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Unsteady convective diffusion in a pulsatile flow through a channel

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Summary. The paper presents an exact analysis of the streamwise dispersion of passive contaminant molecules released in an incompressible viscous fluid flowing through a channel under the influence of a periodic pressure gradient. Using the Aris-Barton method of moments which is valid for all time after the injection of the solute, the dispersion coefficients of a passive contaminant cloud are obtained separately for three different cases: steady, periodic and for comparison the combined effect of steady and periodic currents. Here it is shown how the injected material disperses due to the shear effect caused by the combined effects of flow (steady or periodic) and lateral diffusion about its mean position, and how the centre of gravity of mass moves, when the initial distribution of contaminant is uniform over the cross-section of the channel. The comparison reveals that for all cases the dispersion coefficient asymptotically reaches a stationary state after a certain time, but it changes cyclically with dispersion time even in the stationary state for the case of oscillatory flows. The analysis leads to the interesting result that the dispersion coefficient consists of a steady part and a fluctuating part due to the pulsatility of the flow.

1 Introduction

The dispersion of a cloud of soluble matter injected into a pipe has been extensively studied by many researchers following the classical works of G. I. Taylor [10]. Taylor first presented the idea of "shear effect" for the case of dispersion of passive contaminant in a viscous laminar flow through a circular tube. He also pointed out that the dispersion of a soluble matter in a laminar flow through a capillary tube can be described by means of an apparent diffusion coefficient due to the combined action of convection and molecular diffusion in radial direction. His analysis is applicable at asymptotically large time (after injection). Aris [1] extended Taylor's theory to include longitudinal diffusion and developed an approach "method of moments" to analyze the convection process in steady flow using first few integral moments. He showed from second moment that the effective longitudinal dispersion coefficient D_e is proportional to $D + P_e{}^2 D_a$, where P_e is the Péclet number which measures the relative characteristic time of the diffusion to the convection process, D_a is the apparent dispersion coefficient. Barton [3] presented an approach for steady flow that resolved certain technical difficulties in the Aris method of moments and obtained the solutions of the second and third moment equations which are valid for all time. All the papers mentioned above were based on steady flow.

Aris [2] used his method of moments to analyze the longitudinal dispersion coefficient of a soluble matter in an oscillatory flow of a viscous incompressible fluid within an infinite tube under a periodic pressure gradient. However, his analysis was also limited to asymptotically large time after the injection of the solute and did not throw any idea on the instantaneous variation of the dispersion coefficient with time immediately after the injection of the solute.

Chatwin [4] employed an exact solution of the diffusion equation to study the dispersion in oscillatory flow. He showed that the simple Gaussian form for the cross-sectionally averaged concentration that was proposed by Taylor [10] for a steady flow was retrieved after a sufficiently large time in oscillatory flows, and also pointed out that the contaminant may appear to be periodically expanding and contracting. The effect of flow oscillation (without a time-mean flow) due to a periodic pressure gradient on the axial diffusion of a solute in a pipe was studied by Purtell [8], considering a small perturbation to the oscillation Reynolds number. Smith [9] analyzed the variance and the dispersion coefficient during the initial to the stationary stage in the oscillatory flow. He pointed out that the dispersion coefficient may be sometimes negative due to the reversing flows in oscillatory current, which describes the periodicity in the concentration of the contaminant. Yasuda [11] examined the dispersion process from the initial to the stationary stage in both steady and oscillatory current and proposed a new definition of vertical averaging on variance to escape the negative dispersion coefficient. Some important characteristics of time-dependent laminar flow may be found in Jimnez and Sullivan [6], who studied the rate of growth of variance by using the probabilistic approach. Mukherjee and Mazumder [7] extended the Aris-Barton method of moments for studying the all time evolution of the central moments of dispersion of a passive contaminant cloud in an oscillatory flow. The solution was based on the method of separation of variables which depends upon a certain eigen-value problem with a discrete spectrum of eigen-values. They studied the contribution of the flow oscillation to the longitudinal dispersion coefficient during the initial to stationary stage within a tube, when the initial distribution of contaminant cloud was uniform, and the Péclet number was large. Their analysis was carried out in the flow through a tube in the presence of a periodic pressure gradient with non-zero mean. They confined their analysis only to the case of combined effect of steady and periodic flow within a tube. The dispersion coefficient due to periodic current was not studied separately by Mukherjee and Mazumder [7].

The aim of the present paper is to study the longitudinal dispersion coefficient of passive contaminant in a viscous incompressible fluid flowing through a parallel plate channel under a periodic pressure gradient with a non-zero mean. The solution is based on the method of moments due to Mukherjee and Mazumder [7] by suitably modifying the treatment of Barton [3], who studied only steady flows. Results are shown for all time period how the spreading of tracers is influenced by the combined effect of flow (steady or unsteady) and diffusion in the cross-sectional plane about its mean position, and how the centre of mass of solute moves, when the contaminant is initially uniform over the cross-section of the channel and the Péclet number is large. The analysis has been performed for three different velocity profiles to identify the individual effect on dispersion processes due to steady, periodic and for comparison the combined action of steady and periodic currents. The motivation of the study of diffusion in oscillatory flows stems mainly from important applications, namely, the dispersion of tracers in pulsatile blood flow in a cardiovascular system, the chemical reaction designs, studies on flow transients using probes based on diffusion controlled electrode reactions and the discharge of outfalls in homogeneous tidal estuaries.

2 Mathematical formulation

Consider a two-dimensional fully developed pulsatile laminar flow of a viscous incompressible fluid between two parallel plates of distance 2L apart. We employ a coordinate system with x^*-axis along the flow and y^*-axis perpendicular to the flow and where the plates are at

$y^* = \pm L$. The flow is driven by a periodic axial pressure gradient with a non-zero mean given by

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x^*} = P_{x^*}(1+\varepsilon e^{i\omega t^*}) \tag{1}$$

where ϱ is the density of the fluid (assumed to be homogeneous), P_{x^*} is the mean pressure gradient, εP_{x^*} and ω are respectively the amplitude and frequency of the pressure pulsation.

The velocity distribution $u^*(y^*, t^*)$ parallel to the x^*-axis satisfies the Navier-Stokes equation

$$\frac{\partial u^*}{\partial t^*} = -\frac{1}{\varrho}\frac{\partial p}{\partial x^*} + \nu\left(\frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{2}$$

with no slip conditions at the boundary $u^*(\pm L, t) = 0$, where ν is the kinematic viscosity of the fluid.

When a contaminant of constant molecular diffusivity D is injected into the above mentioned time-dependent flow, the concentration $C(x, y, t)$ of the contaminant satisfies the dimensionless convective-diffusion equation of the form

$$\frac{\partial C}{\partial t} + P_e u(y,t)\,\frac{\partial C}{\partial x} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) C \tag{3}$$

where the dimensionless quantities are given by

$$x = \frac{x^*}{L}, \qquad y = \frac{y^*}{L}, \qquad t = \frac{Dt^*}{L^2}, \qquad u = \frac{u^*}{U}, \qquad P_e = \frac{UL}{D}.$$

U is the reference velocity. The Péclet number P_e, introduced here, measures the relative characteristic time of the diffusion process $\left(\frac{L^2}{D}\right)$ to the convection process $\left(\frac{L}{U}\right)$.

The initial and boundary conditions for the contaminant input are

$$\begin{aligned}
&C(x,y,0) = C(x,y)\\
&\frac{\partial C}{\partial y} = 0 \quad \text{at} \quad y = \pm 1\\
&C \text{ is finite at all points,}\\
&x^n C \to 0 \quad \text{and} \quad x^m \partial_x{}^n C \to 0 \quad \text{as} \quad |x| \to \infty, \quad m,n = 0,1,2,\ldots\\
&\frac{1}{2}\int_{-1}^{+1}\int_{-\infty}^{+\infty} C\,dx\,dy = 1.
\end{aligned} \tag{4}$$

The solution of Eqs. (1) and (2), satisfying the no-slip conditions $u(\pm 1, t) = 0$ at the boundary $y = \pm 1$ is given by

$$u(y,t) = u_0(y) + u_1(y,t) \tag{5}$$

where

$$u_0(y) = \frac{1}{2}(1 - y^2) \tag{6}$$

$$u_1(y,t) = -\mathrm{Re}\left[\frac{i\varepsilon}{\alpha}\left(1 - \frac{\cosh\sqrt{i\alpha}\,y}{\cosh\sqrt{i\alpha}}\right) e^{i\alpha S t}\right], \qquad i = \sqrt{-1} \tag{7}$$

and $u = \frac{u^*}{U}$ is the dimensionless axial velocity $\left(U\right.$ being the time averaged axial velocity $\left.\frac{P_{x^*}L^2}{4\nu}\right)$, $\alpha = \frac{\omega L^2}{\nu}$ is the dimensionless frequency parameter or oscillation Reynolds number, $S = \frac{\nu}{D}$ is the Schmidt number. The first term of the right-hand side of (5) corresponds to plane Poiseuille flow and the second term corresponds to the flow due to pulsation. Here, of course, the physical significance is attributed only to the real part, and the real part of $u(y,t)$ is given by

$$\begin{aligned} u(y,t) = \frac{1}{2}(1-y^2) - \frac{\varepsilon}{\alpha(K_1{}^2 + K_2{}^2)} \Bigg[\Bigg\{ & K_1\left(\cosh\sqrt{\frac{\alpha}{2}}\, y \cos\sqrt{\frac{\alpha}{2}}\, y - K_1\right) \\ & + K_2\left(\sinh\sqrt{\frac{\alpha}{2}}\, y \sin\sqrt{\frac{\alpha}{2}}\, y - K_2\right)\Bigg\} \sin \alpha St \\ & + \left(K_1 \sinh\sqrt{\frac{\alpha}{2}}\, y \sin\sqrt{\frac{\alpha}{2}}\, y - K_2 \cosh\sqrt{\frac{\alpha}{2}}\, y \cos\sqrt{\frac{\alpha}{2}}\, y\right) \cos \alpha st\Bigg] \end{aligned} \tag{8}$$

where $K_1 = \cosh\sqrt{\frac{\alpha}{2}} \cos\sqrt{\frac{\alpha}{2}}$, and $K_2 = \sinh\sqrt{\frac{\alpha}{2}} \sin\sqrt{\frac{\alpha}{2}}$.

Following Aris [1], [2] we define the n-th moment of the concentration distribution through y at time t as

$$C_n(y,t) = \int_{-\infty}^{+\infty} x^n C(x,y,t)\, dx \tag{9}$$

and the n-th moment of the distribution over the cross-section of the channel as

$$M_n(t) = \frac{1}{2}\int_{-1}^{+1} C_n(y,t)\, dy = \bar{C}_n . \tag{10}$$

Using (9) and (10) in Eqs. (3) and (4), we have the following moment equations:

$$\frac{\partial C_n}{\partial t} = \frac{\partial^2 C_n}{\partial y^2} + n(n-1)\, C_{n-2} + nP_e u(y,t)\, C_{n-1} \tag{11}$$

with

$$C_n(y,0) = C_n(y), \qquad \frac{\partial C_n}{\partial y} = 0 \quad \text{at} \quad y = \pm 1, \tag{12}$$

and

$$\frac{\partial M_n}{\partial t} = nP_e \overline{uC}_{n-1} + n(n-1)\, M_{n-2} \tag{13}$$

with

$$M_n(0) = \bar{C}_n . \tag{14}$$

Here an overbar denotes the cross-sectional mean. Also $M_0(t) = 1$, since C has cross-sectional mean unity for all time (see (4)), and $M_1(t)$ is the mean of the distribution and $M_n(0) = 0$ for $n > 0$.

The n-th central moment of the concentration distribution can be defined as

$$\nu_n(t) = \frac{1}{2M_0} \int_{-1}^{+1} \int_{-\infty}^{+\infty} (x - \bar{x}_g)^n C \, dx \, dy \tag{15}$$

where $\bar{x}_g = \frac{1}{2M_0} \int_{-1}^{+1} \int_{-\infty}^{+\infty} xC \, dx \, dy = \frac{M_1}{M_0}$ can be regarded as the 'centroid' of the contaminant distribution which measures the location of the centre of gravity of the cloud movement with the mean velocity of the fluid, initially located at the source, and the second central moment (ν_2) can be related to the dispersion of diffusing substance about its mean position. Thus we have the expression for central moments:

$$\begin{aligned}
\nu_2(t) &= \frac{M_2}{M_0} - \bar{x}_g{}^2 \\
\nu_3(t) &= \frac{M_3}{M_0} - 3\bar{x}_g \nu_2(t) - \bar{x}_g{}^3 \\
\nu_4(t) &= \frac{M_4}{M_0} - 4\bar{x}_g \nu_3(t) - 6\bar{x}_g{}^2 \nu_2 - \bar{x}_g{}^4 .
\end{aligned} \tag{16}$$

Though the third and fourth moments are also important factors during the initial stage, the present study is concentrated only to the dispersion effect (variance).

The aim of the analysis is to solve the moment equations (11) and (13) subject to the initial and boundary conditions (12) and (14) for $n = 0, 1, 2 \ldots$. The method of solution adopted here is the Aris method of moments as modified by Barton [3] for steady flow and later by Mukherjee and Mazumder [7] for oscillatory flow. According to Mukherjee and Mazumder [7] we consider an eigen-value problem

$$\left(\frac{d^2}{dy^2} - \frac{\partial}{\partial t} + \mu_i \right) f_i = 0 \tag{17}$$

$$\frac{\partial f_i}{\partial y} = 0 \quad \text{at} \quad y = \pm 1, \quad f_i \text{ finite} \tag{18}$$

where i runs over the positive integral values.

This gives us a discrete set of eigen-values $\mu_i = \frac{(i\pi)^2}{(1+\varepsilon)}$ and the corresponding eigenfunctions $f_i(y,t) = \sqrt{2} \cos i\pi e^{-\mu_i \varepsilon t}, i = 1, 2, 3, \ldots$, so that

$$\begin{aligned}
&\bar{f}_i = 0 \\
&\overline{f_i f_j} = 0, \quad \text{if} \quad i \neq j \\
&\overline{f_i f_j} = h_i, \quad \text{if} \quad i = j
\end{aligned} \tag{19}$$

where h_i is the function of t alone and $h_i(0) = 1$. To complete the set of eigen-functions we augment this set by setting $f_0 = 1$ corresponding to $\mu_0 = 0$.

Following Mukherjee and Mazumder [7], the expression for variance $\nu_2(t)$ is given by

$$\nu_2(t) = 2t - 2{P_e}^2 \sum_i a^1_{0i}(0) \int \overline{uf}_i e^{-\mu_i t}\, dt + 2{P_e}^2 \sum_i \int a^1_{0i} \overline{uf}_i\, dt$$
$$+ 2{P_e}^2 \sum_i a^1_{0i}(0) \left[\int \overline{uf}_i e^{-\mu_i t}\, dt \right]_{t=0} - 2{P_e}^2 \sum_i \int \left[a^1_{0i} \overline{uf}_i\, dt \right]_{t=0} \tag{20}$$

where $a^1_{0i}(t) = e^{-\mu_i t} \int e^{\mu_i t} \overline{uf}_i h_i^{-1}(t)\, dt$.

The rate of growth of variance which indicates the degree of dispersion effect at any time is given by

$$\frac{1}{2} \frac{d\nu_2}{dt} = 1 + {P_e}^2 D_a \tag{21}$$

where D_a is the apparent dispersion coefficient depending on parameters S, α, ε and t. The first term on the right-hand side comes from longitudinal diffusion and the second term represents the interaction between the convection and lateral diffusion. The analysis is confined to study the behaviour of variance ν_2 and the dispersion coefficient D_a due to shear effects of steady, oscillatory and the combined effect of steady and periodic currents separately.

2.1 Plane Poiseuille flow

The velocity distribution of the plane Poiseuille flow through a parallel plate channel is given (putting $\varepsilon = 0$ in (5)) by

$$u(y) = \frac{1}{2}(1 - y^2). \tag{22}$$

The corresponding results for the plane Poiseuille flow may be retrieved from the eigenvalue problem (17) by putting $\frac{\partial f_i}{\partial t}$ equal to zero and which is same as given by Barton [3]. The eigenvalues and the corresponding normalised eigen-functions for this steady flow are therefore given by

$$\mu_i = (i\pi)^2, \qquad f_i(y) = \sqrt{2}\, \cos i\pi y \quad i = 1, 2, \ldots \tag{23}$$

The corresponding expression for variance ν_2 is

$$\nu_2(t) = 2\left[1 + {P_e}^2 \sum_i \frac{2}{(i\pi)^6}\right] - 2{P_e}^2 \sum_i \frac{2}{(i\pi)^8}\left(1 - e^{-\mu_i t}\right). \tag{24}$$

The rate of change of variance $\frac{1}{2}\frac{d\nu_2}{dt}$ is proportional to the sum of a constant quantity one due to longitudinal diffusion and the apparent dispersion coefficient D_a given by

$$D_a\big|_{\text{steady}} = \sum_i \frac{2}{(i\pi)^6}\left(1 - e^{-\mu_i t}\right). \tag{25}$$

For large time ($t \to \infty$), the longitudinal dispersion coefficient D_e can be written as

$$D_e\bigg|_{t\to\infty} = \frac{1}{2}\frac{d\nu_2}{dt}\bigg|_{t\to\infty} = 1 + {P_e}^2 \sum_i \frac{2}{(i\pi)^6}$$

which is consistent with the asymptotic results of Chatwin [4] and Barton [3].

2.2 Periodic flow

If we consider the flow to be unsteady only due to the periodic pressure gradient, the velocity of the fluid can be obtained form Eq. (5) by setting $u_0 = 0, \varepsilon \neq 0$ and is given by

$$u = u_1(y,t) \tag{26}$$

where $u_1(y,t)$ is given by (7).

Using the general form of $\nu_2(t)$, from Eq. (20) we can easily derive the explicit expression for variance ν_2 taking into account the eigen-values and the corresponding eigen-functions of the eigen-value problem (17).

The expression for variance $\nu_2(t)$ is thus obtained as

$$\begin{aligned}\nu_2(t) = {} & 2\left(1 + {P_e}^2 \sum_i \frac{b_i(D_{1i}^2 + D_{2i}^2)}{2}\right)t - 2{P_e}^2 \sum_i {b_i}^2 \left(D_{2i} - \frac{\alpha S}{i^2\pi^2} D_{1i}\right) \\ & \times \left[\left(D_{2i}\frac{\alpha S}{i^2\pi^2} - D_{1i}\right)\sin\alpha St - \left(D_{2i} + \frac{\alpha S}{i^2\pi^2} D_{1i}\right)(\cos\alpha St - 1)\right] e^{-i^2\pi^2 t} \\ & + 2{P_e}^2 \sum_i \frac{b_i}{4\alpha S}\left[\left(D_{2i}^2 - D_{1i}^2 - 2D_{2i}D_{1i}\frac{\alpha S}{i^2\pi^2}\right)\sin 2\alpha St\right. \\ & \left. + \left\{2D_{2i}D_{1i} + \frac{\alpha S}{i^2\pi^2}(D_{2i}^2 - D_{1i}^2)\right\}(1 - \cos 2\alpha St)\right]\end{aligned} \tag{27}$$

where $b_i = \dfrac{(i\pi)^2}{(\alpha S)^2 + (i\pi)^4}$

$$D_{1i} = -\frac{\varepsilon}{\sqrt{2}\alpha({K_1}^2 + {K_2}^2)}(K_1R_2 + K_2R_1), \quad D_{2i} = -\frac{\varepsilon}{\sqrt{2}\alpha({K_1}^2 + {K_2}^2)}(K_1R_1 - K_2R_2),$$

$$\begin{aligned}R_1 = {} & \frac{2}{\alpha + (\sqrt{\alpha} + i\pi)^2}\left[\sqrt{\alpha/2}\,\sin(\sqrt{\alpha/2} + i\pi)\cosh\sqrt{\alpha/2}\right. \\ & \left. - (\sqrt{\alpha/2} + i\pi)\cos(\sqrt{\alpha/2} + i\pi)\sinh\sqrt{\alpha/2}\right] \\ & + \frac{2}{\alpha - (\sqrt{\alpha} + i\pi)^2}\left[\sqrt{\alpha/2}\,\sin(\sqrt{\alpha/2} - i\pi)\cosh\sqrt{\alpha/2}\right. \\ & \left. - (\sqrt{\alpha/2} + i\pi)\cos(\sqrt{\alpha/2} - i\pi)\sinh\sqrt{\alpha/2}\right],\end{aligned}$$

$$\begin{aligned}R_2 = {} & \frac{2}{\alpha + (\sqrt{\alpha} + i\pi)^2}\left[(\sqrt{\alpha/2} + i\pi)\sin(\sqrt{\alpha/2} + i\pi)\cosh\sqrt{\alpha/2}\right. \\ & \left. + \sqrt{\alpha/2}\,\cos(\sqrt{\alpha/2} + i\pi)\sinh\sqrt{\alpha/2}\right] \\ & + \frac{2}{\alpha - (\sqrt{\alpha} + i\pi)^2}\left[(\sqrt{\alpha/2} - i\pi)\sin(\sqrt{\alpha/2} - i\pi)\cosh\sqrt{\alpha/2}\right. \\ & \left. - \sqrt{\alpha/2}\,\cos(\sqrt{\alpha/2} - i\pi)\sinh\sqrt{\alpha/2}\right].\end{aligned}$$

The apparent dispersion coefficient D_a can be obtained from (21) as

$$\begin{aligned}D_a|_{\text{unst}} = {} & \sum_i \frac{b_i}{2}\left[(D_{1i}^2 + D_{2i}^2) + A_1\cos 2\alpha St + A_2 \sin 2\alpha St\right] \\ & - \sum b_i\left(D_{2i} - \frac{\alpha S}{i\pi} D_{1i}\right)(D_{2i}\cos\alpha St + D_{1i}\sin\alpha St)e^{-i^2\pi^2 t}\end{aligned} \tag{28}$$

where $A_1 = D_{2i}^2 - D_{1i}^2 - 2D_{2i}D_{1i}\dfrac{\alpha S}{i^2\pi^2}$ and $A_2 = 2D_{2i}D_{1i} + \dfrac{\alpha S}{i^2\pi^2}(D_{2i}^2 - D_{1i}^2)$.

For a large time after the release, the expression for $D_a|_{\text{unst}}$ reduces to the form

$$D_a|_{\text{unst}} \approx A_0 + [A_1 \cos 2\alpha St + A_2 \sin 2\alpha St] \tag{29}$$

where the constants A_0, A_1 and A_2 depend on ε, α and S. This result is consistent with the work of Chatwin [5]. It is observed that Eq. (29) consists of a steady A_0 and a fluctuating part within parentheses due to the periodicity in the flow.

As the parameters ε, α and S are involved in the expression of velocity u, the variance ν_2 and D_a are much more important from the physical point of view. The parameter ε indicates the extent to which the velocity profile deviates from the Poiseuille profile after the perturbation is introduced in the steady flow. On the other hand, the parameter $\alpha\left(=\dfrac{\omega L^2}{\nu}=\dfrac{L^2}{\nu}:\dfrac{1}{\omega}\right)$ is a measure of the ratio of the time required for viscous force to diffuse across the channel width (L^2/ν) to the period of imposed oscillation $(1/\omega)$, and the Schmidt number S is a measure of the ratio of the intensities of viscous diffusion and the molecular diffusion. Therefore, $\alpha S\left(=\dfrac{\omega L^2}{\nu}\cdot\dfrac{\nu}{D}\right)$ can be regarded as the ratio of the time taken for transverse variations in concentration to be smoothed out by molecular diffusion (L^2/D) to the period of imposed oscillation.

The effect of the oscillation parameter on the variance ν_2 and the apparent dispersion coefficient D_a will be discussed later on.

2.3 *Flow due to periodic pressure gradient with non-zero mean*

Now we fix our attention to the dispersion phenomena due to the shear effect produced by the combined action of steady and periodic currents as given by the velocity distribution (5). Using the eigen-values and the corresponding eigen-functions of the eigen-value problem (17) in the equation of variance (20), the explicit expression of $\nu_2(t)$ for the combined flow can be obtained and the corresponding expression for D_a is given by

$$\begin{aligned} D_a|_{\text{combined}} = D_a|_{\text{unst}} + \Bigg[\sum_i \bigg\{ \frac{2}{(i\pi)^6} + a_i'(D_1 \sin\alpha St + D_2 \cos\alpha St)\,(1+i^2\pi^2 b_i) \\ + b_i \alpha S\,(D_2 \sin\alpha St - D_1 \cos\alpha St)\bigg\} - \sum_i \bigg\{ a_i b_i \left(D_2 - \frac{\alpha S}{i^2\pi^2} D_1\right) \\ + a_i'(a_i + D_1 \sin\alpha St + D_2 \cos\alpha St)\bigg\}\, e^{-i^2\pi^2 t}\Bigg] \end{aligned} \tag{30}$$

where $a_i = \dfrac{(-1)^{i+1}\sqrt{2}}{i^2\pi^2}$, $\quad a_i' = \dfrac{a_i}{i^2\pi^2}$.

For a large time after the release, the expression for $D_a|_{\text{combined}}$ reduces to

$$D_a|_{\text{combined}} \approx A_0' + A_1' \cos 2\alpha St + A_2' \sin 2\alpha St + A_3' \sin\alpha St + A_4' \cos\alpha St \tag{31}$$

where the constants A_0' and $A's$ are real constants. This result is more general than that obtained by Chatwin [5].

The second term of the right-hand side of Eq. (30) under the parentheses is attributed to longitudinal dispersion due to non-zero mean in the pulsatile flow. The corresponding results for the steady flow can be obtained by putting $\varepsilon = 0$ in the expressions for the variance $\nu_2(t)$ and the apparent dispersion coefficient D_a (Eq. (30)) and are given by Eqs. (24) and (25) in the plane Poiseuille flow.

3 Discussion of results

The results obtained in the previous Section are discussed in the following paragraphs and the effect of oscillation parameter on the longitudinal dispersion has also been widely discussed.

The first order moment of the concentration distribution of solute C_1 can be obtained from Eq. (11) putting $n = 1$ in the case of unsteady flow $(u = u_1(y,t))$ and the combined action of steady and periodic current $(u = u_0(y) + u_1(y,t))$ separately. Following Mukherjee and Mazumder [7], the solution for C_1 in the case of unsteady flow is given by

$$C_1(y,t)|_{\text{unst}} = E_1 - \sqrt{2}\,P_e \sum_i \left(D_2 - D_1 \frac{\alpha S}{i^2\pi^2}\right) b_i \cos i\pi y\, e^{-i^2\pi^2 t} + (E_1 \cos \alpha St + E_2 \sin \alpha St)$$
$$\times \sqrt{2}\,P_e \sum_i \left[\left(D_2 - D_1 \frac{\alpha S}{i^2\pi^2}\right) \cos \alpha St + \left(D_1 + D_2 \frac{\alpha S}{i^2\pi^2} \sin \alpha St\right)\right] b_i \cos i\pi y$$

where $E_1 = -\dfrac{\varepsilon P_e}{2\alpha^2 S(K_1{}^2 + K_2{}^2)}\left(\sinh^2\sqrt{\alpha/2}\,\sin^2\sqrt{\alpha/2} + \cosh^2\sqrt{\alpha/2}\,\cos^2\sqrt{\alpha/2}\right)$ and
$E_2 = -\dfrac{\varepsilon P_e}{2\alpha^2 S(K_1{}^2 + K_2{}^2}\left(\sin\sqrt{\alpha/2}\,\cos\sqrt{\alpha/2} - \sinh\sqrt{\alpha/2}\,\cosh\sqrt{\alpha/2}\right)$.

Similarly, the expression for C_1 in the combined action of steady and periodic flow can be obtained as

$$C_1(y,t)|_{\text{combined}} = C_1|_{\text{unst}} + \left[\frac{1}{3}P_e t - \sqrt{2}\,P_e \sum_i a_i{}' \cos i\pi y e^{-i^2\pi^2 t} + \sqrt{2}\,P_e \sum_i a_i{}' \cos i\pi y\right].$$

The variation of C_1 against y for different phase values $\left(\alpha St = \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right)$ has been plotted in Fig. 1 for $\alpha = 0.5, 4.0$ when $\varepsilon = 1.5$ and $S = P_e = 10^3$. Figure 1 a shows the periodicity of C_1 with flow phases due to the oscillatory nature of the velocity profile. The variation of C_1 against y due to the combined effect of steady and periodic current is shown in Fig. 1 b and it is observed that C_1 oscillates with flow phases and that the effect of steady current in C_1 is more significant than the oscillatory one.

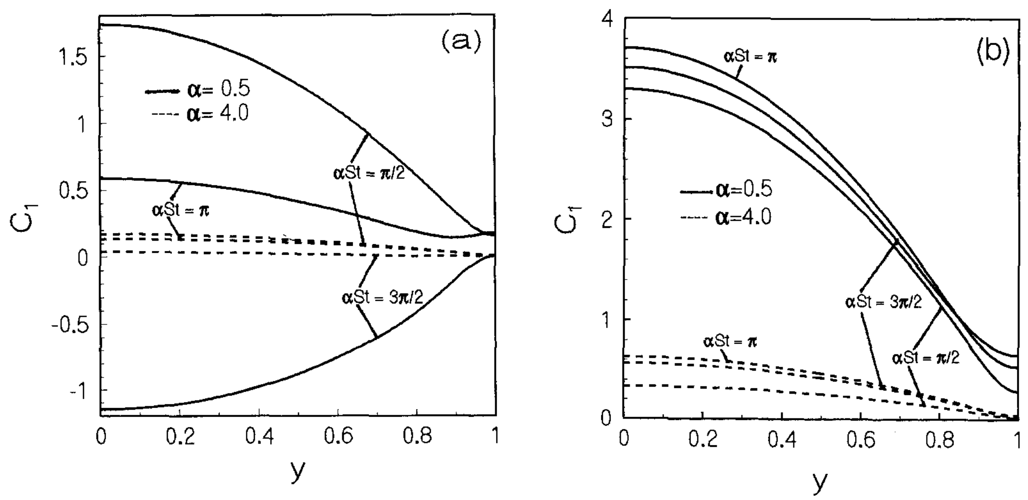

Fig. 1. First order moment C_1 against y (channel width) for different phases, when $S = P_e = 10^3, \varepsilon = 1.5$; **a** periodic flow, **b** combined effect of steady and periodic flow

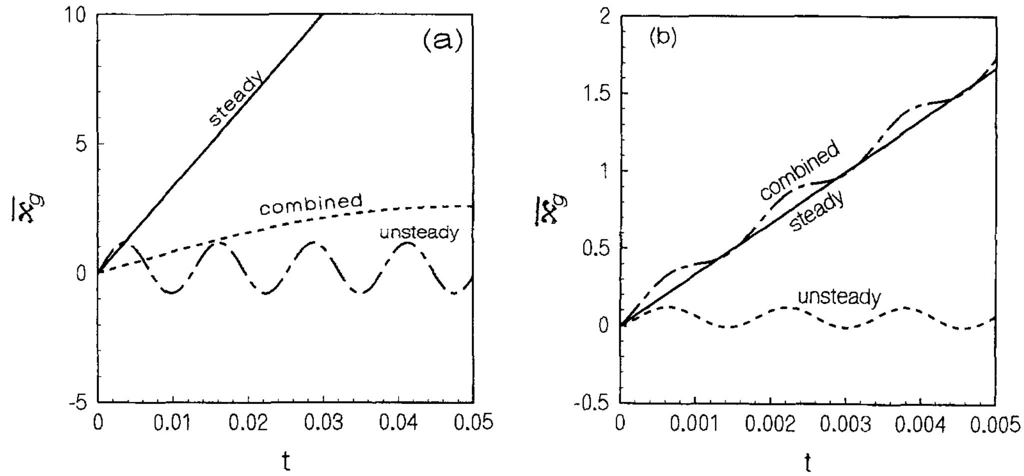

Fig. 2. Comparison of centroid displacement ($\bar{x}_g$) due to steady, periodic and the combined effect of steady and periodic flow, when $S = P_e = 10^3, \varepsilon = 1.5$; **a** $\alpha = 0.5$, **b** $\alpha = 4.0$

The first moment M_1 (putting $n = 1$ in (13)) indicates the mean concentration distribution over the cross-section of the channel. The mean longitudinal displacement $\bar{x}_g = (M_1/M_0)$ of the solute moving with the mean periodic velocity of the fluid mainly depends on α, ε and t. Figure 2 a, b shows the displacement of centroid ($\bar{x}_g$) for three different velocity profiles (steady, periodic and for comparison the combined effect of steady and periodic) and for $\alpha = 0.5, 4.0$. It is seen that for plane Poiseuille flow the centroid of slug increases linearly with time; for periodic flow, it increases with wavy nature and for combined flow, the centroid moves cyclically with the oscillatory nature of the flow, and it changes asymptotically over a period.

Figure 3 a–d represents the plots of variance ν_2 of the longitudinal concentration distribution against the dispersion time (t) for low and high frequency of oscillations ($\alpha = 0.5, 4.0$) due to oscillatory current. In steady current variance ν_2 increases rapidly with dispersion time (t) which agrees well with that of Yasuda [11] in which he has shown the temporal changes of the vertically averaged variance (obtained from averaging the concentration of the diffusion substance). In ocillatory flow for small frequency ($\alpha = 0.5$), it is seen that the variance increases with time in a wavy nature. In one complete period, variance changes cyclically with a double-frequency (Fig. 3 a) and it reaches a stable state after a certain time ($\sim t > 0.3$), whereas for high frequency oscillation ($\alpha = 4.0$) the nature of double-frequency oscillation in the variance almost vanishes (Fig. 3 c) and it increases with a wavy nature (Yasuda [11]). Further, if we fix our attention on the temporal variation of ν_2 with α due to the combined effect of steady and oscillatory current, it is seen from Fig. 4 a, b that the variance for low frequency (ν_2) increases rapidly with time (t) in wavy pattern at small time ($0 < t < 0.05$). This phenomenon is still there for large ($\alpha = 4.0$) at small time (Fig. 4 c, d) and it reaches a steady state after a certain time. The nature of double-frequency oscillation diminishes for both cases. From these observations it may be concluded that the variance (ν_2) due to the oscillatory current was found to be much smaller than that due to the steady and the combined effect of steady and periodic currents. That is, the pulsatility of the flow arising out of a periodic pressure gradient reduces the value of ν_2.

Following Aris [1], we have already described the apparent longitudinal dispersion coefficient D_a as a function of α, ε, S and the dispersion time. This dispersion coefficient D_a will be discussed for each velocity distribution and different frequency of oscillation. In the case of a

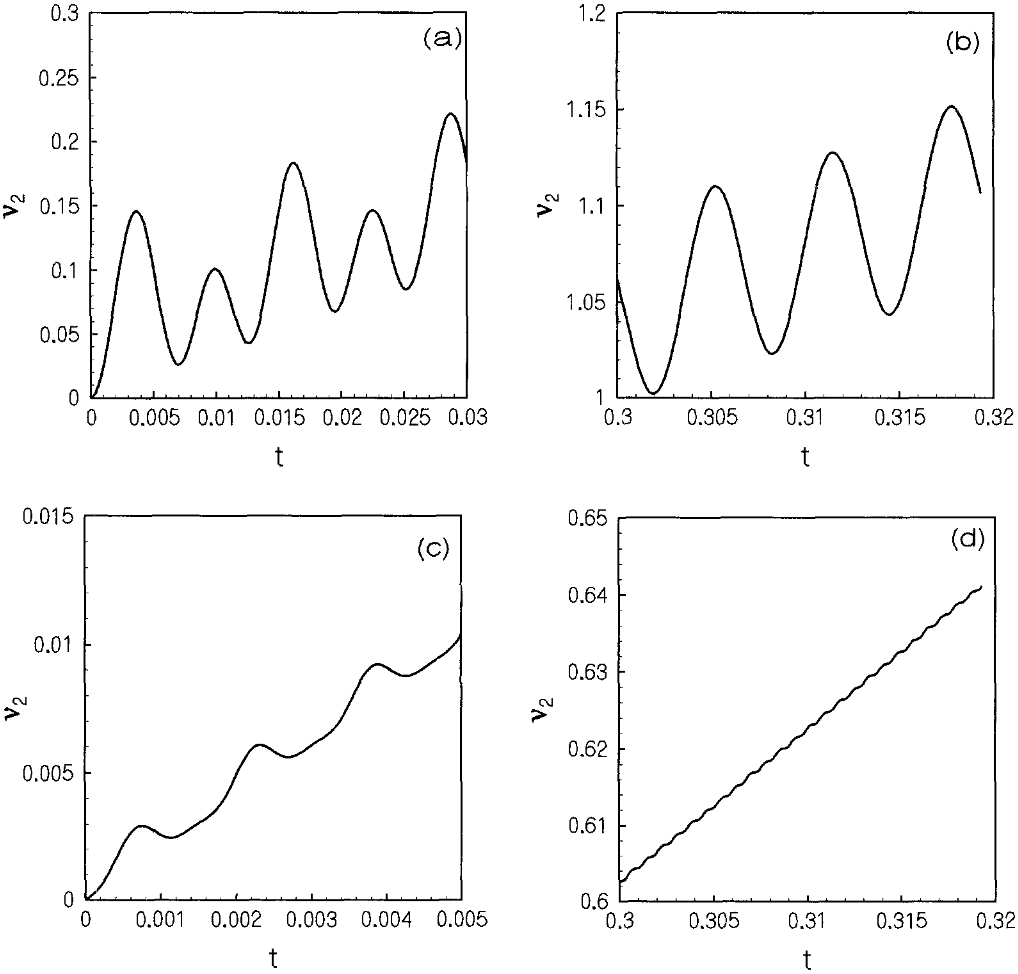

Fig. 3. The temporal variation of variance (ν_2) due to periodic flow for $S = P_e = 10^3, \varepsilon = 1.5$; **a** small time, **b** large time when $\alpha = 0.5$; **c** small time, **d** large time when $\alpha = 4.0$

steady flow ($u = u_0$), the dispersion coefficient D_a increases with time (t) and asymptotically reaches a steady state (~ 0.0023) at dimensionless time ~ 0.34. It is interesting to note that the dispersion coefficient for steady flow through a channel is much smaller than that of the flow through a tube. Smith [9] pointed out that in steady flow the apparent longitudinal dispersion coefficient can exceed molecular diffusivity or eddy diffusivity by many orders of magnitude, and high apparent diffusivities are only achieved after the solute has been mixed right across the flow. The variation of the apparent dispersion coefficient D_a with dispersion time t in the oscillatory flow ($u = u_1(y, t)$) is plotted in Fig. 5 for $\alpha = 0.5$, Fig. 6 for $\alpha = 1.0$ and Fig. 7 for $\alpha = 4.0$. Now $\sqrt{\alpha}$ can also be considered as the ratio of the half of channel width (L) to the thickness of the Stokes boundary layer $\sqrt{\nu/\omega}$. A small value of α implies a large viscous layer near the wall compared with a small inviscid core near the center or, alternatively, a large oscillation period compared with viscous diffusion time and therefore quasi-steady flow and vice-versa for large α. When $\sqrt{\alpha S} = \sqrt{\omega L^2/D}$ is large then the diffusion term in Eq. (3) for C is much less than the time-derivative one and consequently the concentration varies very little in the transverse direction except in the thin boundary layer. When $\sqrt{\alpha S}$ is

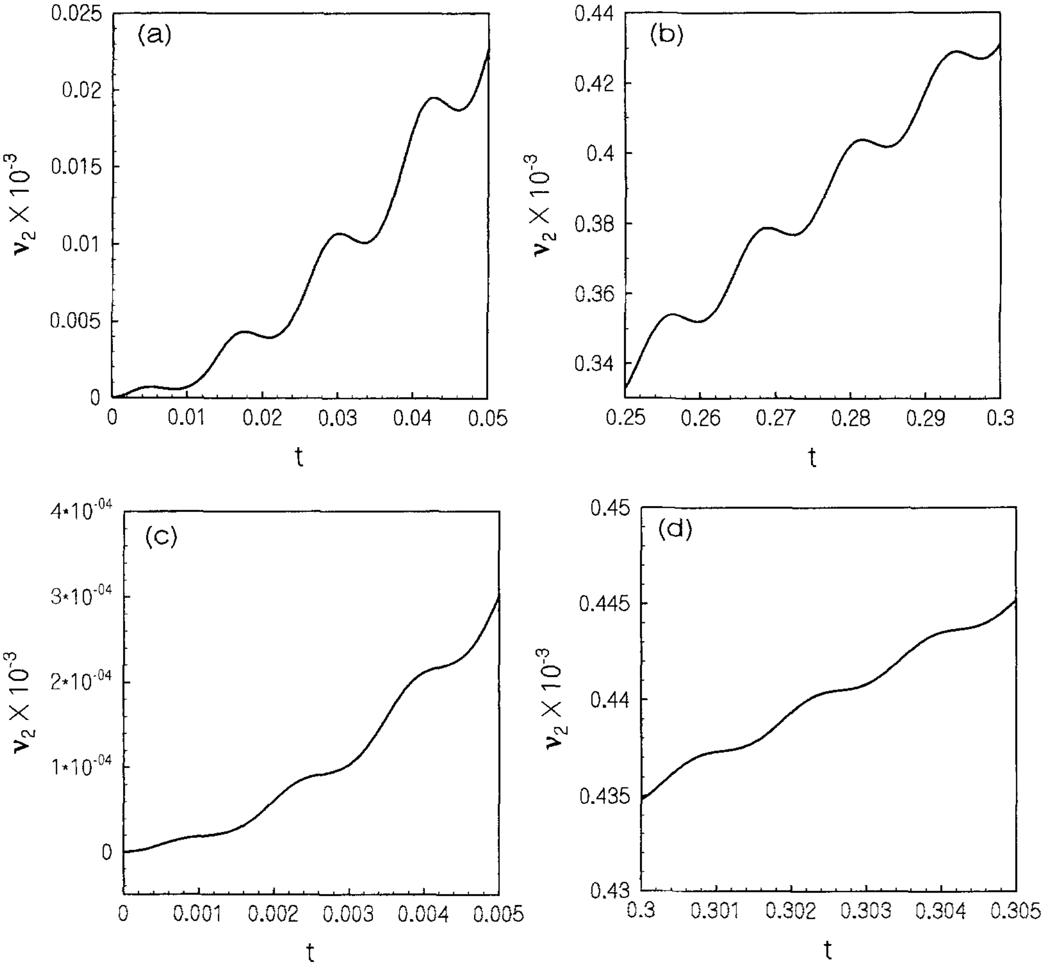

Fig. 4. The temporal variation of variance (ν_2) due to combined shear effects (steady and periodic) for $S = P_e = 10^3, \varepsilon = 1.5$; **a** small time, **b** large time when $\alpha = 0.5$; **c** small time, **d** large time when $\alpha = 4.0$

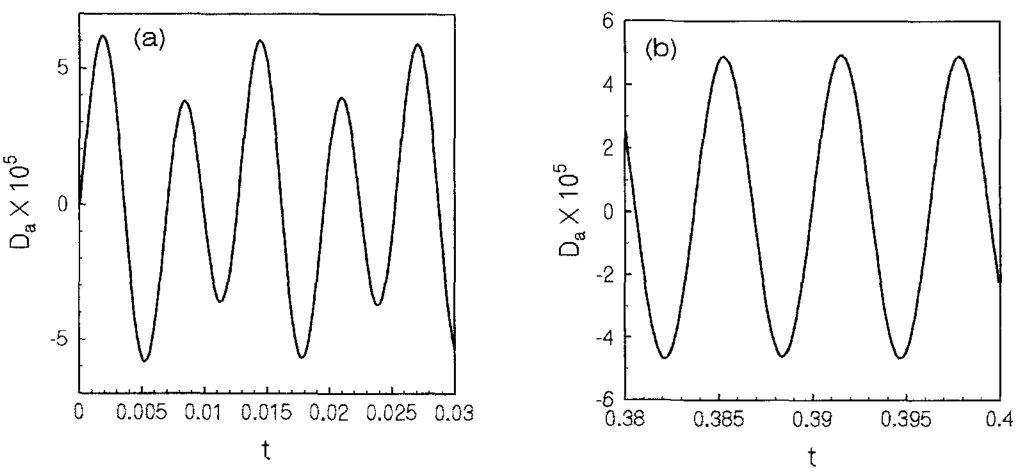

Fig. 5. The dispersion coefficient D_a due to periodic flow; **a** small time, **b** large time when $S = 10^3$, $\varepsilon = 1.5, \alpha = 0.5$

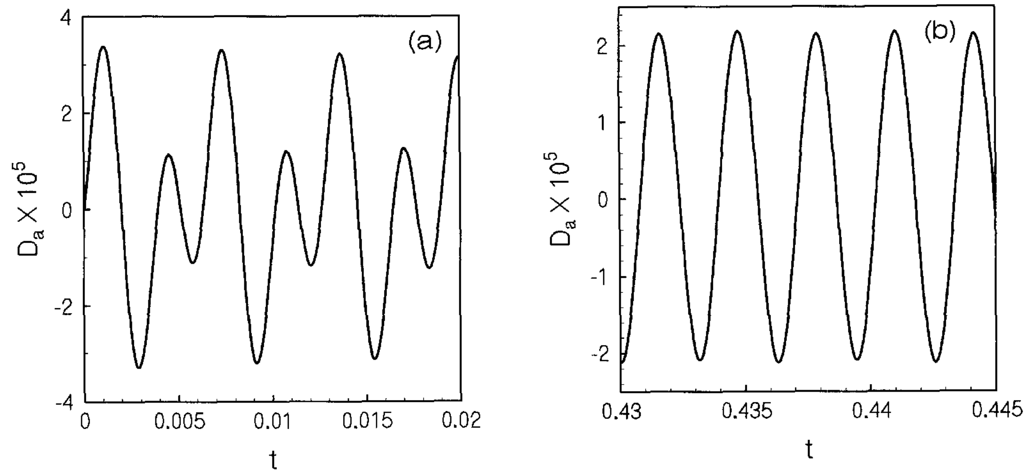

Fig. 6. As Fig. 5 but for $\alpha = 1.0$

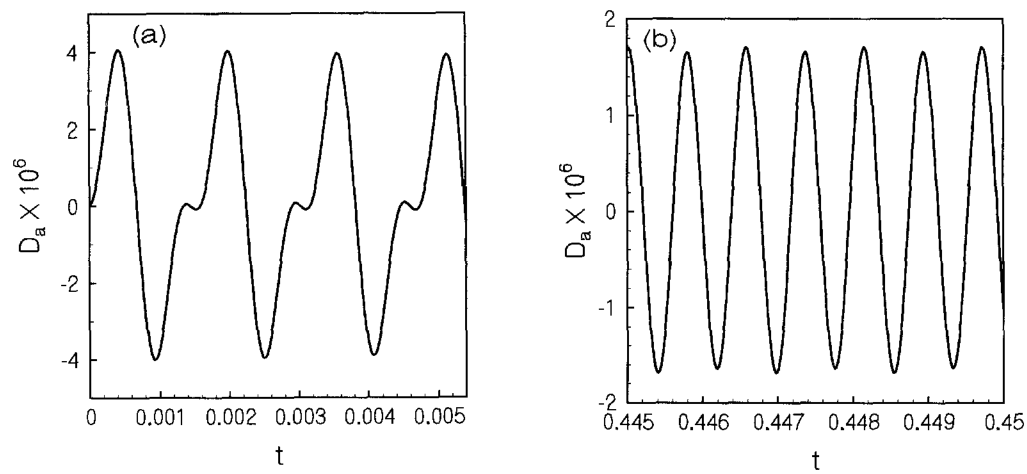

Fig. 7. As Fig. 5 but for $\alpha = 4.0$

small, the diffusion terms are more dominant and concentration is unaffected by the periodic pressure. From Fig. 5 it can be seen that D_a changes cyclically with a double-frequency period in oscillatory flow and after a certain time it reaches a stationary state. At low frequency ($\alpha = 0.5$) of oscillation the amplitudes of D_a are approximately equal for all time (Fig. 5 a) whereas in the case of high frequency ($\alpha = 4.0$) D_a varies cyclically with the same frequency of oscillation as the periodic current during the initial stage and then it fluctuates with a double-frequency oscillation. It is observed from the figures that for small α the dispersion coefficient D_a reaches the steady state earlier than that for large α. Figures 6 and 7 show that for high frequency D_a is more significant during the first half of the period than the second one. The solute disperses at a fairly uniform rate after a certain time ($t \sim 0.4$), that means D_a oscillates steadily (Yasuda [11]). The fluctuations in the velocity profile induce the positive and negative dispersion during the period of oscillation. A negative dispersion coefficient has been obtained due to the reversing flow of oscillatory currents (Smith [9]) at a particular level and D_a decreases with increasing α which shows that due to the high frequency of oscillation D_a becomes negligible although for steady and quasi-steady flow it is more significant. It can

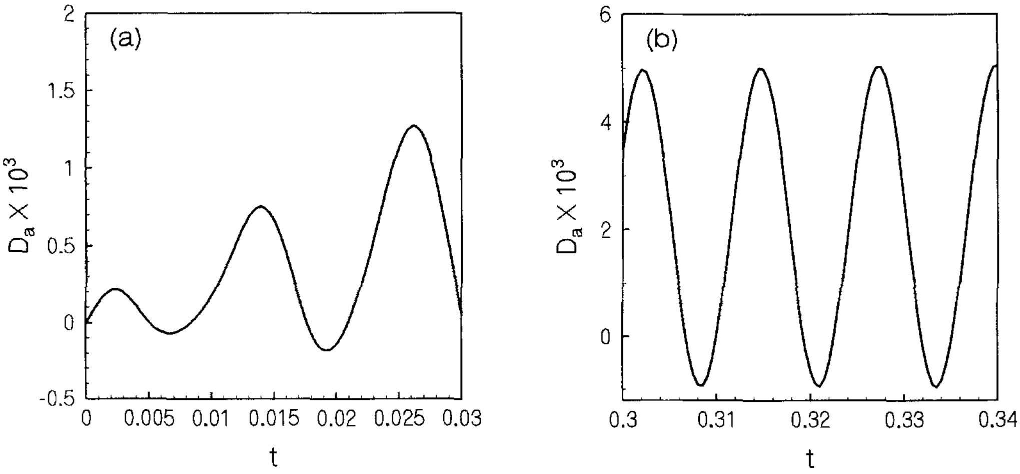

Fig. 8. The dispersion coefficient D_a due to combined shear effects (steady and periodic) for $S = 10^3$, $\varepsilon = 1.5$, **a** small time, **b** large time when $\alpha = 0.5$

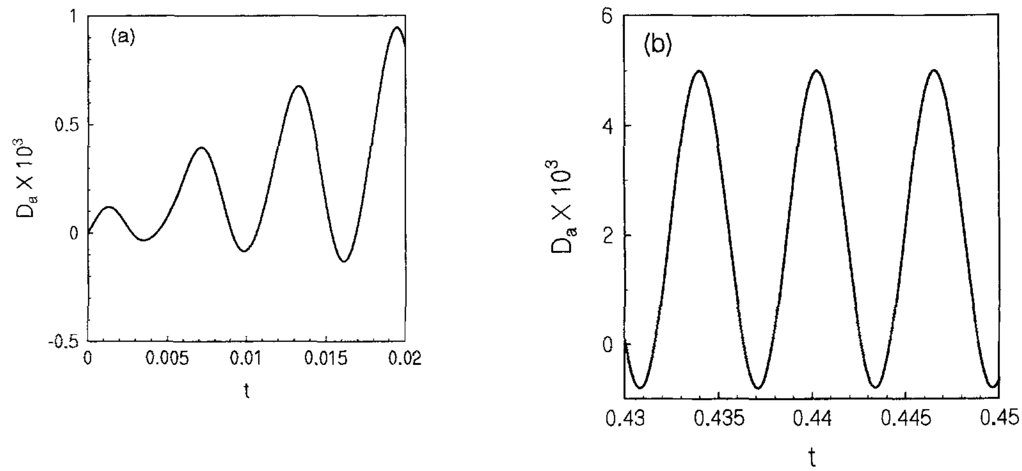

Fig. 9. As Fig. 8 but for $\alpha = 1.0$

be seen that at a fixed instant the amplitude of D_a increases with increase in the amplitude of the pressure pulsation.

For comparison, the plots of the apparent longitudinal dispersion coefficient D_a due to the combined effect of steady and oscillatory current for different frequency of oscillation α ($\varepsilon = 1.5, S = 10^3$) are also shown in Fig. 8 for $\alpha = 0.5$, Fig. 9 for $\alpha = 1.0$ and Fig. 10 for $\alpha = 4.0$. It is observed that D_a in the combined flow is much more significant than D_a in the oscillatory flow and it is also seen that in the combined flow D_a has no longer the double-frequency period. The amplitude of oscillation increases initially upto a certain time and then it stabilizes for a long time which means that the apparent diffusion coefficient D_a due to steady flow plays a more significant role than that due to the periodic flow. The variations of D_a for the frequency parameter $\alpha = 1.0, 4.0$ have been plotted in Fig. 11 a, b for $S = 1.0, \varepsilon = 1.5$ for unsteady and combined flow. It is thus quite clear that both amplitude and frequency of pressure pulsation exert enormous influence on the longitudinal dispersion due to unsteady as well as combined flow. The most interesting fact is that the apparent dispersion coefficient D_a for both unsteady and combined flow consists of a steady part (A_0

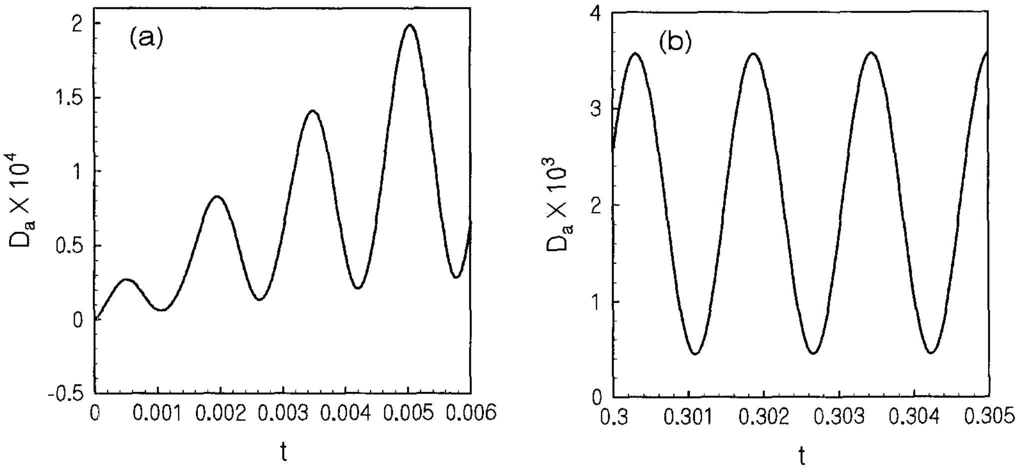

Fig. 10. As Fig. 8 but for $\alpha = 4.0$

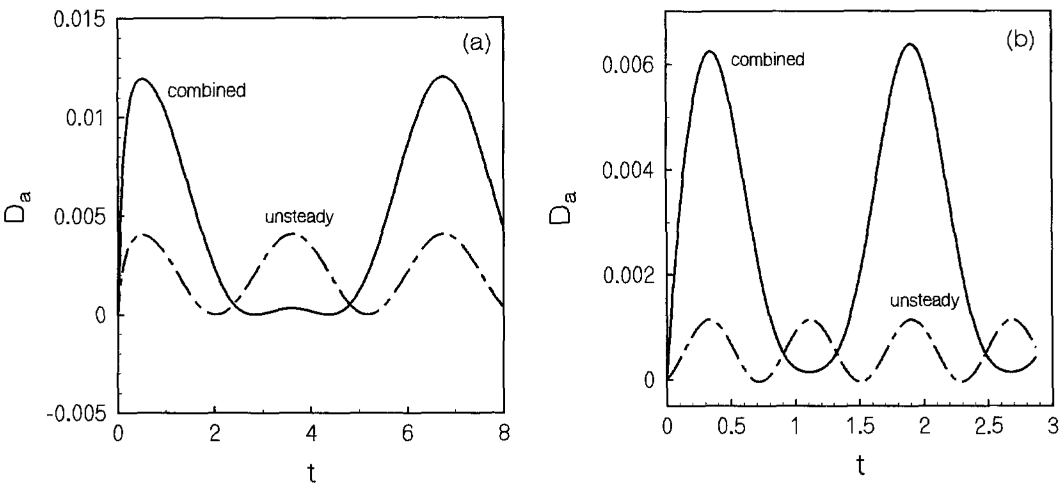

Fig. 11. Comparison of dispersion coefficient D_a against time for $S = 1.0, \varepsilon = 1.5$; **a** $\alpha = 1.0$, **b** $\alpha = 4.0$

in Eq. (29) and A_0' in Eq. (31)) and a fluctuating part due to the pulsatility of the flow. It is also noted that there is a remarkable difference in the behaviour of the longitudinal dispersion coefficient for small and large values of frequency of imposed oscillation.

4 Conclusion

We have focussed our attention to the dispersion processes of contaminant molecules due to the shear effect individually generated by steady, oscillatory and the combined action of steady and oscillatory currents through a parallel plate channel; and also we have compared some specific results with particular emphasis on the role played by non-zero mean flow. All the investigations have been done for flow velocities when the slug is released at the maximum pressure at time (t) equal to zero, given an initially uniform slug over the cross-section of the channel and a large Péclet number.

The temporal variation of centroid displacement ($\bar{x}_g$) of the slug due to the periodic current is much smaller than that for the steady or the combined steady and periodic currents. As the periodic flow moves with the mean flow in the same direction, the effect of frequency of oscillation due to pulsatility in pressure gradient on the central displacement becomes less significant, and hence it behaves like quasi-steady.

The apparent dispersion coefficient (D_a) reaches a stationary state after a certain time for all cases of velocity profiles, though it changes cyclically with time for periodic and combined flow. It is important to note that the dispersion coefficient (D_a) due to oscillatory current is much smaller than that due to steady and combined effect (flow due to periodic with non-zero mean). For low frequency of oscillation, D_a varies cyclically with double-frequency period from the initial stage, whereas for high frequency of oscillation it appears that D_a goes on changing cyclically with the same frequency as the periodic current at the initial stage, and then it oscillates with double-frequency period at large time. When these are compared with D_a due to the combined shearing effect of steady and oscillatory currents, D_a no longer has a double-frequency period, which means that the influence of mean flow plays a significant role in the dispersion process. It is also quite interesting to note from the analysis that the apparent dispersion coefficient due to pulsatility of the flow consists of a steady and a time-dependent part, which is consistent with the results of Chatwin [5]. Both amplitude and frequency of pressure pulsation exert enormous influence on the steady as well as the time-dependent parts of the longitudinal dispersion coefficient.

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