as given in (1.3) we have

$$\mu_{R_1} = E(K)\,E(X) = \frac{\mu_F}{1-p}. \qquad \square$$

Theorem 1 given below provides a sufficient condition for $X_1 + X_2$ to be NBUE when X_1, X_2 are non-negative, independent r.v.'s and X_1 is NBUE.

Theorem 1. *Let F_1 and F_2 be two distribution functions with finite means μ_1 and μ_2, respectively, and $F_1(0^-) = F_2(0^-) = 0$. Let*

$$F_3(t) = \int_0^t F_1(t-u)\,\mathrm{d}F_2(u),$$

and $\bar{F}_3 = 1 - F_3$. If $\bar{F}_1$ is NBUE *and*

$$\frac{1}{\mu_2}\int_t^\infty \bar{F}_2(x)\,\mathrm{d}x \leqslant \bar{F}_3(t) \quad \textit{for all } t \geqslant 0, \tag{2.1}$$

then $\bar{F}_3$ is NBUE.

The L.H.S. of (2.1) is called 'equilibrium' or 'first derived distribution' corresponding to F_2.

Proof. If μ_1 or μ_2 or both are zero then the result follows immediately.

For $\mu_1 > 0$, $\mu_2 > 0$, let

$$\int_0^\infty \bar{F}_3(t+x)\,\mathrm{d}x = \mathrm{I} + \mathrm{II}, \tag{2.2}$$

where

$$\mathrm{I} = \int_0^\infty \int_0^t \bar{F}_1(t+x-u)\,\mathrm{d}F_2(u)\,\mathrm{d}x \quad \text{and} \quad \mathrm{II} = \int_0^\infty \int_t^\infty \bar{F}_1(t+x-u)\,\mathrm{d}F_2(u)\,\mathrm{d}x.$$

Now

$$\mathrm{I} = \int_0^t \left[\int_0^\infty \bar{F}_1(t - x - u)\,\mathrm{d}x \right] \mathrm{d}F_2(u) \leqslant \mu_1(\bar{F}_3(t) - \bar{F}_2(t)) \tag{2.3}$$

and

$$\mathrm{II} = \int_t^\infty \int_0^\infty \bar{F}_1(t + x - u)\,\mathrm{d}x\,\mathrm{d}F_2(u) = \int_t^\infty (u - t + \mu_1)\,\mathrm{d}F_2(u)$$

$$= \mu_1 \bar{F}_2(t) + \int_0^\infty w\,\mathrm{d}F_2(t + w)$$

$$= \mu_1 \bar{F}_3(t) + \int_t^\infty \bar{F}_2(w)\,\mathrm{d}w.$$

Hence, using (2.1) and (2.3) in (2.2), we get

$$\int_t^\infty \bar{F}_3(x)\,\mathrm{d}x \leqslant \mu_1 \bar{F}_3(t) + \int_t^\infty \bar{F}_2(x)\,\mathrm{d}x \leqslant (\mu_1 + \mu_2)\bar{F}_3(t) \quad \text{for all } t \geqslant 0.$$

Therefore, F_3 is NBUE. □

Remark. If F_1 is exponential then the condition (2.1) given in the above theorem is also necessary for F_3 to be NBUE.

3. Aging properties of the system

Theorem 2. *If F is NWUE, then the d.f. R_1 of the time of first system failure, given that initially one component is operative, is NWUE.*

Proof. Taking integral on both sides of (1.4) and using that F is NWUE we get

$$\int_t^\infty \bar{R}_1(x)\,\mathrm{d}x$$

$$\geqslant \mu_F \left[\bar{F}(t) + \sum_{m=1}^\infty p^m \int_0^t \bar{F}(t - u)\,\mathrm{d}H^{(m)}(u) \right] + \sum_{m=1}^\infty p^m \mu_F \int_t^\infty \mathrm{d}H^{(m)}(u).$$

Hence, again using (1.4),

$$\int_t^\infty \bar{R}_1(x)\,\mathrm{d}x \geqslant \mu_F \left[\bar{R}_1(t) + \sum_{m=1}^\infty p^m \bar{H}^{(m)}(t) \right].$$

Also,

$$\begin{aligned}(1-p)\sum_{m=1}^{\infty} p^m H^{(m)}(t) &= p\bar{H}(t) + \sum_{m=1}^{\infty} p^{m+1}\left[\bar{H}^{(m-1)}(t) - \bar{H}^{(m)}(t)\right] \\ &= p\bar{H}(t) + \sum_{m=1}^{\infty} p^{m+1}\int_0^t \bar{H}(t-u)\,\mathrm{d}H^{(m)}(u) \\ &= p\left[\bar{H}(t) + \sum_{m=1}^{\infty} p^{m}\int_0^t \bar{H}(t-u)\,\mathrm{d}H^{(m)}(u)\right] \\ &\geqslant p\left[\bar{F}(t) + \sum_{m=1}^{\infty} p^{m}\int_0^t \bar{F}(t-u)\,\mathrm{d}H^{(m)}(u)\right] \quad \text{(from Lemma 1)} \\ &= p\bar{R}_1(t).\end{aligned}$$

Thus,

$$\begin{aligned}\int_t^{\infty} \bar{R}_1(x)\,\mathrm{d}x &\geqslant \mu_F \bar{R}_1(t)\left[1 + \frac{p}{1-p}\right] \\ &= \frac{\mu_F}{1-p}\,\bar{R}_1(t) = \mu_{R_1}\bar{R}_1(t) \quad \text{(from Lemma 2)}. \qquad \square\end{aligned}$$

Theorem 3. *If F is NBUE, then the d.f. R_0 of the time of first system failure, when initially both components are new and operative, is NBU.*

Proof. As stated in Section 1,

$$\bar{R}_0(t) = P[T_0 > t] = P[X_0 + T_1 > t],$$

where X_0 and T_1 are independent non-negative r.v.'s with survival functions $\bar{F}$ and $\bar{R}_1$ and F is NBUE. Hence, to use Theorem 1 it is enough to prove that

$$\int_t^{\infty} \bar{R}_1(x)\,\mathrm{d}x \leqslant \mu_{R_1}\bar{R}_0(t). \tag{3.1}$$

Since F is NBUE, proceeding as in Theorem 2, we have

$$\int_t^{\infty} \bar{R}_1(x)\,\mathrm{d}x \leqslant \mu_F\left[\bar{F}(t) + \sum_{m=1}^{\infty} p^m P[X_1 + S_m > t]\right],$$

so, in view of Lemma 2, it is enough to prove that

$$(1-p)\left[\bar{F}(t) + \sum_{m=1}^{\infty} p^m P[X_1 + S_m > t]\right] \leqslant \bar{R}_0(t) \tag{3.2}$$

Now

$$
\begin{aligned}
(1-p)\sum_{m=1}^{\infty} p^m P[X_1+S_m>t] &= pP[X_1+S_1>t] \\
&\quad + \sum_{m=1}^{\infty} p^{m+1}\Big[P[X_1+S_{m+1}>t] \\
&\quad - P[X_1+S_m>t]\Big] \\
&= p[P[X_1>t]+P[X_1\leqslant t<X_1+S_1]] \\
&\quad + \sum_{m=1}^{\infty} p^{m+1}\left[\int_0^t \bar{H}(t-x)\,\mathrm{d}(H^{(m)} \circledcirc F)(x)\right],
\end{aligned}
$$

where $\circledcirc$ denotes, the convolution,

$$
\begin{aligned}
&= p\bar{F}(t) - p\int_0^t \bar{H}(t-x)\,\mathrm{d}F(x) \\
&\quad + \sum_{m=1}^{\infty} p^{m+1}\int_0^t \bar{H}(t-x)\,\mathrm{d}(H^{(m)} \circledcirc F)(x). \qquad (3.3)
\end{aligned}
$$

Using (3.3), L.H.S. of (3.2)

$$
= \bar{F}(t) + p\int_0^t \bar{H}(t-x)\,\mathrm{d}F(x) + \sum_{m=1}^{\infty} p^{m+1}\int_0^t \bar{H}(t-x)\,\mathrm{d}(H^{(m)} \circledcirc F)(x). \qquad (3.4)
$$

Now,

$$
\begin{aligned}
R_0(t) &= \bar{F}(t) + \int_0^t \bar{R}_1(t-u)\,\mathrm{d}F(u) \\
&= \bar{F}^{(2)}(t) + \sum_{m=1}^{\infty} p^m \int_0^t \bar{F}(t-x)\,\mathrm{d}(H^{(m)} \circledcirc F)(x). \qquad (3.5)
\end{aligned}
$$

From (3.4) and (3.5) we see that (3.2) is satisfied if

$$
\bar{F}(t) + p\int_0^t \bar{H}(t-x)\,\mathrm{d}F(x) \leqslant \bar{F}^{(2)}(t), \quad \text{for all } t>0
$$

and $p\bar{H}(t-x) \leqslant \bar{F}(t-x)$ for all $t \geqslant 0$, and $0 \leqslant x \leqslant t$.

However, these inequalities are true by Lemma 1. □

Remark. It can be seen that in the special case when both F and G are exponentials with parameters λ and μ, respectively, the reliability functions are given by

$$
R_0(t) = \frac{b}{b-a}\exp(-at) - \frac{a}{b-a}\exp(-bt), \quad t>0
$$

and

$$\bar{R}_1(t) = \frac{b-\lambda}{b-a}\exp(-at) + \frac{\lambda-a}{b-a}\exp(-bt), \quad t > 0.$$

where $b(a) = \frac{1}{2}\{(2\lambda+\mu)\,{}^{+}_{(-)}\,((2\lambda+\mu)^2 - 4\lambda)^{1/2}\}$

and $0 < a < \lambda < b$. Hence, we see that $\bar{R}_0$ is a survival function of the convolution of two independent exponential r.v.'s with parameters b and a while $\bar{R}_1$ is the survival function of proper mixture of same two r.v.'s with mixing probabilities as given in the expression. Thus, R_0 is IFR and R_1 is DFR (see Barlow and Proschan, 1975).

Acknowledgements

This research was done under a joint research programme between Indian Statistical Institute and University of Pune. Research of the first author was partially supported by CSIR, New Delhi, India. The authors are grateful to the referee for valuable suggestions.

References

Barlow, R.E., Proschan, F., 1975. Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, New York.

Barlow, R.E., Proschan, F., 1976. Theory of maintained systems: distribution of time to failure. Maths. Opres. Res. 1(1), 32–42.

Bhattacharjee, M.C., Kandar, R., 1983. Simple bounds on availability in a model with unknown life and repair distributions. J. Statist. Plann. Inference 8, 129–142.

Birolini, A., 1985. On the Use of Stochastic Processes in Modeling Reliability Problems. Lecture Notes in Economics and Mathematical System, Vol. 252, Springer, New York.

Lehmann, E.L., 1986. Testing Statistical Hypotheses. Wiley, New York.

Miller, D.R., 1977. A continuity theorem and some counter examples for the theory of maintained systems. Stochastic Process. Appl. 5, 307–314.