Remarks on the Strong Law of Large Numbers for a Triangular Array of Associated Random Variables

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Abstract: A strong law of large numbers for a triangular array of strictly stationary associated random variables is proved. It is used to derive the pointwise strong consistency of kernel type density estimator of the one-dimensional marginal density function of a strictly stationary sequence of associated random variables, and to obtain an improved version of a result by Van Ryzin (1969) on the strong consistency of density estimator for a sequence of independent and identically distributed random variables.

Key Words: Strong law of large numbers, associated random variables, kernel type density estimation.

1 Introduction

The concept of association was introduced by Esary, Proschan and Walkup (1967); a finite family $\{X_1, \ldots, X_m\}$ of random variables is said to be associated if

$$\mathrm{Cov}(h(X_1, \ldots, X_m), g(X_1, \ldots, X_m)) \geq 0$$

for any componentwise nondecreasing functions h, g on R^m such that the covariance exists. An infinite family of random variables is said to be associated if every finite subfamily is associated.

Independent random variables represent one example of associated random variables (Esary, Proschan and Walkup (1967)). There are several examples in Reliability and Survival Analysis where the random variables of interest are not independent but are associated. Some common examples of associated random variables are: positively correlated normal random variables (Pitt (1982)), components of the multivariate exponential distribution due to Marshall and Olkin (1967) etc.

Some probabilistic and statistical aspects of associated random variables have been extensively discussed in recent years, see for example, Birkel (1988 a,b, 1989), Cox and Grimmett (1984), Newman (1984), Bagai and Prakasa Rao

(1991, 1995), Prakasa Rao (1993) among others. Strong law of large numbers for a stationary sequence and an arbitrary sequence of associated random variables have been obtained by Newman (1984) and Birkel (1989), respectively.

In what follows we prove a strong law of large numbers for a triangular array of associated random variables and give an application to kernel-type density estimation for a strictly stationary associated sequence of random variables. We also prove an improved version of a result by Van Ryzin (1969) on the strong consistency of density estimator for a sequence of independent and identically distributed (i.i.d.) random variables.

2 SLLN for Triangular Array

Consider a triangular array of random variables $\{X_{nj}, 1 \le j \le k_n, n \ge 1\}$ where the random variables in each row are strictly stationary and associated.

Let c denote a generic positive constant which may vary from one step to another. We denote $\mathrm{Var}(X)$ as $\mathrm{Cov}(X, X)$ for convenience.

Lemma 2.1: Let $\{X_{nj}, 1 \le j \le k_n, n \ge 1\}$ be a strictly stationary associated triangular array of random variables with $\mathrm{Var}(X_{n1}) < \infty$ for all n. Let $S_{n,l} = \sum_{j=1}^{l} X_{nj}$. Suppose that there exists $c > 0$ such that for every $n \ge 1$,

$$\sum_{j=1}^{k_n} \mathrm{Cov}(X_{n1}, X_{nj}) \le c \ . \tag{2.1}$$

Then, for $n \ge 1$,

$$\mathrm{Var}(S_{n,k_n}) \le ck_n \ . \tag{2.2}$$

Proof: Note that, for $n \ge 2$,

$$\begin{aligned}\mathrm{Var}(S_{n,k_n}) &= \mathrm{Var}\left[\sum_{j=1}^{k_n} X_{nj}\right] \\ &= \sum_{j=1}^{k_n} \mathrm{Var}(X_{nj}) + 2\sum\sum_{1 \le i < j \le k_n} \mathrm{Cov}(X_{ni}, X_{nj})\end{aligned}$$

$$\leq 2\left[\sum_{j=1}^{k_n} \operatorname{Var}(X_{nj}) + \sum \sum_{1 \leq i < j \leq k_n} \operatorname{Cov}(X_{ni}, X_{nj})\right]$$

$$= 2\left[\sum_{j=1}^{k_n} \operatorname{Var}(X_{nj}) + \sum_{j=2}^{k_n} \operatorname{Cov}(X_{n1}, X_{nj})\right.$$

$$\left. + \sum_{j=3}^{k_n} \operatorname{Cov}(X_{n2}, X_{nj}) + \cdots + \operatorname{Cov}(X_{n,k_n-1}, X_{nk_n})\right]$$

$$= 2\left[\sum_{j=1}^{k_n} \operatorname{Cov}(X_{n1}, X_{nj}) + \sum_{j=2}^{k_n} \operatorname{Cov}(X_{n2}, X_{nj}) + \cdots\right.$$

$$\left. + \sum_{j=k_n-1}^{k_n} \operatorname{Cov}(X_{n,k_n-1}, X_{nj}) + \operatorname{Cov}(X_{n,k_n}, X_{n,k_n})\right]$$

$$= 2\left[\sum_{j=1}^{k_n} \operatorname{Cov}(X_{n1}, X_{nj}) + \sum_{j=1}^{k_n-1} \operatorname{Cov}(X_{n1}, X_{nj}) + \cdots\right.$$

$$\left. + \sum_{j=1}^{2} \operatorname{Cov}(X_{n1}, X_{nj}) + \operatorname{Cov}(X_{n1}, X_{n1})\right] \quad \text{(by stationarity of } X_{nj})$$

$$\leq ck_n \qquad \text{(by (2.1))} .$$

Remark 2.2: The condition that $\{X_{nj}\}$ is associated is not necessary for the above lemma. It is sufficient to assume that $\operatorname{Cov}(X_{ni}, X_{nj}) \geq 0$ for all $1 \leq i < j \leq k_n$, $n \geq 1$.

Theorem 2.3: Let $\{X_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ be a triangular array of strictly stationary associated random variables with $E[X_{n1}] = 0$ and $\operatorname{Var}(X_{n1}) < \infty$ for all n. Suppose that $k_n = O(n^\gamma)$ for some $0 \leq \gamma < \frac{3}{2}$ and the condition (2.1) holds. Further suppose that

$$E\left[\max_{n^2 < j \leq (n+1)^2} |S_{j,k_j} - S_{n^2,k_{n^2}}|\right]^2 = O(n^{4-\delta}) \tag{2.3}$$

for some $\delta > 1$. Then,

$$\frac{S_{n,k_n}}{n} \to 0 \qquad \text{a.s.} \quad \text{as} \quad n \to \infty . \tag{2.4}$$

Proof: Using Chebychev's inequality and Lemma 2.1, we note that

$$\Pr[|S_{n^2,k_{n^2}}| > n^2\varepsilon] \le \frac{ck_{n^2}}{n^4\varepsilon^2} .$$

Note that $\sum_n \Pr[|S_{n^2,k_{n^2}}| > n^2\varepsilon] < \infty$ for all $\varepsilon > 0$, and hence, by the Borel-Cantelli lemma, it follows that

$$\frac{S_{n^2,k_{n^2}}}{n^2} \to 0 \qquad \text{a.s.} \quad \text{as} \quad n \to \infty . \tag{2.5}$$

Let

$$D_n = \max_{n^2 < j \le (n+1)^2} |S_{j,k_j} - S_{n^2,k_{n^2}}| .$$

Then

$$\frac{|S_{j,k_j}|}{j} \le \frac{|S_{n^2,k_{n^2}}| + D_n}{n^2} \qquad \text{for } n^2 < j \le (n+1)^2 .$$

If

$$\frac{D_n}{n^2} \to 0 \qquad \text{a.s.} \quad \text{as} \quad n \to \infty \tag{2.6}$$

then, from (2.5) and (2.6), it would follow that,

$$\frac{S_{n,k_n}}{n} \to 0 \qquad \text{a.s.} \quad \text{as} \quad n \to \infty . \tag{2.7}$$

Thus it is sufficient to prove that

$$\sum_n \Pr\left\{\max_{n^2 < j \le (n+1)^2} |S_{j,k_j} - S_{n^2,k_{n^2}}| \ge n^2\varepsilon\right\} < \infty$$

for all $\varepsilon > 0$. By Chebychev's inequality

$$\Pr\left\{\max_{n^2<j\leq(n+1)^2}|S_{j,k_j}-S_{n^2,k_{n^2}}|\geq n^2\varepsilon\right\}\leq\frac{1}{n^4\varepsilon^2}E\left[\max_{n^2<j\leq(n+1)^2}|S_{j,k_j}-S_{n^2,k_{n^2}}|\right]^2 .$$

The problem is to estimate

$$E\left[\max_{n^2<j\leq(n+1)^2}|S_{j,k_j}-S_{n^2,k_{n^2}}|\right]^2 .$$

Since this term is $O(n^{4-\delta})$ for some $\delta > 1$ by assumption, the result holds by the Borel-Cantelli lemma.

Remark 2.4: Cox and Grimmett (1984) and Birkel (1988a,b), among others, observed that in any asymptotic study of a sequence of associated random variables, the covariance structure plays an important role. Cox and Grimmett (1984), while considering asymptotic normality of a triangular array of associated random variables, assume that there exists a function,

$$u = \{0, 1, \ldots, \} \to R , \qquad u(r) \to 0 \quad \text{as} \quad r \to \infty$$

such that

$$\sum_{j:|\ell-j|\geq r} \mathrm{Cov}(X_{nj}, X_{n\ell}) \leq u(r) \qquad \text{for all } \ell, n, r \geq 0 . \tag{2.8}$$

It can be easily seen that (2.8) implies (2.1) if $\{X_{ni}\}$ is a strictly stationary associated sequence for each n.

3 Applications

Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of associated random variables with one-dimensional unknown marginal density function f for X_1. Let

$$f_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n} K\left(\frac{x-X_j}{h_n}\right) , \qquad x \in R \tag{3.1}$$

be a kernel type estimator for $f(x)$ where h_n is a bandwidth and $K(\cdot)$ is a kernel satisfying the following conditions:

(A_1) $K(\cdot)$ is a bounded density function of bounded variation on R. Hence $K(x) = K_1(x) - K_2(x)$, where $K_1(x)$ and $K_2(x)$ are two monotonic functions. Suppose that (i) $\lim_{|x|\to\infty} |x|K(x) = 0$, (ii) $\int_{-\infty}^{\infty} u^j K_i^2(u)du < \infty$ $i = 1, 2$, $j = 0, 1, 2$;

(A_2) $K_i(x)$ are differentiable with $\sup_x |K_i'(x)| \le c < \infty$, $i = 1, 2$.

Further, assume that the covariance structure of $\{X_n\}$ satisfies the condition:

(A_3) for all ℓ and r

$$\sum_{j: |\ell - j| \ge r} \operatorname{Cov}(X_j, X_\ell) \le u(r)$$

where $u(r) = e^{-\alpha r}$ for some $\alpha > 0$.

(A_4) In addition, assume that f is thrice differentiable and $\sup_x |f'''(x)| < \infty$.

Remark 3.1: The assumption that $K(\cdot)$ is of bounded variation is used to ensure that the transformed variables $K_j\left(\frac{x - X_i}{h_n}\right)$, $1 \le i \le n$ are associated for $j = 1, 2$. This is also used in the classical density estimation for i.i.d. random variables (cf. Prakasa Rao (1983), pp 37). In view of (A_2), Newman's (1980) inequality (see Lemma 3.2) can be used to estimate covariance of functions of associated random variables. Importance of (A_3) is discussed in Remark 2.4 and assumption (A_4) is used to obtain a bound on the variance of the estimator as in the classical i.i.d. case (cf. Praskasa Rao (1983)).

The following inequality due to Newman (1980) is used later in the paper.

Lemma 3.2: (Newman (1980)). Let (Z_1, Z_2) be associated random variables with finite variance. Then, for any two differentiable functions h and g,

$$|\operatorname{Cov}(h(Z_1), g(Z_2))| \le \sup_x |h'(x)| \sup_y |g'(y)| \operatorname{Cov}(Z_1, Z_2) \tag{3.2}$$

where h' and g' denote the derivatives of h and g respectively.

Let

$$
\begin{aligned}
X_{ni} &= \frac{1}{h_n}\left\{K\left(\frac{x - X_i}{h_n}\right) - EK\left(\frac{x - X_i}{h_n}\right)\right\} \\
&= \frac{1}{h_n}\left\{\left[K_1\left(\frac{x - X_i}{h_n}\right) - EK_1\left(\frac{x - X_i}{h_n}\right)\right]\right. \\
&\quad \left. - \left[K_2\left(\frac{x - X_i}{h_n}\right) - EK_2\left(\frac{x - X_i}{h_n}\right)\right]\right\} \\
&= \psi_n^{(1)}(x, X_i) - \psi_n^{(2)}(x, X_i) \quad \text{(say)} . \qquad (3.3)
\end{aligned}
$$

Then, for fixed x, $\{\psi_n^{(j)}(x, X_i)\}$, $j = 1, 2$, being monotonic functions of associated random variables are associated. Using Lemma 3.2 it follows, that for $j = 1, 2$,

$$
\begin{aligned}
&\sum_{j:|\ell-j|>r} \operatorname{Cov}(\psi_n^{(j)}(x, X_j), \psi_n^{(j)}(x, X_\ell)) \\
&\leq \sup_y \left\{\frac{\partial}{\partial y}\psi_n^{(j)}(x, y)\right\}^2 \sum_{j:|\ell-j|>r} \operatorname{Cov}(X_j, X_\ell) . \qquad (3.4)
\end{aligned}
$$

In view of (A_1) to (A_3) and Remark 2.4, the conditions stated in Lemma 2.1 hold for $\psi_n^{(j)}(x, X_i)$, $j = 1, 2$. Using (A_4) it is easy to see that, for $j = 1, 2$,

$$
\operatorname{Var}(\psi_n^{(j)}(x, X_1)) \leq \frac{1}{h_n}[f(x)\beta_0^{(j)} - f'(x)h_n\beta_1^{(j)} + f''(x)h_n^2\beta_2^{(j)} + O(h_n^3)] \qquad (3.5)
$$

where

$$
\beta_i^{(j)} = \int_{-\infty}^{\infty} x^i K_j^2(x)dx , \qquad i = 0, 1, 2 \qquad (3.6)
$$

Furthermore, for $j = 1, 2$,

$$
\sum_{1\leq i} \sum_{<j\leq n} \operatorname{Cov}\{\psi_n^{(j)}(x, X_i), \psi_n^{(j)}(x, X_j)\} \leq \frac{cn}{h_n^4} \qquad (3.7)
$$

by (A_3) and stationarity. Let

$$
T_j^{(n)} = \sum_{i=1}^{j} X_{ni} . \qquad (3.8)
$$

Then

$$
\begin{aligned}
E&\left[\max_{n^2 < j \le (n+1)^2} |T_j^{(n)} - T_{n^2}^{(n)}|^2\right] \\
&\le \sum_{j=n^2+1}^{(n+1)^2} E\left\{\left(\sum_{i=n^2+1}^{j} X_{ni}\right)^2\right\} \\
&= \sum_{j=n^2+1}^{(n+1)^2} E\left\{\left(\sum_{i=n^2+1}^{j} \psi_n^{(1)}(x, X_i) - \sum_{i=n^2+1}^{j} \psi_n^{(2)}(x, X_i)\right)^2\right\} \\
&\le 2 \sum_{j=n^2+1}^{(n+1)^2} \left\{E\left(\sum_{i=n^2+1}^{j} \psi_n^{(1)}(x, X_i)\right)^2 + E\left(\sum_{i=n^2+1}^{j} \psi_n^{(2)}(x, X_i)\right)^2\right\} \\
&= 2 \sum_{j=n^2+1}^{(n+1)^2} \left\{\mathrm{Var}\left(\sum_{i=n^2+1}^{j} \psi_n^{(1)}(x, X_i)\right) + \mathrm{Var}\left(\sum_{i=n^2+1}^{j} \psi_n^{(2)}(x, X_i)\right)\right\} \\
&\le 4n\left[\mathrm{Var}\left\{\sum_{i=n^2+1}^{(n+1)^2} \psi_n^{(1)}(x, X_i)\right\} + \mathrm{Var}\left\{\sum_{i=n^2+1}^{(n+1)^2} \psi_n^{(2)}(x, X_i)\right\}\right] \\
&\qquad \text{(by the associativity of } \{\psi_n^{(1)}\} \text{ and } \{\psi_n^{(2)}\}) \\
&= 4n\left\{O\left(\frac{n}{h_n}\right) + O\left(\frac{n}{h_n^4}\right)\right\} \qquad \text{(using (3.4) and (3.6))} \\
&= O(n^2 h_n^{-4})\,.
\end{aligned}
\tag{3.9}
$$

Note that, if $h_n = O(n^{-\alpha})$, where $\alpha > 0$, then

$$
E\left[\max_{n^2 < j \le (n+1)^2} |T_j^{(n)} - T_{n^2}^{(n)}|^2\right] = O(n^{2+4\alpha}) \tag{3.10}
$$

and the condition (2.3) holds provided $\alpha < \frac{1}{4}$. In particular, for the optimal choice $h_n = O(n^{-1/5})$ (cf. Bagai and Prakasa Rao (1995)) the condition (2.3) holds.

Theorem 3.1: Suppose that the conditions (A1) to (A4) hold and $h_n = O(n^{-\alpha})$ where $0 < \alpha < \frac{1}{4}$. Define $f_n(x)$ by (3.1). Then,

$$
f_n(x) \to f(x) \qquad \text{a.s.} \quad \text{as} \quad n \to \infty \tag{3.11}
$$

at all continuity points x of f.

Proof: It is clear from the calculations given above that the conditions stated in Theorem 2.3 hold. Hence $f_n(x) - Ef_n(x) \to 0$ a.s. as $n \to \infty$. Since $E[f_n(x)] \to f(x)$ as $n \to \infty$ at the continuity points x of $f(\cdot)$ under (A_1)(i), it follows that (3.10) holds, proving Theorem 3.1.

Remark 3.2: Strong consistency for a kernel-type density estimator was proved for a sequence of i.i.d. random variables by Van Ryzin (1969) and for mixing sequences by Roussas (1988).

In the case of a sequence of i.i.d. random variables $\{X_n\}$ with finite second moment, condition that $K(\cdot)$ is of bounded variation in (A_1) and assumption (A_2) can be removed. Assumption (A_3) holds in view of independent $\{X_n\}$. Suppose assumption (A_4) is satisfied. The following result on strong consistency for density estimators for i.i.d. random variables holds improving upon the earlier result of Van Ryzin (1969) in the i.i.d. case.

Theorem 3.3: Let $\{X_i, i \geq 1\}$ be i.i.d. with density f. Suppose $K(\cdot)$ is a bounded probability density function satisfying (i) $\lim_{|x| \to \infty} |x| K(x) = 0$, (ii) $\int_{-\infty}^{\infty} x^j K^2(x)dx < \infty$ $j = 0, 1, 2$ and (A_4) holds. Further assume that $h_n = O(n^{-\alpha})$ where $0 < \alpha < 1$. Then

$$f_n(x) \to f(x) \qquad \text{a.s.} \quad \text{as} \quad n \to \infty \tag{3.12}$$

for all continuity points x of f.

Proof: This follows immediately from the fact that $X_{ni} = \frac{1}{h_n}\left\{K\left(\frac{x - X_i}{h_n}\right) - EK\left(\frac{x - X_i}{h_n}\right)\right\}$, $1 \leq i \leq n$, $n \geq 1$ form a triangular array of strictly stationary associated random variables. Let $T_j^{(n)}$ be as defined in (3.8). It is easy to check that

$$E\left[\max_{n^2 < j \leq (n+1)^2} |T_j^{(n)} - T_{n^2}^{(n)}|^2\right] = O\left(\frac{n^2}{h_n}\right) = O(n^{2+\alpha})$$

if $h_n = O(n^{-\alpha})$ and the condition (2.3) holds for $\alpha < 1$. Hence

$$f_n(x) - E[f_n(x)] \to 0 \qquad \text{a.s.} \quad \text{as} \quad n \to \infty \ .$$

Since $E[f_n(x)] \to f(x)$ as $n \to \infty$ at the continuity points x of $f(\cdot)$ under (A_1)(i), it follows that (3.12) holds, proving Theorem 3.3.

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