SOME STOCHASTIC MODELS IN RELIABILITY

THESIS
SUBMITTED TO THE
INDIAN STATISTICAL INSTITUTE, CALCUTTA,
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
1968

R. NATARAJAN

ACKNOWLEDGEMENTS

The author expresses his gratitude to Dr. J. Sethuraman, Professor of Statistics Research and Training School, Indian Statistical Institute, Calcutta, under whose supervision the work contained in this thesis was carried out. The author has greatly benefitted by his helpful suggestions and constructive criticisms during the course of the work.

The author is extremely grateful to Professor C. R. Rao, Sc.D., F.N.I., F.R.S., Director, Research and Training School, Indian Statistical Institute, Calcutta, for the keen interest he has shown throughout the course of this work, for his valuable advice and help which has enabled the author to submit this thesis.

The author expresses his sincere thanks to the Research and Development Organisation, Ministry of Defence, for providing the necessary facilities and for granting permission to submit this thesis. In particular his sincere thanks are due to Dr. S. S. Srivastava, Director, Scientific Evaluation, for his keen interest and support.

Grateful acknowledgements are also due to Col. R. Rama Rao and Lt. Col. H. S. Subba Rao for their constant encouragement and to Dr. N. K. Jaiswal, Dr. K. Thiruvengadam, Dr. S. Subba Rao and Mr. V. S. Srinivasan for their helpful suggestions and discussions during the preparation of this thesis.

PREFACE

The work presented in this thesis was carried out under the supervision of Dr. J. Sethuraman, Research & Training School, Indian Statistical Institute, Calcutta, and is devoted to the study of some stochastic models of standby and parallel redundant systems. Some of the reliability characteristics studied are the expected time to system failure, the long-run availability, expected number of system failures in a given interval of time, interval reliability etc. These reliability characteristics will be useful in the better design of systems and making management decisions in improving system reliability.

The investigations carried out in this thesis are presented in four chapters which are preceded by an introductory chapter in which a brief history of the development of Reliability Theory and a review of the literature pertaining to the work presented in this thesis have been made.

In chapter 1 is discussed the reliability of a single unit system with (N - 1) units as standby and the units on failure are repaired by a single repair facility. Sections 1 and 2 of this chapter deal respectively with cases of continuous or intermittent usage of the active unit, while section 3 deals with the same system when the spares deteriorate in storage. The phase method, the supplementary variable method and the results of pure birth and death process are used in studying the various models.

The case of N unit standby redundant system with multiple repair facilities with exponential failure time and exponential repair time distributions has been studied in chapter 2. The analysis is carried out by using the compensation function technique. In the first two chapters, the Laplace transform of the distribution of time to system failure and the associated probabilities have been derived first. These are then used to generate the general process probabilities using renewal theoretic arguments by observing that the time to system failure period and the system down-time period that follows it form a renewal process. The general process has been used to discuss the various reliability characteristics.

Chapter 3 deals with the reliability of a standby redundant system with two types of units assigning preemptive resume and head-of-the-line priority repair policies for repair of the failed units. The system consists of two types of units, one type having only one unit the other type having two units - an active unit backed by a standby unit. The two types of units are either series connected or parallel connected. The analysis of this system has been made through the semi-Markov process obtaining the Laplace transform of the distribution of time to system failure etc.

The reliability of parallel redundant systems with two types of units assigning priority repair policies for the repair of the failed units is investigated in chapter 4 using supplementary variable method. In section 1 has been evaluated the Laplace transform of the distribution of the time-to-system failure of a (2,2) - parallel redundant system while in section 2, the long-run availability of a (N_1, N_2) - system.

Analytical inter-model and intra-model comparisons have been effected at suitable places in the thesis. Apart from this in all the chapters, the behaviour of the system has been studied by giving numerical values to the parameters involved and effecting comparisons between the reliability characteristics of standby and parallel redundant systems in chapters 1 and 2 and studying the effect of different priority allocation for repair in chapters 3 and 4.

Lastly, the procedure adopted in numbering the equations requires mentioning. The first figure stands for the chapter, the second figure denotes the section and the last figure the serial number of the equation. Thus (1,3.14) means the fourteenth equation of the third section in the first chapter. However, the equations in the introduction are numbered serially. A list of references of the books and journals which have been consulted during the preparation of this thesis is given at the end.

Directorate of
Scientific Evaluation,
Defence R & D Organisation
New Delhi.

June 2, 1968

R. Natarajan

R. NATARAJAN

C O N T E N T S

Page No

INTRODUCTION

The accelerated rate of technological development since the World War II and especially during the past decade is characterised by the development of more and more complex systems containing large numbers of sub-systems, components and parts. The advent of space crafts, electronic computers, cummunication systems, complex weapon systems, etc. require equipments of ever increasing effectiveness or reliability. The modern systems are so complex that the failure of single inexpensive part or component may cause the failure of the entire system. Therefore, reliability is the concern of all scientists and engineers engaged in developing a system, from the design through the manufacturing to its final use. Reliability is essentially an attribute of the design. If reliability is not built-in at the design stage, no amount of production control will improve the systems reliability. On the other hand, systems designed with high reliability may deteriorate in their performance due to lack of good quality control and maintenance. The systems with high reliability are obtained not by accidents or coincidences but as a result of conscious effort by all concerned.

The theory of reliability has grown out of ever increasing demands of the rapid growth of modern technology and the work in this field derived a great impetous from the applicational needs in missile industry in the early 1950s [Lusser (1952), Carhart (1953)] . Some of the areas of reliability research are: life testing, structural reliability, machine maintenance problems and replacement problems. Of these areas, we restrict ourselves in this thesis only to the study of some stochastic models to evaluate certain reliability characteristics of the system under consideration. An excellent survey of some of these models has been made by Weiss (1962 b) and Barlow, Proschan and Hunter (1965) .

THE DEFINITIONS OF RELIABILITY CHARACTERISTICS

Reliability has been defined in different forms in the literature depending upon the quantities calculated to suit the different reliability problems. These measures have been designated by different names: reliability, availability, interval availability, efficiency, effectiveness and so on. Barlow and Proschan (1965, pp 5-8) have presented a unified treatment of various concepts and the quantities involved in the reliability studies. We shall now define some of these including the ones we shall be using in this thesis.

1. Reliability: Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered *. The period of time intended is also called the "mission time" and is ordinarily assumed to be (o, t). With this definition, if T represents the life of the equipment, the reliability R(t) is given by

$$R(t) = P_r [T > t] \qquad (0.1)$$

As measures of dependability Hosford (1960, p 53) defined the quantities pointwise availability, interval availability and limiting interval availability.

2. Pointwise availability: This is defined as the probability that the system will be able to operate within the tolerences at a given instant of time t. Let this be denoted by $P_A(t)$. The term "availability" is used by Walker and Horne (1960, p 42) for the same quantity.

3. Interval availability: This is defined as the expected fraction of a given interval of time that the system will be able to operate within tolerences. Repair and/or replacement is permitted. Let this quantity be denoted by H (a, b). Then

---

* Radio-Electronics Television Manufacturers Association, 1955.

$$H(a,b) = \frac{1}{b-a} \int_a^b P_A(t)\,dt \qquad (0.2)$$

is the interval availability for the interval (a, b). Barlow and Hunter (1960 a, pp 46-47) refer to essentially the same quantity as "efficiency".

4. Limiting interval availability: This is defined as the expected fraction of time in the long-run that the system operates satisfactorily.

To obtain limiting interval availability simply we compute $\lim_{t \to \infty} H(o, t)$ in item (3) above. Barlow and Hunter (1960 a, p 47) call this quantity "Limiting efficiency". Coleman and Abrams (1962), Bashyam and Jaiswal (1964) call the same quantity "Operational readiness of the system. Gaver (1963) and Natarajan (1967, a, b, c) discuss the same measure under the name "Long-run availability".

5. Interval Reliability: Interval reliability is defined as the probability that at a specified time, the system is operating and it will continue to operate for an interval of duration, say x. This was introduced by Barlow and Hunter (1961, pp 206-207) for the case of a system having single failed state. The combined operation during the interval is, of course, to be achieved without the benefit of repair or replacement. To obtain this quantity, let $X(t) = 1$ if the system is operating in time t, zero otherwise. Then the interval reliability $R(x,t)$

for an interval of duration x starting at time t is given by

$$R(x,t) = P_r\left[X(u) = 1,\ t \leq u \leq t+x\right] \qquad (0\cdot3)$$

"Limiting interval reliability" is simply the limit of R(x, t) as $t \to \infty$. Truelove (1961, p 27) calls this "Strategic reliability". Subsequently, Natarajan (1967 a) and Srinivasan (1967 a) have used this measure to evaluate the reliability of the systems. In chapter 3 of this thesis, the upper and the lower bounds of this measure have been generalised for the case of a system having more than one failed state. It is assumed that repair and/or replacement is permitted.

Drenick (1960 a) also has given a somewhat different general model from which by suitable modifications he derives definitions of certain quantities of interest in reliability. His formulation is in terms of a renewal process. Failures during [0, t] occur at times $t_1, t_2, \cdots t_n$; $t_1 < t_2 < \cdots t_n < t$; replacement is made immediately following failure. A gain function $W_n(\underset{\sim}{t}, t)$ describes the economic gain accruing from this outcome, where $\underset{\sim}{t} = (t_1, t_2, \cdots t_n)$. Thus the expected gain $U(t)$ upto time t is given by

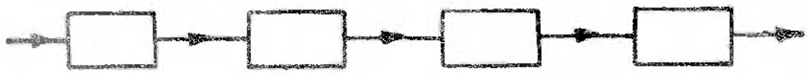
$$U(t) = \sum_{n=0}^{\infty} \int W_n(\underset{\sim}{t}, t)\, f(\underset{\sim}{t}/n, t)\, d_t\, P_r\left[N(t) = n\right] \qquad (0\cdot4)$$

where $f(\underset{\sim}{t} / n, t)$ is the joint conditional density of failure at time $t_1 < t_2 < \cdots t_n$ given that N(t) = n, N(t) being the number of failures in $(0, t)$. Special cases of (0.4) yield quantities that Drenick calls replacement rate, maintenance ratio, mission success ratio and mission survival probability.

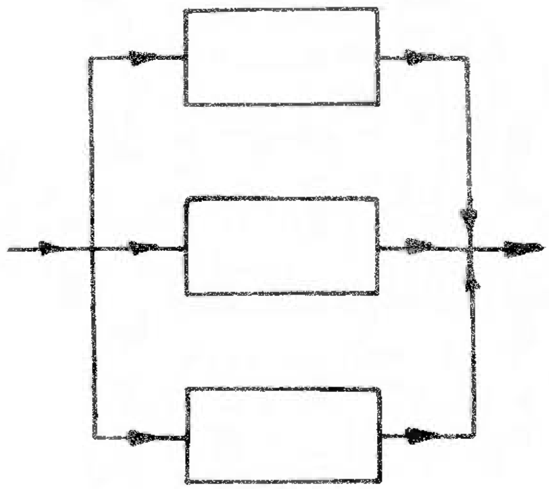
Other reliability characteristics of interest and which have been evaluated in this thesis, are the expected time to system failure, expected duration of system down-time, the expected number of system failures in a given interval of time $(0, t)$. However, for the case of intermittent usage of the system these characteristics are defined slightly differently using the concept of 'disappointment' due to Gaver (1964). For the definitions of these, one can refer to chapter 1. These characteristics are useful in some management decisions, especially, the provisioning of spares, repair facilities etc.

TYPES OF COMPLEX SYSTEMS

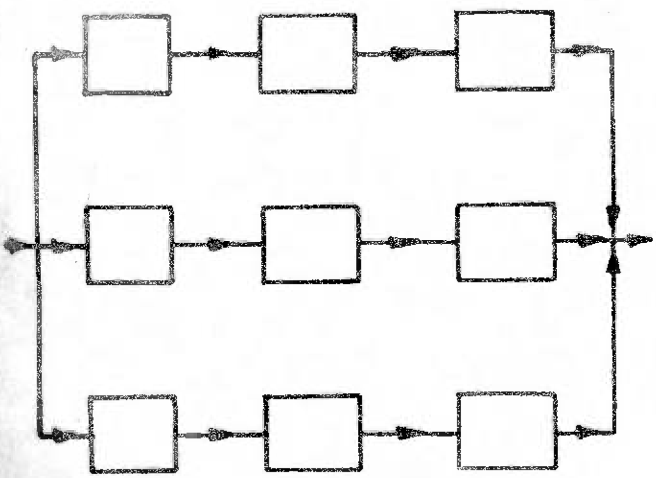
The description of the system which is under investigation, is an essential part of any reliability study. Simplest of all the systems is the single unit system. The unit may be a component like capacitor or resistor, a sub-system like a module of an electronic equipment or an equipment itself like a power generat



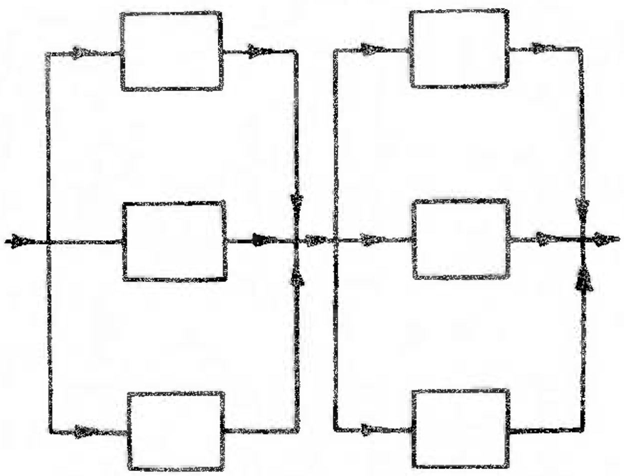
a) A SERIES SYSTEM



b) A PARALLEL SYSTEM



c) PARALLEL CONNECTED SERIES SUBSYSTEM



d) SERIES CONNECTED PAR SUBSYSTEM

FIG. 1: VARIOUS SYSTEM CONFIGURA

a single communication receiver or computer. When there are two or more units in a system, the system becomes complex. In a complex system, the units may be connected in series or in parallel or in a combination of both these arrangements. The various system configurations are shown in fig 1.

Series system: This is defined as a system wherein the failure of one unit will cause failure of the complete system. As shown in fig 1, the input of one unit is the output of just previous unit in the arrangement.

Parallel system: In this system every unit has a separate input and output not affecting the input or the output of the other. Hence, this system is defined as a system wherein the failure of all the units are necessary to cause a failure of the system.

If we know the reliability of the individual units of the system, we can obtain the reliability of the entire system. For these relationships one can refer to some of the recent books by Calabro (1962), Pieruschka (1963), SAE Sub-Committee on Reliability (1963), Barlow and Proschan (1965).

Having known the different types of systems that we commonly meet with in practice, it is necessary to know the method of improving system reliability. One of the well-known techniques of increasing the system reliability is the use of redundant units. These units may be arranged in parallel as above or they may be kept as standbys to be used for replacement when the unit in use fails. The former type of redundancy is called parallel redundancy while the latter is called stand-by redundancy. The redundant units may be identical units or functionally same units. In the case of parallel redundancy all the units in parallel are active, though one may be in use. When the unit in use fails, the function is done by the next unit. At any instant, any one of the units in parallel either in use or otherwise has the same chance of failure as all of them are active. In the case of standby redundancy, the situation is slightly different. Here, the unit in use is the only one which is active and the rest of the units are inactive and when the primary unit fails, immediately one from the stand-by replaces it or is switched on. In these studies, it is assumed that the sensing switching mechanisam is fully reliable and the switch over time is negligible though this assumption may not hold in some practical situations.

THE REQUISITES OF A RELIABILITY MODEL

The reliability studies of these systems mainly comprise of two aspects. The first one: Construction of suitable models to evaluate the reliability characteristics making use of the parameters of the individual units which are affected by the inherent design of the equipment and the manner in which it is used in the field (continuous or intermittent usage) or other environmental factors like deterioration in storage etc. The second one is the optimisation problem i.e.) determination of the number of redundant units so as to optimise a stated reliability characteristic subject to some restrictions like cost, storage space etc. This in itself is a seperate branch of study involving sophisticated techniques like analysis, mathematical programing and information theory. Early work in this line is due to Mine (1959), Moskowitz and Mc Lean (1956), Bellman and Dreyfus (1958) and Moore and Shannon (1956). For a good discussion on redundancy optimisation, reference can be made to Barlow and Proschan (1965), chapter 6.

Our main aim in this thesis has been to study the first aspect of the reliability problem, namely, construction of suitable stochastic models to evaluate the reliability characteristics of some of these systems. No effort has been made to study the optimisation problem.

It is required of any stochastic model in the reliability studies of these system, a complete knowledge of the process involved and relevent assumptions governing it. The description of this stochastic behaviour necessitates the specification of the following:

i) Description of the system;

ii) Definition of suitable reliability characteristic of the system;

iii) Failure distribution of the individual units;

iv) Mode of replacement of the failed units;

v) Repair time distribution of the individual units;

vi) Repair policy;

As the description of different systems and the definitions of different reliability characteristics have already been discussed, we shall now discuss briefly the remaining aspects.

Failure time distributions: As with human life, each population of similar units exhibits life characteristics with respect to time. We suppose that the random variable T denoting the length of life of a unit has a probability density function (p.d.f) $f(t)$ which is zero for negative t. Moreover, the failure times of different units will be assumed to be mutually independent. The distribution function $F(t)$ of the random variable T gives the probability that a unit has failed by time t.

That is

$$F(t) = P_r[T \leq t] = \int_0^t f(u)\,du \qquad (0.5)$$

Evidently F(t) is a non-decreasing function of t with $F(0) = 0$ and $F(+\infty) = 1$. In the reliability studies more often the complementary function $R(t) = 1 - F(t)$ is used. This is also called the "Survivor function" or the reliability function.

Then $$R(t) = P_r[T > t] \qquad (0.6)$$

Clearly R(0) = 1, R (∞) = 0 and R(t) is non-increasing function of t. Also

$$f(t) = -\frac{d}{dt} R(t) \qquad (0.7)$$

assuming that the derivative exists.

Another function of interest is the failure rate function $\lambda(t)$. This is defined for those values of t for which $F(t) < 1$, by

$$\lambda(t) = f(t) \,/\, [1 - F(t)] \qquad (0.8)$$

and is the same as 'age-specific failure rate', instanteneous failure rate' and the 'force of ~~nutrality~~ mortality' as used by the actuaries. In extreme value theory, this is called 'Intensity function' (Gumbel, 1958). This has been widely used in the reliability theory under the name 'hazard rate' as $\lambda(t)\,dt$ represents the conditional probability that a unit will fail in the interval $(t, t+dt)$ given that it has not failed upto time t. Using (0.7) and (0.8) we can write

$$1 - F(t) = \exp\left[-\int_0^t \lambda(u)\,du\right] \tag{0.9}$$

and

$$f(t) = \lambda(t) \exp\left[-\int_0^t \lambda(u)\,du\right] \tag{0.10}$$

In most of the typical situations the life of a unit can be broadly divided into three periods: (i) the period of infant mortality; (ii) the period of useful life having only chance failures; and (iii) the wear-out period. The hazard rate $\lambda(t)$ is very high in the initial stages and sharply decreases as time passes in the first period. During the second period, the hazard rate is constant and failures during this period follow a completely random pattern. This period is normally the longest of the three and the useful period too. Most of the units, equipments are designed for use in this period. Finally, in the wear-out period because of ageing of the unit, the hazard rate increases. The plot of hazard rate $\lambda(t)$ against t would look like a trough.

A number of families of distributions have been used for the life length of electronic and mechanical components and the fatigue failure of materials. A detailed discussion of these distributions are found in Weiss (1962 b), Barlow and Proschan (1965) and Cox (1962). Reference can also be made to ~~Dans~~ Davis (1952) and Mendelhall (1958). Some of the important families of failure distributions are the exponential, the gamma, the Weibull, the normal and the log normal. Of these families, the gamma, the Weibull and the normal have increasing failure rate for some parameter values while the exponential has constant failure rate.

The Weibull distribution is applicable to the large group of problems in which an event in any part of the object affects the object as whole. This distribution has been used to describe fatigue failure (Weibull, 1939), vacuum tube failure (Kao, 1958) and ball-bearing failure (Leiblein and Zelan, 1956). This is most useful to describe wear-out failures. The form of Weibull distribution and its failure rate function are given by, Weibull distributions:

$$f(t) = \lambda \alpha t^{\alpha-1} e^{-\lambda t^{\alpha}}, \quad \lambda(t) = \lambda \alpha t^{\alpha-1}, \quad \lambda, \alpha > 0, \quad t \geq 0 \tag{0.11}$$

The failure rate is increasing for $\alpha > 1$.

The gamma distribution has been widely used to build stochastic models of telephone traffic, queueing, inventory and reliability studies because of its mathe-

~~mathretical~~ tractability. The distribution and its failure rate function are given by, Gamma distribution :

$$f(t) = \frac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{\Gamma(k)} , \quad \lambda(t) = \frac{\lambda^k t^{k-1}}{\Gamma(k)} \cdot \frac{e^{-\lambda t}}{\int_t^\infty f(u)\,du}$$

$$\lambda, k > 0 , \; t \geq 0 \qquad (0.12)$$

This distribution also has increasing failure rate when $k > 1$. A well-known property of this distribution is that it can be generated as k-fold convolution of an exponential distribution whose distribution and failure rate are given by

Exponential distribution:

$$f(t) = \lambda e^{-\lambda t} , \; \lambda(t) = \lambda , \; \lambda > 0 , \; t \geq 0 \qquad (0.13)$$

The exponential distribution has constant failure rate and it characterises that part of the life of an equipment where the failures occur in a purely random fashion. This distribution has been thoroughly studied by a number of people, the basic one being Epstien and Sobel (1953).

Just as central limit theorem plays an important role to provide a basis for the use of normal distribution in many statistical procedures, so does a limit theorem by Drenick(1960 b) for the distribution of positive random variables, appear to favour the use of an exponential life time distribution. On the strength of this argument, exponential failure time distribution has been assumed in a number of models in this thesis.

Mode of Replacement: Replacement of a unit may be as a result of preventive maintenance policy or the failure of the units. A continuous checking is essential to detect the functional deterioration or the failure of the unit. For this purpose, optimum checking procedures and preventive maintenance and replacement policies have been evolved by a number of people of whom reference can be made to Hunter (1962), Weiss (1962 a) ~~and~~ for optimum inspection procedures and to Flehinger (1962), Barlow (1962 c) and Radner and Jorginsen (1962) for preventive maintenance and replacement policies. However, in this thesis, only replacement after the equipment failure is considered. To detect the failure, the system is conceived to have a sensing-switching mechanism which switches on the redundant unit immediately after the failure of the unit in use. It is assumed that the sensing-switching arrangement is fully reliable and the switch-over time is negligible though this assumption is a little restrictive. Recently, Srinivasan (1966 b) has considered the reliability of a single unit system with single standby wherein the switch-over time is not negligible.

Repair Time Distribution: Units on failure are repaired by a single or multiple repair faciltіes. Let $T_1', T_2' \cdots T_i' \cdots$ be the durations of repair on successive units at the service facility. We assume that the T_i' are independent positive random variables and are identically distributed with the probability.

$$P_r\left[t \leq T_i' \leq t+dt\right] = S(t)\,dt$$

S(t) in general has an arbitrary form and for purposes of analysis, it is useful to express S(t) in the following form

$$S(t) = \eta(t)\exp\left[-\int_0^t \eta(u)\,du\right], \quad t \geqslant 0 \qquad (0,14)$$

where $\eta(t)\,dt$ represents the first order conditional probability of repair completion in the interval $(t, t+dt)$ given that the repair has not been completed upto time t.

Some of the commonly used repair time distributions are the exponential, the K-Erlang, the degenerate, the negative bonomial and the geometric. Of these, the most favoured ones being the exponential and the K-Erlang distributions, for the same reasons as mentioned in the case of a failure time distributions.When the repair time is a contant τ , its density function is given by

$$S(t) = \delta(t-\tau)$$

where $\delta(t-\tau)$ is Dirac delta function having the probability concentration at τ. This distribution is obtained as a limit of K-Erland distribution as K tends to infinity.

Recently, Liebowitz (1966) has studied the effect of various repair time distributions in increasing the time to failure of a two element redundant system and came to a conclusion that this increase over that of a single element non-repairable system depends only on the product of the failure rate λ and the mean of the repair time distribution and not on its form.

Repair Policy: The repair policy comes into the picture whenever there is more number of failed units than the repair facilities available. In such a situation, the units on their failure wait at the repair facilities to get repaired. The repair policy followed may be "first failed first repaired" or "random selection of the units" or selection of the units assigning some priority.

The problem of priority assignment arises when there are two or more types of units to be repaired. The normally adopted priority assignment policies are the head-of-the-line, preemptive, alternating and dynamic priority repair policies. In this thesis, only head-of-the-line and preemptive repair policies have been considered and a description of these policies can be found in chapter 3 of this thesis.

MARKOVIAN SYSTEMS

In all the reliability models considered in this thesis, it has been assumed that the failure time distribution and repair time distribution of the units is either exponential or arbitrary distribution. In those models wherein both these distributions are exponential, the process is Markovian. There are a number of techniques available for the solution of these processes; for example, the generating function technique. Moreover, in the models discussed here, the totality of units from which the failures can occur is finite and hence these models are finite source models. These processes can be identified with a birth-death process, a detailed study of which has been made by Karlin and Mc Gregor (1957) by the method of spectral analysis.

Another useful technique, specially useful for finite source problems for analysing Markovian Systems is the compensation technique of Keilson (1962). This method has been made use of in chapter 2 of this thesis in studying the reliability characteristics of a standby redundant system with multiple repair facilities and with exponentially distributed failure and repair times. The compensating function technique is useful when the failure and the repair rates are independent of time and the transitions in the process are skip-free. The description of this technique can be found in chapter 2.

NON-MARKOVIAN SYSTEMS

If, on the other hand, the distribution of the failure time and repair time are other than exponential, the associated stochastic processes are not Markovian in continuous time and hence the study of these non-Markovian processes calls for special techniques. These techniques have been primarily developed for the study of certain queueing processes such as M/G/1, GI/M/R, etc. and are borrowed for the analysis of similar processes in reliability models considered here. The techniques developed are: (i) the Imbedded Markov-Chain method of Kendall (1951, 1953), (ii) the Supplementary variable method due to Cox (1955), (iii) the Phase method of Erlang, Luchak (1956), (iv) the Extended chain method of Gaver (1959) and (v) the Semi-Markov process method of Weiss [(1956 b), (1962 a)] and Fabens (1959).

Kendall (1951) observed that though the queueing process of M/G/1 or GI/M/1 type are non-Markovian in continous time, it is possible to extract a Markov Chain at certain discrete time points called "regeneration points" and this Chain was called the'imbedded Markov-Chain' by Kendall. Using Feller's theory of recurrent events, Kendall discussed the steady-state behaviour of the Chain. This method has extensively been used since then in the study of non-Markovian systems.

The supplementary variable method introduced by Cox (1955) and extended to transient studies by Keilson and Kooharian (1960) consists in carrying along with a process X(t) of motivating interest (such as the number of failed units in the system, the time to system failure etc.) as many supplementary variables $Y_i(t)$ as are required to make the joint process $\{X(t), Y_1(t), Y_2(t), \cdots\}$ Markovian in continuous time. The discussion is then conducted on the state-space or the phase-space $(x, y_1, y_2, \cdots\cdots)$ of the state-space. The formulation leads to a set of difference-differential equations with associated boundary equations. These are then solved to obtain either the transient or the equilibrium solutions. The abstract formulation of this method can be seen in Kendall (1953) who called it the 'augmentation technique', but it was Cox (1955) who applied the method to derive the M/G/1 equilibrium solution. Application of this method can be seen in Jaiswal (1961, 1962, 1965 a,b) Thiruvengadam (1963, 1965 a, b) Subba Rao (1965, 1967), Keilson (1962 c, 1964) and others.

The phase technique, essentially due to Erlang, consists in replacing the failure time and repair time distribution with the Erlang E_k- distribution which is nothing but a k-fold convolution of the exponential distribution with itself. The parameter k provides an extra degree of freedom permitting one to control the variance as well as the mean. This distribution is of great importance by virtue of its analytic simplicity. A failure time or a repair time of an individual unit having this distribution may be regarded as consisting of k successive 'phases' in each of which the time spent has the same exponential distribution. A greater generality is obtained if one divides the failure time or the repair time into equal exponentially distributed phases and assigns to each period a probability C_r of consisting of r-phases. Examples of the use of this method can be seen in Gaver (1954), Luchak (1956), Wishart (1956).

Gaver's extended chain method (1959) is an extension of imbedded Markov-chain method of Kendall which is not in itself suited to the study of processes in continuous time. In the M/G/1 process, Gaver considers 'the state of the system' at regeneration points, which are the service initiation points, is made by T_n the time of the service initiation on the n-th customer and m_n the number of customers left behind in the queue at the instant. The sequence of joint values $\{N(T_n), T_n\}$ constitutes a two dimensional Markov Chain on the co-ordinate space (n, t) described by the set of distributions $P_n(m,t)$ or associated densities. Gaver considers the busy period process first and obtains $P_n(m,t)$ within a busy period in terms of the one-step transition probabilities. From these are deduced the busy period densities and by considering the 'general process' from the Renewal theory point of view, the time dependent distribution in m and waiting time distribution in the general process are obtained.

The semi-Markov process is obtained as a generalisation of the Markov process. Let $P = (p_{ij})$ denote the transition matrix of a time homogeneous Markov-Chain with m + 1 states (that is, $i, j = 0, 1, 2, \ldots m$). We shall now define a stochastic process $\{Z(t), t \geq 0\}$ where $Z(t) = i$ denotes that the process is in the state i at time t. Given that the process has just entered a state, say i, the selection of the next state is made according to the matrix $P = (p_{ij})$. The distribution function for the 'stay' of the process in state i given that the next transition will be to state j is denoted by $F_{ij}(t)$. These $F_{ij}(t)$ are called the waiting-time distributions. Let $\underset{\sim}{F}(t) = (F_{ij}(t))$. The process is Markovian only at certain "Markov points" in time at which the state transitions take place. If we specify the vector of initial probabilities $(a_0, a_1, \cdots \cdots a_m)$, the resulting process is called a semi-Markov process (S.M.P). Thus an SMP is a stochastic process which moves from one to another of a ~~convert~~ count-able number of states with the successive states visited forming a Markov-Chain and that the process stays in a given state a random length of time, the distribution function of which may depend on this state as well as on the one to be visited next. It is thus a Markov-Chain for which time scale has been randomly transformed. The counting process $N_j(t), (j = 0, 1, \cdots m)$ associated with this SMP is called a Markov renewal process (MRP). A time homogeneous Markov-Chain is an S.M.P. where

$$F_{ij}(t) = 0 \quad , \quad t < 1$$
$$= 1 \quad , \quad t \geq 1$$

A stable, continuous time parameter Markov process is an S.M.P. in which all wait time distributions are exponential, that is

$$F_{ij}(t) = 1 - e^{-\lambda_i t} \quad , \quad t \geq 0$$

for constant $\lambda_i > 0$ for every i. If there is only one state, a Markov Renewal process becomes the ordinary renewal process. The method of S.M.P. was applied by Fabens (1959) to queueing, and by Weiss (1956b, 1962a) to inventory models, to problems in reliability and equipment maintenance and by Barlow (1962b) to counter theory and reliability problems. Srinivasan (1967a) has used this method to evaluate certain reliability characteristics of a two-unit standby redundant system with repair.

A combination of above methods have been attempted for solving many of the reliability models considered in this thesis. The general process of some of these models can be viewed as a renewal process with each renewal cycle comprising of a time to system failure (TSF) period and a system down time period. The TSF process has been obtained by any one of the above methods and renewal theoretic argument is then used to obtain various reliability characteristics. A good account of renewal theory can be found in { Feller (1941, 1949, 1957), Smith (1958) and Cox (1962) }.

RELIABILITY OF SYSTEMS WITH REPAIR

In most of the reliability models considered in the literature, it has been assumed that the failed units of the system are replaced by new ones instantaneously. On the contrary, in many of the systems we come across, the failed units are repairable and they become as good as new after repair.

Barlow and Hunter (1961) have investigated the reliability of a single unit system which is repaired upon failure and returned to operation after repair. Their work is an extension of earlier work by Takács (1951a, 1959 b) and Weiss (1956 a), on counter theory problems.

When there are more than one unit in the system, we are confronted with complex systems with a series or parallel redundant or standby redundant configurations. Garg (1962, 1963, 1965) has obtained the measures of dependability as defined by Hosford (1960) for complex systems with units in series and with a single repair facility.

The aspect of repairability of failed units has been further elaborated in this thesis to cover systems with parallel or standby redundancy. It is to be noted that in these cases, a unit after repair is put into operation if the system is parallel redundant and is kept as standby if system is standby redundant.

PARALLEL REDUNDANT SYSTEMS WITH REPAIR

The stochastic behaviour of the N component parallel system can be identified with that of a simple machine interference problem with N machines or finite source models with N units. These models have an immediate application to the reliability problems. For the study of machine interference problems one can refer with advantage to the work of Khintchine (1932), Palm (1947) Benson and Cox (1951), Takács (1951b, 1956, 1957, 1959a, 1962), Cox and Smith (1961) and Thiruvengadam and Jaiswal (1964). Barlow (1962 a) in an expository paper has given a number of repairmen problems as applied to reliability of complex systems. He describes the general repairmen problem as follows.

Suppose we are given m identical units stochastically independent of one another and supported by n spare units. Unless otherwise stated, all units are operating at t = 0. Suppose that each fails according to a distribution F. Furthermore, suppose that we have a repair facility capable of repairing s units simultaneously. Obviously, we could consider the facility as consisting of s repairmen. The following repair policy has been observed. If all repairmen are busy, each new failure joins a waiting line and waits until a repairman becomes free. It has been assumed that the repair times are also independent, identically distributed random variables with distribution G. Figure 2 will be helpful in explaining the various models of repairmen problems.

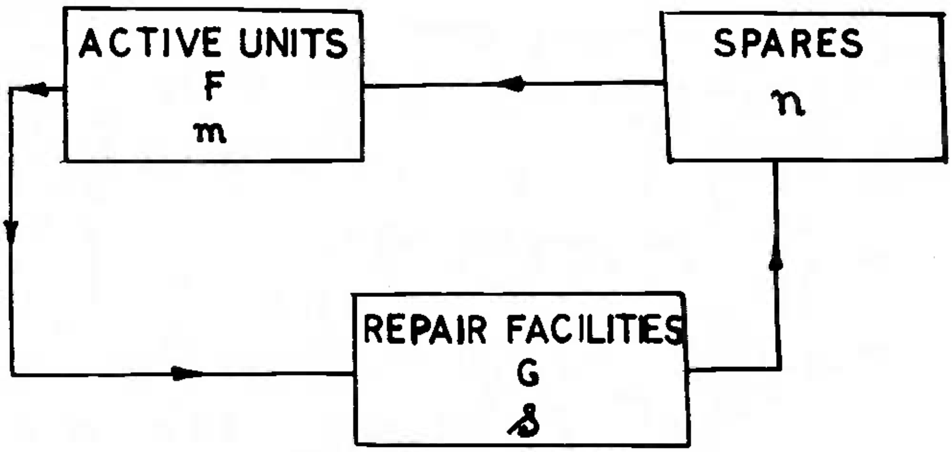


Fig.2 Repairman Problems (Barlow (1962a))

The object of studying repairmen problem is to determine how reliability can be improved by the use of redundant units. In repairmen problem, it is assumed that a total failure of the system occurs when all the machines are in the failed state either undergoing repair or waiting to get repaired. Often we are interested in the distribution of TSF and its moments. Barlow has obtained the solution of these problems by identifying the stochastic behaviour of these models with those of the machine interference problems or telephone trunking problems. In most of these cases, he gives the stationary probabilities and the mean recurrence time to the state of system failure.

Further, when the failure time distribution and the repair time distribution are exponential, the basic process of the repairmen problems is Markovian and corresponds to the birth and death process. For this case, Barlow has given not only the stationary probabilities but also the distribution of TSF. When at least one of the distributions F and G is non-exponential, the distribution of TSF has not yet been obtained. A study of this distribution has been made in chapter 1, section 3, of the present thesis for the N unit parallel system with single repair facility and with exponential failure time and general repair time distributions. This parallel system being one of the repairmen problems, has been discussed as a particular case of standby redundant system with (N-1) spares wherein the spares deteriorate in storage with the same rate as the failure rate of the unit in use. Further, in chapter 4, section 1, is considered a complex system with two types of units such that two units of one type in parallel are connected to two units in parallel of the other type. This sytem is called a (2,2) - system. In general when there are N_1 units of type 1 in parallel and N_2 units of type 2 in parallel in the system, the system is called an (N_1 , N_2) - system. Allocation of priority for repair arises whenever there are more than one type of failed units. In section 1 of chapter 4, the distribution of TSF of a (2,2) - system has been obtained under the assumption of exponential failure time and general repair time distributions assigning the head-of-the-line and preemptive resume priority policies for repair of the failed units. The analysis is carried out through the supplementary variable technique. The effect of interchange of priorities on the expected TSF has also been studied for the particular case when both the failure time and repair time distributions are exponential and it is observed that the adoption of head-of-the-line priority discipline for repair yields higher expected TSF than when preemptive resume discipline is adopted. The material presented in this section is based on a paper published in IEEE Transactions on Reliability [Natarajan, (1967 b)] .

We consider the general (N_1 , N_2) - system in section 2 of chapter 4 and obtain the long-run availability of this system under the same assumptions as in section 1. Investigations are carried out first by obtaining the probabilities associated with the busy period process of the repair facility. Then the general time-dependent process in which busy periods alternate with the idle periods has been studied in terms of the busy period probabilities and the probability of finding the repair facility idle at time t. Finally, the long run availability of the system has been obtained in terms of the steady state probabilities for the two priority repair policies.

STANDBY REDUNDANT SYSTEMS WITH REPAIR

Very little work is available in the literature for the reliability studies of standby redundant systems with repair of the failed units. Srinivasan (1966 a) has studied the TSF of two-unit standby redundant system where both the failure time and repair time distributions are arbitrary. This being a field so little explored that the bulk of the present thesis has been devoted to the reliability studies of standby redundant systems.

Chapter 1 of this thesis discusses the reliability of a system with single unit in use and (N - 1) units as standby and a single repair facility. Sections 1 and 2 deal respectively with the cases of continuous usage or intermittent usage of the active unit. In section 3 has been discussed the reliability of the same system with spares deteriorating in storage. A comparison of the reliability characteristics of the N unit standby redundant systems with those of the N unit parallel systems as well as with the standby system when the spares deteriorate in storage has also been made in this chapter.

The case of N unit standby redundant system with c repair facilities with exponential failure time and exponential repair time distribution has been studied in chapter 2. This analysis can be used to study the effect of increasing the number of spares against the effect of increasing the number of repair facilities to achieve a desired reliability of the system. The results of this chapter are also compared with those of parallel redundant systems with c repair facilities. The material of this chapter is based on a forthcoming paper in Operations Research (Natarajan (1967 c)).

In chapter 3 is considered the reliability of standby redundant system with two types of units assigning head-of-the-line and preemptive resume priority repair policies for repair of the failed units. The system consists of two types of units, one type having only one unit and other type two units - an active unit backed by a standby unit. The two types of units are either series connected or parallel connected. The analysis of the system has been made through the semi-Markov process (SMP) approach.

In all the models considered in this thesis, an attempt has been made to obtain the distribution of time to system failure and its first two moments which are then used to study some of the reliability characteristics of the system such as mean recurrence time to the state of system failure, the expected number of system failures in a given interval of time, the interval reliability and so on. It has also been pointed out in chapters 1 and 2 where there are only one type of units in the system that the expected time to system failure can also be obtained in terms of the long-run availability of the system.

It is hoped that these reliability models will be of use in the better design and utilisation of modern equipments and systems and thus promote the better understanding and utilisation of the advancement of science and technology for the benefit of humanity.

CHAPTER 1

RELIABILITY OF A STANDBY REDUNDANT SYSTEM WITH A SINGLE REPAIR FACILITY

CHAPTER 1

RELIABILITY OF A STANDBY REDUNDANT SYSTEM WITH A SINGLE REPAIR FACILITY

The use of redundancy is a well-known technique of improving system reliability. Redundancy in a system can be achieved either by a parallel arrangement or by a standby arrangement of the units. Units may be components of an equipment, sub-systems or equipment themselves. In a parallel redundant system, all the units of the system are connected in parallel and are in operation to start with. When a unit fails, it is repaired immediately and put into operation. The system failure occurs when all the units are in the failed state simultaneously. On the other hand, in a standby redundant system, besides the unit functioning in the system, some units are kept in reserve as spares to act as standby. That is, when the functioning unit fails, it is replaced by one from the spares and the system continues to function till all the spares are used and system failure occurs when there are no more spares left out when a unit under operation fails. This chapter is devoted to the study of a single-unit system with (N - 1) spares. As soon as the unit fails, it is replaced by a unit from the spares immediately and the system continues to function. In addition, the failed units are repaired by a single repair facility and after repair completion, are returned to the spare pool to act as standby. Failure of the system occurs only when the unit in use fails and no spare is available for replacement. That is when there are (N - 1) failed units and the unit in use also fails. Examples of these types of redundant systems are to be found in power generation systems, communication systems, surviellance systems etc. Figure 1.0a illustrates this system.

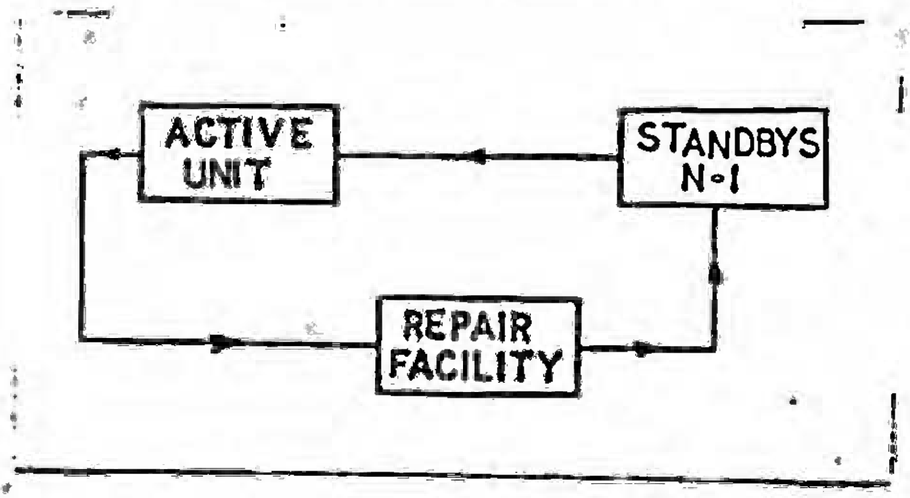


Fig.1.0(a) Standby Redundant System with Repair

As the usage pattern of the system is an important factor affecting the system reliability, we shall study the reliability characteristics of the system when it is in continuous usage and when it is in intermittent usage. If the system is intermittently used, the failure of the system when the demand for its use exists is called "disappointment" [Gaver (1964)]. In both continuous and intermittent usage cases, system non-availability for use is critical and ~~the~~ one aims to keep the system in use as much as possible. In achieving this, some of the important reliability characteristics for making management decisions are; the average duration of time the system remaining in use, the average duration of time the system remaining in the failed state, the mean number of failures of the system in a given interval of time, the expected number of repair completions in a given interval of time (0, t), the long-run availability of the system and so on.

The quality of spares has a considerable effect on the reliability of the system [See Weiss (1962b), Schweitzer (1967)]. When the spares deteriorate in storage to achieve the same reliability we must have more number of spares in storage. We propose to investigate in this chapter the effect of deterioration of spares in storage on system reliability.

This chapter on Standby Redundant System consists of the following three sections.

SECTION 1: RELIABILITY OF A SYSTEM IN CONTINUOUS USAGE

SECTION 2: RELIABILITY OF A SYSTEM IN INTERMITTENT USAGE

SECTION 3: RELIABILITY OF A SYSTEM WHEN THE SPARES DETERIORATE IN STORAGE

In the first section, we study the system under continuous usage and obtain the Laplace transform of the general process probabilities by forming difference-differential equations and solving them. Then, through renewal theoretic arguments, we obtain the distributions of time to system failure and recurrent times. Using the first two moments of these distributions we derive the expression for the expected number of system failures in a given interval (0, t). The long-run availability and the limiting interval availability of the system are also discussed [Natarajan (1967a)] . In the second and the third sections similar discussions are made for the system under the assumptions of intermittent usage and deterioration of spares in storage respectively.

SECTION 1

RELIABILITY OF A SYSTEM IN CONTINUOUS USAGE

In this section, we shall be investigating the reliability characteristics of a system in continuous usage.

The description of the system requires a knowledge of the failure processes, repair processes and mode of replacements. Therefore, these processes are defined below.

1. Failure of Individual Units: We assume that the failure time distribution of the individual unit is arbitrary and is approximated by the well-known phase method {Morse (1958), Luchak (1956), Jaiswal (1961)}. The failure time is assumed to consist of r exponential phases with parameter λ, $(r = 1, \ldots, j)$ with probability C_r such that $\sum_{r=1}^{j} C_r = 1$. That is, if X denotes the time for which a unit has been under operation, then

$$F(t) = P_r [X \leq t] = \int_0^t \sum_{r=1}^{j} C_r \, e^{-\lambda t} \frac{(\lambda t)^{r-1}}{(r-1)!} \cdot \lambda \, dt, \quad t \geq 0, \lambda > 0 \qquad (1,1.1)$$

Further, the failure times of the individual units are identically and independently distributed according to (1,1.1).

2. Mode of Replacement: As soon as the operating unit fails, it is replaced by another one from the spare pool instantaneously. By this, the system is put into operation immediately without any replacement time or switch-over time.

3. Repair Process: A unit on failure is taken for repair immediately by a single repair facility, if the facility is free; otherwise, it waits in a queue. The units are

repaired in their order of failure, i.e. first failed first repaired. Let $\{\theta_n\}$, $n = 1, 2, \cdots$ represent the sequence of repair times. Then we assume that

$$D(t) = P_r[\theta_n \le t] = 1 - e^{-\mu t}, \quad t \ge 0, \ \mu > 0 \tag{1,1.2}$$

That is the repair times are negative exponentially distributed. Further, we introduce the following definitions which will be used in the analysis of the system in this section and later as well.

4. Time-to-System Failure: We define the time-to-system failure (TSF) as the duration of time from the instant the system starts operating till the instant it fails for the first time due to non-availability of a spare unit for replacement. That is, during the TSF period, the system is in the "up state".

5. System Down-time: The system down-time (SDT) is defined as the duration of time the system fails to the time it is restored back into operation by the completion of a repair on a unit. The distribution of SDT is obviously $D(t) = 1 - e^{-\mu t}$.

6. State of the System: Let the random variables n(t) and m(t) represent respectively the number of failed units in the system at time t and the number of repair completions by time t. Then, we define the state of the system to be (m,n) if n(t) = n and m(t) = m, m and n taking the positive

integral values.

7. First Passage Time Distribution: Denote by $G_{ij}(t)$ the distribution of time taken to reach the state $n(t)=j$ for the first time, starting initially in the state $n(0)=i$. Here, we are defining states with respect to n(t) alone. Then $G_{i,N}(t),\ i \neq N$ represents the distribution of TSF, $G_{N,N-1}(t)$ represents the distribution of SDT and is equal to $D(t)$ and $G_{N,N}(t)$ represents the distribution function of the recurrent time to the state N.

8. Finally, we define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad , \qquad Re(s) > 0$$

and the Laplace Stieltjes transform of a distribution function $F(t)$ as

$$\hat{F}(s) = \int_0^\infty e^{-st}\, dF(t) \qquad , \qquad Re(s) > 0$$

Now we proceed to investigate the general process probabilities by forming difference-differential equations and obtaining the Laplace transforms of the state probabilities.

GENERAL PROCESS PROBABILITIES

The process of the system can be viewed as one in which time-to-system failure periods alternate with system down-time periods during which the system is being restored to operable state. When the system starts functioning after restoration, there are still (N-1) failed units in the system. If the system continues to function, repair completion and failure of individual units take place a number of times and the system fails again when all the spares are emptied. These cycles repeat. Thus in this process, transition from any state (m,n) to any other state (m',n') is possible. We call processes permitting such transitions, the 'general process'.

Let us define the following m-step transition probabilities associated with the general process for transition from any state i at time t = 0 to any other state at time t.

1. $P^{(m)}_{i,n}(r,t)$ - the probability that at time t, there are n failed units in the system and the operating unit is in the r^{th} phase of its failure time.

2. $P^{(m)}_{i,N}(t)$ - the probability that at time t, the system is in the failed state as the system failure takes place when all the N units in the system, namely, the single operating unit and all the (N-1) spares are in the

failed state simultaneously. Defining the random variable $\nu(t)$ to denote the phase of the failure time of an operating unit, we have

$$P_{i,n}^{(m)}(r,t) = Pr\left[m(t)=m,\, n(t)=n,\, \nu(t)=r \,/\, n(0)=i,\, m(0)=0 \right] \quad (1,1.3)$$

and

$$0 \le i \le N,\; 0 < n < N,\; 0 < r \le j,\; m \ge 0$$

$$P_{i,N}^{(m)}(t) = Pr\left[m(t)=m,\, n(t)=N \,/\, n(0)=i,\, m(0)=0 \right]$$

$$0 \le i \le N,\quad m \ge 0 \quad (1,1.4)$$

It is observed that the general process is Markovian with respect to the state space over which the set of probabilities (1,1.3) and (1,1.4) have been defined. Therefore, it is easy to construct the difference-differential equations governing the process by connecting the various state probabilities at time t and $t+\Delta$ and taking the limit as $\Delta \to 0$. Thus, we obtain

$$\frac{d}{dt} P_{i,n}^{(m)}(r,t) = -(\lambda+\mu) P_{i,n}^{(m)}(r,t) + \lambda P_{i,n}^{(m)}(r+1,t) + \lambda C_r P_{i,n-1}^{(m)}(1,t) + \mu P_{i,n+1}^{(m-1)}(r,t), \quad (1,1.5)$$

$$m \ge 0,\; 1 \le r < j,\; 0 < n < N-1$$

$$\frac{d}{dt} P_{i,n}^{(m)}(j,t) = -(\lambda+\mu) P_{i,n}^{(m)}(j,t) + \lambda C_j P_{i,n-1}^{(m)}(1,t) + \mu P_{i,n+1}^{(m-1)}(j,t),\quad m \ge 0,\; 0 < n < N-1 \quad (1,1.6)$$

$$\frac{d}{dt} P_{i,N-1}^{(m)}(r,t) = -(\lambda+\mu) P_{i,N-1}^{(m)}(r,t) + \lambda C_r P_{i,N-2}^{(m)}(1,t) + \lambda P_{i,N-1}^{(m)}(r+1,t) + \mu C_r P_{i,N}^{(m-1)}(t) \quad (1,1.7)$$

$$(1 \le r < j)$$

$$\frac{d}{dt} P_{i,N-1}^{(m)}(j,t) = -(\lambda+\mu) P_{i,N-1}^{(m)}(j,t) + \lambda C_j P_{i,N-2}^{(m)}(1,t) + \mu C_j P_{i,N}^{(m-1)}(t) \qquad (1,1.8)$$

$$\frac{d}{dt} P_{i,N}^{(m)}(t) = -\mu P_{i,N}^{(m)}(t) + \lambda P_{i,N-1}^{(m)}(1,t) \qquad (1,1.9)$$

$$\frac{d}{dt} P_{i,0}^{(m)}(r,t) = -\lambda P_{i,0}^{(m)}(r,t) + \lambda P_{i,0}^{(m)}(r+1,t) + \mu P_{i,1}^{(m-1)}(r,t), \; 1 \le r < j \qquad (1,1.10)$$

$$\frac{d}{dt} P_{i,0}^{(m)}(j,t) = -\lambda P_{i,0}^{(m)}(j,t) + \mu P_{i,1}^{(m-1)}(j,t) \qquad (1,1.11)$$

Since the system starts at time t = 0 with i failed units and the newly commissioned unit being in the r^{th} phase, $P_{i,i}^{(m)}(r,0) = C_r$, $(0 \le i \le N)$. Therefore, the initial conditions for this process are

$$P_{i,n}^{(m)}(r,0) = \delta_{0,m}\,\delta_{i,n}\,C_r \qquad (1,1.12)$$

where δ_{ij} is the Kronecker delta function.

To facilitate the solution of the above equations, we define the generating functions

$$P_{i,n}(r,z,t) = \sum_{m=0}^{\infty} z^m P_{i,n}^{(m)}(r,t) \qquad (1,1.13)$$

$$0 < n < N$$

and

$$P_{i,N}(z,t) = \sum_{m=0}^{\infty} z^m P_{i,N}^{(m)}(t) \tag{1,1.14}$$

which are convergent for $|z| \leq 1$

In terms of the generating functions (1,1.13) and (1,1.14) and the Laplace transforms, the equations (1,1.5) to (1.1.11) become

$$-(\lambda+\mu+s)\,\bar{P}_{i,n}(r,z,s) + \lambda \bar{P}_{i,n}(r+1,z,s) + \lambda C_r \bar{P}_{i,n-1}(1,z,s) + z\mu \bar{P}_{i,n+1}(r,z,s) + \delta_{i,n} C_r = 0 \tag{1,1.15}$$

$$-(\lambda+\mu+s)\,\bar{P}_{i,n}(j,z,s) + \lambda C_j \bar{P}_{i,n-1}(1,z,s) + z\mu \bar{P}_{i,n+1}(j,z,s) + \delta_{in} C_j = 0 \tag{1,1.16}$$

$$-(\lambda+\mu+s)\,\bar{P}_{i,N-1}(r,z,s) + \lambda C_r \bar{P}_{i,N-2}(1,z,s) + \lambda \bar{P}_{i,N-1}(r+1,z,s) + z\mu C_r \bar{P}_{i,N}(z,s) = 0 \tag{1,1.17}$$

$$-(\lambda+\mu+s)\,\bar{P}_{i,N-1}(j,z,s) + \lambda C_j \bar{P}_{i,N-2}(1,z,s) + z\mu C_j \bar{P}_{i,N}(z,s) = 0 \tag{1,1.18}$$

$$-(\mu+s)\,\bar{P}_{i,N}(z,s) + \lambda \bar{P}_{i,N-1}(1,z,s) = 0 \tag{1,1.19}$$

$$-(\lambda+s)\,\bar{P}_{i,0}(r,z,s) + \lambda \bar{P}_{i,0}(r+1,z,s) + z\mu \bar{P}_{i,1}(r,z,s) = 0 \tag{1,1.20}$$

$$-(\lambda+s)\,\bar{P}_{i,0}(j,z,s) + z\mu \bar{P}_{i,1}(j,z,s) = 0 \tag{1,1.21}$$

In obtaining the above equations, it was assumed that $\bar{P}^{(m)}_{i,n}(r,s) = 0$ at $m = -1$. Further, let us also define the following generating functions

$$\phi_n(z,x,s) = \sum_{r=1}^{j} x^r \bar{P}_{i,n}(r,z,s) \text{ and } \bar{P}(z,x,y,s) = \sum_{n=0}^{N-1} y^n \phi_n(z,x,s) + y^N \bar{P}_{i,N}(z,s)$$

Multiplying (1,1.15) to (1,1.21) by appropriate powers of x and y and summing over r from $r=1$ to $r=j$ and over n from n = 0 to n = N, we get

$$\left[(\lambda+\mu+s) - \frac{\lambda}{x} - \frac{\mu z}{y}\right] \bar{P}(z,x,y,s)$$

$$= -\lambda\left[1 - y\,C(x)\right] \sum_{n=0}^{N-1} y^n \bar{P}_{i,n}(1,z,s) + y^N \bar{P}_{i,N}(z,s)\left[s+\mu-\frac{\mu z}{y}\right]\left[1-C(x)\right]$$

$$+ \frac{\lambda}{x} \bar{P}_{i,N}(z,s)(x - y^N) + z\mu\left(1-\frac{1}{y}\right)\phi_0(z,x,s) + y^i C(x) \tag{1,1.22}$$

where $C(x) = \sum_{r=1}^{j} x^r C_r$

Choosing x such that the coefficient of $\bar{P}(z,x,y,s)$ in (1,1.22) becomes zero, helps us in evaluating $\sum_{n=0}^{N-1} y^n \bar{P}_{i,n}(1,z,s)$. Accordingly, if we put in (1,1.22)

$$x = \frac{\lambda y}{(\lambda+\mu+s)y - \mu z} = \gamma \text{ (say)}$$

and simplify, we get

$$\bar{G}(z,y,s) = \sum_{n=0}^{N-1} y^n \bar{P}_{i,n}(1,z,s)$$

$$= \left[\mu z(1-\tfrac{1}{y})\sum_{r=1}^{j} Y^r \bar{P}_{i,0}(r,z,s) + \lambda(1-y^N)\bar{P}_{i,N}(z,s)\right.$$

$$\left. + \lambda y^N \bar{P}_{i,N}(z,s)(1-\tfrac{1}{Y})C(Y) + y^i C(Y)\right] \Big/ \lambda\left[1 - yC(Y)\right] \qquad (1,1.23)$$

The denominator of (1,1.23) is a polynomial of $(j+1)$ th degree in y and has, therefore, $(j+1)$ zeros. By Rouche's theorem, it can be easily shown that this equation has j zeros inside and one outside the unit circle. Since the expression on the left hand side of (1,1.23) is regular in the entire y plane, the numerator must vanish at all the $(j+1)$ zeros of the denominator, thus giving rise to $(j+1)$ equations in $(j+1)$ unknowns, namely

$$z\mu(1-\tfrac{1}{y_\ell})\sum_{r=1}^{j} Y_\ell^r \bar{P}_{i,0}(r,z,s) + \lambda(1-y_\ell^N)\bar{P}_{i,N}(z,s)$$

$$+ \lambda y_\ell^N (1-\tfrac{1}{Y_\ell})C(Y_\ell)\bar{P}_{i,N}(z,s) + y_\ell^i C(Y_\ell) = 0, \quad \ell = 1,2,\ldots(j+1) \qquad (1,1.24)$$

where $Y_\ell = \lambda y_\ell / [\lambda+\mu+s-\mu z)$ and y_ℓ are the roots of $1 - y\sum_{r=1}^{j} C_r Y^r = 0$

It can be shown in this case that the $(j+1)$ equations are linearly independent if we assume all the $(j+1)$ zeros of the denominator are simple for, the determinant of the coefficients of $\bar{P}_{i,0}(r,z,s)$ and $\bar{P}_{i,N}(z,s)$ in (1,1.24) can be shown to reduce to

$$\Delta = (-\lambda\mu)^{\frac{j(j-1)}{2}} \prod_{l=1}^{j+1} \left\{ \frac{\lambda\mu(y_l - 1)}{(\lambda+\mu+s)y_l - \mu z} \right\} .$$

$$\cdot \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{j-1} & a_1 \\ 1 & y_2 & y_2^2 & \cdots & y_2^{j-1} & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & y_{j+1} & y_{j+1}^2 & \cdots & y_{j+1}^{j-1} & a_{j+1} \end{vmatrix}$$

(1,1.25)

where

$$a_l = \frac{\lambda(1-y_l^N) + y_l^{N-1}\left[\mu z - \overline{\mu+s}\, y_l\right] \sum_{r=1}^{j} C_r \left\{ \frac{\lambda y_l}{(\lambda+\mu+s)y_l - \mu z} \right\}^r}{\frac{\lambda\mu(y_l-1)}{(\lambda+\mu+s)y_l - \mu z}}$$

From this, it is observed that Δ will be non-vanishing and, therefore, the $(j+1)$ equations are linearly independent provided the roots are not equal to unity and are distinct. In the case, when some roots are repeated, a modified procedure as in Wishart (1956) may be followed. The $(j+1)$ unknowns $\bar{P}_{i,0}(r,z,s)$ and $\bar{P}_{i,N}(z,s)$ are solved by using Cramer's rule. In particular we obtain

$$\bar{P}_{i,N}(z,s) = \frac{\sum\limits_{\ell=1}^{j+1} (-1)^{j+\ell} \frac{y_\ell^{\,i}}{(y_\ell - 1)} \sum\limits_{r=1}^{j} \frac{C_r(\lambda y_\ell)^r}{[(\lambda+\mu+s)y_\ell - \mu z]^{r-1}} \cdot \Delta_{j+1,\ell}}{\sum\limits_{\ell=1}^{j+1} (-1)^{j+\ell+1} \dfrac{\lambda(1-y_\ell^{N}) + y_\ell^{N-1}[\mu z - \overline{\mu+s}\, y_\ell] \sum\limits_{r=1}^{j} \left\{\frac{\lambda y_\ell}{(\lambda+\mu+s)y_\ell - \mu z}\right\}^r C_r}{\dfrac{y_\ell - 1}{(\lambda+\mu+s)y_\ell - \mu z}} \cdot \Delta_{j+1,\ell}}$$

(1, 1·26)

where

$$\Delta_{j+1,\ell} = \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{j-1} \\ 1 & y_2 & y_2^2 & \cdots & y_2^{j-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & y_{\ell-1} & y_{\ell-1}^2 & \cdots & y_{\ell-1}^{j-1} \\ 1 & y_{\ell+1} & y_{\ell+1}^2 & \cdots & y_{\ell+1}^{j-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & y_{j+1} & y_{j+1}^2 & \cdots & y_{j+1}^{j-1} \end{vmatrix} = \prod_{\substack{r=1 \\ r<s;\ r,s\neq \ell}}^{j+1} (y_r - y_s)$$

(1, 1.27)

As $\bar{P}_{i,N}(z,s)$ is the generating function of the Laplace transform of the probabilities $P_{i,N}^{(m)}(t)$, we obtain $P_{i,N}(t)$, the probability that at time t, the system is in the failed state starting initially with i failed units irrespective of the number of repair completions by putting $z=1$ in the generating function $P_{i,N}(z,t)$. That is

$$P_{i,N}(t) = \sum_{m=0}^{\infty} P_{i,N}^{(m)}(t) = \left(P_{i,N}(z,t) \right)_{z=1} \qquad (1,1.28)$$

Hence, the Laplace transform of $P_{i,N}(t)$ can be easily obtained from $\bar{P}_{i,N}(z,s)$ in (1,1.26) by putting $z = 1$.

DISTRIBUTION OF TIME TO SYSTEM FAILURE AND RECURRENCE TIME TO STATE N

Now, we connect the general process probabilities $P_{i,N}(t)$ with the first passage time distributions $G_{i,N}(t)$ by simple probabilistic argument, enumerating the possible ways in which the state N could be reached at time t from the initial state i at time t = 0. The $G_{i,N}(t)$, $i \neq N$ represent the distribution of time-to-system failure and $G_{N,N}(t)$ represents the distribution of recurrent time to the state N. When $i \neq N$ the state n(t) = N at time t starting with the state i initially can be reached in the following way. The system starting in the state i initially reaches the state N for the first time in time $\tau < t$. Then in the interval (τ, t) the system reaches the state N at time t starting in the state N at τ. Obviously, the first event is a first passage event and the second a general process event and, therefore, we can write

$$P_{i,N}(t) = \int_0^t P_{N,N}(t-\tau)\, dG_{i,N}(\tau)$$

$$= G_{i,N}(t) * P_{N,N}(t) \qquad i \neq N \qquad (1,1.29)$$

where * denote the convolution operation. When $i = N$ the system initially starts in the failed state and the state $n(t) = N$ is achieved if a) no transition from the initial state N occurs in time t or b) a transition occurs for the first time from the initial state N, through N-1 etc. to N in time $\tau < t$ and starting at τ in state N the system reaching again N at t. Thus

$$P_{N,N}(t) = [1 - D(t)] + G_{N,N}(t) * P_{N,N}(t) \qquad (1,1.30)$$

Taking the Laplace Stieltjes transform (LST) on both sides of (1,1.29) and using the expressions for $\bar{P}_{i,N}(s)$ and $\bar{P}_{N,N}(s)$ obtained from (1,1.26) by putting $z = 1$ and i = N, we have for the LST of the distribution of time to system failure,

$$\hat{G}_{i,N}(s) = \frac{\hat{P}_{i,N}(s)}{\hat{P}_{N,N}(s)} = \frac{\bar{P}_{i,N}(s)}{\bar{P}_{N,N}(s)}$$

$$= \frac{\sum\limits_{\ell=1}^{j+1} (-1)^{j+\ell} \frac{y_\ell^{i}}{y_\ell - 1} \sum\limits_{r=1}^{j} \frac{c_r (\lambda y_\ell)^r}{[(\lambda+\mu+s)y_\ell - \mu]^{r-1}} \cdot \prod\limits_{\substack{r=1 \\ r<s;\, r,s \neq \ell}}^{j+1} (y_r - y_s)}{\sum\limits_{\ell=1}^{j+1} (-1)^{j+\ell} \frac{y_\ell^{N}}{y_\ell - 1} \sum\limits_{r=1}^{j} \frac{c_r (\lambda y_\ell)^r}{[(\lambda+\mu+s)y_\ell - \mu]^{r-1}} \cdot \prod\limits_{\substack{r=1 \\ r<s;\, r,s \neq \ell}}^{j+1} (y_r - y_s)} \qquad (1,1.31)$$

and the LST of the distribution of recurrence time to the state N obtained from (1,1.30) is given by

$$1 - \hat{G}_{N,N}(s) = \frac{1 - \hat{D}(s)}{\hat{P}_{N,N}(s)} \qquad (1,1.32)$$

Particular cases:

The LST's of the distribution of time-to-system failure in the particular cases; (i) when the failure time distribution of individual units is K-Erlang and (ii) when it is negative exponential, will now be obtained.

(i) K-Erlang failure time distribution:

In this case, $C_r = 1$, $r = k$. and $C_r = 0$, $r \neq k$ so that (1,1.26) becomes

$$\hat{G}_{i,N}(s) = \frac{\sum\limits_{\ell=1}^{k+1} (-1)^{k+\ell} \frac{y_\ell^{i+k}}{(y_\ell - 1)[(\lambda+\mu+s)y_\ell - \mu]^{k-1}} \cdot \prod\limits_{\substack{r=1 \\ r<s;\, r,s \neq \ell}}^{k+1} (y_r - y_s)}{\sum\limits_{\ell=1}^{k+1} (-1)^{k+\ell} \frac{y_\ell^{N+k}}{(y_\ell - 1)[(\lambda+\mu+s)y_\ell - \mu]^{k-1}} \cdot \prod\limits_{\substack{r=1 \\ r<s;\, r,s \neq \ell}}^{k+1} (y_r - y_s)} \qquad (1,1.33)$$

where $y_1, y_2, \ldots y_{k+1}$ are the roots of the equation $1 - y\left[\frac{\lambda y}{(\lambda+\mu+s)y-\mu}\right]^k = 0$

(ii) Exponential failure time distribution:

In this case, $C_r = 1, r = 1$ and $C_r = 0, r \neq 1$ so that (1,1.31) becomes

$$\hat{G}_{i,N}(s) = \frac{y_1^i\left(1-\frac{1}{y_2}\right) - y_2^i\left(1-\frac{1}{y_1}\right)}{y_1^N\left(1-\frac{1}{y_2}\right) - y_2^N\left(1-\frac{1}{y_1}\right)}, \quad (1,1.34)$$

where y_1, y_2 are the roots of the equation $1 - \frac{\lambda y^2}{(\lambda+\mu+s)y-\mu} = 0$

SOME RELIABILITY CHARACTERISTICS

We shall now evaluate some of the important reliability characteristics mentioned at the outset. Let the random variables T_u and T_d represent the time to system failure and the duration of system down-time respectively. Let in the general process, the system start operating at instants $t_0 < t_1 < t_2 \ldots$. Then $\tau_R(k) = t_k - t_{k-1}$ will be comprising of the k th TSF period and the succeeding SDT period. In fact, we can write

$$\tau_R(k) = t_k - t_{k-1} = T_u(k) + T_d(k)$$

We call $\tau_R(k)$ the k th renewal period and the sequence $\{\tau_R(k)\}(k \geq 2)$ constitute a renewal process and $\{\tau_R(k)\}\ (k \geq 1)$ a modified renewal process. [Cox (1962) p.28] . This renewal process is shown in figure 1.0(b).

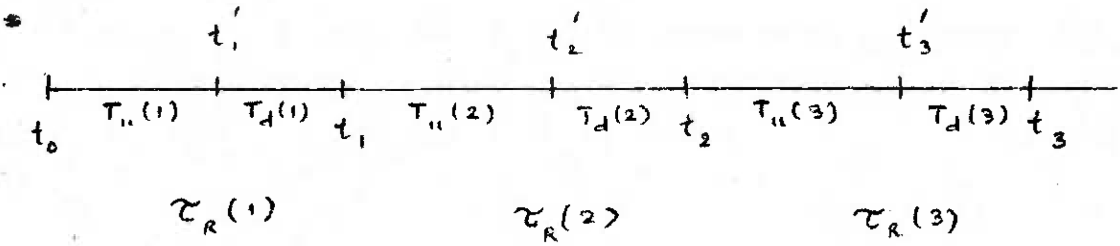


Fig 1.0(b) Renewal Periods

Denoting by $E(T^r)$, the r th moment of the random variable T, we have from the result of renewal theory for the expectation and the second moment of the random variable τ_R

$$E(\tau_R) = E_{N-1}(T_u) + E(T_d) \qquad (1,1.35)$$

$$E(\tau_R^2) = E_{N-1}(T_u^2) + E(T_d^2) \qquad (1,1.36)$$

where the suffix (N-1) stands for the initial state at which the TSF starts within the second and subsequent renewal periods. The moments of the TSF process when it starts initially with any state i, $i \neq N$ can be obtained by differentiation of $\hat{G}_{i,N}(s)$ with respect to s and taking the limit as $s \to 0$. In particular when the failure time distribution of individual units is exponential with parameter λ, we have from (1,1.34)

$$E_i(T_u) = \left[-\frac{d}{ds}\hat{G}_{i,N}(s)\right]_{s=0} = \frac{1}{\mu(1-\rho)}\left[-(N-i) + \frac{1}{1-\rho}\left(\frac{1}{\rho^N} - \frac{1}{\rho^i}\right)\right] \tag{1,1.37}$$

and putting $i = N-1$

$$E_{N-1}(T_u) = \frac{1}{\mu\rho^N}\left[\frac{1-\rho^N}{1-\rho}\right], \quad \rho = \frac{\lambda}{\mu} \tag{1,1.38}$$

Since $E(T_d) = \frac{1}{\mu}$, the expected duration of repair time, the expression for mean recurrence time $E(\tau_R)$ in (1,1.35) becomes

$$E(\tau_R) = \frac{1}{\mu} + \frac{1}{\mu\rho^N}\left[\frac{1-\rho^N}{1-\rho}\right] = \frac{1}{\mu\rho^N}\left[\frac{1-\rho^{N+1}}{1-\rho}\right], \tag{1,1.39}$$

and the expressions for the second moment of T_u and τ_R are given by

$$E_i(T_u^2) = \frac{\rho}{1-\rho}\left[\frac{1}{\mu^2(1-\rho)^2}\left\{(N+i)(N-i) + 2\rho\left(N-i+\frac{1}{1-\rho}\left(\frac{1}{\rho^N}-\frac{1}{\rho^i}\right)\right)\right\} + \frac{2E_i(T_u)}{\mu}\left\{N + \frac{1-\rho^{N+1}}{(1-\rho)\rho^N}\right\}\right] \tag{1,1.40}$$

and

$$E(\tau_R^2) = \frac{2}{\mu^2} + \frac{\rho}{1-\rho}\left[\frac{1}{\mu^2(1-\rho)^2}\left\{(2N-1) + 2\rho\left(1+\frac{1}{\rho^N}\right)\right\} + \frac{2E_{N-1}(T_u)}{\mu}\left\{N + \mu E(\tau_R)\right\}\right]$$

(1,1.41)

Using (1,1.38), the ratios of the expected TSF to the expected SDT (which is also the same as the expected repair time of the failed unit) for various values of ρ and N have been calculated for the case of exponential failure time and repair time distributions. These values are plotted in logarithmic scale against ρ for various values of N in fig.1.1. It may be observed that there is a steep decrease in expected TSF as ρ increases. It also emerges from this figure that when $\rho > 1$, increasing the number of redundant units does not appreciably increase the expected TSF.

LONG-RUN AVAILABILITY OF THE SYSTEM

When the system continues to operate over fairly long period of time, it passes through a number of up-states alternating with down-states. Using the renewal property of the process, we can write $P_{i,N}(t)$ as

$$P_{i,N}(t) = G_{i,N}(t) * [1 - D(t)] + \int_0^t P_{N-1,N}(t-y)\, d_y r(y,i) \quad (1,1.42)$$

where $d_y r(y,i)$ denotes the renewal density giving the probability of occurrance of at least one restoration from the down-state to up-state, i.e. to the state (N-1) at time y, the system having initially started with i failed units. Let the steady-state probability p_N represent the $\lim_{t \to \infty} P_{i,N}(t)$. Then by applying

RATIO OF EXPECTED TIME TO SYSTEM FAILURE TO EXPECTED SYSTEM DOWN TIME FOR VARIOUS ρ AND N OF STANDBY REDUNDANT SYSTEM

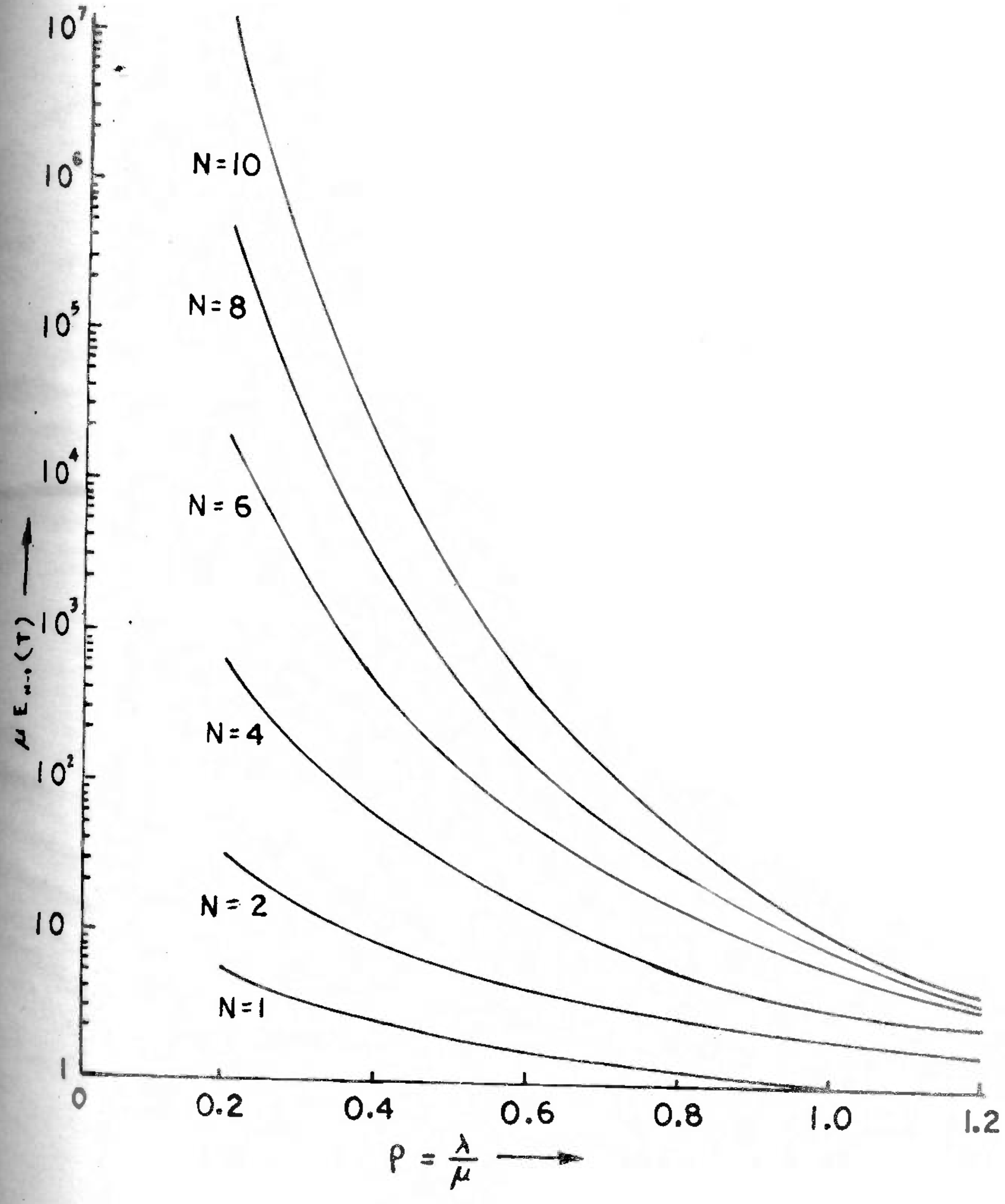


FIG. 1.1

Smith's (1954) theorem to (1,1.42) (for the statement of the theorem see chapter 2, section dealing with ergodic properties) and noting that $\lim_{T\to\infty} \int_0^T P_{N-1,N}(t)\,dt = E(T_d)$ we have

$$p_N = \frac{1}{E(\tau_R)} \int_0^\infty P_{N-1,N}(t)\,dt$$

$$= \frac{E(T_d)}{E(\tau_R)} = \frac{1}{\mu E(\tau_R)}$$

$$= \frac{\rho^N(1-\rho)}{1-\rho^{N+1}} \qquad (1,1.43)$$

for the exponential failure case.

The long-run availability of the system is defined as the probability that the system is in the up-state when it operates over a long period of time and is, therefore, given by $1-p_N$. That is

$$\text{Long-run Availability} = 1-p_N$$

$$= 1-\frac{E(T_d)}{E(\tau_R)} = \frac{E_{N-1}(T_u)}{E(\tau_R)}$$

$$= \frac{1-\rho^N}{1-\rho^{N+1}} \qquad (1,1.44)$$

for the case of exponential failures.

Thus, we observe that the long-run availability of the system is also equal to the ratio of the expected time to system failure to the mean renewal period.

LONG-RUN AVAILABILITY OF THE STANDBY REDUNDANT SYSTEM FOR VARIOUS N AND ρ

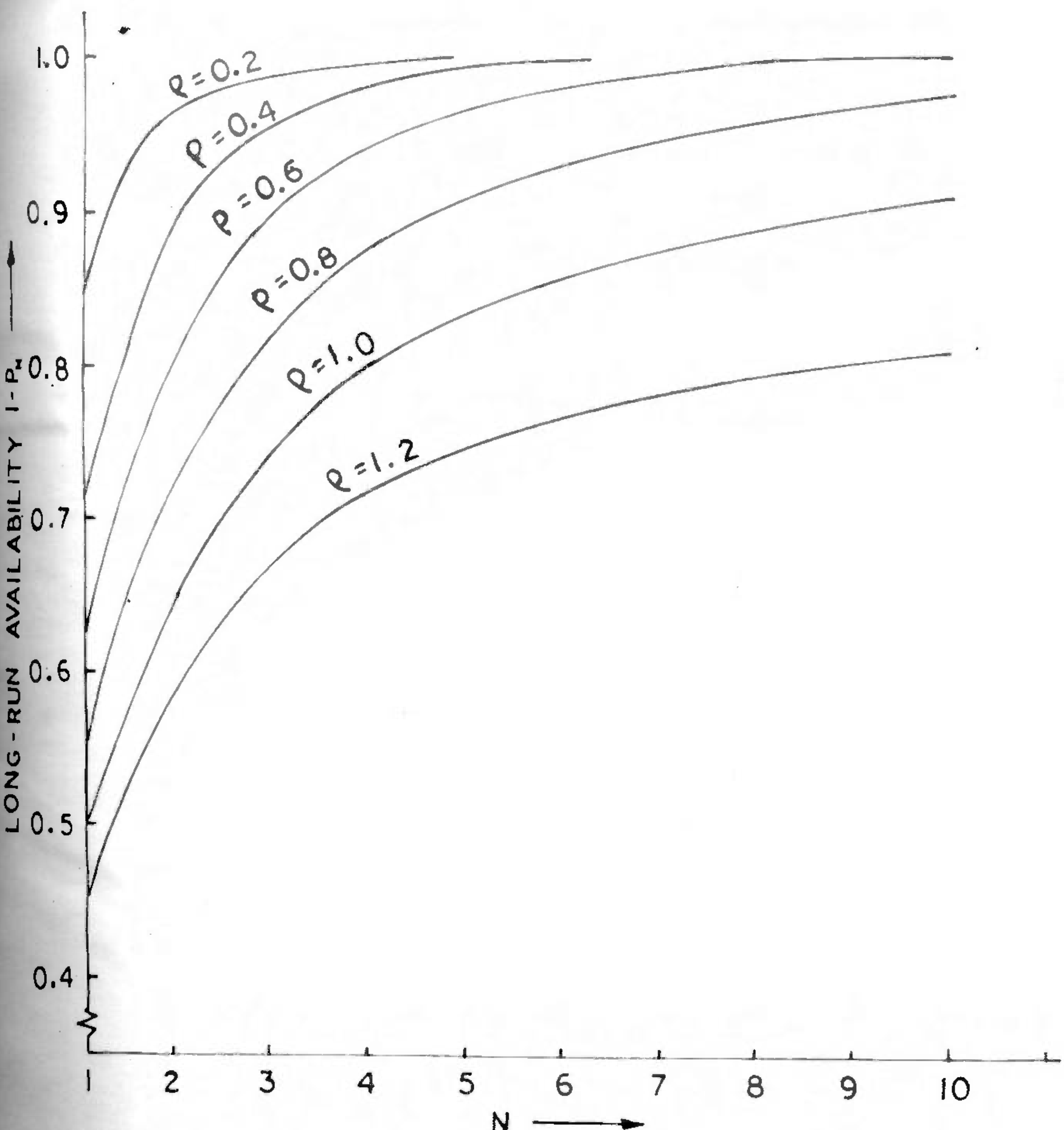


FIG. 1.2

The long-run availability of this system for various combinations of ρ and N has been obtained by using (1,1.43) and is graphically shown at fig. 1.2. It will be observed that the curves for long-run availability increase sharply at the beginning stages and tend to flatten out as N increases. It is evident that there is almost complete system availability for small values of ρ and for practically small number of standby units. However, for large values of ρ $(\rho \geq 1)$ as the curves flatten out, it is practically not possible to achieve any arbitrarily high degree of system availability as it requires very large number of units. Fig. 1.2 can also be used to determine the number of standby units to be used for a given ρ to achieve 90, 95 and 99 per cent of system availability.

Number of system failures in the interval (0, t)

The successive points of occurrance of system failure form a renewal process. Define the random variable N_t as the number of system failures in (0, t). We note that the system failure states alternate with the system up-states and the commencement of system down-state at $t = 0$ is not counted as a renewal. The random variable N_t forms one of the most interesting characteristics of the process. When the system starts initially with the down state, the renewal process is an 'ordinary renewal process' and when it initially starts with up-state, it is a modified renewal process. Denoting the renewal function by H(t), the expected number of system failures by $E(N_t)$, the probability density of $G_{N,N}(t)$ and $G_{i,N}(t)$ by $g_{N,N}(t)$ and $g_{i,N}(t)$, $i \neq N$ we have

$$H(t) = E(N_t) \qquad (1,1.45)$$

The Laplace transform of H(t) for the modified renewal process is given by [Cox (1962) p.46]

$$\bar{H}_m(s) = \frac{\bar{g}_{i,N}(s)}{s[1-\bar{g}_{N,N}(s)]} \qquad (1,1.46)$$

and for the ordinary renewal process is given by

$$\bar{H}_o(s) = \frac{\bar{g}_{N,N}(s)}{s[1-\bar{g}_{N,N}(s)]} \qquad (1,1.47)$$

Now, we study the form of H(t) for large t through its Laplace transform using Tauberian arguments. The Laplace transform of $g_{N,N}(t)$ and $g_{i,N}(t)$ as $s \to 0$ can be expressed as

$$\bar{g}_{N,N}(s) = 1 - s\,E(\tau_R) + s^2\,E(\tau_R^2) + o(s^2) \qquad (1,1.48)$$

and

$$\bar{g}_{i,N}(s) = 1 - s\,E_i(T_u) + s^2\,E_i(T_u^2) + o(s^2) \qquad (1,1.49)$$

Using (1,1.48) and (1,1.49) in (1,1.46) and (1,1.47) and expressing in terms of powers of $\frac{1}{s}$ we obtain

$$\bar{H}_{m}(s) = \frac{1}{s^2 E(\tau_R)} + \frac{E(\tau_R^2)}{2s\{E(\tau_R)\}^2} - \frac{1}{s}\,\frac{E_1(T_u)}{E(\tau_R)} + o\left(\frac{1}{s}\right) \quad (1,1.50)$$

$$\bar{H}_{o}(s) = \frac{1}{s^2 E(\tau_R)} + \frac{E(\tau_R^2)}{2s\{E(\tau_R)\}^2} - \frac{1}{s} + o\left(\frac{1}{s}\right) \quad (1,1.51)$$

Formal inversion of (1,1.50) and (1,1.51) gives the expected number of system failures in (0, t) as $t \to \infty$:

$$E_m(N_t) = \frac{t}{E(\tau_R)} + \frac{E(\tau_R^2)}{2[E(\tau_R)]^2} - \frac{E_1(T_u)}{E(\tau_R)} + o(1) \quad (1,1.52)$$

$$E_o(N_t) = \frac{t}{E(\tau_R)} + \frac{E(\tau_R^2)}{2[E(\tau_R)]^2} - 1 + o(1) \quad (1,1.53)$$

where the suffix m and o for E on the left side stand for the modified and ordinary renewal process respectively.

Interval Reliability

Interval reliability R (x,t) is defined [Barlow and Hunter (1961) , Barlow et al (1965)] as the probability that given the system is in the up-state at time t, it will continue to be in the upstate without reaching the down-state for an interval of duration of x. The limiting interval reliability is simply the limit of R (x,t) as $t \to \infty$. Then it is easily seen that

$$R(x,t) = [1 - G_{i,N}(t+x)] + \int_0^t [1 - G_{N-1,N}(t-y+x)]\, d_y Y(y,i) \qquad (1,1.54)$$

where $d_y Y(y,i)$ denotes the probability of a recurrènce of restoration from the down-state to up-state, i.e. to state (Ñ-1) at y. The limit of R(x,t) as $t \to \infty$ follows from Smith's theorem and is given by

$$\lim_{t \to \infty} R(x,t) = \frac{\int_x^\infty [1 - G_{N-1,N}(y)]\, dy}{E(\tau_R)} \qquad (1,1.55)$$

$E(\tau_R)$ being the mean recurrence time.

Integrating by parts the numerator of (1,1.54), the upper and lower bounds of the limiting interval reliability are obtained as

$$\frac{E_{N-1}(T_u) - x}{E(\tau_R)} \le \lim_{t \to \infty} R(x,t) \le \frac{E_{N-1}(T_u)}{E(\tau_R)} \qquad (1,1.56)$$

which is always true.

When the failure time and repair time distributions are exponential, the lower bounds of limiting interval reliability have been calculated using (1,1.56) with $\mu = 1.0$ and $\rho = 0.2$ for various values of N. These values have been plotted against the interval length x and are shown in fig. 1.3. From this figure, one can

THE LOWER BOUNDS OF LIMITING INTERVAL RELIABILITY FOR VARYING LENGTHS x OF THE INTERVALS (STANDBY REDUNDANT SYSTEM)

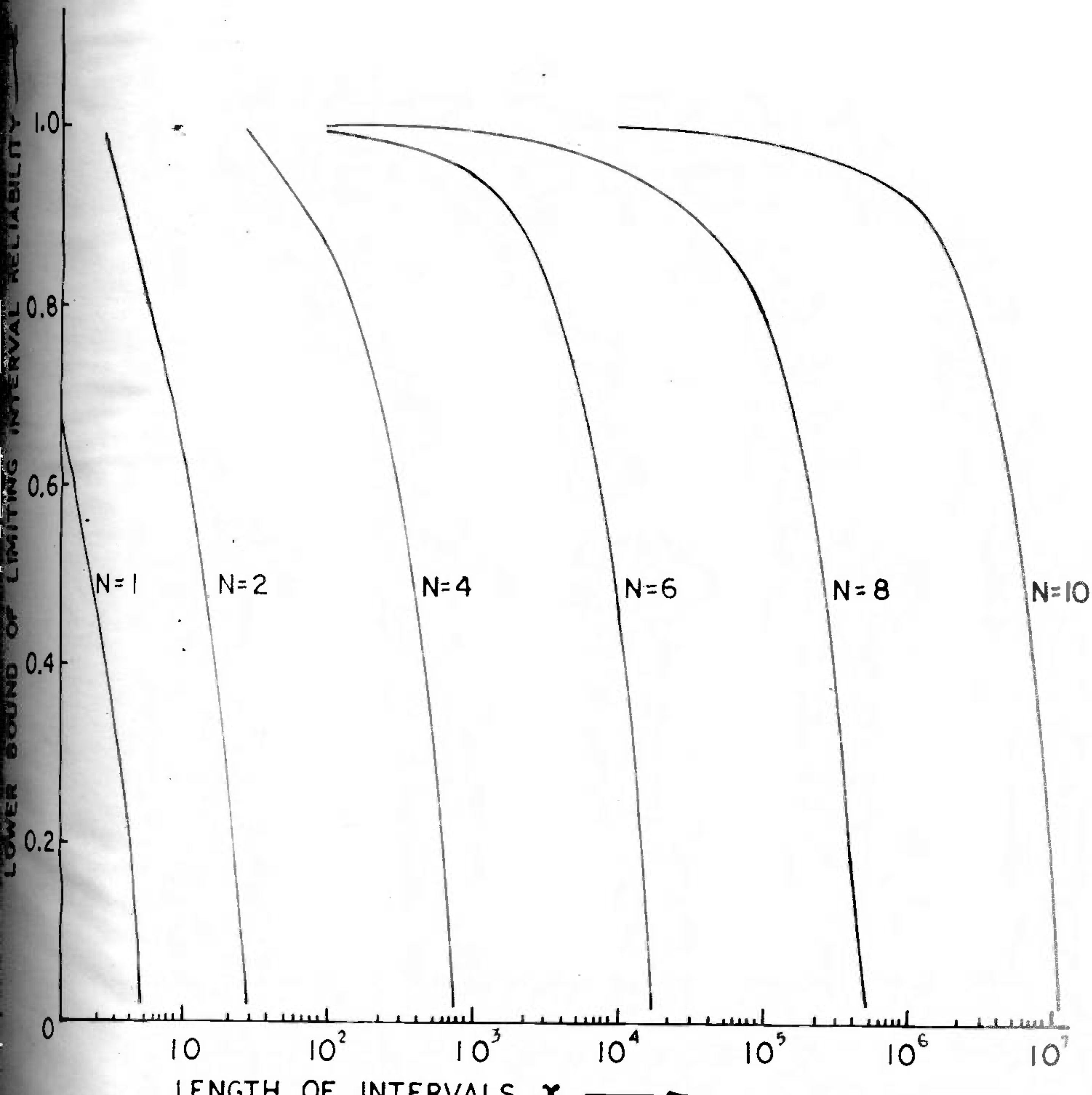


FIG. 1.3

determine the number of standby redundant units required to achieve a stated interval reliability in a given interval of time x.

SECTION 2

RELIABILITY OF A SYSTEM IN INTERMITTENT USAGE

Unlike the case where the system is used continuously as in section 1, in this section, we consider a system which is used only during the period when the demand for its operation exists even though it may be functioning. The disappointment occurs when the system fails within a period of existance of its demand or it is already in the failed state when the demand occurs. Naturally, the states of this process depend upon the existance of demand for the use of the system. Corresponding to each state of the constant usage process, we can associate a state of the intermittent usage process. The state of the system is described by the random variable n(t), the number of failed units at time t. In the constant usage process, n(t) varies over 0, 1, 2... N whereas in the intermittent usage process, it is assumed that n(t) takes the value j when the demand for the system does not exist and takes the value N + 1 + j when the demand exists, j denoting the number of failed units in the system varying from 0, 1, 2 ... N. We distinguish between the probabilities corresponding to the intermittent usage process and the continuous usage process by placing an asterik (*) over the former.

In order to relate the probabilities corresponding to the intermittent usage process with those of the continuous usage process, we define two probabilities $P_1(t)$ and $P(t)$. Let $P_1(t)$ denote the probability that the demand for the system exists at time t given that it existed at an initial moment. Let $\lambda_1 \Delta$ be the probability of demand occuring in time (t, t + Δ) and $\mu_1 \Delta$ be the probability that the demand terminates in (t, t + Δ). Then connecting the probabilities at t and t + Δ, we have

$$P_1(t+\Delta) = P_1(t)\left[1-\mu_1\Delta\right] + \left[1-P_1(t)\right]\lambda_1\Delta$$

Transposing and taking the limit as $\Delta \to 0$ we obtain

$$\frac{d}{dt}P_1(t) = -\overline{\lambda_1+\mu_1}\,P_1(t) + \lambda_1$$

Integrating this linear differential equation, we have

$$P_1(t) = \frac{\lambda_1}{\lambda_1+\mu_1} + \frac{\mu_1}{\lambda_1+\mu_1}\,e^{-(\lambda_1+\mu_1)t} \qquad (1,2\cdot1)$$

Similarly defining P(t) as the probability that the demand does not exist at time t, given that it did not exist at an initial moment, we obtain

$$P(t) = \frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t} \qquad (1,2.2)$$

Let $P_{ij}(t)$ denote when the system is in continuous usage, the probability that at time t there are j failed units in the system given that it started with i failed units at time t = 0 and let $P^*_{ij}(t)$ denote when the system is intermittently used, the probability that at time t there are j failed units in the system given that it started with i failed units initially. Now using $P_1(t)$ and $P(t)$ we derive a set of relations between $P^*_{ij}(t)$ and $P_{ij}(t)$. These relations are

$$\left.\begin{aligned}
P^*_{i,j}(t) &= P(t)\, P_{ij}(t) \\
P^*_{i,N+1+j}(t) &= [1-P(t)]\, P_{ij}(t) \\
P^*_{N+1+i,j}(t) &= [1-P_1(t)]\, P_{ij}(t) \\
P^*_{N+i+i,N+1+j}(t) &= P_1(t)\, P_{ij}(t)
\end{aligned}\right\} \quad \begin{aligned} &i = 0,1,2\ldots N \\ &j = 0,1,2\ldots N \\ &(1,2.3)\end{aligned}$$

Denoting by $G^*_{ij}(t)$ the distribution of first-passage time from the state i to the state j in time t in the intermittent usage process, the Laplace Stieltjes transform of the distribution of time-to-disappointment and the LST of the distribution of recurrence time to the disappointment state, the disappointment state being the state 2N + 1, are obtained by applying (1,1.29) and (1,1.32) with suitable modifications. From the state possibilities in (1,2.3) and using enumerative arguments as in section 1, we have

$$P^*_{\alpha,2N+1}(t) = \int_0^t G_{\alpha,2N+1}(t-\tau)\,d\,P^*_{2N+1,2N+1}(\tau)\,,\quad \alpha \neq 2N+1$$

$$P^*_{2N+1,2N+1}(t) = \left[1 - D^*(t)\right] + \int_0^t G_{2N+1,2N+1}(t-\tau)\,d\,P^*_{2N+1,2N+1}(\tau)$$

which on taking LST yield

$$\hat{P}^*_{\alpha,2N+1}(s) = \hat{G}^*_{\alpha,2N+1}(s)\,.\,\hat{P}^*_{2N+1,2N+1}(s)\,,\quad \alpha \neq 2N+1 \qquad (1,2.4)$$

and

$$\hat{P}^*_{2N+1,2N+1}(s) = \frac{1 - \hat{D}^*(s)}{1 - \hat{G}^*_{2N+1,2N+1}(s)} \qquad (1,2.5)$$

where $D^*(t) = 1 - e^{-(\lambda_1+\mu_1)t}$, the distribution of the duration of disappointment time. Here the suffix α takes the value i + N + 1 or i according as initially the demand for the system exists or does not exist.

If β_1 is the probability that initially the demand for the system use does not exist and $\beta_2 = 1-\beta_1$ is the probability that initially the demand for the system use exist, then the distribution of time-to-disappointment can be written as

$$\overset{*}{G}_{\alpha,2N+1}(t) = \beta_1 \overset{*}{G}_{i,2N+1}(t) + \beta_2 \overset{*}{G}_{N+1+i,2N+1}(t) \qquad (1,2.6)$$

the LST's of which using (1,2.4) reduce to

$$\hat{G}^{*}_{\alpha,2N+1}(s) = \beta_1 \frac{\hat{P}^{*}_{i,2N+1}(s)}{\hat{P}^{*}_{2N+1,2N+1}(s)} + \beta_2 \frac{\hat{P}^{*}_{N+1+i,2N+1}(s)}{\hat{P}^{*}_{2N+1,2N+1}(s)} \qquad (1,2.7)$$

Applying the appropriate relations from (1,2.3) we can now obtain $\overset{*}{P}_{i,2N+1}(t)$, $\overset{*}{P}_{N+1+i,2N+1}(t)$ and $\overset{*}{P}_{2N+1,2N+1}(t)$ in terms of $P_1(t)$, $P(t)$ and $P_{i,N}(t)$

Hence

$$\overset{*}{P}_{i,2N+1}(t) = [1-P(t)]\, P_{i,N}(t) = \frac{\lambda_1}{\lambda_1+\mu_1}\left[1-e^{-(\lambda_1+\mu_1)t}\right] P_{i,N}(t)$$

$$\overset{*}{P}_{N+1+i,2N+1}(t) = P_1(t)\, P_{i,N}(t) = \frac{\lambda_1}{\lambda_1+\mu_1}\left[1+\frac{\mu_1}{\lambda_1} e^{-(\lambda_1+\mu_1)t}\right] P_{i,N}(t)$$

$$(1,2.8)$$

and

$$\overset{*}{P}_{2N+1,2N+1}(t) = P_1(t)\, P_{N,N}(t) = \frac{\lambda_1}{\lambda_1+\mu_1}\left[1+\frac{\mu_1}{\lambda_1} e^{-(\lambda_1+\mu_1)t}\right] P_{N,N}(t)$$

Taking the LST on both sides of the above relations, we have

$$\hat{P}^{*}_{i,2N+1}(s) = \frac{\lambda_1}{\lambda_1+\mu_1}\left[\hat{P}_{i,N}(s) - \frac{s}{\lambda_1+\mu_1+s}\hat{P}_{i,N}(\lambda_1+\mu_1+s)\right]$$

$$\hat{P}^{*}_{N+1+i,2N+1}(s) = \frac{\lambda_1}{\lambda_1+\mu_1}\left[\hat{P}_{i,N}(s) + \frac{\mu_1}{\lambda_1}\cdot\frac{s}{\lambda_1+\mu_1+s}\hat{P}_{i,N}(\lambda_1+\mu_1+s)\right]$$

$$(1,2.9)$$

$$\hat{P}^{*}_{2N+1,2N+1}(s) = \frac{\lambda_1}{\lambda_1+\mu_1}\left[\hat{P}_{N,N}(s) + \frac{\mu_1}{\lambda_1}\cdot\frac{s}{\lambda_1+\mu_1+s}\hat{P}_{N,N}(\lambda_1+\mu_1+s)\right]$$

Therefore, making use of the relations (1,2.9), the expression for $\hat{G}^{*}_{\alpha,2N+1}(s)$ becomes

$$\hat{G}^{*}_{\alpha,2N+1}(s) = \left[\hat{P}_{N,N}(s) + \frac{\mu_1}{\lambda_1}\cdot\frac{s}{\lambda_1+\mu_1+s}\hat{P}_{N,N}(\lambda_1+\mu_1+s)\right]^{-1}\cdot$$

$$\cdot\left[\beta_1\left\{\hat{P}_{i,N}(s) - \frac{s}{\lambda_1+\mu_1+s}\hat{P}_{i,N}(\lambda_1+\mu_1+s)\right\}\right.$$

$$\left.+\beta_2\left\{\hat{P}_{i,N}(s) + \frac{\mu_1}{\lambda_1}\cdot\frac{s}{\lambda_1+\mu_1+s}\hat{P}_{i,N}(\lambda_1+\mu_1+s)\right\}\right] \qquad (1,2.10)$$

where $\hat{P}_{i,N}(s)$ is obtained from (1,1.26) by putting $z = 1$ and applying the relation $\hat{P}_{i,N}(s) = s\bar{P}_{i,N}(s)$

Next, we shall derive the expression for the expected time-to-disappointment. Let $E^{*}_{\alpha}(T)$ denote the expectation of the time-to-disappointment, T, irrespective of whether the demand for system use initially exists or not. Let $E^{*}_{i}(T)$ and $E^{*}_{N+1+i}(T)$ denote the expected time-to-disappointment when initially the demand for use does not and does exist respectively. Then

$$E^{*}_{\alpha}(T) = \beta_1 E^{*}_{i}(T) + \beta_2 E^{*}_{N+1+i}(T) \qquad (1,2.11)$$

and is obtained by differentiating the LST of $G^{*}_{\alpha,2N+1}(t)$ in (1,2.10) with respect to s and taking limit as $s \to 0$. Therefore,

$$E_i^{*}(T) = - \lim_{s \to 0} \frac{d}{ds} \hat{G}^{*}_{i,2N+1}(s)$$

$$= - \lim_{s \to 0} \frac{d}{ds} \left[\frac{\hat{P}_{i,N}(s) - \frac{s}{\lambda_1+\mu_1+s} \hat{P}_{i,N}(\lambda_1+\mu_1+s)}{\hat{P}_{N,N}(s) + \frac{\mu_1}{\lambda_1} \cdot \frac{s}{\lambda_1+\mu_1+s} \hat{P}_{N,N}(\lambda_1+\mu_1+s)} \right]$$

$$= E_i(T_u) + \frac{E_N(\tau_R)}{E(T_d)} \cdot \frac{1}{\lambda_1(\lambda_1+\mu_1)} \left[\mu_1 \hat{P}_{N,N}(\lambda_1+\mu_1) + \lambda_1 \hat{P}_{i,N}(\lambda_1+\mu_1) \right]$$

(1, 2.12)

Similarly,

$$E^{*}_{N+1+i}(T) = E_i(T_u) + \frac{E_N(\tau_R)}{E(T_d)} \cdot \frac{\mu_1}{\lambda_1(\lambda_1+\mu_1)} \left[\hat{P}_{i,N}(\lambda_1+\mu_1) + \hat{P}_{N,N}(\lambda_1+\mu_1) \right]$$

(1,2.13)

Multiplying (1,2.12) by β_1 and (1,2.13) by β_2 and adding we obtain

$$E^{*}_{\alpha}(T) = E_i(T_u) + \frac{E_N(\tau_R)}{E(T_d)} \cdot \frac{1}{\lambda_1(\lambda_1+\mu_1)} \Big[\beta_1 \{ \lambda_1 \hat{P}_{i,N}(\lambda_1+\mu_1) + \mu_1 \hat{P}_{N,N}(\lambda_1+\mu_1) \}$$

$$+ \mu_1 \beta_2 \{ \hat{P}_{i,N}(\lambda_1+\mu_1) + \hat{P}_{N,N}(\lambda_1+\mu_1) \} \Big]$$

(1,2.14)

where $E(T_d)$ represents the expected down-time, $E_i(T_u)$ the expected time-to-system failure from the state i and $E_N(\tau_R)$, the expected recurrence time to the state N, all pertaining to the case of continuous usage of the system, already obtained in section 1. The ratio $E(T_d) / E_N(\tau_R)$ has also been obtained as $\lim_{t \to \infty} P_{N,N}(t)$ using Smith's theorem and is given by (1,1.43).

It is interesting to observe that in (1,2.15) the expected time-to-disappointment is readily expressible as the sum of three terms, the first one being the expected time to system failure in the case of continuous usage of the system and the second and the third term giving the additional contribution due to the use and non-use pattern of the system.

It also follows from the above result that $E_d^*(T) \to E_i(T_N)$ as $\frac{1}{\mu_1} \to \infty$ and $\frac{1}{\lambda_1} \to 0$, that is when the mean period of use is infinite and the mean period of non-use is zero - a fact which is intutively obvious.

In (1,2.5) substituting for $\hat{P}^*_{2N+1,2N+1}(s)$ from (1,2.9), we obtain the expression for $\hat{G}^*_{2N+1,2N+1}(s)$ which uniquely determines the distribution of recurrence time to disappointment and is given by

$$1 - \hat{G}^*_{2N+1,2N+1}(s) = \frac{1 - \hat{D}^*(s)}{\frac{\lambda_1}{\lambda_1+\mu_1} \hat{P}_{N,N}(s) + \frac{\mu_1}{\lambda_1+\mu_1} \cdot \frac{s}{\lambda_1+\mu_1+s} \hat{P}(\lambda_1+\mu_1+s)} \qquad (1,2.15)$$

The moments of the distribution of recurrence time to disappointment are obtained by differentiating (1,2.15) with respect to s at $s=0$.

Denoting by $E^*_{2N+1}(\tau'_R)$ and $E^*_{2N+1}(\tau'^2_R)$ the mean and the second moment of recurrence time τ'_R respectively, we have

$$E^*_{2N+1}(\tau'_R) = \lim_{s \to 0} \frac{d}{ds}\left[1 - \hat{G}^*_{2N+1,\,2N+1}(s)\right] = \left(\frac{1+\rho_1}{1+\rho_0}\right) E(\tau_R) \tag{1, 2.16}$$

and

$$E^*_{2N+1}(\tau'^2_R) = \lim_{s \to 0} -\frac{d^2}{ds^2}\left[1 - \hat{G}^*_{2N+1,2N+1}(s)\right]$$

$$= \left[\frac{\lambda_1}{\lambda_1+\mu_1}\hat{P}_{N,N}(0)\right]^{-2} \cdot \left[\eta^* \left\{\frac{\lambda_1}{\lambda_1+\mu_1}\left(\frac{d}{ds}\hat{P}_{N,N}(s)\right)_{s=0} + \frac{\mu_1}{(\lambda_1+\mu_1)^2}\hat{P}_{N,N}(\lambda_1+\mu_1)\right\} + \frac{\lambda_1}{\lambda_1+\mu_1}\hat{P}_{N,N}(0)\,\nu^*\right]$$

where (1,2.17)

$\eta^* = \int_0^\infty t\, d(1-e^{-\overline{\lambda_1+\mu_1}\,t}) =$ the expected duration of disappointment $= \dfrac{1}{\lambda_1+\mu_1}$

$\nu^* = \int_0^\infty t^2\, d(1-e^{-\overline{\lambda_1+\mu_1}\,t}) =$ the second moment of duration of disappointment $= \dfrac{2}{(\lambda_1+\mu_1)^2}$

$\hat{P}_{N,N}(0) = \dfrac{1}{\mu\, E(\tau_R)} = \dfrac{1}{\mu\, E_N(\tau_R)}$; $\hat{P}_{N,N}(\lambda_1+\mu_1) = \int_0^\infty e^{-(\lambda_1+\mu_1)t}\, d P_{N,N}(t)$

$$\left(\frac{d}{ds} \hat{P}_{N,N}(s) \right)_{s=0} = \frac{1}{[\mu \, E_N(\tau_R)]^2} \left[- E_N(\tau_R) + \frac{\mu}{2} E_N(\tau_R^2) \right]$$

$$\rho_0 = \frac{\mu_1}{\mu} \quad \text{and} \quad \rho_1 = \frac{\mu_1}{\lambda_1}$$

Disappointment free System Availability

In the intermittent usage case, the reliability characteristics corresponding to system availability is the disappointment free system availability which is explained now. When the system is operating over fairly a long period of time, it passes through a number of "up" states alternating with "down" states with demand for system use arising intermittently. The system disappointment occurs at time t when the system reaches the state 2N + 1 whatever be its initial state. Let the steady state probability p^*_{2N+1} represent lim $t \longrightarrow \infty$ $P^*_{\alpha,\,2N+1}(t)$. Then by Smith's theorem [Smith (1954)]

$$p^*_{2N+1} = \frac{\eta^*}{E^*_{2N+1}(\tau_R')} = \frac{1}{\mu_1 + \mu} \cdot \left(\frac{1+\rho_0}{1+\rho_1} \right) \frac{1}{E(\tau_R)} \qquad (1,2\cdot18)$$

Defining the long-run disappointment free system availability to be $1 - p^*_{2N+1}$, Long-run disappointment-free System Availability

$$= 1 - p^{*}_{2N+1}$$

$$= 1 - \frac{1}{\mu_1 + \mu} \cdot \left(\frac{1+\rho_0}{1+\rho_1} \right) \frac{1}{E(\tau_R)}$$

$$= 1 - \frac{1}{1+\rho_1} \cdot \frac{1}{1+\mu \, E_{N-1}(T_N)} \qquad (1, 2.19)$$

since

$$E(\tau_R) = \frac{1}{\mu} + E_{N-1}(T_N)$$

obtained in (1,1.39).

Again, using (1,1.43), the expression for $1 - p^{*}_{2N+1}$ in (1,2.19) can be rewritten as

$$1 - p^{*}_{2N+1} = (1 - p_N) + \frac{\rho_1}{1+\rho_1} p_N$$

where $1 - p_N$ is the long-run availability of the system under continuous usage. Hence,

[Long-run disappointment free System Availability
- long-run Availability of System under continuous usage] $= \frac{\rho_1}{1+\rho_1} p_N$ (1,2.20)

showing thereby that this difference depends only on p_N corresponding to the continuous usage of the system and ρ_1, a parameter indicating the usage pattern of the system. It may be observed that the quantity on the right hand side of (1,2.20) tends to zero or p_N according as ρ_1 tends to zero or is very large. Therefore, the disappointment free system availability is equal to the long-run availability of the system under continuous usage when $\rho_1 = 0$ and tends to unity when $\rho_1 \to \infty$. The long-run availability of the system has been calculated for various combinations of N, ρ and ρ_1 and are given in table 1.1. An examination of this table reveals the following points:

(i) the long-run availability increases with increasing ρ_1 as well as with increasing N;

(ii) as ρ increases, the long-run availability decreases;

(iii) when both ρ and ρ_1 are large (> 1) increasing the number of standby units does not appreciably increase long-run availability of the system.

<u>Number of disappointments in the interval (0, t)</u>

Define the random variable, N_t^* as the number of disappointments ~~according~~ occurring in the interval (0, t). Using exactly the same argument as in section 1 to find the expression for the expected number of system failures in the interval (0, t), we obtain the expected number of

TABLE 1.1

Long-Run Availability of A Standby Redundant System With Intermittent Usage For Various Values of ρ And ρ_1 .

	N \ ρ	$\rho=0.2$	$\rho=0.4$	$\rho=0.6$	$\rho=0.8$	$\rho=1.0$	$\rho=1.2$
$\rho_1=.01$	1	.83498	.71712	.62877	.55996	.50496	.45897
	2	.96806	.89846	.81666	.73945	.66997	.60710
	4	.99873	.98462	.94438	.88005	.80198	.72344
	6	.99995	.99756	.98096	.93399	.85856	.77081
	8	.99999	.99961	.99318	.96162	.89999	.79501
	10	.99999	.99994	.99759	.97672	.90999	.80923
$\rho_1=.05$	1	.84127	.72790	.64291	.57671	.52381	.45957
	2	.96928	.90232	.82363	.74937	.68254	.62207
	4	.99878	.98521	.94650	.88462	.80952	.73397
	6	.99995	.99765	.98169	.93651	.86394	.77954
	8	.99999	.99963	.99344	.96309	.89419	.80282
	10	.99999	.99994	.99769	.97765	.91342	.81650
$\rho_1=0.1$	1	.84848	.74026	.65914	.59595	.54546	.51323
	2	.97067	.90676	.83165	.76076	.69730	.63925
	4	.99884	.98588	.94593	.88986	.81818	.74606
	6	.99996	.99775	.98252	.93938	.87013	.78956
	8	.99999	.99965	.99374	.96476	.89899	.81178
	10	.99999	.99995	.99779	.97865	.91735	.82484
$\rho_1=0.5$	1	.88889	.80953	.75003	.70370	.66667	.63570
	2	.97849	.93163	.87655	.82456	.77778	.73545
	4	.99915	.98965	.96255	.91923	.86667	.81378
	6	.99997	.99835	.98718	.95554	.90476	.84568
	8	.99999	.99974	.99541	.97416	.92593	.86397
	10	.99999	.99996	.99838	.98435	.93939	.87155

TABLE 1.1 (Contd.)

	N \ ρ	ρ = 0·2	ρ = 0·4	ρ = 0·6	ρ = 0·8	ρ = 1·0	ρ = 1·2
$\rho_1 = 1{\cdot}0$	1	.91667	.85714	.81252	.77778	.75000	.72678
	2	.98387	.94872	.90741	.86842	.83333	.80159
	4	.99934	.99224	.97191	.93943	.90000	.86033
	6	.99998	.99876	.99139	.96667	.92857	.88426
	8	.99999	.99981	.99656	.98062	.94444	.89648
	10	.99999	.99997	.99879	.98826	.95454	.90366
$\rho_1 = 2{\cdot}0$	1	.94444	.90476	.87602	.85186	.83333	.81786
	2	.98925	.96915	.93827	.91228	.88889	.86773
	4	.99957	.99483	.98122	.95962	.93333	.90689
	6	.99998	.99918	.99359	.97778	.95238	.92284
	8	.99999	.99987	.99771	.98708	.96297	.93099
	10	.99999	.99998	.99919	.99218	.96970	.93578
$\rho_1 = 5{\cdot}0$	1	.97222	.95218	.93751	.92592	.91667	.90893
	2	.99463	.98291	.93914	.95614	.93333	.93387
	4	.99979	.99741	.99064	.97981	.96667	.95345
	6	.99999	.99959	.99679	.98889	.97639	.96142
	8	.99999	.99993	.99886	.99354	.98148	.96383
	10	.99999	.99999	.99960	.99609	.98485	.96789

disappointments in (0, t) as t $\to \infty$ as

$$E_m(N_t^*) = \frac{t}{E_{2N+1}^*(\tau_R')} + \frac{E_{2N+1}^*(\tau_R'^2)}{2[E_{2N+1}^*(\tau_R')]^2} - \frac{E_\alpha^*(T)}{E_{2N+1}^*(\tau_R')} + o(1) \quad (1,2\cdot 21)$$

$$E_o(N_t^*) = \frac{t}{E_{2N+1}^*(\tau_R')} + \frac{E_{2N+1}^*(\tau_R'^2)}{2[E_{2N+1}^*(\tau_R')]^2} - 1 + o(1) \quad (1,2\cdot 22)$$

where the suffix m and o as in section 1 stand for the modified and ordinary renewal processes respectively.

INTERVAL RELIABILITY

The definition of "interval reliability" is modified slightly for the case of intermittent usage of the system. Here, we define this as the probability that the system is in the disappointment-free state at time t, and will continue to be disappointment-free for a further period of x and we denote the same by $R^*(x,t)$. The "limiting interval reliability" is simply the limit of $R^*(x,t)$ as t $\to \infty$. It can be easily seen that

$$R^*(x,t) = [1 - G^*_{i,2N+1}(t+x)] + \int_0^t [1 - G^*_{2N,2N+1}(t-y+x)]\, d_y \Upsilon^*(y,i) \quad 1,2\cdot 23$$

where $d_y \Upsilon^*(y,i)$ denotes the probability of a recurrence of disappointment-free state at y. The limit of $R^*(x,t)$ as t $\to \infty$ follows from Smith's theorem and is given by

$$\lim_{t \to \infty} \cdot \; R^*(x,t) = \frac{\int_x^\infty [1 - G^*_{2N,2N+1}(y)]dy}{E^*_{2N+1}(\tau_R')} \qquad (1, 2\cdot 24)$$

$E^*_{2N+1}(\tau_R')$ being the mean recurrence time to the disappointment state.

Integrating by parts, the upper and the lower bounds of the limiting interval reliability are obtained as

$$\frac{E^*_{2N}(T) - x}{E^*_{2N+1}(\tau_R')} \leq \lim_{t \to \infty} R^*(x,t) \leq \frac{E^*_{2N}(T)}{E^*_{2N+1}(\tau_R')}$$

$(1, 2\cdot 25)$

SECTION 3

RELIABILITY OF A SYSTEM WHEN THE SPARES DETERIORATE IN STORAGE

The first two sections of this chapter dealt with the reliability characteristics of a single unit system with (N-1) spares in continuous as well as intermittent usage. In these models, when the unit constituting the system fails, it was immediately replaced by a spare and the failed one was repaired by a single repair facility and returned to the spare pool to act as standby. In this section, we impose an additional condition that the redundant units can fail in storage and study the effect of this deterioration of spares. The reliability of a standby redundant system above in which the units are not repaired on failure has been investigated by Weiss (1962c) and Schweitzer (1967). Weiss has obtained the distribution of TSF and its first and second moment for a single unit stand-by redundant system and has shown that even a small amount of spare deterioration results in serious degradation of system reliability. Schweitzer, on the other hand, considers a N unit series connected system with m spares and gives expressions for system reliability and its asymptotic expressions in powers of spare failure rate. Also, he gives graphs that yield for arbitrary failure rates of spares and parts in use, the minimum number of spares necessary for achieving system reliabilities of 90, 95 and 99 per cent. Similar models in which failures can be one of two types have been also considered before in the literature, particularly by Proschan and Collaborators (1959), (1960) who were interested in determining the optimal spare parts kit for a system of components in series, given a fixed amount of capital to provide the spare parts and by Barlow et al (1963) in determining the number of series sub-systems in parallel to maximise system reliability.

The model considered in this section differs from the models of Weiss and Schweitzer, in that, we have introduced one more factor, namely, the repair of the failed units. Figure 1.0(c) will be helpful in illustrating this system. This naturally increases the system reliability. We have evaluated for this system the expected time to system failure, expected system down-time, the expected number of failures of the system in a given interval of time (0, t) and the interval reliability of the system

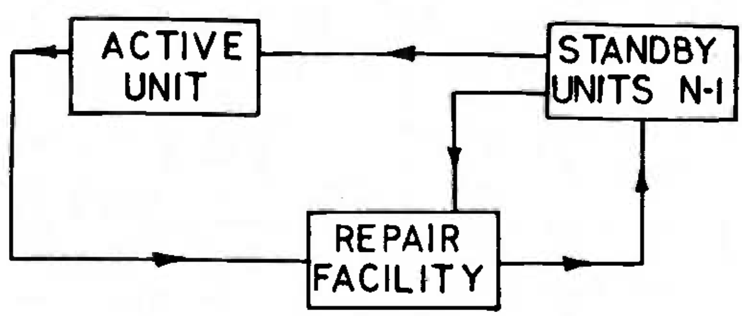


Fig. 1.0(c)

Standby Redundant System with spares
Deteriorating in Storage and with Repair

We now describe the failure process associated with the units in use and storage, the replacement mode, repair policy and the repair process.

1. Failure of individual units: The individual units fail while in use as well as in storage. The failure-time distribution of units in use is negative exponential with mean rate of failure λ_1 and of those in storage is also negative exponential but with rate λ_2.

2. Mode of replacement: When the operating unit fails it is replaced by a unit from the spare pool instantaneously.

3. Repair: The operating unit on failure is taken for repair immediately if the repair facility is free, otherwise it waits in a queue. The units which fail while in storage are also repaired by the same facility and for repair, they also join the queue. The repair is strictly according to first failed first repaired. After repair the unit is returned to the spare pool.

Let $\theta_1, \theta_2, \dots \theta_n, \dots$ be the durations of repair of successive units undergoing repair. The sequence $\{\theta_n\}$ is assumed to be a sequence of positive independent random variables identically distributed with the density function $S(x)$ such that

$$D(x) = P_r[\theta_n \leq x] = \int_0^x S(u)\,du \qquad (4,3.1)$$

Further, let $\eta(x)\,dx$ be the first order conditional probability of repair completion in $(x, x+dx)$ given that repair has not been completed upto x. Evidently, we have

$$S(x) = \eta(x) \exp\left\{-\int_0^x \eta(u)\,du\right\} \qquad (1,3.2)$$

It is also assumed that the repair process is independent of all other processes.

We define the random variable n(t) as the number of failed units in the system at time t. The state of the system is said to be n at time t if $n(t) = n$.

The definitions and assumptions regarding time-to-system failure, system down-time, first passage time distributions are the same as in section 1.

GENERAL PROCESS PROBABILITIES

The general process in this case, as in section 1, is one in which the TSF periods alternate with SDT periods. Transitions from any state i to any other state j are possible in time t. Again as in section 1, at instants where the system is restored to operation from the down-state, the state of the system is N - 1.

The general process has the following associated state probabilities for transition from state i at time $t = 0$ to state n at time t.

$$P_{i,n}(t) = P_r\left[n(t) = n \,/\, n(o) = i\right], \quad 0 \leq n, i \leq N \qquad (1,3.3)$$

$$P_{i,n}(x,t)\,dx = P_r\left[n(t) = n,\ x \leq T^e \leq x + dx \,/\, n(o) = i\right] \qquad (1,3.4)$$

$$0 \leq n, i \leq N$$

where T^e is a random variable denoting the elapsed repair time of the unit under repair at time t. The probability $P_{i,n}(x,t)\,dx$ represents the probability that at time t there are n failed units in the system with the elapsed repair time of the unit under repair lying between x and $x + dx$, the system having started at t = 0 with i failed units. It may be noted that the general process so described is Markovian with respect to the state space over which the set of probabilities (1,3.4) have been defined.

We derive the difference-differential equations governing the general process through continuity arguments. Connecting the probabilities at time t + Δ with those at time t and taking limits as $\Delta \to o$, we obtain

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + (N-1-n)\lambda_2 + \eta(x)\right\} P_{i,n}(x,t)$$

$$= \left[\lambda_1 + \lambda_2(\overline{N-1} - \overline{n-1})\right] P_{i,n-1}(x,t), \quad 0 \leq i \leq N,\ 1 < n < N$$

$$(1,3.5)$$

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \eta(x)\right\} P_{i,N}(x,t) = \lambda_1 P_{i,N-1}(x,t) \qquad (1,3.6)$$

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + (\overline{N-1} - 1)\lambda_2 + \eta(x)\right\} P_{i,1}(x,t) = 0 \qquad (1, 3.7)$$

$$\left\{\frac{\partial}{\partial t} + \lambda_1 + (N-1)\lambda_2 \right\} P_{i,0}(t) = \int_0^\infty P_{i,1}(x,t)\,\eta(x)\,dx \qquad (1, 3.8)$$

The coefficients (N-1-n) λ_2 on the left hand side and $(\overline{N-1} - \overline{n-1})\lambda_2$ on the right hand side of the equation (1,3.5) arise because of the failure behaviour assumed for the units in storage. For, when there are n failed units in the system, any one of $(\overline{N-1} - n)$ spares may fail and the density function for the same is

$$(N-1-n)\lambda_2 \exp\{-(\overline{N-1} - n)\lambda_2 t\}$$

In deriving these equations, the existance of right hand derivatives at $x = 0$ is assumed. Also, the probabilities are assumed to be zero beyond the specified values of n.

Considerations of the movement of the system in the state-space when a repair completion occurs, leads to the boundary conditions

$$P_{i,n}(0,t) = \int_0^\infty P_{i,n+1}(x,t)\,\eta(x)\,dx \qquad (1, 3.9)$$

$$P_{i,1}(0,t) = \int_0^\infty P_{i,2}(x,t)\,\eta(x)\,dx + (\lambda_1 + \overline{N-1}\,\lambda_2) P_{i,0}(t) \qquad (1, 3.10)$$

$$P_{i,N}(0,t) = 0 \qquad (1,3.11)$$

Since the process starts with i failed units initially

$$P_{i,n}(x,0) = \delta_{i,n}\,\delta(x) \qquad (1,3.12)$$

where $\delta_{i,n}$ is the Kronecker delta function and $\delta(x)$ the Dirac delta function.

The case of $\lambda_1 = \lambda_2$:

The solutions of these equations when $\lambda_1 \neq \lambda_2$ is difficult to analyse. As such, we investigate the case when $\lambda_1 = \lambda_2$. This problem has the same structure as that of a N unit parallel system or a machine interference problem with N machines studied by Thiruvengadam and Jaiswal (1963, 1964) and Thiruvengadam (~~1963~~, 1965) ~~1966~~) . Recently, Downton (1966) studied the distribution of TSF for a N component parallel system with general repair time distribution through the Semi-Markov process method. Here, we consider this case for two reasons: (1) we give an alternative method based on discrete transforms for deriving the Laplace transform of the distribution of TSF and (2) this analysis will enable us to study the effect of failure of the spare units while in storage on the reliability characteristics of the system. Let

$\lambda_1 = \lambda_2 = \lambda$ (say). Then the difference-differential equations governing the general process probabilities are given by

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + (N-n)\lambda + \eta(x)\right\} \bar{P}_{i,n}(x,t) = (N-n+1)\lambda P_{i,n-1}(x,t) \qquad (1,3.13)$$

$$\left[\frac{d}{dt} + N\lambda\right] P_{i,0}(t) = \int_0^\infty P_{i,1}(x,t)\,\eta(x)\,dx \qquad (1,3.14)$$

with the boundary conditions

$$P_{i,n}(0,t) = \int_0^\infty P_{i,n+1}(x,t)\,\eta(x)\,dx + \delta_{1,n} N\lambda P_0(t) \qquad (1,3.15)$$

However, the modifications $\lambda_1 = \lambda_2$ does not affect the initial conditions and, therefore, they are still the same as given in (1,3.12).

Taking the Laplace transform of the equations (1,3.13) to (1,3.15) and using the initial condition at (1,3.12), we obtain

$$\left[\frac{\partial}{\partial x} + (N-n)\lambda + \eta(x) + s\right] \bar{P}_{i,n}(x,s) = (N-n+1)\lambda \bar{P}_{i,n-1}(x,s) + \delta_{i,n}\delta(x) \qquad n = 1,2,\ldots N \qquad (1,3.16)$$

$$[N\lambda + s]\, \bar{P}_{i,0}(s) = \int_0^\infty \bar{P}_{i,1}(x,s)\,\eta(x)\,dx \qquad (1,3.17)$$

$$\bar{P}_{i,n}(0,s) = \int_0^\infty \bar{P}_{i,n+1}(x,s)\,\eta(x)\,dx + \delta_{1,n} N\lambda \bar{P}_0(s) \qquad (1,3.18)$$

We observe that the set of equations (1,3.16) are differential-difference equations with variable coefficients. The usual generating function technique leads to partial differential equations (in x and the variable of the generating function) which are very difficult to solve. In order to solve such types of equations, we make use of the following discrete transforms [Thiruvengadam & Jaiswal (1964)].

$$\bar{B}_{i,n}(x,s) = \sum_{j=n}^{N-1} \binom{j}{n} \bar{P}_{i,N-j}(x,s) \quad , \quad 0 \le n < N \qquad (1,3.19)$$

The inverse transforms expressing $\bar{P}_{i,n}(x,s)$ in terms of $\bar{B}_{i,n}(x,s)$ are given by

$$\bar{P}_{i,n}(x,s) = \sum_{j=0}^{n-1} (-1)^{j} \binom{N-n+j}{j} \bar{B}_{i,N-n+j}(x,s) \qquad (1,3.20)$$
$$0 < n \le N$$

which can be verified by substitution.

By means of the discrete transforms (1,3.19) the equations (1,3.16) are reduced to a form in which they can be easily solved. To apply this transform first change n to N-j in (1,3.16) and multiply throughout by $\binom{j}{n}$ and sum over j from j = n to j = N - 1. After simplification, we obtain

$$\left[\frac{\partial}{\partial x} + n\lambda + \eta(x) + s \right] \bar{B}_{i,n}(x,s) = \binom{N-i}{n} \delta(x) \qquad (1,3.21)$$

where $\binom{j}{n} = 0$ if $n > j$ by definition .

The boundary conditions get transformed into

$$\bar{B}_{i,n}(0,s) = \int_0^\infty \left[\bar{B}_{i,n}(x,s) + \bar{B}_{i,n-1}(x,s)\right]\eta(x)\,dx$$
$$- \binom{N}{n}(N\lambda+s)\,\bar{P}_{i,0}(s) + \binom{N-1}{n} N\lambda\,\bar{P}_{i,0}(s) \qquad (1,\ 3.22)$$

The equation (1,3.21) has the solution

$$\bar{B}_{i,n}(x,s) = \bar{A}_{i,n}(0,s)\cdot\exp\left[-(n\lambda+s)x - \int_0^x \eta(u)\,du\right] \qquad (1,\ 3.23)$$

where $\bar{A}_{i,n}(0,s) = \bar{B}_{i,n}(0,s) + \binom{N-i}{n}$

Using (1,3.23) in (1,3.22) and integrating, we have

$$\bar{A}_{i,n}(0,s)\left[1 - \bar{S}(n\lambda+s)\right] = \bar{A}_{i,n-1}(0,s)\,\bar{S}(\overline{n-1}\,\lambda+s) + \binom{N-i}{n}$$
$$- \binom{N}{n}(N\lambda+s)\,\bar{P}_{i,0}(s) + \binom{N-1}{n} N\lambda\,\bar{P}_{i,0}(s) \qquad (1,\ 3.24)$$

Now, we observe that (1,3.24) is a difference equation. To solve this difference equation, we define the following products

$$C'_\ell(s) = \begin{cases} \displaystyle\prod_{r=0}^{\ell} \frac{\bar{S}[(r-1)\lambda+s]}{1-\bar{S}(r\lambda+s)}, & \ell > 0 \\[2ex] \dfrac{1}{1-\bar{S}(s)}, & \ell = 0 \\[2ex] 1, & \ell = -1 \end{cases}$$

and

$$C_\ell(s) = \begin{cases} \prod_{r=0}^{\ell} \dfrac{\bar{S}(r\lambda+s)}{1-\bar{S}(r\lambda+s)} & , \ell \geq 0 \\ 1 & , \ell = -1 \end{cases} \qquad (1,3.25)$$

It can be easily verified that

$$C_{\ell-1}(s) = \bar{S}[(\ell-1)\lambda+s]\,C'_{\ell-1}(s) = [1-\bar{S}(\ell\lambda+s)]\,C'_\ell(s) \quad (1,3.26)$$

Dividing (1,3.24) throughout by $C_{n-1}(s)$, there results

$$\frac{\bar{A}_{i,n}(0,s)}{C'_n(s)} = \frac{\bar{A}_{i,n-1}(0,s)}{C'_{n-1}(s)} + \binom{N-i}{n}\frac{1}{C_{n-1}(s)} - (N\lambda+s)\bar{P}_{i,0}(s)\binom{N}{n}\frac{1}{C_{n-1}(s)}$$
$$+ N\lambda\,\bar{P}_{i,0}(s)\binom{N-1}{n}\frac{1}{C_{n-1}(s)} \qquad (1,3.27)$$

Changing n to 0, 1, 2,... upto n and adding all the resulting equations, we obtain

$$\frac{\bar{A}_{i,n}(0,s)}{C'_n(s)} = \Bigg\{ \sum_{\ell=0}^{n}\binom{N-i}{\ell}\frac{1}{C_{\ell-1}(s)} - (N\lambda+s)\bar{P}_{i,0}(s)\sum_{\ell=0}^{n}\binom{N}{\ell}\frac{1}{C_{\ell-1}(s)}$$
$$+ N\lambda\,\bar{P}_{i,0}(s)\sum_{\ell=0}^{n}\binom{N-1}{\ell}\frac{1}{C_{\ell-1}(s)}\Bigg\} \qquad (1,3.28)$$

We define the values of B's and, therefore, A's to be zero beyond the specified values of n, namely at $n = N$ and $n = -1$. Using (1,3.26), the equation (1,3.28) can be rewritten as

$$\bar{A}_{i,n}(o,s) = \frac{C_{n-1}(s)}{1-\bar{S}(n\lambda+s)}\left[\sum_{l=0}^{n}\binom{N-i}{l}\frac{1}{C_{l-1}(s)} - (N\lambda+s)\,\bar{P}_{i,0}(s)\sum_{l=0}^{n}\binom{N}{l}\frac{1}{C_{l-1}(s)}\right.$$

$$\left. + N\lambda\,\bar{P}_{i,0}(s)\sum_{l=0}^{n}\binom{N-1}{l}\frac{1}{C_{l-1}(s)}\right] \qquad (1,3.29)$$

Also, changing n to N in (1,3.29) and noting that $\bar{A}_{i,N}(o,s) = 0$ by definition, we obtain by rearranging the resulting terms

$$\bar{P}_{i,0}(s) = \frac{\bar{b}_i(s)}{N\lambda[1-\bar{b}(s)]+s} \qquad (1,3.30)$$

where

$$\bar{b}_i(s) = \left[\sum_{l=0}^{N-i}\binom{N-i}{l}\frac{1}{C_{l-1}(s)}\right]\left[\sum_{l=0}^{N}\binom{N}{l}\frac{1}{C_{l-1}(s)}\right]^{-1}$$

and

$$\bar{b}(s) = \left[\sum_{l=0}^{N-1}\binom{N-1}{l}\frac{1}{C_{l-1}(s)}\right]\left[\sum_{l=0}^{N}\binom{N}{l}\frac{1}{C_{l-1}(s)}\right]^{-1}$$

It may be remarked that $\bar{b}_i(s)$ represents the L T of initial busy period distribution of the repair facility the busy period starting initially with i failed units in the system and $\bar{b}(s)$, the busy period distribution starting with the failure of a unit at t = 0.

The expression (1,3.30) for $\bar{P}_{i,0}(s)$ can also be obtained through simple probabilistic arguments by considering the alternating busy and idle periods of the repair facility with initial busy period starting with i failed units in the system at t = 0.

Now let

$$\bar{B}_{ij}(s) = \int_0^\infty \bar{B}_{ij}(x,s)\,dx$$

Then

$$\bar{B}_{i,n}(s) = \int_0^\infty \bar{B}_{i,n}(x,s)\,dx = \frac{1-\bar{S}(n\lambda+s)}{n\lambda+s}\,\bar{A}_{i,n}(0,s) \qquad (1,3.31)$$

Therefore, using (1,3.31) in (1,3.29) we have

$$\bar{B}_{i,n}(s) = \frac{C_{n-1}(s)}{n\lambda+s}\left[\sum_{l=0}^{n}\binom{N-i}{l}\frac{1}{C_{l-1}(s)} - (N\lambda+s)\bar{P}_{i,0}(s)\sum_{l=0}^{n}\binom{N}{l}\frac{1}{C_{l-1}(s)} + N\lambda\bar{P}_{i,0}(s)\sum_{l=0}^{n}\binom{N-1}{l}\frac{1}{C_{l-1}(s)}\right] \qquad (1,3.32)$$

The Laplace transform of the general process probability $P_{i,N}(t)$ can now be obtained by using the inverse transforms expressing P's in terms of B's as in (1,3.20) and is given by

$$\bar{P}_{i,N}(s) = \sum_{j=0}^{N-1}(-1)^j\,\bar{B}_{ij}(s) \qquad (1,3.33)$$

DISTRIBUTION OF TIME TO SYSTEM FAILURE

The relationships connecting the general process probability $P_{i,N}(t)$ with the first passage time distribution $G_{i,N}(t)$ which is in fact the distribution of the TSF, have already been obtained in (1,1.29) and (1,1.30) by simple probabilistic arguments. The distribution D(t) of SDT for this case is given by $D(t) = \int_0^t S(x)\,dx$, which is nothing but the distribution function of the repair times. Now, taking the Laplace Stieltjes transform on both sides of (1,1.29) and (1,1.30) we obtain the LST of the distribution of TSF as

$$\hat{G}_{i,N}(s) = \frac{\hat{P}_{i,N}(s)}{\hat{P}_{N,N}(s)} = \frac{\bar{P}_{i,N}(s)}{\bar{P}_{N,N}(s)} \qquad (1,3.34)$$

and that of the recurrence time to the system failure state as

$$1 - \hat{G}_{N,N}(s) = \frac{1 - \hat{D}(s)}{\hat{P}_{N,N}(s)} \qquad (1,3.35)$$

where the expressions for $\bar{P}_{i,N}(s)$ is obtained by using (1,3.32) in (1,3.33) and that for $\bar{P}_{N,N}(s)$ is obtained by changing i into N in the expression for $\bar{P}_{i,N}(s)$ so obtained. We have, therefore, from these relations

$$\bar{P}_{i,N}(s) = \sum_{j=0}^{N-1} (-1)^j \frac{C_{j-1}(s)}{j\lambda + s} \left[\sum_{\ell=0}^{j} \binom{N-i}{\ell} \frac{1}{C_{\ell-1}(s)} - (N\lambda + s)\,\bar{P}_{i,0} \sum_{\ell=0}^{j} \binom{N}{\ell} \frac{1}{C_{\ell-1}(s)} + N\lambda\,\bar{P}_{i,0}(s) \sum_{\ell=0}^{j} \binom{N-1}{\ell} \frac{1}{C_{\ell-1}(s)} \right] \quad (1,\ 3\cdot36)$$

$$\bar{P}_{N,N}(s) = \sum_{j=0}^{N-1} (-1)^j \frac{C_{j-1}(s)}{j\lambda + s} \left[1 - (N\lambda+s)\,\bar{P}_{N,0}(s) \sum_{\ell=0}^{j} \binom{N}{\ell} \frac{1}{C_{\ell-1}(s)} + N\lambda\,\bar{P}_{N,0}(s) \sum_{\ell=0}^{j} \binom{N-1}{\ell} \frac{1}{C_{\ell-1}(s)} \right] \quad (1,3\cdot37)$$

Substituting for $\bar{P}_{i,N}(s)$ and $\bar{P}_{N,N}(s)$ from (1,3.36) and (1,3.37) in (1,3.34) we obtain the $\hat{G}_{i,N}(s)$, the LST of TSF. The moments of the distribution of TSF can now be obtained by successive differentiation of $\hat{G}_{i,N}(s)$ with respect to s at $s = 0$. Denote by $E_i(T_u)$ and $E_i(T_u^2)$, the first and second moment of the TSF distribution when the system starts with i failed units.

Also, define

$$C_\ell = \begin{cases} \prod_{r=1}^{\ell} \left[\dfrac{\bar{S}(r\lambda)}{1 - \bar{S}(r\lambda)} \right] & , \ \ell > 0 \\ 1 & , \ \ell = 0 \end{cases} \quad (1,3\cdot38)$$

Then making use of the following limits

$$\text{i)} \quad \lim_{s \to 0} \frac{1}{C_\ell(s)} = \begin{cases} 1 & \text{if } \ell = -1 \\ 0 & \text{otherwise} \end{cases}$$

ii) $$\lim_{s \to 0} \frac{1}{s\,C_\ell(s)} = \frac{\eta}{C_\ell}, \quad \eta = \int_0^\infty x\, S(x)\, dx$$

iii) $$\lim_{s \to 0} \frac{d}{ds}\left(\frac{1}{C_\ell(s)}\right) = \begin{cases} 0 & \text{if } \ell = -1 \\ \dfrac{\eta}{C_\ell} & \text{otherwise} \end{cases}$$

iv) $$\lim_{s \to 0} \frac{d^2}{ds^2}\left(\frac{1}{C_\ell(s)}\right) = \begin{cases} 0 & \text{if } \ell = -1 \\ -\dfrac{2\eta^2}{C_\ell}\left[\dfrac{\eta^{(2)}}{2\eta^2} - 1 - \displaystyle\sum_{r=1}^{\ell} \dfrac{1}{\bar{S}(r\lambda)[1-\bar{S}(r\lambda)]}\right] & \text{otherwise} \end{cases}$$

where $$\eta^{(2)} = \int_0^\infty x^2\, S(x)\, dx \qquad (1,\ 3.39)$$

v) $$\lim_{s \to 0} \frac{d}{ds}\left(s\, C_\ell(s)\right) = \begin{cases} 0 & \text{if } \ell = -1 \\ C_\ell\left[\dfrac{\eta^{(2)}}{2\eta^2} - 1 - \displaystyle\sum_{r=1}^{\ell} \dfrac{1}{\bar{S}(r\lambda)[1-\bar{S}(r\lambda)]}\right] & \text{otherwise} \end{cases}$$

vi) $$\lim_{s \to 0} \frac{d}{ds}\left(\frac{1}{s\, C_\ell(s)}\right) = \begin{cases} 0 & \text{for } \ell = -1 \\ -\dfrac{\eta^2}{C_\ell}\left[\dfrac{\eta^{(2)}}{2\eta^2} - 1 - \displaystyle\sum_{r=1}^{\ell} \dfrac{1}{\bar{S}(r\lambda)[1-\bar{S}(r\lambda)]}\right] & \text{otherwise} \end{cases}$$

We obtain the expressions for $E_i(T_{11})$ and $E_i(T_{11}^2)$.

The expected time to system failure, $E_i(T_{11})$ is given by

$$E_i(T_u) = -\left(\frac{d}{ds}\hat{G}_{i,N}(s)\right)_{s=0} = \frac{1}{\hat{P}_{N,N}(0)}\left[\hat{P}'_{N,N}(0) - \hat{P}'_{i,N}(0)\right] \tag{1,3.40}$$

where $\hat{P}'_{i,N}(0) = \left[\frac{d}{ds}\left(s\bar{P}_{i,N}(s)\right)\right]_{s=0}$

and $\hat{P}_{N,N}(0) = \lim_{s\to 0} s\bar{P}_{N,N}(s) = p_N$

the steady state probability of the state N. Hence,

$$E_i(T_u) = \frac{1}{p_N}\left[p_0\eta\sum_{\ell=1}^{N-i}\binom{N-i}{\ell}\frac{1}{C_{\ell-1}} + \sum_{j=1}^{N-1}(-1)^j\frac{C_{j-1}}{j\lambda}\left\{-\sum_{\ell=1}^{j}\binom{N-i}{\ell}\frac{1}{C_{\ell-1}} + p_0\sum_{\ell=1}^{N-i}\binom{N-i}{\ell}\frac{1}{C_{\ell-1}}\left(1+N\lambda\eta\sum_{\ell=0}^{j-1}\binom{N-1}{\ell}\frac{1}{C_\ell}\right)\right\}\right] \tag{1,3.41}$$

The steady state probabilities p_0 and p_N can be shown to be independent of the initial state of the system and the expressions for p_0 and p_N are obtained from (1,3.30) and (1,3.36). These are

$$p_N = \lim_{s\to 0} s\bar{P}_{i,N}(s) = p_0\left[N\lambda\eta\sum_{\ell=0}^{N-1}\binom{N-1}{\ell}\frac{1}{C_\ell} + \sum_{j=1}^{N-1}(-1)^j\frac{N}{j}C_{j-1}\sum_{\ell=j}^{N-1}\binom{N-1}{\ell}\frac{1}{C_\ell}\right] \tag{1,3.42}$$

$$p_0 = \lim_{s\to 0} s\bar{P}_{i,0}(s) = \left[1+N\lambda\eta\sum_{\ell=0}^{N-1}\binom{N-1}{\ell}\frac{1}{C_\ell}\right]^{-1} \tag{1,3.43}$$

In the case when the initial state $i = N - 1$, after a little simplification, we have for the expected TSF

$$E_{N-1}(T_u) = \frac{1}{p_N}\left[p_0 \eta \left\{ 1 + \sum_{j=1}^{N-1} (-1)^j C_{j-1} \frac{N}{j} \sum_{\ell=0}^{j-1} \binom{N-1}{\ell} \frac{1}{C_\ell} \right\} + \frac{1-p_0}{\lambda} \sum_{j=1}^{N-1} (-1)^{j-1} \frac{C_{j-1}}{j} \right] \quad (1,3.44)$$

Here, we observe that by using the expressions for p_0 and p_N in (1,3.43) and (1,3.42), the value of $E_{N-1}(T_u)$ reduces to

$$E_{N-1}(T_u) = \frac{\eta(1-p_N)}{p_N} \quad (1,3.45)$$

which can also be proved alternatively by the use of Smith's theorem on renewal theory. For, the TSF period followed by a SDT period constitutes a renewal period.

Using (1,3.45) the ratio of the expected TSF to the expected SDT can be calculated for any given repair time distribution and the effect of repair time distribution on the reliability of the system can be studied. Fig.1.4 depicts the ratio of the expected TSF to expected SDT when the repair times are: (i) exponentially distributed and (ii) constant. The variation with λ = 0.4 and 0.8 and η = 1 has been shown against N . It is observed that the ratio $\frac{1}{\eta} E_{N-1}(T_u)$ is higher for constant repair time case than for exponential case. This fact is to be expected as in the constant repair time

THE EXPECTED TIME TO SYSTEM FAILURE $E_{N-1}(T_W)$ FOR PARALLEL REDUNDANT SYSTEM WITH CONSTANT REPAIR TIME AND EXPONENTIAL REPAIR TIME DISTRIBUTIONS

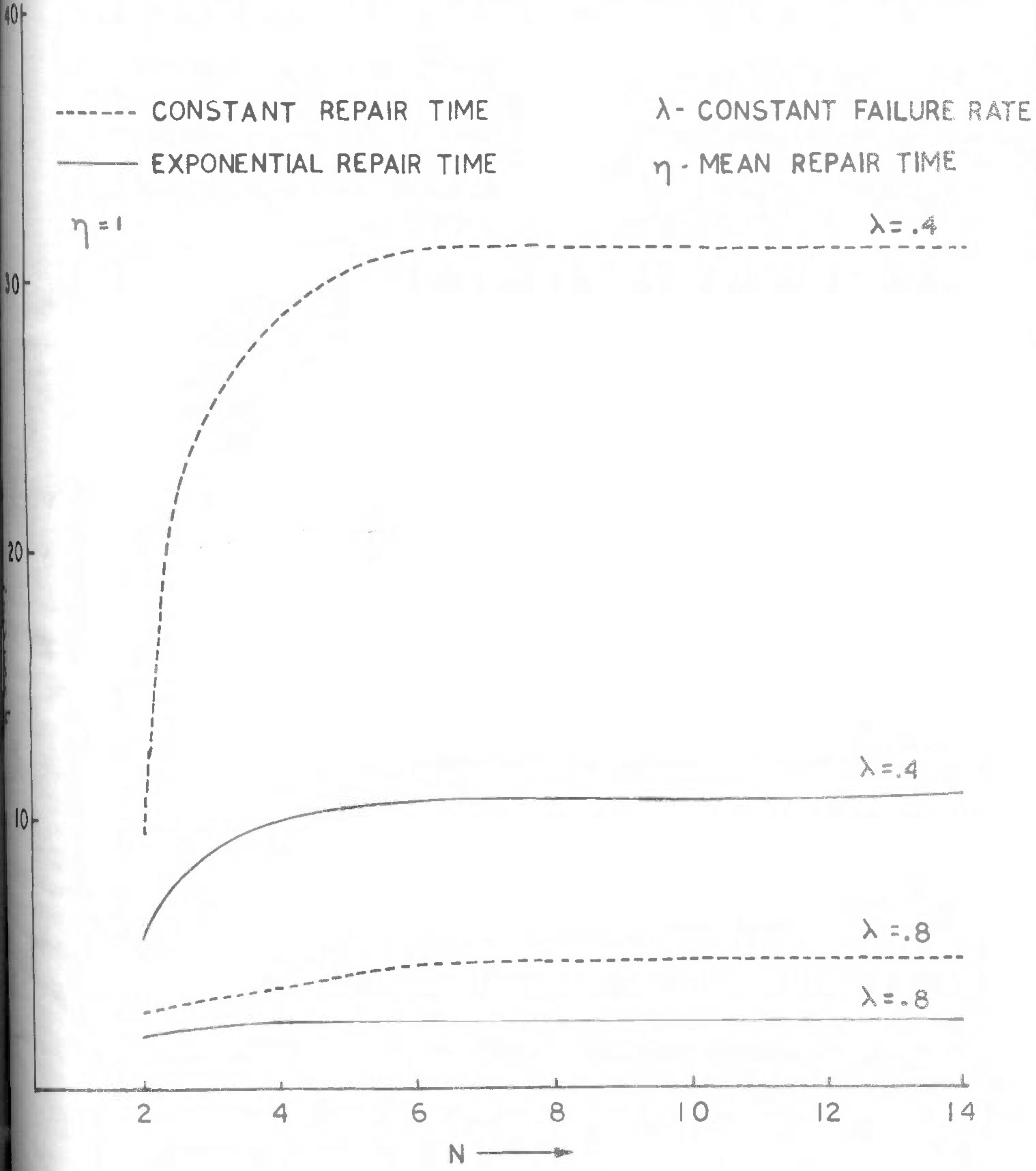


FIG. 1.4

case, at least the variability of the repair times is eliminated. Further, it may be noted that not much gain in the reliability of the system is achieved by increasing the number of parallel redundant units beyond a certain limit, e.g. beyond N = 6, in this case.

Now, the second moment of the time-to-system failure distribution when initially the system starts with i failed units is given by

$$E_i(T_u^2) = \left[-\frac{d^2}{ds^2} \hat{G}_{i,N}(s) \right]_{s=0} = \frac{1}{p_N} \left[\hat{P}''_{i,N}(0) - \hat{P}''_{N,N}(0) + 2\, E_i(T_u)\, \hat{P}'_{N,N}(0) \right] \tag{1,3.46}$$

where

$$\hat{P}'_{N,N}(0) = \left[\frac{d}{ds} \hat{P}_{N,N}(s) \right]_{s=0} = \left(\frac{d}{ds} \left\{ s\, \bar{B}_{N,0}(s) + s \sum_{j=1}^{N-1} (-1)^j \bar{B}_{N_j}(s) \right\} \right)_{s=0}$$

$$= p_0^2\, \eta \left[1 + \sum_{j=1}^{N-1} (-1)^j \frac{C_{j-1}}{j\lambda\eta} \left\{ 1 + N\lambda\eta \sum_{l=0}^{j-1} \binom{N-1}{l} \frac{1}{C_l} \right\} \right] .$$

$$\cdot \left[\sum_{l=1}^{N} \binom{N}{l} \frac{1}{C_{l-1}} - N\lambda\eta \sum_{l=0}^{N-1} \binom{N-1}{l} \frac{1}{C_l} \left\{ \frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{r=1}^{l} \frac{1}{\bar{S}(r\lambda)[1-\bar{S}(r\lambda)]} \right\} \right]$$

$$+ \sum_{j=1}^{N-1} (-1)^{j-1} \frac{C_{j-1}}{j\lambda} \left[\left\{ \frac{1}{j\lambda\eta} - \left[\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{r=1}^{j-1} \frac{1}{\bar{S}(r\lambda)[1-\bar{S}(r\lambda)]} \right] \right\} . \right.$$

$$\left\{ 1 - p_0 \left(1 + N\lambda\eta \sum_{l=0}^{j-1} \binom{N-1}{l} \frac{1}{C_l} \right) \right\}$$

$$\left. + p_0 \left\{ \sum_{l=1}^{j} \binom{N}{l} \frac{1}{C_{l-1}} - N\lambda\eta \sum_{l=0}^{j-1} \binom{N-1}{l} \frac{1}{C_l} \left[\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{r=1}^{l} \frac{1}{\bar{S}(r\lambda)[1-\bar{S}(r\lambda)]} \right] \right\} \right]$$

$$\hat{P}''_{i,N}(0) - \hat{P}''_{N,N}(0)$$

and

$$= \left(\frac{d^2}{ds^2} \left[s \left(\bar{B}_{i,0}(s) - \bar{B}_{N,0}(s) + \sum_{j=1}^{N-1} (-1)^j (B_{ij}(s) - \bar{B}_{Nj}(s)) \right) \right] \right)_{s=0}$$

$$= 2\eta \left\{ \eta + \sum_{j=1}^{N-1} (-1)^j \frac{C_{j-1}}{\lambda j} \left(1 + N\lambda\eta \sum_{\ell=0}^{j-1} \binom{N-1}{\ell} \frac{1}{C_\ell} \right) \right\} \left\{ p_0 \left[\sum_{\ell=1}^{N} \binom{N}{\ell} \frac{1}{C_{\ell-1}} \right.\right.$$

$$\left. - N\lambda\eta \sum_{\ell=0}^{N-1} \binom{N-1}{\ell} \frac{1}{C_\ell} \left[\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{\nu=1}^{\ell} \frac{1}{\bar{S}(\nu\lambda)[1-\bar{S}(\nu\lambda)]} \right] \right] \sum_{\ell=1}^{N-i} \binom{N-i}{\ell} \frac{1}{C_{\ell-1}}$$

$$\left. + \sum_{\ell=1}^{N-i} \binom{N-i}{\ell} \frac{1}{C_{\ell-1}} \left(\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{\nu=1}^{\ell-1} \frac{1}{\bar{S}(\nu\lambda)[1-\bar{S}(\nu\lambda)]} \right) \right\}$$

$$+ 2\eta \sum_{j=1}^{N-1} (-1)^{j-1} \frac{C_{j-1}}{\lambda j} \left\{ p_0 \sum_{\ell=1}^{N-i} \binom{N-i}{\ell} \frac{1}{C_\ell} \left[\sum_{\ell=1}^{j} \binom{N}{\ell} \frac{1}{C_{\ell-1}} \right.\right.$$

$$\left. - N\lambda\eta \sum_{\ell=0}^{j-1} \binom{N-1}{\ell} \frac{1}{C_\ell} \left(\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{\nu=1}^{\ell} \frac{1}{\bar{S}(\nu\lambda)[1-\bar{S}(\nu\lambda)]} \right) \right]$$

$$+ \sum_{\ell=1}^{j} \binom{N-i}{\ell} \frac{1}{C_{\ell-1}} \left(\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{\nu=1}^{\ell-1} \frac{1}{\bar{S}(\nu\lambda)[1-\bar{S}(\nu\lambda)]} \right)$$

$$+ \left[\frac{1}{j\lambda\eta} - \left(\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{\nu=1}^{j-1} \frac{1}{\bar{S}(\nu\lambda)[1-\bar{S}(\nu\lambda)]} \right) \right] \left[\sum_{\ell=1}^{j} \binom{N-i}{\ell} \frac{1}{C_{\ell-1}} \right.$$

$$\left.\left. - p_0 \sum_{\ell=1}^{N-i} \binom{N-i}{\ell} \frac{1}{C_{\ell-1}} \left(1 + N\lambda\eta \sum_{\ell=0}^{j-1} \binom{N-1}{\ell} \frac{1}{C_\ell} \right) \right] \right\}$$

and in the case when $i = N-1$, $E_{N-1}(T_u^2)$ is given by

$$E_{N-1}(T_u^2) = \frac{1}{p_N}\left[\hat{P}''_{N-1,N}(s) - \hat{P}''_{N,N}(s) + 2E_{N-1}(T_u)\hat{P}'_{N,N}(s)\right]_{s=0} \qquad (1,3.47)$$

where $E_{N-1}(T_u)$ and $\hat{P}'_{N,N}(0)$ have already been defined and

$$\left[\hat{P}''_{N-1,N}(s) - \hat{P}''_{N,N}(s)\right]_{s=0} = \left[\frac{d^2}{ds^2}s\left(\bar{P}_{N-1,N}(s) - \bar{P}_{N,N}(s)\right)\right]_{s=0}$$

$$= 2\eta^2\left\{1 + \sum_{j=1}^{N-1}(-1)^j\frac{C_{j-1}}{j\lambda\eta}\left(1 + N\lambda\eta\sum_{\ell=0}^{j-1}\binom{N-1}{\ell}\frac{1}{C_\ell}\right)\right\}\left\{p_0\left[\sum_{\ell=1}^{N}\binom{N}{\ell}\frac{1}{C_{\ell-1}}\right.\right.$$

$$\left.\left. - N\lambda\eta\sum_{\ell=0}^{N-1}\binom{N-1}{\ell}\frac{1}{C_\ell}\left(\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{\gamma=1}^{\ell}\frac{1}{\bar{S}(\gamma\lambda)[1-\bar{S}(\gamma\lambda)]}\right)\right] + \frac{\eta^{(2)}}{2\eta^2} - 1\right\}$$

$$+ 2\eta\sum_{j=1}^{N-1}(-1)^{j-1}\frac{C_{j-1}}{j\lambda}\left\{p_0\left[\sum_{\ell=1}^{j}\binom{N}{\ell}\frac{1}{C_{\ell-1}}\right.\right.$$

$$- N\lambda\eta\sum_{\ell=0}^{j-1}\binom{N-1}{\ell}\frac{1}{C_\ell}\left(\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{\gamma=1}^{\ell}\frac{1}{\bar{S}(\gamma\lambda)[1-\bar{S}(\gamma\lambda)]}\right) + \frac{\eta^{(2)}}{2\eta^2} - 1$$

$$\left.\left. + \left[\frac{1}{j\lambda\eta} - \left(\frac{\eta^{(2)}}{2\eta^2} - 1 - \sum_{\gamma=1}^{j-1}\frac{1}{\bar{S}(\gamma\lambda)[1-\bar{S}(\gamma\lambda)]}\right)\right]\left[1 - p_0\left(1 + N\lambda\eta\sum_{\ell=0}^{j-1}\binom{N-1}{\ell}\frac{1}{C_\ell}\right)\right]\right]\right\}$$

The mean and the second moment of the recurrence time to the state N can be obtained by making use of the renewal property of the process as in sections 1 and 2. As the renewal period τ_R can be expressed as a sum of the independent random variables, namely, the TSF T_u and the SDT T_d, the expectation and the second moment of the random variable τ_R are given by (1,1.35) and (1,1.36) respectively. We use (1,3.44) and (1,3.47) for $E_{N-1}(T_u)$ and $E_{N-1}(T_u^2)$ and $\eta = \int_0^\infty x\, S(x)\, dx$, $\eta^{(2)} = \int_0^\infty x^2\, S(x)\, dx$ for $E(T_d)$ and $E(T_d^2)$ respectively.

RELIABILITY CHARACTERISTICS

The reliability characteristics we are interested in are the same as in sections 1 and 2, namely, the long-run availability of the system, the expected number of system failures in a given interval of time (0, t) and the interval reliability of the system.

The long-run Availability $= 1 - p_N$

$$= p_0 \left[1 + \sum_{j=1}^{N-1} (-1)^j C_{j-1} \frac{N}{j} \sum_{l=0}^{j-1} \binom{N-1}{l} \frac{1}{C_l} \right] + \frac{1-p_0}{\lambda\eta} \sum_{j=1}^{N-1} (-1)^{j-1} \frac{C_{j-1}}{j} \qquad (1, 3\cdot 48)$$

where p_0 is given in (1,3.43). In the particular case when the failure time distribution of individual unit is negative exponential with mean life $1/\lambda$ and the repair time distribution also is negative exponential with mean repair time $1/\mu$, the long-run availability of the system is given by

The long-run availability $= 1 - p_N$ $\qquad (1, 3\cdot 49)$

$$= 1 - \frac{1}{\sum_{r=0}^{N} \frac{1}{r!} \cdot \frac{1}{\rho^r}} = 1 - \frac{e^{-\frac{1}{\rho}}}{\sum_{r=0}^{N} \frac{(\frac{1}{\rho})^r}{r!} e^{-\frac{1}{\rho}}}, \quad \rho = \frac{\lambda}{\mu}$$

which can be easily tabulated by using Molina's Tables for Poisson probabilities [Molina (1942)] .

As we are interested in studying the effect of failure of the units in storage on the reliability characteristics of the system, this can easily be done, for example, by comparing the expression for the long-run availability of the standby redundant system whose spares do not fail in storage with that of the standby redundant system whose spares fail in storage. In the particular case, when the failure rate λ_1 of the unit in use is equal to the failure rate λ_2 of the spares in storage, this system as mentioned earlier is equivalant to the N unit parallel system. To this end, we rewrite (1,3.49) of this section and (1,1.44) of section 1, as follows:

Long-run availability of the standby redundant system when spares do not fail in storage as in (1,1.44)

$$= 1 - \frac{\rho^N}{\dfrac{1-\rho^{N+1}}{1-\rho}} = 1 - \frac{\rho^N}{\sum_{r=0}^{N} \rho^r} \qquad (1,3.50)$$

Long-run availability of the standby redundant system when spares fail in storage with the same rate as the unit in use, which is equivalent to that of N unit parallel system as in (1,3.49)

$$= 1 - \frac{\rho^N}{\sum_{r=0}^{N} \frac{1}{(N-r)!} \cdot \rho^r} \qquad (1,3.51)$$

Since for a given value of N and ρ, $\rho^r > \frac{\rho^r}{(N-r)!}$ the failure of the units in storage reduces the long-run

TABLE 1.2

LONG-RUN AVAILABILITY OF STANDBY AND PARALLEL REDUNDANT SYSTEMS FOR VARYING ρ AND N

N	Long-run Availability of Standby Redundant System					
	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 1.0$	$\rho = 1.2$
1	.833333	.714286	.625046	.555555	.500000	.453552
2	.967742	.897436	.814815	.736842	.666667	.603174
4	.998720	.984472	.943820	.878048	.800000	.720670
6	.999949	.997538	.980769	.933333	.857142	.768518
8	.999997	.999607	.993211	.961240	.888889	.792960
10	.999999	.999937	.997572	.976525	.909090	.807321

N	Long-run Availability of Parallel Redundant System					
	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 1.0$	$\rho = 1.2$
1	.833333	.714286	.625046	.555555	.500000	.453552
2	.945946	.849057	.753683	.669627	.600000	.484831
4	.984704	.907892	.810238	.710414	.630770	.562760
6	.991160	.916734	.811229	.713023	.632090	.563487
8	.992770	.917821	.811540	.713135	.632121	.563499
10	.993168	.917910	.811552	.713137	.632121	.563499

availability of the system. As the case $\lambda_1 = \lambda_2$ considered here corresponds to a parallel redundant system, this leads to the following important conclusion about parallel and standby redundant systems: The long-run availability of N-unit parallel system is less than that of the standby redundant system with (N-1) spares.

Table 1.2 exhibits the long-run availability of parallel and standby redundant systems for varying values of ρ and N. In the standby system, there is no deterioration of spares in storage. And the parallel system is also equivalent to a standby system in which the failure rate of units in storage is the same as the unit in use. Thus, table 1.2 can also provide a measure of the effect of deterioration of spares in storage.

It can be seen from table 1.2 that the long-run availability is very high for small ρ and progressively decreases with the increase in ρ for both parallel and standby systems. The long-run availability of the standby system is higher for given ρ and N than the corresponding values for the parallel system - a point which was demonstrated analytically earlier.

Further, for these two systems, a comparison of $\mu E_{N-1}(T_u)$ has been made in fig.1.5 for various values of ρ and N . It is observed that the increase in $\mu E_{N-1}(T_u)$ is not appreciable for an increase in the

THE VALUES OF $\mu E_{N-1}(T_u)$ FOR VARIOUS VALUES OF ρ AND N OF A STANDBY REDUNDANT AND PARALLEL REDUNDANT SYSTEMS

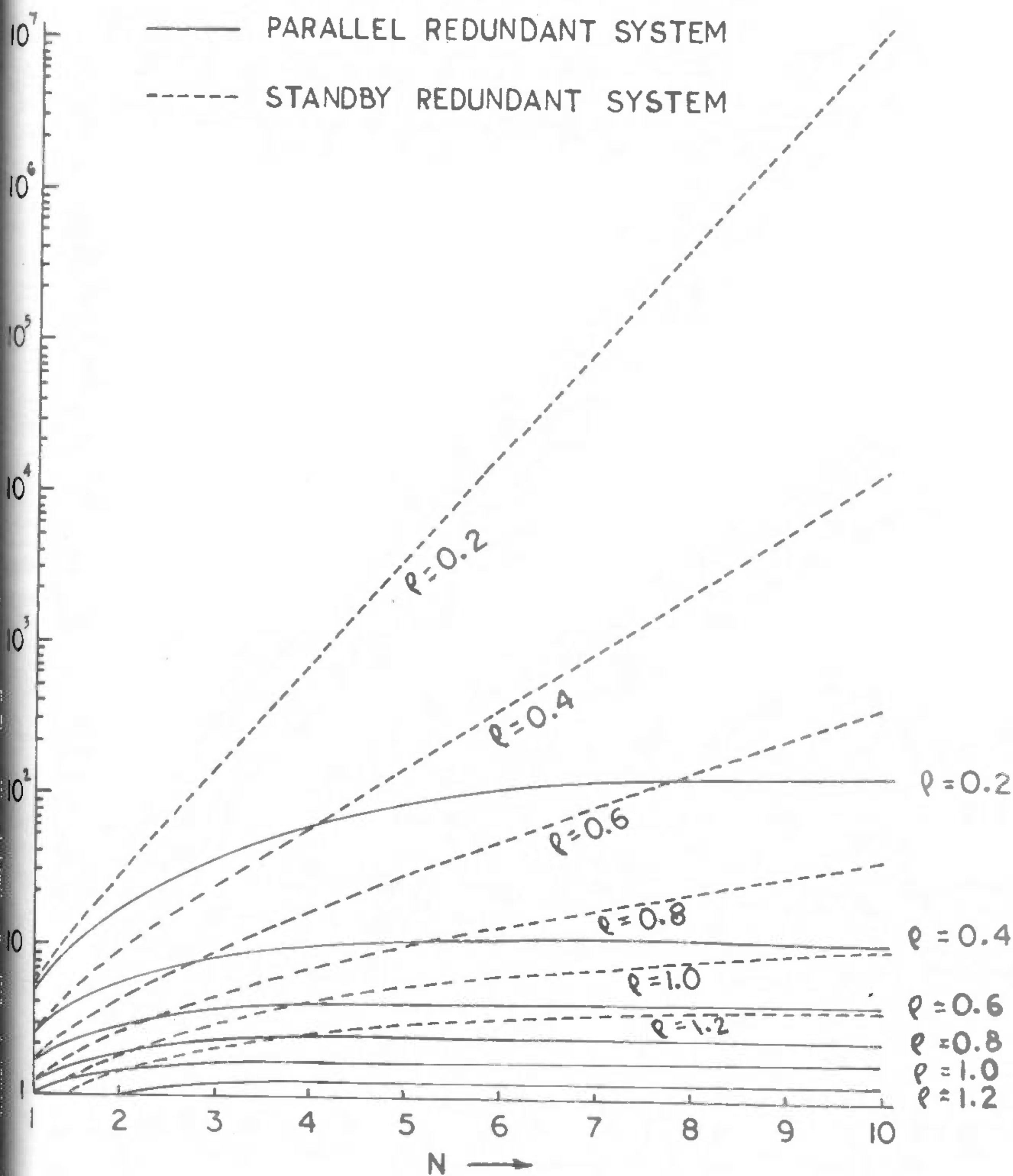


FIG. 1.5

parallel redundant units beyond 6 or 7. Besides, higher values of $\mu E_{N-1}(T_u)$ are obtained by decreasing ρ, preferably for $\rho < 0.5$, for both the parallel and standby redundant systems. In all the cases, $\mu E_{N-1}(T_u)$ is higher for the standby redundant system than for the corresponding parallel redundant system.

Similar comparisons can also be made with the expected number of failures of the system in a given interval of time (0, t) and the interval reliability of the system which involve, as may be seen from the expressions in (1,1.52), (1,1.53) and (1,1.56), only the expectation and the second moment of the time to system failure and the recurrence time to the state N, the expressions for which have already been derived.

<u>The case of $\lambda_1 \neq \lambda_2$:</u>

Though the general problem when the failure rate λ_1 of the unit in use and the failure rate λ_2 of the units in storage are different and the repair time distribution assumed to be arbitrary is quite difficult to solve, the particular case when the failure time distributions of the units in use and in storage and the repair time distribution are exponential can be identified with a birth and death process. A very elegant treatment of this important class of processes has been given by [Karlin and Mc Gregor (1957)] and Karlin (1966). This has been excellently summarised by Barlow (1962 a), emphasizing its application to repairman problems [Feller (1957), Cox and Smith (1961) and Takacs (1962)]. Many reliability problems of systems with redundant units have the same stochastic behaviour as these repairman problems except for the fact that the characteristics under study are different. As in the previous case, we are interested in the expected TSF, the expected SDT, the long-run availability of the system, the expected number of failures of the system in a given interval of time (0, t) and the interval reliability.

As usual, let the random variable $n(t)$ represent the number of failed units in the system. Then $\{n(t) = n\}, n = 0, 1, 2, \ldots N$ are the non-absorbing, non-reflecting states of a pure birth-death process and the state -1 and N + 1 will be the reflecting states of this process. And for this birth-death process let

$$P_{ij}(t) = P_r\left[\, n(t) = j \;/\; n(0) = i \,\right]$$

be the state probabilities of finding j failed units at time t given initially there were i failed units. For this process, the one-step transition probabilities $P_{ij}(t) = P_r\left[n(t) = j / n(0) = i \right]$ in an infinitesimal interval Δt are given by

$$P_{ij}(\Delta t) = \begin{cases} \sigma_i \, \Delta t + o(\Delta t) & , \quad j = i+1 \\ \mu_i \, \Delta t + o(\Delta t) & , \quad j = i-1 \\ 1-(\sigma_i+\mu_i)\,\Delta t + o(\Delta t) & , \quad j = i \end{cases} \tag{1,3.52}$$

where $\sigma_i = \lambda_1 + (\overline{N-1} - i)\lambda_2$, $\mu_i = \mu$, $\lambda_1, \lambda_2, \mu \geq 0$ for $i \geq 0$

It is well-known that the transition probability matrix $\underset{\sim}{P}(t) = [P_{ij}(t)]$ satisfies the differential equations

$$\underset{\sim}{P}'(t) = A\,\underset{\sim}{P}(t) \quad \text{and} \quad \underset{\sim}{P}'(t) = \underset{\sim}{P}(t)\,A \tag{1,3.53}$$

with the initial condition $\underset{\sim}{P}(0) = I$, where A is (N + 1) x (N + 1) order matrix given by

$$A = \begin{bmatrix} -\sigma_0 & \sigma_0 & 0 & 0 \cdots 0 & 0 \\ \mu_1 & -(\sigma_1+\mu_1) & \sigma_1 & 0 \cdots 0 & 0 \\ 0 & \mu_2 & -(\sigma_2+\mu_2) & \sigma_2 \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots -(\sigma_{N-1}+\mu_{N-1}) & \sigma_{N-1} \\ 0 & 0 & 0 & 0 \cdots \mu_N & -\mu_N \end{bmatrix}$$

and I is the identity matrix.

The Laplace transform of (1,3,53) can be written as

$$\bar{A}\,\bar{\underset{\sim}{P}}(s) = I \tag{1,3.54}$$

where $\bar{A}$ is given by

$$\bar{A} = \begin{bmatrix} \sigma_0+s & -\mu_1 & 0 & 0 \cdots\cdots 0 & 0 \\ -\sigma_0 & (\sigma_1+\mu_1+s) & -\mu_2 & 0 \cdots\cdots 0 & 0 \\ 0 & -\sigma_1 & (\sigma_2+\mu_2+s) & -\mu_3 \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \quad (\sigma_{N-1}+\mu_{N-1}+s) & -\mu_N \\ 0 & 0 & 0 & 0 \quad \cdots -\sigma_{N-1} & (\mu_N+s) \end{bmatrix}$$

and $\bar{\underset{\sim}{P}}(s)$ is the Laplace transform of the matrix $\underset{\sim}{P}(t)$

We know that the LST of the distribution of time-to-system failure, namely, $\hat{G}_{i,N}(s)$ is given by

$$\hat{G}_{i,N}(s) = \frac{\hat{P}_{i,N}(s)}{\hat{P}_{N,N}(s)} = \frac{\bar{P}_{i,N}(s)}{\bar{P}_{N,N}(s)}$$

where $\bar{P}_{i,N}(s)$ and $\bar{P}_{N,N}(s)$ can be obtained by solving the matrix equation (1,3.54). Thus, denoting by $|\bar{A}|$, the determinant of $\bar{A}$ and by $|\bar{A}_{ij}|$, the co-factor of the (i,j) th element of $\bar{A}$ and i standing for the row number and j standing for the column number of $\bar{A}$, the values of $\bar{P}_{i,N}(s)$ and $\bar{P}_{N,N}(s)$ are found by applying Cramer's rule to solve (1,3.54). These are given by

$$\bar{P}_{i,N}(s) = \frac{|\bar{A}_{N+1,i+1}|}{|\bar{A}|} \qquad 1,3.55$$

and
$$\bar{P}_{N,N}(s) = \frac{|\bar{A}_{N+1,N+1}|}{|\bar{A}|} \qquad (1, 3.56)$$

Hence, we obtain the LST of the TSF to be
$$\hat{G}_{i,N}(s) = \frac{|\bar{A}_{N+1,i+1}|}{|\bar{A}_{N+1,N+1}|} \qquad (1, 3.57)$$

As the expectation and the second-moment of TSF are obtained by differentiating $\hat{G}_{i,N}(s)$ with respect to s and taking the limit as $s \to 0$, we have

$$E_i(T_u) = \left[\frac{1}{|\bar{A}_{N+1,N+1}|} \left[\frac{d}{ds} \left(|\bar{A}_{N+1,N+1}| - |\bar{A}_{N+1,i+1}| \right) \right] \right]_{s=0} \qquad (1, 3.58)$$

$$E_{N-1}(T_u) = \left[\frac{1}{|\bar{A}_{N+1,N+1}|} \left[\frac{d}{ds} \left(|\bar{A}_{N+1,N+1}| - |\bar{A}_{N+1,N}| \right) \right] \right]_{s=0} \qquad (1, 3.59)$$

and

$$E_{N-1}(T_u^2) = \left[\frac{1}{|\bar{A}_{N+1,N+1}|} \left[\frac{d^2}{ds^2} \left(|\bar{A}_{N+1,N}| - |\bar{A}_{N+1,N+1}| \right) + 2\, E_{N-1}(T_u) \frac{d}{ds} |\bar{A}_{N+1,N+1}| \right] \right]_{s=0} \qquad (1, 3.60)$$

We shall now obtain these expressions, for example, when N = 4. In this case, there is one unit in operation and the other three units are in spare pool. We obtain the expressions for the LST of the distribution of TSF ~~failure~~, its expectation and the second moment with initially no failed units in the system and with initially 3 failed units in the system as

$$\hat{G}_{0,4}(s) = \frac{\lambda_1(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)(\lambda_1+3\lambda_2)}{\begin{aligned}&[\{(s+\lambda_1+2\lambda_2+\mu)(s+\lambda_1+\lambda_2+\mu)-\mu(\lambda_1+2\lambda_2)\}(\lambda_1+2\lambda_2+s)\\ &-\mu(\lambda_1+3\lambda_2)(s+\lambda_1+\lambda_2+\mu)](\lambda_1+\mu+s)\\ &-\mu(\lambda_1+\lambda_2)\{(s+\lambda_1+3\lambda_2)(s+\lambda_1+2\lambda_2+\mu)-\mu(\lambda_1+3\lambda_2)\}\end{aligned}} \qquad (1,3.61)$$

$$\hat{G}_{3,4}(s) = \frac{\begin{aligned}&\lambda_1[(s+\lambda_1+\lambda_2)\{(s+\lambda_1+3\lambda_2)(s+\lambda_1+2\lambda_2)+s\mu\}\\ &+\mu(s+\lambda_1+3\lambda_2)(s+\lambda_1+2\lambda_2)+s\mu^2-\mu(s+\lambda_1+3\lambda_2)(\lambda_1+2\lambda_2)]\end{aligned}}{\begin{aligned}&[(s+\lambda_1+2\lambda_2)(s+\lambda_1+\lambda_2)(s+\lambda_1)+(s+\lambda_1+3\lambda_2)(s+\lambda_1+\lambda_2)(s+\lambda_1)\\ &+(s+\lambda_1+3\lambda_2)(s+\lambda_1+2\lambda_2)(s+\lambda_1)+(s+\lambda_1+3\lambda_2)(s+\lambda_1+2\lambda_2)(s+\lambda_1+\lambda_2)\\ &+\mu(s+\lambda_1+3\lambda_2)(s+\lambda_1+2\lambda_2)+\mu^2(s+\lambda_1)+\mu^3]\end{aligned}} \qquad (1,3.62)$$

$$E_{0,4}(T_u) = \frac{1}{\mu\rho_1}\left[1+\frac{\rho_1}{\rho_1+3\rho_2}+\frac{\rho_2}{\rho_1+3\rho_2}\left(1+\frac{1}{1+3\rho_2}\right) + \frac{\rho_1+1}{\rho_1+\rho_2}\left(1+\frac{1}{\rho_1+2\rho_2}+\frac{1}{(\rho_1+2\rho_2)(\rho_1+3\rho_2)}\right)\right] \qquad (1,3.63)$$

$$E_{3,4}(T_u) = \frac{1}{\mu\rho_1}\left[1+\frac{1}{\rho_1+\rho_2}\left\{1+\frac{1}{\rho_1+2\rho_2}\left(1+\frac{1}{\rho_1+3\rho_2}\right)\right\}\right] \qquad (1,3.64)$$

and

$$E_{3,4}(T_u^2) = \frac{2}{\mu^2}\Big[\mu E_{3,4}(T_u)\Big\{\frac{1}{\rho_1}+\frac{1}{\rho_1+\rho_2}+\frac{1}{\rho_1+2\rho_2}+\frac{1}{\rho_1+3\rho_2} + \frac{\rho_1+1}{\rho_1(\rho_1+\rho_2)(\rho_1+2\rho_2)}+\frac{1}{\rho_1(\rho_1+\rho_2)}+\frac{1}{(\rho_1+\rho_2)(\rho_1+3\rho_2)}$$

$$+ \frac{1}{(\rho_1+\rho_2)(\rho_1+2\rho_2)(\rho_1+3\rho_2)} + \frac{1}{\rho_1(\rho_1+\rho_2)(\rho_1+2\rho_2)(\rho_1+3\rho_2)} \Bigg\}$$

$$- \frac{1}{\rho_1} \Bigg\{ \frac{1}{\rho_1+\rho_2} + \frac{1}{\rho_1+2\rho_2} + \frac{1}{\rho_1+3\rho_2} + \frac{1}{(\rho_1+\rho_2)(\rho_1+3\rho_2)}$$

$$+ \frac{1}{(\rho_1+2\rho_2)(\rho_1+3\rho_2)} + \frac{2}{(\rho_1+\rho_2)(\rho_1+2\rho_2)} + \frac{3}{(\rho_1+\rho_2)(\rho_1+2\rho_2)(\rho_1+3\rho_2)} \Bigg\} \Bigg] \qquad (1,3.65)$$

where

$$\rho_1 = \frac{\lambda_1}{\mu}, \quad \rho_2 = \frac{\lambda_2}{\mu}$$

These expressions of the expectation and the second moment of time to system failure can be used in (1,1.35) and (1,1.36) to obtain the expectation and the second moment of recurrence time τ_R to the state N, which are in turn used in the expressions (1,1.52), (1,1.53) and (1,1.56) for the expected number of failures of the system in a given interval of time (0, t) and the interval reliability of the system respectively. However, to study the long-run availability of the system, which is equal to $1-p_N$, we have to evaluate p_N, the steady state probability of the system being in state N.

The steady state probabilities of the various states can be readily evaluated by solving the difference equations representing the equilibrium state of the process. Let p_j be the steady state probability that there are j failed units in the system. Since the process is independent of time in the equilibrium state, putting the derivatives with respect to t in the equations (1,3.53) equal to zero and rearranging the resulting terms, we have

$$p_1 = \frac{\lambda_1 + (N-1)\lambda_2}{\mu} \cdot p_0 = \frac{\sigma_0}{\mu} p_0 \qquad (1,3.66)$$

$$\sigma_n p_n - \sigma_{n-1} p_{n-1} = \mu (p_{n+1} - p_n) \qquad (1,3.67)$$

and $$p_N = \frac{\sigma_{N-1}}{\mu} p_{N-1} \quad \text{since } \lambda_1 = \sigma_{N-1} \qquad (1,3.68)$$

Adding the set of equations (1,3.67) for $n = 0$ to j we obtain

$$\sigma_j p_j - \sigma_0 p_0 = \mu (p_{j+1} - p_1)$$

$$= \mu p_{j+1} - \sigma_0 p_0 \quad \text{by } (1,3.66)$$

we have, therefore,

$$p_{j+1} = \frac{\sigma_j}{\mu} p_j \qquad (1,3.69)$$

from which it follows that

$$p_n = \frac{\sigma_{n-1}\sigma_{n-2}\cdots\sigma_0}{\mu^n} \cdot p_0 \qquad (1,3.70)$$

Define

$$\pi_\ell = \begin{cases} \dfrac{\sigma_0 \sigma_1 \sigma_2 \cdots \sigma_{\ell-1}}{\mu^\ell}, & \ell > 0 \\ 1, & \ell = 0 \end{cases} \qquad (1,3.71)$$

Then, the expression for p_n can be written as

$$p_n = \pi_n \, p_o \qquad (1,3.72)$$

Since $\sum_{n=0}^{N} p_n = 1$ we have $p_o \left(\sum_{n=0}^{N} \pi_n \right) = 1$ whence

$$p_o = \frac{1}{\sum_{n=0}^{N} \pi_n} \qquad (1,3.73)$$

Using (1,3.72), the expression for p_N is obtained as

$$p_N = \pi_N \, p_o = \frac{\pi_N}{\sum_{n=0}^{N} \pi_n} \qquad (1,3.74)$$

Hence, the long-availability of the system

$$= 1 - p_N = 1 - \frac{\pi_N}{\sum_{n=0}^{N} \pi_n} = \frac{\sum_{n=0}^{N-1} \pi_n}{\sum_{n=0}^{N} \pi_n} \qquad (1,3.75)$$

From (1,3.74), using the renewal property of the process, namely, the renewal period consists of a TSF period followed by a SDT period, we can find the expression for the expected TSF within a renewal period, which in fact is $E_{N-1}(T_u)$. From Smith's theorem, as in section 1, we have

$$p_N = \lim_{t \to \infty} P_{i,N}(t) = \frac{1}{\mu \, E(\tau_R)}$$

Now equating the value of p_N to that given by (1,3.74) one gets

$$p_N = \frac{1}{\mu \, E(\tau_R)} = \frac{\pi_N}{\sum_{n=0}^{N} \pi_n} \qquad (1,3.76)$$

NUMBER OF UNITS FOR 90% AVAILABILITY

OF A STANDBY REDUNDANT SYSTEM

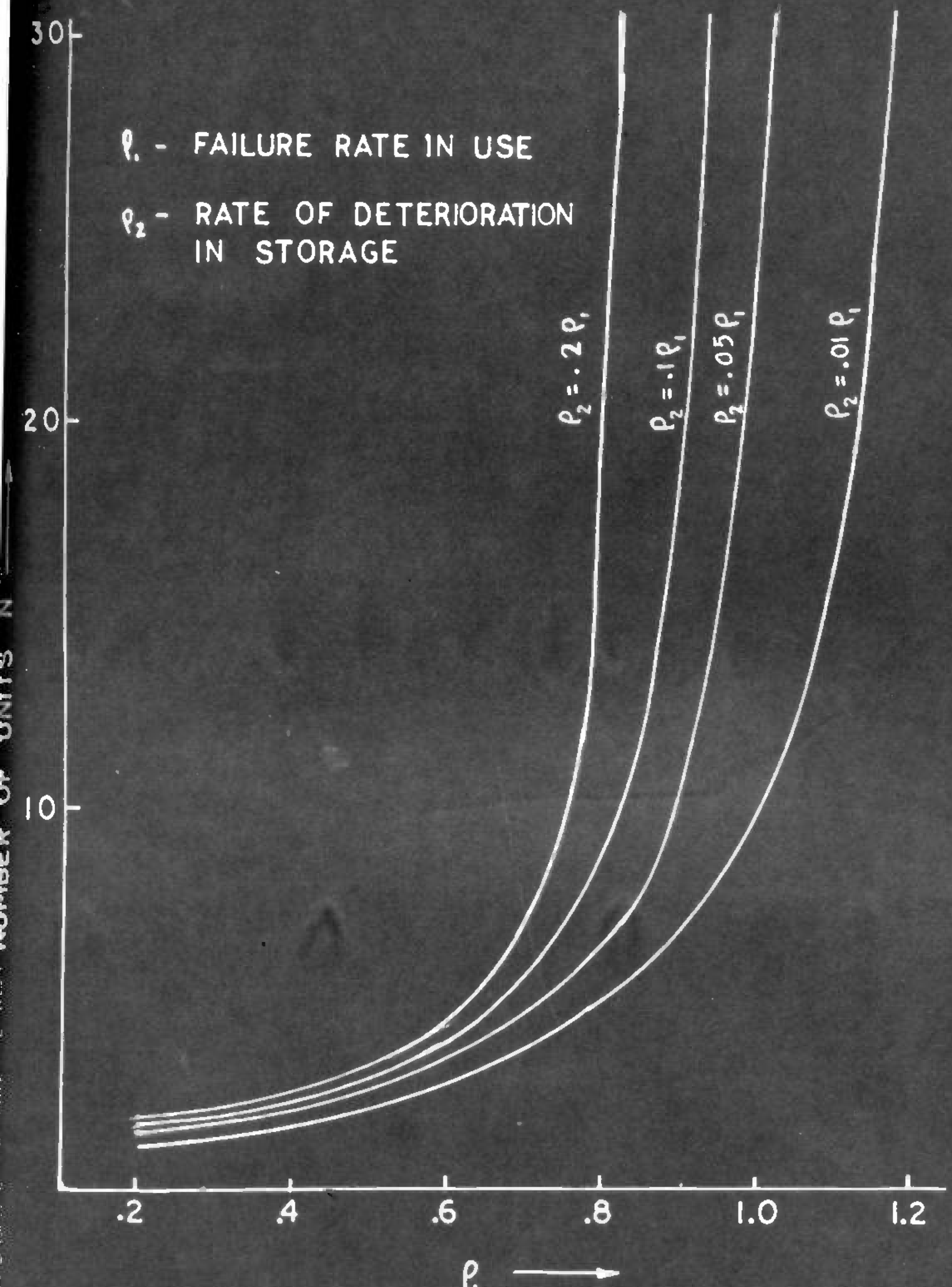


FIG. 1.6

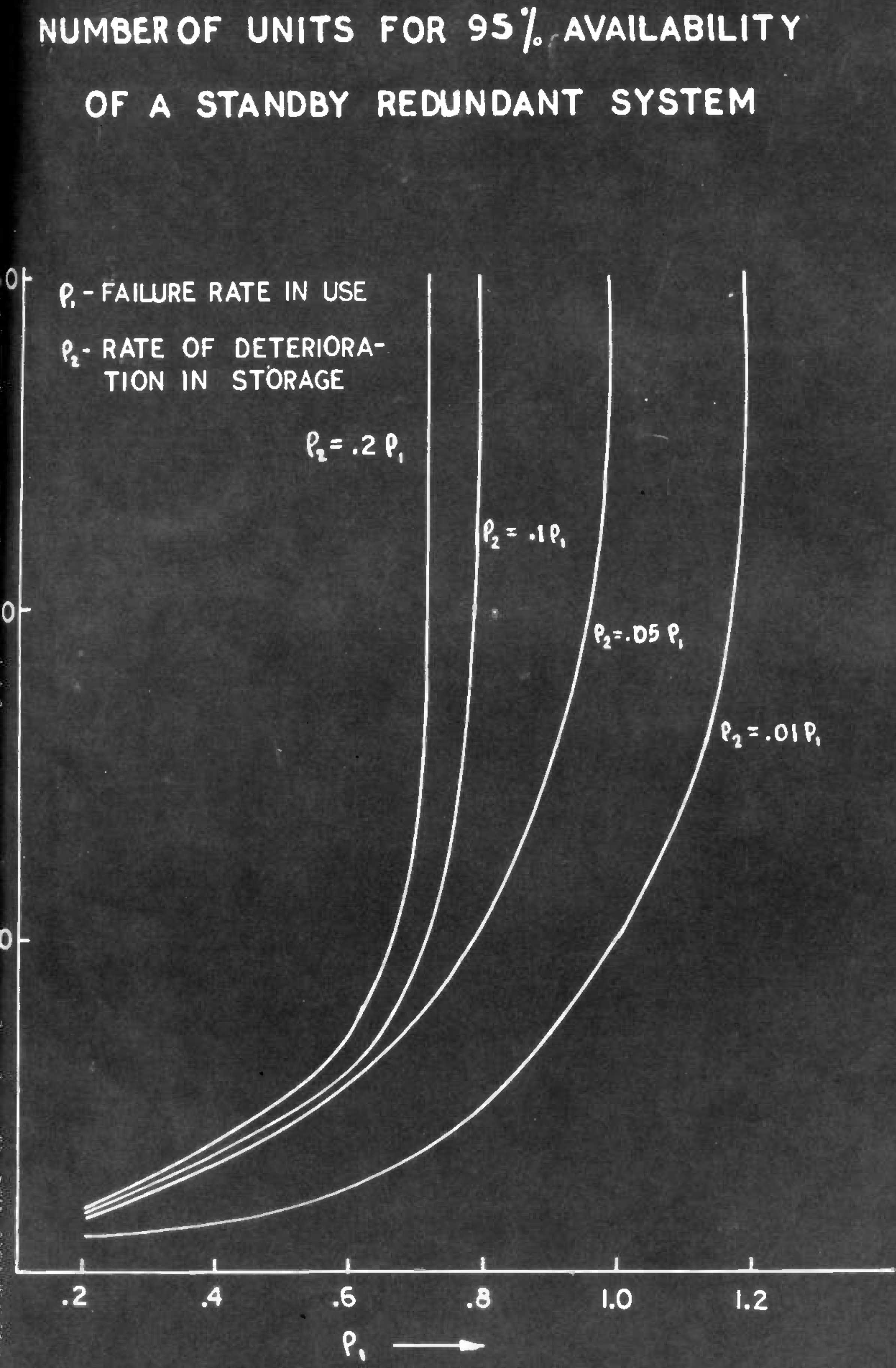

NUMBER OF UNITS FOR 95% AVAILABILITY
OF A STANDBY REDUNDANT SYSTEM
ρ1 - FAILURE RATE IN USE
ρ2 - RATE OF DETERIORATION IN STORAGE
ρ2 = .2 ρ1
ρ2 = .1 ρ1
ρ2 = .05 ρ1
ρ2 = .01 ρ1
.2
.4
.6
.8
1.0
1.2
ρ1

FIG. 1.7

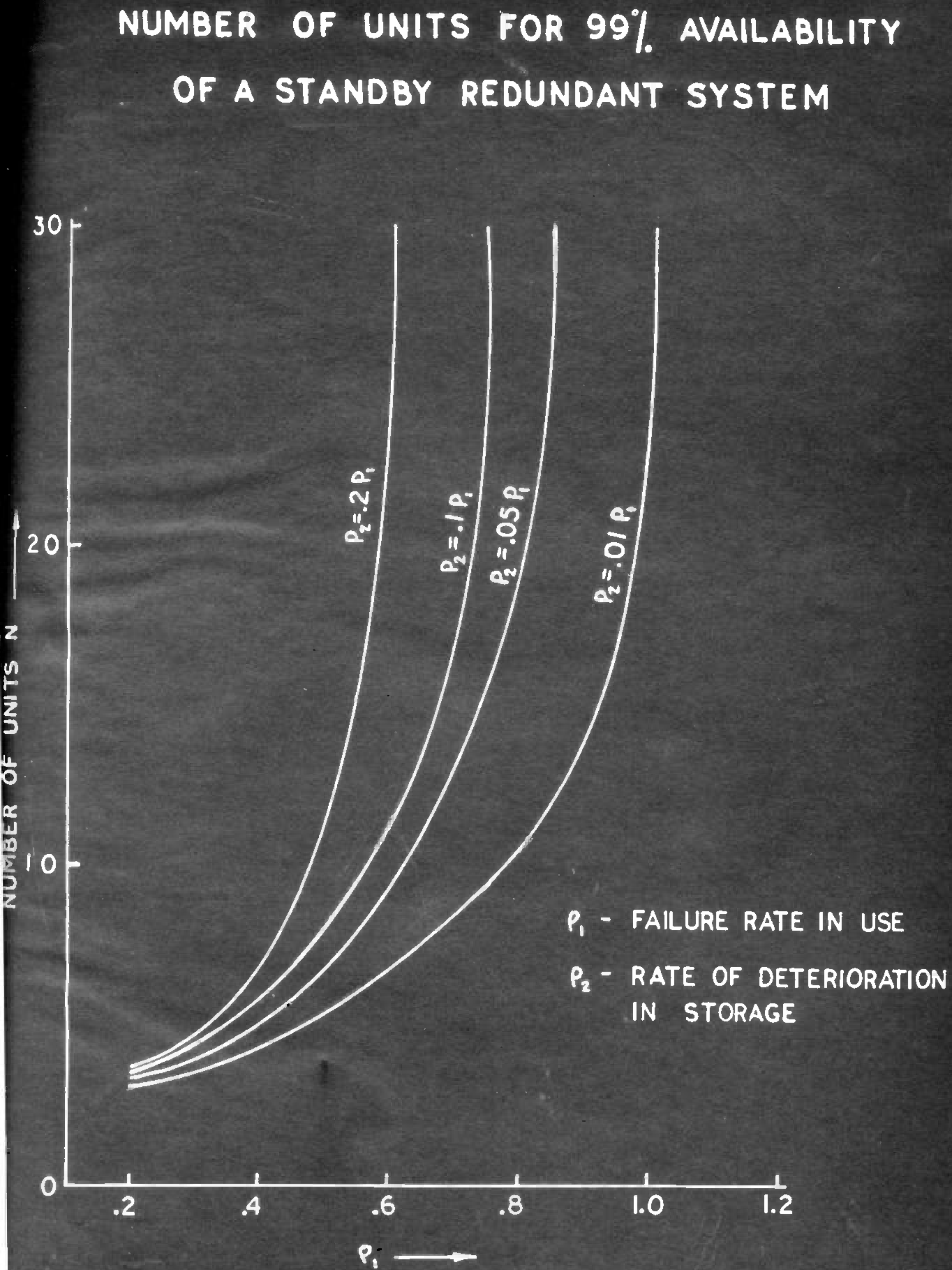

NUMBER OF UNITS FOR 99% AVAILABILITY
OF A STANDBY REDUNDANT SYSTEM
30
20
10
0
NUMBER OF UNITS N
.2
.4
.6
.8
1.0
1.2
P_1
$P_2 = .2 P_1$
$P_2 = .1 P_1$
$P_2 = .05 P_1$
$P_2 = .01 P_1$
P_1 - FAILURE RATE IN USE
P_2 - RATE OF DETERIORATION IN STORAGE

FIG. 1. 8

Since $E(\tau_R) = E_{N-1}(T_u) + E(T_d)$ and since $E(T_d) = \frac{1}{\mu}$,

(1,3.76) can be written as

$$\mu E(\tau_R) = \frac{1}{\pi_N} \sum_{l=0}^{N} \pi_l = 1 + \frac{1}{\pi_N} \sum_{l=0}^{N-1} \pi_l$$

$$= 1 + \mu E_{N-1}(T_u)$$

whence

$$E_{N-1}(T_u) = \frac{1}{\mu \pi_N} \sum_{l=0}^{N-1} \pi_l \qquad (1,3.77)$$

By this method, we are able to find only $E_{N-1}(T_u)$. However, to evaluate $E_{N-1}(T_u^2)$ and higher moments, we require a knowledge of the distribution of TSF

It is also observed that $E_{N-1}(T_u)$ can be expressed in terms of p_N and the mean repair time η. Using (1,3.77) in (1,3.74) and simplifying, we have

$$E_{N-1}(T_u) = \frac{\eta(1-p_N)}{p_N} \qquad [\text{cf } 1,3.45] \qquad (1,3.78)$$

An interesting information that can be of use in the design of standby systems and which can be derived from (1,3.75) is the number of standby redundant units required to achieve a stated level of system reliability when the spares deteriorate in storage. To this end, the number of standby units required to achieve 90, 95 and 99 per cent availability of the system have been computed using (1,3.75) for various ρ_1 and deterioration intensities $\rho_2 = 0.01\,\rho_1$, $0.05\,\rho_1$, $0.1\,\rho_1$ and $0.2\,\rho_1$.

TABLE 1.3

THE RATIO OF EXPECTED TIME TO SYSTEM FAILURE TO EXPECTED REPAIR TIME OF A STANDBY REDUNDANT SYSTEM WHEN SPARES DETERIORATE IN STORAGE FOR VARIOUS VALUES OF ρ_1, ρ_2 AND N.

$$\frac{\text{Expected Time to System Failure}}{\text{Expected Repair Time}} = \mu\, E_{N-1}(T_{11})$$

..of ..its for the ..stem	ρ_1	SPARES DETERIORATION IN STORAGE				No deterioration in storage
		$\rho_2 = 0.01\,\rho_1$	$\rho_2 = .05\,\rho_1$	$\rho_2 = .10\,\rho_1$	$\rho_2 = .20\,\rho_1$	$\rho_2 = 0$
	0.2	2.92971.10	2.88095.10	2.77273.10	2.58333.10	3.00000.10
	0.4	8.68812	8.45239	8.18182	7.70833	8.75000
:2	0.6	4.41694	4.31216	4.19192	3.98148	4.44445
	0.8	2.79703	2.73809	2.67045	2.55208	2.81251
	1.0	1.99009	1.96238	1.90909	1.83333	2.00000
	1.2	1.52090	1.49471	1.46464	1.41203	1.52755
	0.2	$7.39741.10^2$	$6.03569.10^2$	$4.86662.10^2$	$3.32756.10^2$	$7.79982.10^2$
	0.4	6.06682.10	5.13897.10	4.27926.10	3.15412.10	6.34378.10
:4	0.6	1.61826.10	1.41297.10	1.21957.10	9.60776	1.67901.10
	0.8	6.99371	6.27717	5.57282	4.62563	7.20700
	1.0	3.90316	3.57105	3.24936	2.80059	4.00000
	1.2	2.53710	2.35882	2.18407	1.93591	2.58870
	0.2	$1.70638.10^4$	$1.04048.10^4$	$6.12270.10^3$	$2.59371.10^3$	$1.95300.10^4$
	0.4	$3.59867.10^2$	$2.29782.10^2$	$1.51183.10^2$	7.69538.10	$4.05239.10^2$
:6	0.6	4.03831.10	3.29554.10	2.34956.10	1.44809.10	5.10844.10
	0.8	1.30512.10	1.02074.10	7.90169	5.64789	1.40736.10
	1.0	5.67257	4.70036	3.94315	2.43270	6.00000
	1.2	3.19039	2.77899	2.44430	2.05358	3.32550
	0.2	$3.70369.10^5$	$1.53854.10^5$	$5.95228.10^4$	$1.39072.10^4$	$4.89280.10^5$
	0.4	$2.01797.10^3$	$8.88326.10^2$	$4.12719.10^2$	$1.35489.10^2$	$2.54144.10^3$
:8	0.6	$1.20720.10^2$	6.29828.10	3.57663.10	1.73239.10	$1.46341.10^2$
	0.8	2.14688.10	1.29567.10	9.33677	5.98852	2.48015.10
	1.0	7.24761	5.37247	4.27356	3.17704	8.00000
	1.2	3.59449	2.95829	2.51636	2.07124	3.83720
	0.2	$4.99999.10^6$	$1.99999.10^6$	$4.76189.10^5$	$5.49441.10^4$	$1.22070.10^7$
	0.4	$1.08686.10^4$	$3.03023.10^3$	$9.13411.10^2$	$1.89843.10^2$	$1.58930.10^4$
:10	0.6	$2.97516.10^2$	$1.07340.10^2$	4.61627.10	1.85163.10	$4.10942.10^2$
	0.8	3.28582.10	1.64154.10	9.75911	6.06964	4.15654.10
	1.0	8.59834	5.72192	4.30504	3.18738	2.00000.10
	1.2	3.83545	3.01564	2.53226	2.07314	4.19250

These are exhibited in fig. 1.6, 1.7 and 1.8. In these figures, the ordinate gives the total number of units in the system so that the required number of spares to be kept as standby is one less than the number given by the graphs.

The values of $\mu \cdot E_{N-1}(T_{u})$ for various values of ρ_1, ρ_2 and N are calculated using (1,3.78) and are given in table 1.3. Given in the last column are the values of $\mu E_{N-1}(T_u)$ when there is no deterioration in storage. Comparison of these values with those in other columns brings out clearly as to how deterioration in storage causes an increase in the initial provisioning of the number of standbys to achieve a specified reliability of the system.

Case of unlimited spares and limited space at the repair facility:

In this case also, we assume that the failure time distribution of the unit in use and that of the spare in storage are exponential with mean failure rates λ_1 and λ_2 respectively and the repair time distribution is exponential with the mean repair rate μ. Though the number of spares available is unlimited, we assume that because of the limited space in the repair facility, the input of spares is cut off as soon as N units are in the failed state at the repair facility.

The system failure is said to occur when all the N units including the one in use are in the failed state and are either undergoing repair or waiting for repair.

The differential-difference equations describing the general process probabilities are given by

$$\left[\frac{d}{dt} + \lambda_1 + \lambda_2 + \mu\right] P_{i,n}(t) = (\lambda_1 + \lambda_2)\, P_{i,n-1}(t) + \mu\, P_{i,n+1}(t)$$

$$n = 0, 1, 2, \ldots, N.$$

with the necessary modifications for the end values $n = 0, N$ and with the initial condition $P_{i,n}(0) = \delta_{i,n}$ where $\delta_{i,n}$ is Kronecker delta.

Comparing this with the process when the spares do not deteriorate in storage, we observe that this process is the same as that when spares do not deteriorate in storage where λ is replaced by $(\lambda_1 + \lambda_2)$ Therefore, all the results of the reliability characteristics such as long-run availability, the expected time to system failure, the expected number of failure of the system in a given interval of time (0, t) and the interval reliability can be found by changing λ into $(\lambda_1 + \lambda_2)$ of the corresponding results of section 1.

Case of Intermittent usage of the system:

The discussions in this section upto now have been made when the system is in continuous usage. As we have seen in section 2, the state probabilities and the reliability characteristics of the intermittent usage case can be expressed in terms of the general process probabilities of the continuous usage case and the probability of existance or otherwise of demand for the use of the system. Therefore, we obtain the results for the intermittent usage case when spares deteriorate in storage by proceeding in exactly the same lines as in section 2 and using the general process probabilities of the case of continuous usage of this system.

CHAPTER 2

RELIABILITY OF A STANDBY REDUNDANT SYSTEM WITH MULTIPLE REPAIR FACILITIES

CHAPTER 2

RELIABILITY OF A STANDBY REDUNDANT SYSTEM WITH MULTIPLE REPAIR FACILITY

INTRODUCTION

While in the previous chapter we considered systems having a single repair facility, in this chapter, we study the reliability characteristics of a single unit system with (N - 1) spares and multiple repair facilities. Whenever a unit in use fails, it is replaced by another one from the spares. If one of the c repair facilities is free at that moment, the failed unit is taken up for repair; otherwise it waits for repair in a queue. The system failures ~~occurs~~ occur when the unit in use fails and no spare is available for replacement. On repair completion, the failed units are returned to the spare pool for use as standby. Examples of such system are many: two navigation computers on each Polaris submarine, search radars, power generators in operation theatres of hospitals etc.

In the types of situations as described above, the aim is to keep the system in the operating state as long as possible with minimum system down-time —

the duration of time the system is in the failed state. At the same time, proper reliability management calls for a maintenance of the system with minimum costs. A desired level of system reliability can be achieved either by increasing the number of redundant units or by increasing the number of repair facilities. If we can determine the additional numbers of units or repair facilities then depending on their respective costs, the total costs of maintaining the system can be minimised. The important reliability characteristics useful for making such management decisions are, as in the last chapter, the average duration of time the system is in the operating state, the average duration of time the system is in the failed state, the mean number of failures of the system in a given interval of time $(0, t)$, the expected number of repair completions in a given interval of time and so on.

The distribution of time-to-system failure (TSF) for parallel systems with repair has been studied by Barlow (1962 a), Gaver (1963), Mc Gregar (1963) and Natarajan (1967 b). In a recent paper, Downton (1966) considers the distribution of TSF of N unit parallel system with c repair facilities. He derives the results under exponential failure time and repair time distributions through the results of birth and death process. Weiss (1962c) and Schweitzer (1967) have considered systems in which failures occur when the units are in use as well as when they are in storeage and have derived some of their reliability characteristics. Further, Srinivasan (1967 b) has investigated the distribution of TSF of a system with spares and ~~random~~ repairable components.

In this chapter, we devote our attention to the standby redundant system described at the outset and investigate the process during a TSF period. Then we describe the general process by means of renewal theoretical arguments using the TSF period probabilities, distributions of TSF and of system down-time (SDT). The ergodic (steady-state) probabilities have been obtained from the general process probabilities by using some well-known results of renewal theory. Towards the end, an analysis of the TSF and the long-run availability of the system is made by a numerical illustration where the mean TSF and the long-run availability of the system are worked out for various combinations of the number of repair facilities, the number of redundant units and traffic intensities. It is pointed out how the management decisions could be made based on this numerical illustration.

DESCRIPTION OF THE SYSTEM

A formal description of the system is given now by defining the failure, repair and replacement processes.

1. Failures: Let x be the time for which a unit is in operation. Then the failure-time distribution of the unit is

$$F(t) = P_r[X \leq t] = 1 - e^{-\lambda t} . \quad (t \geq 0) \qquad (2,1.1)$$

ie, the failure times are exponentially distributed. We further assume that the failure times of individual units are independently and identically distributed according to (2,1.1).

2. Replacement: When a unit fails, it is replaced immediately by a spare. There is no switch-over time involved.

3. Repair: A unit on failure is taken up for repair by one of the c (> 1) repair facilities if any one of them is free; otherwise the unit waits for repair in a queue. Let $\{\theta_{i,r}\}$ (i = 1,2,...c) be the sequence of repair time at the i-th facility. We assume that the

$\{\theta_{i,r}\}$ are identically and independently distributed random variables with the distribution

$$D(t) = P_r[\theta_{i,r} \leq t] = 1 - e^{-\mu t}, \quad i = 1, 2, \cdots c \qquad (2,1.2)$$

ie, the repair time at each of the c facilities are exponentially distributed.

4. The units are taken for repair in the order of their failure.

5. State of the system: Let the random variable $n(t)$ denote the number of failed units in the system at time t. Then we define the state of the system to be n if $n(t) = n$.

TIME-TO-SYSTEM FAILURE PROCESS

During the TSF period, the system is in the up-state. The TSF period may begin with i ($i \leq N$) failed units initially. When $i < N$, a new unit from the spare pool starts operating at $t = o$, thus initiating a TSF period. When $i = N$, the TSF period is initiated by the repair completion of one of the failed units at $t = o-$. The TSF period ends with failure of the N-th unit leaving the system in the down-state. Obviously, during a TSF period, n(t) the number of failed units in the system is never N. The

TSF process has the following associated transition probabilities for transition from state i at time $t = o$ to state n at time t.

$$P_{i,n}(t) = P_r\left[n(t) = n,\ n(t') < N\ (o \leq t' \leq t) \,/\, n(o) = i\right] \qquad (2,1.3)$$
$$(o \leq n, i < N)$$

The set of probabilities (2,1.3) characterises the process whose states are mutually exclusive and exhaustive. It may be noted in this case that the process is Markovian.

We derive the difference-differential equations governing the TSF process through continuity arguments. Connecting the probabilities at time $t + \Delta$ with those at time t and taking limits as $\Delta \to o$, we obtain

$$\frac{d}{dt} P_{i,n}(t) = -\left[\lambda + n(1-\delta_{o,n})\mu\right] P_{i,n}(t) + \lambda P_{i,n-1}(t)$$
$$+ (n+1)\mu\, P_{i,n+1}(t), \quad (o \leq n \leq c-1) \qquad (2,1.4)$$

and

$$\frac{d}{dt} P_{i,n}(t) = -(\lambda + c\mu) P_{i,n}(t) + \lambda P_{i,n-1}(t) \qquad (2,1.5)$$
$$+ c\mu(1-\delta_{a,N-1}) P_{i,n+1}(t), (c \leq n \leq N-1)$$

where $\delta_{i,n}$ is the Kronecker delta.

The system failure occurs in the interval $(t, t+\Delta)$ if the system reaches the state N during this interval. If $g_{i,n}(t)$ denotes the density of the TSF when the system started initially with i units in the failed state, then

$$g_{i,n}(t) = \lambda P_{i,N-1}(t) \qquad (2,1.6)$$

It is observed that the repair rate goes on changing when the failed units in the system are less than the number of repair facilities and it remains constant when they are greater than or equal to c, the total number of repair facilities available. Accordingly the solution of the set of equations (2,1.4) and (2,1.5) must be dealt with separately.

Of the many methods available for the solution of these set of equations, we make use of the method of compensating function [Keilson (1962 a)]. By this method, a bounded process may be effectively discussed by imbedding it in the unbounded process and providing fictitious sources or compensations at the boundaries to satisfy the boundary conditions. This technique is applicable principally to "skip-free" transitions, ie, transitions over an ordered set of points which are required to pass through all intermediate points when going from one point to another.

Now, we are given a process consisting of a discrete finite set of states indexed by n = 0, 1, 2, ...N . To solve this problem, we imbed our finite array into two infinite arrays, one over n = 0, 1, 2... and the other over n = ...-2, -1, 0, 1, 2,... Now we consider the set of auxiliary equations, valid for all n, describing the associated ~~im~~ unbounded process, modified however by the presence of unknown compensating functions. To this end we define new set of probabilities P_n and Q_n with single suffix such that P_n varies over all non-negative lattice points and Q_n varies over all integral lattice points. Then we have

$$\frac{d}{dt} P_n(t) = -(\lambda + n\mu) P_n(t) + \lambda(1-\delta_{n,0}) P_{n-1}(t) + (n+1)\mu P_{n+1}(t) + \epsilon_n(t), \quad n \geq 0 \tag{2,1.7}$$

$$\frac{d}{dt} Q_n(t) = -(\lambda + c\mu) Q_n(t) + \lambda Q_{n-1}(t) + c\mu Q_{n+1}(t) + d_n(t)$$

$$n = 0, \pm 1, \pm 2, \dots \tag{2,1.8}$$

with the appropriate initial conditions

$$\begin{aligned} P_n(0) &= \delta_{i,n} \qquad \text{for} \quad n \leq c-1 \\ Q_n(0) &= \delta_{i,n} \qquad \text{for} \quad n \geq c \end{aligned} \tag{2,1.9}$$

Here $\epsilon_n(t)$ and $d_n(t)$ are compensation functions so chosen that

$$P_n(t) = P_{i,n}(t) \quad \text{for } n = 0, 1, 2, \cdots c-1$$

and

$$Q_n(t) = P_{i,n}(t) \quad \text{for } n = c, c+1, \cdots (N-1)$$

Also, because of the skip-free characteristic of the random walk

$$P_n(t) = 0 \quad \text{for } n \geq c \,, \quad Q_n = 0 \quad \text{for } n \leq c-1, n > N$$

these will be satisfied if the compensation functions are chosen to obey

$$\begin{aligned}
P_c(t) = 0 \quad &: \quad \epsilon_c(t) + \lambda P_{c-1}(t) = 0 \\
&\qquad \epsilon_{c-1}(t) = c\mu\, Q_c(t) \\
Q_{c-1}(t) = 0 \quad &: \quad d_{c-1}(t) + c\mu\, Q_c(t) = 0 \\
&\qquad d_c(t) = \lambda P_{c-1}(t) \\
Q_N(t) = 0 \quad &: \quad \lambda Q_{N-1}(t) + d_N(t) = 0
\end{aligned} \qquad (2,1.10)$$

with the assumption $\epsilon_n(t) = 0$, $d_n(t) = 0$ for other values of n.

It may be noted that the choice of compensation functions are similar to those used by Keilson (1962 b) for the case of M/M/K queueing process except for the compensation at state N .

We observe that the set of equations (2,1.7) are differential-difference equations with variable coefficients. We reduce them to an easily solvable form by introducing binomial moments defined by

$$A_n(t) = \sum_{j=n}^{c-1} \binom{j}{n} P_j(t) \qquad n = 0, 1, \ldots (c-1) \qquad (2,1.11)$$

Transforming the set of equations (2,1.7) by means of (2,1.11) and using conditions (2,1.10), we have

$$\frac{d}{dt} A_n(t) + n\mu A_n(t) = \lambda A_{n-1}(t) - \binom{c}{n} d_c(t) + \binom{c-1}{n} \epsilon_{c-1}(t) \qquad (2,1.12)$$

$$\frac{d}{dt} A_0(t) = -d_c(t) + \epsilon_{c-1}(t) \qquad (2,1.13)$$

The set of equations (2,1.8) can be readily solved by generating functions. Define the generating function of the probabilities $Q_n(t)$ by

$$Q(z,t) = \sum_{n=c}^{N-1} z^n Q_n(t) \qquad (2,1.14)$$

which is convergent for $|z| \leq 1$

Then (2,1.8) becomes

$$\frac{d}{dt} Q(z,t) + [\lambda(1-z) + c\mu(1-\tfrac{1}{z})] Q(z,t) = -\lambda z^N Q_{N-1}(t) - c\mu z^{c-1} Q_c(t) + z^c d_c(t) \qquad (2,1.15)$$

This on using condition (2,1.10) becomes

$$\frac{d}{dt} Q(z,t) + [\lambda(1-z) + c\mu(1-\tfrac{1}{z})] Q(z,t) = d_N(t) z^N - z^{c-1} \epsilon_{c-1}(t) + z^c d_c(t) \qquad (2,1.16)$$

We now discuss the solutions of (2,1.12), (2,1.13) and (2,1.16) under two cases: (i) when $i < c$ and (ii) when $i \geq c$.

Case i: When the system starts initially with i failed units, $i < c$, we have $P_n(0) = \delta_{i,n}$ and therefore,

$$A_n(0) = \sum_{j=n}^{c-1} \binom{j}{n} \delta_{i,n} = \begin{cases} 0 & \text{for } n < i \\ \binom{i}{n} & \text{for } n \leq i \end{cases} \qquad (2,1.17)$$

On taking the Laplace transform of (2,1.12) and using the initial condition (2,1,17) we get

$$(n\mu+s)\bar{A}_n(s) = \lambda \bar{A}_{n-1}(s) - \binom{c}{n}\bar{d}_c(s) + \binom{c-1}{n}\bar{\epsilon}_{c-1}(s) + \binom{i}{n}, (n \leq i)$$

and (2,1.18)

$$(n\mu+s)\bar{A}_n(s) = \lambda \bar{A}_{n-1}(s) - \binom{c}{n}\bar{d}_c(s) + \binom{c-1}{n}\bar{\epsilon}_{c-1}(s), (n > i) \qquad (2,1.19)$$

In order to solve the difference equations (2,1.18) and (2,1.19), we define the following products

$$\phi_n(s) = \begin{cases} \prod_{r=0}^{n} \left[\frac{\lambda}{r\mu+s}\right] & \text{for } n \geq 0 \\ 1 & \text{for } n = -1 \end{cases} \qquad (2,1.20)$$

from which we have

$$(n\mu+s)\phi_n(s) = \lambda\, \phi_{n-1}(s)$$

On dividing the left hand side of (2,1.18) and (2,1.19) by $(n\mu+s)\ \phi_n(s)$ and the right hand side by $\lambda\ \phi_{n-1}(s)$ we obtain

$$\frac{\bar{A}_n(s)}{\phi_n(s)} = \frac{\bar{A}_{n-1}(s)}{\phi_{n-1}(s)} - \frac{\bar{d}_c(s)}{\lambda}\binom{c}{n}\frac{1}{\phi_{n-1}(s)} + \frac{\bar{\epsilon}_{c-1}(s)}{\lambda}\binom{c-1}{n}\frac{1}{\phi_{n-1}(s)} + \frac{1}{\lambda}\binom{i}{n}\frac{1}{\phi_{n-1}(s)} \qquad (2,1.21)$$

Now changing n to n - 1, n - 2, n - 3, . . . 0 and noting that $\bar{A}_{-1}(s) = 0$ by definition and adding all these equations including (2,1.21) we have

$$\frac{\bar{A}_n(s)}{\phi_n(s)} = -\frac{\bar{d}_c(s)}{\lambda}\sum_{l=0}^{n}\binom{c}{l}\frac{1}{\phi_{l-1}(s)} + \frac{\bar{\epsilon}_{c-1}(s)}{\lambda}\sum_{l=0}^{n}\binom{c-1}{l}\frac{1}{\phi_{l-1}(s)} + \frac{1}{\lambda}\sum_{l=0}^{n}\binom{i}{l}\frac{1}{\phi_{l-1}(s)}, \quad n < i \qquad (2,1.22)$$

and

$$\frac{\bar{A}_n(s)}{\phi_n(s)} = -\frac{\bar{d}_c(s)}{\lambda}\sum_{l=0}^{n}\binom{c}{l}\frac{1}{\phi_{l-1}(s)} + \frac{\bar{\epsilon}_{c-1}(s)}{\lambda}\sum_{l=0}^{n}\binom{c-1}{l}\frac{1}{\phi_{l-1}(s)} + \frac{1}{\lambda}\sum_{l=0}^{i}\binom{i}{l}\frac{1}{\phi_{l-1}(s)}, \quad n \geqslant i \qquad (2,1.23)$$

It may be noted that in (2,1.22) and (2,1.23) the maximum value that i can take is only (c - 1) and

$$\lambda\bar{P}_{c-1}(s) = \lambda\,\bar{A}_{c-1}(s) = \bar{d}_c(s) \qquad (2,1.24)$$

Changing n to c - 1 and using (2,1.24) in (2,1.23) and solving for $\bar{d}_c(s)$ there results

$$\bar{d}_c(s) = \left[\bar{\epsilon}_{c-1}(s) \sum_{l=0}^{c-1} \binom{c-1}{l} \frac{1}{\phi_{l-1}(s)} + \sum_{l=0}^{i} \binom{i}{l} \frac{1}{\phi_{l-1}(s)} \right] \left[\sum_{l=0}^{c} \binom{c}{l} \frac{1}{\phi_{l-1}(s)} \right]^{-1} \quad (2,1.25)$$

which we rewrite as

$$\bar{d}_c(s) = \bar{\epsilon}_{c-1}(s)\, G^*(s) + G_i^*(s) \quad (2,1.26)$$

where

$$G^*(s) = \left[\sum_{l=0}^{c-1} \binom{c-1}{l} \frac{1}{\phi_{l-1}(s)} \right] \left[\sum_{l=0}^{c} \binom{c}{l} \frac{1}{\phi_{l-1}(s)} \right]^{-1} \quad (2,1.27)$$

and

$$G_i^*(s) = \left[\sum_{l=0}^{i} \binom{i}{l} \frac{1}{\phi_{l-1}(s)} \right] \left[\sum_{l=0}^{c} \binom{c}{l} \frac{1}{\phi_{l-1}(s)} \right]^{-1} \quad (2,1.28)$$

Now, from (2,1.16), we have for the Laplace transform of the generating function $Q(z,t)$

$$\bar{Q}(z,s) = \frac{\bar{d}_N(s)\, z^N - \bar{\epsilon}_{c-1}(s)\, z^{c-1} + \bar{d}_c(s)\, z^c}{\lambda(1-z) + c\mu\left(1-\frac{1}{z}\right) + s} \quad (2,1.29)$$

Substituting the value of $\bar{d}_c(s)$ from (2,1.26) in (2,1.29) we obtain

$$\bar{Q}(z,s) = \frac{\bar{d}_N(s)\, z^N + z^{c-1}\, \bar{\epsilon}_{c-1}(s)\,(z\, G^*(s) - 1) + z^c\, G_i^*(s)}{\lambda(1-z) + c\mu(1-\frac{1}{z}) + s} \qquad (2,1.30)$$

Since $\bar{Q}(z,s)$ is analytic in z, the numerator of (2,1.30) should vanish at the zeros of the denominator. Let z_1 and z_2 be the zeros of the denominator, then

$$z_r^N\, \bar{d}_N(s) + z_r^{c-1}\, \bar{\epsilon}_{c-1}(s)\,(z_r\, G^*(s) - 1) + z_r\, G_i^*(s) = 0 \qquad (2,1.31)$$

$$r = 1, 2$$

from the last equation in (2,1.10) and the definition of $g_{i,N}(t)$, the density function of the TSF at (2,1.6), it is clear that

$$-\bar{d}_N(s) = \bar{g}_{i,N}(s) = \lambda\, \bar{P}_{i,N-1}(s) \qquad (2,1.32)$$

where $\bar{g}_{i,N}(s)$ is the Laplace transform of $g_{i,N}(t)$ Therefore, solving for $-\bar{d}_N(s)$ from the two equations in (2,1.31) and using (2,1.32) we obtain

$$\bar{g}_{i,N}(s) = \frac{(z_1 - z_2)\, G_i^*(s)}{(z_1^{N-c+1} - z_2^{N-c+1}) - z_1 z_2 (z_1^{N-c} - z_2^{N-c})\, G^*(s)}$$

$$(2,1.33)$$

Case ii: When the system starts with i failed units $i \geq c$, we have $Q_n(0) = \delta_{i,n}$. Therefore using (2,1.17), $A_n(0) = 0$ for $n \geq c$. Then (2,1.22) and (2,1.23) reduce to the single equation

$$\frac{\bar{A}_n(s)}{\phi_n(s)} = -\frac{1}{\lambda}\,\bar{d}_c(s)\sum_{l=0}^{n}\binom{c}{l}\frac{1}{\phi_{l-1}(s)} + \frac{1}{\lambda}\sum_{l=0}^{n}\binom{c-1}{l}\frac{1}{\phi_{l-1}(s)}\cdot\bar{\epsilon}_{c-1}(s)$$

Setting $n = c - 1$, using (2,1.24) and rearranging the terms, we obtain

$$\bar{d}_c(s) = \bar{\epsilon}_{c-1}(s)\, G^{*}(s) \tag{2,1.34}$$

From (2,1.16), the Laplace transform of the generating function for this case becomes,

$$\bar{Q}(z,s) = \frac{\bar{d}_N(s)\, z^{N} + \bar{\epsilon}_{c-1}(s)\left(z\, G^{*}(s) - 1\right) z^{c-1} + z^{i}}{\lambda(1-z) + c\mu\left(1-\frac{1}{z}\right) + s} \tag{2,1.35}$$

As in case (i) we make use of the analyticity condition on $\bar{Q}(z,s)$ to obtain

$$\bar{d}_N(s)\, z_r^{N} + z_r^{c-1}\,\bar{\epsilon}_{c-1}(s)\left(z_r\, G^{*}(s) - 1\right) + z_r^{i} = 0 \qquad r = 1, 2 \tag{2,1.36}$$

where z_r ($r = 1, 2$) are the zeros of the denominator of (2,1.35).

We obtain $\bar{g}_{i,N}(s)$ by solving for $-\bar{d}_N(s)$ from the two equations in (2,1.36). Whence

$$\bar{g}_{i,N}(s) = \frac{z_1^{\,i-c+1}\left(1 - z_2 G^*(s)\right) - z_2^{\,i-c+1}\left(1 - z_1 G^*(s)\right)}{z_1^{\,N-c+1}\left(1 - z_2 G^*(s)\right) - z_2^{\,N-c+1}\left(1 - z_1 G^*(s)\right)} \qquad (2,1.37)$$

Further, the values of z_1 and z_2 in (2,1.33) and (2,1.37) are given by

$$z_1, z_2 = \frac{(\lambda + c\mu + s) \pm \left[(\lambda + c\mu + s)^2 - 4c\lambda\mu\right]^{\frac{1}{2}}}{2\lambda} \qquad (2,1.38)$$

<u>A Particular case:</u> <u>When c = N</u>

This case arises when the total number of repair facilities is equal to the total number of units in the system, the one in use and the (N - 1) units in the spare pool. In this situation as soon as a unit fails, the repair on the failed one is started and the failed units do not wait in a line to get repaired. Now putting c = N in (2,1.33) we get

$$\bar{g}_{i,N}(s) = G_i^*(s) = \left[\sum_{l=0}^{i} \binom{i}{l} \frac{1}{\phi_{l-1}(s)}\right] \left[\sum_{l=0}^{N} \binom{N}{l} \frac{1}{\phi_{l-1}(s)}\right]^{-1} \qquad (2,1.39)$$

If the system starts initially with no failed units, i = 0. Then

$$\bar{g}_{0,N}(s) = \left[\sum_{l=0}^{N} \binom{N}{l} \frac{1}{\phi_{l-1}(s)}\right]^{-1} \qquad (2,1.40)$$

which is in agreement with Srinivasan's (1967) result. Denoting by the random variable T_u, the time to system failure of process, the moments of T_u can be obtained from the Laplace transform $\bar{g}_{i,N}(s)$ by successive differentiation with respect to s at $s=0$. Let $E_i(T_u)$ denote the expected value of T_u when the system initially starts with i failed units. Then

$$E_i(T_u) = \begin{cases} \dfrac{1}{c\mu(1-\rho)}\Big[-(N-c+1) + \dfrac{\rho}{1-\rho}\Big(\dfrac{1}{\rho^{N-c+1}} - 1\Big) \\ \quad + \dfrac{1}{\rho}\Big(\dfrac{1}{\rho^{N-c}} - 1\Big)\Big(\sum_{\ell=0}^{c-1}\binom{c-1}{\ell}\dfrac{1}{\phi_\ell}\Big) \\ \quad + \dfrac{1-\rho}{\rho}\Big(\sum_{\ell=1}^{c}\binom{c}{\ell}\dfrac{1}{\phi_{\ell-1}} - \sum_{\ell=1}^{i}\binom{i}{\ell}\dfrac{1}{\phi_{\ell-1}}\Big)\Big] & i < c \\ \\ \dfrac{1}{c\mu(1-\rho)}\Big[-(N-i) + \dfrac{\rho}{1-\rho}\Big(1 + \dfrac{1-\rho}{\rho}\sum_{\ell=0}^{c-1}\binom{c-1}{\ell}\dfrac{1}{\phi_\ell}\Big) \\ \quad \cdot \Big(\dfrac{1}{\rho^{N-c+1}} - \dfrac{1}{\rho^{i-c+1}}\Big)\Big] & i \geq c \end{cases}$$

(2, 1.41)

where

$$\phi_\ell = \begin{cases} \prod_{r=1}^{\ell} \left(\frac{\lambda}{r\mu} \right) & , \ \ell > 0 \\ 1 & , \ \ell = 0 \end{cases} \quad , \quad \rho = \frac{\lambda}{c\mu} \qquad (2,1\cdot42)$$

If we have only a single repair facility, then

c = 1 and

$$E_i(T_u) = \frac{1}{\mu(1-\rho)} \left[-(N-i) + \frac{1}{1-\rho} \left(\frac{1}{\rho^N} - \frac{1}{\rho^i} \right) \right] \qquad (2,1\cdot43)$$

As was mentioned in the last chapter, the expected TSF when the system starts initially with (N - 1) failed units, ie, $E_{N-1}(T_u)$ has a special significance. This is the average duration of time the system will be in the up-state after it has been restored through a repair completion from the down-state. This is given by

$$E_{N-1}(T_u) = \frac{1}{c\mu\,\rho^{N-c}} \left[\frac{1-\rho^{N-c}}{1-\rho} + \frac{1}{\rho} \sum_{\ell=0}^{c-1} \binom{c-1}{\ell} \frac{1}{\phi_\ell} \right] \qquad (2,1\cdot44)$$

The expression for the variance of T_u in the case when i > c is given by

$$\begin{aligned} Var_i(T_u) &= E_i(T_u^2) - \left[E_i(T_u) \right]^2 \\ &= -\frac{\rho}{1-\rho} \Big[E_i(T_u) \left(a_N'(0) + a_i'(0) \right) \\ &\qquad - \left(a_N''(0) - a_i''(0) \right) \Big] \end{aligned}$$

(2,1·45)

where $a_i(s)$ and $a_N(s)$ are the numerator and denominator of (2,1.37), $a'(s)$ and $a''(s)$ are the first and the second derivative of $a(s)$ and $a(0) = \lim_{s \to 0} a(s)$

And

$$a'_N(0) + a'_i(0) = \Big[\big((N-c+1) + (i-c+1) \big) \frac{1}{c\mu\rho}$$
$$- \frac{1}{c\mu\rho(1-\rho)} \left(\frac{1}{\rho^{N-c}} + \frac{1}{\rho^{i-c}} + 2 \right)$$
$$- \frac{1}{\rho\lambda} \left(\sum_{l=0}^{c-1} \binom{c-1}{l} \frac{1}{\Phi_l} \right) \left(\frac{1}{\rho^{N-c}} + \frac{1}{\rho^{i-c}} - 2 \right) \Big]$$

and

$$a''_N(0) - a''_i(0) = \Big[\big\{ (N-c+1)(N-c) - (i-c+1)(i-c) \big\} \frac{1}{(c\mu)^3 \rho (1-\rho)^2}$$
$$- \frac{2}{c\mu\lambda(1-\rho)} \left(\sum_{l=0}^{c-1} \binom{c-1}{l} \frac{1}{\Phi_l} \right) \Big\{ (N-i) \frac{1}{\rho}$$
$$+ (N-c+1) \frac{1}{\rho^{N-c+1}} - (i-c+1) \frac{1}{\rho^{i-c+1}}$$
$$- \left(\frac{1}{\rho^{N-c+1}} - \frac{1}{\rho^{i-c+1}} \right) \left(1 + \frac{1-\rho}{\rho} \sum_{l=1}^{c} \binom{c-1}{l} \frac{1}{\Phi_l} \right) \Big\}$$
$$- \frac{2}{(c\mu)^2 (1-\rho)^2} \left\{ \frac{1}{\rho^{N-c+1}} \left(N-c+1 - \frac{1}{1-\rho} \right) - \frac{1}{\rho^{i-c+1}} \left(i-c+1 - \frac{1}{1-\rho} \right) \right\}$$
$$- \frac{2}{\lambda\mu} \left(\frac{1}{\rho^{N-c+1}} - \frac{1}{\rho^{i-c+1}} \right) \sum_{l=0}^{c-1} \binom{c-1}{l} \frac{\zeta_l}{\Phi_l} \Big]$$

where

$$\zeta_l = \sum_{r=1}^{l} \frac{1}{r} \quad \text{and} \quad \zeta_0 = 0$$

We note that $Var_{N-1}(T_u)$ can be obtained from (2,1.45) by putting $i = N - 1$.

THE GENERAL PROCESS

In the previous section, we investigated the TSF process wherein the transitions from any state i to any other state j are permitted without an intermediate passage to the state N, when the system reaches the failed state. However, in the general process TSF periods alternate with the system down-time (SDT) periods during which the system is being restored to the operable state. As stated in chapter 1, when the system starts operating after restoration, there will be (N - 1) failed units in the system either undergoing repair or waiting to get repaired. We shall now proceed to relate the general process to the TSF process, thereby deriving the distribution of n(t) in the general process.

In a realisation of the general process, let us suppose that the new TSF periods begin at the sequence of random instants of time ($0 < t_1 < t_2 < \cdots$) where time is measured from an arbitrary initial instant. We call t_k the time of beginning of the k th TSF period and the interval (t_{k-1}, t_k) the k th 'renewal period'. Then the random variable $\tau_R(k) = t_k - t_{k-1}$ will be the length of the k th renewal period.

In the general process one has

$$\tau_R(k) = T_u(k) + T_d(k) \quad , (k \geq 2) \tag{2,1.46}$$

where $T_u(k)$ is the length of the TSF period that immediately follows the restoration of the system to the state N - 1 by a repair completion at t_{k-1} and $T_d(k)$ is the length of the SDT period that immediately follows this TSF period and preceeds restoration of the system at t_k. Since the length of a SDT period is that length of time during which repair completion takes place, it follows from the nature of the repair-time distribution that

$$D(t) = P_r\,[\,T_d(k) \le t\,] = 1 - e^{-c\mu t} \tag{2,1.47}$$

Now, we observe that $\{T_u(k)\},(k \ge 2)$ is a sequence of independent positive random variables identically distributed with the density function $g_{N-1,N}(t)$. Let $G_{N-1,N}(t)$ be the distribution function corresponding to $g_{N-1,N}(t)$ Then $G_{N-1,N}(t) = \int_0^t g_{N-1,N}(u)\,du$. Further, we observe that $\{T_d(k)\},(k \ge 1)$ is a sequence of independent positive random variables each having the distribution specified at (2,1.47). So, it follows from (2,1.46) that the sequence $\{\tau_R(k)\},(k \ge 2)$ of renewal periods forms a sequence of independent identically distributed positive random variables with the distribution function

$$R(t) = P_r\,[\,\tau_R(k) \le t\,] = G_{N-1,N}(t) * D(t) \tag{2,1.48}$$

the $*$ denoting the convolution operation. The above sequence of renewal periods constitute a 'renewal process' [C.f Smith (1954), Gaver (1959), Subba Rao (1967) Jaiswal (1965 b)] . It may be added that the sequence $\{\tau_R(k)\}$, $(k \geq 1)$ constitutes a 'modified renewal process' [C.f Cox (1962)] .

Because of the initial conditions (that at time t = o the process starts with i units in the failed state) we note that the distribution function of $\tau_R(1) = t_1$, the time of beginning of the first TSF period is given by

$$\begin{aligned} R(t,i) &= G_{i,N}(t) * D(t) & (i < N) \\ &= D(t) & (i = N) \end{aligned} \qquad (2,1\cdot 49)$$

where $G_{i,N}(t) = \int_0^t g_{i,N}(u)\,du$

Now we shall obtain the distribution function $\gamma_k(t,i)$ of t_k , the time of beginning of the k th TSF period taking into account the initial condition that the process started with i failed units at time t = 0 .

$$\begin{aligned} \gamma_k(t,i) &= P_r\,[\,t_k \leq t \,/\, n(o) = i\,] \\ &= R(t,i) * R^{(k-1)*}(t) \qquad (k \geq 1,\ i \leq N) \end{aligned} \qquad (2,1\cdot 50)$$

where $R^{k*}(t)$ is the k th iterated convolution of R(t) with itself and $R^{0*}(t)$ is interpreted as the unit step function at the origin. We shall call $\gamma_k(t,i)$ the renewal distribution.

Denoting the general process probabilities by script letters, let

$$\mathcal{P}_{i,n}(t) = P_r\left[n(t) = n \,/\, n(o) = i\right] \quad (0 \leq n, i \leq N) \tag{2,1.51}$$

These provide us the distribution of n(t) in the general process, that is, the probabilities that there are n units in the failed state in the system at time t given that the process started with i units in the failed state at time t = o. Now in order to obtain the probability that n(t) = n < N in the general process, given that at time t = o, n (o) = i, (i ≤ N) i.e. initially there are i failed units in the system we observe that either a) n(t) = n < N and n(t') < N ($0 \leq t' \leq t$) ie, that at time t the number of failed units present in the system is n and the first TSF period has not terminated, or b) n(t) = n < N and n(t') = N at least once in $(0 \leq t' \leq t)$; ie, that at time t the number of failed units present is n and at least one TSF period has elapsed. The probability of the event (a) is $P_{i,n}(t)$ and the probability of the event (b) is easily seen to be

$$\sum_{k=1}^{\infty} P_{N-1,n}(t) * \gamma_k(t,i)$$

Summing the probabilities of the mutually exclusive events (a) and (b) we obtain

$$\mathcal{P}_{i,n}(t) = P_{i,n}(t) + \sum_{k=1}^{\infty} P_{N-1,n}(t) * r_k(t,i) \;, (0 \le n, i < N) \quad (2,1.52)$$

$$\mathcal{P}_{i,N}(t) = G_{i,N}(t) * [1 - D(t)] + \sum_{k=1}^{\infty} G_{N-1,N}(t) * [1 - D(t)] * r_k(t,i) \quad (0 \le i < N) \quad (2,1.53)$$

and

$$\mathcal{P}_{N,n}(t) = \sum_{k=1}^{\infty} P_{N-1,n}(t) * r_k(t,N) \;, (0 \le n < N) \quad (2,1.54)$$

$$\mathcal{P}_{N,N}(t) = [1 - D(t)] + \sum_{k=1}^{\infty} G_{N-1,N}(t) * [1 - D(t)] * r_k(t,N) \quad (2,1.55)$$

After deriving the general process probabilities, we shall now proceed to investigate the ergodic properties of n(t) in the general process with the aid of some results in renewal theory.

ERGODIC PROPERTIES OF THE GENERAL PROCESS

The expression $r'(t,i) = \sum_{k=1}^{\infty} \frac{d}{dt} r_k(t,i)$ will be called the 'renewal density' [C.f Smith (1954), Gaver (1959)]. In words, $r'(t,i)\,dt + o(dt)$ is the probability that the TSF process begins in the interval $(t, t+dt)$. We observe that $r'(t,i)$ exists for all t since R(t) is a convolution of $G_{N-1,N}(t)$ with D(t), the latter being absolutely continuous.

Under broad conditions, the renewal density converges to a constant as $t \to \infty$. To this end, we make use of the following theorem of Smith (1954).

Theorem (W. L. Smith): If

(i) the reneal periods $\{ \tau_R(k) \}$ are non-negative and $E(\tau_R(k)) < \infty$

(ii) $\frac{d}{dt} R(t) \in L_{1+\delta}$ for some $\delta > 0$

(iii) $\frac{d}{dt} R(t)$ tends to zero as t tends to infinity

then

$$\lim_{t \to \infty} r'(t, i) = \frac{1}{E(\tau_R)} \qquad (2,1.57)$$

where $E(\tau_R)$ is the expected length of any but the first renewal period.

Referring to the definition of renewal periods, the expressions (2,1.7) - (2,1.8) and the results of TSF process, it can be seen that the conditions of the Smith's theorem are satisfied. Referring to the definition of τ_R and to the expression for the expected duration of TSF which starts with (N - 1 failed units in the system, we have

$$E(\tau_R) = \frac{1}{c\mu} \left[1 + \frac{1}{\rho^{N-c}} \left(\frac{1-\rho^{N-C}}{1-\rho} + \frac{1}{\rho} \sum_{\ell=0}^{c-1} \binom{c-1}{\ell} \frac{1}{\phi_\ell} \right) \right]$$

(2,1.58)

where ϕ_ℓ has been defined in (2,1.42). It can be seen that $E(\tau_R)$ is always positive and finite.

Usually in reliability theory, we define 'dependability ratio' d as the reciprocal of ρ , that is $d = \frac{1}{\rho}$. In the light of this (2,1.58) can be written as

$$E(\tau_R) = \frac{1}{c\mu}\left[1 + d\left(\frac{d^{N-c}-1}{d-1}\right) + d^{N-c+1}\sum_{\ell=0}^{c-1}\binom{c-1}{\ell}\frac{1}{\phi_\ell}\right]$$

We need the following lemma of Smith (1954) pp 14-15 to discuss the limiting form of $P_{i,n}(t)$ as $t \to \infty$.

<u>Lemma</u>: If K(t) is of bounded total variation and if $b(t)$ is any bounded function which tends to a limit λ as $t \to \infty$, then

$$b(t) * K(t) = \int_{-\infty}^{\infty} b(t-z)\, dK(z)$$

is bounded and tends to the limit $\lambda\left[K(+\infty) - K(-\infty)\right]$ as $t \to \infty$.

Now, using the above lemma, (2,1.57) and (2,1.52) - (2,1.53) by identifying $P_{N-1,n}(t)$ with K(t) and $\sum \gamma_k(t,i)$ with $b(t)$, it can be shown that the general process probabilities $P_{i,n}(t)$ tend to limits independent of initial conditions as $t \to \infty$.

Let $\{p_n\}, n=0,1,\cdots N$ denote these steady-state probabilities and $p(z)=\sum_{n=0}^{N} p_n z^n$, their generating function. Then

$$p_n = \lim_{t\to\infty} P_{i,n}(t) = \begin{cases} \dfrac{1}{E(\tau_R)}\displaystyle\int_0^\infty P_n(t)\,dt \;, & n<c \\[2ex] \dfrac{1}{E(\tau_R)}\displaystyle\int_0^\infty Q_n(t)\,dt \;, & n\geq c \end{cases} \tag{2,1.59}$$

and

$$p_N = \lim_{t\to\infty} P_{i,N}(t) = \frac{1}{c\mu\, E(\tau_R)}$$

$$= \left[1+\frac{1}{\rho^{N-c}}\left(\frac{1-\rho^{N-c}}{1-\rho}+\frac{1}{\rho}\sum_{\ell=0}^{c-1}\binom{c-1}{\ell}\frac{1}{\phi_\ell}\right)\right]^{-1} \tag{2,1.60}$$

where $P_n(t)$ and $Q_n(t)$ are defined by the equations (2,1.7) and (2,1.8). Now the generating function $p(z)$ can be written as

$$p(z) = \sum_{n=0}^{c-1} p_n z^n + \sum_{n=c}^{N} q_n z^n \tag{2,1.61}$$

The first term of (2,1.61) on using (2,1.59) becomes

$$\sum_{n=0}^{c-1} p_n z^n = \frac{1}{E(\tau_R)}\sum_{n=0}^{c-1} z^n \int_0^\infty P_n(t)\,dt \tag{2,1.62}$$

Expressing $P's$ in terms of A's by means of inverse transform of (2,1.11), namely

$$P_j(t) = \sum_{n=j}^{c-1}(-1)^{n-j}\binom{n}{j} A_n(t) \;,\quad j=0,1,2,\cdots c-1 \tag{2,1.63}$$

and changing the order of summation and simplifying (2,1.62) reduces to

$$\sum_{n=0}^{c-1} p_n z^n = \frac{1}{E(\tau_R)} \sum_{n=0}^{c-1} (-1)^n (1-z)^n \int_0^\infty A_n(t)\,dt \qquad (2,1.64)$$

We observe that the integrals in (2,1.64) are obtained as the limit of $\bar{A}_n(s)$ in (2,1.22) as $s \to 0$. Using the limiting values and the relation (2,1.59) we get

$$\sum_{n=0}^{c-1} p_n z^n = \left(\frac{1}{\rho}\right)^{N-c} p_N \Big[\frac{1}{\rho} \sum_{\ell=0}^{c-1} \binom{c-1}{\ell} \frac{1}{\phi_\ell} + \sum_{n=1}^{c-1} (-1)^n (1-z)^n \Big(\frac{c}{n} \phi_{n-1} \sum_{\ell=1}^{c-1} \binom{c-1}{\ell} \frac{1}{\phi_\ell} \Big) \Big] \qquad (2,1.65)$$

Similarly, the second term of $p(z)$ in (2,1.61) using (2,1.60) and (2,1.59) in (2,1.35) and taking limit as $s \to 0$, reduces to

$$\sum_{n=c}^{N} p_n z^n = \frac{\bar{Q}(z,0)}{E(\tau_R)} + p_N z^N = p_N z^c \left(\frac{1}{\rho}\right)^{N-c} \frac{(\rho z)^{N-c+1} - 1}{\rho z - 1} \qquad (2,1.66)$$

Combining (2,1.65) and (2,1.66) and simplifying, we obtain the generating function $p(z)$ as

$$p(z) = \frac{\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda z}{\mu}\right)^{n} + \frac{1}{c!}\left(\frac{\lambda z}{\mu}\right)^{c} \frac{(\rho z)^{N-c+1} - 1}{\rho z - 1}}{\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^{n} + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^{c} \frac{\rho^{N-c+1} - 1}{\rho - 1}} \qquad (2,1.67)$$

SOME RELIABILITY CHARACTERISTICS

Besides the expectation of TSF and the expectation of SDT, other reliability characteristics we are interested in, are the same as in chapter 1, namely, the long-run availability of the system, the expected number of system failures in a given interval of time (o, t) and the interval reliability of the system.

The Long-run availability $= 1 - p_N$

$$= 1 - \frac{1}{c\mu E(\tau_R)}$$

$$= \frac{\frac{1}{\rho^{N-c}}\left(\frac{1-\rho^{N-c}}{1-\rho} + \frac{1}{\rho}\sum_{\ell=0}^{c-1}\binom{c-1}{\ell}\frac{1}{\phi_\ell}\right)}{1 + \frac{1}{\rho^{N-c}}\left(\frac{1-\rho^{N-c}}{1-\rho} + \frac{1}{\rho}\sum_{\ell=0}^{c-1}\binom{c-1}{\ell}\frac{1}{\phi_\ell}\right)} = \frac{E_{N-1}(T_u)}{E(\tau_R)} \qquad (2,1.68)$$

From (2,1.68), it is observed that the knowledge of expected duration of TSF and the expected duration of SDT alone are sufficient to study the long-run availability of the system.

If, instead of a standby redundant system, we consider a system with all the N units in parallel redundancy with multiple repair facilities, the long-run availability of the system from finite queueing theory [see Wohl (1966)] is given by

long-run availability $= 1 - p_N$

where

$$p_N = \frac{N!\,c^c}{c!}\cdot\rho^N \Big/ \left[\sum_{\ell=0}^{c-1}\frac{N!\,c^\ell}{(N-\ell)!\,\ell!}\cdot\rho^\ell + \sum_{\ell=c}^{N}\frac{N!\,c^c}{(N-\ell)!\,c!}\cdot\rho^\ell\right] \qquad (2,1.69)$$

Now, rewriting the value of p_N in (2,1.60), we have

$$p_N = \frac{c^c}{c!}\,\rho^N \Big/ \left[\sum_{\ell=0}^{c-1}\frac{c^\ell}{\ell!}\,\rho^\ell + \frac{c^c}{c!}\cdot\rho^c\,\frac{1-\rho^{N-c+1}}{1-\rho}\right] \qquad (2,1.70)$$

We note that, after cancelling N! in the numerator and the denominator of (2,1.69), each term of the denominator of (2,1.69) is less than the corresponding term of the denominator of (2,1.70), thereby leaving p_N in (2,1.69) greater than the p_N in (2,1.70). Therefore, we conclude that for a given value of traffic intensity ρ and a given number of repair facilities the long-run availability of a single-unit system with (N - 1) spares is greater than that of a system with N units in parallel redundancy. The same conclusion applies to the standby redundant case wherein spares deteriorate in storage, as this model, when $\lambda_1 = \lambda_2$ corresponds to the parallel redundant case as seen in chapter 1, section 3.

We shall next obtain the expected number of system failures in a given interval of time (o, t) as $t \longrightarrow \infty$. This is obtained by the use of the expressions in (1,1.52) and (1,1.53). These involve the knowledge of the mean and the second moment of the recurrence time τ_R to the state N. $E(\tau_R)$ has already been obtained in (2,1.58). As $E(\tau_R^2) = E(T_d^2) + E_{N-1}(T_u^2)$ where $E(T_d^2) = \frac{2}{(c\mu)^2}$ and $E_{N-1}(T_u^2) = Var_{N-1}(T_u) + [E_{N-1}(T_u)]^2$ we can evaluate $E(\tau_R^2)$ by applying the result (2,1.44) and that obtained by changing i into (N - 1) in (2,1.45).

The interval reliability $R(x,t)$ has also been defined earlier in chapter 1, section 1 and its limits have been obtained in (1,1.56). The upper and lower bounds of $R(x,t)$ as $t \longrightarrow \infty$ as given in (1,1.56) involve only $E_{N-1}(T_u)$ and $E(\tau_R)$ which are given by (2,1.44) and (2,1.58).

A comparison similar to that mentioned for long-run availability of the system, between the standby redundant system and the parallel redundant system can be made using the expected number of system failures in a given interval of time (o, t) and the interval reliability of the system would lead to similar conclusions as arrived at earlier.

A NUMERICAL ILLUSTRATION

To facilitate computations, the equation (2,1.44) giving the expected time-to-system failure can be rewritten as

$$c\mu E_{N-1}(T_u) = \frac{\frac{1-\rho^{N-c}}{1-\rho} e^{-c\rho} \frac{(c\rho)^c}{c!} + \sum_{r=0}^{c-1} e^{-c\rho} \frac{(c\rho)^r}{r!}}{e^{-c\rho} \frac{(c\rho)^c}{c!} \cdot \rho^{N-c}} \qquad (2,1.71)$$

We observe that in (2,1.71), the probabilities are Poisson probabilities. The individual terms and cumulated terms of Poisson distribution can be readily read from Molina's Table (1942). From (2,1.69) putting $E(\tau_R) = \frac{1}{c\mu} + E_{N-1}(T_u)$ we have

$$\text{Long-run availability} = 1 - p_N = \frac{c\mu E_{N-1}(T_u)}{1 + c\mu E_{N-1}(T_u)} \qquad (2,1.72)$$

where $c\mu E_{N-1}(T_u)$ is the ratio of the expected time-to-system failure to the expected system down-time.

TABLE 2.1

Ratio of Expected Time-to-System Failure to Expected System Down-time for various λ/μ, Number of repair facilities c and Number of Units N

No. of repair facilities c	$\frac{\lambda}{\mu} = c\rho$	$c\mu E_{N-1}(T_u) = \frac{\text{Expected Time-to-System Failure}}{\text{Expected System Down Time}}$					
		N=2	N=4	N=6	N=8	N=10	N=C
c = 1	0.2	30.00	779.98	$1.95.10^{4}$	$4.88.10^{5}$	$1.22.10^{7}$	5.00
	0.4	8.75	63.44	405.24	2541.44	$1.57.10^{4}$	2.5
	0.6	4.44	16.79	51.08	146.34	410.94	1.67
	0.8	2.81	7.21	14.07	24.80	41.57	1.25
	1.0	2.00	4.00	6.00	8.00	20.00	1.00
	1.2	1.53	2.59	3.33	3.84	4.19	0.83
c = 3	0.2		$1.37.10^{4}$	$3.00.10^{6}$	$6.69.10^{8}$	$1.49.10^{11}$	915.03
	0.4		1050.76	$5.50.10^{4}$	$3.36.10^{6}$	$1.90.10^{8}$	138.75
	0.6		252.22	6335.62	$1.58.10^{5}$	$3.96.10^{6}$	49.44
	0.8		96.80	1375.53	$1.93.10^{4}$	$2.71.10^{5}$	24.84
	1.0		48.05	445.32	$4.03.10^{3}$	$3.63.10^{4}$	15.00
	1.2		27.84	182.79^{3}	1.15.10	$7.20.10^{3}$	10.14
c = 5	0.2			$1.14.10^{7}$	$7.10.10^{8}$	$4.44.10^{12}$	$4.55.10^{5}$
	0.4			$2.19.10^{5}$	$3.41.10^{7}$	$5.34.10^{9}$	$1.75.10^{4}$
	0.6			$2.34.10^{4}$	$1.63.10^{6}$	$1.13.10^{8}$	$2.81.10^{3}$
	0.8			$5.12.10^{3}$	$1.99.10^{5}$	$7.77.10^{6}$	$8.14.10^{2}$
	1.0			$1.63.10^{3}$	$4.08.10^{4}$	$1.62.10^{6}$	$3.25.10^{2}$
	1.2			$6.66.10^{2}$	$1.16.10^{4}$	$2.01.10^{5}$	$1.59.10^{2}$
c = 8	0.2					$1.60.10^{11}$	$1.0.10^{8}$
	0.4					$1.10.10^{10}$	$2.5.10^{7}$
	0.6					$6.44.10^{8}$	$2.5.10^{6}$
	0.8					$5.00.10^{7}$	$5.0.10^{5}$
	1.0					$7.12.10^{6}$	$1.1.10^{5}$
	1.2					$1.39.10^{6}$	$3.1.10^{4}$
c = 10	0.2						$5.0.10^{8}$
	0.4						$1.0.10^{8}$
	0.6						$2.5.10^{7}$
	0.8						$1.0.10^{7}$
	1.0						$2.5.10^{6}$
	1.2						$1.0.10^{5}$

TABLE 2.2.

Long-run Availability of the System for various $\frac{\lambda}{\mu}$, Number of repair facilities c and the number of units N

Number of Repair Facilities c	$\frac{\lambda}{\mu} = c\rho$	Long-run Availability = $\dfrac{c\mu E_{N-1}(T_{11})}{1 + c\mu E_{N-1}(T_{11})}$					
		N = 2	N = 4	N = 6	N = 8	N = 10	N = c
c = 1	0.2	.967742	.998720	.999949	.999997	.999999	0.833333
	0.4	.897436	.984472	.997538	.999607	.999937	0.714286
	0.6	.814815	.943820	.980769	.993211	.997572	0.625046
	0.8	.736842	.878048	.933333	.961240	.976525	0.555555
	1.0	.666667	.800000	.857142	.888689	.909090	0.500000
	1.2	.603174	.720670	.768518	.792960	.807321	0.453552
c = 3	0.2		.999926	.999999	.999999	.999999	.990708
	0.4		.999049	.999983	.999999	.999999	.992841
	0.6		.996050	.999936	.999999	.999999	.980174
	0.8		.989775	.999273	.999948	.999999	.961240
	1.0		.979591	.997759	.999751	.999972	.937500
	1.2		.965277	.994556	.999132	.999861	.910217
c = 5	0.2			.999999	.999999	.999999	.999997
	0.4			.999995	.999999	.999999	.999943
	0.6			.999957	.999999	.999999	.999644
	0.8			.999805	.999995	.999999	.998773
	1.0			.999387	.999975	.999999	.996932
	1.2			.998501	.999913	.999995	.993742
c = 8	0.2					.999999	.999999
	0.4					.999999	.999999
	0.6					.999999	.999999
	0.8					.999999	.999998
	1.0					.999999	.999991
	1.2					.999999	.999968
c = 10	0.2						.999999
	0.4						.999999
	0.6						.999999
	0.8						.999999
	1.0						.999999
	1.2						.999990

The values of $c\mu E_{N-1}(T_u)$ and long-run availability of the system are tabulated for the values of $\frac{\lambda}{\mu}$ varying from 0.2 to 1.2, the number of channels c varying from 1 to 10 and the number of units N varying from 1 to 10 (ie, the number of spare units vary from 0 to 9) in tables 2.1 and 2.2 respectively. An inspection of the tables reveals that the ratio $c\mu E_{N-1}(T_u)$ and the long-run availability of the system increase with the reduction of $\frac{\lambda}{\mu}$, with the increase of the number of spare units and with the increase in the number of repair facilities.

As mentioned earlier, it is of interest to know the number of spares and the number of repair facilities required to achieve a pre-assigned long-run availability for a given value of $\frac{\lambda}{\mu}$. For example, from table 2.2 it is seen that for $\frac{\lambda}{\mu} = 0.6$, to achieve a long-run availability of 0.98, we require 6 units (ie., 5 spares) and a single repair facility or 3 units (ie., 2 spares) and 3 repair facilities. Naturally, the choice between the number of spares and the additional repair facilities will depend upon their relative costs. In the above example, if the cost of providing additional repair facilities is higher than the cost of procuring three additional spares, then the obvious choice will be to provide more number of spares rather than repair facilities.

CHAPTER 3

RELIABILITY OF A STANDBY REDUNDANT SYSTEM

WITH PRIORITY REPAIR POLICIES

A SEMI-MARKOV PROCESS APPROACH

CHAPTER 3

RELIABILITY OF A STANDBY REDUNDANT SYSTEM UNDER PRIORITY REPAIR POLICY- A SEMI-MARKOV PROCESS (SMP) APPROACH

INTRODUCTION

Until now we have considered a system with only one type of units (units being components of an equipment, sub-systems or equipments themselves) with standby redundancy or parallel redundancy. In this and the next chapter will be discussed the reliability characteristics of a system with two types of units, each type having units either in standby redundancy or parallel redundancy with a single repair facility to repair the failed units. This chapter will be confined to the study of standby redundant system while the next will be devoted to the study of parallel redundant systems. There are a number of systems having more than one type of units. For example in a surviellance system, a radar may constitute a type 1 unit and a computer working in conjunction with the radar may constitute the type 2 unit. There arises a necessity for imposition of some priority repair policy for repair of the failed units when there are more than one type of units in the system and having different degrees of criticality. The units on failure are repaired by a single repair facility and after repair completion, they are either kept as standby in the case of standby redundant systems or put into operation in the case of parallel redundant systems. A unit not having any additional supporting units is always put into operation after repair completion. The question arises as to which type of units should be assigned priority and which priority repair policy is to be followed. This will naturally depend upon how these policies affect the reliability characteristics of the system such as time to system failure, long-run availability and so on.

So far the concept of priority allocation for repair has been confined only to the study of Queues, on which a large literature is available. Jaiswal (1968) among others has made an elaborate study on Priority Queues and has discussed various queueing models under different priority disciplines. In this and the next chapter, an attempt has been made to study the reliability characteristics of some of the systems applying two types of priority repair policies, namely, the pre-emptive resume and the head-of-the-line priority repair policies. Under the preemptive resume priority repair policy as soon as a unit with higher priority for repair fails, it pre-empts the lower priority unit being repaired and is taken for repair at once. The preempted unit is taken back for repair only when there are no higher priority units awaiting repair and its repair is resumed from the point where it was preempted. On the other hand, under the head-of-the-line priority repair policy, once the lower priority unit is taken for repair, the repair on it is completed before taking up for repair the higher priority units failed during this time. That is the higher priority units wait till the repair completion of the lower priority unit. As there are only two types of units in the system, we call the higher priority units "priority units" and the lower priority units "ordinary units".

Discussing the allocation of priorities for repair, Morse (1958) has pointed out that the imposition of priorities increases the average number of failed ordinary units present and makes their average waiting time longer before getting repaired, whereas it usually reduces the number of failed units in the system and delay for repair of the priority unit. If the over-riding requirement is to reduce the delay for repair of one particular class of units, then this class should be assigned priority. This is particularly profitable if the non-urgent units take longer to be repaired on the average. Throughout this and the next chapter, the discussions are carried out by assigning priority for repair of the type 1 unit and the type 2 units are assumed to be non-priority units. Our aim in these two chapters is to study the effect of allocation of the two types of priority repair policies mentioned earlier on the reliability characteristics of the system such as time to first failure of the system, long-run availability of the system etc. and this chapter is devoted to the study of these characteristics in a standby redundant systems.

This chapter comprises of the following sections.

SECTION 1 a: SERIES CONNECTED STANDBY REDUNDANT SYSTEM WITH PREEMPTIVE RESUME PRIORITY REPAIR POLICY;

SECTION 1 b: PARALLEL CONNECTED STANDBY REDUNDANT SYSTEM WITH PREEMPTIVE RESUME PRIORITY REPAIR POLICY;

SECTION 2 a: SERIES CONNECTED STANDBY REDUNDANT SYSTEM WITH HEAD-OF-THE-LINE PRIORITY REPAIR POLICY;

SECTION 2 b: PARALLEL CONNECTED STANDBY REDUNDANT SYSTEM WITH HEAD-OF-THE-LINE PRIORITY REPAIR POLICY.

In these four sections, we investigate the reliability characteristics of a system with two types of units, the first type having only one unit and the second type having two units - one unit operating and other a standby. These two types are connected together either in series or in parallel. When they are connected in series, the system is called "Series Connected Standby Redundant System" (fig.3.0 a) and when they are connected in parallel, it is called "Parallel connected Standby Redundant System" (fig 3.0 b).

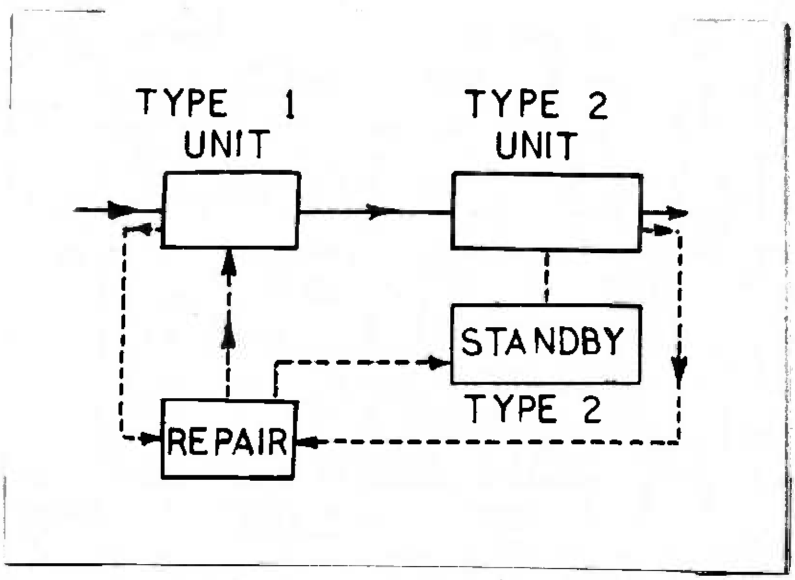


Fig.3.0 a

Series Connected Standby Redundant System

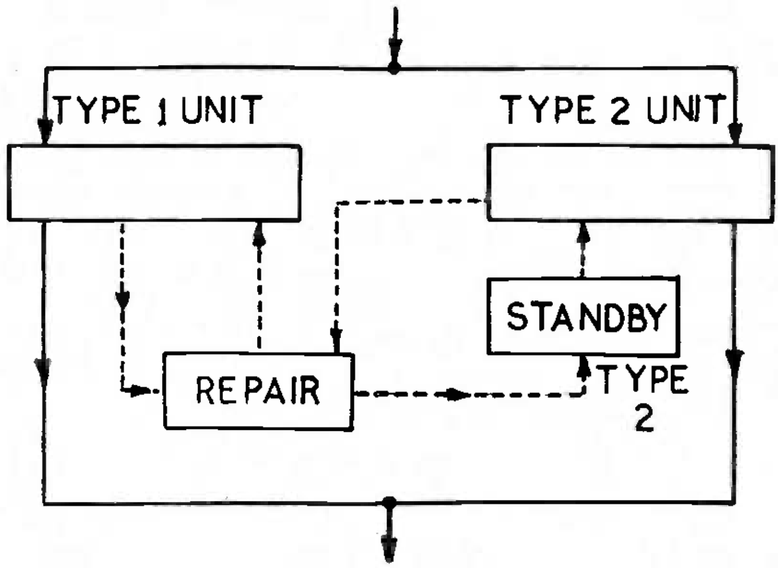


Fig. 3.0 b

Parallel Connected Standby Redundant System

The operations of repair and replacement of the failed units are shown by dotted lines in the figures. We assume that the failure time distribution of the individual unit as well as the repair time distributions are exponential. In sections 1(a) and 1(b), we have derived the reliability characteristics such as the expected time-to-system failure, long-run availability of the system, the expected number of failures of the system in a given interval of time (0, t) and interval reliability under the assumption that the type 1 unit having no standby has a preemptive resume priority for repair over the type 2 unit having one standby. The analysis covers the two different cases one when the type 1 unit and the type 2 unit are connected in series and the other when they are connected in parallel. A similar treatment is given in sections 2(a) and 2(b) for the case when the type 1 unit is assigned head-of-the-line priority for repair. The analysis of these four sections is based on the semi-Markov process (SMP) technique.

SECTION 1 (a)

SERIES CONNECTED STANDBY REDUNDANT SYSTEM WITH PRE-EMPTIVE RESUME PRIORITY REPAIR

The two types of units being connected in series in this case, the system failure occurs whenever the failure of the type 1 unit or of both the type 2 units occurs. As the type 1 unit has been assigned preemptive resume priority for repair, it is taken for repair as soon as it fails. If at this time, a type 2 unit (non-priority unit) is already under repair, its repair is interrupted and the type 1 unit is taken up for repair. The repair on the type 2 unit is resumed from the point where it has been interrupted when the type 1 unit leaves the repair facility. The stochastic model of this sytem is based on the following assumptions and definitions.

1. Failure of Individual Units: Let $F_1(t)$ and $F_2(t)$ be the distribution ~~of~~ functions of failure times of type 1 and type 2 unit respectively. It is assumed that they are negative exponential with mean rates λ_1 and λ_2 so that $F_1(t) = 1 - e^{-\lambda_1 t}$ and $F_2(t) = 1 - e^{-\lambda_2 t}$ $t \geq 0$ and $\lambda_1, \lambda_2 \geq 0$

2. Replacement Process: Replacement arises only in the case of type 2 unit. As soon as the operating type 2 unit fails, the standby unit is put into operation immediately. It is assumed that the replacement is instantaneous so that the system is operative immediately without any replacement time or switchover time.

3. Repair Process: The failed units are repaired by a single repair facility, the repair policy being preemptive resume priority for type 1 unit and 'repair in the order of failure' for the type 2 units. The distribution functions of the repair time $G_1(t)$ and $G_2(t)$ of type 1 and type 2 unit respectively are assumed to be negative exponential with parameters μ_1 and μ_2 i,e, $G_1(t) = 1 - e^{-\mu_1 t}$ and $G_2(t)$ $1 - e^{-\mu_2 t}$, $t \geq 0, \mu_1, \mu_2 \geq 0$. It is also assumed

that the repair process is independent of all other processes.

The definitions of time to system failure (TSF) and the system down time (SDT) are the same as in chapter 1, section 1. During the TSF period, the system is in 'up' state and during the SDT, the system is restored to 'up' state by the repair completion of the failed units. Whenever the system failure occurs by the failure of type 1 unit, the system is restored to 'up' state by the repair completion of that unit and whenever the system failure is caused by the failure of type 2 unit when already a unit of the same type is in the failed state, it is restored to operation by completing the repair on this type of unit. Now we define the following random variables.

$N^{(1)}(t)$: Number of type 1 units (priority units) in the failed state at time t being repaired or waiting for repair;

$N^{(2)}(t)$: Number of type 2 units (non-priority units) in the failed state at time t being repaired or waiting for repair;

Then the state of the system at any time t is defined by the ordered pair of numbers (n_1, n_2) denoting the value of $N^{(1)}(t)$ and $N^{(2)}(t)$ at time t. For various values of n_1 and n_2 , these states are redesignated by a state number i within square brackets, i.e. [i] varying over the sequence of non-negative integers 0, 1, 2, 3, ...

The definitions of distribution of first passage time, time to system failure, recurrence time to any state [i] follow the same lines as in chapter 1, section 1. Hence, $G_{i,j}$ (t) represents the distribution function of time to first passage from the state [i] to state [j] at time t, $G_{i,i}$ (t) represents the distribution function of recurrence time to the state [i] and if the state [d] denotes the failed state of the system, then $G_{i,d}$ (t) denotes the distribution function of the time to system failure starting initially in the state [i] .

We now proceed to investigate the general process probabilities, distribution function of the time to system failure and recurrence time to state [i] by evaluating the Laplace Stieltjes transform of the state probabilities viewing the whole process as a Semi-Markov Process (SMP).

GENERAL PROCESS PROBABILITIES

It may be seen that the process under consideration which is a continuous time Markov Process and which can be viewed as a Semi-Markov Process in which the distribution function of wait-time between successive transition is negative exponential [Pyke (1961a, 1961b) and Barlow (1962b)]. In order to have a complete description of the process, it is necessary to enumerate all the possible states of the process. As the two types of units are connected in series to each other, the failure of the type 2 unit cannot occur, when once the failure of the type 1 unit has caused the system failure. Similarly, when the system failure is caused by the failure of both the type 2 units, the type 1 unit cannot fail. Therefore, the ordered pair (n_1, n_2) representing the state of the system at any time t can only be any of (0, 0), (0,1), (0,2), (1,0) and (1,1). These states are given the state designations as follows.

Serial No.	State of Process at time t	State designation
1	(0, 0)	[0]
2	(0, 1)	[1]
3	(0, 2)	[2]
4	(1, 0)	[3]
5	(1, 1)	[4]

From this, it can be seen that the system will be in the failed state when it is at time t in any of the states [2], [3], and [4].

In order to identify this process with the SMP, we start with the definition of the transition of the process. A transition is said to take place at an instant of occurance of a failure of a unit or repair completion of the failed one. Let us denote the successive transitions by the sequence of non-negative integers $\{ n, n = 0, 1, 2, \ldots \}$. Following Pyke (1961a) we note by J_n, the state designation of the system at the time of the n^{th} transition. Let X_n denote the time between $(n - 1)^{th}$ and n^{th} transition and let $S_n = \sum_{i=1}^{n} X_i$. It can be verified that the two-dimensional stochastic process (J,S) is a Markov Process and the J-process is a Markov-Chain. Moreover, Pyke defines another process $Z_t = J_{N(t)}$ which represents the state of the system at any instant of time t, where $N(t) = \operatorname{Sup}\{ n \geqslant 0;\ S_n \leqslant t \}$. The process Z_t has transitions only at the instants of failure or repair completions. Between transitions, the value of Z_t is its value at the last transition [Fabens (1959)]. The process Z_t is a Semi-Markov Process and the Markov-Chain underlying the SMP is ergodic since all states communicate with each other.

Before proceeding to obtain the expressions for Laplace Stieltjes transform (LST) of the general process probabilities, we require the definition of some basic quantities. Let the matrix $Q(t) = (Q_{ij}(t))$ denote the matrix of transition distributions, whose elements $Q_{ij}(t)$ represent the probability of occurance of a transition from the state [i] at $t = 0$, to a state [j] before time t i.e. in time $\leq t$. The transition probability matrix $P = (p_{ij})$ is obtained from $Q(t)$ as $Q_{ij}(\infty) = p_{ij}$ where p_{ij} represent the probability of occurance of a transition from the state [i] to state [j]. If there are m states in the process, then (i) $Q_{ij}(t) = 0$ for $t \leq 0$ and (ii) $\sum_{j=1}^{m} Q_{ij}(+\infty) = 1, (1 \leq i \leq m)$ For each i and every real t set $H_i(t) = \sum_{j=1}^{m} Q_{ij}(t)$

which by (ii) is a distribution function representing the probability that the process will leave the state [i] in time $\leq t$. In the particular case of the model of this section, totoal number of states is five and i and j vary from 0 to 4. Moreover, $Q_{ij}(t) = p_{ij} F_{ij}(t)$ where $F_{ij}(t)$ is the conditional probability that the transition will occur in time $\leq t$ given that the transition is into the state [j]. $F_{ij}(t)$ is a conditional wait time distribution. Naturally, the d.f. $F_{ij}(t)$ depends on the two states [i] and [j] between which the transition is being made. Further, if $p_{ij} > 0$, $F_{ij}(t) = Q_{ij}(t)/p_{ij}$ while if $p_{ij} = 0$ set $F_{ij}(t) = U_0(t)$, a d.f. having unit step at $t = 0$. In fact, when $p_{ij} = 0$, $F_{ij}(t)$ may be chosen arbitrarily.

Since the process is a continuous time parameter Markov-Process, the wait time d.f. $F_{ij}(t)$ is exponential, that is $F_{ij}(t) = 1 - e^{-\lambda_i t}$, $t \geq 0$ for constants $\lambda_i > 0$ for every i.

The non-zero elements of the matrix of transition distributions $Q_{ij}(t)$ associated with the SMP Z_t in this case are given by

$$Q_{0,1}(t) = \int_0^t e^{-\lambda_1 t} . \lambda_2 e^{-\lambda_2 t} dt = \frac{\lambda_2}{\lambda_1 + \lambda_2} [1 - e^{-(\lambda_1+\lambda_2)t}]$$

$$Q_{0,3}(t) = \int_0^t e^{-\lambda_2 t} . \lambda_1 e^{-\lambda_1 t} dt = \frac{\lambda_1}{\lambda_1 + \lambda_2} [1 - e^{-(\lambda_1+\lambda_2)t}]$$

$$Q_{1,0}(t) = \int_0^t e^{-(\lambda_1+\lambda_2)t} . \mu_2 e^{-\mu_2 t} dt = \frac{\mu_2}{\lambda_1 + \lambda_2 + \mu_2} [1 - e^{-(\lambda_1+\lambda_2+\mu_2)t}]$$

$$Q_{1,2}(t) = \int_0^t e^{-(\lambda_1+\mu_2)t} . \lambda_2 e^{-\lambda_2 t} dt = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \mu_2} [1 - e^{-(\lambda_1+\lambda_2+\mu_2)t}]$$

$$Q_{1,4}(t) = \int_0^t e^{-(\lambda_2+\mu_2)t} . \lambda_1 e^{-\lambda_1 t} dt = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \mu_2} [1 - e^{-(\lambda_1+\lambda_2+\mu_2)t}]$$

$$Q_{2,1}(t) = \int_0^t \mu_2 e^{-\mu_2 t} dt = 1 - e^{-\mu_2 t}$$

$$Q_{3,0}(t) = \int_0^t \mu_1 e^{-\mu_1 t} dt = 1 - e^{-\mu_1 t}$$

and finally

$$Q_{4,1}(t) = \int_0^t \mu_1 e^{-\mu_1 t}\, dt = 1 - e^{-\mu_1 t}$$

For example, $Q_{1,0}$ (t) represents the probability of occurence of a transition from state [1] at t = 0 to state [0] in time $\leq$ t. This is nothing but the probability that in duration of time $\leq$ t, the repair of the non-priority unit is completed and none of the operating units fails during that period. Similarly other $Q_{i,j}$ (t)'s are obtained.

The LST's of these $Q_{i,j}(t)$'s are given by

$$\hat{Q}_{0,1}(s) = \frac{\lambda_2}{\lambda_1+\lambda_2+s} = \alpha_1 \ (say), \quad \hat{Q}_{0,3}(s) = \frac{\lambda_1}{\lambda_1+\lambda_2+s} = \alpha_2$$

$$\hat{Q}_{1,0}(s) = \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2+s} = \alpha_3, \quad \hat{Q}_{1,2}(s) = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2+s} = \alpha_4$$

$$\hat{Q}_{1,4}(s) = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu_2+s} = \alpha_5, \quad \hat{Q}_{2,1}(s) = \frac{\mu_2}{\mu_2+s} = \alpha_6$$

and $$\hat{Q}_{3,0}(s) = \frac{\mu_1}{\mu_1+s} = \alpha_7 = \hat{Q}_{4,1}(s)$$

It may be observed that the transition probability p_{ij} is obtained by letting $s \to 0$ in $\hat{Q}_{i,j}(s)$ as $p_{ij} = Q_{ij}(+\infty) = \lim_{s \to 0} \hat{Q}_{ij}(s)$. Therefore, the non-zero elements of the transition probability matrix $P = (p_{ij})$ are obtained by letting $s \to 0$ in the matrix $\hat{Q}(s)$. Denote by a's the corresponding values of α's at $s = 0$. Then,

$$p_{0,1} = \frac{\lambda_2}{\lambda_1+\lambda_2} = -a_1 \,,\; p_{0,3} = \frac{\lambda_1}{\lambda_1+\lambda_2} = -a_2 \,,\; p_{1,0} = \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2} = -a_3$$

$$p_{1,2} = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2} = -a_4 \,,\; p_{1,4} = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu_2} = -a_5 \,,\; p_{2,1} = 1 = -a_6$$

and

$$p_{3,0} = 1 = -a_7 = p_{4,1}$$

In the general process, the transition from any state [i] to any other state [j] is possible and let $P_{i,j}(t)$ give the probability that the process is in state [j] at an instant of time t, starting in the state [i] at t = 0. That is

$$P_{ij}(t) = P_r [Z_t = [j] \,/\, Z_0 = [i]]$$

The relationship between the general process probabilities $P_{i,j}(t)$ and the transition distribution $Q_{i,j}(t)$ can be obtained [Pyke (1961b)] by simple probabilistic arguments. Considering the state of the process at time t after the n^{th} transition and summing the probabilities over all n, n = 1 to ∞, Pyke obtains the matrix equation giving the relationship between

$P_{i,j}(t)$ and $Q_{i,j}(t)$. From this relationship $\hat{P}_{ij}(s)$ can be solved when $\hat{Q}_{i,j}(s)$ is known. This matrix equation is given by Theorem 1.

Theorem 1: (Pyke). Let $\hat{Q}(s) = (\hat{Q}_{ij}(s))$ be matrix of LST of transition distributions and let $\hat{H}(s) = (\delta_{ij}\hat{H}_i(s))$ Then,

$$\hat{P}(s) = (\hat{P}_{i,j}(s)) = (I - \hat{Q}(s))^{-1}.(I - \hat{H}(s)) \qquad (3,1a.1)$$

$\hat{P}(s)$ being defined over $(0, \infty)$ and δ_{ij} is the Kronecker delta and the matrix I is the unit matrix.

From (3,1a.1), the expression for $\hat{P}_{i,j}(s)$ becomes

$$\hat{P}_{ij}(s) = (I - \hat{Q}(s))^{-1}_{i,j}\ (1 - \hat{H}_j(s)) \qquad (3,1a.2)$$

where $(I - \hat{Q}(s))^{-1}_{i,j}$ represents (i, j) the element of matrix $(I - \hat{Q}(s))^{-1}$, that is, the element of the row corresponding to the state [i] and column corresponding to the state [j] in $(I - \hat{Q}(s))^{-1}$.

A similar relationship exists between $P_{i,j}(t)$ and the distribution of the first passage times $G_{i,j}(t)$ This is also obtained [Pyke (1961b)] through simple probabilistic arguments by enumerating ways by which the process can reach state [j] at time t, starting in state [i] initially. These relationships are given by Theorem 2.

Theorem 2: (Pyke). For $t \geq 0$, $s > 0$

$$P_{ij}(t) = G_{ij}(t) * P_{jj}(t) + \delta_{ij}\left[1 - H_i(t)\right], \tag{3,1a.3}$$

$$\left.\begin{aligned} \hat{P}_{ij}(s) &= \hat{G}_{ij}(s)\,\hat{P}_{jj}(s) && , i \neq j \\ \hat{P}_{jj}(s) &= \frac{1 - \hat{H}_j(s)}{1 - \hat{G}_{jj}(s)} && , i = j \end{aligned}\right\} \tag{3,1a.4}$$

where $*$ denote the convolution operation.

From theorem 1 and theorem 2, it is clear that explicit expressions for $\hat{P}_{i,j}(s)$ and $\hat{G}_{i,j}(s)$ are in terms of the (i,j) th element of the inverse of the matrix $(I - \hat{Q}(s))$. Hence $(I - \hat{Q}(s))$ is the <u>basic matrix</u> of the process and is given by

$$(I - \hat{Q}(s)) = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & \alpha_1 & 0 & \alpha_2 & 0 \\ \alpha_3 & 1 & \alpha_4 & 0 & \alpha_5 \\ 0 & \alpha_6 & 1 & 0 & 0 \\ \alpha_7 & 0 & 0 & 1 & 0 \\ 0 & \alpha_7 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{3,1a.5}$$

where the numbers given along the border of the matrix denote the state numbers. Just as the fundamental matrix $(I - P)^{-1}$ plays an important role in Markov-Chain theory, $(I - \hat{Q}(s))^{-1}$ plays an important role in SMP theory.

DISTRIBUTION OF TIME TO SYSTEM FAILURE

The relationship between $P_{ij}(t)$ and the first passage time distribution $G_{i,j}(t)$ has been obtained in the equation (3,1a.4). If the state $[j]$ is the 'down' state of the system, then $G_{i,j}(t)$ is nothing but the distribution of time to system failure. This distribution can be found by considering the system failure states as absorbing states [Barlow and Proschan (1965) p.135] and thus making the SMP under study an absorbing one since the Marko-Chain underlying the process becomes an absorbing Markov-Chain. As we are interested in obtaining the distribution of TSF through any one of the three states [2], [3] and [4]. we make these states as absorbing states. This is done by making

$$
\begin{aligned}
p_{ij} &= 1 \qquad , \; i = j \\
&= 0 \qquad , \; i \neq j \, , \; i = 2,3,4 \, ; \; j = 0,1,2,3,4
\end{aligned}
$$

in the transition probability matrix P. Also when the state [j] is an absorbing one, we define

$$
\begin{aligned}
F_{jj}(t) &= 0 \quad , \quad 0 \leq t < \infty \\
&= 1 \quad , \quad t = \infty
\end{aligned}
$$

which leads to the definition

$$
\begin{aligned}
H_{j}(t) &= 0 \quad , \quad 0 \leq t < \infty \\
&= 1 \quad , \quad t = \infty
\end{aligned}
$$

Because of this definition, the transition distribution $Q_{jj}(t)$ corresponding to the absorbing state [j] becomes zero and its LST $\hat{Q}_{jj}(s)$ vanishes.

When the states [2] , [3] and [4] are made absorbing states, the basic matrix $I - \hat{Q}(s)$ of the SMP is denoted by $I - \hat{Q}_0(s)$ and is given by

$$I - \hat{Q}_0(s) = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & \alpha_1 & 0 & \alpha_2 & 0 \\ \alpha_3 & 1 & \alpha_4 & 0 & \alpha_5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (3, 1a.6)$$

As we are interested in the LST of the combined distribution of TSF through any one of the states [2] , [3] and [4] , we can obtain this by lumping these three states as a single absorbing state denoted by [d] [Kemeny and Snell (1960) p.45] . Now denoting the new basic matrix as $I - \hat{Q}_d(s)$, we obtain from $I - \hat{Q}_0(s)$

$$I - \hat{Q}_d(s) = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & d \end{array} \\ \begin{array}{c} 0 \\ 1 \\ d \end{array} & \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ \alpha_3 & 1 & \alpha_4+\alpha_5 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (3, 1a.7)$$

This is obtained from $I - \hat{Q}_0(s)$ by lumping elements of the rows and the columns corresponding to the absorbing states when $I - \hat{Q}_o(s)$ is in the canonical form. If it is not in the canonical form, it can always be brought into the following form

$$\begin{array}{cc} k & m-k \\ \left[\begin{array}{c|c} I & O \\ \hline R & Q \end{array}\right] \end{array} \begin{array}{c} k \\ m-k \end{array} \quad \text{or} \quad \begin{array}{cc} m-k & k \\ \left[\begin{array}{c|c} Q & R \\ \hline O & I \end{array}\right] \end{array} \begin{array}{c} m-k \\ k \end{array}$$

by suitable renumbering of the states. Till now, this technique has been used { Kemeny and Snell (1960) and Barlow and Proschan (1965) } to find the expected number of visits to state [j] starting in the state [i] before reaching the state [k] and the mean first passage time only from the transition probability matrix P. In this section, the same technique has been applied to the basic matrix of the SMP, viz, $I - \hat{Q}(s)$ to obtain the LST of the first passage time distributions and distributions of TSF.

When the state [j] becomes an absorbing one, it is seen from the relationships (3,1a.4) & (3,1a.2) that

$$\hat{G}_{ij}(s) = \left(I - \hat{Q}_o(s) \right)^{-1}_{i,j} \qquad (3,1a.8)$$

where $I - \hat{Q}_o(s)$ is the basic matrix of absorbing SMP. From this we obtain $\hat{G}_{ij}(s)$, the LST of the distribution function of TSF, as the elements of the column corresponding to state [j] in the matrix $\left(I - \hat{Q}_o(s) \right)^{-1}$ for all the non-absorbing states [i]. Therefore, $\hat{G}_{o,d}(s)$

and $\hat{G}_{1,d}(s)$, the LSTs of the TSF are obtained as the elements of column corresponding to the state [d] and the row states [0] and [1] in the matrix $\left(I - \hat{Q}_d(s)\right)^{-1}$ which is given by

$$\left(I - \hat{Q}_d(s)\right)^{-1} = \frac{1}{\left|I - \hat{Q}_d(s)\right|} \begin{array}{cc} \begin{array}{ccc} 0 & 1 & d \end{array} & \\ \begin{bmatrix} 1 & -\alpha_1 & \alpha_1(\alpha_4+\alpha_5)-\alpha_2 \\ -\alpha_3 & 1 & \alpha_2\alpha_3-(\alpha_4+\alpha_5) \\ 0 & 0 & 1-\alpha_1\alpha_3 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ d \end{array} \end{array} \qquad (3,1a.9)$$

where $\left|I - \hat{Q}_d(s)\right| = 1-\alpha_1\alpha_3$ denotes the determinant of the matrix $\left(I - \hat{Q}_d(s)\right)$. Thus we have

$$\hat{G}_{0,d}(s) = \frac{\alpha_1(\alpha_4+\alpha_5)-\alpha_2}{1-\alpha_1\alpha_3} \qquad (3,1a.10)$$

and

$$\hat{G}_{1,d}(s) = \frac{\alpha_2\alpha_3-(\alpha_4+\alpha_5)}{1-\alpha_1\alpha_3} \qquad (3,1a.11)$$

The moments of these distributions can be derived by differentiating $\hat{G}_{i,d}(s)$ with respect to s at $s=0$. Now, let the random variable T_u denote the time to system failure and let $E_{i,d}(T_u)$ and $E_{i,d}(T_u^2)$ denote the mean and the second moment of $G_{i,d}(t)$, the distribution of TSF, Then

$$E_{i,d}(T_u) = \left[\frac{\frac{d}{ds}[D_r - N_r]}{D_r} \right]_{s=0} \qquad (3,1a.12)$$

and

$$E_{i,d}(T_u^2) = \left[\frac{2\, E_{i,d}(T_u) \cdot \frac{d}{ds}(D_r) + \frac{d^2}{ds^2}(N_r - D_r)}{D_r} \right]_{s=0} \qquad (3,1a.13)$$

where N_r and D_r stand for the numerator and denominator in the expression for $\hat{G}_{i,d}(s)$. Thus we obtain

$$E_{o,d}(T_u) = \frac{\frac{1}{\lambda_1+\lambda_2}\left(1 + \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2}\right)}{1 - \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2}} \qquad (3,1a.14)$$

$$E_{1,d}(T_u) = \frac{\frac{1}{\lambda_1+\lambda_2+\mu_2}\left(1 + \frac{\mu_2}{\lambda_1+\lambda_2}\right)}{1 - \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2}} \qquad (3,1a.15)$$

and defining $\eta_j = \int_0^\infty t\, dH_j(t)$ and $\eta_j^{(2)} = \int_0^\infty t^2\, dH_j(t)$ the second moments can be written as

$$E_{o,d}(T_u^2) = \frac{2\, E_{o,d}(T_u)\, C_{o,1} + C_{o,2}}{1 - a_1 a_3}, \qquad (3,1a.16)$$

where a's stand for the values of α's at $s=0$ and

$$C_{o,1} = a_1 a_3 \left(\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1+\lambda_2+\mu_2} \right)$$

$$C_{o2} = 2\left[\eta_o^2 - a_1 \eta_1^2 - \eta_o \eta_1 a_1 \right]$$

Similarly

$$E_{1,d}(T_u^2) = \frac{2\,E_{1,d}(T_u)\,C_{01} + C_{12}}{1 - a_1 a_3} \qquad (3,1a.17)$$

where $\quad C_{12} = 2\,[\,\eta_1^2 - a_3\eta_0^2 - a_3\eta_0\eta_1\,]$

$$\eta_0 = \frac{1}{\lambda_1+\lambda_2},\ \eta_1 = \frac{1}{\lambda_1+\lambda_2+\mu_2},\ \eta_2 = \frac{1}{\mu_2}\ \text{and}\ \eta_3 = \frac{1}{\mu_1} = \eta_4 \qquad (3,1a.18)$$

However, the expression for $E_{i,d}(T_u)$ can alternatively be obtained from a knowledge of the transition probability matrix P and η_j, the expectation of the unconditional wait time in the state [j] , j varying over non-absorbing states. This method does not assume a knowledge of the distribution of TSF. When [d] is an absorbing state P can be written in the form

$$P = \begin{array}{c|cc} & 0\ \ 1 & d \\ \hline 0,1 & Q & R \\ d & O & I \end{array}$$

Using a result due to Barlow, [Barlow and Proschan (1965) Theorem 2.5 p.135)] we obtain the mean time to system failure starting in the state [i] to be

$$E_{i,d}(T_u) = \sum_{j=0}^{1} m_{ij}\,\eta_j \quad , \quad i = 0,1 \qquad (3,1a,19)$$

where

$$(m_{ij}) = (I - Q)^{-1}$$

DISTRIBUTION OF RECURRENCE TIME TO STATE [j]

From the relationship (3,1a.4) between $\hat{P}_{ij}(s)$ and $\hat{G}_{ij}(s)$ and the relationship (3,1a.1) between $\hat{P}_{ij}(s)$ and $(I - \hat{Q}(s))^{-1}_{ij}$ we obtain

$$1 - \hat{G}_{jj}(s) = \frac{1 - \hat{H}_j(s)}{(I - \hat{Q}(s))^{-1}_{jj}\ (1 - \hat{H}_j(s))} = \frac{1}{(I - \hat{Q}(s))^{-1}_{jj}} \qquad (3,1a.20)$$

where $\hat{G}_{jj}(s)$ is the LST of distribution of recurrence time to state [j] . Now this is in terms of the (j,j) th element of $(I - \hat{Q}(s))^{-1}$ where j stands for the state designation [j] . Denote by Δ_{jj}, the cofactor of (j,j) th element in $(I - \hat{Q}(s))^{-1}$ and by D_j , the value of Δ_{jj} at $s = 0$. Then (3,1a.14) reduces to

$$1 - \hat{G}_{jj}(s) = \frac{|I - \hat{Q}(s)|}{\Delta_{jj}} \qquad (3,1a.21)$$

and hence the LST of recurrence time distribution to the states [2] , [3] and [4] are obtained from

$$1 - \hat{G}_{22}(s) = \frac{|I - \hat{Q}(s)|}{\Delta_{22}}$$

$$1 - \hat{G}_{33}(s) = \frac{|I - \hat{Q}(s)|}{\Delta_{33}}$$

and

$$1-\hat{G}_{44}(s) = \frac{|I-\hat{Q}(s)|}{\Delta_{44}}$$

where the determinant $|I-\hat{Q}(s)| = (1-\alpha_2\alpha_7)(1-\alpha_6\alpha_4-\alpha_5\alpha_7)-\alpha_1\alpha_3$

$$\Delta_{22} = (1-\alpha_2\alpha_7)(1-\alpha_5\alpha_7)-\alpha_1\alpha_3$$

$$\Delta_{33} = (1-\alpha_4\alpha_6-\alpha_5\alpha_7)-\alpha_1\alpha_3$$

and

$$\Delta_{44} = (1-\alpha_2\alpha_7)(1-\alpha_6\alpha_4)-\alpha_1\alpha_3$$

The moments of the recurrence time distribution can be obtained by successive differentiation of (3,1a.21) with respect to s and letting $s \to 0$. Let the random variable τ_R denote the reccurence time and let $E_{jj}(\tau_R)$ and $E_{jj}(\tau_R^2)$ denote respectively the expectation and the second moment of the recurrence time to the state [j] , then

$$E_{jj}(\tau_R) = \left[\frac{\frac{d}{ds}|I-\hat{Q}(s)|}{\Delta_{jj}}\right]_{s=0} \qquad (3,1a.22)$$

$$E_{jj}(\tau_R^2) = \left[\frac{2\,E_{jj}(\tau_R)\frac{d}{ds}\Delta_{jj} + \frac{d^2}{ds^2}|I-\hat{Q}(s)|}{\Delta_{jj}}\right]_{s=0} \qquad (3,1a.23)$$

Observing that at $s=0$, the sum of the row elements in the matrix $I-\hat{Q}(s)$ is zero. And noting that $D_0 = -a_3$, $D_1 = -a_1$ $D_2 = a_1a_4$, $D_3 = a_2a_3$ and $D_4 = a_1a_5$ we obtain

$$E_{22}(\tau_R) = \frac{\sum_{k=0}^{4} \eta_k D_k}{D_2} \qquad (3,1a.24)$$

$$E_{33}(\tau_R) = \frac{\sum_{k=0}^{4} \eta_k D_k}{D_3} \qquad (3,1a.25)$$

$$E_{44}(\tau_R) = \frac{\sum_{k=0}^{4} \eta_k D_k}{D_4} \qquad (3,1a,26)$$

and the second moments

$$E_{22}(\tau_R^2) = \frac{2\, E_{22}(\tau_R)\, C_{22} + C_{21}}{D_2} \qquad (3,1a,27)$$

$$E_{33}(\tau_R^2) = \frac{2\, E_{33}(\tau_R)\, C_{23} + C_{21}}{D_3} \qquad (3,1a.28)$$

and

$$E_{44}(\tau_R^2) = \frac{2\, E_{44}(\tau_R)\, C_{24} + C_{21}}{D_4} \qquad (3,1a.29)$$

where

$$C_{22} = \eta_0 (D_0 + a_2 a_4) + \eta_1 (D_1 - D_2) + \eta_3 (D_3 + a_2 a_4) + \eta_4 D_4$$

$$C_{23} = \eta_0 a_1 a_3 + \eta_1 (1 - a_2 a_3) - \eta_2 a_4 - \eta_4 a_5$$

$$C_{24} = \eta_0 (a_2 a_7 + a_1 a_3) + \eta_1 (D_2 + a_1 a_3) + \eta_2 D_2 + \eta_3 a_2 a_7$$

$$c_{21} = \sum_{k=0}^{4} \eta_k^{(2)} D_k + 2\eta_0 \left[\eta_1 (D_1 + D_0 - 1) + \eta_2 (D_2 + a_4) \right.$$

$$\left. + \eta_3 D_3 + \eta_4 (D_4 + a_5) \right] + 2\eta_1 \left[\eta_2 D_2 + \eta_3 (D_3 + a_2) + \eta_4 D_4 \right]$$

$$+ 2\eta_2 \eta_3 (D_2 + a_4) + 2\eta_3 \eta_4 (D_4 + a_5)$$

and

$$\eta_0^{(2)} = \frac{2}{(\lambda_1 + \lambda_2)^2}, \quad \eta_1^{(2)} = \frac{2}{(\lambda_1 + \lambda_2 + \mu_2)^2}, \quad \eta_2^{(2)} = \frac{2}{\mu_2^2}, \quad \eta_3^{(2)} = \frac{2}{\mu_1^2} = \eta_4^{(2)}$$

So far we have discussed the reliability characteristics viz., mean time to system failure and the mean recurrence time. Now, we shall obtain an expression for the long-run availability of the system.

LONG-RUN AVAILABILITY OF THE SYSTEM

Since the general process is ergodic, let p_j denote the steady state probability of the state $[j]$. Then $p_j = \lim_{t \to \infty} P_{ij}(t)$ is independent of the initial state $[i]$. The system is 'down' when it reaches the state $[2]$, $[3]$, or $[4]$. Therefore, when system continues to operate over fairly a long time, the proportion of time the system will be in the 'down' state is given by $p_2 + p_3 + p_4$. Let the random variable T_d denote duration of down-time of the system and let $E(T_d; T)$ denote its expectation in an interval (o, T). when T is large

$$E(T_d, T) \sim (p_2 + p_3 + p_4).T \qquad (3,1a.30)$$

From Smith's theorems

$$p_j = \lim_{t \to \infty} P_{ij}(t) = \frac{\eta_j}{E_{jj}(\tau_R)}$$

$$= \frac{D_j \eta_j}{\sum_{k=0}^{4} D_k \eta_k} \qquad (3,1a.31)$$

Therefore, by (3,1a.24) an explicit expression for the expected duration of system down-time in a large interval (O,T) is given by

$$E(T_d; T) \sim \left(\frac{D_2 \eta_2 + D_3 \eta_3 + D_4 \eta_4}{\sum_{k=0}^{4} D_k \eta_k} \right).T \qquad (3,1a.32)$$

Since the long-run availability of the system is the probability that the system is in the up-state when it operates over a long period of time, it is given by $1 - (p_2 + p_3 + p_4)$. Therefore,

$$\text{Long-run Availability} = 1 - (p_2 + p_3 + p_4)$$

$$= 1 - \frac{\sum_{k=2}^{4} D_k \eta_k}{\sum_{k=0}^{4} D_k \eta_k}$$

$$= \frac{D_0 \eta_0 + D_1 \eta_1}{\sum_{k=0}^{4} D_k \eta_k} \qquad (3,1a.33)$$

EXPECTED NUMBER OF SYSTEM FAILURES IN THE INTERVAL (0, t)

Another quantity of interest from the theory of Renewal process (R P) and Markov Renewal Process (MRP) is the expected number of times the process visits a given state in (0, t). This has an important application in reliability studies, when the given state is a system 'down' state. Then this gives the expected number of system failures due to that state in time (0, t). Let the state [j] denote the system down state and let $N_{ij}(t)$ denote the number of failures of the system in (0, t) starting initially in the state [i]. Define for $t \geq 0$ and $s > 0$

$$M_{ij}(t) = E\left[N_{ij}(t) / Z_0 = i \right] \qquad (3,1a.34)$$

$$\hat{M}_{ij}(s) = \int_0^\infty e^{-st} \, d\, M_{ij}(t) \qquad (3,1a.35)$$

Also set $M(t) = (M_{ij}(t))$ and $\hat{M}(s) = (\hat{M}_{ij}(s))$. This is called the Renewal function of the process. $M_{ij}(t)$

gives the expected number of system failures due to the state [j] in (0,t) given that initially the process is in state [i] . From theorem 5.2 of Pyke (1961b) it follows

$$\hat{M}(s) = \left(I - \hat{Q}(s)\right)^{-1} - I \qquad \text{on } (0,\infty) \qquad (3,1a.36)$$

where $\left(I - \hat{Q}(s)\right)^{-1}$ is already known explicitly.

Next, we shall consider the asymptotic value of the renewal function. Using the theorem of Smith [See Smith (1954), Cox (1962), Barlow (1962b)] the expected number of system failures due to the state [j] in the interval (0,t) as $t \to \infty$ is obtained as

$$M_{ij}(t) = \frac{t}{E_{jj}(\tau_R)} + \frac{E_{jj}(\tau_R^2)}{2\left[E_{jj}(\tau_R)\right]^2} - \frac{E_{ij}(T_u)}{E_{jj}(\tau_R)} + o(1) \qquad (3,1a.37)$$

and

$$M_{jj}(t) = \frac{t}{E_{jj}(\tau_R)} + \frac{E_{jj}(\tau_R^2)}{2\left[E_{jj}(\tau_R)\right]^2} - 1 + o(1) \qquad (3,1a.38)$$

Corresponding to the 'modified' and 'ordinary' renewal process respectively. Since the states [2] , [3] and [4] constitute the system failure states, expected number of total failures in the interval (0,t) as $t \to \infty$ is given by the sum of $M_{02}(t)$, $M_{03}(t)$ and $M_{04}(t)$.

INTERVAL RELIABILITY

In chapter 1, section 1, an expression for the interval reliability has been obtained when there was only one 'down' state of the system. In this section, a more general expression for the interval reliability has been obtained when there are a number of states constituting the system 'down' state. Let there be m states in the system and let the state $[\ell]$ $(\ell = k+1, k+2, \ldots m)$ represent the $(m-k)$ 'down' states. Then as explained earlier, all the $(m-k)$ 'down' states can be lumped into a single 'down' state $[d]$. Let $P_{ij}(y,t)$ denote the probability that at time t the system is in the state $[j]$, starting in the state $[i]$ initially and the elapsed time since last transition into this state $[j]$ is y. Then the interval reliability $R(x,t)$ is given by

$$R(x,t) = \sum_{j=1}^{k} \int_0^t P_{ij}(o,t-y)\left[1 - G_{j,d}(y+x)\right]dy \qquad (3,1a.39)$$

Since the process enters one of the 'up' states $[j]$ at an instant $t-y$ and continues to be 'up' during an interval of length $y+x$. This can also be written alternatively

$$R(x,t) = \sum_{j=1}^{k} \left[1 - G_{j,d}(t+x)\right] + \sum_{j=1}^{k} \int_0^t \left[1 - G_{j,d}(t-y+x)\right] dy\, M_{ij}(y)$$

$$(3.1a.40)$$

where $d_y M_{ij}(y)$ denote the probability of occurence of the state [j] in time $(y, y+dy)$

The limit of this quantity as t approaches infinity follows from Smith's theorem and is given by

$$\lim_{t \to \infty} R(x,t) = \sum_{j=1}^{k} \frac{\int_x^\infty [1 - G_{jd}(y)]\,dy}{E_{jj}(\tau_R)} \qquad (3,1a.41)$$

where the summation extends over all the 'up' states of the system and the Renewal function is assumed to be non-lattice.

It can easily be seen that the upper and the lower bounds of the limiting interval reliability are obtained as

$$\sum_{j=1}^{k} \frac{E_{jd}(T_u) - x}{E_{jj}(\tau_R)} \leq \lim_{t \to \infty} R(x,t) \leq \sum_{j=1}^{k} \frac{E_{jd}(T_u)}{E_{jj}(\tau_R)} \qquad (3,1a.42)$$

SECTION 1 (b)

PARALLEL CONNECTED STANDBY REDUNDANT SYSTEM WITH PREEMPTIVE RESUME PRIORITY REPAIR

In this system, type 1 and type 2 units are connected in parallel acting as parallel redundant units. Type 1 and type 2 units can accomplish the assigned task independently and the use of such systems are not uncommon in the sophisticated equipments used in modern warfare, communication systems and space research. For example, type 1 and type 2 units may constitute two types of computers capable of carrying out the same task connected in parallel or two types of sub-assemblies connected in parallel forming part of a major equipment or weapon system.

In this section, we consider the case where the type 2 unit has a standby. The system will continue to function even when the type 1 unit fails or all the type 2 units fail and the failure of the system occurs only when all the type 1 and type 2 units are in the failed state simultaneously. As in the previous case, when the units fail, type 1 unit is assigned preemptive resume priority for repair.

The assumptions regarding the failure process of the individual unit, replacement process, repair process and definitions of the TSF, SDT, state of the process, distributions of TSF and recurrence times are the same as in section 1 (a). As the description and analysis of the process also is almost the same as in the previous case, only differences wherever they occur will be mentioned and the final results given.

GENERAL PROCESS PROBABILITIES

As the system failure occurs when all the units of both the types have failed, the process passes through the state (1, 2) and this constitutes the system 'down' state. The different states and their state designations are:

Serial No.	State of the process at tim t	State designation
1	(0, 0)	[0]
2	(0, 1)	[1]
3	(0, 2)	[2]
4	(1, 0)	[3]
5	(1, 1)	[4]
6	(1, 2)	[5]

Treating the process as a SMP, the non-zero elements of Q(t), the matrix of transition distributions are

$$Q_{0,1}(t) = \frac{\lambda_2}{\lambda_1+\lambda_2}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right], \quad Q_{0,3}(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$Q_{1,0}(t) = \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2}\left[1 - e^{-(\lambda_1+\lambda_2+\mu_2)t}\right], \quad Q_{1,2}(t) = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2}\left[1 - e^{-(\lambda_1+\lambda_2+\mu_2)t}\right]$$

$$Q_{1,4}(t) = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu_2}\left[1-e^{-(\lambda_1+\lambda_2+\mu_2)t}\right], \quad Q_{2,1}(t) = \frac{\mu_2}{\lambda_1+\mu_2}\left[1-e^{-(\lambda_1+\mu_2)t}\right]$$

$$Q_{2,5}(t) = \frac{\lambda_1}{\lambda_1+\mu_2}\left[1-e^{-(\lambda_1+\mu_2)t}\right], \quad Q_{3,0}(t) = \frac{\mu_1}{\mu_1+\lambda_2}\left[1-e^{-(\mu_1+\lambda_2)t}\right]$$

$$Q_{3,4}(t) = \frac{\lambda_2}{\mu_1+\lambda_2}\left[1-e^{-(\mu_1+\lambda_2)t}\right], \quad Q_{4,1}(t) = \frac{\mu_1}{\mu_1+\lambda_2}\left[1-e^{-(\mu_1+\lambda_2)t}\right]$$

$$Q_{4,5}(t) = \frac{\lambda_2}{\lambda_2+\mu_1}\left[1-e^{-(\lambda_2+\mu_1)t}\right], \quad Q_{5,2}(t) = 1-e^{-\mu_1 t}$$

As in the previous case denoting the LST's of the $Q_{i\,j}(t)$'s by α's we have

$$\hat{Q}(s) = \frac{\lambda_2}{\lambda_1+\lambda_2+s} = -\alpha_1, \quad \hat{Q}_{0,3}(s) = \frac{\lambda_1}{\lambda_1+\lambda_2+s} = -\alpha_2$$

$$\hat{Q}_{1,0}(s) = \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_3, \quad \hat{Q}_{1,2}(s) = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_4$$

$$\hat{Q}_{1,4}(s) = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_5, \quad \hat{Q}_{2,1}(s) = \frac{\mu_2}{\lambda_1+\mu_2+s} = -\alpha_6$$

$$\hat{Q}_{2,5}(s) = \frac{\lambda_1}{\lambda_1+\mu_2+s} = -\alpha_7 \ , \ \hat{Q}_{3,0}(s) = \frac{\mu_1}{\mu_1+\lambda_2+s} = -\alpha_8 = \hat{Q}_{4,1}(s)$$

$$\hat{Q}_{3,4}(s) = \frac{\lambda_2}{\mu_1+\lambda_2+s} = -\alpha_9 = \hat{Q}_{4,5}(s) \ , \ \hat{Q}_{5,2}(s) = \frac{\mu_1}{\mu_1+s} = -\alpha_{10}$$

Since all the states communicate with each other, the SMP is ergodic and the basic matrix is obtained as

$$I - \hat{Q}(s) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & \alpha_1 & 0 & \alpha_2 & 0 & 0 \\ \alpha_3 & 1 & \alpha_4 & 0 & \alpha_5 & 0 \\ 0 & \alpha_6 & 1 & 0 & 0 & \alpha_7 \\ \alpha_8 & 0 & 0 & 1 & \alpha_9 & 0 \\ 0 & \alpha_8 & 0 & 0 & 1 & \alpha_9 \\ 0 & 0 & \alpha_{10} & 0 & 0 & 1 \end{bmatrix} \end{array}$$

As we have seen in section 1(a), the first passage time distribution, recurrence time distribution and the Renewal function depend on the inverse of the matrix $I - \hat{Q}(s)$. This can be easily obtained by partitioning $I - \hat{Q}(s)$ and obtaining the inverse of the partitioned matrix. Denoting the partitions of $I - \hat{Q}(s)$ as α, β, γ and δ and the corresponding partitions of $\left(I - \hat{Q}(s) \right)^{-1}$ as A, B, C and D we have, if

$$I - \hat{Q}(s) = \left[\begin{array}{ccc|ccc} 1 & \alpha_1 & 0 & \alpha_2 & 0 & 0 \\ \alpha_3 & 1 & \alpha_4 & 0 & \alpha_5 & 0 \\ 0 & \alpha_6 & 1 & 0 & 0 & \alpha_7 \\ \hline \alpha_8 & 0 & 0 & 1 & \alpha_9 & 0 \\ 0 & \alpha_8 & 0 & 0 & 1 & \alpha_9 \\ 0 & 0 & \alpha_{10} & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

then [Hadley (1961)]

$$\left(I - \hat{Q}(s)\right)^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where -

$$A = \left(\alpha - \beta\delta^{-1}\gamma\right)^{-1}, \qquad B = -A\beta\delta^{-1}$$

$$C = -\delta^{-1}\gamma A, \qquad D = \delta^{-1} - \delta^{-1}\gamma B$$

The state [5] represents the system 'down' state and, therefore, the distributions of TSF are given by $G_{i,5}(t)$ for $i = 0, 1, 2, 3, 4$ and the distribution of recurrence time to the state [5] is given by $G_{5,5}(t)$. If we are interested only in the distributions of TSF, they can be obtained as in the last case by making the state [5] as an absorbing state. However, as we are interested in other system characteristics also, we will carry out the analysis considering the process as a general process in which all the states are ergodic. We may point out here that the LST of the general process probabilities $P_{ij}(t)$ are given by (3,1a.2).

DISTRIBUTION OF TSF AND RECURRENCE TIME

Using the relations (3,1a.1) in (3,1a.4) we obtain the LST of the distribution of TSF as

$$\hat{G}_{i,5}(s) = \frac{(I - \hat{Q}(s))^{-1}_{i,5}}{(I - \hat{Q}(s))^{-1}_{5,5}} \quad , \quad i = 0, 1, 2, 3, 4 \tag{3,1b.1}$$

and the LST of the distribution of recurrence time to the state [5] is given by

$$1 - \hat{G}_{5,5}(s) = \frac{1}{(I - \hat{Q}(s))^{-1}_{5,5}} \tag{3,1b.2}$$

It is clear from (3,1b.1) and (3,1b.2) that LST's of the distribution of TSF and the recurrence time to the 'down' state are expressed in terms of elements of the column corresponding to the 'down' state in the matrix $(I - \hat{Q}(s))^{-1}$. Therefore, using the cofactors $\Delta_{5,i}$ of elements in the row corresponding to the state [5] and column corresponding to the state [i] of $I - \hat{Q}(s)$, $\hat{G}_{i,5}(s)$ and $\hat{G}_{5,5}(s)$ can be written as

$$\hat{G}_{i,5}(s) = \frac{\Delta_{5,i}}{\Delta_{5,5}} \quad , \quad i = 1, 2, 3, 4 \tag{3,1b.3}$$

and

$$1 - \hat{G}_{5,5}(s) = \frac{|I - \hat{Q}(s)|}{\Delta_{5,5}} \qquad (3,1b.4)$$

where

$$|I - \hat{Q}(s)| = (1-\alpha_2\alpha_8)\left\{(1-\alpha_8\alpha_5)(1-\alpha_7\alpha_{10}) - \alpha_6(\alpha_4+\alpha_5\alpha_{10}\alpha_9)\right\}$$
$$-\alpha_3\left\{(\alpha_1+\alpha_2\alpha_8\alpha_9)(1-\alpha_7\alpha_{10}) + \alpha_2\alpha_6\alpha_9^2\alpha_{10}\right\}$$

$$\Delta_{5,0} = -\left[\alpha_2\alpha_9^2(1-\alpha_5\alpha_8-\alpha_6\alpha_4) + (\alpha_1+\alpha_2\alpha_8\alpha_9)(\alpha_4\alpha_7+\alpha_5\alpha_9)\right]$$

$$\Delta_{5,1} = (1-\alpha_2\alpha_8)(\alpha_4\alpha_7+\alpha_5\alpha_9) + \alpha_3\alpha_2\alpha_9^2$$

$$\Delta_{5,2} = -(1-\alpha_8\alpha_2)\left[\alpha_7(1-\alpha_5\alpha_8)+\alpha_6\alpha_5\alpha_9\right]$$
$$+\alpha_3\left[\alpha_7(\alpha_1+\alpha_2\alpha_9\alpha_8) - \alpha_6\alpha_2\alpha_9^2\right]$$

$$\Delta_{5,3} = \alpha_9^2\left[(1-\alpha_6\alpha_4-\alpha_1\alpha_3)+\alpha_1\alpha_5\alpha_8\right] + \alpha_4\alpha_7\alpha_8(\alpha_9+\alpha_1)$$

$$\Delta_{5,4} = -\alpha_9\left[(1-\alpha_6\alpha_4+\alpha_4\alpha_7\alpha_8)(1-\alpha_2\alpha_8)-\alpha_1\alpha_3\right]$$

$$\Delta_{5,5} = (1-\alpha_2\alpha_8)(1-\alpha_6\alpha_4-\alpha_5\alpha_8) - \alpha_3(\alpha_1+\alpha_9\alpha_2\alpha_8)$$

The mean and the second moment of the distribution of TSF and recurrence time to state [5] are obtained by applying with necessary modifications (3.1a.12), (3,1a.19), (3,1a.13) to $\hat{G}_{1,5}(s)$ given in (3,1b.3) and (3,1a.22), (3,1a.23) to $1 - \hat{G}_{5,5}(s)$ given in (3,1b.4) respectively. For instance

$$
\begin{aligned}
E_{0,5}(T_u) = \frac{1}{D_5} \Big[& \eta_0 \{ 1 - a_8 a_5 - a_6 a_4 + (1 - a_4)(a_1 + a_9 a_8 a_2) \} \\
& + \eta_1 \{ - a_4 (a_1 + a_9 a_8 a_2) \} \\
& + \eta_2 \{ - a_6 a_4 (a_1 + a_9 a_8 a_2) \} \\
& + \eta_3 \{ a_8 a_5 - a_1 a_9 a_5 + (1 - a_6 a_4) a_8 a_2 \\
& - a_2 a_9 (1 - a_6 a_4) - 2 a_6 a_4 a_8^2 a_2 \} \Big]
\end{aligned}
$$

and the mean recurrence time to state [5] is given by

$$
E_{5,5}(\tau_R) = \sum_{k=0}^{5} D_k \eta_k \Big/ D_5 \qquad (3,1b.5)
$$

where a's and $D's$ stand for the values of $\alpha's$ and $\Delta_{jj}'s$ at $s = 0$. Making use of the fact that the sum of the row elements of the matrix $\left(I - \hat{Q}(s) \right)_{s=0}$ is zero, we have

$$D_0 = a_3 a_6$$

$$D_1 = - a_6 (1 - a_8 a_2)$$

$$D_2 = (1 - a_8 a_2)(1 - a_8 a_5) - a_3 (a_1 + a_9 a_8 a_2)$$

$$D_3 = - a_6 a_3 a_2$$

$$D_4 = -a_6\{-(1-a_8a_2)a_5 + a_3a_2 + a_3a_2a_8\}$$

$$D_5 = (1-a_8a_2)(1-a_8a_5-a_6a_4) - a_3(a_1+a_9a_8a_2)$$

and $$\eta_0 = \frac{1}{\lambda_1+\lambda_2}, \quad \eta_1 = \frac{1}{\lambda_1+\lambda_2+\mu_2}, \quad \eta_2 = \frac{1}{\lambda_1+\mu_2}$$

$$\eta_3 = \eta_4 = \frac{1}{\mu_1+\lambda_2}, \quad \eta_5 = \frac{1}{\mu_1}$$

are the means of the unconditional wait time distributions in the different states.

The expressions for the second moments become very lengthy and so we will restrict ourselves to that of the distribution of recurrence time to state [5] as this will be required in the evaluation of the expected number of system failures in an interval of time (0, t) as $t \to \infty$. We have for the second moment

$$E_{5,5}(\tau_R^2) = \frac{2\, E_{55}(\tau_R).C_1 + C_2}{D_5} \qquad (3,1b.6)$$

where

$$C_1 = \eta_0\left[(1-a_8a_5-a_6a_4)a_8a_2 + a_3(a_1+a_9a_2a_8)\right]$$

$$+ \eta_1\left[(1-a_8a_2)(a_8a_5+a_6a_4) + a_3(a_1+a_9a_2a_8)\right]$$

$$+ \eta_2 (1 - a_8 a_2) a_6 a_4$$

$$+ \eta_3 [(1 - a_8 a_2) a_8 a_5 + (1 - a_8 a_5 - a_6 a_4) a_8 a_2 + 2 a_3 a_8 a_9 a_2]$$

$$C_2 = \sum_{k=0}^{5} D_k \eta_k^{(2)} + 2 \eta_0 \eta_1 [D_0 + a_6 a_8 a_2]$$

$$+ 2 \eta_0 \eta_2 [D_0 + D_2 - (1 - a_8 a_5)]$$

$$+ 2 \eta_0 \eta_3 [D_3 + D_4 + a_6 a_5 - a_3 a_6 a_8 a_2]$$

$$+ 2 \eta_0 \eta_5 [D_5 + (1 - a_8 a_2) a_7 + a_6 a_9 a_5]$$

$$+ 2 \eta_1 \eta_2 [D_2 + (1 - a_8 a_2) a_7]$$

$$+ 2 \eta_1 \eta_3 [D_3 + D_4 + a_6 a_8 a_2]$$

$$+ 2 \eta_1 \eta_5 [D_5 + a_7 (1 - a_8 a_6)]$$

$$+ 2 \eta_2 \eta_3 [D_3 + D_4 - (1 - a_8 a_2) a_8 a_5$$

$$- (1 - a_8 a_5) a_8 a_2 - 2 a_3 a_9 a_8 a_2]$$

$$+ 2 \eta_2 \eta_5 D_5 + 2 \eta_3 \eta_5 [D_3 + D_4 - (1 - a_8 a_2) a_8 a_5$$

$$- (1 - a_8 a_5 - a_6 a_4) a_8 a_2 - 2 a_3 a_9 a_8 a_2]$$

$$+ 2 \eta_3 \eta_4 [D_4 - a_6 a_5 + a_3 a_6 a_8 a_9 a_2]$$

OTHER RELIABILITY CHARACTERISTICS

Following the analysis of section 1(a), the expected duration of the system down-time in a large interval (0, T) is given by

$$E(T_d ; T) \sim \frac{D_5 \eta_5}{\sum_{k=0}^{5} D_k \eta_k} \cdot T$$

the long-run availability $= 1 - p_5$

$$= 1 - \frac{D_5 \eta_5}{\sum_{k=0}^{5} D_k \eta_k}$$

and the expected number of system failures in an interval (0, t) as $t \to \infty$ is given by

$$M_{i,5}(t) = \frac{t}{E_{55}(\tau_R)} + \frac{E_{55}(\tau_R^2)}{2\left[E_{55}(\tau_R)\right]^2} - \frac{E_{i,5}(T_R)}{E_{55}(\tau_R)} + o(1)$$

and

$$M_{5,5}(t) = \frac{t}{E_{55}(\tau_R)} + \frac{E_{55}(\tau_R^2)}{2\left[E_{55}(\tau_R)\right]^2} - 1 + o(1)$$

for the modified and ordinary Renewal processes respectively.

The evaluation of the expression for interval reliability follows that of chapter 1, section 1, as in this case, there is only one system down-state and the process can be identified with a Renewal process. For,

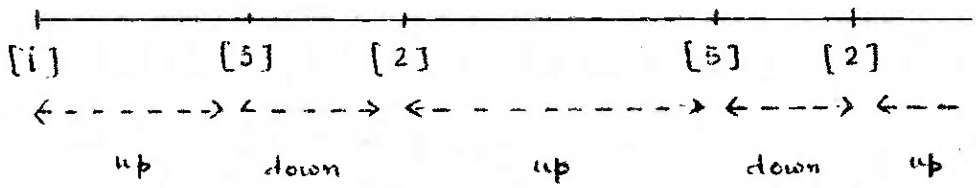


Fig.3.0c Renewal Periods

as in fig (3.0c) we observe that the TSF periods alternating with SDT periods form a Renewal process and whenever a TSF period starts after an SDT period through repair completion, it always starts with the state [2] . Therefore, the interval reliability $R(x,t)$ is given by

$$R(x,t) = [1 - G_{i,5}(t+x)] + \int_0^t [1 - G_{2,5}(t-y+x)]\, d\nu(y,i) \qquad (3,1b.7)$$

where $d\nu(y,i)$ denotes the probability of a recurrence of restoration from 'down' state to 'up' state, i.e. to state [2] , at y . The limiting interval reliability is then given as

$$\lim_{t \to \infty} R(x,t) = \frac{\int_x^\infty [1 - G_{2,5}(y)]\, dy}{E_{55}(\tau_R)}$$

and its upper and lower bounds are given by

$$\frac{E_{2,5}(T_u) - x}{E_{5,5}(\tau_R)} \leq \lim_{t \to \infty} R(x,t) \leq \frac{E_{2,5}(T_u)}{E_{55}(\tau_R)}$$

SECTION 2 (a)

SERIES CONNECTED STANDBY REDUNDANT SYSTEM WITH HEAD-OF-THE-LINE PRIORITY FOR REPAIR

This system is exactly the same as in section 1(a) except for the fact that when the type 1 unit fails it is assigned head-of-the-line priority for repair. Also, there is a slight difference in the description of the state of the process. When the process is in the state (1, 1), we know that a type 1 unit and a type 2 unit are in the failed state; but we do not know which type of unit is undergoing repair as in this case the type 1 unit is not taken up for repair preempting the type 2 unit. The type 1 unit, when fails, has to wait till the repair completion of the type 2 unit if it is already undergoing repair at that instant. In order to distinguish this, denote by $(1, \bar{1})$ the state of the system when type 1 unit is undergoing repair and the type 2 unit is waiting for repair and similarly $(\bar{1}, 1)$ denoting the state when the type 2 unit is undergoing repair and the type 1 unit is waiting for repair. With this definition, the states of the process and their state designations are:

Serial No.	State of the process at time t	State designation
1	(0, 0)	[0]
2	(0, 1)	[1]
3	(0, 2)	[2]
4	(1, 0)	[3]
5	$(\bar{1}, 1)$	[4]

It may be observed that in the series connected system the state $(1, \bar{1})$ does not appear in the general process.

GENERAL PROCESS PROBABILITIES

The process is a SMP in which all the states communicate with each other. The non-zero elements of the matrix of transition distribution Q(t) are given by

$$Q_{0,1}(t) = \frac{\lambda_2}{\lambda_1+\lambda_2}\left[1-e^{-(\lambda_1+\lambda_2)t}\right], \quad Q_{0,3}(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\left[1-e^{-(\lambda_1+\lambda_2)t}\right]$$

$$Q_{1,0}(t) = \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2}\left[1-e^{-(\lambda_1+\lambda_2+\mu_2)t}\right], \quad Q_{1,2}(t) = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2}\left[1-e^{-(\lambda_1+\lambda_2+\mu_2)t}\right]$$

$$Q_{14}(t) = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu_2}\left[1-e^{-(\lambda_1+\lambda_2+\mu_2)t}\right], \quad Q_{2,1}(t) = 1-e^{-\mu_2 t}$$

$$Q_{3,0}(t) = 1-e^{-\mu_1 t}, \quad Q_{4,3}(t) = 1-e^{-\mu_2 t}$$

As before denoting by α's the LST's of the transition distributions $Q_{ij}(t)$'s we have

$$\hat{Q}_{0,1}(s) = \frac{\lambda_2}{\lambda_1+\lambda_2+s} = -\alpha_1, \quad \hat{Q}_{0,3}(s) = \frac{\lambda_1}{\lambda_1+\lambda_2+s} = -\alpha_2$$

$$\hat{Q}_{1,0}(s) = \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_3, \quad \hat{Q}_{1,2}(s) = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_4$$

$$\hat{Q}_{1,4}(s) = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_5, \quad \hat{Q}_{21}(s) = \frac{\mu_2}{\mu_2+s} = -\alpha_6 = \hat{Q}_{43}(s)$$

and $$\hat{Q}_{3,0}(s) = \frac{\mu_1}{\mu_1+s} = -\alpha_7$$

Hence, the basic matrix $I-\hat{Q}(s)$ of the process is given by

$$I-\hat{Q}(s) = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} 1 & \alpha_1 & 0 & \alpha_2 & 0 \\ \alpha_3 & 1 & \alpha_4 & 0 & \alpha_5 \\ 0 & \alpha_6 & 1 & 0 & 0 \\ \alpha_7 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha_6 & 1 \end{array}\right] \end{array}$$

Now, LST's of the general process probabilities $P_{ij}(t)$ can be easily obtained by inverting $I-\hat{Q}(s)$ and using (3, 1a.2) and the $\hat{P}_{ij}(s)$ thus obtained can be used in (3,1a.4) to obtain the LST's of the first passage time distributions $G_{ij}(t)$ and the recurrence time distributions $G_{jj}(t)$.

DISTRIBUTION OF TSF AND RECURRENCE TIMES TO FAILURE STATES

We observe that the system failure occurs when the process reaches the state [2] , [3] or [4] and hence as in section 1(a), the LST of the distribution of TSF can be obtained by making these three states absorbing ones and lumping them together into a new state [d], the matrix of this absorbing SMP is given by

$$I-\hat{Q}_d(s) = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & d \end{array} \\ \begin{array}{c} 0 \\ 1 \\ d \end{array} & \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ \alpha_3 & 1 & \alpha_4+\alpha_5 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (3,2a.1)$$

As the matrix of (3,2a.1) is in structure identical with that of (3,1a.7), the entire analysis of the matrix of (3,2a.1) follows the analysis of the matrix of (3,1a.7) verbatim. The LST of the distributions of TSF, namely of $G_{0d}(t)$ and $G_{1d}(t)$ have been obtained in (3,1a.10) and (3,1a.11). And their expectations $E_{id}(T_{11})$ and the second moments $E_{id}(T_{11}^2)$ for $i=0,1$ are given by

Table 3.1

The Values of $\mu_2 E_{o,d}(T_u)$ for Series connected Stand-by Redundant System under Premeptive Resume and Head-of-the-line Priority Repair Policies

$\rho_1^* = \frac{\lambda_1}{\mu_2}$	$\rho_2 = \frac{\lambda_2}{\mu_2}$	$\mu_2 E_{o,d}(T_u)$	$\rho_1^* = \frac{\lambda_1}{\mu_2}$	$\rho_2 = \frac{\lambda_2}{\mu_2}$	$\mu_2 E_{o,d}(T_u)$
0.05	0.05	19.166	1.0	0.05	0.999
	0.1	17.242		0.1	0.995
	0.2	12.839		0.2	0.984
	0.5	5.815		0.5	0.923
	1.0	2.646		1.0	0.800
	2.0	1.188		2.0	0.600
	5.0	0.432		5.0	0.324
0.1	0.05	9.796	2.0	0.05	0.4997
	0.1	9.285		0.1	0.4992
	0.2	7.894		0.2	0.4970
	0.5	4.565		0.5	0.4848
	1.0	2.366		1.0	0.4545
	2.0	1.131		2.0	0.3888
	5.0	0.425		5.0	0.2540
0.2	0.05	4.952	5.0	0.05	0.1999
	0.1	4.827		0.1	0.1999
	0.2	4.444		0.2	0.1997
	0.5	3.188		0.5	0.1985
	1.0	1.951		1.0	0.1931
	2.0	1.032		2.0	0.1851
	5.0	0.411		5.0	0.1523
0.5	0.05	1.994			
	0.1	1.977			
	0.2	1.919			
	0.5	1.667			
	1.0	1.273			
	2.0	0.815			
	5.0	0.374			

(3,1a.14), (3,1a.15) and (3,1a.16), (3,1a.17) respectively.

For the ease of computation, the expected TSF $E_{o,d}(T_u)$ given by (3,1a.14), which is the same for both head-of-the-line and preemptive resume priority repair policies is rewritten as

$$\mu_2 E_{o,d}(T_u) = \frac{\frac{1}{\rho_1^* + \rho_2}\left(1 + \frac{\rho_2}{\rho_1^* + \rho_2 + 1}\right)}{1 - \frac{\rho_2}{\rho_1^* + \rho_2} \cdot \frac{1}{1 + \rho_1^* + \rho_2}}$$

where $\rho_1^* = \frac{\lambda_1}{\mu_2}$ and $\rho_2 = \frac{\lambda_2}{\mu_2}$. The values of $\mu_2 E_{o,d}(T_u)$ are given in table 3.1 for ρ_1^*, ρ_2 = 0.05, 0.1, 0.2, 0.5, 1.0, 2.0 and 5.0. It can be seen from this table that $\mu_2 E_{o,d}(T_u)$ has higher values when ρ_1^*, ρ_2 are small and they sharply decrease for an increase in these two parameters.

The LST's of the distribution of recurrence time to the system 'down' states {2}, {3} and {4} are obtained with necessary modifications using (3,1a.21) Since α's in the basic matrix $I - \hat{Q}(s)$ are different in this case, we have

$$1-\hat{G}_{jj}(s) = \frac{|I-\hat{Q}(s)|}{\Delta_{jj}} \quad , \quad j = 2,3,4$$

where

$$\Delta_{22} = (1-\alpha_2\alpha_7) - \alpha_1(\alpha_3+\alpha_7\alpha_6\alpha_5)$$

$$\Delta_{33} = (1-\alpha_6\alpha_4) - \alpha_1\alpha_3$$

$$\Delta_{44} = (1-\alpha_2\alpha_7)(1-\alpha_6\alpha_4) - \alpha_1\alpha_3$$

and

$$|I-\hat{Q}(s)| = (1-\alpha_2\alpha_7)(1-\alpha_6\alpha_4) - \alpha_1(\alpha_3+\alpha_7\alpha_6\alpha_5)$$

The expressions for the mean recurrence time to these three states are

$$E_{22}(\tau_R) = \sum_{k=0}^{4} D_k \eta_k \Big/ D_2$$

$$E_{33}(\tau_R) = \sum_{k=0}^{4} D_k \eta_k \Big/ D_3$$

$$E_{44}(\tau_R) = \sum_{k=0}^{4} D_k \eta_k \Big/ D_4$$

where

$$D_0 = 1 + a_4$$

$$D_1 = -a_1$$

$$D_2 = a_1 a_4$$

$$D_3 = (1 + a_4) - a_1 a_3$$

$$D_4 = a_1 a_5$$

$$\eta_0 = \frac{1}{\lambda_1 + \lambda_2}, \quad \eta_1 = \frac{1}{\lambda_1 + \lambda_2 + \mu_2}, \quad \eta_2 = \frac{1}{\mu_2} = \eta_4, \quad \eta_3 = \frac{1}{\mu_1}$$

The second moments of the recurrence time distribution to the states [2], [3] and [4] have the same form as those in section 1(a) and are given by

$$E_{22}(\tau_R^2) = \frac{2\, E_{22}(\tau_R)\, C_{32} + C_{31}}{D_2}$$

$$E_{33}(\tau_R^2) = \frac{2\, E_{33}(\tau_R)\, C_{33} + C_{31}}{D_3}$$

$$E_{44}(\tau_R^2) = \frac{2\, E_{44}(\tau_R)\, C_{34} + C_{31}}{D_4}$$

where

$$C_{32} = \eta_0 (1 - D_2) + \eta_1 (D_1 - D_2) + \eta_2 D_4 + \eta_3 D_4$$

$$C_{33} = \eta_0 a_1 a_3 + \eta_1 (a_1 a_3 - a_4) - \eta_2 a_4$$

$$C_{34} = \eta_0 (D_0 - D_4) + \eta_1 (D_1 - D_4) + \eta_2 D_2 + \eta_3 (D_0 + D_2 - D_1)$$

$$C_{31} = \sum_{k=0}^{4} \eta_k^{(2)} D_k$$

$$+ 2\eta_0 [\eta_1 (D_1 + D_0 - 1) + \eta_2 (D_0 + D_2 + D_4 - 1) + \eta_3 D_3]$$

$$+ 2\eta_1 [\eta_2 (D_2 + D_4) + \eta_3 (D_3 + a_2)]$$

$$+ 2\eta_2 \eta_3 (D_4 + D_2 + D_0 - 1)$$

OTHER RELIABILITY CHARACTERISTICS

Other reliability characteristics such as expected duration of system down-time in a large interval of time (0, T), Long-run availability of the system, expected number of system failures in a given interval of time (0, t) as $t \rightarrow \infty$ and finally the interval reliability are obtained by using the same relations as in section 1(a) with only substituting the relevent system parameters of this process.

The long-run availability of this system for the head-of-the-line priority repair policy can be expressed as

$$\text{Long-run Availability} = 1 \Big/ \left(1 + \rho_1 + \rho_2 + \frac{\rho_1^* + \rho_2}{1 + \rho_1^* + \rho_2} \right)$$

where $\rho_1 = \frac{\lambda_1}{\mu_1}$, $\rho_1^* = \frac{\lambda_1}{\mu_2}$, $\rho_2 = \frac{\lambda_2}{\mu_2}$ for the ease of computation. The values of long-run availability have been plotted against $\frac{1}{\rho_2}$ for different values of ρ_1 and fixing the value of ρ_1^* at 0.2 in fig 3.1 and 0.8 in fig 3.2. It will be observed that the curves for the long-run availability increase sharply in the beginning stages and tend to flatten out as $\frac{1}{\rho_2}$ increases. The sharp increase is marked for values $\frac{1}{\rho_2} < 10.0$ and beyond this value of $\frac{1}{\rho_2}$, the curves flatten out. These values also increase for smaller values of ρ_1 fixing the other two parameters.

The long-run availability of this system for preemptive resume priority repair policy given by expression (3,1a.33) can be rewritten as

Long-run Availability

$$= 1 \Big/ \left(1 + \rho_1 + \frac{\rho_2^2}{1 + \rho_2} \right)$$

LONG-RUN AVAILABILITY OF SERIES CONNECTED STANDBY REDUNDANT SYSTEM UNDER HEAD-OF-THE-LINE PRIORITY REPAIR POLICY FOR VARIOUS ρ_1, ρ_2 AND $\rho_1^* = 0.2$

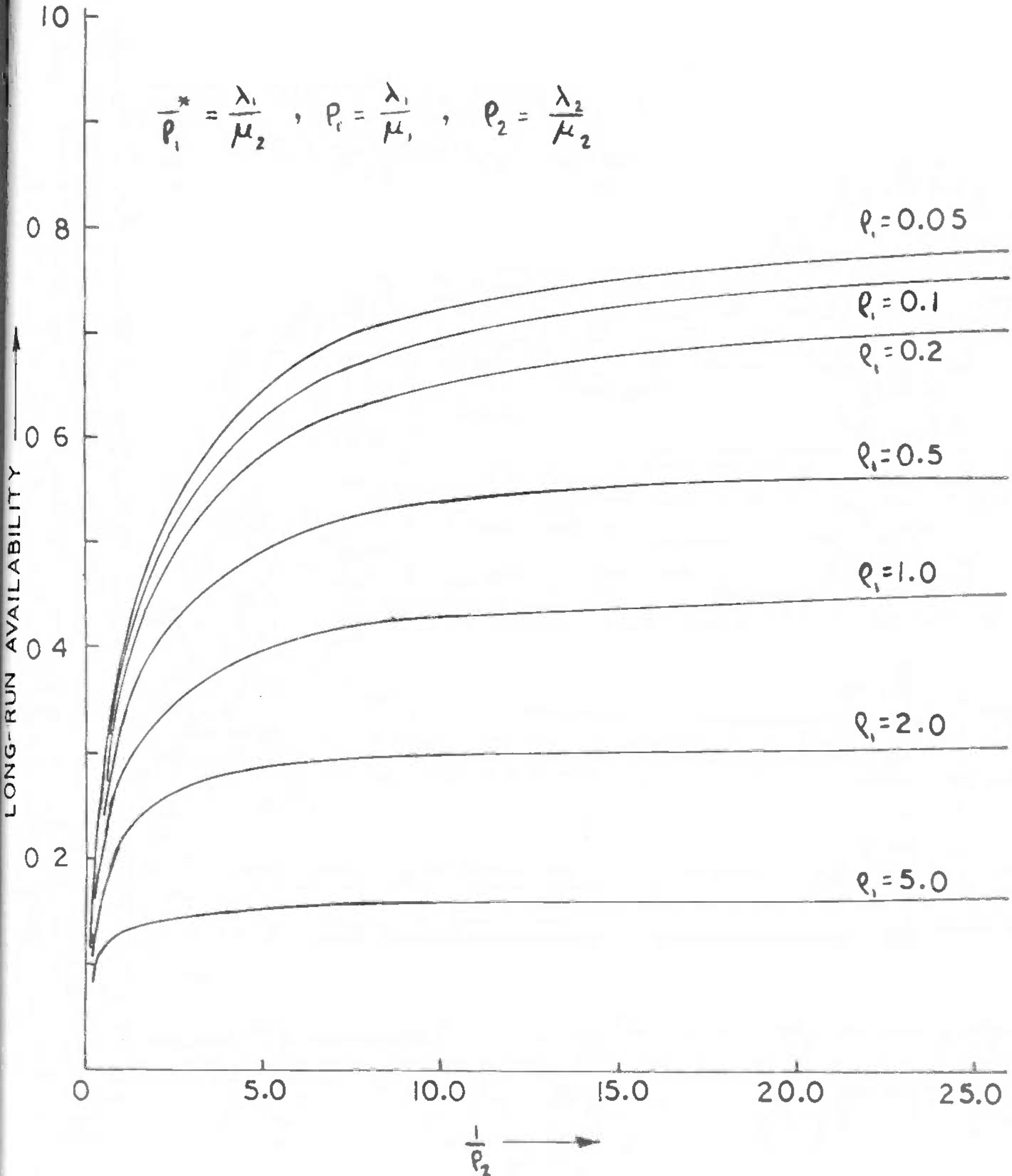


FIG. 3.1

LONG-RUN AVAILABILITY OF SERIES CONNECTED STANDBY REDUNDANT SYSTEM UNDER HEAD-OF-THE-LINE PRIORITY REPAIR POLICY FOR VARIOUS ρ_1, ρ_2 AND $\rho_1^* = 0.8$

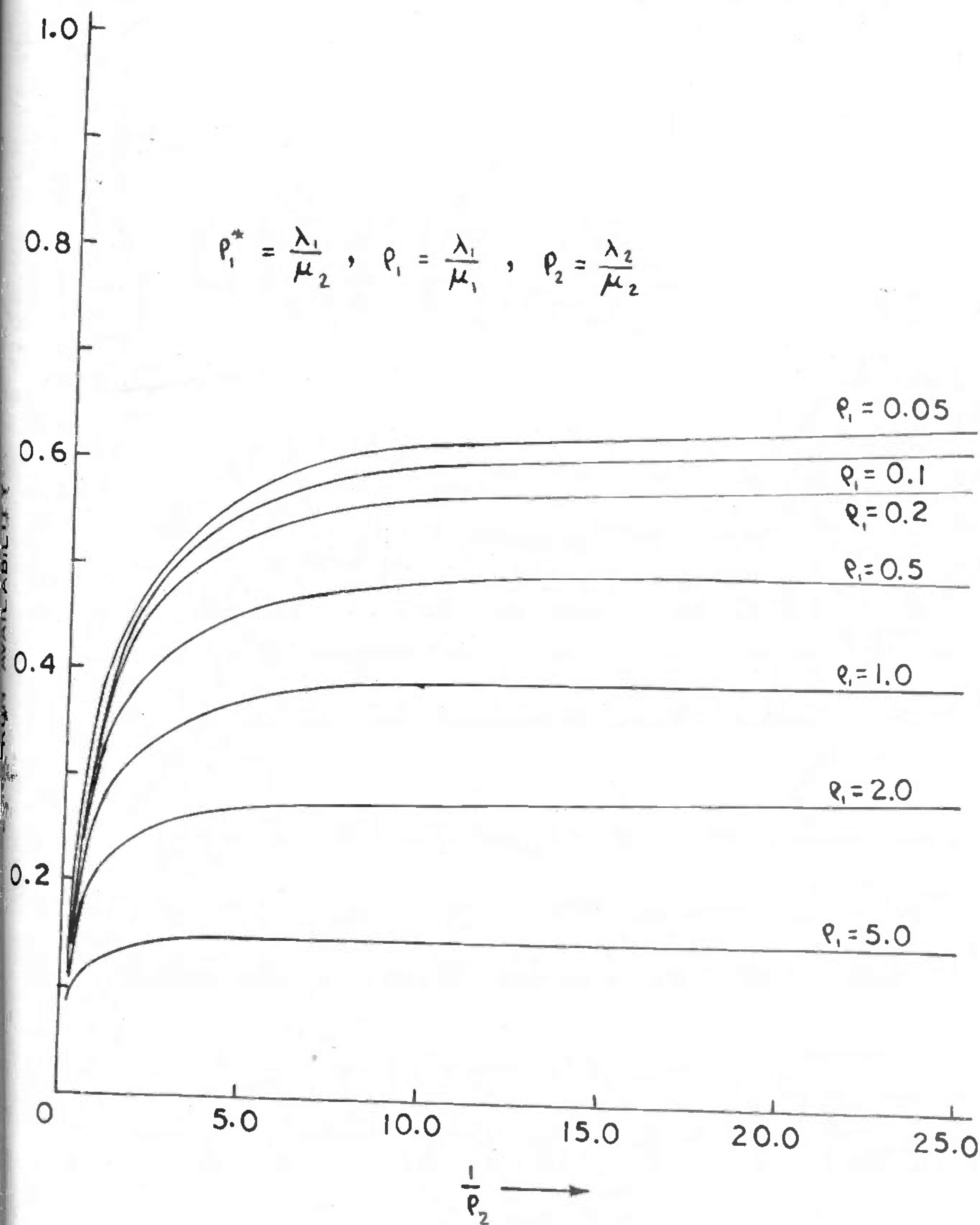


FIG. 3.2

ONG-RUN AVAILABILITY OF SERIES-CONNECTED STANDBY EDUNDANT SYSTEM UNDER PRE-EMPTIVE RESUME PRIORITY REPAIR POLICY FOR VARIOUS ρ_1 AND ρ_2

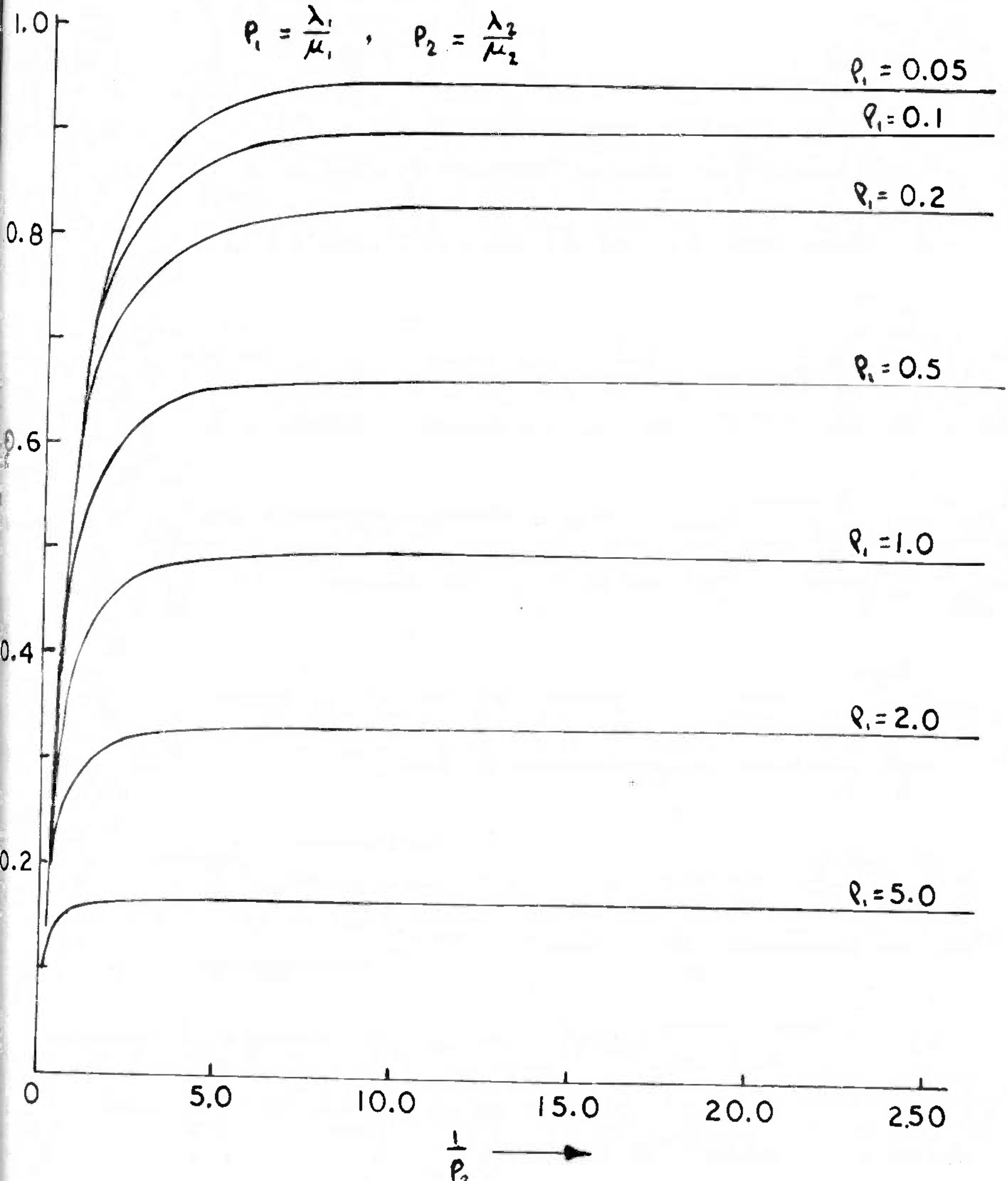


FIG. 3.3

where $\rho_1 = \frac{\lambda_1}{\mu_1}$ and $\rho_2 = \frac{\lambda_2}{\mu_2}$. These values are plotted against $\frac{1}{\rho_2}$ for various values of ρ_1 in fig.3.3. It will be observed that the curve has a sharp increase when the values of $\frac{1}{\rho_2}$ are small (say $<$ 0.5) and beyond which they tend to flatten out. These values also show an increase when the values of ρ_1 are smaller.

Another important point emerging out of this discussion is that the adoption of preemptive resume priority repair policy yields higher long-run availability than when head-of-the-line priority repair policy is adopted. This fact can easily be mathematically verified by comparing the corresponding expressions.

SECTION 2 (b)

PARALLEL CONNECTED STANDBY REDUNDANT SYSTEM WITH HEAD-OF-THE-LINE PRIORITY FOR REPAIR

The same system as in section 1(b) has been considered in this sub-section with the difference that the type 1 unit on failure is given head-of-the-line priority for repair. The definition of the states is the same as in section 2(a) which specifies which type of unit is undergoing repair when both types of units are in the failed state, i.e. the state (1,1). With this modification, the states of the process and their state designations are:

Serial No.	State of the process at time t	State designation
1	(0, 0)	[0]
2	(0, 1)	[1]
3	(0, 2)	[2]
4	(1, 0)	[3]
5	$(1, \bar{1})$	[4]
6	$(\bar{1}, 1)$	[5]
7	$(1, \bar{2})$	[6]
8	$(\bar{1}, 2)$	[7]

Compare these 8 states with 6 of section 1(b) and note that the imposition of the head-of-the-line priority increases the number of states required for description of the process. As in the previous case, the process is a SMP in which all the states communicate with each other. The non-zero elements of the matrix of transition distributions Q(t) are given by

$$Q_{0,1}(t) = \frac{\lambda_2}{\lambda_1+\lambda_2}\left[1-e^{-(\lambda_1+\lambda_2)t}\right], \quad Q_{0,3}(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\left[1-e^{-(\lambda_1+\lambda_2)t}\right]$$

$$Q_{1,0}(t) = \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2}\left[1-e^{-(\lambda_1+\lambda_2+\mu_2)t}\right], \quad Q_{1,2}(t) = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2}\left[1-e^{-(\lambda_1+\lambda_2+\mu_2)t}\right]$$

$$Q_{1,5}(t) = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu_2}[1-e^{-(\lambda_1+\lambda_2+\mu_2)t}], \quad Q_{2,1}(t) = \frac{\mu_2}{\lambda_1+\mu_2}[1-e^{-(\lambda_1+\mu_2)t}]$$

$$Q_{2,6}(t) = \frac{\lambda_1}{\lambda_1+\mu_2}[1-e^{-(\lambda_1+\mu_2)t}], \quad Q_{3,0}(t) = \frac{\mu_1}{\mu_1+\lambda_2}[1-e^{-(\mu_1+\lambda_2)t}]$$

$$Q_{3,4}(t) = \frac{\lambda_2}{\mu_1+\lambda_2}[1-e^{-(\mu_1+\lambda_2)t}], \quad Q_{4,1}(t) = \frac{\mu_1}{\mu_1+\lambda_2}[1-e^{-(\mu_1+\lambda_2)t}]$$

$$Q_{4,6}(t) = \frac{\lambda_2}{\mu_1+\lambda_2}[1-e^{-(\mu_1+\lambda_2)t}], \quad Q_{5,3}(t) = \frac{\mu_2}{\lambda_2+\mu_2}[1-e^{-(\lambda_2+\mu_2)t}]$$

$$Q_{5,7}(t) = \frac{\lambda_2}{\mu_2+\lambda_2}[1-e^{-(\mu_2+\lambda_2)t}], \quad Q_{6,2}(t) = 1-e^{-\mu_1 t}$$

$$Q_{7,4}(t) = 1-e^{-\mu_2 t}$$

and denoting by $\alpha' s$, the LST's of the corresponding transition distributions, we have

$$\hat{Q}_{0,1}(s) = \frac{\lambda_2}{\lambda_1+\lambda_2+s} = -\alpha_1, \quad \hat{Q}_{0,3}(s) = \frac{\lambda_1}{\lambda_1+\lambda_2+s} = -\alpha_2$$

$$\hat{Q}_{1,0}(s) = \frac{\mu_2}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_3, \quad \hat{Q}_{1,2}(s) = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_4$$

$$\hat{Q}_{1,5}(s) = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu_2+s} = -\alpha_5 \quad , \quad \hat{Q}_{2,1}(s) = \frac{\mu_2}{\lambda_1+\mu_2+s} = -\alpha_6$$

$$\hat{Q}_{2,6}(s) = \frac{\lambda_1}{\lambda_1+\mu_2+s} = -\alpha_7 \quad , \quad \hat{Q}_{3,0}(s) = \frac{\mu_1}{\mu_1+\lambda_2+s} = -\alpha_8$$

$$\hat{Q}_{3,4}(s) = \frac{\lambda_2}{\mu_1+\lambda_2+s} = -\alpha_9 \quad , \quad \hat{Q}_{4,1}(s) = \frac{\mu_1}{\mu_1+\lambda_2+s} = -\alpha_8$$

$$\hat{Q}_{4,6}(s) = \frac{\lambda_2}{\mu_1+\lambda_2+s} = -\alpha_9 \quad , \quad \hat{Q}_{5,3}(s) = \frac{\mu_2}{\lambda_2+\mu_2+s} = -\alpha_{10}$$

$$\hat{Q}_{5,7}(s) = \frac{\lambda_2}{\lambda_2+\mu_2+s} = -\alpha_{11} \quad , \quad \hat{Q}_{6,2}(s) = \frac{\mu_1}{\mu_1+s} = -\alpha_{12}$$

$$\hat{Q}_{7,4}(s) = \frac{\mu_2}{\mu_2+s} = -\alpha_{13}$$

GENERAL PROCESS PROBABILITIES

The basic matrix $I-\hat{Q}(s)$ of the general process is then

$$
I - \hat{Q}(s) = \begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 1 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \\
1 & \alpha_3 & 1 & \alpha_4 & 0 & 0 & \alpha_5 & 0 & 0 \\
2 & 0 & \alpha_6 & 1 & 0 & 0 & 0 & 0 & \alpha_7 \\
3 & \alpha_8 & 0 & 0 & 1 & \alpha_9 & 0 & 0 & 0 \\
4 & 0 & \alpha_8 & 0 & 0 & 1 & 0 & \alpha_9 & 0 \\
5 & 0 & 0 & 0 & \alpha_{10} & 0 & 1 & 0 & \alpha_{11} \\
6 & 0 & 0 & \alpha_{12} & 0 & 0 & 0 & 1 & 0 \\
7 & 0 & 0 & 0 & 0 & \alpha_{13} & 0 & 0 & 1
\end{array}
$$

The LST's of the general process probabilities $P_{ij}(t)$ are readily obtained by inverting $I - \hat{Q}(s)$ and using (3,1a.2) and these values of $\hat{P}_{ij}(s)$ can then be used in (3,1a.4) to obtain the LST's of the first passage time distributions $G_{ij}(t)$ and the recurrence time distributions $G_{jj}(t)$.

DISTRIBUTION OF TSF AND RECURRENCE TIME TO FAILURE STATES

The system failure occurs when the process reaches either the state [6] or the state [7] and hence as in section 1(a), the LST of the distribution of TSF can be obtained by making these two states absorbing ones and lumping them together into one state [d]. The basic matrix of this absorbing SMP is given by

$$
I - \hat{Q}_d(s) \quad = \quad
\begin{array}{c|ccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & d \\
\hline
0 & & & & & & & \\
\end{array}
$$

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ d
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & d \end{array} \\
\begin{bmatrix}
1 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 \\
\alpha_3 & 1 & \alpha_4 & 0 & 0 & \alpha_5 & 0 \\
0 & \alpha_6 & 1 & 0 & 0 & 0 & \alpha_7 \\
\alpha_8 & 0 & 0 & 1 & \alpha_9 & 0 & 0 \\
0 & \alpha_8 & 0 & 0 & 1 & 0 & \alpha_9 \\
0 & 0 & 0 & \alpha_{10} & 0 & 1 & \alpha_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

The LST's of the distributions of TSF $G_{id}(t)$ are obtained by (3,1a.8) in terms of the elements in the column corresponding to state [d] in $\{I - \hat{Q}_d(s)\}^{-1}$. As before, denoting the cofactor of (d, i) th element of $(I - \hat{Q}_d(s))$ by $\Delta_{d,i}$ and since $|I - \hat{Q}_d(s)| = \Delta_{d,d}$ we have

$$
\hat{G}_{i,d}(s) = \frac{\Delta_{d,i}}{\Delta_{d,d}} \quad , \quad i = 0, 1, 2, 3, 4, 5 \qquad (3,2b.1)
$$

where

$$
|I - \hat{Q}_d(s)| = \Delta_{d,d} = [(1 - \alpha_6\alpha_4 - \alpha_1\alpha_3) - \alpha_2\alpha_8(1 - \alpha_6\alpha_4)
$$

$$
- \alpha_3\alpha_2\alpha_8\alpha_4 - (\alpha_8\alpha_9\alpha_5 + \alpha_1\alpha_8\alpha_5)\alpha_{10}]
$$

$$
\Delta_{d,0} = -[\alpha_9^2\{\alpha_2(1 - \alpha_6\alpha_4) + \alpha_1\alpha_5\alpha_4\} + \alpha_4\alpha_7(\alpha_1 + \alpha_2\alpha_8\alpha_9)
$$

$$
+ \alpha_2\alpha_8\alpha_5\alpha_9\alpha_{11}]
$$

$$
\Delta_{d,1} = \alpha_9^2(\alpha_5\alpha_{10} + \alpha_2\alpha_3) + (\alpha_4\alpha_7 + \alpha_5\alpha_{11})(1 - \alpha_2\alpha_8)
$$

$$\Delta_{d,2} = -\left[\alpha_9^2 \cdot \alpha_6(\alpha_5\alpha_{10}+\alpha_2\alpha_3)+\alpha_7(1-\alpha_1\alpha_3-\alpha_2\alpha_8)\right.$$
$$\left.+\alpha_6\alpha_5\alpha_{11}(1-\alpha_2\alpha_8)-\alpha_8\alpha_7\alpha_{10}(\alpha_9+\alpha_1\alpha_5)-\alpha_2\alpha_3\alpha_7\alpha_8\alpha_9\right]$$

$$\Delta_{d,3} = \alpha_1\left[\alpha_9(1-\alpha_6\alpha_4-\alpha_1\alpha_3)+2\alpha_8(\alpha_{11}\alpha_5+\alpha_9\alpha_7)\right]$$

$$\Delta_{d,4} = -\left[\alpha_9(1-\alpha_6\alpha_4-\alpha_1\alpha_3)+\alpha_5\alpha_8(\alpha_{11}-\alpha_2\alpha_8-\alpha_1\alpha_{10}\alpha_9)\right.$$
$$\left.+\alpha_4\alpha_8\alpha_7(1-\alpha_2\alpha_8)-\alpha_2\alpha_8\alpha_7(1-\alpha_6\alpha_4)\right]$$

$$\Delta_{d,5} = -\alpha_{11}\left[(1-\alpha_6\alpha_4-\alpha_1\alpha_3)-\alpha_2\alpha_8(1-\alpha_6\alpha_4+\alpha_9\alpha_3)\right]$$
$$+\alpha_{10}\left[\alpha_9^2(1-\alpha_6\alpha_4-\alpha_1\alpha_3)+\alpha_7\alpha_8\alpha_9\alpha_4\right.$$
$$\left.+\alpha_7\alpha_1\alpha_5\alpha_4\right]$$

On the other hand, the LST's of the distributions of recurrence time to the failure states [6] and [7] can be obtained from the basic matrix of the general process. As such

$$1-\hat{G}_{jj}(s) = \frac{|I-\hat{Q}(s)|}{\Delta_{jj}}, \quad j = 6,7 \qquad (3,2b.2)$$

where

$$|I-\hat{Q}(s)| = (1-\alpha_6\alpha_4-\alpha_1\alpha_3)-\alpha_2\alpha_8(1-\alpha_6\alpha_4) -$$

$$- \alpha_9 \alpha_2 \alpha_3 \alpha_8 - \alpha_5 \alpha_{10} \alpha_8 (\alpha_1 + \alpha_9)$$

$$- \alpha_9^2 \alpha_6 \alpha_{12} (\alpha_2 \alpha_3 + \alpha_5 \alpha_{10})$$

$$- \alpha_{13} \alpha_9 \alpha_7 \alpha_{12} (1 - \alpha_1 \alpha_3 - \alpha_2 \alpha_8 - \alpha_8 \alpha_1 \alpha_5 \alpha_{10})$$

$$- \alpha_{13} \alpha_8 (1 - \alpha_2 \alpha_8)(\alpha_5 \alpha_{11} + \alpha_4 \alpha_7)$$

$$\Delta_{6,6} = (1 - \alpha_6 \alpha_4 - \alpha_1 \alpha_3) - \alpha_2 \alpha_8 (1 - \alpha_6 \alpha_4) - \alpha_9 \alpha_2 \alpha_3 \alpha_8$$
$$- \alpha_5 \alpha_{10} \alpha_8 (\alpha_1 + \alpha_9) - \alpha_{13} \alpha_8 (1 - \alpha_2 \alpha_8)(\alpha_5 \alpha_{11} + \alpha_4 \alpha_7)$$

and

$$\Delta_{7,7} = (1 - \alpha_6 \alpha_4 - \alpha_1 \alpha_3) - \alpha_2 \alpha_8 (1 - \alpha_6 \alpha_4) - \alpha_9 \alpha_2 \alpha_3 \alpha_8$$
$$- \alpha_5 \alpha_{10} \alpha_8 (\alpha_1 + \alpha_9) - \alpha_9^2 \alpha_6 \alpha_{12} (\alpha_2 \alpha_3 + \alpha_5 \alpha_{10})$$

As before the mean and the second moments of TSF and recurrence times can easily be obtained from (3,2b.1) and (3,2b.2) by differentiation. For instance

$$E_{jj}(\tau_R) = \sum_{k=0}^{7} D_k \eta_k \Big/ D_j \qquad , \quad j = 6,7$$

where the means of the unconditional wait time distributions are given by

$$\eta_0 = \frac{1}{\lambda_1 + \lambda_2}, \quad \eta_1 = \frac{1}{\lambda_1 + \lambda_2 + \mu_2}, \quad \eta_2 = \frac{1}{\lambda_1 + \mu_2}$$

$$\eta_3 = \frac{1}{\mu_1 + \lambda_2} = \eta_4, \quad \eta_5 = \frac{1}{\lambda_2 + \mu_2}, \quad \eta_6 = \frac{1}{\mu_1}, \quad \eta_7 = \frac{1}{\mu_2}$$

and

$$D_0 = (1 - a_6 a_4) - a_5 a_{10} a_9 a_8 - a_9^2 a_6 a_{12} a_5 a_{10}$$
$$- a_{13} a_9 a_7 a_{12} - a_{13} a_8 (a_5 a_{11} + a_4 a_7)$$

$$D_1 = (1 - a_2 a_8)(1 - a_{13} a_9 a_7 a_{12})$$

$$D_2 = (1 - a_1 a_3) - a_2 a_8 - a_9 a_2 a_3 a_8 - a_5 a_{10} a_8 (a_1 + a_9)$$
$$- a_{13} a_8 (1 - a_2 a_8) a_5 a_{11}$$

$$D_3 = (1 - a_6 a_4 - a_1 a_3) - a_{13} a_9 a_7 a_{12} (1 - a_1 a_3)$$
$$- a_{13} a_8 (a_5 a_{11} + a_4 a_7)$$

$$D_4 = (1 - a_6 a_4 - a_1 a_3) - a_2 a_8 (1 - a_6 a_4) - a_5 a_{10} a_1 a_8$$

$$D_5 = (1 - a_6 a_4 - a_1 a_3) - a_2 a_8 (1 - a_6 a_4) - a_9 a_2 a_3 a_8$$
$$- a_9^2 a_6 a_{12} a_2 a_3 - a_{13} a_9 a_7 a_{12} (1 - a_1 a_3 - a_2 a_8)$$
$$- a_{13} a_8 (1 - a_2 a_8) a_4 a_7$$

$$D_6 = (1 - a_6 a_4 - a_1 a_3) - a_2 a_8 (1 - a_6 a_4) - a_9 a_2 a_3 a_8$$
$$- a_5 a_{10} a_8 (a_1 + a_9) - a_{13} a_8 (1 - a_2 a_8)(a_5 a_{11} + a_4 a_7)$$

and

$$D_7 = (1 - a_6 a_4 - a_1 a_3) - a_2 a_8 (1 - a_6 a_4) - a_9 a_2 a_3 a_8 - a_5 a_{10} a_8 (a_1 + a_9) - a_9^2 a_6 a_{12} (a_2 a_3 + a_5 a_{10})$$

RELIABILITY CHARACTERISTICS

The derivation of other reliability characteristics follow exactly the same pattern as in section 1(b) and the corresponding results can be obtained by appropriate substitution of the system parameters.

Before we close this section, we shall make two remarks.

Remark 1: In this study, we have assumed negative exponential distribution for the repair time of the individual units. The analysis can, however, be made even if the repair time distribution is arbitrary. Since the process will be a SMP in this case also, the same type of analysis can be carried out.

Remark 2: As in section 3, chapter 1, we can also evaluate the system characteristics when it is used intermittently by making use of the general process probabilities.

CHAPTER 4

RELIABILITY OF A PARALLEL REDUNDANT SYSTEM WITH PRIORITY REPAIR POLICIES.

CHAPTER 4

RELIABILITY OF A PARALLEL REDUNDANT SYSTEM UNDER PRIORITY REPAIR POLICIES

INTRODUCTION

Last chapter was devoted to the study of some of the reliability characteristics of a standby redundant system. In many situations there is a necessity for the use of parallel redundant systems, for example, electronic equipments in a communication network, surviellance radars, power generators in an operation theatre of a hospital and so on. In this chapter, a study has been made of some ~~of~~ reliability characteristics such as time to system failure and long-run availability of a parallel redundant system. A parallel redundant systems in its most general form may be conceived of having in series k types of units, the type i consisting of N_i units in parallel. We designate such a system as "A series connected $(N_1, N_2, ..N_k)$ - Parallel Redundant System" or simply "$(N_1, N_2, ..N_k)$ - Parallel System" for shortness. This system continues to function so long as at least one unit from each type is active. The system failure happens when all the units of any one type are in the failed state. The units on failure are repaired by a single repair facility following a certain repair policy and put back into operation after repair completion. As there are more than one type of units, the repair is carried out following one of the two types of priority repair policies outlined in the last chapter, namely, preemptive resume priority repair policy and head-of-the-line priority repair policy. Normally, we come across only systems with two types of units and each type having not more than two or three units in parallel redundancy.

In this chapter, the investigations of the reliability characteristics have been divided into the following two sections.

SECTION 1: TIME TO SYSTEM FAILURE OF A PARALELL REDUNDANT SYSTEM UNDER PRIORITY REPAIR POLICIES.

SECTION 2: LONG-RUN AVAILABILITY OF A PARALLEL REDUNDANT SYSTEM UNDER PRIORITY REPAIR POLICIES.

Section 1 deals with a "(2,2) - parallel system" in which two paralleled units of type 1 and two paralleled units of type 2 are connected in series. This system is illustrated in fig.4.0. The distribution of the time to system failure has been studied by deriving its Laplace transform when the failure times of the individual unit are exponentially distributed and their repair times follow an arbitrary distribution.

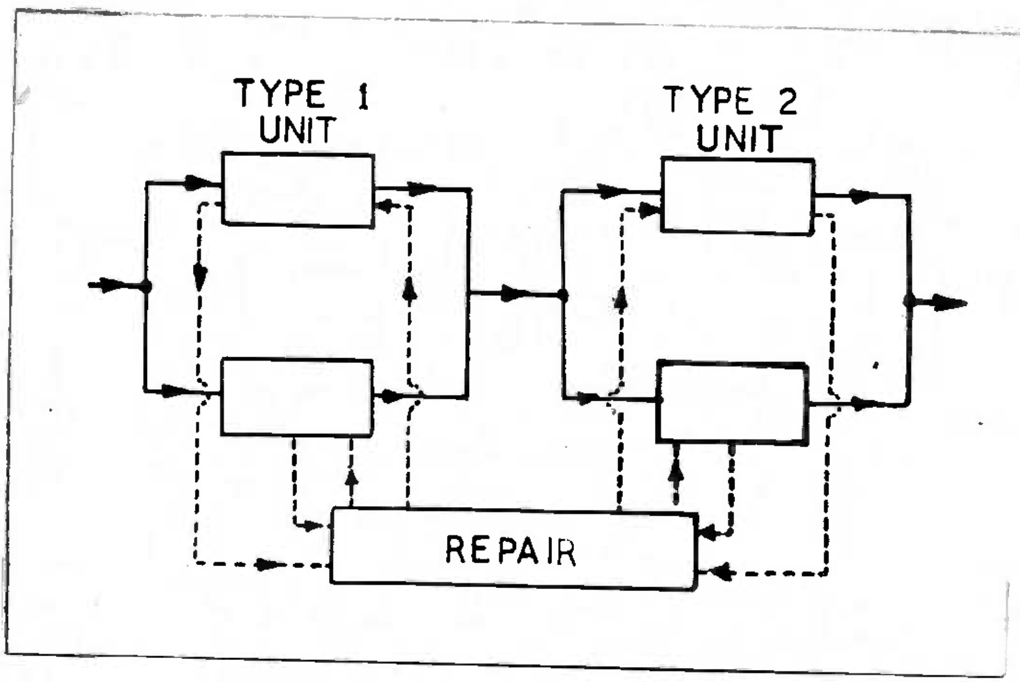


Fig.4.0. A Series Connected (2,2) - parallel System.

The repair of the failed units are carried out by assigning the two priority repair policies, namely, preemptive resume and the head-of-the-line, to the type 1 units. The type 2 units are treated as non-priority units. The discussions are carried out separately for these two cases of priority repair policies and at the end, the effect of interchange of priorities for repair on the time to system failure has been studied when the failure time distributions as well as the repair time distributions are exponential.

The results presented in this section are based on a paper (Natarajan, 1967b) published in IEEE Transactions on Reliability.

In section 2, an analysis is made of an (N_1 , N_2) - parallel system in which N_1 paralleled units of type 1 and N_2 paralleled units of type 2 are connected in series. The main aim of this section is to evaluate the long-run availability of this sytem. This system fails when all the units of any one type fail. To derive the long run availability of the system, the busy period process of the repair facility is investigated first using the supplementary variable technique under the two priority repair policies. Then extending the method of Gaver (1959) the general time-dependent process in which the busy periods of the repair facility alternate with its idle periods has been studied in terms of the busy period probabilities and the probability of finding the repair facility idle at time t, using renewal theoretic arguments. Finally, the long-run availability of the system in terms of the steady state probabilities for the two cases of priority repair policies has been obtained. This section is based on a paper (Natarajan, 1966) to be published in Metrika.

SECTION 1

TIME TO SYSTEM FAILURE OF A PARALLEL REDUNDANT SYSTEM UNDER PRIORITY REPAIR POLICIES

In this section we study the TSF of a (2,2) - parallel system. The type 1 units may constitute two radars and the type 2 units may constitute two computers.

Earlier, Mc Gregar (1963) and Halperin (1964) have studied the time to system failure of a N component parallel system of the same type of units under the assumption that the distributions of failure time and repair time of individual components are negative exponential. Gaver (1963) has considered a two component parallel redundant system with arbitrary distribution of repair time. It is evident that the problem considered here is a generalisation of Gaver's problem as we are having two types of units which enables us to allocate some priority disciplines in repairing the units.

In the (2,2) - parallel system considered here, it is assumed that the failure times of individual units of type i are independently and identically distributed having the distribution function $A_i(x) = 1 - e^{-\lambda_i x}$, $\lambda_i > 0$ And the repair times are distributed according to an arbitrary distribution having the density $S_i(x)$ given by

$$S_i(x) = \eta_i(x) \exp\left[-\int_0^x \eta_i(u)\, du\right], \quad i = 1,2 \qquad (4.1.1)$$

As before, the state of the system is characterised by the number of failed units in the system.

PREEMPTIVE RESUME PRIORITY DISCIPLINE

As a first step towards analysing the system under this priority discipline, we shall define the following state probabilities associated with the time to system failure process of this system.

1. $P_{1,1}(x_1, x_2, t)\, dx_1 dx_2$ - the probability that at time t, there is one priority (type 1) unit under repair with elapsed repair time lying between x_1 and $x_1 + dx_1$, and there is one ordinary (type 2) unit which was preempted earlier when its elapsed repair time was between x_2 and $x_2 + dx_2$

2. $Q_{1,n}(x_1, t)\, dx_1 \; (n = 0,1)$ - the probability that at time t, there is one priority unit and n ordinary units are in the failed state and the priority unit is under repair with elapsed repair time lying between x_1 and $x_1 + dx_1$ while none of the ordinary units has been preempted earlier. Obviously, the ordinaryunits have failed during the repair of the priority unit.

3. $U_1(x_2, t)\, dx_2$ - the probability that at time t, there

is one ordinary unit and no priority units in the failed state and the ordinary unit is undergoing repair with an elapsed repair time lying between x_2 and $x_2 + dx_2$

4. e(t) - the empty state probability, i.e. the probability that at time t, the system is in the 'up' state with no failed units.

The states of this process over which these probabilities have been defined are mutually exclusive and totally exhaustive and provide a Markovian characterisation of the process under discussion. Considering the process at the epochs when the priority units and the ordinary units are taken for repair and the failure and repair process of the individual units during time x_1 and x_2 of the elapsed repair times, we obtain by direct probabilistic arguments

$$P_{1,1}(x_1, x_2, t) = P_{1,1}(0, x_2, t - x_1) \exp\left\{-(\lambda_1 + \lambda_2)x_1 - \int_0^{x_1} \eta_1(u)\,du\right\}, \quad (4,1.2)$$

For, at $t - x_1$ when the ordinary unit has undergone a repair upto x_2 , a priority unit failed and in the interval $(t - x_1, t)$ none of the type 1 and type 2 units working failed and the repair on the type 1 unit is not completed. The combined probability of these events is given by the right hand members of (4,1.2). By similar arguments we can obtain

$$Q_{1,0}(x_1,t) = Q_{1,0}(0,t-x_1)\exp\left\{-(\lambda_1+2\lambda_2)x_1 - \int_0^{x_1}\eta_1(u)\,du\right\}, \quad (4,1\cdot3)$$

$$Q_{1,1}(x_1,t) = Q_{1,0}(0,t-x_1)\binom{2}{1}\left[1-\exp(-\lambda_2 x_1)\right]\exp\left\{-(\lambda_1+\lambda_2)x_1 - \int_0^{x_1}\eta_1(u)\,du\right\}$$

$$= 2\,Q_{1,0}(0,t-x_1)\left(1-\exp(-\lambda_2 x_1)\right)\exp\left\{-(\lambda_1+\lambda_2)x_1 - \int_0^{x_1}\eta_1(u)\,du\right\} \quad (4,1\cdot4)$$

Because of the preemptive nature of the priority discipline, the repair process of the ordinary unit is slightly complicated. While the ordinary unit is under repair, it may be preempted a number of times by the priority units failing during this repair time. Since the number of interruptions occuring in a given interval of time is a Poisson process with parameter $2\lambda_1$, we have

$$U_1(x_2,t) = \int U_1(0,t-\tau)\cdot P_r\left[\text{the repair of the ordinary unit is not completed in time } x_2 \text{ and there are interruptions of priority units, given that these interruptions are cleared in duration of time } t-\tau-x_2\right]d\tau$$

$$= U_1(0,t) * \left[e^{-\int_0^{x_2}\eta_2(y)\,dy} \cdot \sum_{n=0}^{\infty} e^{-2\lambda_1 x_2}\frac{(2\lambda_1 x_2)^n}{n!}\, e^{-\lambda_2 x_2} \left\{\int_0^{t-x_2} e^{-(\lambda_1+\lambda_2)(u+x_2)} s_1(u)\,du\right\}^{n*}\right]$$

where f^{n*} denotes the n-th iterated convolution of a function f with itself. Also, we have the relations

$$P_{1,1}(0, x_2, t) = 2\lambda_1 U_1(x_2, t) \quad (4,1.6)$$

$$Q_{1,0}(0, t) = 2\lambda_1 e(t) \quad (4,1.7)$$

$$U_1(0, t) = \int_0^\infty Q_{1,1}(x_1, t)\,\eta_1(x_1)\,dx_1 + 2\lambda_2\, e(t) \quad (4,1.8)$$

Further, by connecting the empty state probability at time t with that at time $t+\Delta$ and taking the limit as Δ tends to zero, we have

$$\frac{d}{dt} e(t) + 2(\lambda_1+\lambda_2)\, e(t) = \int_0^\infty Q_{1,0}(x_1, t)\,\eta_1(x_1)\,dx_1 + \int_0^\infty U_1(x_2, t)\,\eta_2(x_2)\,dx_2 \quad (4,1.9)$$

Taking the Laplace transform with the initial condition that at time t = 0, the system starts operation with no failed units in the system, the equations (4,1.2) to (4,1.9) become

$$\bar{P}_{11}(x_1, x_2, s) = \bar{P}_{1,1}(0, x_2, s)\, \exp\left\{-(\lambda_1+\lambda_2+s)x_1 - \int_0^{x_1} \eta_1(u)\,du\right\} \quad (4,1.10)$$

$$\bar{Q}_{1,0}(x_1, s) = \bar{Q}_{1,0}(0, s)\, \exp\left\{-(\lambda_1+2\lambda_2+s)x_1 - \int_0^{x_1} \eta_1(u)\,du\right\} \quad (4,1.11)$$

$$\bar{Q}_{1,1}(x_1, s) = 2\,\bar{Q}_{1,0}(0,s)\,(1-\exp(-\lambda_2 x_1)) \cdot \exp\left\{-(\lambda_1+\lambda_2+s)x_1 - \int_0^{x_1} \eta_1(u)\,du\right\} \quad (4,1.12)$$

$$\bar{U}_1(x_2, s) = \bar{U}_1(0, s)\, \exp\left\{-\left[2\lambda_1(1-A_{11}(s)) + \lambda_2 + s\right]x_2 - \int_0^{x_2} \eta_2(y)\,dy\right\} \quad (4,1.13)$$

where $A_{\ell,m}(s) = \bar{S}_1\left[\ell\lambda_1 + m\lambda_2 + s\right]$ and

$$\bar{P}_{11}(0, x_2, s) = 2\lambda_1 \bar{U}_1(x_2, s) \quad (4,1.14)$$

$$\bar{Q}_{1,0}(0,s) = 2\lambda_1 \bar{e}(s) \qquad (4,1.15)$$

$$\bar{U}_1(0,s) = \int_0^\infty \bar{Q}_{1,1}(x_1,s)\,\eta_1(x_1)\,dx_1 + 2\lambda_2 \bar{e}(s) \qquad (4,1.16)$$

On substitution of the value of $\bar{Q}_{1,1}(x_1,s)$ from (4,1.12) in (4,1.16) and simplifying we obtain $\bar{U}_1(0,s)$ as

$$\bar{U}_1(0,s) = 2\,\bar{e}(s)\left[2\lambda_1\left(A_{11}(s) - A_{12}(s)\right) + \lambda_2\right] \qquad (4,1.17)$$

and from (4,1.9)

$$(2\lambda_1 + 2\lambda_2 + s)\,\bar{e}(s) - 1 = 2\lambda_1 \bar{e}(s) A_{12}(s) + \bar{U}_1(0,s) B(s) \qquad (4,1.18)$$

where $\quad B(s) = \bar{S}_2\left[2\lambda_1(1 - A_{11}(s)) + \lambda_2 + s\right]$

Solving for $\bar{e}(s)$ from (4,1.17) and (4,1.18) we obtain

$$\bar{e}(s) = \frac{1}{2}\left[\lambda_1(1 - A_{12}(s)) + \lambda_2(1 - B(s)) - 2\lambda_1\left(A_{11}(s) - A_{12}(s)\right)B(s) + s\right]^{-1} \qquad (4,1.19)$$

Let F(t) denote the survival function of the system and G(t) be the distribution function of TSF. That is, F(t) = $P_r[T_u > t]$, where T_u is the time to failure of the system and

$$G(t) = P_r[T_u \le t] = 1 - F(t) \qquad (4,1.20)$$

Differentiating both sides, we have

$$\frac{d}{dt}G(t) = -\frac{d}{dt}F(t) \qquad (4,1.21)$$

For any system of the type discussed in this section F(t) is the sum of all the probabilities contributing to keep the system in the 'up' or operating state, that is, the probability that at least one unit of each type is in the 'up' state. Hence

$$F(t) = e(t) + \int_0^\infty \int_0^\infty P_{1,1}(x_1, x_2, t)\, dx_1 dx_2 + \int_0^\infty Q_{1,1}(x_1, t)\, dx_1 + \int_0^\infty Q_{1,0}(x_1, t)\, dx_1 + \int_0^\infty U_1(x_2, t)\, dx_2 \tag{4,1.22}$$

Denoting by g(t), the density of the distribution function G(t), we have from (4,1.22)

$$g(t) = -\frac{d}{dt}\left\{ e(t) + \int_0^\infty \int_0^\infty P_{11}(x_1, x_2, t)\, dx_1 dx_2 + \int_0^\infty Q_{1,1}(x_1, t)\, dx_1 + \int_0^\infty Q_{1,0}(x_1, t)\, dx_1 + \int_0^\infty U_1(x_2, t)\, dx_2 \right\} \tag{4,1.23}$$

Alternatively, g(t) can also be written as

$$g(t) = (\lambda_1 + \lambda_2)\left[\int_0^\infty \int_0^\infty P_{1,1}(x_1, x_2, t)\, dx_1 dx_2 + \int_0^\infty Q_{11}(x_1, t)\, dx_1 \right] + \lambda_1 \int_0^\infty Q_{1,0}(x_1, t)\, dx_1 + \lambda_2 \int_0^\infty U_1(x_2, t)\, dx_2 \tag{4,1.24}$$

for failure of either the type 1 or type 2 unit will lead to system failure from the system states considered in the definition of the probabilities.

Taking the Laplace transform of (4,1.23) and noting that the initial condition is, $e(o)=1$, there results

$$\bar{g}(s) = \text{L.T. of } \frac{d}{dt}G(t) = \text{L.T. of } \left[-\frac{d}{dt}F(t)\right] \tag{4,1.25}$$

$$= 1 - s\Big[\bar{e}(s) + \int_0^\infty\int_0^\infty \bar{P}_{11}(x_1,x_2,s)\,dx_1\,dx_2 + \int_0^\infty \bar{Q}_{11}(x_1,s)\,dx_1 + \int_0^\infty \bar{Q}_{1,0}(x_1,s)\,dx_1 + \int_0^\infty \bar{U}_1(x_2,s)\,dx_2\Big] \tag{4,1.26}$$

From (4,1.24) we have for

$$\bar{g}(s) = (\lambda_1+\lambda_2)\Big[\int_0^\infty\int_0^\infty \bar{P}_{11}(x_1,x_2,s)\,dx_1\,dx_2 + \int_0^\infty \bar{Q}_{11}(x_1,s)\,dx_1 + \lambda_1\int_0^\infty \bar{Q}_{1,0}(x_1,s)\,dx_1 + \lambda_2\int_0^\infty \bar{U}_1(x_2,s)\,dx_2 \tag{4,1.27}$$

when $s=0$, (4,1.26) reduces to unity showing thereby G(t) is the honest probability distribution of the TSF. Now, the value of $\bar{g}(s)$ can be obtained by substituting the values of $\bar{P}_{11}(x_1,x_2,s)$, $\bar{Q}_{1,1}(x_1,s)$, $\bar{Q}_{1,0}(x_1,s)$ and $\bar{U}_1(x_2,s)$ in either (4,1.26) or (4,1.27) and simplifying, thus

$$\bar{g}(s) = 2\,\bar{e}(s)\Bigg[\left(\lambda_2 + \frac{2\lambda_1(\lambda_1+\lambda_2)(1-A_{11}(s))}{\lambda_1+\lambda_2+s}\right) \cdot \left(\frac{\lambda_2 + 2\lambda_1(A_{11}(s)-A_{12}(s))}{2\lambda_1(1-A_{11}(s))+\lambda_2}\right)(1-B(s)) + \frac{2\lambda_1(\lambda_1+\lambda_2)(1-A_{11}(s))}{\lambda_1+\lambda_2+s} - \frac{\lambda_1(\lambda_1+2\lambda_2)(1-A_{12}(s))}{\lambda_1+2\lambda_2+s}\Bigg] \tag{4,1.28}$$

where $\bar{e}(s)$ is given by (4,1.19).

The expected time to system failure, $E(T_u)$ is given by

$$E(T_u) = -\frac{d}{ds}\bar{g}(s)\Big|_{s=0} = \int_0^\infty F(t)\,dt = \bar{F}(s)\Big|_{s=0} \qquad (4,1.29)$$

Differentiating $\bar{g}(s)$ and after necessary simplifications we have

$$E(T_u) = \Bigg[\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2+2\lambda_1(1-A_{11}(0))}+1\right) \cdot \Big(2\lambda_1(1-A_{12}(0))+\lambda_2\Big)\big(1-B(0)\big) - \frac{\lambda_1(1-A_{12}(0))}{\lambda_1+2\lambda_2}\Bigg] \Bigg/ \Big[\lambda_1(1-A_{12}(0))+\lambda_2(1-B(0)) - 2\lambda_1(A_{11}(0)-A_{12}(0))B(0)\Big] \qquad (4,1.30)$$

HEAD-OF-THE-LINE PRIORITY DISCIPLINE

Under the head of the line priority discipline the following are the state probabilities associated with the process:

1. $P_{1,n}(x_1,t)\,dx_1 \; (n=0,1)$ - the probability that at time t there are one priority unit and n ordinary units in the failed state and the priority unit is under repair with elapsed repair time lying between x_1 and x_1+dx_1.
2. $Q_{m,1}(x_2,t)\,dx_2 \; (m=0,1)$ - the probability that at time t there are m priority units and one ordinary unit in the failed state and the ordinary unit is undergoing

repair with an elapsed repair time lying between x_2 and $x_2 + dx_2$

3. e(t) - the probability that at time t, the system is in the 'up' state with no failed units.

As we have seen earlier the states of this process over which the probabilities are defined are mutually exclusive and totally exhaustive and provide a Markovian characterisation of the process under discussion. Considering the process at the epochs when the priority units and the ordinary units are taken for repair and the failure and repair process of the individual units during time x_1 and x_2 of the elapsed repair time, we obtain by direct probabilistic arguments, as in the pre-emptive case

$$P_{11}(x_1,t) = P_{1,0}(0,t-x_1)\left[\binom{2}{1}\left(1-\exp(-\lambda_2 x_1)\right)\exp(-\lambda_2 x_1)\exp\left\{-\lambda_1 x_1 - \int_0^{x_1}\eta_1(u)\,du\right\}\right] \tag{4,1.31}$$

Note the second factor on the right hand side of (4,1.31) within the square bracket gives the binomial probability of occurrance of a failure of an ordinary unit, out of two ordinary units functioning in the system in the interval $(t - x_1, t)$ and the third factor gives the probability that in the same interval the remaining priority unit does not fail and the repair of the priority unit is not completed. Similar arguments lead to

$$P_{1,0}(x_1,t) = P_{1,0}(0,t-x_1)\exp\left\{-(\lambda_1+2\lambda_2)x_1 - \int_0^{x_1}\eta_1(u)\,du\right\} \tag{4,1.32}$$

$$Q_{1,1}(x_2,t) = Q_{0,1}(0,t-x_2)\left[\binom{2}{1}(1-\exp(-\lambda_1 x_2))\exp(-\lambda_1 x_2)\right] \cdot \exp\left\{-\lambda_2 x_2 - \int_0^{x_2}\eta_2(y)\,dy\right\} \quad (4,1.33)$$

$$Q_{0,1}(x_2,t) = Q_{0,1}(0,t-x_2)\exp\left\{-(2\lambda_1+\lambda_2)x_2 - \int_0^{x_2}\eta_2(y)\,dy\right\} \quad (4,1.34)$$

Also we have the relations

$$P_{1,0}(0,t) = 2\lambda_1 e(t) + \int_0^\infty Q_{1,1}(x_2,t)\,\eta_2(x_2)\,dx_2 \quad (4,1.35)$$

$$Q_{0,1}(0,t) = 2\lambda_2 e(t) + \int_0^\infty P_{1,1}(x_1,t)\,\eta_1(x_1)\,dx_1 \quad (4,1.36)$$

By continuity arguments as in (4,1.9) we obtain

$$\frac{d}{dt}e(t) + 2(\lambda_1+\lambda_2)e(t) = \int_0^\infty P_{1,0}(x_1,t)\,\eta_1(x_1)\,dx_1 + \int_0^\infty Q_{0,1}(x_2,t)\,\eta_2(x_2)\,dx_2 \quad (4,1.37)$$

Taking Laplace transforms above equations become

$$\bar{P}_{1,1}(x_1,s) = 2\bar{P}_{1,0}(0,s)(1-\exp(-\lambda_2 x_1))\exp\left\{-(\lambda_1+\lambda_2+s)x_1 - \int_0^{x_1}\eta_1(u)\,du\right\} \quad (4,1.38)$$

$$\bar{P}_{1,0}(x_1,s) = \bar{P}_{1,0}(0,s)\exp\left\{-(\lambda_1+2\lambda_2+s)x_1 - \int_0^{x_1}\eta_1(u)\,du\right\} \quad (4,1.39)$$

$$\bar{Q}_{1,1}(x_2,s) = 2\bar{Q}_{0,1}(0,s)(1-\exp(-\lambda_1 x_2))\exp\left\{-(\lambda_1+\lambda_2+s)x_2 - \int_0^{x_2}\eta_2(y)\,dy\right\} \quad (4,1.40)$$

$$\bar{Q}_{0,1}(x_2,s) = \bar{Q}_{0,1}(0,s)\exp\left\{-(2\lambda_1+\lambda_2+s)x_2 - \int_0^{x_2}\eta_2(y)\,dy\right\} \quad (4,1.41)$$

and

$$\bar{P}_{1,0}(0,s) = 2\lambda_1\bar{e}(s) + \int_0^\infty \bar{Q}_{1,1}(x_2,s)\,\eta_2(x_2)\,dx_2 \quad (4,1.42)$$

$$\bar{Q}_{0,1}(0,s) = 2\lambda_2\bar{e}(s) + \int_0^\infty \bar{P}_{1,1}(x_1,s)\,\eta_1(x_1)\,dx_1 \quad (4,1.43)$$

Substituting the values of $\bar{P}_{1,1}(x_1, s)$ and $\bar{Q}_{1,1}(x_2, s)$ from (4,1.38) and (4,1.40) in (4,1.42) and (4,1.43) respectively we get

$$\bar{P}_{1,0}(0,s) = 2\lambda_1 \bar{e}(s) + 2\bar{Q}_{0,1}(0,s)\left[B_{1,1}(s) - B_{2,1}(s)\right], \quad (4,1.44)$$

$$\bar{Q}_{0,1}(0,s) = 2\lambda_2 \bar{e}(s) + 2\bar{P}_{1,0}(0,s)\left[A_{11}(s) - A_{12}(s)\right] \quad (4,1.45)$$

where

$$A_{\ell,m}(s) = \bar{S}_1(\ell\lambda_1 + m\lambda_2 + s) \quad (4,1.46)$$

$$B_{\ell,m}(s) = \bar{S}_2(\ell\lambda_1 + m\lambda_2 + s) \quad (4,1.47)$$

Solving for $\bar{P}_{1,0}(0,s)$ and $\bar{Q}_{0,1}(0,s)$ from these two equations

$$\bar{P}_{1,0}(0,s) = \frac{2}{\Delta}\left[\lambda_1 + 2\lambda_2\left(B_{11}(s) - B_{21}(s)\right)\right]\bar{e}(s), \quad (4,1.48)$$

$$\bar{Q}_{0,1}(0,s) = \frac{2}{\Delta}\left[\lambda_2 + 2\lambda_1\left(A_{11}(s) - A_{12}(s)\right)\right]\bar{e}(s), \quad (4,1.49)$$

where

$$\Delta = 1 - 4\left[A_{11}(s) - A_{12}(s)\right]\left[B_{11}(s) - B_{21}(s)\right] \quad (4,1.50)$$

Using the initial condition that $e(0)=1$, (4,1.37) becomes after taking the Laplace transform

$$\left[2(\lambda_1+\lambda_2)+s\right]\bar{e}(s) - 1$$

$$= \int_0^\infty \bar{Q}_{0,1}(x_2,s)\eta_2(x_2)\,dx_2 + \int_0^\infty \bar{P}_{1,0}(x_1,s)\eta_1(x_1)\,dx_1 \quad (4,1.51)$$

The equation (4,1.51) on substitution of the values of $\bar{P}_{1,0}(x_1,s)$ and $\bar{Q}_{0,1}(x_2,s)$ and simplification reduces to

$$(2\lambda_1+2\lambda_2+s)\,\bar{e}(s)-1 = \frac{\bar{e}(s)}{\Delta}\left[A(s)A_{12}(s)+B(s)B_{21}(s)\right] \qquad (4,1.52)$$

where

$$A(s) = 2\left[\lambda_1+2\lambda_2\left(B_{11}(s)-B_{21}(s)\right)\right] \qquad (4.1.53)$$

$$B(s) = 2\left[\lambda_2+2\lambda_1\left(A_{11}(s)-A_{12}(s)\right)\right] \qquad (4,1,54)$$

Solving for $\bar{e}(s)$ we get

$$\bar{e}(s) = \left[(2\lambda_1+2\lambda_2+s)\Delta - A(s)A_{12}(s)-B(s)B_{21}(s)\right]^{-1}\cdot\Delta \qquad (4,1.55)$$

Following the arguments given for deriving $\bar{g}(s)$ for the preemptive case, the Laplace transform of the density g(t) of the TSF is given by

$$\bar{g}(s) = \lambda_1\int_0^\infty \bar{P}_{1,0}(x_1,s)\,dx_1 + \lambda_2\int_0^\infty \bar{Q}_{0,1}(x_2,s)\,dx_2$$
$$+(\lambda_1+\lambda_2)\left[\int_0^\infty \bar{P}_{1,1}(x_1,s)\,dx_1 + \int_0^\infty \bar{Q}_{1,1}(x_2,s)\,dx_2\right]$$
$$(4,1.56)$$

On substitution of the values of $\bar{P}_{1,0}(x_1,s)$, $\bar{P}_{1,1}(x_1,s)$ $\bar{Q}_{0,1}(x_2,s)$ and $\bar{Q}_{1,1}(x_2,s)$ and simplification (4,1.56) becomes

$$\bar{g}(s) = \frac{\bar{e}(s)}{\Delta}\left[2(\lambda_1+\lambda_2)\frac{1-A_{11}(s)}{\lambda_1+\lambda_2+s}\cdot A(s) + 2(\lambda_1+\lambda_2)\frac{1-B_{11}(s)}{\lambda_1+\lambda_2+s}B(s)\right.$$
$$\left. -(\lambda_1+2\lambda_2)\frac{1-A_{12}(s)}{\lambda_1+2\lambda_2+s}A(s) - \frac{(2\lambda_1+\lambda_2)(1-B_{21}(s))}{2\lambda_1+\lambda_2+s}B(s)\right]$$
$$(4,1.57)$$

$E(T_u)$, the expected TSF is obtained as

$$E(T_u) = -\frac{d}{dt}\,\bar{g}(s)\Big|_{s=0} = \frac{1}{L(0)}\Bigg[\Big\{1-4\,(A_{11}(0)-A_{12}(0))\,(B_{11}(0)-B_{21}(0))\Big) + \frac{2}{\lambda_1+\lambda_2}\Big\{(1-A_{11}(0))A(0) + (1-B_{11}(0))B(0)\Big\} - \frac{1}{\lambda_1+2\lambda_2}(1-A_{12}(0))A(0) - \frac{1}{2\lambda_1+\lambda_2}(1-B_{21}(0))B(0)\Bigg]$$

where

$$L(0) = 2\,(\lambda_1+\lambda_2)\Big\{1-4\,(A_{11}(0)-A_{12}(0))(B_{11}(0)-B_{21}(0))\Big\} - A_{12}(0)\,A(0) - B_{21}(0)\,B(0) \qquad (4,1.58)$$

So far we have considered the derivation of the Laplace transform of the distribution of time to system failure and its expectation under the preemptive resume and head-of-the-line priority disciplines for repair of the failed units. Next, we consider the case when there is no repair, in order to have a comparison of the expected time to system failure in both the situations. For this case, putting $\bar{S}_1(s) = \bar{S}_2(s) = 0$, we have from (4,1.58)

$$\bar{g}(s) = \frac{1}{2\lambda_1+2\lambda_2+s}\left[\frac{4(\lambda_1+\lambda_2)^2}{\lambda_1+\lambda_2+s} - \frac{2\lambda_1(\lambda_1+2\lambda_2)}{\lambda_1+2\lambda_2+s} - \frac{2\lambda_2(2\lambda_1+\lambda_2)}{2\lambda_1+\lambda_2+s}\right] \quad (4,1.59)$$

and

$$E(T_u) = \frac{1}{2(\lambda_1+\lambda_2)}\left[5 - \frac{2\lambda_1}{\lambda_1+2\lambda_2} - \frac{2\lambda_2}{2\lambda_1+\lambda_2}\right] \quad (4,1.60)$$

DISCUSSION OF RESULTS

The main problem confronted in practical situations is(1) which priority should be assigned and (2) which type of unit should be assigned priority. This can be done on the basis of mean time to system failure, $E(T_u)$. We have considered this when $S_i(x) = \mu_i e^{-\mu_i x}, \mu_i > 0$ [Natarajan (1967b)] . To compare the variations exhibited by the mean time to system failure with the two types of repair policies, it is desirable to examine the system with units having

i) longer mean time to failure with longer mean repair times;

ii) longer mean time to failure and shorter mean repair times;

Keeping this in view we have distinguished the following four cases under the preemptive resume and head-of-the-line priority repair policies.

Longer mean repair times:

Case (i): $\lambda_1 = 0.10$, $\lambda_2 = .04$, and
$\mu_2 = 0.1$ for various values
of μ_1

Case (ii): $\lambda_1 = 0.04$, $\lambda_2 = 0.1$, and
$\mu_1 = 0.1$ for various values
of μ_2

Shorter mean repair times:

Case (iii) $\lambda_1 = 0.1$, $\lambda_2 = 0.04$, and
$\mu_2 = 1.0$ for various values
of μ_1

Case (iv): $\lambda_1 = 0.04$, $\lambda_2 = 0.1$ and
$\mu_1 = 1.0$ for various values
of μ_2

Graphs have been drawn (fig.4.1) giving mean TSF for the different cases. To illustrate the use of these graphs, consider the following examples.

Example 1:

When $\lambda_1 = 0.1$, $\lambda_2 = 0.04$, $\mu_1 = 1.0$, and $\mu_2 = 0.1$, the type 1 unit fails more often than type 2 unit and the repair time of type 1 unit is less than that of the type 2. When the priorities for repairing the different types of units are interchanged, the following values of the parameters are obtained, i.e., $\lambda_1 = 0.04$, $\mu_1 = 0.1$,

EXPECTED TIME TO FAILURE FOR DIFFERENT PRIORITY REPAIR POLICIES

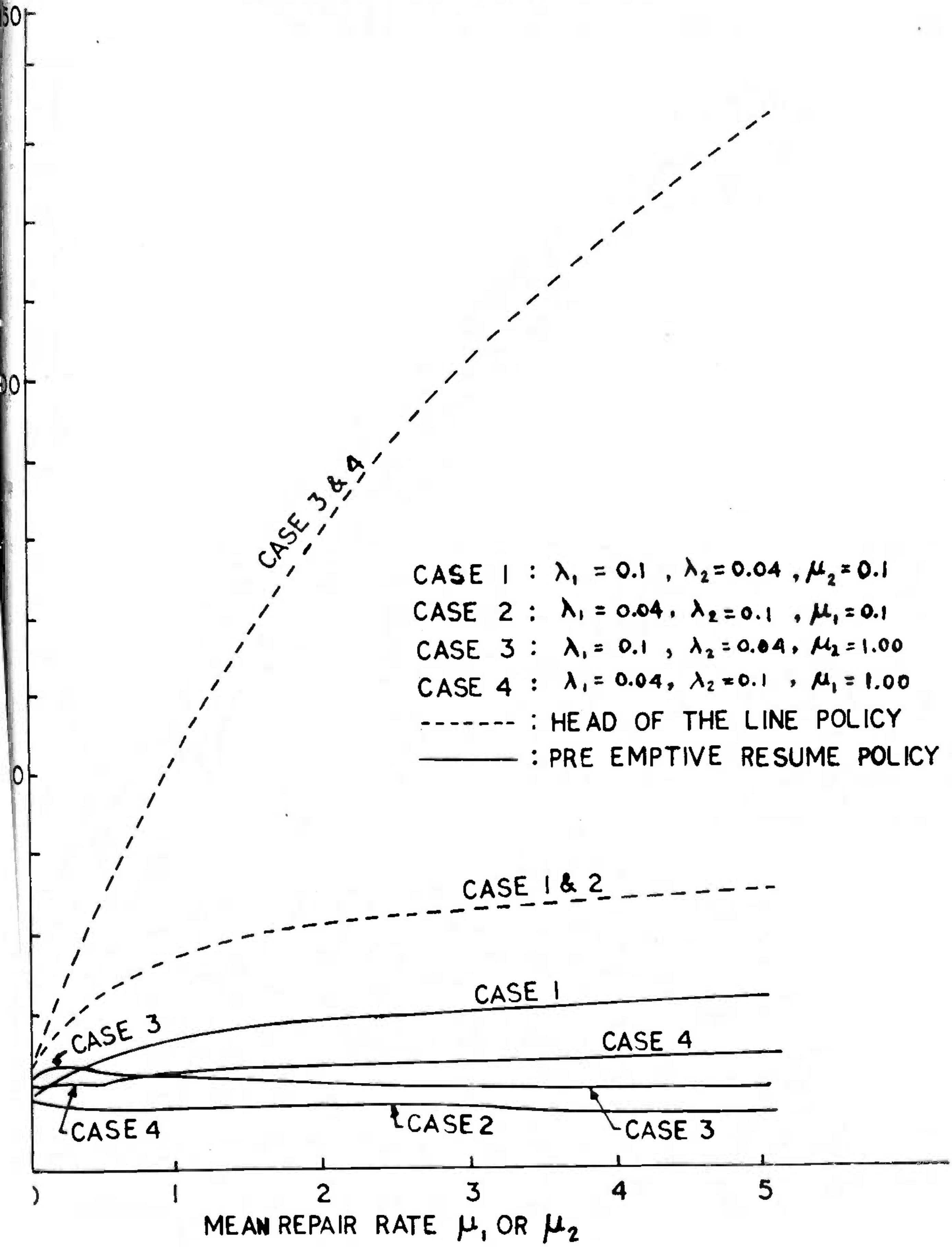


FIG. 4.1

$\lambda_2 = 0.1$, and $\mu_2 = 1.0$. From fig 4.1 we obtain the values given in table 4.1 for the mean time to system failure under various priority disciplines.

TABLE 4. 1

MEAN TIME TO SYSTEM FAILURE FOR CASES (i) & (ii)*

Priority Discipline	For given Values Case (i)	After interchanging Values Case (ii)
Preemptive resume	16.6	7.9
Head-of-the-line	26.8	26.8

* Case (i) $\lambda_1 = 0.1, \lambda_2 = 0.04, \mu_2 = 0.1$ with $\mu_1 = 1.0$, and

Case (ii) $\lambda_1 = 0.04, \lambda_2 = 0.1, \mu_1 = 0.1$ with $\mu_2 = 1.0$.

Table 4.1 reveals that: (1) The interchange of priorities has no effect under the head-of-the-line discipline whether the mean time to failure of the individual unit is short or long. The contrary is found for the preemptive resume discipline. (2) The mean TSF is longer when the head-of-the-line discipline is adopted for repairing the units than when preemptive resume discipline is used.

Example 2:

When $\lambda_1 = 0.1, \lambda_2 = 0.04, \mu_1 = 0.1$, and $\mu_2 = 1$, a type 1 unit fails more often than a type 2 unit and the repair time of the type 2 unit is shorter than that of type 1. That is, the units having longer mean time to failure have shorter mean repair time. The mean time to system failure for different priority disciplines before and after interchaning the priorities is given in Table 4.2.

TABLE 4.2

MEAN TIME TO SYSTEM FAILURE FOR CASES (iii) & (iv)*

Priority Discipline	For given Values Case (iii)	After Interchanging Values Case (iv)
Preemptive resume	12.77	10.59
Head-of-the-line	18	18

* Case (iii): $\lambda_1 = 0.1, \lambda_2 = 0.04, \mu_2 = 1.00$ with $\mu_1 = 0.1$; and

Case (iv): $\lambda_1 = 0.04, \lambda_2 = 0.1, \mu_1 = 1.00$ with $\mu_2 = 0.1$.

From Table 4.2, it is apparent that the head-of-the-line discipline does not affect the mean TSF whereas there is difference in the case of preemptive resume in repairing the two types of units. It may be noted that the difference is not as marked as in the previous example.

CONCLUSIONS

From the study of these two groups, the following points can be observed:

(1) Head-of-the-line policy for repair of priority units is more effective whenever the mean time to failure of the unit is longer and its mean repair times are shorter, as is evident from the higher values of the mean TSF in fig.4.1.

(2) Even when the longer mean time to failure of the unit is associated with longer repair time, the head-of-the-line discipline for repair of the units is still better than the corresponding preemptive resume case.

Hence, it is desirable to repair the units as and when they fail adopting the head-of-the-line discipline for the priority units rather than using preemptive resume discipline. Preemptive resume discipline must be used only under special circumstances governed by emergency or risk considerations.

(3) Comparison of the mean TSF in fig.4.1 when preemptive resume policy is imposed shows that better values

are obtained when longer time to failure is associated with shorter time for repair.

SECTION 2

LONG-RUN AVAILABILITY OF A PARALLEL REDUNDANT SYSTEM UNDER PRIORITY REPAIR POLICIES

In the last section was studied the effect of priority repair policies in increasing the expected TSF of a (2,2) - parallel system. Now, in this section, we shall study the long-run availability of a more general system, namely, the (N_1, N_2) - parallel system. The description of the failure process of the individual units, the repair process and the assignment of priority repair policies are the same as in section 1.

The problem discussed in this section is a generalisation of the problem considered by Gaver (1963) as we have introduced priority repair policies for repairing the failed units. The method adopted here is the same as that of the solution of machine interference problems studied by Takács (1962) and Thiruvengadam (1965).

Now we shall define the following random variables in order to describe the process. Let at time t measured from the start of the system

$m(t)$: the number of priority units in the failed state in the system;

$n(t)$: the number of ordinary units in the failed state in the system;

$X(t)$: the elapsed repair time of the priority unit under repair;

$Y(t)$: the elapsed repair time of the ordinary unit under repair;

$Y_p(t)$: the elapsed repair time of the ordinary unit undergoing repair when it was pre-empted by the priority unit.

Further, it may be noted that the term "unit" is used in a more generalised sense to denote components of an equipment, sub-systems or equipments themselves.

THE BUSY PERIOD PROCESS

Head-of-the-line Case

The busy period of the repair facility is that length of time the facility is busy repairing the ordinary units or priority units. The busy period is initiated by the failure of an ordinary unit, a priority unit, and ends with the departure of an ordinary unit or a priority unit that leaves the repair facility idle for the first time. Obviously, in such a process transitions from any state to any other state are permissible without the intervention of the empty state. A complete Markovian characterisation of the busy period process is provided by the set of mutually exclusive and totally exhaustive states of the process over which the following probabilities are defined

$$P_{m,n}(x,t)\,dx = P_r\left[m(t)=m,\ n(t)=n,\ m(t')+n(t')>0\ (0\le t'\le t),\ x \le X(t) \le x+dx \,/\, m(0)+n(0)=1 \right]$$
$$(1 \le m \le N_1,\ 0 \le n \le N_2)$$

$$Q_{m,n}(y,t)\,dt = P_r\left[m(t)=m,\ n(t)=n,\ m(t')+n(t')>0\ (0\le t'\le t),\ y \le Y(t) \le y+dy \,/\, m(0)+n(0)=1 \right]$$
$$(0 \le m \le N_1,\ 1 \le n \le N_2)$$

It is clear that $P_{m,n}(x,t)dx$ represents the probability at time t during the busy period process there are m priority failed units of which one is undergoing repair with elapsed repair time lying between x and $x+dx$ and n ordinary failed units waiting to get repaired. Similarly, it may be understood that $Q_{m,n}(y,t)dy$ represents the probability that at time t there are m priority failed units waiting for repair while there are n ordinary failed units in the system of which one is undergoing repair with elapsed repair time lying between y and $y+dy$

Now, it is easy to construct the differential difference equations for the process by connecting the various state probabilities at time t and $t+\Delta$ and using the continuity argument as in Keilson and Kooharian (1960). Thus we have

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+(N_1-m)\lambda_1+(N_2-n)\lambda_2+\eta_1(x)\right\}P_{m,n}(x,t)$$
$$=(N_1-m+1)\lambda_1 P_{m-1,n}(x,t)+(N_2-n+1)\lambda_2 P_{m,n-1}(x,t) \qquad (4,2\cdot1)$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+(N_1-m)\lambda_1+(N_2-n)\lambda_2+\eta_2(y)\right\}Q_{m,n}(y,t)$$
$$=(N_1-m+1)\lambda_1 Q_{m-1,n}(y,t)+(N_2-n+1)\lambda_2 Q_{m,n-1}(y,t) \qquad (4,2\cdot2)$$

Let $\gamma(t)$ denote the density of the busy period initiated by the failure of an ordinary unit or a priority unit at time t = 0. Then

$$\gamma(t)=\int_0^\infty P_{1,0}(x,t)\,\eta_1(x)\,dx+\int_0^\infty Q_{0,1}(y,t)\,\eta_2(y)\,dy \qquad (4,2\cdot3)$$

The equations hold with appropriate modifications for the range of values of m and n given at the definitions of the various probabilities. Consideration of repair completions of priority and ordinary units and the failure of a priority unit when an ordinary unit is under repair lead to the boundary conditions

$$P_{m,n}(0,t)=\int_0^\infty P_{m+1,n}(x,t)\,\eta_1(x)\,dx+\int_0^\infty Q_{m,n+1}(y,t)\,\eta_2(y)\,dy$$
$$(1\le m\le N_1,\ 0\le n\le N_2)$$
$$(4,2.4)$$

$$P_{m,n}(0,t) = 0 \qquad \text{for all } n \text{ and } m \geq 1 \tag{4,2.5}$$

$$Q_{0,n}(0,t) = \int_0^\infty Q_{0,n+1}(y,t)\,\eta_2(y)\,dy + \int_0^\infty P_{1,n}(x,t)\,\eta_1(x)\,dx \qquad (1 \leq n \leq N_2) \tag{4,2.6}$$

As for initial conditions, we shall have

$$P_{m,n}(x,0) = \delta_{m,1}\,\delta_{n,0}\,\delta(x)\left[N_1\lambda_1/(N_1\lambda_1 + N_2\lambda_2)\right] \tag{4,2.7}$$

$$Q_{m,n}(y,0) = \delta_{m,0}\,\delta_{n,1}\,\delta(y)\left[N_2\lambda_2/(N_1\lambda_1 + N_2\lambda_2)\right] \tag{4,2.8}$$

where δ_{ij} is the Kronecker delta and $\delta(x)$, $\delta(y)$ are the Dirac delta functions. We choose the initial conditions in this form as they facilitate the simultaneous study of busy period processes initiated by the ordinary unit alone or by the priority unit alone by choosing $N_1\lambda_1 = 0$ or $N_2\lambda_2 = 0$ respectively.

We observe that the equations (4,2.1) and (4,2.2) are linear differential-difference equations in the variables x, y and t. As a first step towards the solution, we take Laplace transform of these equations. Making use of the initial conditions given at (4,2.7) and (4,2.8) the Laplace transforms of (4,2.1) and (4.2.2) become

$$\left\{\frac{\partial}{\partial x} + (N_1 - m)\lambda_1 + (N_2 - n)\lambda_2 + s + \eta_1(x)\right\}\bar{P}_{m,n}(x,s)$$
$$= (N_1 - m + 1)\lambda_1 \bar{P}_{m-1,n}(x,s) + (N_2 - n + 1)\lambda_2 \bar{P}_{m,n-1}(x,s)$$
$$+ \delta_{m,1}\,\delta_{n,0}\,\delta(x)\left(N_1\lambda_1/[N_1\lambda_1 + N_2\lambda_2]\right) \tag{4,2.9}$$

$$\left\{\frac{\partial}{\partial y}+(N_1-m)\lambda_1+(N_2-n)\lambda_2+s+\eta_2(y)\right\}\bar{Q}_{m,n}(y,s)$$
$$=(N_1-m+1)\lambda_1\bar{Q}_{m-1,n}(y,s)+(N_2-n+1)\lambda_2\bar{Q}_{m,n-1}(y,s)$$
$$+\delta_{m,0}\,\delta_{n,1}\,\delta(y)\left[N_2\lambda_2/(N_1\lambda_1+N_2\lambda_2)\right] \qquad (4,2.10)$$

The density of the busy period duration at (4,2.3) is transformed to

$$\bar{\gamma}(s)=\int_0^\infty \bar{P}_{1,0}(x,s)\,\eta_1(x)\,dx+\int_0^\infty \bar{Q}_{0,1}(y,s)\,\eta_2(y)\,dy \qquad (4,2,11)$$

The boundary conditions given by the equations (4,2.4) - (4,2.6) on taking Laplace transform become

$$\bar{P}_{m,n}(0,s)=\int_0^\infty \bar{P}_{m+1,n}(x,s)\,\eta_1(x)\,dx+\int_0^\infty \bar{Q}_{m,n+1}(y,s)\,\eta_2(y)\,dy$$
$$(1\le m\le N_1,\ 0\le n\le N_2) \qquad (4,2.12)$$

$$\bar{Q}_{m,n}(0,s)=0 \quad \text{for all } n \text{ and } m\ge 1 \qquad (4,2.13)$$

$$\bar{Q}_{0,n}(0,s)=\int_0^\infty \bar{Q}_{0,m+1}(y,s)\,\eta_2(y)\,dy+\int_0^\infty \bar{P}_{1,n}(x,s)\,\eta_1(x)\,dx$$
$$(1\le n\le N_2) \qquad (4,2.14)$$

As the set of equations (4,2.9) and (4,2.10) are differential-difference equations with variable coefficients in m and n, the usual generating function technique leads to partial differential equations which are very difficult to solve. To facilitate solution of such types of equations the following discrete transforms [see Thiruvengadam and Jaiswal (1964a)] which transforms P and Q into a new set of quantities A and B are introduced. If $\bar{A}$ and $\bar{B}$ denote the Laplace transform of A and B, then we define

$$\bar{A}_{m,n}(x,s) = \sum_{i=m}^{N_1-1} \binom{i}{m} \sum_{j=n}^{N_2} \binom{j}{n} \bar{P}_{N_1-i,\,N_2-j}(x,s)$$

$$(0 \le m \le N_1-1,\; 0 \le n \le N_2) \qquad (4,2.15)$$

$$\bar{B}_{m,n}(y,s) = \sum_{i=m}^{N_1} \binom{i}{m} \sum_{j=n}^{N_2-1} \binom{j}{n} \bar{Q}_{N_1-i,\,N_2-j}(y,s)$$

$$(0 \le m \le N_1,\; 0 \le n \le N_2-1) \qquad (4,2.16)$$

and we can obtain the busy period probabilities P's and Q's by means of the following inverse transforms,

$$\bar{P}_{m,n}(x,s) = \sum_{i=N_1-m}^{N_1-1} (-1)^{i+m-N_1} \binom{i}{i+m-N_1} \sum_{j=N_2-n}^{N_2} (-1)^{j+n-N_2} \binom{j}{j+n-N_2} \bar{A}_{i,j}(x,s)$$

$$(4,2.17)$$

$$\bar{Q}_{m,n}(y,s) = \sum_{i=N_1-m}^{N_1} (-1)^{i+m-N_1} \binom{i}{i+m-N_1} \sum_{j=N_2-n}^{N_2-1} (-1)^{j+n-N_2} \binom{j}{j+n-N_2} \bar{B}_{i,j}(y,s)$$

$$(4,2.18)$$

Using the discrete transforms, the equations (4,2.9) - (4,2.11) reduce to

$$\left\{ \frac{\partial}{\partial x} + m\lambda_1 + n\lambda_2 + s + \eta_1(x) \right\} \bar{A}_{m,n}(x,s)$$

$$= \binom{N_1-1}{m} \binom{N_2}{n} \delta(x) \left(N_1\lambda_1 / [N_1\lambda_1 + N_2\lambda_2] \right)$$

$$(4,2.19)$$

$$\left\{\frac{\partial}{\partial y}+m\lambda_1+n\lambda_2+s+\eta_2(y)\right\}\bar{B}_{m,n}(y,s)$$

$$=\binom{N_1}{m}\binom{N_2-1}{n}\delta(y)\left(N_2\lambda_2/[N_1\lambda_1+N_2\lambda_2]\right) \qquad (4,2\cdot 20)$$

$$\bar{\gamma}(s)=\int_0^\infty \bar{A}_{N_1-1,N_2}(x,s)\,\eta_1(x)\,dx$$

$$+\int_0^\infty \bar{B}_{N_1,N_2-1}(y,s)\,\eta_2(y)\,dy \qquad (4,2\cdot 21)$$

and the boundary conditions (4.2.12) - (4.2.14) become in terms of transforms

$$\bar{A}_{m,n}(0,s)=\int_0^\infty\Big[\bar{A}_{m,n}(x,s)+\bar{A}_{m-1,n}(x,s)$$

$$-\binom{N_1}{m}\bar{A}_{N_1-1,m}(x,s)\Big]\eta_1(x)\,dx$$

$$+\int_0^\infty\Big[\bar{B}_{m,n}(y,s)-\binom{N_1}{m}\bar{B}_{N_1,n}(y,s) \qquad (4,2\cdot 22)$$

$$+\bar{B}_{m,n-1}(y,s)-\binom{N_1}{m}\bar{B}_{N_1,n-1}(y,s)\Big]\eta_2(y)\,dy$$

$$\bar{B}_{m,n}(0,s)-\binom{N_1}{n}\bar{B}_{N_1,n}(0,s)=0 \quad \text{for all } n \qquad (4,2\cdot 23)$$

$$\bar{B}_{N_1,n}(0,s)=\int_0^\infty\Big[\bar{B}_{N_1,n}(y,s)+\bar{B}_{N_1,n-1}(y,s)\Big]\eta_2(y)\,dy$$

$$+\int_0^\infty \bar{A}_{N_1-1,n}(x,s)\,\eta_1(x)\,dx-\binom{N_2}{n}\bar{\gamma}(s)$$

$$(4,2\cdot 24)$$

The equations (4,2.19) and (4,2.20) obtained through the discrete transforms are in a form in which they can be easily solved. The solution of these linear differential equations are given by

$$\bar{A}_{m,n}(x,s) = \bar{A}'_{m,n}(0,s)\exp\left[-(m\lambda_1+n\lambda_2+s)x - \int_0^x \eta_1(u)\,du\right] \qquad (4,2\cdot25)$$

$$\bar{B}_{m,n}(y,s) = \bar{B}'_{m,n}(0,s)\exp\left[-(m\lambda_1+n\lambda_2+s)y - \int_0^y \eta_2(u)\,du\right] \qquad (4,2\cdot26)$$

where

$$\bar{A}'_{m,n}(0,s) = \bar{A}_{m,n}(0,s) + \binom{N_1-1}{m}\binom{N_2}{n}\left(N_1\lambda_1 / [N_1\lambda_1+N_2\lambda_2]\right)$$

$$\bar{B}'_{m,n}(0,s) = \bar{B}_{m,n}(0,s) + \binom{N_1}{m}\binom{N_2-1}{n}\left(N_2\lambda_2 / [N_1\lambda_1+N_2\lambda_2]\right)$$

Using (4,2.25) and (4,2.26) in (4,2.24) and integrating we obtain

$$\bar{B}'_{N_1,n}(0,s) = \bar{B}'_{N_1,n}(0,s)\,\theta_{N_1,n}(s) + \bar{B}'_{N_1,n-1}(0,s)\,\theta_{N_1,n-1}(s)$$
$$+ \bar{A}'_{N_1-1,n}(0,s)\,\xi_{N_1-1,n}(s) - \binom{N_2}{n}\bar{\gamma}(s)$$
$$+ \binom{N_2-1}{n}\left(N_2\lambda_2 / [N_1\lambda_1+N_2\lambda_2]\right) \qquad (4,2\cdot27)$$

where

$$\theta_{m,n}(s) = \bar{S}_2(m\lambda_1+n\lambda_2+s)$$

$$\xi_{m,n}(s) = \bar{S}_1(m\lambda_1+n\lambda_2+s)$$

Similarly, from (4,2.22) using the relation (4,2.23) and solving

$$\bar{A}'_{m,n}(0,s)\left[1-\xi_{m,n}(s)\right] = \bar{A}'_{m-1,n}(0,s)\,\xi_{m-1,n}(s) -$$

$$- \binom{N_1}{m} \bar{A}'_{N_1-1,n}(0,s)\,\xi_{N_1-1,n}(s) + \bar{B}'_{N_1,n}(0,s)\binom{N_1}{m}\left\{\theta_{m,n}(s) - \theta_{N_1,n}(s)\right\}$$

$$+ \bar{B}'_{N_1,n-1}(0,s)\binom{N_1}{m}\left\{\theta_{m,n-1}(s) - \theta_{N_1,n-1}(s)\right\}$$

$$+ \binom{N_1-1}{m}\binom{N_2}{n}\left(N_1\lambda_1 / [N_1\lambda_1 + N_2\lambda_2]\right) \qquad (4,2.28)$$

Defining the products

$$\phi'_m(n,s) = \begin{cases} \prod_{r=0}^{m} \left[\dfrac{\xi_{r-1,n}(s)}{1 - \xi_{r,n}(s)} \right], & m > 0 \\ \dfrac{1}{1 - \xi_{0,n}(s)}, & m = 0 \\ 1, & m = -1 \end{cases}$$

and

$$(4,2.29)$$

$$\phi_m(n,s) = \begin{cases} \prod_{r=0}^{m} \left[\dfrac{\xi_{r,n}(s)}{1 - \xi_{r,n}(s)} \right], & m \geq 0 \\ 1, & m = -1 \end{cases}$$

$$(4,2.30)$$

We note

$$\phi_{m-1}(n,s) = [1 - \xi_{m,n}(s)]\, \phi'_m(n,s) = \xi_{m-1,n}(s)\, \phi'_{m-1}(n,s) \qquad (4,2\cdot31)$$

Dividing (4,2.28) throughout by $\phi_{m-1}(n,s)$ and using (4,2.31) we have

$$\frac{\bar{A}'_{m,n}(0,s)}{\phi'_m(n,s)} = \frac{\bar{A}'_{m-1,n}(0,s)}{\phi'_{m-1}(n,s)} - \bar{A}'_{N_1-1,n}(0,s)\, \xi_{N_1-1,n}(s) \binom{N_1}{m} \frac{1}{\phi_{m-1}(n,s)}$$

$$+ \bar{B}'_{N_1,n}(0,s) \binom{N_1}{m} \frac{\theta_{m,n}(s) - \theta_{N_1,n}(s)}{\phi_{m-1}(n,s)} + \bar{B}'_{N_1,n-1}(0,s) \binom{N_1}{m} \frac{\theta_{m,n-1}(s) - \theta_{N_1,n-1}(s)}{\phi_{m-1}(n,s)}$$

$$+ \binom{N_1-1}{m} \binom{N_2}{n} \left(N_1\lambda_1 / [N_1\lambda_1 + N_2\lambda_2] \right) \qquad (4,2\cdot32)$$

The equation (4,2.32) holds good for the values of m from 0 to N_1 as $\bar{A}_{m,n}(0,s) = 0$ for m = -1 and N_1 by definition. Adding the equations (4,2.32) for the value of m from 0 to N_1, we obtain

$$\bar{A}'_{N_1-1,n}(0,s)\, \xi_{N_1-1,n}(s)$$

$$= \left[\sum_{r=0}^{N_1} \binom{N_1}{r} \frac{1}{\phi_{r-1}(n,s)} \right]^{-1} \left[\bar{B}'_{N_1,n}(0,s) \sum_{r=0}^{N_1} \binom{N_1}{r} \frac{\theta_{r,n}(s) - \theta_{N_1,n}(s)}{\phi_{r-1}(n,s)} \right.$$

$$+ \bar{B}'_{N_1,n-1}(0,s) \sum_{r=0}^{N_1} \binom{N_1}{r} \frac{\theta_{r,n-1}(s) - \theta_{N_1,n-1}(s)}{\phi_{r-1}(n,s)}$$

$$\left. + \left(N_1\lambda_1 / [N_1\lambda_1 + N_2\lambda_2] \right) \binom{N_2}{n} \sum_{r=0}^{N_1-1} \binom{N_1-1}{r} \frac{1}{\phi_{r-1}(n,s)} \right] \qquad (4,2\cdot33)$$

Substituting (4,2.33) in (4,2.27) and simplifying we get

$$\bar{B}'_{N_1,n}(0,s)\left\{1-\bar{U}(n\lambda_2+s)\right\} = \bar{B}'_{N_1,n-1}(0,s)\,K_n(s)$$
$$+\,(N_1\lambda_1/[N_1\lambda_1+N_2\lambda_2])\binom{N_2}{n}\bar{G}(n\lambda_2+s) - \binom{N_2}{n}\bar{\gamma}(s)$$
$$+\,(N_2\lambda_2/[N_1\lambda_1+N_2\lambda_2])\binom{N_2-1}{n} \qquad (4,2\cdot 34)$$

where

$$\bar{U}(n\lambda_2+s) = \left[\sum_{r=0}^{N_1}\binom{N_1}{r}\frac{\theta_{r,n}(s)}{\phi_{r-1}(n,s)} \Big/ \sum_{r=0}^{N_1}\binom{N_1}{r}\frac{1}{\phi_{r-1}(n,s)}\right] \qquad (4,2\cdot 35)$$

$$K_n(s) = \begin{cases} \left[\sum_{r=0}^{N_1}\binom{N_1}{r}\frac{\theta_{r,n-1}(s)}{\phi_{r-1}(n,s)} \Big/ \sum_{r=0}^{N_1}\binom{N_1}{r}\frac{1}{\phi_{r-1}(n,s)}\right], & n>0 \\ 1 & n=0 \end{cases} \qquad (4,2\cdot 36)$$

and

$$\bar{G}(n\lambda_2+s) = \left[\sum_{r=0}^{N_1-1}\binom{N_1-1}{r}\frac{1}{\phi_{r-1}(n,s)} \Big/ \sum_{r=0}^{N_1}\binom{N_1}{r}\frac{1}{\phi_{r-1}(n,s)}\right] \qquad (4,2\cdot 37)$$

Now we define the product

$$\psi_n(s) = \begin{cases} \prod_{r=0}^{n} \left[\dfrac{K_r(s)}{1-\bar{U}(r\lambda_2+s)} \right] & , \ n \geq 0 \\ 1 & , \ n = -1 \end{cases} \qquad (4,2\cdot38)$$

and note

$$\psi_r(s)\left[1-\bar{U}(r\lambda_2+s)\right] = K_r(s)\,\psi_{r-1}(s) \ , \qquad (4,2\cdot39)$$

Changing n to r, dividing by $\psi_r(s)\left[1-\bar{U}(r\lambda_2+s)\right]$ throughout and using (4,2.39), the equation (4,2.34) becomes

$$\frac{\bar{B}'_{N_1,r}(0,s)}{\psi_r(s)} = \frac{\bar{B}'_{N_1,r-1}(0,s)}{\psi_{r-1}(s)} + \frac{N_1\lambda_1}{N_1\lambda_1+N_2\lambda_2}\binom{N_2}{r}\frac{\bar{G}(r\lambda_2+s)}{K_r(s)\,\psi_{r-1}(s)}$$

$$-\bar{\gamma}(s)\binom{N_2}{r}\frac{1}{K_r(s)\,\psi_{r-1}(s)} + \frac{N_2\lambda_2}{N_1\lambda_1+N_2\lambda_2}\binom{N_2-1}{r}\frac{1}{K_r(s)\,\psi_{r-1}(s)}$$

$$(4,2\cdot40)$$

This difference equation (4,2.40) holds for r from 0 to N_2 as $\bar{B}_{N_1,n}(0,s) = 0$ for n = - 1 and N_2 by definition. This equation can be recursively solved to obtain

$$\frac{\bar{B}'_{N_1,n}(0,s)}{\psi_n(s)} = \frac{N_1\lambda_1}{N_1\lambda_1+N_2\lambda_2}\sum_{r=0}^{n}\binom{N_2}{r}\frac{\bar{G}(r\lambda_2+s)}{K_r(s)\,\psi_{r-1}(s)}$$

$$+\frac{N_2\lambda_2}{N_1\lambda_1+N_2\lambda_2}\sum_{r=0}^{n}\binom{N_2-1}{r}\frac{1}{K_r(s)\,\psi_{r-1}(s)} - \bar{\gamma}(s)\sum_{r=0}^{n}\binom{N_2}{r}\frac{1}{K_r(s)\,\psi_{r-1}(s)}$$

$$(4,2.41)$$

Changing n to N_2 in (4,2.41) and noting that $\bar{B}'_{N_1,N_2}(o,s)=0$, we get $\bar{\gamma}(s)$ as

$$\bar{\gamma}(s) = \frac{\left\{\frac{N_1\lambda_1}{N_1\lambda_1+N_2\lambda_2}\sum_{r=0}^{N_2}\binom{N_2}{r}\frac{\bar{G}(r\lambda_2+s)}{K_r(s)\psi_{r-1}(s)} + \frac{N_2\lambda_2}{N_1\lambda_1+N_2\lambda_2}\sum_{r=0}^{N_2-1}\binom{N_2-1}{r}\frac{1}{K_r(s)\psi_{r-1}(s)}\right\}}{\sum_{r=0}^{N_2}\binom{N_2}{r}\frac{1}{K_r(s)\psi_{r-1}(s)}}$$

(4,2.42)

We have now the Laplace transform of the probability density $\gamma(t)$ of the busy period duration given by the equation (4,2.42).

The mean duration of busy period of the repair facility is obtained by differentiating $\bar{\gamma}(s)$ at s = 0. Hence

$$-\bar{\gamma}'(0) = -\frac{\partial}{\partial s}\bar{\gamma}(s)\Big|_{s=0} = \int_0^\infty t\,\gamma(t)\,dt$$

$$= \frac{N_1\lambda_1}{N_1\lambda_1+N_2\lambda_2}\left(-\bar{G}'(0)\right) + \left(-\bar{U}'(0)\right)\left\{\frac{N_1\lambda_1}{N_1\lambda_1+N_2\lambda_2}\sum_{r=1}^{N_2}\binom{N_2}{r}\frac{1-\bar{G}(r\lambda_2)}{K_r\psi_{r-1}} + \frac{N_2\lambda_2}{N_1\lambda_1+N_2\lambda_2}\sum_{r=0}^{N_2-1}\binom{N_2-1}{r}\frac{1}{K_{r+1}\psi_r}\right\}$$

(4,2.43)

where

$$-\bar{G}'(0) = -\frac{\partial}{\partial s}\bar{G}(s)\Big|_{s=0} = \eta_1\sum_{r=0}^{N_1-1}\binom{N_1-1}{r}\frac{1}{\phi_r}$$

(4,2.44)

$$-\bar{U}'(0) = -\frac{\partial}{\partial s}\bar{U}(s)\Big|_{s=0} = \eta_1 \sum_{r=1}^{N_1} \binom{N_1}{r} \frac{1-\bar{S}_2(r\lambda_2)}{\phi_{r-1}} + \eta_2 \qquad (4,2.45)$$

and η_1 and η_2 are the mean repair times of the priority and the ordinary units respectively. That is

$$\eta_1 = \int_0^\infty x\, S_1(x)\, dx \;,\quad \eta_2 = \int_0^\infty x\, S_2(x)\, dx \qquad (4,2.46)$$

and

$$K_r = K_r(0)\,,\quad \psi_n = \prod_{r=1}^{n} \left\{ K_r(0) \big/ \left[1-\bar{U}(r\lambda_2)\right]\right\} \;, \qquad (4,2.47)$$

$$\phi_m = \prod_{r=1}^{m} \left\{ \xi_{r,0}(0) \big/ \left[1-\xi_{r,0}(0)\right]\right\} \;, \qquad (4,2.48)$$

We may observe here that $-G'(0)$ is nothing but the mean duration of the busy period of the repair facility engaged in repairing the priority units alone and $-\bar{U}'(0)$ is the average completion time of the ordinary units [See Thiruvengadam and Jaiswal (1964a)] , the completion time of the ordinary unit being defined as the length of time from the time at which an ordinary unit is taken ~~from~~ for repair to the time at which the repair facility is free to start repair on the next ordinary unit.

<u>Pre-emptive Resume Case</u>

As in the head-of-the-line case, the following probabilities are defined over a set of mutually exclusive and totally exhaustive states which provide a complete Markovian characterisation of the busy period process

$$\begin{aligned} P_{m,n}(x,y,t)\,dx\,dy = P_r\big[\, & m(t)=m,\ n(t)=n,\ m(t')+n(t')>0 \\ & (0 \le t' \le t),\ x \le X(t) \le x+dx \\ & y \le Y_p(t) \le y+dy \,/\, m(0)+n(0)=1 \big] \\ & (1 \le m \le N_1,\ 1 \le n \le N_2) \end{aligned}$$

$$\begin{aligned} Q_{m,n}(x,t)\,dx = P_r\big[\, & m(t)=m,\ n(t)=n,\ m(t')+n(t')>0 \\ & (0 \le t' \le t),\ x \le X(t) \le x+dx \,/ \\ & m(0)+n(0)=1 \big] \\ & (1 \le m \le N_1,\ 0 \le n \le N_2) \end{aligned}$$

$$\begin{aligned} R_n(y,t)\,dy = P_r\big[\, & m(t)=0,\ n(t)=n,\ m(t')+n(t')>0\ (0 \le t' \le t), \\ & y \le Y(t) \le y+dy \,/\, m(0)+n(0)=1 \big] \\ & (1 \le n \le N_2) \end{aligned}$$

Naturally, with these definitions, $P_{m,n}(x,y,t)dx\,dy$ represents the probability that at time t, there are m priority units and n non-priority units in the failed state and a priority unit is being repaired having an elapsed repair time lying between x and x + dx after preempting an ordinary unit and at the time of preemption, the ordinary unit had an elapsed repair time lying between y and y + dy. Similar meanings could be understood for the probabilities $Q_{m,n}(x,t)\,dx$ and $R_n(y,t)\,dy$

The differential-difference equations describing the process are obtained by connecting the various state probabilities at time t and $t+\Delta$ and using the continuity argument as in the head-of-the-line case. These are

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+(N_1-m)\lambda_1+(N_2-n)\lambda_2+\eta_1(x)\right\}P_{m,n}(x,y,t)$$
$$=(N_1-m+1)\lambda_1 P_{m-1,n}(x,y,t)+(N_2-n+1)\lambda_2 P_{m,n-1}(x,y,t) \qquad (4,2\cdot49)$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+(N_1-m)\lambda_1+(N_2-n)\lambda_2+\eta_1(x)\right\}Q_{m,n}(x,t)$$
$$=(N_1-m+1)\lambda_1 Q_{m-1,n}(x,t)+(N_2-n+1)\lambda_2 Q_{m,n-1}(x,t) \qquad (4,2\cdot50)$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+N_1\lambda_1+(N_2-n)\lambda_2+\eta_2(y)\right\}R_n(y,t)$$
$$=(N_2-n+1)\lambda_2 R_{n-1}(y,t)+\int_0^\infty P_{1,n}(x,y,t)\,\eta_1(x)\,dx \qquad (4,2\cdot51)$$

Obviously, the busy period density $\gamma(t)$ is

$$\gamma(t)=\int_0^\infty R_1(y,t)\,\eta_2(y)\,dy+\int_0^\infty Q_{1,0}(x,t)\,\eta_1(x)\,dx \qquad (4,2\cdot52)$$

The boundary conditions are

$$P_{m,n}(0,y,t)=\int_0^\infty P_{m+1,n}(x,y,t)\,\eta_1(x)\,dx$$
$$+\delta_{1,m}N_1\lambda_1 R_n(y,t) \qquad (4,2\cdot53)$$

$$P_{N_1,n}(0,y,t) = 0 \quad \text{for all } n \tag{4,2.54}$$

$$Q_{m,n}(0,t) = \int_0^\infty Q_{m+1,n}(x,t)\,\eta_1(x)\,dx \tag{4,2.55}$$

$$Q_{N_1,n}(0,t) = 0 \quad \text{for all } n \tag{4,2.56}$$

$$R_n(0,t) = \int_0^\infty R_{n+1}(y,t)\,\eta_2(y)\,dy + \int_0^\infty Q_{1,n}(x,t)\,\eta_1(x)\,dx \tag{4,2.57}$$

with initial conditions

$$Q_{1,0}(x,0) = \delta_{m,1}\,\delta_{n,0}\,\delta(x)\,(N_1\lambda_1/[N_1\lambda_1+N_2\lambda_2]) \tag{4,2.58}$$

$$R_1(y,0) = \delta_{n,1}\,\delta(y)\,(N_2\lambda_2/[N_1\lambda_1+N_2\lambda_2]) \tag{4,2.59}$$

We observe that the set of equations (4,2.49) - (4,2.51) are differential difference equations with variable coefficients as in the head-of-the-line priority case. The method of solution of these equations along with the boundary conditions (4,2.53) - (4,2.57) proceeds on the same lines as in that case, namely, through Laplace transforms and discrete transforms. The discrete transforms that are used in this case are defined as

$$\bar{A}_{m,n}(x,y,s) = \sum_{i=m}^{N_1-1} \binom{i}{m} \sum_{j=n}^{N_2-1} \binom{j}{n} \bar{P}_{N_1-i,\,N_2-j}(x,y,s)$$

$$(0 \le m \le N_1-1,\ 0 \le n \le N_2-1) \tag{4,2.60}$$

$$\bar{B}_{m,n}(x,s) = \sum_{i=m}^{N_1-1} \binom{i}{m} \sum_{j=n}^{N_2} \binom{j}{n} \bar{Q}_{N_1-i,\, N_2-j}(x,s)$$

$$(0 \le m \le N_1-1,\ 0 \le n \le N_2) \qquad (4,2.61)$$

$$\bar{C}_n(y,s) = \sum_{j=n}^{N_2-1} \binom{j}{n} \bar{R}_{N_2-j}(y,s) \qquad (4,2.62)$$

$$(0 \le n \le N_2-1)$$

The inverse transforms expressing P, Q, R in terms of A, B and C are

$$\bar{P}_{m,n}(x,y,s) = \sum_{i=N_1-m}^{N_1-1} (-1)^{i+m-N_1} \binom{i}{i+m-N_1} \sum_{j=N_2-n}^{N_2-1} (-1)^{j+n-N_2} \binom{j}{j+n-N_2} \bar{A}_{ij}(x,y,s)$$

$$(4,2.63)$$

$$\bar{Q}_{m,n}(x,s) = \sum_{i=N_1-m}^{N_1-1} (-1)^{i+m-N_1} \binom{i}{i+m-N_1} \sum_{j=N_2-n}^{N_2} (-1)^{j+n-N_2} \binom{j}{j+n-N_2} \bar{B}_{ij}(x,s)$$

$$(4,2.64)$$

$$\bar{R}_n(y,s) = \sum_{j=N_2-n}^{N_2-1} (-1)^{j+n-N_2} \binom{j}{j+n-N_2} \bar{C}_j(y,s) \qquad (4,2.65)$$

Using the above transforms (4,2.60) - (4,2.62) to transform (4,2.49) - (4,2.57) and solving the equations so transformed, we obtain

$$\bar{A}_{m,n}(x,y,s) = \bar{A}_{m,n}(0,y,s) \exp\left[-(m\lambda_1+n\lambda_2+s)x - \int_0^x \eta_1(u)\,du\right]$$

$$(4,2.66)$$

$$\bar{B}_{m,n}(x,s) = \bar{B}'_{m,n}(0,s) \exp\left[-(m\lambda_1+n\lambda_2+s)x - \int_0^x \eta_1(u)\,du\right]$$

$$(4,2.67)$$

$$\bar{C}_n(y,s) = \bar{C}'_n(0,s) \exp\left[-\left\{N_1\lambda_1(1-\bar{G}(n\lambda_2+s)) + n\lambda_2 + s\right\}y - \int_0^y \eta_2(u)\,du\right] \qquad (4,2\cdot 68)$$

where

$$\frac{\bar{A}_{m,n}(0,y,s)}{\phi'_m(n,s)} = N_1\lambda_1 \bar{C}_n(y,s)\left[\sum_{r=0}^{m}\binom{N_1-1}{r}\frac{1}{\phi_{r-1}(n,s)} - \bar{G}(n\lambda_2+s)\sum_{r=0}^{m}\binom{N_1}{r}\frac{1}{\phi_{r-1}(n,s)}\right] \qquad (4,2\cdot 69)$$

$$\frac{\bar{B}'_{m,n}(0,s)}{\phi'_m(n,s)} = \frac{N_1\lambda_1}{N_1\lambda_1+N_2\lambda_2}\binom{N_2}{n}\left[\sum_{r=0}^{m}\binom{N_1-1}{r}\frac{1}{\phi_{r-1}(n,s)} - \bar{G}(n\lambda_2+s)\sum_{r=0}^{m}\binom{N_1}{r}\frac{1}{\phi_{r-1}(n,s)}\right] \qquad (4,2\cdot 70)$$

$$\frac{\bar{C}'_n(0,s)}{\omega'_n(s)} = \left[\frac{N_1\lambda_1}{N_1\lambda_1+N_2\lambda_2}\sum_{r=0}^{n}\binom{N_2}{r}\frac{\bar{G}(r\lambda_2+s)}{\omega_{r-1}(s)} + \frac{N_2\lambda_2}{N_1\lambda_1+N_2\lambda_2}\sum_{r=0}^{n}\binom{N_2-1}{r}\frac{1}{\omega_{r-1}(s)}\right] - \bar{\gamma}(s)\sum_{r=0}^{n}\binom{N_2}{r}\frac{1}{\omega_{r-1}(s)} \qquad (4,2\cdot 71)$$

And the Laplace transform of the busy period distribution is given by

$$\bar{\gamma}(s) = \frac{\dfrac{N_1\lambda_1}{N_1\lambda_1+N_2\lambda_2}\displaystyle\sum_{r=0}^{N_2}\binom{N_2}{r}\frac{\bar{G}(r\lambda_2+s)}{\omega_{r-1}(s)} + \dfrac{N_2\lambda_2}{N_1\lambda_1+N_2\lambda_2}\displaystyle\sum_{r=0}^{N_2-1}\binom{N_2-1}{r}\frac{1}{\omega_{r-1}(s)}}{\displaystyle\sum_{r=0}^{N_2}\binom{N_2}{r}\frac{1}{\omega_{r-1}(s)}}$$

$$(4,2\cdot 72)$$

where

$$\omega_m(s) = \prod_{r=0}^{m} \frac{V(r\lambda_2+s)}{1-V(r\lambda_2+s)}, \quad \omega_{-1}(s) = 1 \qquad (4,2.73)$$

$$\omega'_m(s) = \prod_{r=0}^{m} \frac{V(\overline{r-1}\,\lambda_2+s)}{1-V(r\lambda_2+s)}, \quad \omega'_0(s) = \frac{1}{1-V(s)}, \quad \omega'_{-1}(s) = 1 \qquad (4,2.74)$$

and

$$V(r\lambda_2+s) = \bar{S}_2\left[N_1\lambda_1(1-\bar{G}(r\lambda_2+s)) + r\lambda_2+s\right] \qquad (4,2.75)$$

The expressions for $\phi'_m(n,s)$ and $\phi_m(n,s)$ have already been defined by the equations (4,2.29) and (4,2.30).

As before, the mean duration of busy period of the repair facility is obtained by differentiating $\bar{\gamma}(s)$ at $s = 0$. Therefore,

$$-\bar{\gamma}'(0) = -\frac{\partial}{\partial s}\bar{\gamma}(s)\Big|_{s=0} = \int_0^\infty t\,\gamma(t)\,dt = (N_1\lambda_1+N_2\lambda_2)^{-1}\Big\{N_1\lambda_1\eta_1\sum_{r=0}^{N_1-1}\binom{N_1-1}{r}\frac{1}{\phi_r}$$
$$+\eta_2\Big(1+N_1\lambda_1\eta_1\sum_{r=0}^{N_1-1}\binom{N_1-1}{r}\frac{1}{\phi_r}\Big)\Big(N_1\lambda_1\sum_{r=1}^{N_2}\frac{1-\bar{G}(r\lambda_2)}{\omega_{r-1}} + N_2\lambda_2\sum_{r=0}^{N_2-1}\binom{N_2-1}{r}\frac{1}{\omega_r}\Big)\Big\}$$

where $\qquad (4,2.76)$

$$\omega_m = \prod_{r=1}^{m}\frac{V(r\lambda_2)}{1-V(r\lambda_2)}, \quad \omega_0 = 1, \quad (1 \le m \le N_2-1) \qquad (4,2.77)$$

THE GENERAL PROCESS

In chapters 1 and 2, we considered the general process as generated by a sequence of Renewal periods, each period comprising of a TSF period and SDT period that followed it. Here, we shall view the general process from a different angle, in which busy periods alternate with idle periods of the repair facility. As such, the general process is again a sequence of Renewal periods, the Renewal periods comprising of a busy period and the idle period that follows it. Naturally, when the repair facility is idle, there is no failed unit in the system. Let us denote by e(t), the probability of the empty state. That is

$$e(t) = P_r\left[m(t)=0,\ n(t)=0 \,/\, m(0)=0,\ n(0)=0\right] \qquad (4,2.78)$$

where m(t) and n(t) are the random variables denoting the number of failed units of priority and the ordinary type at time t.

By a direct probabilistic argument we obtain

$$\frac{d}{dt} e(t) = -(N_1\lambda_1 + N_2\lambda_2)\, e(t) + (N_1\lambda_1 + N_2\lambda_2)\, e(t) * Y(t) \qquad (4,2.79)$$

where $*$ denotes convolution. On taking Laplace transform of (4,2.79) we have

$$\bar{e}(s) = 1 \,/\, \left[s + (N_1\lambda_1 + N_2\lambda_2)(1 - \bar{\gamma}(s)) \right] \qquad (4,2.80)$$

under the assumption that the system starts initially with the repair facility empty.

By Tauberian arguments, we can show that $\lim_{t \to \infty} e(t) = e_0$ exists and is independent of the initial conditions; for

$$e_0 = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\,\bar{e}(s) = 1 \,/\, \left[1 - (N_1\lambda_1 + N_2\lambda_2)\, \bar{\gamma}'(0) \right] \qquad (4,2.81)$$

provided $-\bar{Y}'(0)$, the mean busy period is finite.

We shall now proceed to obtain the general process probabilities in terms of the busy period process probabilities and e(t). Let $\mathcal{P}_{m,n}(x,y,t)\,dx\,dy$, $\mathcal{Q}_{m,n}(x,t)\,dx$ and $\mathcal{R}_n(y,t)\,dy$ denote the general process probabilities at any time t for the preemptive resume case. Then

$$\mathcal{P}_{m,n}(x,y,t)\,dx\,dy = P_r\left[m(t)=m,\ n(t)=n,\ x \le X(t) \le x+dx,\ y \le Y_p(t) \le y+dy \,/\, m(0)=0,\ n(0)=0\right]$$

$$\mathcal{Q}_{m,n}(x,t)\,dx = P_r\left[m(t)=m,\ n(t)=n,\ y \le Y(t) \le y+dy \,/\, m(0)=0,\ n(0)=0\right]$$

$$\mathcal{R}_n(y,t)\,dy = P_r\left[m(t)=0,\ n(t)=n,\ y \le Y(t) \le y+dy \quad m(0)=0,\ n(0)=0\right]$$

where $X(t)$, $Y(t)$ and $Y_p(t)$ have already been defined at the outset and the range of m and n being the same as for the corresponding busy period probabilities defined earlier. During a renewal cycle of the general process, again by simple probabilistic arguments we have

$$\mathcal{P}_{m,n}(x,y,t) = (N_1\lambda_1 + N_2\lambda_2)\,e(t) * P_{m,n}(x,y,t)$$

$$\mathcal{Q}_{m,n}(x,t) = (N_1\lambda_1 + N_2\lambda_2)\, e(t) * Q_{m,n}(x,t)$$

$$\mathcal{R}_n(y,t) = (N_1\lambda_1 + N_2\lambda_2)\, e(t) * R_n(y,t)$$

Now making use of the lemma of Smith (1954) pp 14-15 and the result $\lim_{t\to\infty} e(t) = e_0$ obtained earlier in this section, we show that the limiting forms of $\mathcal{P}_{m,n}(x,y,t)$, $\mathcal{Q}_{m,n}(x,t)$ and $\mathcal{R}_n(y,t)$ as $t \to \infty$ exist and are given by

$$p_{m,n}(x,y) = \lim_{t\to\infty} \mathcal{P}_{m,n}(x,y,t)$$

$$= (N_1\lambda_1 + N_2\lambda_2)\, e_0 \cdot \int_0^\infty P_{m,n}(x,y,t)\, dt \tag{4,2.82}$$

$$q_{m,n}(x) = \lim_{t\to\infty} \mathcal{Q}_{m,n}(x,t)$$

$$= (N_1\lambda_1 + N_2\lambda_2)\, e_0 \cdot \int_0^\infty Q_{m,n}(x,t)\, dt \tag{4,2.83}$$

$$r_n(y) = \lim_{t\to\infty} \mathcal{R}_n(y,t)$$

$$= (N_1\lambda_1 + N_2\lambda_2)\, e_0 \cdot \int_0^\infty R_n(y,t)\, dt \tag{4,2.84}$$

Noting that $\int_0^\infty P_{m,n}(x,y,t)\, dt = \bar{P}_{m,n}(x,y,0)$ and so on the steady-state probabilities $p_{m,n}(x,y)$, $q_{m,n}(x)$ and $r_n(y)$ for the preemptive resume case can be easily evaluated from the transforms (4,2.63) to (4,2.65) since

$$p_{m,n}(x,y) = (N_1\lambda_1 + N_2\lambda_2)\, e_0 \cdot \bar{P}_{m,n}(x,y,0)$$

$$q_{m,n}(x) = (N_1\lambda_1 + N_2\lambda_2)\, e_0 \cdot \bar{Q}_{m,n}(x,0)$$

$$r_n(y) = (N_1\lambda_1 + N_2\lambda_2)\, e_0 \cdot \bar{R}_n(y,0)$$

Similarly, we can obtain the steady-state probabilities corresponding to the head-of-the-line case.

LONG-RUN AVAILABILITY OF THE SYSTEM

We have discussed above a system with N_1 units (units may be components of an equipment, subsystems or equipments themselves) of type 1 and N_2 units of type 2. assigning priority to the type 1 units for repair. We are interested in the limiting probability as $t \to \infty$ that the system will be in the operative condition. And the probability is the average fraction of time, over a long period, during which the system is available. If we stipulate that the system fails when r_1 of the type 1 units and r_2 of type 2 units fail, then the long-run availability of the system in the preemptive resume case is given by

$$\lim_{t \to \infty} P_r\,[\text{ System is 'up' at time } t\,] =$$

$$= e_0 + \sum_{m=1}^{r_1-1} \sum_{n=0}^{r_2-1} \int_0^{\infty}\int_0^{\infty} p_{m,n}(x,y)\,dx\,dy + \sum_{m=1}^{r_1-1} \sum_{n=0}^{r_2-1} \int_0^{\infty} q_{m,n}(x)\,dx + \sum_{n=1}^{r_2-1} \int_0^{\infty} r_n(y)\,dy \qquad (4,2.85)$$

and for the head-of-the-line case is given by

$$\lim_{t \to \infty} P_r[\text{ System is 'up' at time } t\,]$$

$$= e_0 + \sum_{m=1}^{r_1-1} \sum_{n=0}^{r_2-1} \int_0^{\infty} p_{m,n}(x)\,dx + \sum_{m=0}^{r_1-1} \sum_{n=1}^{r_2-1} \int_0^{\infty} q_{m,n}(x)\,dx \qquad (4,2.86)$$

where the steady-state probabilities have already been evaluated.

In practice, we may not require N_1 and N_2 to be more than 2 or 3. If the distributions of failure time and repair time of the individual component are assumed to be negative exponential with parameters λ_i and μ_i $(i = 1,2)$ respectively, the results for the (2,2) - parallel system of the previous section can be obtained from (4,2,85) and (4,2.86) noting that $\bar{S}_i(s) = \mu_i/(\mu_i+s)$

Remark 1: This system is basically equivalent to a priority allocation problem of a machine interference model considered by Thiruvengadam (1965) in which the analysis is carried out on similar lines. The analysis of this section differs from Thiruvengadam's in that, the general process probabilities are evaluated in terms of the busy period probabilities and the empty state probability by the use of renewal property of the process.

Remark 2: Later the same model has been studied by Jaiswal and Thiruvengadam (1967) and Jaiswal (1968) making use of the concept of "completion times" introduced by Gaver (1962) which was termed the "basic server sojourn process" by Keilson (1962c).

The basic methodology of the completion times method of Jaiswal and Thiruvengadam to study finite source priority queueing process is outlined below.

(i) Defining completion times - The completion time is defined as the length of time from the time at which an ordinary unit is taken for repair to the time at which the repair facility is free to start repair on the next ordinary unit;

(ii) The time points at which repair starts on the non-priority units are identified with the repair beginning points of units in a basic finite-source process, namely, M/G/1/N in the usual queueing theory terminology.

(iii) Then the densities associated with the completion time process of the non-priority units are combined with the results of basic finite-source process to generate the probability densities of the busy period process.

(iv) From this the general process probabilities are obtained by renewal arguments.

Remark 3: The characteristics studied in this section are different from those considered in the finite-source queueing model outlined in Remarks (1) and (2) above.

REFERENCES

Barlow, R. E. (1962a) Repairman Problems, Studies in Applied Probability and Management Science, Chap.2, edited by Arrow, Karlin, and Scarf, Stanford University Press, Stanford, Calif.

________ (1962b) Applications of semi-Markov processes to counter problems, Studies in Applied Probability and Management Science, Chap.3, edited by Arrow, Karlin, and Scarf, Stanford University Press, Stanford, Calif.

________ (1962c) Maintenance and Replacement Policies, Statistical Theory of Reliability, edited by M. Zelen Proc. of Advanced Seminar held in May 8-10, 1962. The University of Wisconsin Press, Madison (1963).

________ and L. C. Hunter (1960a) System efficiency and reliability, Technometrics, V.2, No.1, pp.43-53.

________ (1961) Reliability analysis of a one-unit system, Operations Res. V.9, No.2, pp.200-208.

Barlow, R. E., L. C. Hunter, and F. Proschan (1963) Optimum Redundancy when components are subject to two kinds of failure, J.Soc. Indust.Appl. Math., V.11, No.1, pp.64-73.

Barlow, R. E., F. Proschan, and L. C. Hunter (1965) Mathematical Theory of Reliability, John Wiley, New York.

Bellman, R., and S. Dreyfus (1958) Dynamic Programming and the reliability of multicomponent devices, Operations Res., V.6 No.2, pp.200-206.

Benson. F., and D. R. Cox (1951) The Productivity of Machines Requiring attention at random intervals, J.Roy.Statis.Soc. (B), 13, pp.65-82.

Black, G., and F. Proschan (1959) Spare parts at minimum cost, Proc. Fifth National Symposium On Reliability and Quality Control.

Calabro, S. R. (1962) Reliability principles and practice, Mc-Graw Hill Book Co. Inc., New York.

Carhart, R. R. (1953) A survey of the current status of the electronic reliability program - Rand Corporation Report RM-1131.

Coleman, J. J., and I. J. Abrams (1962) Mathematical Models for operational readiness, Operations, Res. 10, pp.126-138.

Cox, D. R. (1955) The analysis of non-Markovian stochastic processes by inclusion of supplementary variables, Proc. Camb. Phil. Soc., 51 pp.433-441.

________ (1962) Renewal Theory, Methuen Monograph, Methuen, London. John Wiley, New York.

________ and W. L. Smith (1961) Queues, Methuen & Co., London.

Davis, D. J. (1952) An analysis of some failure data, J. Amer. Statist. Assoc., V.47, No.258, pp.113-150.

Downton, F. (1966) The reliability of multiplex systems with repair, J.Roy. Statist. Soc., Ser.B., 28 pp.459-477.

Drenick, R. F. (1960a) Mathematical aspects of the reliability problem, J.Soc. Indust. Appl.Math. V.8, No.1, pp.125-149.

Drenick, R. F. (1960b) The failure law of complex equipment, J.Soc.Indust.Appl. Math., V.8, No.4, pp.680-690

Epstein, B., and M. Sobel (1953) Life testing, J.Amer.Statist. Assoc., V.48, No.263, pp.486-503.

Fabens, A. J. (1959) The solution of queueing and inventory models by semi-Markov processes, Technical Report No. 20, Stanford University, Contract Nonr-225(28) (NR-047-019). Also published in J.Roy.Statist.Soc. Series B, V.23, No.1, pp.113-127 (1961)

Feller, W (1941) On the integral equation of renewal theory, Ann.Math.Soc.V.12 pp.243-267.

________ (1949) Fluctuation theory of recurrent events, Trans.Amer.Math.Soc. V.67, pp.98-119.

________ (1957) An Introduction to Probability Theory and its Applications, V.1, 2nd ed.John Wiley and Sons, New York.

Flehinger, B. J. (1962) A general model for the Reliability analysis of systems under various preventive maintenance policies, Ann.Math.Statist.V.33 No.1, pp.137-156.

Garg, R. C. (1962) Dependability of a system with general failure time and the exponential repair time distribution, Opnl.Res.Quart. V.13, No.3, pp.243-246.

________ (1963) Dependability of a complex system with general repair time distributions, Def.Sc.Jour.V.13 No.2, pp.209-214.

________ (1965) Dependability and disposal decision problems, Ph.D., Thesis submitted to the University of Delhi, India.

Gaver, D. P. (1959) Imbedded Markov-Chain analysis of a waiting line process in continuous time, Ann. Math. Statist., V.30, pp.698-720.

__________ (1962) A waiting line with interrupted service, including priorities, J. Roy, Statist. Soc., Series B 24, pp.73-90.

__________ (1963) Time to failure and availability of paralleled systems with repair, I E E E Transactions on Reliability, R.12, 2, pp.30-38.

__________ (1964) A probability problem arising in reliability and traffic studies, Opns.Res., V.12, pp.534-542.

Gumbel, E. J. (1958) Statistics of Extremes, Columbia University Press, New York.

Halperin, M. (1964) Some waiting time distributions for redundant systems with repair, Technometrics, V.6, No.1, pp.27-40.

Hadley, G. (1961) Linear Algebra, Addison-Wesley Publishing Co., Massachusetts.

Hosford, J. E (1960) Measures of dependability, Opns.Res., V.8, No.1, pp.53-64.

Hunter, L. C. (1962) Optimum checking procedures, Statistical Theory of Reliability chap.4, edited by Marvin Zelen, The University of Wisconsin Press Madison (1963). Proc. Advanced Seminar, May 8-10, 1962.

Jaiswal, N. K. (1961) The Queueing System GI/M/1 with finite waiting space, Metrika, V.4, pp.107-125.

__________ (1962) Time dependent solution of the head-of-the-line priority queue, J. Roy, Statist. Soc.B,24, pp.91-101.

Jaiswal, N. K. (1965a) On some waiting line problems, Opsearch, V.2, pp.27-43.

__________ (1965b) Finite source Queueing models, Tech.Memo.No.45, OR Group, Case Institute of Technology.

__________ (1968) Priority Queues, Academic Press, New York. (In print).

__________ and K. Thiruvengadam (1967) Finite source priority queues, SIAM Journal of Appl.Math. V.15, No.5, pp.1278-1293.

Kao, J. H. K. (1958) Computer methods for estimating Weibull parameters in reliability studies, IRE Transaction on Reliability and Quality Control, PGRQC-13, pp.15-22.

Karlin, S. (1966) First Course in Stochastic Processes, Academic Press, New York and London.

__________ and J. Mc Gregor, (1957) The classification of birth and death processes, Trans. Amer. Math. Soc., 86, pp.366-400.

Keilson, J. (1962a) The use of Green's functions in the study of bounded random walks with application to queueing theory, J.Math. and Physics, 41, pp.42-52.

__________ (1962b) Non-stationary Markov walks on the Lattice, J.Math. and Physics 41, pp.205-211.

__________ (1962c) Queues subject to Service interruption, Ann.Math.Statist. V.33, pp.1314-1322.

__________ (1964) Some comments on single-server Queueing methods and some new results, Proc.Camb.Phil.Soc.60 pp.237-251.

__________ and A. Kooharian (1960) On time dependent queueing processes, Ann. Math. Statist., V.31, pp.104-113.

Kemeny, J., and J. Snell (1960) Finite Markov Chains, D. Van Nostrand Co., Princeton, New Jersey.

Kendall, D. G. (1951) On some problems in the theory of queues, J. Roy. Statist.Soc. B.13, pp.151-185.

__________ (1953) Stochastic processes occuring in the theory of queues and their analysis by the method of imbedded Markov-Chain, Ann.Math.Statist., V.24, pp.338-354.

Khintchine, A, Ya. (1932) Mathematisches uber die Erwartung von einen offentlicher Schalter, Matem, Sbornik.

Liebowitz, B. H. (1966) Reliability considerations for a two element redundant system with generalised repair times, Opns. Res. V.14, No.2, pp. 233-241.

Lieblein, J. and M. Zelen (1956) Statistical investigation of the fatigue life of deep-groove ball bearings, J. Res. Nat. Bureau Stand., 57, pp.273-316.

Luchak, G. (1956) The solution of the single channel queueing equations characterised by a time-dependent Poisson distributed arrival rate and a general class of holding time, Opns. Res., V.4, pp.711-731

Lusser, R. (1952) Reliability through safety margins, Redstone Arsenal, USA.

Mc Gregar, M. A. (1963) Approximate formulas for reliability with repair, I E E E Transactions on reliability, R-12, 4, pp.64-92.

Mendelhall, W. (1958) A bibliography on Life Testing and Related Topics, Biometrika, V.45, pp.521

Mine, H. (1959) Reliability of Physical System, Transactions of the 1959 International Symposium on Circuit and Information Theory.

Molina, E. C. (1942) Poisson's Exponential Binomial Limit, D. Van Nostrand Co., Princeton, New Jersey.

Moore, E. F., and C. E. Shannon (1956) Reliable circuits using less reliable relays, J. of the Franklin Institute, 262, Pt.I, pp.191-208, 262, Pt.II, pp.281-297.

Morse, P. M. (1958) Queues, inventories and Maintenance, John Wiley and Sons, New York.

Moskowitz, F., and J. B. Mc Lean (1956) Some reliability aspects of system design, I R E Transactions on Reliability and Quality Control, PGRQC-8, pp.7-35.

Natarajan, R. (1966) Long-run availability of a complete system, accepted for publication in Metrika in mid 1968.

__________ (1967a) Reliability of a single unit system with spares and single repair facility, International Seminar on Operations Research (India) Aug. 1967.

__________ (1967b) Assignment of priority in improving system reliability, I E E E Transactions on Reliability, R-16, No.3, pp.104-110 Dec. 1967.

__________ (1967c) A reliability problem with spares and multiple repair facilities, accepted for publication in Opns. Res.

Palm, C. (1947) Arbetskraftens Fordelning vid betjaning av automatskiner, Industritidningen Norden 75

Pieruschka, E. (1963) Principles of Reliability Prentice-Hall Inc. Englewood Cliffs, New Jersey.

Proschan, F. (1960) Optimal system supply, Naval Res. Logistics Quart., v.7, pp.609-

Pyke, R. (1961a) Markov-Renewal processes: Definitions and Preliminary properties, Ann. Math. Statist. V.32, No.4, pp.1231-1242.

________ (1961b) Markov-Renewal Processes with finetely many states, Ann. Math. Statist. V.32, No.4 pp.1243-1259.

Radner, R., and D. W. Jorgenson (1962) Optimal Replacement and Inspection of stochastically failing equipment, Studies in Applied Probability and Management Science, edited by Arrow, Karlin and Scarf, Stanford University Press, Stanford, Calif.

SAE Sub Committee on Reliability (1963) Reliability Control in Aerospace Equipment Development The Macmillan Co., New York, and Pergamen Press Ltd., Oxford, England.

Schweitzer, P. J. (1967) Initial Provisioning with spare deterioration, Opns.Res. V.15, pp.513-539.

Smith, W. L. (1954) Asymptotic renewal theorems, Proc.Roy.Soc. Edinburgh, Sect. A, 64, Pt.1 (No.2) pp.9-48.

________ (1958) Renewal theory and its ramifications, J. Roy. Statist.Soc. Series B, V.20, No.2, pp.243-302.

Srinivasan, V. S. (1966a) The effect of standby redundancy in system's failure with repair maintenance, Opns.Res. V.14, pp.1024-1036.

________ (1966b) A standby redundant model when switch-over is not instantaneous, communicated to IEEE Transactions on Reliability.

Srinivasan, V. S. (1967a) An application of semi-Markov Process to a problem in reliability, International Seminar on Operational Research, New Delhi, Aug. 7-10, 1967.

______ (1967b) First emptiness in the spare parts problem for repairable components, Opns.Res.V.15, Nov.-Dec.1967.

Subba Rao, S. (1965) Queueing models with balking, reneging and interruptions, Opns. Res. V.13, pp.596-608

______ (1967) A Queueing model with balking and reneging, Ann, Inst. Statist. Math. (Japan), V.19, No.1, pp.55-71.

Takacs, L. (1951a) Occurrence and coincidence phenomena in case of happenings with arbitrary distribution law of duration, Acta Math. Acad, Sci. Hungar, pp.275-298.

______ (1951b) Probabilistic Treatment of the simultaneous stoppage of machines with consideration of the waiting times, Magyar Tudomanyos Akad III (Mat. Fiz) Oszt Kozlemenyn, 1, pp.228-234. (Hungarian)

______ (1956) On the generalisation of Erlang's Formula, Acta Math. Acad, Sc. Hunger, 7, pp.419-433.

______ (1957) On stochastic process concerning some waiting time problems, Tenv. Verovatnost i Primenen, 2, pp. 92-105

______ (1959a) On the limiting distribution of the number of coincidences concerning telephone traffic, Ann. Math.Statist. V.30, pp.134-142.

______ (1959b) On a sojourn time problem in the theory of stochastic processes, Trans.Amer.Math.Soc. V.93, pp.531-540.

Takacs, L. (1962) <u>Introduction to the Theory of Queues</u>, Oxford University Press, New York.

Thiruvengadam, K. and N. K. Jaiswal (1963) Stochastic law of the busy period for the simple machine interference problem, <u>Def. Science Journal, V.13</u>, pp.263-270.

________ (1964) Application of discrete transforms to a Queueing Process of servicing machines, <u>Opsearch</u> (India), V.1. No.2, pp.87-105.

________ (1964a) A priority assignment in machine interference problem, <u>Opsearch</u> (India), V. 1, No.4, pp.197-216.

Thiruvengadam, K. (1965) Studies in waiting line problems, <u>Doctoral Thesis</u>, University of Delhi, Delhi, India.

________ (1965a) Machine Interference problems with limited server's availability, Opsearch, V. 2, pp.65-84.

________ (1965b) On a head-of-the-line priority queue with a finite priority source, (To appear in <u>Ann. Math. Stat.</u>)

Truelove, A. J. (1961) Strategic reliability and preventive maintenance, <u>Opns. Res.</u> V. 9, No.1, pp.22-29.

Weibull, W. (1939) A statistical theory of the strength of materials, <u>Ing.Vetenskaps, Akad</u>, Handl. No.151

Weiss, G. H. (1956a) A note on the coincidence of random functions, <u>Quart. Appl.</u> Math., V.14, 1, pp.103-111.

________ (1956b) On a Semi-Markovian Process with a particular application to reliability theory, <u>NAVORD Report</u> 4351, White Oak, Maryland.

Weiss, G. H. (1962a) A problem in equipment maintenance, Management Science, V. 8, No.3, pp. 266-277.

__________ (1962b) A survey of some mathematical models in the theory of reliability Statistical Theory of Reliability chap.1, edited by M. Zelen, Proc. of advanced Seminar held in May 8-10, 1962, The University of Wisconsin Press, Madison (1963).

__________ (1962c) Reliability of a system in which spare parts deteriorate in storage, J. Res. of the Nat. Bureau of Standards, B. Math. and Mathl.Phys. 66B, 4, pp.157-160.

Welker, E. L., and R. C. Horne (1960) Concepts associated with system effectiveness, ARINC monograph No.9, Aeronautical Radio Inc., Washington, D. C.

Wishart, D. M. G. (1956) A queueing system with χ^2 Service time distribution, Ann. Math. Statist., 27, pp.768-779

Wohl, J. G. (1966) System operational readiness and equipment dependability, IEEE Transactions on reliability, R-15, No.1, pp.1