EXTREMAL GRAPH THEORETIC PROBLEMS
WITH
APPLICATIONS

[to communication networks] (18)

by

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CONTENTS

INTRODUCTION

Graphs have now become recognized models for a wide variety of situations. Whenever we have a collection of objects with a binary relation defined on them graphs serve as excellent tools to study the combinatorial properties of the collection with respect to the binary relation. D. Konig was the first person to recognize the usefulness of graph theoretic models. He conceived of a unified study of graphs under an abstract set up. His book was a pioneering work in this field.

The graph theoretic problems embodied in this thesis have been motivated by situations arising in communication networks. In terms of graphs these problems ask for the extremal structures with preassigned diameters and their variations under suppression of vertices and edges. The motivation for the problems and their applications is deferred to the penultimate chapter. This has been found reasonable, in any case not disorderly, as the appropriate combinatorial problems in terms of graphs seem to be of

great interest on their own. Perhaps we have the cu
to study the extremal structures of graphs satisfyin
given property and retaining the same property after
portions of the graphs have been suppressed.

All the graph theoretic problems considered her
round the distance metric defined for graphs. An at
made in the last chapter to define a distance between
two columns of a (0, 1)-matrix. In light of this it
to be possible to carry the analogy from graphs to ma

The contributions of this thesis have been divid
into six chapters and an appendix. At the beginning
each chapter a detailed summary of the results in tha
chapter is provided. Berge [1] has been followed th
for notation and terminology. We give below a summar
the work done in various chapters of the thesis.

In chapter I the structures of the graphs with mi
possible number of edges with prescribed diametral var
under suppression of vertices are studied. It has bee

proved that if the number of vertices of graph is sufficiently large compared to the stipulated number of suppressible vertices and if the diameter should remain two throughout, then the extremal graph will be an appropriate complete bipartite graph. This is followed by the results concerning the extremal structures when the diameter is allowed to vary from 2 to ℓ ($\ell \geq 3$) under the suppression of a single vertex. Several upper bounds have been given and some conjectures have been made about the extremal numbers that have not been determined here.

In chapter 2 problems similar to those in chapter 1 are considered with regard to edge suppression. Here again the most general result is when the diameter remains two throughout. We prove that the extremal structures in this case are obtained by completing one of the sets of the extremal complete bipartite graph of chapter 1. As in chapter 1, structures for other diametral variations have also been studied and some conjectures presented. The dual

aspects of problems of chapter 1 and problems of chapte
are discussed in the appendix.

In chapter 3 structures of extremal graphs with pr
considered in the earlier two chapters are studied with
additional constraint of an upper bound on the degrees
vertices. Some impossible configurations of number of
number of suppressible vertices, maximum degree have be
established and some extremal structures presented.

In chapter 4 the problems are considered for direc
graphs of two kinds, the one in which a pair of vertice
be joined by two oppositely oriented arcs and the other
which at most one arc is allowed between any pair of ve
For the structures considered here, the extremal direct
graphs of first kind are obtained by replacing an edge b
two oppositely oriented arcs in the corresponding extrem
unoriented graphs. In the case of directed graphs of se
kind some lower bounds for number of edges in the extrem
graphs have been obtained and some further discussion a

impossible configurations has been presented.

In chapter 5 the applications of the results in the first four chapters to problems in communication networks are discussed. The general problem for the weighted graphs has been mentioned and has been formulated for a particular case as a Pseudo-Boolean Programming problem. Extremal problems with given probabilities of breakdown have also been mentioned.

In the last chapter a distance between any two columns of a (0, 1)-matrix has been defined. This generalizes the distance metric from graphs to general (0, 1)-matrices. In the light of this distance, connectedness, diameter and several other terms can be assigned appropriate meaning in terms of (0, 1)-matrices. The necessary and sufficient condition for the existence of a connected matrix in the class U(R, S) has been established. A condition for the diameter to be finitely bounded for matrices in U(R, S) has also been derived.

Chapter I

ACCESSIBILITY AND ITS VERTEX ORDER

Introduction and Summary

We consider unoriented graphs without loops and mult edges. Let $G = (V, E)$ be such a graph. V denotes the of vertices and E denotes the set of edges. If x and be any two vertices belonging to V, then $d(x, y)$, called distance between x and y, is the length of the shortest chain between x and y in G. The diameter of the graph denoted by $d(G)$ is defined as

$$d(G) = \max_{x,y \in V} d(x,y)$$

If the graph is disconnected we say that the diameter is infinite. It is easy to observe that the diameter of a subgraph need not be equal to the diameter of the original graph. The extremal graphs considered in this chapter hav the property that all their subgraphs with preassigned ord have their diameters ranging over prescribed sets of inte

A graph is called k-accessible if the diameter of the graph $\leq$ k.

A k-accessible graph is said to be k to λ accessible ($\lambda \geq k$) of vertex order s if all the sub-graphs obtained by suppressing any s or less number of vertices are λ-accessible.

If n, k, λ and s be positive integers such that $n > \lambda \geq k \geq 1$ and $n > s \geq 0$, then G_n denotes a graph on n-vertices, $G_V(n, k, \lambda, s)$ denotes a k to λ accessible graph of vertex order s on n-vertices.

It is the aim of this chapter to study the minimal graphs in the class $G_V(n, k, \lambda, s)$ in the sense of number of edges.

A graph with minimum possible number of edges within the class $G_V(n, k, \lambda, s)$ is denoted by Min $G_V(n, k, \lambda, s)$ and the minimum possible number of edges is denoted by $M_V(n, k, \lambda, s)$.

In the next section we shall make some preliminary observations and we shall prove the theorems concerning the extremal structures in § 3. Under theorem 1 in § 3 we prove that when n is sufficiently large compared to s, then

$$M_V(n, 2, 2, s) = (s+1)(n-s-1)$$

and the Min $G_V(n, 2, 2, s)$ are complete bipartite graphs with s+1 vertices in one set and n-s-1 vertices in the other set. Under theorem 2 we prove that if $n > 5$

$$M_V(n, 2, \ell, 1) = 2n - 5$$

for all $\ell \geq 3$. We also describe the extremal structures. We end up the chapter with a number of conjectures and remarks regarding the extremal cases that have not been established here.

2. Some Preliminary Observations

(1) $M_V(n, 1, \ell, s) = \frac{n(n-1)}{2}$ and the only graph in the class $G_V(n, 1, \ell, s)$ is the complete graph on n vertices.

(2) $M_V(n, k, \lambda, 0) = n-1$ and Min $G_V(n, k, \lambda, 0)$ is some appropriate tree, for all $k \geq 2$.

(3) In a $G_V(n, k, \lambda, s)$ the degree of each vertex is at least $s+1$. Further there are at least $s+1$ disjoint chains of length at most λ, out of which at least one is of length $\leq k$, between any two distinct non-adjacent vertices. Note that we do not have any demand about the number of chains between adjacent pair of vertices.

3. The Main Theorems

Before we state and prove the first theorem we shall describe a class of graphs which plays an important role in theorem 1. $A_n(s)$ $(n \geq 2s+2)$ denotes the class of complete bipartite graphs on n-vertices with $s+1$ vertices in one set and $n-s-1$ vertices in the other set. Figure 1 represents a member of $A_9(2)$

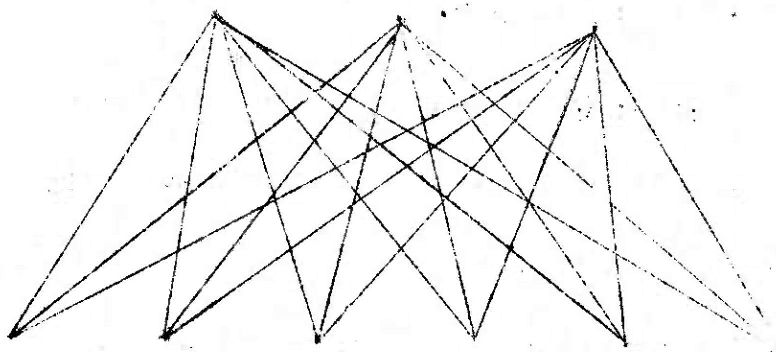

Figure 1.

Any member of the class $A_n(s)$ contains $(s+1)(n-s-1)$ edges and the class $A_n(s)$ is contained within the class $G_V(n, 2, 2, s)$. Hence

$M_V(n, 2, 2, s) \leq (s+1)(n-s-1)$ when $n \geq 2s+2$. In fact for $n >> s$, we shall prove under theorem 1 that $A_n(s)$ provide the structures for Min $G_V(n, 2, 2, s)$.

We shall now prove a lemma that we shall use for the proof of theorem 1.

<u>Lemma 1</u> : If $n > (2 + \sqrt{2})\, s+2$ then any Min $G_V(n,2,2,s)$ will contain at least one vertex x, such that $|\Gamma x| = s+1$.

Proof : Consider any Min $G_V(n, 2, 2, s)$. If possible let

$$\min_{x \in V} |\Gamma x| = m > s+1$$

where V is the set vertices of the graph.

Consider any vertex x for which $|\Gamma x| = m$ and let $x_1, x_2, \ldots, x_m$ be the vertices adjacent to x. Each of the remaining vertices would be joined to at least $s+1$ of the vertices from among $x_1, x_2, \ldots, x_m$.

(as there must be $s+1$ disjoint chains from each of the remaining $n-m-1$ vertices to x, each of these should be joined to at least $s+1$ vertices of Γx),thus accounting for a total of $(n-m-1)(s+1)$ edges. Also, since the degree of each vertex is at least m, there must at least be $m-s-1$ more edges incident with each of these $n-m-1$ vertices and clearly the number of edges required for this would be at least $\frac{(n-m-1)(m-s-1)}{2}$. Hence a Min $G_V(n,2,2,s)$ must contain at least

$$m + (n-m-1)(s+1) + \frac{(n-m-1)(m-s-1)}{2} \text{ edges}$$

i.e. $$m + (n-m-1)(s+1) + \frac{(n-m-1)(m-s-1)}{2}$$

$$\leq M_V(n, 2, 2, s)$$

$$\leq (s+1)(n-s-1)$$

i.e. $$\frac{(n-m-1)(m-s-1)}{2} \leq s(m-s-1)$$

Since $m > s + 1$, we have

$$n-m-1 \leq 2s$$

or $$m \geq n-2s-1 \qquad (1)$$

Again, as each of the vertices has a degree m, the $\text{Min}\,G_V(n, 2, 2, s)$ must have at least $\frac{nm}{2}$ edges and hence

$$\frac{nm}{2} \leq (s+1)(n-s-1)$$

or $$m \leq \frac{2(s+1)(n-s-1)}{n} \qquad (2)$$

From (1) and (2) we get

$$n-2s-1 \leq m \leq \frac{2(s+1)(n-s-1)}{n}$$

$$\text{or} \quad n^2 - n(4s+3) + 2(s+1) \leq 0 \qquad (3)$$

From this quadratic inequality it follows that

$$n \leq (2 + \sqrt{2})\, s+2$$

This contradiction proves the lemma.

<u>Theorem 1</u> : If $n > (2 + \sqrt{2})\ s+2$ then

$$M_V\ (n, 2, 2, s) = (s+1)\ (n-s-1)$$

and the class Min $G_V\ (n, 2, 2, s)$ coincides with the class $A_n(s)$.

<u>Proof of the theorem</u> : From lemma 1 it follows that under the conditions stated in the theorem there exists a vertex x in Min $G_V(n, 2, 2, s)$ such that $|\Gamma x| = s+1$. Let $x_1, x_2, \ldots x_{s+1}$ be the vertices in Γx. From the observations made earlier all the remaining vertices are joined to each of $x_1, x_2, \ldots x_{s+1}$. The graph does not need any

further edges to become a G_V (n, 2, 2, s) and all these edges are essential. Hence the theorem.

We will now take up the discussion about the extremal structures Min $G_V(n, 2, \ell, 1)$ for $\ell \geq 3$. We shall first observe by actual construction that $M_V(n, 2, \ell, 1) \leq 2n-5$ for $n \geq 5$ and $\ell \geq 3$.

$B_n^{(1)}$ denotes the class of graphs on n vertices $n \geq 5$ which are obtained in the following way. We start with a cycle $(x_1, x_2, x_3, x_4, x_5, x_1)$ of length five. Some of the other (possibly none) n-5 vertices are simultaneously joined to both x_2 and x_5, and all others (possibly none) remaining are joined to both x_1 and x_4 or both x_3 and x_5. Figures 2 and 3 represent two typical members of $B_7^{(1)}$.

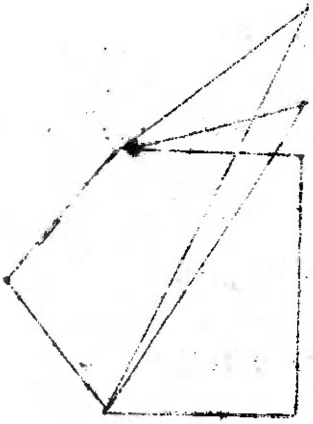

Figure 2.

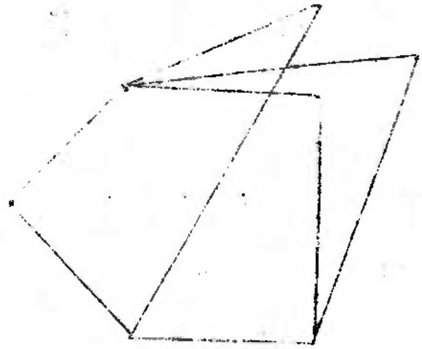

Figure 3.

It is easy to observe that a member of $B_n^{(1)}$ contains $2n-5$ edges and the class $B_n^{(1)}$ is contained within the class $G_V(n, 2, \ell, 1)$ for all $\ell \geq 3$. Hence

$$M_V(n, 2, \ell, 1) \leq 2n - 5 \quad \text{for all } \ell \geq 3 \text{ and } n \geq 5$$

We will in fact prove under theorem 2 that

$$M_V(n, 2, 3, 1) = 2n - 5 \quad \text{for all } n \geq 5,$$

but $B_n^{(1)}$ will not be the unique extremal structures.

We may also note that if u and v are two vertices of degree 2 in a $G_V(n, 2, 3, 1)$, both connected to p and q, then $X^1 - v$ generates a $G_V(n-1, 2, 3, 1)$; and conversely, if u is of degree 2 with $\Gamma u = \{p, q\}$ in a $G_V(n,2,3,1)$, then by adding a new vertex z with $\Gamma z = \{p, q\}$, we obtain a $G_V(n+1, 2, 3, 1)$. Therefore we can restrict our structural study to graphs which do not have configurations like $\Gamma u = \Gamma v = \{p, q\}$, for, in that case we can omit a vertex like v and if we have a basic solution, we can add new vertices like z. During the proof of the theorem 2 concerning the minimal structures in $G_V(n, 2, 3, 1)$ we shall assume that the minimal graphs are free from configurations of the above type. We shall call this the assumption regarding the absence of rectangles.

The circuit of length five is the first basic graph that will prove useful in the discussion of extremal structures Min $G_V(n, 2, 3, 1)$. The other basic graph is the one described below

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[1] X is the set of vertices of the original $G_V(n, 2, 3, 1)$

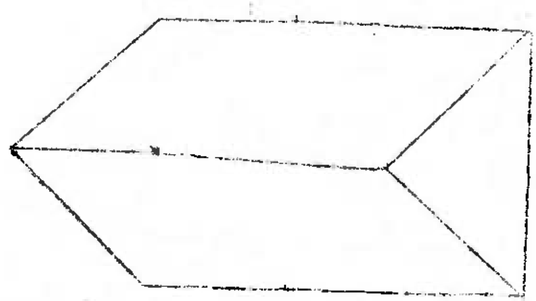

Figure 4.

This graph on 7 vertices has 9 edges and $G_n (n \geq 7)$ obtained by sequentially adding $n-7$ new vertices and joining the new vertex at every stage to two of the old vertices that happen to be the vertices adjacent to a vertex of degree two, we obtain the class $B_n^{(2)}$. Clearly $B_n^{(2)}$ are also $G_V(n, 2, 3, 1)$. We shall in fact prove under theorem 2 that $B_n^{(1)}$ and $B_n^{(2)}$ provide the extremal structures Min $G_V(n, 2, 3, 1)$.

We have a lemma before we proceed to the theorem.

<u>Lemma 2</u> : If $n \geq 5$ then any Min $G_V(n, 2, 3, 1)$ will contain a vertex of degree 2.

<u>Proof</u> : From the observation made earlier it is clear that $M_V(n, 2, 3, 1) \leq 2n - 5$. Now if there is no vertex of degree 2 in a Min $G_V(n, 2, 3, 1)$, then let

$$\min_{x \in X} |\Gamma x| = m > 2$$

where X is the set of vertices of the Min $G_V(n, 2, 3, 1)$.

Since the degree of each vertex is at least m, we have

$$\frac{nm}{2} \leq 2n-5 \quad \text{or} \quad m \leq \frac{4n-10}{n} \qquad \text{(A)}$$

As $n \geq 5$ (A) implies that $m \leq 3$ so if $m > 2$, $m=3$.
If $n < 10$, then

$$\frac{3n}{2} > 2n - 5$$

and hence Min $G_V(n, 2, 3, 1)$ cannot but contain a vertex of degree two for $n < 10$. If $n \geq 10$, then let x be any vertex of degree 3. $(x, \Gamma x)$ accounts for 3 edges. Each vertex belonging to $X - x - \Gamma x$ should be joined to some vertex in Γx. These account for $n-4$ edges. Further, the degree of each vertex in $X - x - \Gamma x$ being at least 3, the graph would at least have to have $n-4$ more edges. But with this bare minimum we have a cycle in $X - x - \Gamma x$ and we do not yet have a $G_V(n, 2, 3, 1)$ although we have already been forced to use $2n-5$ edges. And m should therefore be equal to 2 in the extremal case.

Theorem 1 : $M_V(n, 2, 3, 1) = 2n - 5$ and $B_n^{(1)}$ and $B_n^{(2)}$ provide the extremal structures Min $G_V(n, 2, 3, 1)$ when $n \geq 5$.

Proof : By the lemma just proved, there is a vertex of degree 2 in a Min $G_V(n, 2, 3, 1)$, whenever $n \geq 5$. Let x be any vertex of degree 2. Let a and b be the vertices in Γx. Let Y denote the set $X-x-a-b$. Let Y_a, Y_b and Y_{ab} be the disjoint subsets of Y that are joined to a, to b , to both a and b, respectively. We shall write

$$|Y_{ab}| = r, \quad |Y_a| = k, \quad |Y_b| = n-r-3-k.$$

First let us examine the possibility $|Y_a| = 0$. In this case each vertex $y_b \in Y_b$ would have to be joined to some vertex belonging to Y_{ab} and in this case we have to have at least $2n-4$ edges and hence would not lead us to the best. Similarly we cannot have $|Y_b| = 0$.

Hence we may suppose that $|Y_a| \neq 0$ and $|Y_b| \neq 0$. In this case $y_a \in Y_a$ should be joined to some $y \in Y - Y_a$. There should be a chain of length two between y_a and any $y_b \in Y_b$ and these edges should be in Y. This means that there must be a connected component of Y, containing $Y_a \cup Y_b$. But if $Y_a \cup Y_b$ is a connected component then we have already accounted for at least $2n-5$ edges and any further edge will take us away from the best. Hence a and b cannot be joined and $Y_a \cup Y_b$ must be by itself a connected component and in fact a tree in $Y_a \cup Y_b$. Now because of the assumption regarding the absence of rectangles in the minimal structure we will have $|Y_{ab}|=r=0$. The tree in $Y_a \cup Y_b$ cannot have a vertex d connected to two pendent vertices e and f of the tree, for (1) if e and f belong to the same group, namely Y_a or Y_b, then by our assumption e could be omitted (2) if d, e $\in Y_a$ and f $\in Y_b$, then the distance between e and b is 3, and the diameter of the graph would not any more be 2. Consequently the tree $Y_a \cup Y_b$ is a simple chain. This chain cannot be of length two, for if it is of length two then we have a diameter 3 for the graph. The length of the chain cannot be more than three either, for if it is so we again have a diameter strictly greater than 2. Therefore there are two possibilities : either the chain is of length 1 or length 3, giving the following two basic groups.

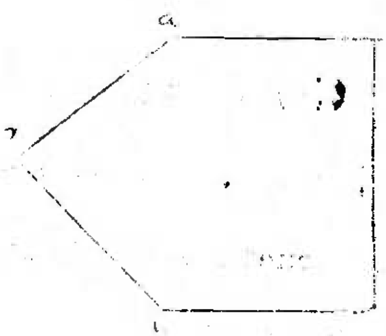

Figure 5.

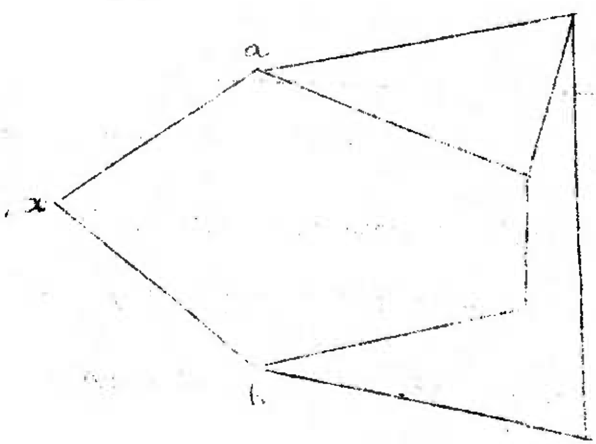

Figure 6.

The graph in figure 6 is same as the graph in figure 4. Now from these two basic graphs we develop the minimal structures. This completes the proof that $B_n^{(1)}$ and $B_n^{(2)}$ provide the extremal structures Min $G_V(n, 2, 3, 1)$ for $n \geq 5$.

<u>Remark</u> : Even when we consider Min $G_V(n, 2, \lambda, 1)$ $\lambda > 3$, we would have to have a connected component in Y, containing $Y_a \cup Y_b$. Therefore

$$M_V\ (n, 2, \lambda, 1) = 2n - 5 \quad \text{for all } \lambda \geq 3 \text{ and } n \geq 5$$

<u>This means that a biconnected graph of diameter two would</u> have to have at least 2n-5 edges when $n \geq 5$.

<u>4 Remarks and Conjectures</u>

In theorem 1 we proved that $A_n(s)$ provide the extremal sturctures when $n > (2+\sqrt{2})\,s + 2$. It appears true that $A_n(s)$ provide the extremal structures as long as $n \geq 2s+2$. This difference may be due to the method we adopted.

Very little information is known regarding other configurations of n, k, ℓ and s. It however appears reasonable to make the following conjectures.

(1) $M_V(n, 3, 3, 1) = 2n - 6$ when $n \geq 8$

The following figure represents a graph belonging to the class $G_V(n, 3, 3, 1)$

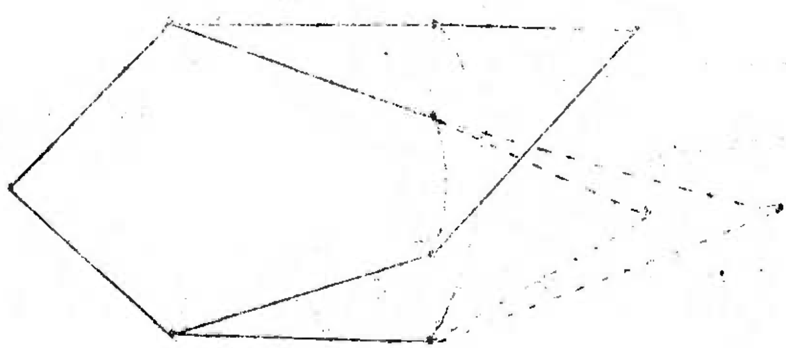

Figure 7.

(The vertices from which the dotted lines emerge are n-8 in number). The graph in figure 7 has 2n-6 edges. This proves that

$$M_V(n, 3, 3, 1) \leq 2n-6.$$

I can in fact prove that if the minimum degree in a

Min $G_V(n, 3, 3, 1)$ is equal to 2 then $M_V(n, 3, 3, 1) = 2n-6$ But if the minimum degree happens to be 3 (the other possibility) I know nothing. This means that we can show that

$$\frac{3n}{2} \leq M_V(n, 3, 3, 1) \leq 2n-6$$

In general it can be seen by actual construction that

$$(2) \quad M_V(n, 3, 3, s) \leq (s+1)(n - 2s - 1)$$

Perhaps the equality holds in (2) if $n >> s$.

Two more candidates with regard to three accessible graphs are the following

$$(3) \quad M_V(n,3,4,1) = n + 1 + \left[\frac{n-4}{2}\right], \; n \geq 7$$

$$(4) \quad M_V(n,3,\ell,1) = n - 1 + \left[\frac{n-4}{2}\right]$$

for all $\ell \geq 5$, $n \geq 7$

In each case I can prove by actual construction that the left hand side is less than or equal to the right hand side expression.

Chapter II

ACCESSIBILITY AND ITS EDGE ORDER

Introduction and Summary: In the last chapter we considered the variations of the diameter over certain subgraphs of a given graph. In this chapter partial graphs play the role played by subgraphs in the previous chapter. A k-accessible graph is said to be k to λ accessible ($\lambda \geq k$) of edge order[1] s if all the partial graphs obtained by suppressing any s or less number of edges are λ-accessible. If n, k, λ and s be positive integers such that $n > \lambda \geq k$ and $n > s$, then $G_E(n, k, \lambda, s)$ denotes a k to λ accessible graph of edge order s on n vertices. It is the aim of this chapter to study the minimal graphs in the class $G_E(n, k, \lambda, s)$. A graph with minimum possible number of edges within the class $G_E(n, k, \lambda, s)$ is denoted by Min $G_E(n, k, \lambda, s)$ and the minimum possible number of edges is denoted by $M_E(n, k, \lambda, s)$. Theorem 1 states that

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1 At the outset edge order of accessibility and vertex order of accessibility are dual concepts. In Appendix 1 we shall examine the relation between the edge order accessibility of a graph and the vertex order accessibility of its dual or adjoint graph.

if $n > \frac{(3+\sqrt{5})(s+1)}{2}$ then

$$M_E(n, 2, 2, s) = (s+1)(n-s-1) + \frac{s(s+1)}{2}$$

and describes the unique extremal structure. Under theorem 2 we prove that if $n > 8$

$$M_E(n, 2, \lambda, 1) = n - 1 + \left[\frac{n}{2}\right] \text{ for all } \lambda \geq 3.$$

In fact, we shall prove under theorem 3 that $M_E(n, 3, 3, 1)$ is also equal to $n - 1 + \left[\frac{n}{2}\right]$ when n is sufficiently large. In the next section we make some preliminary observations and in § 3 prove the theorems. We end up the chapter with some remarks and conjectures.

Some Preliminary Observations

(1) In a $G_E(n, k, \lambda, s)$ the degree of each vertex is at least $s+1$. Further, the chains between any two distinct vertices have the property that at least one of them is of length $\leq k$ and either there are $s+1$ edge

disjoint chains of length $\leq \lambda$ or the suppression of any $t(t \leq s)$ common edges leaves at least $s+1-t$ chains of length $\leq \lambda$.

Now we shall define two classes of graphs which would play important roles in the next section of this chapter. $F_n(s)$ $(n \geq 2s + 2)$ denotes the class of all graphs obtained from $A_n(s)$ by completing the set of $s+1$ vertices. Figure 8 represents a member of $F_9(2)$.

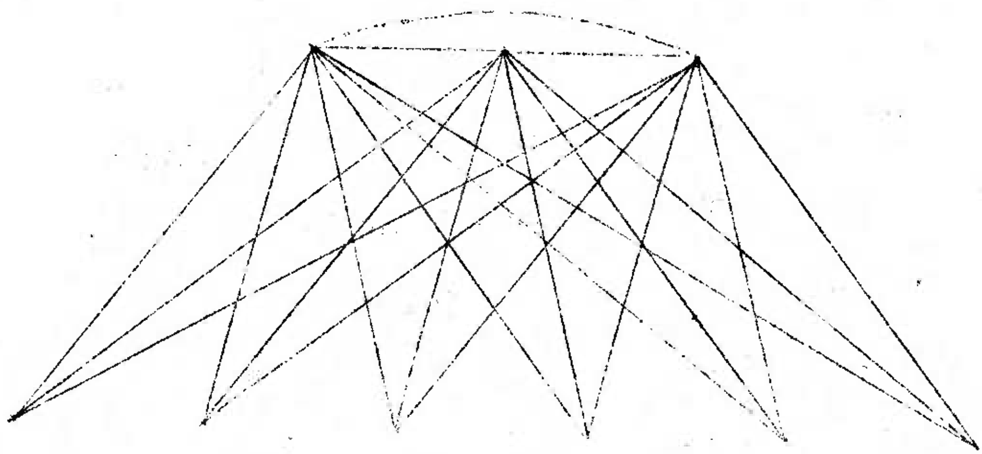


Figure 8.

Any member of $F_n(s)$ contains $(s+1)(n-s-1) + \frac{s(s+1)}{2}$ edges and the class $F_n(s)$ is contained within the class $G_E(n,2,2,s)$.

Hence

$$(2) \qquad M_E(n, 2, 2, s) \leq (s+1)(n-s-1) + \frac{s(s+1)}{2}$$

In fact for $n >> s$, we shall prove under theorem 1 that $F_n(s)$ provide the extremal structures Min $G_E(n, 2, 2, s)$.

$H_n (n > 6)$ is the class of graphs on n vertices defined in the following manner. We arbitrarily divide the set of vertices into two sets, one set containing a single vertex and the other set containing the remaining $n-1$ vertices. The single vertex is joined to each vertex of the other set. Further the vertices of the larger set are joined within themselves by a set of $\left[\frac{n}{2}\right]$ edges.

Figure 9 represents member of H_7 and H_8.

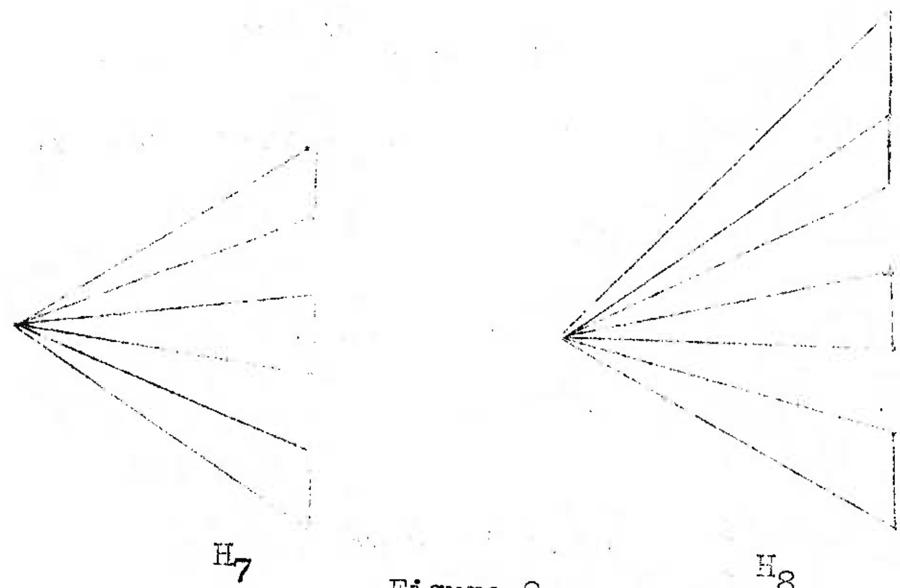


Figure 9.

A member of H_n contains $n - 1 + \left[\frac{n}{2}\right]$ edges and the class H_n is contained within the class $G_E(n, 2, \lambda, 1)$, for all $\lambda \geq 3$. Under theorem 2 we shall prove that H_n in fact coincides with Min $G_E(n, 2, \lambda, 1)$ for $\lambda \geq 3$ and n sufficiently large. It can also be observed that H_n is contained within the class $G_E(n, 3, 3, 1)$ and as it happens H_n provide the extremal structures Min $G_E(n, 3, 3, 1)$ too.

3 Theorems Concerning Extremal Structures

We start this section with a lemma which is crucial for the proof of theorem 1.

<u>Lemma 1</u> : If $n > f_2(s) = \frac{(3 + \sqrt{5})(s+1)}{2}$ then any Min $G_E(n, 2, 2, s)$ will have at least one vertex x, such that $| \Gamma x | \doteq s+1$.

<u>Proof</u> : Consider a Min $G_E(n, 2, 2, s)$. Let V denote the set of its vertices. If possible let

$$\min_{x \in V} | \Gamma x | = m > s+1$$

Consider any vertex x for which $| \Gamma x | = m$, and let $x_1, x_2, \ldots x_m$ be the vertices adjacent to x. By the observations made in § 2 there would be at least $s+1$ distinct chains of length two between X and the $n-m-1$ vertices not adjacent to x. Therefore each of these $n-m-1$ vertices would be joined to at least $s+1$ of the vertices from $x_1, x_2 \ldots x_m$. This would account for $(s+1)(n-m-1)$ edges.

Again, from x to each of $x_1, x_2 \ldots x_m$, there would have to be at least $s+1$ edge-disjoint chains of length less than or equal to two, out of which one would be the link already existing. Other chains would be of length two and are at least s in number. Therefore each of the vertices of the set Γx would be joined to at least s other vertices of the same set. This accounts for at least another $\frac{m.s}{2}$ edges.

Since the degree of each vertex of the set $V - x - \Gamma x$ of $n - m - 1$ vertices must at least be m, there would at least be $m - s - 1$ more edges incident with each of these vertices and clearly this would require at least another $\frac{(n-m-1)\ (m-s-1)}{2}$ edges. Hence a Min $G_E(n, 2, 2, s)$ must contain at least

$$m + (n-m-1)\ (s+1) + \frac{m.s}{2} + \frac{(n-m-1)\cdot(m-s-1)}{2}$$

edges. We therefore have

$$m + (n-m-1)(s+1) + \frac{m \cdot s}{2} + \frac{(n-m-1)(m-s-1)}{2}$$

$$\leq M_E(n, 2, 2, s)$$

$$\leq \frac{s(s+1)}{2} + (s+1)(n-s-1) \qquad (1)$$

(This follows again from the observations made in § 2).

Simplifying (1), we have

$$(n-m-1)(m-s-1) \leq s(m-s-1) \qquad (2)$$

Since $m > s + 1$, we have

$$n-m-1 \leq s$$

$$\text{or} \quad m \geq n-s-1 \qquad (3)$$

Again as each of the vertices has a degree m, the Min $G_E(n, 2, 2, s)$ must have at least $\frac{n \cdot m}{2}$ edges and hence

$$\frac{n \cdot m}{2} \leq \frac{s(s+1)}{2} + (s+1)(n-s-1)$$

$$\text{or} \quad m \leq \frac{(s+1)(2n-s-2)}{n} \qquad (4)$$

From 3 and 4 we have

$$n-s-1 \leq m \leq \frac{(s+1)\ (2n-s-2)}{n}$$

$$\text{or} \quad n^2 - n(3s+3) + s^2 + 3s + 2 \leq 0 \qquad (5)$$

From 5 it follows that

$$n < \frac{(3 + \sqrt{5})\ (s+1)}{2} = f_2(s)$$

This contradiction proves the lemma.

Theorem 1 : If $n > f_2(s)$, then

$$M_E(n, 2, 2, s) = \frac{s(s+1)}{2} + (s+1)\ (n-s-1)$$

and the class Min G_E (n, 2, 2, s) coincides with the class F_n.

Proof : From lemma 1 it follows that if $n > f_2(s)$ there exists a vertex x of degree s+1 in every Min $G_E(n, 2, 2, s)$.

Suppose $x_1, x_2, \ldots, x_{s+1}$ be the vertices in Γx. By the observation 1 of § 2, it follows that all the remaining vertices, namely $V-x-\Gamma x$ are joined to each of the vertices in Γx. Also, all the vertices of Γx would be joined to each other. The graph does not need any further edges to become a $G_E(n, 2, 2, s)$ and no edge can be deleted.

<u>Theorem 2</u>: $M_E(n, 2, \lambda, 1) = n-1+\left[\frac{n}{2}\right]$ and Min $G_E(n, 2, \lambda, 1)$ coincide with H_n, for all $\lambda \geq 3$ and $n > 8$.

<u>Proof</u> First we observe that there is a vertex of degree two in a Min $G_E(n, 2, \lambda, 1)$, for all $\lambda \geq 3$, for, if this is not the case there would have to be at least $\frac{3n}{2}$ edges contradicting the observation that $M_E(n, 2, \lambda, 1) \leq n-1+\left[\frac{n}{2}\right]$. Consider a Min $G_E(n, 2, \lambda, 1)$ for some $\lambda \geq 3$ and $n > 8$. Let V denote the set of its vertices. Let x be a vertex with degree 2 and let a and b be the two vertices in Γx. Let Y denote the set $V-x-a-b$. Let Y_a, Y_b and Y_{ab}

be the subsets of Y that are joined to a, to b, to both a and b, respectively. We shall write

$$|\ddot{Y}_{ab}| = r, \ |\dot{Y}_a| = k, \ |\ddot{Y}_b| = n-r-3-k ,$$

without loss of generality we can assume that $k \leq \frac{n-r-3}{2}$.

The proof of theorem 2 will be divided into two parts. In the first part we shall suppose that a and b are not joined and in the second part the assumption will be that a and b are joined.

Part I: Here a and b are not joined. In this case it is not possible to have $|Y_{ab}| = 0$. For if $|Y_{ab}| = 0$, Y would have to be connected in the subgraph spanned by itself and this requires $|Y| - 1 = n - 4$ edges, and in all we require at least 2n-5 edges. But $2n-5 > n-1+\left[\frac{n}{2}\right]$ when $n > 8$. Again, $|Y_a|$ and $|Y_b|$ cannot be simultaneously zero, for, when $|Y_a|$ and $|Y_b| = 0$, the number of edges is at least 2n-4 and $2n - 4 > n - 1 + \left[\frac{n}{2}\right]$ when

when $n > 8$. Now let $|Y_a| = 0$, $|Y_b| \neq 0$, $|Y_{ab}| \neq 0$. Each Y_b should be joined to some Y_{ab} and this would mean that the graph would require at least $(n+r-1) + (n-r-3) = 2n-4$ edges, which does not lead us to the best when $n > 8$. By symmetry we can rule out the possibility $|Y_a| \neq 0$, $|Y_b| = 0$ and $|Y_{ab}| \neq 0$. Finally suppose that none of $|Y_a|$, $|Y_b|$ and $|Y_{ab}|$ is zero. Here, again each Y_a should be joined to some vertex in $Y - Y_a$ and each Y_b should be joined to some vertex in $Y - Y_b$. Using the edges within Y we should be able to trace chains of length two between any Y_a and any Y_b. This would mean that the edges within Y would at least be $n-r-4$ in number and therefore the graph would at least have to have $(n+r-1) + (n-r-4) = 2n-5$ edges, and it cannot be the best for $n > 8$. Hence we can conclude that in a Min $G_E(n, 2, \lambda, 1)$ the descendents of a vertex of degree two are joined to each other.

Part II : Here we examine the situation where a and b are joined. From the conclusion of the previous part of the proof we know that in a Min $G_E(n, 2, \lambda, 1)$ a and b are joined. If $M_E(n, 2, \lambda, 1) \leq n - 1 + \left[\frac{n}{2}\right]$, then the edges in the subgraph spanned by Y together with the edges in $(Y, \{a, b\})$ should not be more than $n - 3 + \left[\frac{n-2}{2}\right]$ in number. And it can be seen that at least these many edges are always necessary and exactly these many lead to a $G_V(n, 2, \lambda, 1)$ when $|Y_a| = 0$, $|Y_{ab}| = 0$ and $|Y_b| = n-3$ or $|Y_b| = 0$, $|Y_{ab}| = 0$ and $|Y_a| = n-3$ or when n is even $|Y_a| = n-4$, $|Y_{ab}| = 1$ and $|Y_b| = 0$ or $|Y_b| = n-4$, $|Y_{ab}| = 1$ and $|Y_a| = 0$. These are no other graphs than the members of H_n. This completes the proof that Min $G_E(n, 2, \lambda, 1) = H_n$ for all $\lambda \geq 3$ when $n > 8$.

<u>Theorem 3</u> :

$$M_E(n, 3, 3, 1) = n - 1 + \left[\frac{n}{2}\right] \text{ and}$$

Min $G_E(n, 3, 3, 1)$ coincide with H_n, for all sufficiently large n.

<u>Proof</u> First we observe that there is a vertex of degree two in a Min $G_E(n, 3, 3, 1)$. Consider a Min $G_E(n, 3, 3, 1)$. Let V denote the set of its vertices. Let x be a vertex with degree 2 and let a and b be the two vertices in Γx. Let Y_2 denote the set $\Gamma^2 x - \Gamma x - x$ and let Y_3 denote the set $\Gamma^3 x - Y_2 - \Gamma x - x$. Let Y_a, Y_b and Y_{ab} be the subsets of Y that are joined to a, to b, to both a and b, respectively. Y_a, Y_b and Y_{ab} are disjoint sets.

Here again we shall divide the proof of the theorem into two parts. In the first part we shall suppose that a and b are not joined and in the second part the assumption will be that a and b are joined.

Part I In this case if $|Y_{ab}|$ or $|Y_a|$ or $|Y_b|$ is zero then the graph will require at least 2n-5 edges. Therefore let $|Y_a| \neq 0$, $|Y_b| \neq 0$, $|Y_{ab}| \neq 0$. At this stage we shall consider two cases.

Case 1: $|Y_3| = 0$. In this case each $y \varepsilon Y_a$ should be joined to some $Y_2 - Y_a$. If $y \varepsilon Y_a$ is not joined to some vertex in Y_{ab}, then y is either joined to a vertex in Y_a in addition to being joined to some vertex in Y_b or y is joined to a vertex in Y_b from which another edge comes into Y_a. This is to fulfil the requirement of a second chain of length ≤ 3 between y and a. But if this were to be satisfied and if n is sufficiently large, then clearly the requirement on the number of edges is greater than $n - 1 + \left[\frac{n}{2}\right]$.

Case 2: $|Y_3| \neq 0$. Let $|Y_{ab}| + |Y_3| = t$. $(x, \{a, b\})$ and the edges from Y_{ab} and Y_3 account for at least $n+t-1$ edges. If this case should lead to the extremal case then we can have at most $\left[\frac{n}{2}\right] - t$ more edges. But when n is large it would not be possible to have a $G_V(n, 3, 3, 1)$ with just these many additional edges.

Hence we conclude that a and b are joined to each other.

Part II Here we examine the situation where a and b are joined. From Part I we know that in a Min $G_E(n,3,3,1)$ a and b are joined. Now by the same type of arguments used for the proof of Part II of theorem 2, we can see that H_n provides the extremal structures $\mathrm{Min}_E(n,3,3,1)$.

Although we allowed for a diameter 3, we find $|Y_3|=0$. If $|Y_3| \neq 0$ then for every vertex in Y_3 we should have two additional edges, whereas if $|Y_3| = 0$, for every two vertices in Y_2 we need have just three additional edges. This explains the result.

4. Remarks and conjectures.

Under theorem 1 we proved that $F_n(s)$ provide the extremal structures Min $G_E(n, 2, 2, s)$ when

$n > \frac{(3 + \sqrt{5})\ (s+1)}{2}$. It appears true that $F_n(s)$ provide the extremal structures as long as $n \geq 2s + 2$.

It is interesting to note that

$M_E(n, 2, 2, 1) - M_E(n, 2, 3, 1) = \frac{n-5}{2}$ (n taken to be even) whereas

$$M_V(n, 2, 2, 1) - M_V(n, 2, 3, 1) = 1$$

This curious revelation is perhaps due to the fact that we do not demand in the case of edge order accessibility that the relevant s+1 chains between a pair of vertices be disjoint. We just demand that they should be edge disjoint. In the case of chains of length two edge disjointness is equivalent to disjointness, and there is no saving in the case of 2 to 2 accessibility. In fact in this case edge order accessibility demands more, for there is a requirement on the number of chains between an adjacent pair of vertices, which is not required in the case of vertex order accessibility.

It is perhaps true that

$$M_E(n, 2, 3, s) = s(n-s) + \left[\frac{n - s + 1}{2}\right] \quad \text{for } n >> s$$

It can be proved by actual construction that the left hand side is less than or equal to the right hand side when $n >> s$. The following figure 10 represents a $G_E(6, 2, 3, 2)$.

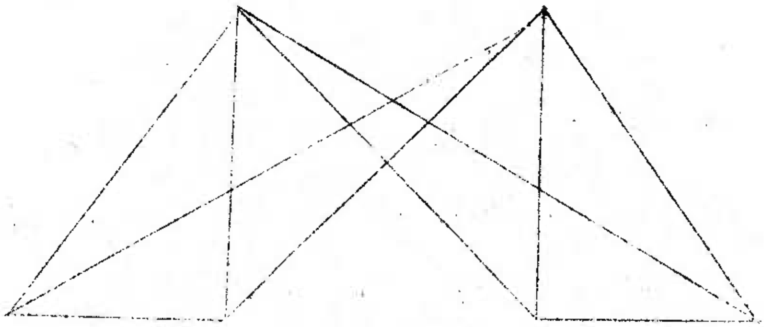


Figure 10.

We shall observe later that the above figure provides an extremal structure when we have bounds on degrees. We move over to the next chapter wondering how uncanny can be the properties of these mere points and lines.

Chapter 3

EXTREMAL GRAPHS WITH BOUNDS ON DEGREES

1 Introduction and Summary In the earlier chapters we investigated the structures of extremal graphs with some restrictions on the variations of diameter over subgraphs and partial graphs. In this chapter we shall study extremal structures with an additional restriction of a preimposed upper bound on the degrees of vertices.

Erdos and Renyi [4] have investigated the structures of two accessible graphs with a given maximum degree for any vertex and with the least possible number of edges. Here we attempt to investigate the structures of $G_V(n, k, \lambda, s)$ and $G_E(n, k, \lambda, s)$ with a given maximum degree r for any vertex, and with minimum possible number of edges. We simply use r as a superscript to indicate the additional restriction. For example, $G_V^r(n, k, \lambda, s)$ will denote a $G_V(n, k, \lambda, s)$ with maximum degree r for any vertex. In our notation the results of Erdos and Renyi are as follows.

$$M^r_{V(E)}(n, 2, 2, 0) \geq \frac{n(n-1)}{2r} \qquad (E)$$

$$M^r_{V(E)}(n, 2, 2, 0) \geq \frac{n(n-1)}{r + \frac{8n}{r}}$$

$$\text{if } r^2 > 8n \qquad (R)$$

Further they proved that E is asymptotically best possible. Exact results are not yet available.

In § 2 we shall derive some bounds for n for which the $G^r_V(n, k, \lambda, s)$ and $G^r_E(n, k, \lambda, s)$ are non-null. This gives us the impossible configurations. Under the assumption that the upper bound for degrees is actually attained by some vertex, we derive two extremal structures.

In § 3 we derive some bounds for $M^r_V(n,2,2,s)$ and $M^r_E(n, 2, 2, s)$ and make a few remarks regarding the exact values.

§ 2 Extremal Graphs with bounds on degrees.

For $r < s+1$, we know that the class $G_V^r(n, 2, 2, s)$ $\{G_E^r(n, 2, 2, s)\}$ is empty. We have the following theorems

Theorem 1: If $r \geq n-s-1$ and if $n > f_1(s)$ then the structure defined for Min $G_V(n, 2, 2, s)$ is also the structure for the class Min $G_V^r(n, 2, 2, s)$.

Proof: It is easy to see that

$$M_V(n, 2, 2, s) \leq M_V^r(n, 2, 2, s)$$

So, if there is a $G_V^r(n, 2, 2, s)$ with $M_V(n, 2, 2, s)$ edges, then it would mean that it is a Min $G_V(n, 2, 2, s)$.

In a similar way we have

Theorem 2: If $r = n-1$ and if $n > f_2(s)$ then the structure defined for Min $G_E(n, 2, 2, 3)$ is also structure for the class Min $G_V^r(n, 2, 2, s)$.

Theorem 3: $G_V^r(n, 2, 2, s)$

is empty if $n > \frac{r^2+sr+s+1}{2}$.

Proof Suppose that the maximum degree in a $G_V(n, 2, 2, s)$ is t. Consider a vertex x of degree t. Each of the vertices in $V-x-\Gamma x$ would have to be joined to Γx and therefore the load on Γx would at least be $t + (s+1)(n-t-1)$ edges. Since t is an upper bound on the degrees of each of the vertices in Γx, we have

$$t + (s+1)(n-t-1). \leq t^2.$$

$$\text{or} \quad n \leq \frac{t^2+st+s+1}{s+1}$$

The right hand side expression of the above inequality is an increasing function in t (for non-negative t) and therefore the class $G_V^r(n, 2, 2, s)$ is empty if

$$n > \frac{r^2+sr+s+1}{s+1}$$

or, equivalently, $G_V^r(n, 2, 2, s)$ is empty if

$$r < \frac{\sqrt{s^2+4(s+1)(n-1)} - s}{2}$$

<u>Theorem 4</u>: $G_E^r(n, 2, 2, s)$ is empty if

$$n > \frac{r^2 + r(\frac{s}{2}) + s + 1}{s + 1}$$

or $$r < \frac{2\sqrt{(\frac{s}{2})^2 + 4(s+1)(n-1)} - s}{4}.$$

The proof is similar to that of theorem 3.

Although we can see that

$$M_V^{r+1}(n, 2, 2, s) \leq M_V^r(n, 2, 2, s)$$

and $$M_E^{r+1}(n, 2, 2, s) \leq M_E^r(n, 2, 2, s)$$

the strict inequality may not hold in the above inequalities. We in fact know that

$$M_V^{n-1}(n, 2, 2, 1) = M_V^{n-2}(n, 2, 2, 1).$$

If we can assume that the upper bound for degrees is attained by some vertex we can prove the following theorems.

<u>Theorem 5</u> : Under the above assumption

$$M_V^{n-3}(n, 2, 2, 1) = 3n-9 \quad \text{when} \quad n \geq 6.$$

<u>Proof</u> : Under the assumption of the theorem there exists a vertex of degree n-3 in a Min $G_V^{n-3}(n, 2, 2, 1)$. Let x be a vertex with $|\Gamma x| = n-3$. Apart from x and Γx, there will be another two vertices in the graph, let they be named y and z

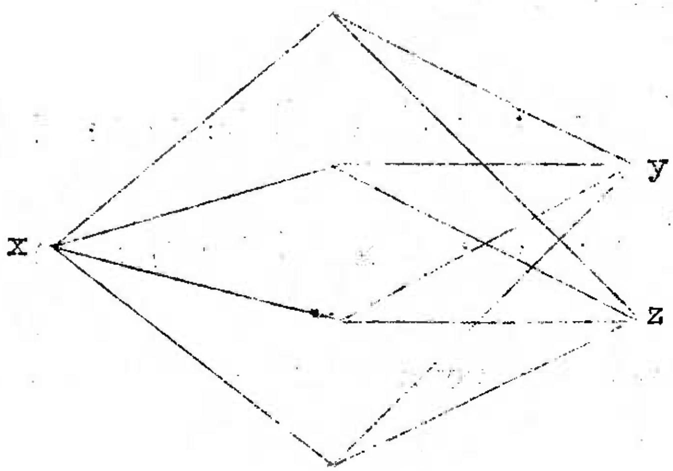


Figure 11.

Let y be joined to $\Gamma y \subseteq \Gamma x$ and z be joined to $\Gamma z \subseteq \Gamma x$. If $\Gamma x - \{\Gamma y \cup \Gamma z\}$ is non-null, there will be at least two edges from each vertex in this set to the set $\Gamma y \cap \Gamma z$. Each vertex in Γy, Γz is joined to two vertices in Γz and each vertex in $\Gamma z - \Gamma y$ is joined to two vertices in Γy. Hence, in addition to the edges in $(x, \Gamma x)$, $(y, \Gamma y)$ and $(z, \Gamma z)$ we should have at least

$$2 \left| \Gamma x - \{\Gamma y \cup \Gamma z\} \right| + \left| \Gamma y - \Gamma z \right| + \left| \Gamma z - \Gamma y \right|$$

edges. This gives us a minimum requirement of 3n-9 edges. Hence $M_V^{n-3}(n, 2, 2, 1) = 3n - 9$. Figure 11 provides the example of an extremal graph Min $G_V^{n-3}(n, 2, 2, 1)$ for $n = 7$.

<u>Theorem 6</u> : Under the above assumption

$$M_E^{n-2}(n, 2, 2, 1) = 3n-6-\left[\frac{n-2}{2}\right] \text{ when } n \geq 6.$$

<u>Proof</u> : Under the assumption of theorem there exists a vertex of degree n-2 in a Min $G_E^{n-2}(n, 2, 2, 1)$. Let x be the vertex with $|\Gamma x| = n-2$. Apart from x and Γx there will be another vertex, let it be y.

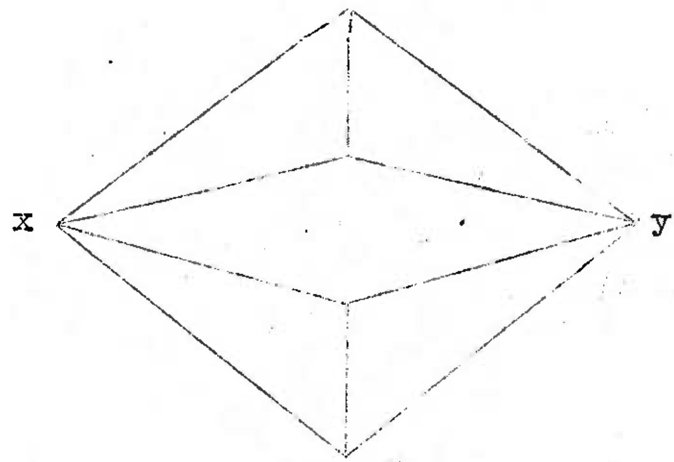


Figure 12.

Let y be joined to $\Gamma y \subseteq \Gamma x$. If $\Gamma x - \Gamma y$ is non-null, then each of the vertices in this set will have to be joined to two distinct vertices in Γy and each vertex in Γy will have to be joined to some other vertex in Γy. This gives us

$$M_E^{n-2}(n, 2, 2, s) \geq 3n - 6 - \left[\frac{|\Gamma y|}{2}\right]$$

$$\geq \min_{2 \leq |\Gamma y| \leq n-2} \left\{ 3n-6-\left[\frac{|\Gamma y|}{}\right] \right\}$$

$$= 3n - 6 - \left[\frac{n-2}{2}\right]$$

Therefore if there exists a $G_E^{n-2}(n, 2, 2, 1)$ with $3n-6 \left[\underline{n-2}\right]$ it has to be the best. **And** in fact it does.

Figure 12 provides an example of each an extremal graph when $n = 6$.

In general it is not known if the assumptions would hold and far less known are the values of $M_V^r(n, 2, 2, s)$ and $M_E^r(n, 2, 2, s)$.

§ 3 Bounds for extremal numbers and remarks.

Using the result E we can very easily establish the following lower bounds for $M_V^r(n, 2, 2, s)$ and $M_E^r(n, 2, 2, s)$.

Theorem 7 : $M_V^r(n, 2, 2, s)$

$$\geq \frac{(n-s)(n-s-1)}{2r} + \sum_{i=1}^{s} \frac{\sqrt{i^2 + 4(i+1)(n-s+i-1)} - i}{2}$$

Proof : The proof of the theorem is by induction on s. We know that the theorem is true for $s=0$. Suppose that the theorem is true for $s < t - 1$. Consider any

Min G_V^r (n, 2, 2, t). We know that there exists a vertex with degree

$$\frac{\sqrt{t^2 + 4(t+1)(n-1)} - t}{2}$$

By dropping this vertex we obtain a subgraph which is a $G_V^r(n-1, 2, 2, t-1)$ and we have

$$M_V^r(n,2,2,t) \geq M_V^r(n-1, 2, 2, t-1) + \frac{\sqrt{t^2+4(t+1)(n-1)} - t}{2}$$

which shows that the theorem is true for $s = t$ also. Hence the theorem.

By similar arguments we have

<u>Theorem 8</u> :

$$M_E^r(n, 2, 2, s)$$

$$\geq \frac{(n-s)(n-s-1)}{2r}$$

$$+ \sum_{i=1}^{s} \frac{2\sqrt{\left(\frac{i}{2}\right)^2 + 4(i+1)(n-s+i-1)} - 1}{4}$$

By actual construction we have the following upper bounds

$$M_V^r(n, 2, 2, s) \leq r(n-r) \qquad (B_1)$$

when $\frac{n}{2} \leq r$, and

$$M_E^r(n, 2, 2, 1) \leq r(n-r) + \left[\frac{r}{2}\right]^{*}\ [1] \qquad (B_2)$$

when $\frac{n+2}{2} \leq r$.

The class of bipartite graphs with r vertices in one set and $n-r$ in the other set form a sub-class of $G_V^r(n, 2, 2, s)$ and hence the bound B_1. The class of graphs obtained by adjoining $\left[\frac{r}{2}\right]^*$ edges to the set of r vertices in the above bipartite graphs such that all the r vertices are covered by these $\left[\frac{r}{2}\right]^*$ edges form a sub-class of $G_E^r(n, 2, 2, 1)$ and hence the bound B_2.

---

1. $\left[\frac{r}{2}\right]^*$ denotes the least integer greater than $\frac{r}{2}$.

How good are the bounds is as yet an unsettled question. Prof. Erdos communicated to me that the bound B_1 does not fare very well. He writes that he could prove by probabilistic arguments that if $r > c_1 n$ then $M_V^r(n, 2, 2, s)$ is less than $f(c_1)n$ for every $c_1 > 0$. At this stage no exact formulae, even asymptotic, seem to be possible.

Chapter 4

EXTREMAL DIRECTED GRAPHS

1 Introduction and Summary: In this Chapter we study the structures of extremal directed graphs with reference to the properties similar to those considered in earlier chapters for undirected graphs. We shall distinguish between two kinds of directed graphs. In the first kind of directed graphs, a pair of vertices is joined by at most one arc in a given direction but a pair of vertices can be joined by two oppositely oriented arcs. In the second kind of directed graphs two ~~of~~ vertices are joined by at most one arc and if (x, y) is an arc (y, x) is not an arc. The second kind can be obtained by orienting the edges of an unoriented graph without multiple edges.

The following definitions hold for both kinds of directed graphs.

A directed graph to be simply referred to as a graph in the sequel, is called k-accessible (k being a positive integer) if the diameter of the graph is $\leq k$. A k-accessible graph is called k to ℓ accessible (ℓ being a positive integer $\geq k$) of vertex order s (s being a non-negative integer) if all the subgraphs obtained by suppressing any s or less number of arcs are ℓ-accessible. A k to ℓ accessible graph of vertex order s on n-vertices is denoted by $D_V(n,k,\ell,s)$. A graph with minimum possible of arcs within the class $D_V(n, k, \ell, s)$ is denoted by Min $D_V(n, k, \ell, s)$ and the minimum number of arcs is denoted by $d_V(n, k, \ell, s)$. A k-accessible graph is called k to ℓ accessible of arc order s if all the partial graphs obtained by suppressing any s or less number of arcs are ℓ-accessible. For the notation corresponding to arc order accessibility we replace V by A.

In § 2 we shall consider the extremal structures for Min $D_V(n, 2, 2, s)$ in the case of directed graphs of first kind. We shall prove under theorem 1 that the extremal graphs Min $D_V(n, 2, 2, s)$ are complete directed bipartite graphs with $s+1$ vertices in one set and $n-s-1$ vertices in the other set. It means that a Min $D_V(n, 2, 2, s)$ is obtained by replacing each edge of Min $G_V(n, 2, 2, s)$ by two oppositely oriented edges. Of course we demand that $n >> s$. We prove a similar theorem for Min $D_A(n, 2, 2, s)$. If § 3 we have a discussion about the problems with regard to directed graphs of second kind. Katono proved that

$$d_V(n, 2, 2, 0) \geq c_1 \; n \; \log \; n.$$

We do not know any thing about $d_V(n, 2, 2, s,)$ for $s \geq 1$. For given n and s a $D_V(n, 2, 2, s)$ need not exist. We present some discussion in this direction. In the last section we shall mention the famous Bratton's conjecture.

§ 2 Directed Graphs of first kind

Before we prove the main results of this section we shall make some preliminary observations.

(1) In a $D_V(n, k, \ell, s)$ and $D_A(n, k, \ell, s)$ the internal as well as external demi-degrees of any vertex is at least $s+1$. In a $D_V(n, k, \ell, s)$ if x and y are two distinct vertices and if (x, y) is not any arc, then there would at least be $s+1$ disjoint paths of length at most ℓ out of which at least one is of length $\leq k$. In a $D_A(n, k, \ell, s)$ if x and y are two distinct vertices, then the paths from x to y would have the property that at least one of them is of length $\leq k$ and there are at least $s+1$ arc disjoint paths of length $\leq \ell$ or removal of any $t (t \leq s)$ arcs leaves at least $s+1-t$ paths of length $\leq \ell$.

(2) $d_V(n, 1, \ell, s) = n(n-1)$

We shall now describe two classes of graphs which, as we shall prove later, provide the structures for the extremal graphs that we set ourselves to seek. The first of these classes of graphs, denoted by $\alpha_n(s)$ $(n \geq 2s+2)$ are complete directed bipartite graphs with $s+1$ vertices in one set and $n-s-1$ in the other set. Figure 13 represents a $\alpha_6(1)$.

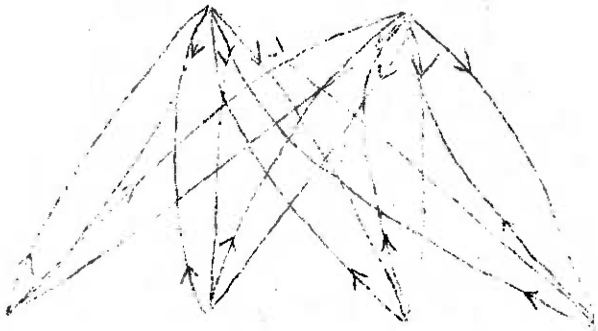


Figure 13.

Any member of $\alpha_n(s)$ contains $2(s+1)(n-s-1)$ arcs and the class $\alpha_n(s)$ is contained within the class $D_V(n, 2, 2, s)$, hence

$$(3) \quad d_V(n, 2, 2, s) \leq 2(s+1)(n-s-1)$$

Under theorem we shall in fact prove that equality holds in (3) for $n \gg s$.

$\beta_n(s)$ is the class of graphs obtained from $A_n(s)$ by completing the set of $s+1$ vertices (i.e. every vertex belonging to the set of $s+1$ vertices is joined to all other vertices of the same, in both directions, Figure 14 represents a $\beta_6(1)$.

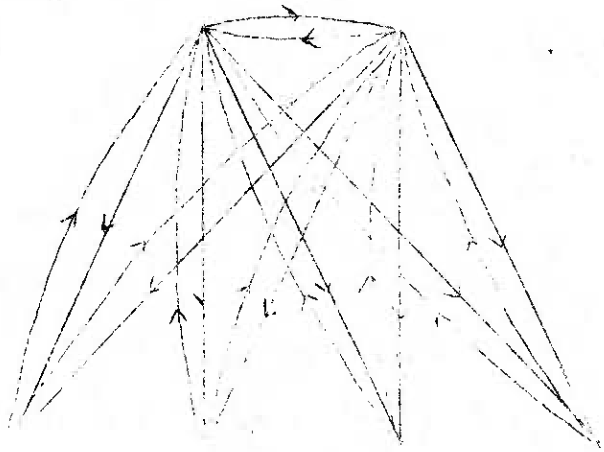


Figure 14.

A graph in the class $\beta_6(1)$ has $(s+1)(2n-s-2)$ arcs. The class $\beta_n(s)$ is contained within the class Min $D_A(n, 2, 2, s)$ and hence

$$(4) \quad d_A(n, 2, 2, s) \leq (s+1)(2n-s-2)$$

Theorem 2 states that in fact equality holds in (4) when $n >> s$.

<u>Lemma 1</u> : There exists a function $\phi_1(s)$ of s, such that whenever $n > \phi_1(s)$ any Min $D_V(n, 2, 2, s)$ would contain a vertex x with $|\Gamma^+ x| + |\Gamma^- x| = 2(s+1)$.

<u>Proof</u> : Consider any Min $D_V(n, 2, 2, s)$. If there is no vertex $x \in V$ with $| \Gamma^{+}x | + | \Gamma^{-}x | = 2(s+1)$, then let us suppose that

$$\min_{x \in V} | \Gamma^{+} | + | \Gamma^{-}x | = m > 2(s+1)$$

Under this assumption, we obtain two lower bounds for $d_V(n, 2, 2, s)$ and make use of the inequality of $d_V(n, 2, 2, s) \leq 2(s+1)(n-s-1)$ to obtain an inequality of the form $n \leq \phi_1(s)$, giving us that if $n > f_1(s)$ it cannot happen that

$$\min_{x \in V} | \Gamma^{+}x | + | \Gamma^{-}x | > 2(s+1).$$

Let $| \Gamma^{+}x \cap \Gamma^{-}x | = d$.

One lower bound for $d_V(n, 2, 2, s)$ is obtained by observing that the degree of each vertex is at least m, we have

$$\frac{n\,m}{2} \leq d_V(n, 2, 2, s)$$

$$\leq 2(s+1)(n-s-1)$$

$$\text{or } m \leq \frac{4(s+1)(n-s-1)}{n} \qquad (1)$$

The second lower bound for $d_V(n, 2, 2, s)$ is obtaine through the following observations : x, the vertex with $|\Gamma^+ x| + |\Gamma^- x| = m$ is joined to each of $\Gamma^+ x \cup \Gamma^- x$, this accounts for m arcs. Each of the vertices in $V - x - \Gamma^+ x - \Gamma^- x$ would have to be joined to $\Gamma^+ x \cup \Gamma^- x$, with at least s+1 arces in each direction. This accounts for $2(s+1)(n-m+d-1)$ arcs. The degrees of each of these vertices is at least m. Therefore we would have at least another

$$\frac{(m-2s-s)(n-m+d-1)}{2} \quad \text{arcs.}$$

This gives us the inequality

$$m + 2(s+1)(n-m+d-1) + \frac{(m-2s-1)(n-m+d-1)}{2}$$

$$\leq d_V(n, 2, 2, s)$$

$$\leq 2(s+1)(n-s-1)$$

or $(m-2s-2)(n-m+d-1) \leq 4(s+1)(m-d-s) - 2m$ (2)

But $n - m + d - 1 \geq 0$, and we have, as $m > 2s + 2$

$$n-m+d-1 \leq 4(s+1)\,(m-d-s) - 2s$$

or $$m \geq \frac{n + (4s+5)d + 4s(s+1) - 1}{4s+3} \qquad (3)$$

Combining (1) and (3) we have

$$\frac{n+(4s+5)d + 4s(s+1) - 1}{4s+3} \leq \frac{4(s+1)\,(n-s-1)}{n} \qquad (4)$$

or $$n^2 - n[12s + 13] + 4(s+1)^2\,(4s+3) \leq 0 \qquad (5)$$

Let $\phi_1(s)$ denote the largest root of the quadratic equation.

$$n^2 - n[12s + 13] + 4(s+1)^2\,(4s+3) = 0$$

then (5) implies that

$$n \leq \phi_1(s).$$

So, if $n > \phi_1(s)$, minimum degree of a vertex in a Min $D_V(n, 2, 2, s)$ cannot be greater than $2(s+1)$.

Theorem 1 :

$d_V(n, 2, 2, s) = 2(s+1)(n-s-1)$ and the class Min $D_V(n, 2, 2, s)$ coincides with the class $\alpha_n(s)$, if $n \gg s$.

Proof : By lemma 1 we know that when $n > \phi_1(s)$ there exists a vertex x in a Min $D_V(n, 2, 2, s)$ with $|\Gamma^+ x| + |\Gamma^- x| = 2(s+1)$. Therefore $|\Gamma^+ x \cap \Gamma^- x| = d \leq s+1$. Write A for $\Gamma^+ x - \Gamma^- x$, B for $\Gamma^+ x \cap \Gamma^- x$, C for $\Gamma^- x - \Gamma^+ x$, and Y for $V-x-A-B-C$. We have $|A| = s+1-d$, $|B| = d$, $|C| = s+1-d$ and $|Y| = n-2s+d-3$. The vertex x is joined to each vertex of A and the arcs are directed away from x, it is joined to each vertex of B in both directions and each vertex of C is joined to x and the arcs are directed towards x. Now, since the graph is a $D_V(n, 2, 2, s)$ each vertex of Y would have to be joined to $\Gamma^- x$, with arcs directed away from Y and all vertices of $\Gamma^+ x$ would have to be joined to all vertices in Y, with arcs directed away from $\Gamma^+ x$. All these account for $2(s+1)(n-2s+d-2)$ arcs, but to establish the necessary accessibility and order of accessibility we need to introduce some more arcs. These can be only between and within $\Gamma^+x \cup \Gamma^-x$ and Y. Consider any vertex $a \in A$. With arcs already existing there is no path of length ≤ 2 between a and x, and a would compulsorily have to be joined to all the vertices in Γ^-x. This accounts for a must of another $(s+1)(s-d+1)$ arcs. Again, to establish the accessibility with the necessary order between C and Y we need to have some more arcs leaving C. If n is sufficiently large compared to s, then the number of arcs we would need greater than $(s+1)(s-d+1)$ arcs. And in all we have already been forced to use more than

$$2(s+1)(n-2s+d-2) + 2(s+1)(s-d+1) = 2(s+1)(n-s-1)$$

arcs.

Therefore as long as $d < s+1$, we do not come across the extremal graph. But when $d = s+1$, we have $\Gamma^+x = \Gamma^-x$ and we have the structure $\alpha_n(s)$. This completes the proof of the theorem.

<u>Theorem 2</u> : $d_A(n, 2, 2, s) = (s+1)(2n-s-2)$ and the class Min $D_A(n, 2, 2, s)$ coincides with the class $\beta_n(s)$, when $n \gg s$.

The proof of this theorem follows the same type of arguments adopted for the proof of theorem 1 and is omitted here.

<u>Remarks</u> : Perhaps it would be true that all extremal directed graphs of first kind are obtained by having two oppositely oriented arcs for each edge in the corresponding undirected extremal graphs.

§3. Directed Graphs of Second Kind

In this section we shall consider directed graphs in which at most one arc is permitted between two vertices. Let $G = (V, A)$ be such a graph. V denotes the set of vertices and A denotes the set of arcs. Katono proved that $d_V(n, 2, 2, 0) \geq C_1\ n \log n$. We obtain an upperbound for $d_V\,(n, 2, 2, 0)$ which can be sequentially improved.

<u>Theorem 3</u> : The complete undirected graph on n vertices can be oriented in such a way that the resulting directed graph is 2-accessible, except when $n = 2, 4$.

The proof of the theorem will be evident from the construction in the following theorem.

<u>Theorem 4</u> : $d_V(n, 2, 2, 0) \le \frac{n^2-3n+10}{2}$ for $n \ge 5$ and $\lambda \ge 2$.

<u>Proof</u> : The proof of the theorem is constructive. For $n = 3, 5, 6$ and 7 the result holds and we have the following graphs

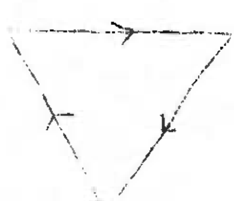


Figure 15.

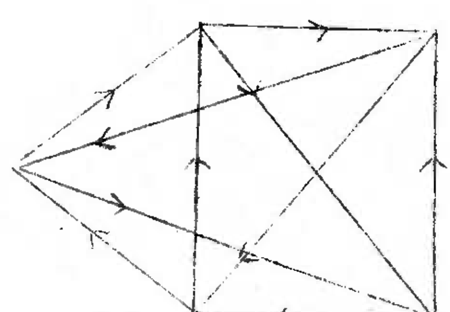


Figure 16.

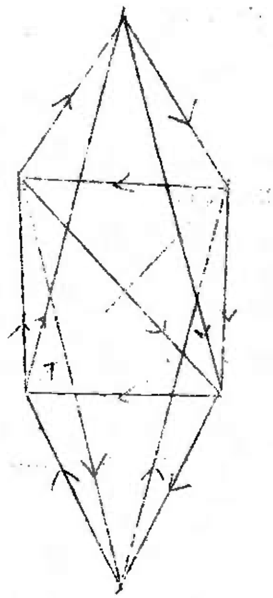


Figure 17.

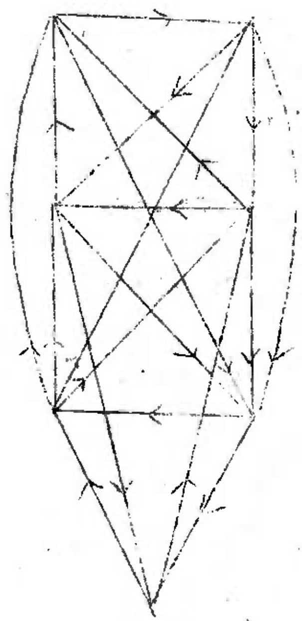


Figure 18.

In general when $n \geq 8$ we start with a graph D_{n-3} of diameter two on $n-3$ vertices. To this graph we adjoin a graph of three vertices $D_{(3)}$ which is shown in the figure below

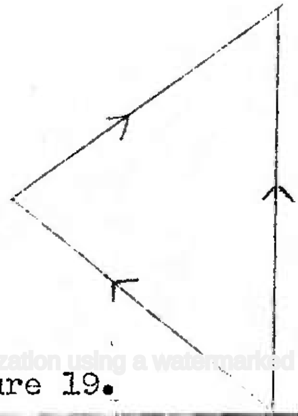


Figure 19.

The vertex b is joined to all the vertices of the D_{n-3} graph with arcs directed away from b. All the vertices of the D_{n-3} are joined to c with arcs directed towards c. The vertex a is joined to two different vertices in D_{n-3}. One of these arcs is directed away from a and the other directed towards a. It is easy to verify that we obtain a 2-accessible graph on n-vertices by this process. D_{n-3} has at most $\frac{(n-3)(n-4)}{2}$ arcs and 2n-1 arcs are incident to D_3. Hence we have

$$d_V(n, 2, \ell, 0) \leq 2n-1+ \frac{(n-3)(n-4)}{2} = \frac{n^2-3n+10}{2}$$

and from the way this upper bound is obtained it is clear that this upper bound can be improved.

In general it is not known for what values of n it is possible to orient the complete n-graph in such a way that it becomes a $D_V(n, 2, 2, s)$ or $D_A(n, 2, 2, s)$. If g(2, s) denotes the least value of n, then g(2, s) is perhaps $\geq 2^{s+2}-1$. For s=0 it has been observed to be true and for $s=1$ it can be observed to be true. The figure below is a $D_V(n, 2, 2, 1)$.

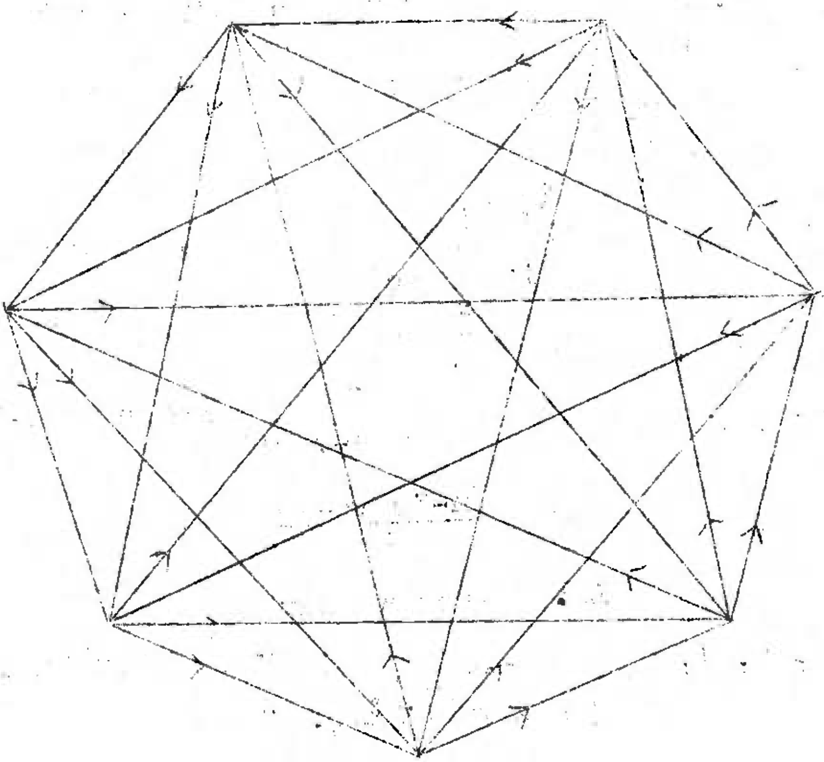


Figure 20.

This graph is from one of Professor Erdös's papers. Our problem has an alluring similarity with the problem which Prof. Erdös considered.

We now obtain a lower bound for $d_V(n, 2, 2, s)$ and $d_A(n, 2, 2, s)$ when $n \gg s$.

<u>Theorem 5</u> : When $n \gg s$

$$d_V(n, 2, 2, s) \geq (2s+2)(n-2s-2)$$

and $$d_A(n, 2, 2, s), \geq \frac{(2s+2)(2s-1)}{2} + (2s+2)(n-2s-2)$$

<u>Proof</u> : In a $D_V(n, 2, 2, s)$ if the directions are dropped there will be at least $2s+2$ disjoint chains of length ≤ 2 between any non-adjacent vertices or in other words it will be a $G_V(n, 2, 2, 2s+1)$. Hence we have

$$d_V(n, 2, 2, s) \geq (2s+2)(n-2s-2)$$

Again, if the directions are dropped in a $D_A(n, 2, 2, s)$ there will be $2s+2$ edge disjoint chains of length ≤ 2 between any two vertices or in otherwords it will be a $G_E(n, 2, 2, s)$. Hence we have

$$d_A(n, 2, 2, s) \geq \frac{(2s+1)(2s+2)}{2} + (2s+2)(n-2s-2)$$

§ 4 Bratton's Conjecture

In this section we shall once again consider the directed graphs of first kind. If such a graph is strongly connected, we have

$$n \leq m \qquad (6)$$

$$m \leq n(n-1) \qquad (7)$$

and $$d \leq n-1 \qquad (8)$$

(n stands for the number of vertices, m for the number of arcs and d for the diameter). But there need not correspond a strongly connected graph with every set of numbers satisfying the inequalities (6), (7) and (8). In otherwords given n, m, d which satisfy (6), (7) and (8) there may not exist a strongly connected graph $G = (X, U)$ with $|X| = n$, $|U| = m$ and $d(G) = d$. This impossibility arises whenever too small a number is chosen for d.

This situation led to the demand for the lower bound f(m,n) of the diameters of a strongly connected graph without loops with n vertices and m arcs, and to complete the above system of inequalities by the inequality :

$$d \geq f(m, n) \tag{9}$$

We state below a very famous, yet undecided, conjecture of Bratton.

Let

$$n-1 = q(m-n+1) + r, \quad r \leq m-n+1$$

Put :

$$\partial(m, n) = \begin{cases} 2q & \text{if } r = 0 \\ 2q + 1 & \text{if } r = 1 \\ 2q + 2 & \text{if } r \geq 2 \end{cases} \tag{10}$$

Bratton's conjecture states that : every strongly connected directed graph of first kind with m arcs and n vertices has a diameter which is greater than or equal to $\partial(m, n) - 1$.

A similar question that can be asked in this connection is given n and d, what is the maximum m for which there exists a directed graph of first kind with n vertices and diameter $\geq d+1$. (If $d=1$, this maximum m is equal to $n(n-1)-1$).

Chapter 5

Communication Networks and Graphs

1 Introduction and Summary : In this chapter we shall discuss the applications of some of the results obtained in the earlier chapters to communication networks. Further we shall discuss some programming problems related with weighted graphs and their applications to problems in communication networks.

A communication network is any complex of centres dispatching and receiving information together with a set of links that convey information. A problem of constructing a communication network is one of establishing the links between the different centres. The pattern of linking is usually dictated by the requirements of the inter-communicative readiness on the network and cost considerations. We would like to set up the links in such a way that the cost is minimum subject to the dictates of the requirements on the network.

A communication network can be identified with a graph, by identifying the centres and links respectively with vertices and edges (or arcs). When the links are two way channels we identify the network with an unoriented graph and when the links are one way channels we identify with an oriented graph. A number of practical problems concerning the construction of communication network can be translated into familiar graph theoretic language and can be dealt with with the usual ease and elegance of graph theoretic methods. We shall use the graph theoretic and network theoretic terms interchangeably.

In the next section we shall discuss some problems in communication networks and point out some uses of the results we already obtained. In § 3 we shall discuss the programming problems related with weighted networks and in § 4 we shall mention the problems of constructing communication networks with probabilities of breakdown.

§2 PROBLEMS IN COMMUNICATION NETWORKS

The diametral structure of a graph is of significance with reference to the applications of graph theory to problems of construction of communication networks. A large diameter has several disadvantages. In the first instance a large number of relays cause delay and if there is a possibility of errors creeping in to the messages then each relay would mean more errors. A small diameter is always desirable. The most ideal situation would be to have a complete graph for a network. But when it is impracticable to have such a situation we would prefer to have as small a diameter as possible. Figures 21, 22, 23 and 24 show examples of 2-accessible networks with 5 centres.

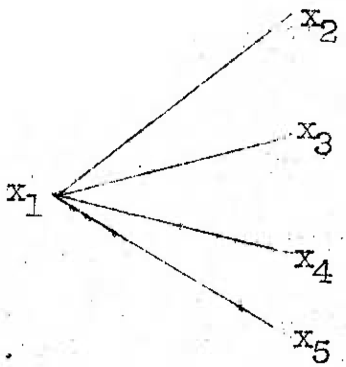


Figure 21.

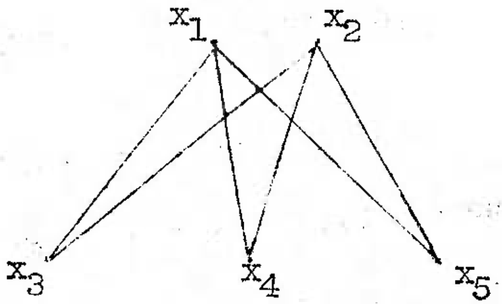


Figure 22.

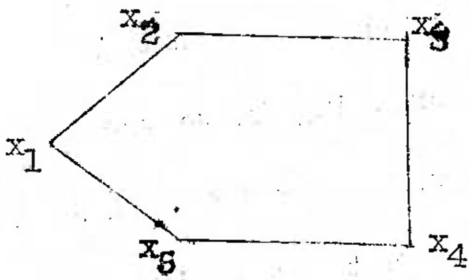


Figure 23.

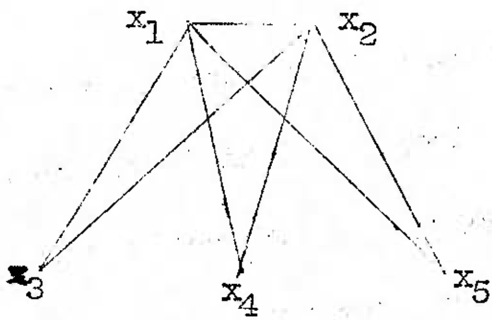


Figure 24.

Suppose that in the network corresponding to figure 21, the centre x_1 goes out of order or breaks down by an accident ; then the whole communication system would be in a jeopardy. There would not be any possibility of communication. But in the communication network corresponding to figure 22, the situation is not so very precarious. Even if a centre breaks down by accident the *remaining centres can communicate with each other and* the resulting sub-network is also 2-accessible. But we notice that we would have to have two more links. In the network of figure 23, a breakdown would not lead to complete jeopardy but the diameter would be increased. Hence the vertex order accessibility of the network emerges as a natural proof against breakdowns of centres. In a similar way the edge order accessibility of the network is a proof against breakdowns of links.

We define the capacity of a centre as the maximum number of links which can be incident to it. If suppose 3 is an upperbound on the capacities of the five centres, $x_1, x_2, \ldots, x_5$. Then networks in figure 21 and 24 violate the capacity restrictions and the networks of 22 and 23 do not violate. Thus we see that bounds on degrees is also a desirable criterion.

Having defined the desirable criteria, we can now think of the following problems. Given n centres of communication $x_1, x_2, \ldots, x_n$ and $n \times n$ matrix $c = \{c_{ij}\}$, which describes the costs of connecting the centres x_i with x_j in the communication network, find out the optimum set of links which minimizes the total establishment cost and such that the resulting network is

(1) a $G_V(n, k, \ell, s)$ for some predetermined values of k, ℓ and s.

(2) a $G_E(n, k, \ell, s)$ for some predetermined values of k, ℓ and s.

(3) a $G_E^r(n, k, \ell, s)$ for some predetermined values of k, ℓ, s and r.

If we can assume that $c_{ij} = c$, a constant, then the above three problems reduce to finding Min $G_V(n,k,\ell,s)$, Min $G_E(n, k, \ell, s)$ and Min $G_E^r(n, k, \ell, s)$ respectively. Some of these problems for certain values of the parameters have already been solved by us in the earlier chapters. But when the above assumption is not valid then the elegant combinatorial methods that we used in the earlier chapters would no more be useful. We have to take recourse to programming methods, which are essentially computational in nature. We would have no ready made answer. When all

c_{ij}'s are not equal, we call the corresponding graph a weighted graph. We shall discuss the problems of construction of minimal cost weighted networks belonging to (1), (2) and (3) in the next section.

§ 3 WEIGHTED GRAPHS

We are required to establish a network on 'n' communication centres. The cost of establishing a link between the i-th and j-th centres is c_{ij} ($c_{ii} = 0$, $c_{ij} \geq 0$). The $n \times n$ matrix $\{c_{ij}\}$ is given. Any network on these n vertices is completely described a $n \times n$ (0, 1)-matrix $\{x_{ij}\}$. If $x_{ij}=1$ we can think of i-th and j-th centres as being connected and otherwise not connected. So our aim is to find that (0, 1)-matrix the graph corresponding to which has the desirable properties and the cost of establishing the network is minimum among all those graphs satisfying the same properties. If the graph is a directed graph of second kind then $x_{ij} = 1 \Rightarrow x_{ji} = 0$; and when the graph is an unoriented graph then $\{x_{ij}\}$ should be a symmetric matrix.

Problem 1 : To find a 2-accessible directed graph of first kind with minimum total cost.

Formulation : If $\{x_{ij}\}$ described such a network, we would have to find a 2-accessible graph for which $\Sigma\, \Sigma\, c_{ij}\, x_{ij}$ is minimum. Two accessibility means that there is either an arc joining two vertices or there is a path of length two (or both). So if $x_{ij} = 0 (i \neq j)$ (i.e. if there is no direct arc between x_i and x_j) there should exist a k such that $x_{ij} . x_{kj} = 1$. (i.e. there should be a path of length 2). If we set $x_{ii} = 1$ we can write down the condition for two accessibility as : For any given pair i, j there exists a k such that $x_{ik} . x_{kj} = 1$ or for all i and j $1 - x_{ik}\, x_{kj} = 0$ for some k,

or for all i and j $\prod_{k=1}^{n} (1 - x_{ik}\, x_{kj}) = 0$

or $\sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{k=1}^{n} (1 - x_{ik}\, x_{kj}) = 0$

Hence problem 1 is equivalent to minimizing

$$\sum_i \sum_j x_{ij} \; c_{ij} \qquad (A)$$

Subject to the conditions

$$\sum_i \sum_j \prod_{k=1}^{n} (1 - x_{ik} \, x_{kj}) = 0 \qquad (B)$$

$$x_{ii} = 1, \; x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j.$$

Theorem 1 : Problem 1 is equivalent to minimizing the Pseudo-Boolean function

$$F_1(X_i) = \Sigma \Sigma c_{ij} \, x_{ij} + (\Sigma \Sigma c_{ij} + 1) \Sigma \Sigma \prod (1 - x_{ik} \, x_{kj}) \qquad (C)$$

where $x_{ii} = 1$ and c_{ij}'s are the given constants.

Proof : Suppose $\{x^o_{ij}\}$ with $x^o_{ii} = 1$ and $x^o_{ij} = 0$ or 1 minimizes (A) subject to (B), then it minimizes (C).

Indeed, if

$$\Sigma \Sigma c_{ij} \, x_{ij} + (\Sigma \Sigma c_{ij} + 1) \Sigma \Sigma \prod (1 - x_{ik} \, x_{kj})$$

$$< \Sigma \Sigma c_{ij} \, x^o_{ij} + (\Sigma \Sigma c_{ij} + 1) \Sigma \Sigma \prod (1 - x^o_{ik} \, x^o_{kj})$$

then

$$(\Sigma\,\Sigma\, c_{ij}+1)\,\Sigma\,\Sigma \prod (1-x_{ik}\, x_{kj}) \leq \Sigma\,\Sigma\, c_{ij}\, x^{o}_{ij} - \Sigma\,\Sigma\, c_{ij}\, x_{ij}$$

$$= \Sigma\,\Sigma\, c_{ij}\,(x^{o}_{ij} - x_{ij})$$

$$< \Sigma\,\Sigma\, c_{ij} + 1$$

This implies that $\Sigma\,\Sigma \prod (1 - x_{ik}\, x_{kj}) = 0$ which in turn implies that $\{x_{ij}\}$ minimizes (A) subject to (B) contradicting the definition of $\{x^{o}_{ij}\}$.

Conversely if $\{x^{*}_{ij}\}$, $x^{*}_{ii} = 0$, $x^{*}_{ij} = 0$ or 1 minimizes (C) then it is subject to (B). If this is not true, then

$$\Sigma\,\Sigma \prod (1 - x^{*}_{ik}\, x^{*}_{kj}) \geq 1 .$$

and

$$F_1\,(X^{*}_{ij}) \geq \Sigma\,\Sigma\, c_{ij} + 1 \quad \text{contradicting}$$

$$F_1\,(J) = \Sigma\,\Sigma\, c_{ij}$$

$$< \Sigma\,\Sigma\, c_{ij} + 1 \leq F_1(X^{*}_{ij})$$

where J is matrix of all ones.

Remark : When we consider unoriented graphs we have the additional condition that $x_{ij} = x_{ji}$. If we consider oriented graphs of second kind we have a condition of the type $x_{ij} \cdot x_{ji} = 0$.

<u>Problem 2</u> : To find a $G_V(n, 2, 2, 1)$ with minimum total cost.

<u>Problem 3</u> : To find a $D_A(n, 2, 2, 1)$ (of second kind) with minimum total cost.

With some labour the Pseudo-Boolean expression to be minimized for obtaining solution to Problems 2 and 3 can be written down. But they would be far too ghastly looking and no simplication seems to be possible. Even (σ), the Pseudo-Boolean function to be minimized for obtaining a solution to Problem 1 is intricate, but electronic computers should prove to be useful in simplifying the expression and finding its minimu. Pseudo-Boolean Programming of Invanescu [5] would perhaps provide a fairly quick solution. It is not conceivable that any other programming method is quicker. But perhaps it is more straight forward to compute the values of (C) for all 2^{n^2-n} matrices. In practical situations we usually face with an impossibility of connecting two centres x_i, x_j (say) in which case we can put $c_{ij} = \alpha$ and $x_{ij} = 0$. This can greatly simplify the problem.

§ 4 PROBABILISTIC PROBLEMS

In § 2 we considered vertex order and edge order of the accessibility in the networks as proofs against accidents or breakdowns. There is another, and perhaps more natural way of making the accessibility proof against accidents. Suppose that a certain probability of breakdown p_i is associated with each vertex p_i. Now we can demand for a minimal cost k-accessible network such that the probability that the network remains k-accessible under breakdowns is greater than or equal to a preassigned number α. It is also possible to associate a probability of breakdown with each pair of vertices and demand for a minimal cost network in a similar way. These problems do not seem to be easily tractable, but a solution to them would be of great practical value.

We presented in this chapter problems of great complexity. We did not succeed much in simplifying the matters. But the applications that we pointed seem to be very very relevant.

Chapter 6

A DISTANCE FOR (0, 1)-MATRICES

§ 1 Introduction and Summary : In all the earlier chapters the concept of distance on graphs played the central role. As incidence matrices of graphs form particular classes of (0, 1)-Matrices, it would be reasonable to seek for a definition of a distance between any two columns of a (0, 1)-Matrix. We succeed, in this chapter, in defining a distance between the columns of a (0, 1)-Matrix. This gives a great scope for carrying the analogy from graphs to matrices. A wide range of problems spring up - but most of them seem to defy any easy solution. We succeed in proving some specific theorems.

Let A be an $m \times n$ (0,1)-Matrix. Let $R_1, R_2 \ldots R_m$ denote its rows and $C_1, C_2 \ldots C_n$ denote its columns. A finite sequence of distinct columns $C_1, C_2, \ldots C_k$ is said to form a chain if C_i and C_{i+1} (for $i = 1$ to $k - 1$) have a one in the same row. k-1 is called the length of the chain and this chain is said to connect C_1 and C_k. As can be easily seen connectedness is an equivalence relation. Equivalence classes of columns are called connected components. A matrix is called connected if all its columns belong to the same equivalence class. We define the distance between any two columns C_i and C_j, to be denoted by $d(C_i, C_j)$, in the following manner :

$$d(C_i, C_i) = 0$$

$$i \neq j \qquad d(C_i, C_j) = \begin{cases} t & \text{if } C_i \text{ and } C_j \text{ belong to the same equivalence class} \\ \infty & \text{otherwise} \end{cases}$$

where t is the length of the shortest chain starting at C_i and ending at C_j.

The diameter $d(A)$ of a (0, 1)-Matrix A is defined in the following manner

$$d(A) = \max_{\substack{i = 1 \text{ to } n \\ j = 1 \text{ to } n}} d(C_i, C_j)$$

The diameter of a disconnected matrix is ∞. Given R and S it would be of interest to find the formulae for $\bar{d}$ and $\tilde{d}$ where $\bar{d}$ and $\tilde{d}$ stand for $\max_{A \in U(R,S)^1} d(A)$ and $\min_{A \in U(R,S)} d(A)$ respectively. We do not succeed in providing these formulae, but we find necessary and sufficient conditions under which $\bar{d}$ is finite. We mention the formula for $\bar{d}$ in the particular case $U(k, k)$.

§ 2 THE RESULTS

Lemma 1 : A connected (0, 1)-Matrix of size m by n with given row sums $r_1, r_2, \ldots, r_m$ $0 < r_i \leq n$ exists if and only if $\Sigma r_i \geq m + n - 1$.

Lemma 2 : Let A be a connected matrix with each $r_i \geq 2$ and $\Sigma r_i > n+m-1$ then there exists a row in the matrix which has distinct intersection with two other rows, or there is a row which has double link with at least one row.

---

1. U(R,S) denotes the class of (0,1)-Matrices with row sum vector R and column sum vector S. H.J. Ryser discussed the necessary and sufficient conditions on R and S for U(R,S) to be non-empty.

The proofs of these two lemmas are simple and are omitted here.

<u>Theorem 1</u>: Given $R = (r_1, r_2 \dots r_m)$, $r_i \geq 2$, $i = 1,2,\dots m$ and $S = (s_1, s_2 \dots s_n)$, $s_j > 0$, $j = 1,2, \dots n$; such that the class $U(R, S)$ is non-empty, then the necessary and sufficient condition for the existence of a connected matrix in the class $U(R, S)$ is that $\Sigma r_i \geq m + n - 1$.

<u>Pfoof</u> : The necessity follows from lemma 1. The proof of sufficiency is as follows.

If there is no connected matrix in the class $U(R, s)$, then let k, $k > 1$ be the minimum number of connected components in $U(R, S)$. Let A be the matrix with k-connected components. Then, we can rearrange the columns and rows and partition the matrix in to k^2 sub-matrices so that the connected components are along the diagonal and all non-diagonal blocks are zero matrices. Let T_i denote the number of ones in the i-th connected component . Let m_i and n_i denote respectively the number of rows and columns of the i-th connected component. Then, by lemma 1, $T_i \geq m_i + n_i - 1$, for $i = 1, 2, \ldots k$. But $T_i = m_i + n_i - 1$ cannot hold for all $i = 1, 2, \ldots k$. For, if $T_i = m_i + n_i - 1$ for all $i = 1, 2, \ldots k$ then $\Sigma T_i = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} n_i - k$ or $\Sigma r_i = m + n - k < m + n - 1$ contradicting the hypothesis. Therefore there is a component, say the r-th component, with $T_n > m_r + n_r - 1$. By lemma 2, the r-th component has a row which has distinct intersection with the other rows or has a double link with some row. Now we can apply an interchange[1] using a one of this row and a one from some other connected component. This interchange reduces the number of connected components. Thus the assumption that $k > 1$ is not correct. This completes the proof of the theorem.

<u>Theorem 2</u> : Given $R = (r_1, r_2 \ldots r_m)$ $r_i \geq 2$ for $i \in I = 1, 2, \ldots m$ and $S = (s_1, s_2, \ldots, s_n)$, $s_j > 0$ for $j \in I = 1, 2, \ldots n$ and that $U(R, S)$ is non-empty then $\bar{d}$ of this class of matrices is finite if $\Sigma r_i \geq m_i + n_i - 1$ and for every partitioning of I as $I_1 \cup I_2, \ldots I_k$; J as $J_1 \cup J_2 \ldots \cup J_k$; one of the following should be violated

(1) $\sum_{i \in I_p} r_i = \sum_{j \in J_p} s_j, \quad p = 1 \text{ to } k$

(2) $\sum_{i \in I'_p} r_i \geq |I_p| + |J_p| - 1, \quad p = t \text{ to } k.$

This theorem follows from the previous theorem.

$U(\tilde{k}, \tilde{k})$ denotes the class of all $n \times n$ (0, 1)-Matrices with $\tilde{k} = (k, k, k, \ldots, k)$ both for row sum and column sum vectors. We have the following theorem for the class $U(\tilde{k}, \tilde{k})$.

Theorem 3 :

$$\bar{d} = \begin{cases} 1 & k \leq n \leq 2k - 1 \\ \infty & \text{if } n \geq 2k \end{cases}$$

Proof : When $n \leq 2k - 1$ we cannot have two columns without a one in some common row and so $d(A) = 1$ for all $A \in U(\tilde{k}, \tilde{k})$, therefore $\bar{d}=1$. When $n \geq 2k$, $U(\tilde{k}, \tilde{k})$ contains a disconneted matrix and hence $\bar{d} = \infty$.

When $n = q \cdot k + r$ it is perhaps true that $\tilde{d} \leq q$.

The formulae for $\bar{d}$ and $\tilde{d}$ in the general case would be desirable but not yet available.

In view of this definition of a distance for (0,1)-Matrices, ever so many questions can be asked, carrying the analogy from graphs to matrices.

Appendix

ACCESSIBILITY AND ADJOINT GRAPHS

Let $G = (X, U)$ be a finite unoriented graph. The adjoint (or interchanbe graph) $I(G)$ of G is defined as follows. The edges of G form the vertex set of $I(G)$, and two vertices in $I(G)$ are joined by an edge if and only if the corresponding edges in G have a vertex in common.

The study of the behaviour of graphs under repeated interchanges has been a topic of many investigations. In this appendix we shall study the interchange graphs from the point of view of accessibility and order. Before going further we define a quadrilateral. A quadrilateral in a graph is defined as a set of four vertices. There are shortest chains between six different pairs. Out of these six shortest chains four are named diagonals. The two diagonals do not intersect at any of these four vertices.

Theorem 1 Let G be a finite connected graph and let its diameter be k. Then the diameter of $I(G)$ is either $k-1$,

k or $k+1$.

<u>Lemma 1</u> If two edges are the terminal edges of a simple chain of length t in G, then in $I(G)$ the vertices corresponding to these edges will have a simple chain of length $t-1$ between them.

<u>Proof</u> This follows from the definition of $I(G)$.

<u>Proof of theorem 1</u> Since the diameter of G is k, there exist two vertices a_o and b_o such that the shortest chain between them is of length k.

$$\text{i.e.} \quad d(a_o, b_o) = k \qquad \ldots \quad (1)$$

Let (a_o, a_1) and (b_o, b_1) be the two edges having a_o and b_o as their terminal vertices. (If $k = 1$, (a_o, a_1) and (b_o, b_1) will coincide). Therefore (a_o, a_1) and (b_o, b_1) are the terminal edges of a chain of length at most $k+2$. Since k is the maximum distance between any two vertices in G, any two edges of G lie on a chain of length at most $k+2$. And hence by lemma 1 the shortest chain between any two vertices in $I(G)$ is of length at most $k+1$. In other words the diameter of $I(G)$ is less than or equal to $k+1$.

The four vertices a_o, a_1, b_o and b_1 form a quadrilateral in G. We shall call the shortest chains between a_o and b_1 and a_1 and b_o as the diagonals and the other shortest chains as sides. The diagonals have to be at least $k-1$ in length. For, say, the diagonal from a_o is of length less than $k-1$, then there would exist a chain of length less than k between a_o and b_o. This is contradictory to the assumption that the distance between a_1 and b_o or a_o and b_1 can at most be equal to k.

We therefore have

$$k - 1 \leq d(a_1, b_o) \leq k \qquad \text{.... (2)}$$

$$k - 1 \leq d(a_o, b_o) \leq k \qquad \text{.... (3)}$$

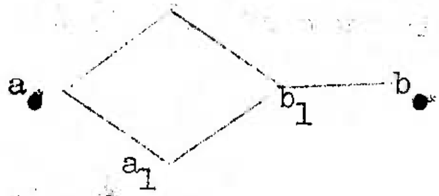
By similar arguments we can see that the side (a_1, b_1) is not greater than $k-2$ in length and we have

$$k - 2 \leq d(a_1, b_1) \leq k. \qquad ....(4)$$

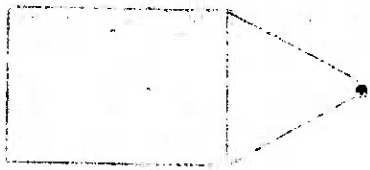
Considering 1, 2, 3 and 4, we observe that any quadrilateral having two edges for two sides and containing a diametral chain as one of the other sides can have twelve possible structures. If in any of these quadrilaterals the fourth side is of length $k-2$, then the diameter of $I(G)$ would be $k-1$. For, the shortest simple chain having (a_0, a_1) and (b_0, b_1) as terminal edges is of length k. And by the lemma the vertices in $I(G)$ corresponding to these edges of G will lie on a simple chain of length $k-1$. If the fourth side or one of the diagonals is of length $k-1$, then the diameter of $I(G)$ would be k. If the fourth side and both the diagonals are of length k, then the diameter of $I(G)$ would be $k+1$.

To summarize the above discussion, the diametral structure of $I(G)$ depends upon the existence of certain types of quarilaterals in G. The interchange graphs of

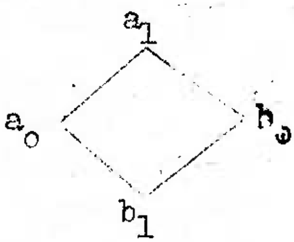
G_1, G_2 and G_3 in Figure 1 illustrate the three possibilities. The relevant quadrilaterals are indicated.



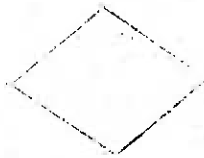
$G_1 : d = 3.$



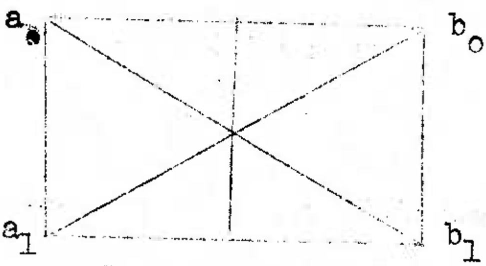
$I(G_1) : d = 2.$



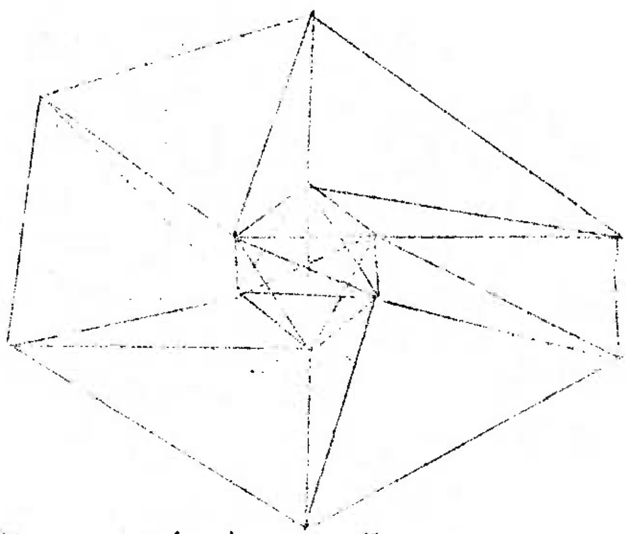
$G_2 : d = 2.$



$I(G_2) : d =$ [illegible].



$G_3 : d = 2.$



$I(G_3) : d = 3.$

Theorem 2 Let G be a finite connected graph with diameter k. If $I^{-1}(G)$ is defined, then the diameter of $I^{-1}(G)$ is either k-1 or k or k+1.

This follows from theorem 1.

Theorem 3 If G is a k to λ accessible graph of edge order s, then I(G) is a $G_V(N_1, k+i, \lambda + j, s)$ where N is the no of vertices in I(G) and i and j take the values -1 or 0 or 1.

There is a one to one correspondence between the partial graphs of G and subgraphs of I(G). Every partial graph of G with more than $|U| - s - 1$ edges (where $|U|$ denotes the number of edges of G) is λ-accessible and the corresponding subgraphs of I(G) with more than $|U| - s - 1$ vertices are $\lambda + j$-accessible, where $j = -1$ or 0 or +1. I(G) itself will be k + i accessible where i takes one of the values -1, 0 or +1. Hence the theorem.

Theorem 4 Let G be a k to λ accessible graph of vertex order s. Then, if $I^{-1}(G)$ is defined it will be a $G_E(N_2, k+i, \ell+j, s)$ where N is the number of vertices of $I^{-1}(G)$ and i and j take one of the values -1, 0 or $+1$.

The proof of this theorem is similar to that of Theorem 3.

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