Some Inequalities for Commutators and an Application to Spectral Variation. II

RAJENDRA BHATIA[a], FUAD KITTANEH[b] and REN-CANG LI[c,*,†]

[a]*Indian Statistical Institute, New Delhi 110 016, India;* [b]*Department of Mathematics, University of Jordan, Amman, Jordan;* [c]*Mathematical Science Section, Oak Ridge National Laboratory, P. O. Box 2008, Bldg 6012, Oak Ridge, TN 37831-6367*

Communicated by T. Ando

Dedicated to the memory of R. C. Thompson

(Received 28 June 1996; Revised 18 December 1996)

Inequalities that compare unitarily invariant norms of $A-B$ and those of $A\Gamma-\Gamma B$ and $\Gamma^{-1}A-B\Gamma^{-1}$ are obtained, where both A and B are either Hermitian or unitary or normal operators and Γ is a positive definite operator in a complex separable Hilbert space. These inequalities are then applied to derive bounds for spectral variation of diagonalisable matrices. Our new bounds improve substantially previously published bounds.

Keywords: Linear operator; commutator; unitarily invariant norm; diagonalisable matrix; spectral variation

AMS Subject Classification: 15A42, 15A60, 65F99

* The third author gratefully acknowledges the support of a Householder Fellowship in Scientific Computing at Oak Ridge National Laboratory, supported by the Applied Mathematical Sciences Research Program, Office of Energy Research, United States Department of Energy contract DE-AC05-96OR22464 with Lockheed Martin Energy Research Corp.

† Present address: Department of Mathematics, University of Kentucky, Lexington, KY 40506.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the space of bounded linear operators on $\mathcal{H}$. This paper concerns bounding various norms of $A-B$ by those of $A\Gamma-\Gamma B$ and of $\Gamma^{-1}A-B\Gamma^{-1}$, where A, B, and Γ are in $\mathcal{B}(\mathcal{H})$. Often an expression like $A\Gamma-\Gamma B$ is called a *generalised commutator*, or simply a *commutator*.

Apart from the usual operator norm $\|\cdot\|$ on $\mathcal{B}(\mathcal{H})$, we are interested in other *symmetric (unitarily invariant) norms*. We denote such norms by $|||\cdot|||$. Each such norm is defined on a *norm ideal* contained in $\mathcal{B}(\mathcal{H})$. When we use the symbol $|||T|||$ we tacitly assume that T is in the appropriate ideal. Specially interesting are the *Schatten p-norms*

$$\|T\|_p \stackrel{def}{=} \sqrt[p]{\sum_j s_j^p(T)} \quad \text{for } 1 \leq p \leq \infty,$$

where $s_1(T) \geq s_2(T) \geq \cdots$ are the singular values of the compact operator T. In particular, $\|T\|_\infty$ coincides with the operator norm $\|T\|$. For $p=2$, $\|\cdot\|_2$, called the *Hilbert-Schmidt norm*, will receive special attention. Another particularly useful class of norms is the *Ky Fan* k-norms:

$$\|T\|_{(k)} \stackrel{def}{=} \sum_{j=1}^{k} s_j(T) \quad \text{for } k = 1, 2, \ldots.$$

For basic facts about unitarily invariant norms used below see [3], [9] or [14].

2. NORM INEQUALITIES INVOLVING COMMUTATORS

There is a wide body of striking and useful inequalities that compare norms of $A\Gamma-\Gamma B$ and those of $A-B$; see [5] for some references. The theorems below are of this type.

THEOREM 2.1 *Let Γ be a positive definite operator and let A and B be Hermitian operators. Then*

$$|||A - B|||^2 \leq |||A\Gamma - \Gamma B|||\ |||\Gamma^{-1}A - B\Gamma^{-1}|||, \tag{2.1}$$

for every unitarily invariant norm.

THEOREM 2.2 *Let Γ be a positive definite operator and let A and B be unitary operators. Then*

$$|||A - B|||^2 \leq |||A\Gamma - \Gamma B|||\ |||\Gamma^{-1}A - B\Gamma^{-1}|||, \tag{2.2}$$

for every unitarily invariant norm.

THEOREM 2.3 *Let Γ be a positive definite operator and let A and B be normal operators. Then*

$$\|A - B\|_2^2 \leq \|A\Gamma - \Gamma B\|_2 \|\Gamma^{-1}A - B\Gamma^{-1}\|_2. \tag{2.3}$$

Later in §4 we shall present a counterexample showing that inequality (2.3) may fail for other Schatten norms.

Our proof will use three inequalities stated below. The first is an operator version of the Cauchy-Schwarz inequality proved in [2]: for all operators X and Y

$$|||\ |XY|^{1/2}|||\ \leq (|||X|||\ |||Y|||)^{1/2}, \tag{2.4}$$

where $|T|$ stands for the operator absolute value, $|T| = (T^*T)^{1/2}$.

The other two inequalities we need are related to each other. If X and Y are two operators such that the product XY is normal, then

$$|||XY||| \leq |||YX|||. \tag{2.5}$$

This inequality has been used in earlier papers. Here is an outline of its proof. Let $r(T)$ denote the spectral radius of T. Then $r(T) \leq \|T\|$, and $r(T) = \|T\|$ if T is normal. Further, $r(ST) = r(TS)$ for any two operators S and T. So, if XY is normal, we have

$$\|XY\| = r(XY) = r(YX) \leq \|YX\|.$$

This proves (2.5) in the special case of the operator norm. It can be extended to other norms using a standard argument involving antisymmetric tensor products and majorisation; see, e.g., [2]. By the same argument we can prove that if XY is a normal operator then

$$\left|\left|\left| \, |XY|^{1/2} \right|\right|\right| \leq \left|\left|\left| \, |YX|^{1/2} \right|\right|\right|. \tag{2.6}$$

If XY is Hermitian, inequality (2.5) can be strengthened. In this case we have

$$|||XY||| \leq |||Re(YX)|||, \tag{2.7}$$

where $Re T = \frac{1}{2}(T + T^*)$ [10, Lemma 1].

Proof of Theorem 2.1 Since A and B are Hermitian, we can write

$$|||A-B||| = \left|\left|\left| \, \left|(A-B)^2\right|^{1/2} \right|\right|\right| = \left|\left|\left| \, \left|(A-B)\Gamma^{-1/2}\Gamma^{1/2}(A-B)\right|^{1/2} \right|\right|\right|.$$

Using inequalities (2.6), (2.4) and (2.7) successively, we obtain from this

$$\begin{aligned} |||A-B||| &\leq \left|\left|\left| \, \left|\Gamma^{1/2}(A-B)^2\Gamma^{-1/2}\right|^{1/2} \right|\right|\right| \\ &\leq \left(\left|\left|\left|\Gamma^{1/2}(A-B)\Gamma^{1/2}\right|\right|\right| \; \left|\left|\left|\Gamma^{-1/2}(A-B)\Gamma^{-1/2}\right|\right|\right|\right)^{1/2} \\ &\leq \left(|||Re[(A-B)\Gamma]||| \; |||Re[\Gamma^{-1}(A-B)]|||\right)^{1/2}. \end{aligned}$$

Now note that $B\Gamma - \Gamma B$ and $B\Gamma^{-1} - \Gamma^{-1}B$ are skew-Hermitian. Hence,

$$\begin{aligned} Re[(A-B)\Gamma] &= Re[(A-B)\Gamma + (B\Gamma - \Gamma B)] = Re(A\Gamma - \Gamma B), \\ Re[\Gamma^{-1}(A-B)] &= Re[\Gamma^{-1}(A-B) - (B\Gamma^{-1} - \Gamma^{-1}B)] \\ &= Re(\Gamma^{-1}A - B\Gamma^{-1}). \end{aligned}$$

Since $|||Re\,T||| \leq |||T|||$ for any operator T, we have

$$|||A-B||| \leq \left(|||A\Gamma - \Gamma B||| \; |||\Gamma^{-1}A - B\Gamma^{-1}|||\right)^{1/2}.$$

This proves inequality (2.1). ∎

Now suppose A and B are any two operators. Consider the operators on $\mathcal{H}\oplus\mathcal{H}$ that correspond to the block matrices

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}, \quad \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}.$$

The first two are Hermitian and the third is positive definite. So using the result already proved, we have

$$\left|\left|\left| \begin{bmatrix} 0 & A-B \\ A^*-B^* & 0 \end{bmatrix} \right|\right|\right|^2 \leq \left|\left|\left| \begin{bmatrix} 0 & A\Gamma-\Gamma B \\ A^*\Gamma-\Gamma B^* & 0 \end{bmatrix} \right|\right|\right| \left|\left|\left| \begin{bmatrix} 0 & \Gamma^{-1}A-B\Gamma^{-1} \\ \Gamma^{-1}A^*-B^*\Gamma^{-1} & 0 \end{bmatrix} \right|\right|\right|. \tag{2.8}$$

Proof of Theorem 2.2 Let $s(T)$ denote the vector whose coordinates are the singular values of T arranged in decreasing order. If A and B are unitary, we have

$$s(A^*\Gamma-\Gamma B^*) = s(\Gamma - A\Gamma B^*) = s(A\Gamma B^* - \Gamma) = s(A\Gamma - \Gamma B).$$

By the same argument

$$s(\Gamma^{-1}A^* - B^*\Gamma^{-1}) = s(\Gamma^{-1}A - B\Gamma^{-1}).$$

Inequality (2.2) for all Schatten p-norms ($1 \leq p \leq \infty$) and for all Ky Fan k-norms ($k=1, 2,\ldots$) now follows from (2.8). This is what the authors proved in the first version of this paper. For all unitarily invariant norms, inequality (2.2) is a consequence of Proposition 2.1 given below. The proof given here is due to Referees Profs. T. Ando and F. Hiai. Another proof may be found in [11]. ∎

PROPOSITION 2.1 (ANDO, HIAI, LI, AND MATHIAS) *Let X, Y, and Z be linear operators. Then*

$$|||X|||^2 \leq |||Y|||\,|||Z||| \tag{2.9}$$

for every unitarily invariant norm if and only if

$$\|X\|_{(k)}^2 \leq \|Y\|_{(k)}\|Z\|_{(k)} \tag{2.10}$$

for every Ky Fan k-norm ($k=1, 2,\ldots$).

Proof It suffices to show that (2.10) implies (2.9). The following proof is due to T. Ando and F. Hiai. For fixed nonnegative numbers y and z, we have

$$\min_{\lambda>0}\left(\frac{\lambda y+\lambda^{-1}z}{2}\right)=(yz)^{1/2}.$$

Inequality (2.10) is, therefore, equivalent to

$$\|X\|_{(k)}\le\frac{\lambda\|Y\|_{(k)}+\lambda^{-1}\|Z\|_{(k)}}{2}\quad\text{for all }\lambda>0,$$

or, more explicitly,

$$\sum_{i=1}^{k}s_i(X)\le\sum_{i=1}^{k}\frac{\lambda s_i(Y)+\lambda^{-1}s_i(Z)}{2}\quad\text{for all }\lambda>0.\tag{2.11}$$

For fixed $\lambda>0$, let $s_{i,\lambda}\overset{def}{=}[\lambda s_i(Y)+\lambda^{-1}s_i(Z)]/2$ for $i=1,2,\ldots$ Since $s_{1,\lambda}\ge s_{2,\lambda}\ge\cdots$, inequalities (2.11) mean that the sequence $\{s_i(X), i=1,2,\ldots\}$ is *weakly majorized* by the sequence $\{s_{i,\lambda}, i=1,2,\ldots\}$; see, e.g., [3]. This implies that for every *symmetric gauge function* Φ,

$$\Phi(s_1(X),s_2(X),\ldots)\le\Phi(s_{1,\lambda},s_{2,\lambda},\ldots).$$

Since Φ is convex and positively homogeneous, we have

$$\Phi(s_1(X),s_2(X),\ldots)\le\frac{\lambda\Phi(s_1(Y),s_2(Y),\ldots)+\lambda^{-1}\Phi(s_1(Z),s_2(Z),\ldots)}{2}.$$

Taking minimum over $\lambda>0$ on the right-hand side leads to

$$\Phi(s_1(X),s_2(X),\ldots)\le[\Phi(s_1(Y),s_2(Y),\ldots)\cdot\Phi(s_1(Z),s_2(Z),\ldots)]^{1/2},$$

as was to be shown. ∎

A generalisation of Proposition 2.1 is given in §4.

Proof of Theorem 2.3 If A and B are normal, then by the Fuglede-Putnam Theorem modulo the Hilbert-Schmidt operators [17], we have

$$\|AX-XB\|_2=\|A^*X-XB^*\|_2\quad\text{and}\quad\|XA-BX\|_2=\|XA^*-B^*X\|_2$$

for all X. Using this we can derive inequality (2.3) from (2.8). ∎

In [5] it was proved that if A and B are Hermitian then

$$|||A - B||| \leq ||\Gamma^{-1}|| \, |||A\Gamma - \Gamma B|||.$$

From this we get

$$|||A - B|||^2 \leq ||\Gamma|| \, ||\Gamma^{-1}|| \, |||A\Gamma - \Gamma B||| \; |||A\Gamma^{-1} - \Gamma^{-1}B|||,$$

an inequality weaker than (2.1) above.

3. BOUNDS FOR EIGENVALUE VARIATION

Let A and B be $n\times n$ Hermitian matrices with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_n$, respectively. Let $\mathrm{Eig}^{\downarrow}(A)$ and $\mathrm{Eig}^{\downarrow}(B)$ be the diagonal matrices with diagonal entries $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively. A celebrated theorem of Lidskii, Wielandt and Mirsky says that

$$|||\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(B)||| \leq |||A - B|||, \tag{3.12}$$

for every unitarily invariant norm; see [1, p.45], [3, p.101], [15, p.204]. This inequality has been a model for other spectral variation bounds and has been generalised in various directions.

Suppose A and B are two diagonalisable matrices, i.e., there exist invertible matrices X and Y and diagonal matrices Λ and Ω such that

$$A = X\Lambda X^{-1} \quad \text{and} \quad B = Y\Omega Y^{-1}. \tag{3.13}$$

For such A and B, we have

$$A - B = X\Lambda X^{-1} - Y\Omega Y^{-1} = X(\Lambda X^{-1}Y - X^{-1}Y\Omega)Y^{-1}.$$

Hence for any untiarily invariant norm $|||\cdot|||$

$$|||\Lambda X^{-1}Y - X^{-1}Y\Omega||| = |||X^{-1}(A - B)Y||| \leq ||X^{-1}|| \, |||A - B||| \, ||Y||. \tag{3.14}$$

We could also write $A - B = Y(Y^{-1}X\Lambda - \Omega Y^{-1}X)X^{-1}$ to get

$$|||Y^{-1}X\Lambda - \Omega Y^{-1}X||| \leq ||Y^{-1}|| \, |||A - B||| \, ||X||. \tag{3.15}$$

Let $X^{-1}Y$ have the singular value decomposition $X^{-1}Y = U\Gamma V^*$. Then

$$\begin{aligned}|||\Lambda X^{-1}Y - X^{-1}Y\Omega||| &= |||\Lambda U\Gamma V^* - U\Gamma V^*\Omega||| \\ &= |||U^*\Lambda U\Gamma - \Gamma V^*\Omega V||| \\ &= |||\tilde{A}\Gamma - \Gamma\tilde{B}|||,\end{aligned}$$

where

$$\tilde{A} = U^*\Lambda U \quad \text{and} \quad \tilde{B} = V^*\Omega V. \tag{3.16}$$

Note that $Y^{-1}X = V\Gamma^{-1}U^*$. So, by the same argument

$$|||Y^{-1}X\Lambda - \Omega Y^{-1}X||| = |||\Gamma^{-1}\tilde{A} - \tilde{B}\Gamma^{-1}|||.$$

From (3.14) and (3.15), we have

$$|||\tilde{A}\Gamma - \Gamma\tilde{B}||| \leq \|X^{-1}\|\,\|Y\|\;|||A - B|||, \tag{3.17}$$

$$|||\Gamma^{-1}\tilde{A} - \tilde{B}\Gamma^{-1}||| \leq \|X\|\,\|Y^{-1}\|\;|||A - B||| \tag{3.18}$$

which imply

$$|||\tilde{A}\Gamma - \Gamma\tilde{B}|||\;|||\Gamma^{-1}\tilde{A} - \tilde{B}\Gamma^{-1}||| \leq c(X)c(Y)|||A - B|||^2, \tag{3.19}$$

where $c(X)$ is the *spectral condition number* of X defined as

$$c(X) = \|X\|\,\|X^{-1}\|.$$

In [5] it was proved that if A and B are diagonalisable and have real eigenvalues then

$$|||\text{Eig}^{\downarrow}(A) - \text{Eig}^{\downarrow}(B)||| \leq c(X)c(Y)|||A - B|||. \tag{3.20}$$

Note that when A and B are Hermitian this reduces to (3.12). Our next theorem improves upon this inequality.

THEOREM 3.1 *Let A and B be diagonalisable matrices as in* (3.13). *Supose A and B have only real eigenvalues. Then, for every unitarily* *invariant norm*

$$|||\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(B)||| \leq [c(X)c(Y)]^{1/2}|||A - B|||. \tag{3.21}$$

Proof Using Theorem 2.1, we have by (3.19)

$$|||\tilde{A} - \tilde{B}||| \leq [c(X)c(Y)]^{1/2}|||A - B|||. \tag{3.22}$$

Notice that in the present case both $\tilde{A}$ and $\tilde{B}$ defined by (3.16) are Hermitian, and have the same eigenvalues as those of A and B, respectively. Inequality (3.21) now follows from (3.12). ■

For the operator norm alone inequality (3.21) has been proved recently by Lu [3]. This paper has motivated our work.

Theorems 2.2 and 2.3 lead to two more results of this type.

THEOREM 3.2 *Let A and B be diagonalisable matrices as in* (3.13). *Suppose the eigenvalues of A and B lie on the unit circle. Then, there exists a permutation matrix P such that*

$$|||\Lambda - P\Omega P^{-1}||| \leq \frac{\pi}{2}[c(X)c(Y)]^{1/2}|||A - B|||. \tag{3.23}$$

Proof Using Theorem 2.2 we obtain inequality (3.22) now with unitary $\tilde{A}$ and $\tilde{B}$. Inequality (3.23) now follows from known results on eigenvalue variation of unitary matrices [6], [1, p.71], [3, p.178]. ■

The factor $\frac{\pi}{2}$ in inequality (3.22) can be replaced by 1 in two special cases. For the operator norm this is a consequence of results in [4]. For the Hilbert-Schmidt norm this is subsumed in the following more general theorem.

THEOREM 3.3 *Let A and B be diagonalisable matrices as in* (3.13). *Then there exists a permutation matrix P such that*

$$\|\Lambda - P\Omega P^{-1}\|_2 \leq [c(X)c(Y)]^{1/2}\|A - B\|_2. \tag{3.24}$$

Proof Use Theorem 2.3. The proof is exactly as above. Now matrices $\tilde{A}$ and $\tilde{B}$ in (3.22) are normal, and for such pairs we have the Hoffman-Wielandt inequality [1, p.74], [3, p.165]. ■

Inequality (3.24) improves upon a result of Sun [16] and Zhang [18].

4. CONCLUDING REMARKS

Remark 1 In [8] and [12] it was shown how the perturbation theory for diagonalisable matrix pencils with real spectra can be reduced to the one for matrices similar to unitary matrices. Theorem 3.2 would lead to improvements of the inequalities in [8] and [12].

Remark 2 There is another way of proving the bounds in Theorems 3.1–3.3. This does not use the new commutator inequalities of §2, but exploits the ones in [5]. Let us show this for Theorem 3.1. Let $\alpha = \|X^{-1}\|\,\|Y\|$ and $\beta = \|Y^{-1}\|\,\|X\|$. Since both $\tilde{A}$ and $\tilde{B}$ are Hermitian and Γ is real diagonal,

$$|||\Gamma^{-1}\tilde{A} - \tilde{B}\Gamma^{-1}||| = |||(\Gamma^{-1}\tilde{A} - \tilde{B}\Gamma^{-1})^*||| = |||\tilde{A}\Gamma^{-1} - \Gamma^{-1}\tilde{B}|||.$$

Now from (3.17) and (3.18),

$$\begin{aligned} 2|||A - B||| &\geq \left|\left|\left|\tilde{A}\,\frac{\Gamma}{\alpha} - \frac{\Gamma}{\alpha}\,\tilde{B}\right|\right|\right| + \left|\left|\left|\tilde{A}\,\frac{\Gamma^{-1}}{\beta} - \frac{\Gamma^{-1}}{\beta}\,\tilde{B}\right|\right|\right| \\ &\geq \left|\left|\left|\tilde{A}\left(\frac{\Gamma}{\alpha} + \frac{\Gamma^{-1}}{\beta}\right) - \left(\frac{\Gamma}{\alpha} + \frac{\Gamma^{-1}}{\beta}\right)\tilde{B}\right|\right|\right|. \end{aligned}$$

Since $\left[\left(\frac{\Gamma}{\alpha}\right)^{1/2} - \left(\frac{\Gamma^{-1}}{\beta}\right)^{1/2}\right]^2 \geq 0$, we have $\frac{\Gamma}{\alpha} + \frac{\Gamma^{-1}}{\beta} \geq \frac{2}{\sqrt{\alpha\beta}} I$, where I is the identity matrix. So, using [5, Theorem 1] we obtain

$$|||A - B||| \geq \frac{1}{\sqrt{\alpha\beta}}|||\tilde{A} - \tilde{B}|||.$$

From this inequality (3.21) follows as before.

Remark 3 When A and B are arbitrary normal matrices inequality (2.3) of Theorem 2.3 need not be valid for other p-norms. This is illustrated by the following example borrowed from [7]. Let

$$\Gamma = \begin{pmatrix} 0.6384 & 0 & 0 \\ 0 & 0.6384 & 0 \\ 0 & 0 & 1.0000 \end{pmatrix}$$

and let A and B be the normal matrices[1]

$$A = \begin{pmatrix} -0.5205 - 0.1642i & 0.1042 - 0.3618i & -0.1326 - 0.0260i \\ -0.1299 + 0.1709i & 0.4218 + 0.4685i & -0.5692 - 0.3178i \\ 0.2850 - 0.1808i & -0.3850 - 0.4257i & -0.2973 - 0.1715i \end{pmatrix},$$

$$B = \begin{pmatrix} -0.6040 + 0.1760i & 0.5128 - 0.2865i & 0.1306 + 0.0154i \\ 0.0582 + 0.2850i & 0.0154 + 0.4497i & -0.5001 - 0.2833i \\ 0.4081 - 0.3333i & -0.0721 - 0.2545i & -0.2686 + 0.0247i \end{pmatrix}.$$

Then $\|A - B\|^2 = \mathbf{0.5378}$ and $\|A\Gamma - \Gamma B\| \, \|\Gamma^{-1}A - B\Gamma^{-1}\| = \mathbf{0.4132}$.

Remark 4 Proposition 2.1 has a generalisation. This is given below.

PROPOSITION 4.1 *Let $X, Y_1, \ldots, Y_m$ be linear operators and let p_1, $p_2, \ldots, p_m$ be positive numbers such that $\sum 1/p_j = 1$. Then we have*

$$|||X||| \leq |||Y_1|||^{1/p_1} \cdots |||Y_m|||^{1/p_m}$$

for every unitarily invariant norm if and only if

$$\|X\|_{(k)} \leq \|Y_1\|_{(k)}^{1/p_1} \cdots \|Y_m\|_{(k)}^{1/p_m}$$

for every Ky Fan k-norm, $k = 1, 2, \ldots$.

---

[1]Rigorously, they are *not* normal, but close to. In fact, $\|AA^* - A^*A\| \approx \mathbf{2.393E-4}$ and $\|BB^* - B^*B\| \approx \mathbf{1.489E-4}$. Better A and B could be obtained by doing, e.g., $A = VDV^{-1}$ (eigen-decomposition), $V = QR$ (QR decomposition) and then updating $A = QDQ^*$; in MATLAB notation:

$$[V, D] = \mathrm{eig}(A); [Q, R] = \mathrm{qr}(V); A = Q * D * Q'; [V, D] = \mathrm{eig}(B); [Q, R] = \mathrm{qr}(V); \\ B = Q * D * Q';$$

Then new A and B are much closer to normal. In fact, this time $\|AA^* - A^*A\| \approx \mathbf{2.429E-16}$ and $\|BB^* - B^*B\| \approx \mathbf{5.406E-16}$. Fortunately, both $\|A - B\|^2$ and $\|A\Gamma - \Gamma B\| \, \|\Gamma^{-1}A - B\Gamma^{-1}\|$ with new and old A and B agree up to 4 decimal digits.

Proof Let $y_1, \ldots, y_m$ be nonnegative numbers. Then using the weighted arithmetic-geometric mean inequality one can see that

$$y_1^{1/p_1} \cdots y_m^{1/p_m} = \min_{\substack{\lambda_j > 0 \\ \prod \lambda_j = 1}} \frac{\lambda_1^{p_1}}{p_1} y_1 + \cdots + \frac{\lambda_m^{p_m}}{p_m} y_m.$$

The rest of the argument is similar to the one used in proving Proposition 2.1 ∎

Acknowledgement

We thank Referees T. Ando and F. Hiai for their elegant and simple argument – Proposition 2.1 – that extends to all unitarily invariant norms our original version of Theorem 2.2 which we proved for all Schatten p-norms and all *Ky* Fan k-norms. Li and Mathias [11] have also proved Proposition 2.1 by an entirely different argument. We thank them for sending us their manuscript before its publication.

References

[1] Bhatia, R. (1987). *Perturbation Bounds for Matrix Eigenvalues*. Pitman Research Notes in Mathematics. Longman Scientific & Technical, Harlow, Essex, Published in the USA by John Wiley.
[2] Bhatia, R. (1988). Perturbation inequalities for the absolute value map in norm ideals of operators. *J. Operator Theory*, **19**, 129–136.
[3] Bhatia, R.(1996). *Matrix Analysis, Graduate Texts in Math.*, **169**, Springer, New York.
[4] Bhatia, R. and Davis, C. (1984). A bound for the spectral variation of a unitary operator. *Linear and Multilinear Algebra*, **15**, 71–76.
[5] Bhatia, R., Davis, C. and Kittaneh, F. (1991). Some inequalities for commutators and an application to spectral variation. *Aequationes Math.*, **41**, 70–78.
[6] Bhatia, R., Davis, C. and McIntosh, A. (1983). Perturbation of spectral subspaces and solution of linear operator equations. *Linear Algebra Appl.*, **52-53**, 45–67.
[7] Bhatia, R., Elsner, L. and Krause, G. M. (1994). Spectral variation bounds for diagonalisable matrices. *Aequationes Math.* (to appear).
[8] Bhatia, R., and Li, R.-C. (1996). On perturbations of matrix pencils with real spectra. II. *Math. Comp.*, **65**(214), 637–645.
[9] Gohberg, I. C. and Krein, M. G. (1969). *Introduction to the Theory of Linear Nonselfadjoint Operators*. American Mathematical Society, Providence, RI, *Transl. Math. Monographs*, **18**.
[10] Kittaneh, F. (1992). A note on the arithmetic-geometric mean inequality for matrices. *Linear Algebra Appl.*, **172**, 1–8.
[11] Chi-Kwong Li, and Mathias, R. (1996). Generalizations of *Ky* Kan's dominance theorem. Manuscript, Department of Mathematics, College of William & Mary. Submitted to *SIAM J. Matrix Anal. Appl.*

[12] Li, R.-C. (1994). On perturbations of matrix pencils with real spectra. *Math. Comp.*, **62**, 231–265.
[13] Lu, T.-X. Perturbation bounds of eigenvalues of symmetrizable matrices. *Numer. Math. J. Chinese Univ.*, **16**, 177–185, In Chinese.
[14] Simon, B. (1982). *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge.
[15] Stewart, G. W. and Sun, J.-G. *Matrix Perturbation Theory*, Academic Press, Boston.
[16] Sun, J.-G. (1984). On the perturbation of the eigenvalues of a normal matrix. *Math. Numer. Sinica*, **6**, 334–336, In Chinese.
[17] Weiss, G. (1981). The Fuglede commutativity theorem modulo the Hilbert-Schmidt class and generating functions for matrix operators. *J. Operator Theory*, **5**, 3–16.
[18] Zhang, Z. (1986). On the perturbation of the eigenvalues of a non-defective matrix. *Math. Numer. Sinica*, **6**, 106–108, In Chinese.