Conditionally exactly solvable problems and non-linear algebras

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Abstract

Using ideas of supersymmetric quantum mechanics we construct a class of conditionally exactly solvable potentials which are supersymmetric partners of the linear and radial harmonic oscillator. Furthermore we show that this class of problems possesses some symmetry structures which belong to non-linear algebras.

1. Introduction

In quantum mechanics, there are only a few potentials for which the Schrödinger equation is exactly solvable [1]. The class of exactly solvable problems can however be enlarged by using the method of generating isospectral Hamiltonians [2–6]. Recently a new type of problems has been added to the class of exactly solvable ones. These are called conditionally exactly solvable (CES) problems [7,8]. The main characteristic of the CES problems is that they are exactly solvable when the potential parameters assume some specific numerical values. In this Letter our aim is to construct a number of CES potentials using ideas of supersymmetric (SUSY) quantum mechanics [9,10]. We shall also study the symmetry structure of these problems and it will be shown that these symmetries are related to some closed non-linear algebras.

In the next section we will briefly review some algebraic properties of SUSY quantum mechanics which are relevant in our construction of CES potentials. In Section 3 we construct CES potentials which are partner potentials of the one-dimensional harmonic oscillator. Besides the unbroken SUSY these problems also possess a non-linear algebra which is quadratic. Section 4 discusses the same approach for the radial harmonic oscillator. Here SUSY is broken and the resulting non-linear algebra for the CES problems is of cubic type. Finally, in Section 4 we outline a general construction principle for CES potentials which are the SUSY partners of exactly solvable ones.

2. Supersymmetric quantum mechanics

In supersymmetric quantum mechanics [9,10] one considers a pair of so-called partner Hamiltonians $H_\pm$ which are defined by

$$\begin{aligned} H_+ &= AA^\dagger = -\tfrac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x)\,, \\ H_- &= A^\dagger A = -\tfrac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x)\,, \end{aligned} \tag{1}$$

$$V_\pm(x) = \tfrac{1}{2}W^2(x) \pm \tfrac{1}{2}W'(x)\,, \tag{2}$$

where $W(x)$ denotes the SUSY potential, $W'(x) = (d/dx)W(x)$, and the operators A and $A^\dagger$ are given by

$$A = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + W(x)\right),$$
$$A^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + W(x)\right). \tag{3}$$

Possible zero-energy eigenfunctions of $H_\pm$ are necessarily of the form

$$\Psi_0^{(\pm)}(x) = C\exp\left(\pm\int^x dt\, W(t)\right), \tag{4}$$

with C being the normalisation constant. If either of the functions (4) is normalisable then supersymmetry is said to be unbroken, while if none of them is normalisable supersymmetry is broken.

In the case of unbroken SUSY (let us assume that $\Psi_0^{(-)}$ is normalisable) we have the following spectral properties among the eigenvalues $E_n^{(\pm)}$ and eigenfunctions $\Psi_n^{(\pm)}$ of the partner Hamiltonians $H_\pm$, $n = 0,1,2,\ldots$,

$$E_0^{(-)} = 0, \quad E_{n+1}^{(-)} = E_n^{(+)} > 0, \tag{5}$$

$$\Psi_n^{(+)} = \frac{1}{\sqrt{E_{n+1}^{(-)}}} A\Psi_{n+1}^{(-)},$$
$$\Psi_{n+1}^{(-)} = \frac{1}{\sqrt{E_n^{(+)}}} A^\dagger\Psi_n^{(+)}. \tag{6}$$

On the other hand in the case of broken SUSY we have the relations

$$E_n^{(+)} = E_n^{(-)} > 0, \tag{7}$$

$$\Psi_n^{(+)} = \frac{1}{\sqrt{E_n^{(-)}}} A\Psi_n^{(-)},$$
$$\Psi_n^{(-)} = \frac{1}{\sqrt{E_n^{(+)}}} A^\dagger\Psi_n^{(+)}. \tag{8}$$

3. Construction of CES potentials for unbroken SUSY

Let us now turn to the construction of the CES potentials which are SUSY partners of the harmonic oscillator potential on the real line. To this end we consider the following family of SUSY potentials [11],

$$W(x) = x + \sum_{i=1}^{N}\frac{d}{dx}\ln(1+g_i x^2)$$
$$= x + \sum_{i=1}^{N}\frac{2g_i x}{1+g_i x^2}, \quad g_i \geqslant 0, \tag{9}$$

which reduces to the SUSY potential of the linear harmonic oscillator for $g_1 = g_2 = \ldots = g_N = 0$. The corresponding ground-state wavefunction for H_- reads according to (4)

$$\Psi_0^{(-)}(x) = C\exp(-x^2/2)\prod_{i=1}^{N}(1+g_i x^2)^{-1} \tag{10}$$

and is normalisable. Hence, SUSY is unbroken.

In order to demonstrate our principle for constructing CES potentials with the help of SUSY quantum mechanics let us first consider the case $N = 1$. In this case the partner potentials (2) read

$$V_+(x) = \frac{x^2}{2} + \frac{g_1 - 2}{1+g_1x^2} + \frac{5}{2}, \tag{11}$$

$$V_-(x) = \frac{x^2}{2} - \frac{g_1 + 2}{1+g_1x^2} + \frac{4g_1^2x^2}{(1+g_1x^2)^2} + \frac{3}{2}. \tag{12}$$

Now if we put $g_1 = 2$ then the first of these potentials, V_+, reduces to that of the linear harmonic oscillator,

$$V_+(x) = \frac{x^2}{2} + \frac{5}{2}, \tag{13}$$

$$V_-(x) = \frac{x^2}{2} - \frac{4}{(1+2x^2)} + \frac{16x^2}{(1+2x^2)^2} + \frac{3}{2}. \tag{14}$$

In this case the energy eigenvalues of the Hamiltonian H_+ are obviously given by $E_n^{(+)} = n + 3$ and the corresponding eigenfunctions (H_n denotes the Hermite polynomial of degree n)

$$\Psi_n^{(+)}(x) = \pi^{-1/4}(2^n n!)^{-1/2}H_n(x)\exp(-x^2/2) \tag{15}$$

are the well-known eigenfunctions of the linear harmonic oscillator [1,12]. Because of this knowledge we can now obtain the eigenvalues and eigenfunctions of H_- associated with the potential (14) via the SUSY relations (5) and (6). The ground-state wavefunction follows from (10) and the ground-state energy vanishes because of unbroken SUSY. To summarise,

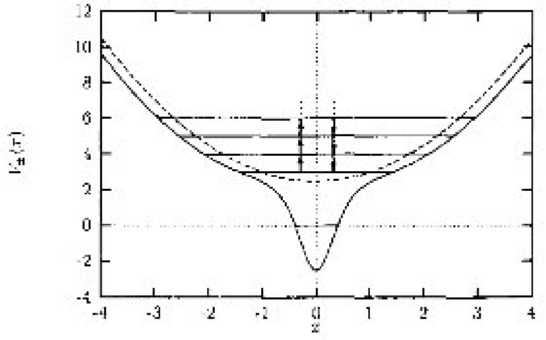

Fig. 1. The CES potential (14) (solid line) and its SUSY partner (13) (dashed line). The horizontal lines indicate the first five eigenvalues of H_- associated with this CES potential and the up and down arrows sketch the action of the creation and annihilation operators $B^\dagger$ and B on eigenstates of H_-. Note that the ground state ($E_0^{(-)}=0$) is isolated due to unbroken SUSY.

the complete spectral properties of H_- with potential (14) are given by ($n=0,1,2,\ldots$)

$$\begin{aligned} E_0^{(-)} &= 0\,, \quad E_{n+1}^{(-)} = n+3\,, \\ \Psi_0^{(-)}(x) &= C(1+2x^2)^{-1}\exp(-x^2/2)\,, \\ \Psi_{n+1}^{(-)}(x) &= (n+3)^{-1/2}A^\dagger\Psi_n^{(+)}(x) \\ &= \frac{\exp(-x^2/2)}{\sqrt{2^{n+1}n!(n+3)\sqrt{\pi}}} \\ &\times\left(H_{n+1}(x)+\frac{4x}{1+2x^2}H_n(x)\right). \end{aligned} \tag{16}$$

Here we would like to stress that the potential in (12) is not exactly solvable unless $g_1=2$. That is, (12) is indeed a CES potential. In Fig. 1 we have shown graphs of the two potentials (13) and (14).

Here naturally arises the question whether this conditionally exact solvability of H_- can be related to some underlying symmetry structure. This expectation is supported by the fact that its SUSY partner H_+ does have a well-known Lie-algebra structure which allows for a complete and exact solution of its eigenvalue problem by pure algebraic methods. Introducing the standard harmonic-oscillator annihilation and creation operators

$$a=\frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}+x\right), \qquad a^\dagger=\frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x}+x\right), \tag{17}$$

and noting that $H_+=a^\dagger a+3$ one immediately verifies that

$$[H_+,a]=-a\,, \quad [H_+,a^\dagger]=a^\dagger\,, \quad [a,a^\dagger]=1. \tag{18}$$

The annihilation and creation operators act on the eigenstates of H_+ as follows,

$$a\Psi_n^{(+)}=\sqrt{n}\,\Psi_{n-1}^{(+)}\,, \qquad a^\dagger\Psi_n^{(+)}=\sqrt{n+1}\,\Psi_{n+1}^{(+)}. \tag{19}$$

Now the SUSY relation (5) and (6) suggests to construct similar annihilation and creation operators for the system H_- [3],

$$B=A^\dagger aA\,, \quad B^\dagger=A^\dagger a^\dagger A\,. \tag{20}$$

Actually, from (6) with $E_{n+1}^{(-)}=E_n^{(+)}=n+3$ and (19) one finds ($n=0,1,2,\ldots$)

$$\begin{aligned} B\Psi_{n+1}^{(-)} &= \sqrt{n(n+2)(n+3)}\,\Psi_n^{(-)}\,, \\ B^\dagger\Psi_{n+1}^{(-)} &= \sqrt{(n+1)(n+3)(n+4)}\,\Psi_{n+2}^{(-)} \end{aligned} \tag{21}$$

and $B\Psi_0^{(-)}=B^\dagger\Psi_0^{(-)}=0$. Hence, by repeated application of the operator $B^\dagger$ we can create all states above the first excited state $\Psi_1^{(-)}$ of H_-,

$$\Psi_{n+1}^{(-)}=\sqrt{\frac{2!3!}{n!(n+2)!(n+3)!}}\,\left(B^\dagger\right)^n\Psi_1^{(-)}, \tag{22}$$

where the normalised state $\Psi_1^{(-)}$ is given in (16). With the help of (21) it can be shown that H_-, B and $B^\dagger$ satisfy the following commutation relations,

$$\begin{aligned} &[H_-,B]=-B\,, \quad [H_-,B^\dagger]=B^\dagger\,, \\ &[B^\dagger,B]=5H_- - 3H_-^2 \end{aligned} \tag{23}$$

and thus the symmetry algebra of the CES problem H_- is a quadratic one. Let us note that the algebra (23) as it stands is only defined on the orthogonal complement of the kernel of H_-. That is, on the space spanned by the excited energy eigenfunctions of H_- (cf. Fig. 1). However, because of the relations $H_-\Psi_0^{(-)}=B\Psi_0^{(-)}=B^\dagger\Psi_0^{(-)}=0$ the domain for this algebra may trivially be extended to the full Hilbert space $L^2(\mathbb{R})$. Nevertheless, the state $\Psi_1^{(-)}$ cannot be created via $B^\dagger$ from the ground state $\Psi_0^{(-)}$ and vice versa $\Psi_0^{(-)}$ cannot be reached via an application of B on $\Psi_1^{(-)}$ because $B\Psi_1^{(-)} = 0$.

Next we consider the case $N = 2$ in (9). As in the previous case we find here that for a particular choice of the now two parameters g_1 and g_2 the potential V_+ reduced to that of the harmonic oscillator. To be more explicit we have

$$V_+(x) = \frac{x^2}{2} + \frac{9}{2}, \tag{24}$$

$$V_-(x) = \frac{x^2}{2} + \frac{4g_1g_2/(g_2-g_1) - 2 - g_1}{1+g_1x^2} - \frac{4g_1g_2/(g_2-g_1) + 2 + g_2}{1+g_2x^2} + \frac{4g_1x^2}{(1+g_1x^2)^2} + \frac{4g_2x^2}{(1+g_2x^2)^2} + \frac{7}{2}, \tag{25}$$

where we have used the following parameter set in V_+,

$$g_1 = 2 - \sqrt{8/3}, \quad g_2 = 2 + \sqrt{8/3}. \tag{26}$$

For this set of parameters we can now obtain (also via SUSY) the spectral properties of H_- with potential V_- as given in (25). Because of unbroken SUSY we have ($n = 0, 1, 2, \ldots$)

$$\begin{aligned} &E_0^{(-)} = 0, \quad E_{n+1}^{(-)} = n + 5, \\ &\Psi_0^{(-)}(x) = C(1+g_1x^2)^{-1} \\ &\quad\times (1+g_2x^2)^{-1}\exp(-x^2/2), \\ &\Psi_{n+1}^{(-)}(x) = (n+5)^{-1/2}A^\dagger\Psi_n^{(+)}(x) \\ &\quad= \frac{\exp(-x^2/2)}{\sqrt{2^{n+1}n!(n+5)\sqrt{\pi}}} \\ &\quad\times\left[H_{n+1}(x) + \left(\frac{2g_1x}{1+g_1x^2} + \frac{2g_2x}{1+g_2x^2}\right)H_n(x)\right]. \end{aligned} \tag{27}$$

Thus V_- as given in (25) is yet another CES potential, that is, exactly solvable for the particular choice (26) of the two parameters g_1 and g_2.

In this case the algebra corresponding to (23) is given by

$$\begin{aligned} &[H_-, B] = -B, \quad [H_-, B^\dagger] = B^\dagger, \\ &[B^\dagger, B] = 9H_- - 3H_-^2, \end{aligned} \tag{28}$$

where B and $B^\dagger$ are defined as in (20) but now with

$$A = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x + \frac{2g_1x}{1+g_1x^2} + \frac{2g_2x}{1+g_2x^2}\right). \tag{29}$$

Similarly, we have relations analogous to those given in (21) where the factors $(n-2)$, $(n+3)$ and $(n+4)$ have to be replaced by $(n+4)$, $(n+5)$ and $(n+6)$, respectively. Finally the relation analogous to (22) reads

$$\Psi_{n+1}^{(-)} = \sqrt{\frac{4!5!}{n!(n+4)!(n+5)!}}\,(B^\dagger)^n\Psi_1^{(-)}, \quad n = 0, 1, \ldots, \tag{30}$$

and the ground-state wavefunction is given by (10) with $N = 2$ and parameters (26).

Clearly if we consider higher values of N, we can obtain further CES potentials which are SUSY partners of the harmonic oscillator in one dimension. The underlying symmetry algebra for the excited states will be of the form

$$\begin{aligned} &[H_-, B] = -B, \quad [H_-, B^\dagger] = B^\dagger, \\ &[B^\dagger, B] = \alpha H_- - 3H_-^2, \end{aligned} \tag{31}$$

where α is some constant depending on the N potential parameters $g_1, g_2, \ldots, g_N$.

4. Construction of CES potentials for broken SUSY

So far we have studied CES systems for unbroken supersymmetry. In this section we shall construct CES potentials which are SUSY partners of the radial harmonic oscillator with broken SUSY. In other words we will now consider quantum mechanics on the half line $x \geqslant 0$. The family of SUSY potentials which we will consider is a generalisation of the previous one,

$$W(x) = x + \sum_{i=1}^{N}\frac{2g_ix}{1+g_ix^2} + \frac{\gamma+1}{x}, \quad \gamma \geqslant 0. \tag{32}$$

The corresponding zero-energy solutions read according to (4)

$$\Psi_0^{(\pm)}(x) = Cx^{\pm(\gamma+1)}\exp(\pm x^2/2)\prod_{i=1}^{N}(1+g_ix^2)^{\pm 1}. \tag{33}$$

Clearly, because of $\gamma \geqslant 0$, neither $\Psi_0^{(+)}$ nor $\Psi_0^{(-)}$ are normalisable and hence SUSY is broken.

Here let us only consider the simplest case $N = 1$ for which the partner potentials read

$$V_-(x) = \frac{x^2}{2} + \frac{\gamma(\gamma+1)}{2x^2} + \frac{2\gamma g_1 + 3g_1 - 2}{1 + g_1 x^2} + \frac{2\gamma + 7}{2}, \tag{34}$$

$$V_+(x) = \frac{x^2}{2} + \frac{(\gamma+1)(\gamma+2)}{2x^2} + \frac{2\gamma+5}{2} + \frac{4g_1^2 x^2}{(1+g_1x^2)^2} + \frac{2\gamma g_1 + g_1 - 2}{1 + g_1 x^2}. \tag{35}$$

Obviously, for $g_1 = 2/(2\gamma + 3)$ the potential V_+ reduces to that of the "radial harmonic oscillator". That is, the effective radial potential of a three-dimensional isotropic harmonic oscillator for a given angular momentum γ. Note that we do not limit ourselves to integer values of γ but allow for all non-negative real values $\gamma \geqslant 0$. The spectral properties of this one-dimensional quantum system are well known [1,12],

$$E_n^{(+)} = 2n + 2\gamma + 5, \quad n = 0, 1, 2, \ldots, \tag{36}$$

$$\Psi_n^{(+)}(x) = \sqrt{\frac{2\,n!}{\Gamma(n+\gamma+\frac{3}{2})}} \times L_n^{\gamma+1/2}(x^2)\,x^{\gamma+1}\exp(-x^2/2), \tag{37}$$

with L_n^α denoting the Laguerre polynomial of degree n and parameter α [12]. Therefore, because of broken SUSY by (7) and (8) we have

$$E_n^{(-)} = 2n + 2\gamma + 5, \qquad \Psi_n^{(-)} = \frac{1}{\sqrt{2n + 2\gamma + 5}} A^\dagger \Psi_n^{(+)}, \tag{38}$$

where

$$A^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{\gamma+1}{x} + \frac{2g_1 x}{1+g_1x^2}\right).$$

Thus we have found yet another CES potential, note that $g_1 = 2/(2\gamma - 3)$, which is associated with the radial harmonic oscillator. See Fig. 2 for the graphs of the two partner potentials (34) and (35).

With the help of the annihilation and creation operators (19) for the harmonic oscillator on the real line we may introduce annihilation and creation operators for the radial harmonic oscillator [12],

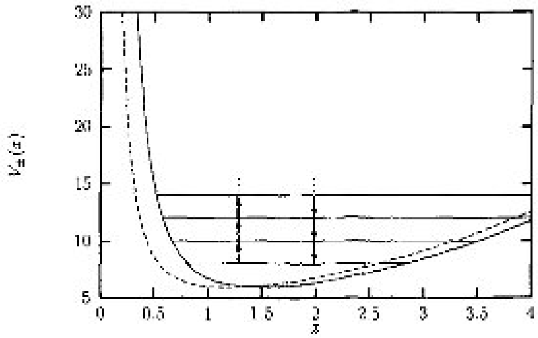

Fig. 2. The CES potential (35) (solid line) and its SUSY partner (34) (dashed line) for $\gamma = 1$ and $g_1 = 2/5$. The horizontal lines indicate the first four eigenvalues of the corresponding SUSY Hamiltonians $H_\pm$. Note that due to broken SUSY the creation and annihilation operators $D^\dagger$ and D act on all eigenstates of H_- as indicated by the up and down arrows.

$$c = a^2 - \frac{\gamma(\gamma+1)}{2x^2}, \qquad c^\dagger = (a^\dagger)^2 - \frac{\gamma(\gamma+1)}{2x^2}, \tag{39}$$

which together with H_+ for (34) obey the following linear algebra [12],

$$[H_+, c] = -2c, \quad [H_+, c^\dagger] = 2c^\dagger, \qquad [c, c^\dagger] = 4(H_+ - \gamma - 7/2). \tag{40}$$

The operators c and $c^\dagger$ can be shown [12] to act on the eigenstates of H_+ as follows[3],

$$c\Psi_n^{(+)} = -2\sqrt{n(n+\gamma+1/2)}\,\Psi_{n-1}^{(+)}, \tag{41}$$

$$c^\dagger\Psi_n^{(+)} = -2\sqrt{(n+1)(n+\gamma+3/2)}\,\Psi_{n+1}^{(+)}. \tag{42}$$

To determine the algebraic structure associated with the CES potential (35) we now consider the following operators,

$$D = A^\dagger c A, \quad D^\dagger = A^\dagger c^\dagger A \tag{43}$$

where

$$A = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + x + \frac{\gamma+1}{x} + \frac{2g_1 x}{1+g_1x^2}\right)$$

with $g_1 = 2/(2\gamma+3)$. The operators D and $D^\dagger$ act like annihilation and creation operators on the eigenstates of the partner Hamiltonian H_-.

[3] Note that in Eq. (18.3.13') of Ref. [12] the prefactor on the right-hand side should read $-2\sqrt{n(n+\alpha-\frac{1}{2})}$.

$$D\Psi_n^{(-)} = -4[n(n+\gamma+1/2)(n+\gamma+3/2)$$
$$\times (n+\gamma+5/2)]^{1/2}\Psi_{n-1}^{(-)},$$
$$D^\dagger\Psi_n^{(-)} = -4[(n+1)(n+\gamma+3/2)$$
$$\times (n+\gamma+5/2)(n+\gamma+7/2)]^{1/2}\Psi_{n+1}^{(-)}. \quad (44)$$

Hence, we also have

$$\Psi_n^{(-)} = \left(-\tfrac{1}{4}\right)^n \sqrt{\frac{1}{n!}\left(\gamma+\tfrac{3}{2}\right)_n\left(\gamma+\tfrac{5}{2}\right)_n\left(\gamma+\tfrac{7}{2}\right)_n}$$
$$\times (D^\dagger)^n\Psi_0^{(-)}, \quad (45)$$

with $(z)_n = \Gamma(z+n)/\Gamma(z)$ being Pochhammer's symbol. The normalised ground-state wavefunction explicitly reads

$$\Psi_0^{(-)}(x) = \frac{1}{\sqrt{2\gamma+5}} A^\dagger\Psi_0^{(+)}(x)$$
$$= \sqrt{\frac{2}{\Gamma(\gamma+5/2)}}$$
$$\times \left(1+\frac{2g_1}{1+g_1x^2}\right) x^{\gamma+2}\exp(-x^2/2). \quad (46)$$

With the help of (44) it can be shown that $D, D^\dagger$ and H_- obey the following non-linear algebra which is of cubic type,

$$[H_-, D] = -2D, \quad [H_-, D^\dagger] = 2D^\dagger,$$
$$[D^\dagger, D] = -8H_-^3 + (12\gamma+42)H_-^2 - (24\gamma+52)H_-. \quad (47)$$

In contrast to the unbroken SUSY cases this symmetry algebra is realised over all the states (cf. Fig. 2). As in the example of unbroken SUSY, in this case also we can obtain further CES potentials by considering $N = 2, 3, \ldots$.

5. Concluding remarks

In this work we have constructed a number of CES potentials, which are the partner potentials of the solvable linear and radial harmonic oscillator. Obviously, the present approach can also be applied to other solvable problems characterised by a shape-invariant SUSY potential Φ. See, for example, Ref. [10]. Using then the ansatz $W(x) = \Phi(x) + f(x)$ the potential

$$V_+(x) = \tfrac{1}{2}[\Phi^2(x) + \Phi'(x) - f^2(x)$$
$$+ 2\Phi(x)f(x) + f'(x)] \quad (48)$$

will be exactly solvable if the function f obeys the generalised Riccati equation

$$f^2(x) + 2\Phi(x)f(x) + f'(x) = \text{const.} \quad (49)$$

for certain values of parameters contained in Φ and f. Then the partner potential

$$V_-(x) = \tfrac{1}{2}[\Phi^2(x) - \Phi'(x) - 2f'(x) + \text{const.}] \quad (50)$$

will become a CES potential. The corresponding spectral properties of H_- are easily obtainable via the SUSY relations (5)–(8). Note that the present approach differs from the usual one [4,6], which also leads to Eq. (49), however, with a vanishing constant. Whereas in the usual approach one seeks for a solution of (49) for a given Φ and vanishing constant [4], the present approach looks for solutions of (49) with a non-vanishing constant but particular values of parameters contained in Φ. Another difference between the present approach and the usual one is that the former is also applicable to cases with broken SUSY.

In this paper we have also obtained the algebraic structure associated with the newly found CES problems. In contrast to the usual exactly solvable problems which are associated with linear Lie algebras, these CES problems can be related to non-linear algebras. To be more explicit, the CES potentials which are SUSY partners of the linear oscillator will give rise to a quadratic algebra. In essence, this is a consequence of the cubic type of the ladder operators (20) for the CES potential. In contrast to this, the ladder operators (43) for the SUSY partners of the radial harmonic oscillator are of quartic type (a fourth-order differential operator) and hence will lead to a cubic algebra. For other shape-invariant potentials there are no such ladder operators known and, hence, we do not expect to find for the corresponding CES potentials, obtained via the general method outlined above, some non-linear algebraic structure. Clearly, the linear and radial harmonic oscillator treated in the present work are very special in this respect.

[4] In this case, (49) actually reduces to Bernoulli's equation, which is much easier to solve.

Finally, let us note that these types of symmetry algebras are not totally unknown and have been studied before [13–16] in connection with other quantum models. However, to our knowledge, the present work is the first to related such non-linear algebras with SUSY and CES problems.

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