STUDIES IN THE THEORY OF GRAPHS

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PREFACE

The research work, on which the present thesis is based, was carried out in part, during my stay at the Indian Statistical Institute, Calcutta, from July 1962 to February 1965, and mostly thereafter, at the Tata Institute of Fundamental Research, Bombay.

The thesis consists of four chapters which are, more or less, independent of each other. A detailed summary of the results contained in each chapter is provided at the beginning of that chapter. A brief account of the main results presented in the thesis is given below.

In Chapter 1, the concept of the cover index of a graph is introduced. Let G be a graph (undirected, without loops) with set of vertices V(G) and set of edges E(G). Any $C \subseteq E(G)$ is said to be a cover of G if each vertex $v \in V(G)$ is incident with at least one edge in C. The cover index k(G) of G is the maximum number k such that there exists a partition of E(G) into k sets each of which is a cover of G. The greatest lower bound of the degrees of the vertices of G is denoted by $\tilde{d}(G)$. We prove (Theorem 1. 3. 1) that if G is a locally finite bipartite graph, then, $k(G) = \tilde{d}(G)$. This theorem solves, in particular, a problem suggested by O. Ore [18], and implies a well-known theorem of J. Petersen [20] and D. Konig [14]. A graph G is called an s-graph if no two of its vertices are joined by more than s edges. We prove (Theorem 1. 4. 1), in general, that if G is an s-graph which is locally finite, then, $\tilde{d}(G) \geq k(G) \geq \tilde{d}(G) - s$. It is shown that the bounds cannot be improved if $s \geq 1$ and $\tilde{d}(G) = 2ms - r$ where $r \geq 0$, $m > \left[\frac{r-1}{2} \right] + s$.

In Chapter 2, the problem of determining the chromatic index of a graph is considered. For a graph G, any (nonempty) $M \subseteq E(G)$ is said to be a matching of G if each vertex $v \in V(G)$ is incident with at most one edge in M. The chromatic index $q(G)$ of G is the minimum number q such that there exists a partition of $E(G)$ into q sets each of which is a matching of G. The least upper bound of the degrees of the vertices of G is denoted by $\bar{d}(G)$. The main result (Theorem 2.3.1) proved is that if G is an s-graph which is locally bounded, then, $\bar{d}(G) \leq q(G) \leq \bar{d}(G)+s$. The bounds are more exact than the previously known bounds due to C. E. Shannon [22] and are shown to be best possible if $s \geq 1$ and $\bar{d}(G) = 2ms-r$ where $r \geq 0$, $m > \left[\frac{r+1}{2}\right]$.

In Chapter 3, we consider the problem of determining the largest number of arcs in any basis digraph consisting of n vertices. It is proved that any basis digraph with n vertices and k strong components can have at most $2(n-k)+\left[\frac{k^2}{4}\right]$ arcs. Further, the bound is shown to be best possible and the structure of extremal basis digraphs is completely determined. This solves a problem proposed by O. Ore [19].

Chapter 4 is based on the results published by the author in [5], [6] and [7]. The proof of a theorem, which is stronger than an earlier theorem due to A. Kotzig [16] and the author [5] is included, and some remarks on the problem of analysis of a digraph are added.

A few conjectures, which are related to the problems considered in the thesis, have also been stated.

For terminology, C. Berge (1962) has been followed, in general, throughout. However, to make the work self-contained, and to avoid any possible confusion, definitions of all the necessary terms and concepts are incorporated.

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TABLE OF CONTENTS

CHAPTER 4

ON PSEUDOSYMMETRIC DIGRAPHS

CHAPTER 1

THE COVER INDEX OF A GRAPH

1.1. Introduction and Summary

Let G be a graph (undirected, without loops) with set of vertices V(G) and set of edges E(G). Any subset C of E(G) is said to be a cover of G if each vertex $v \in V(G)$ is an endvertex of at least one edge in C. The cover index of a graph G, denoted by k(G), is defined to be the maximum number k such that there exists a decomposition of E(G) into k mutually disjoint sets each of which is a cover of G. The lower degree $\tilde{d}(G)$ of G is the greatest lower bound of the degrees of its vertices. (For the definitions of the cover index k(D) and the lower degree $\tilde{d}(D)$ of a directed graph D, see section 1.2.) A graph G is called an s-graph if no two of its vertices are joined by more than s edges. The least number s for which G is an s-graph is called its multiplicity.

In the present chapter, we propose to investigate the problem of determining the cover index of a graph in terms of its lower degree and multiplicity. Evidently, for any graph G we must have $k(G) \leq \tilde{d}(G)$. Also, it is obvious that $k(G) \geq 1$ if and only if $\tilde{d}(G) \geq 1$. O. Ore (1962) proved that "for a graph G, $k(G) \geq 2$ if and only if $\tilde{d}(G) \geq 2$ and no connected component of G is a cycle with odd number of edges". He then proposed [18] to establish an analogue of this result for directed graphs. In section 1.3, the following general theorem is proved. Theorem 1.3.1

"If G is any locally finite bipartite graph then $k(G) = \tilde{d}(G)$". As a corollary to Theorem 1.3.1, we then obtain Theorem 1.3.2 "For any locally finite directed graph D, $k(D) = \tilde{d}(D)$". Evidently, Theorem 1.3.2 solves a more general problem than that suggested by O. Ore. Further, it is observed that Theorem 1.3.2, and hence, <u>a priori</u> Theorem 1.3.1, implies a well-known theorem of J. Petersen [20] and D. Konig [14].

In section 1.4, considering the problem for arbitrary graphs (not necessarily bipartite) the following theorem is proved. Theorem 1.4.1 "If G is any s-graph which is locally finite then $\tilde{d}(G) \geq k(G) \geq \tilde{d}(G) - s$. The bounds are best possible in the following cases: $s \geq 1$ and $\tilde{d}(G) = 2ms - r$ where $r \geq 0$, $m > \left[\frac{r-1}{2}\right] + s$".

A few problems for further investigation are also suggested and some conjectures are made.

It may be remarked that the theorems proved in this chapter may also be stated, for the special case of graphs of multiplicity 1, in terms of (0, 1)-matrices studied extensively by H. J. Ryser [21] and others, or in terms of families of subsets of a given set.

<u>1.2. Definitions and Notations; Colorings; The Concept of Alternating Chains</u>

The method used in proving our results in this chapter is based on the concept of alternating chains introduced by Petersen (1891) in his investigations on the existence of subgraphs which has been an effective tool for many problems in graph theory. In this section, we first explain some of the basic concepts and notations which are used throughout this work, and then define the concepts of colorings and alternating chains. For terminology, C. Berge [1] is followed, in general.

Definitions and Notations: An undirected graph or simply a graph G is defined by a nonempty set V(G) of vertices and a set E(G) of edges together with a relationship which identifies each edge $e \in E(G)$ with an unordered pair (v, u) of distinct vertices $v, u \in V(G)$, called its endvertices, which the edge is said to join. In the following, the letter G denotes, without any further specification, a graph. (It should be noted that by definition we exclude from our consideration graphs which have 'loops', i. e., edges with coincident endvertices). Two vertices are adjacent if they are joined by an edge; two edges are adjacent if they have a common endvertex. A vertex v and an edge e are incident with each other if v is an endvertex of e. (This terminology is suggested by geometric considerations.)

As a notational convention, if e is an edge joining the vertices v and u then we write $e = (v, u)$. We note that if $e = (v, u)$ and $e' = (v', u')$ are two distinct edges, then, unless specifically stated, we may have $v = v'$ or $u = u'$ or both $v = v'$ and $u = u'$.

If no two vertices of a graph G are joined by more than s edges, $s \geq 0$, then G is called an s-graph. The least number s for which G is an s-graph is called the multiplicity of G.

A graph G is finite if both V(G) and E(G) are finite; otherwise G is infinite.

Any subset E' of E(G) defines a partial graph G' of G such that V(G') = V(G) and E(G') = E'. Any (nonempty) subset V' of V(G) defines a subgraph G' of G such that V(G') = V', E(G') $\subseteq$ E(G) and an edge of G is an edge of G' if and only if both of its endvertices are in V'. A partial graph of a subgraph of G is called a partial subgraph of G. (It is implied that every edge of a partial subgraph G' of G has the same endvertices in G' as in G).

Two graphs are said to be edge-disjoint if they share no edge in common. A nonempty family $\{G_0, G_1, \ldots, G_{k-1}\}$ of mutually edge-disjoint partial graphs of a graph G is said to form a decomposition of G if $E(G) = \bigcup_{i=0}^{k-1} E(G_i)$; we then write

(1.2.1) $$G = G_0 + G_1 + \ldots + G_{k-1}.$$

A chain in a graph G is a finite sequence (we shall have no occasion to consider infinite chains) of the form

(1.2.2) $$\mu(v_0, v_r) = (v_0, e_1, v_1, e_2, \ldots, e_r, v_r), \; r \geq 0,$$

where (1) $v_i \in V(G)$, (2) $e_i \in E(G)$, and (3) e_i joins v_{i-1} and v_i for each $i = 1, 2, \ldots, r$. (If all the vertices and edges of a graph G can be arranged in a chain then G is itself called a chain). The chain μ (v_0, v_r) is said to connect its 'terminal vertices' v_0 and v_r. A graph is connected if every two of its vertices are connected by a chain. The relation of being connected in G is obviously an equivalence relation. It, therefore, 'partitions' G into a family $\{G_a\}$ of graphs each of which is a maximal connected subgraph of G and is called a connected component of G. A chain (1. 2. 2) is called a cycle if (1) it has at least one edge, (2) its edges $e_1, e_2, \ldots, e_r$ are all distinct, and (3) its last vertex v_r coincides with its first vertex v_0.

In a graph G, the degree $d(G, v)$ of a vertex v is the number of edges that are incident with v. The lower degree of G, denoted by $\tilde{d}(G)$, is the greatest lower bound of the degrees of its vertices. Thus, the lower degree of a graph is the largest number k such that there are at least k edges incident with each of its vertices.

A graph is said to be locally finite if the degree $d(G, v)$ of each vertex $v \in V(G)$ is finite. Trivially, if G is locally finite, then its lower degree $\tilde{d}(G)$ is finite.

Any set of edges C of a graph G, $C \subseteq E(G)$, is called a cover or an edge-cover (for the vertices) of G if every vertex $v \in V(G)$ is incident with at least one edge in C. Obviously, if G has an 'isolated vertex' (a vertex is isolated if its degree is zero) or if $\tilde{d}(G) = 0$, then there can exist no cover of G.

The cover index of a graph G, denoted by k(G), is defined as follows: k(G) = 0 if $\tilde{d}(G) = 0$; otherwise, k(G) is the maximum number k such that there exists a partition of E(G) into k sets

(1.2.3) $$E_0, E_1, \ldots, E_{k-1}; \quad E(G) = \bigcup E_i, \quad E_i \cap E_j = \emptyset$$

where each of the sets E_i ($i = 0, 1, \ldots, k-1$) is a cover of G.

A partial graph G' of G is called a covering graph of G if $\tilde{d}(G') \geq 1$ or, in other words, if its defining set of edges E(G') is a cover of G. The cover index k(G) of a graph G may equivalently be defined to be the maximum number k such that there exists a decomposition (1.2.1) of G where each of the graphs G_i ($i = 0, 1, \ldots, k-1$) is a covering graph of G. Evidently, if $\tilde{d}(G) = 0$, then no such decomposition of G exists and in that case we have, by definition, k(G) = 0. Henceforth, we may disregard the trivial case $\tilde{d}(G) = 0$ and assume tacitly, if necessary, that $\tilde{d}(G) \geq 1$.

Remark: It may be observed that the concept of the 'cover index' of a graph, as we have introduced above, is apparently dual to the well-known concept of the 'chromatic index' of a graph. This duality is made more apparent by the comparison of the results obtained in this chapter and the next.

A directed graph or briefly digraph D is defined by a nonempty set V(D) of vertices and a set A(D) of arcs together with a relationship which identifies each arc with an ordered pair of (not necessarily distinct) vertices which the arc is said to join. For each arc (v, u), v is called its initial vertex and u its terminal vertex, both v and u being called its endvertices.

A digraph D is finite if both V(D) and A(D) are finite; otherwise, D is infinite.

In a digraph D, the outward degree $d^+(D, v)$ of a vertex v is the number of arcs in A(D) with initial vertex v and the inward degree $d^-(D, v)$ of v is the number of arcs in A(D) with terminal vertex v. A digraph D is locally finite if for every vertex $v \in V(D)$, the sum $d^+(D, v) + d^-(D, v)$ is finite.

The lower degree of D, denoted by $\tilde{d}(D)$, is the largest number k such that $d^+(D, v) \geqslant k$, $d^-(D, v) \geqslant k$ for every vertex $v \in V(D)$. Obviously, if D is locally finite then its lower degree $\tilde{d}(D)$ is finite.

For a digraph D, the notions of partial graph and subgraph are defined analogously as for undirected graphs. Thus, a partial graph of D is a digraph D' such that $V(D') = V(D)$ and $A(D') \subseteq A(D)$. Two digraphs are arc-disjoint if they have no arc in common. A nonempty family $\{D_0, D_1, \ldots, D_{k-1}\}$ of mutually arc-disjoint partial graphs of a digraph D is said to form a decomposition of D if $A(D) = \bigcup_{i=0}^{k-1} A(D_i)$, and we write $D = D_0 + D_1 + \ldots + D_{k-1}$.

A set of arcs C of a digraph D, $C \subseteq A(D)$, is called a cover of D if for each vertex $v \in V(D)$, there is in C at least one arc with initial vertex v and at least one arc with terminal vertex v. A partial graph D' of D is defined, as suggested by O. Ore [18], to be a covering graph of D if $\tilde{d}(D') \geq 1$ or, equivalently, if its defining set of arcs A(D') is a cover of D.

The cover index of a digraph D, denoted by k(D), is defined as follows: k(D) = 0 if $\tilde{d}(D) = 0$; otherwise, k(D) is the maximum number k such that there exists a decomposition of D into k partial graphs each of which is a covering graph of D; or, equivalently, k(D) is the maximum number k such that there exists a partition of A(D) into k sets each of which is a cover of D. In the following, we may disregard the case $\tilde{d}(D) = 0$ and assume that $\tilde{d}(D) \geq 1$.

Colorings: Our objects in this chapter necessitate the consideration of partitions of the set of edges of a given graph. Consider, therefore, for an undirected graph G, an arbitrary decomposition of its set of edges E(G):

(1.2.5) $$E_0, E_1, \ldots, E_{k-1}; \quad E(G) = \cup E_i, \quad E_i \cap E_j = \emptyset$$

It is convenient for us to think of each set E_i as representing a color which we denote by the integer i. Further, we say that each edge $e \in E_i$ is colored with the color i thus obtaining a coloring of the edges of G by means of k distinct colors. Formally, any function f which assigns to each edge $e \in E(G)$ an integer $f(e) \in \{0, 1, \ldots, k-1\}$ is called a coloring or, more specifically, a k-coloring (of the edges) of G. The integers 0, 1, ..., k-1 are referred to as colors and if $f(e) = i$ we say that the edge e is colored with the color i or briefly that e is an i-edge. It is easily seen that there is a natural one-to-one correspondence between k-colorings of G and decompositions of E(G) into k sets.

Any k-coloring of G is called admissible if every vertex $v \in V(G)$ is an endvertex of at least one i-edge for each $i = 0, 1, \ldots, k-1$. It is immediately seen that in a decomposition (1.2.5) of E(G), each of the sets $E_i (i = 0, 1, \ldots, k-1)$ is a cover of G if and only if its corresponding k-coloring of G is admissible. Thus, the cover index of a graph G is the maximum number k for which an admissible k-coloring of G exists.

Let f be any k-coloring of G. For any vertex $v \in V(G)$, we denote by $c_f(v)$ or simply by c(v), if no confusion is possible, the set of colors which are assigned by f to the edges incident with v. Clearly, $c_f(v) \subseteq \{0, 1, \ldots, k-1\}$. The difference

$$\Delta_f(v) = (k - |c_f(v)|) \tag{1.2.6}$$

is called the deficiency of f at the vertex v. ($|S|$ denotes the number of elements in the set S.) For any subset V' of V(G), the sum

$$\Delta_f(V') = \sum_{v \in V'} (k - |c_f(v)|) \tag{1.2.7}$$

is called the deficiency of f at V'. (By definition, $\Delta_f(V') = 0$ if $V' = \emptyset$.) The total deficiency of f, denoted by $\Delta_f(G)$, is obtained when the sum

(1.2.7) is extended over all vertices of G, i.e., when $V' = V(G)$. It is clear that a k-coloring f of a graph G is admissible if and only if its total deficiency $\Delta_f(G) = 0$. Thus, in order to prove that $k(G) \geq k$, where k is some positive integer, it is sufficient to show that G possesses a k-coloring f whose total deficiency is zero.

The Concept of Alternating Chains: Any chain considered in the following is assumed to have at least one edge and all of its terms are to be distinct except that its last vertex may coincide with one of the vertices preceding it.

Consider an arbitrary k-coloring ($k \geq 2$) of a graph G and let i and j be any two fixed colors. A chain

(1.2.8) $$\mu(v_0, v_r) = (v_0, e_1, v_1, e_2, \ldots, v_{r-1}, e_r, v_r), \quad r \geq 1,$$

in G is called an (i, j)-alternating chain if its edges $e_1, e_2, \ldots, e_r$ are alternately colored with i and j, the first one being an i-edge.

An (i, j)-alternating chain (1.2.8) is called suitable if it satisfies any one of the following conditions:

(i) $v_0 = v_r$;

(ii) e_r is an i-edge (respectively, j-edge) and there is no j-edge (respectively, i-edge) incident with v_r;

(iii) e_r is an i-edge (respectively, j-edge) and there is another i-edge (respectively, j-edge) incident with v_r.

A suitable (i, j)-alternating chain (1. 2. 8) is called minimal if

$\mu(v_0, v_t) = (v_0, e_1, v_1, \ldots, e_t, v_t)$ is not suitable for any t, $1 \leqslant t \leqslant r-1$.

It is easy to observe that if G is finite then, given any i-edge $e_1 = (v_0, v_1) \in E(G)$, a (minimal) suitable (i, j)-alternating chain (1. 2. 8) in G can always be found. (In fact, this can be done by following along an (i, j)-alternating chain starting with e_1 till one of the conditions (i), (ii) or (iii) is satisfied).

The method of alternating chains as used in this chapter, consists in the following. Suppose G is a finite graph and we want to show that G possesses a k-coloring ($k \geqslant 2$) which is admissible. We start with an arbitrary k-coloring f of G. If $\Delta_f(G) = 0$, we are through. Otherwise, choosing a vertex $v_0 \in V(G)$ and the colors i and j properly, a suitable (i, j)-alternating chain starting with the vertex v_0 is found and the colors i and j interchanged on all edges belonging to this chain. Such a step may have to be taken more than once, till we obtain a k-coloring g (say), such that $\Delta_g(G) < \Delta_f(G)$. The assertion is then proved by a proper use of finite induction.

1. 3. The Cover Index of a Bipartite Graph

A graph G is said to be bipartite (simple, C. Berge [1]) if it contains no cycle with odd number of edges. Alternatively, G is bipartite if (and only if) its set of vertices V(G) can be decomposed into two disjoint sets V' and V'' such that each edge $e = (v', v'') \in E(G)$ joins a vertex $v' \in V'$ with a vertex $v'' \in V''$. These special kind of graphs are of particular significance in graph theory and play a considerable role in various applications. For instance, any (0, 1)-matrix or a family of subsets of a given set can be represented by a bipartite graph and graph theory becomes a handy tool for several combinatorial problems relating to such systems. Also, it is seen that to any directed graph D, there corresponds a unique bipartite graph G_D (an arbitrary bipartite graph may not correspond to any directed graph) defined as follows: to each vertex $v \in V(D)$ there correspond two vertices $v', v'' \in V(G_D)$ and to each arc $(v, u) \in A(D)$ there corresponds an edge $(v', u'') \in E(G)$. Thus, any problem for directed graphs may be formulated as a problem for bipartite graphs.

Let G be any graph. Clearly, if $\tilde{d}(G) = 0$ or 1, then $k(G) = 0$ or 1 respectively. Oystein Ore (1962) proved that "for any graph G, $k(G) \geq 2$ if and only if $\tilde{d}(G) \geq 2$ and no connected component of G is a cycle with odd number of edges." He then suggested [18] to establish an analogue of this theorem for directed graphs. The solution to this problem is provided by the following theorem "for any digraph D, $k(D) \geq 2$ if and only if $\tilde{d}(D) \geq 2$," which we can now prove. Indeed, this theorem is most conveniently derived as a corollary to the above theorem of O. Ore. (The details of the proof which are simple, are omitted here).

We consider below the general problem of determining the cover index of a bipartite graph which, as is easily shown, includes the problem of determining the cover index of a directed graph. We shall now prove the following theorem which may be called a <u>Decomposition Theorem for Bipartite Graphs</u>.

Theorem 1.3.1. Let G be a bipartite graph which is locally finite and let $\tilde{d}(G) = k$. Then, there exists a decomposition

(1.3.1) $$G = G_0 + G_1 + \ldots + G_{k-1}$$

such that each of the graphs $G_i (i = 0, 1, \ldots, k-1)$ is a covering graph of G. In other words, if G is a locally finite bipartite graph then $k(G) = \tilde{d}(G)$.

Proof: If $\tilde{d}(G) = 0$ then the theorem is trivially true. Let therefore, $\tilde{d}(G) = k \geq 1$. Evidently, we must have $k(G) \leq k$. Hence, to prove the theorem, it is sufficient to show that $k(G) \geq k$ or equivalently, that there exists a k-coloring f of G which is admissible, i.e., $\Delta_f(G) = 0$. We shall, infact, prove the following lemma which is apparently stronger than the above statement.

Lemma 1.3.1. Let G be a locally finite bipartite graph and let V' be any subset of V(G) such that $d(G, v) \geq k$, $k \geq 1$, for all $v \in V'$. Then, there exists a k-coloring f of G such that $\Delta_f(V') = 0$.

Proof: The lemma will be proved first for the case in which G is finite; then, assuming the lemma to be true for all finite graphs, it will be proved, by using a well-known argument due to D. Konig and S. Valko (1926), when G is infinite (but locally finite).

G is finite: If k = 1, the lemma holds obviously. Suppose, therefore, that $k \geq 2$. Consider k-colorings of the graph G. Since G is finite, the number of k-colorings of G is finite. Hence, there must be a k-coloring f of G such that $\Delta_f(V')$ is minimal, i.e., $\Delta_f(V') \leqslant \Delta_g(V')$ for any other k-coloring g of G. We shall show that $\Delta_f(V') = 0$.

Suppose, if possible, that $\Delta_f(V') > 0$. Then, there will be a vertex $v_0 \in V'$ such that $\Delta_f(v_0) > 0$ which implies that there is a color j such that $j \notin c_f(v_0)$. Since, now, at most, k-1 colors are assigned to the edges incident with v_0 and by hypothesis $d(G, v_0) \geqslant k$, there must be a color i such that there are (at least) two i-edges incident with v_0. Now, we determine a suitable (i, j)-alternating chain $\mu(v_0, v_r)$ starting with the vertex v_0. Let g denote the k-coloring of G obtained from f by interchanging the colors i and j on all edges belonging to the chain $\mu(v_0, v_r)$. Since the chain is suitable, it is easily seen that $|c_g(v)| \nless \geqslant |c_f(v)|$ for all vertices v except possibly when $v = v_0$. We now observe that v_r cannot coincide with v_0. Infact, if $v_r = v_0$, then, since $j \notin c_f(v_0)$, $\mu(v_0, v_r)$ would be a cycle in G with odd number of edges contradicting the assumption that G is bipartite. Hence, it is seen that, since there were two i-edges incident with v_0 with respect to f, $c_g(v_0) = c_f(v_0) \cup \{j\} \supsetneqq c_f(v_0)$. Hence, by definition (1. 2. 7), it follows that $\Delta_g(V') < \Delta_f(V')$ which is contradictory to the choice of f. Hence, we must have $\Delta_f(V') = 0$. This prove the lemma for finite graphs.

<u>G is infinite:</u> Let us first assume that G is connected. Since, moreover, G is locally finite, the set of vertices V(G) of G is enumerable. Let $v_1, v_2, v_3, \ldots$ be an enumeration of V(G) in some order. Let V_n (n = 1, 2, ...) denote the

set of all vertices v such that either $v \in \{v_1, v_2, \ldots, v_n\}$ or v is adjacent to some vertex v_i, $i = 1, 2, \ldots, n$. Denote by G_n the subgraph of G defined by the set of vertices V_n and let $V_n' = V' \cap \{v_1, v_2, \ldots, v_n\}$. Now, it is evident that the graphs G_n are finite and $d(G_n, v) \geq k$ for each $v \in V_n'$. Since the lemma holds for finite graphs, it follows that G_n has k-colorings f_n such that $\Delta_{f_n}(V_n') = 0$, but the number of such k-colorings of G_n is obviously finite. Let any such k-coloring f_n of G_n be called 'proper'. Now, if $n < m$, G_n is a subgraph of G_m, and each proper k-coloring f_m of G_m implies a proper k-coloring f_n of G_n which is called the restriction of the coloring of G_m; f_m is called an extension of f_n and we write $f_n \prec f_m$.

Now, the proper k-colorings of the graphs $G_2, G_3, \ldots$ imply proper k-colorings of G_1 and since G_1 has only a finite number of proper k-colorings, there must be one among them which is restriction of infinitely many proper k-colorings. Let us fix one such coloring f_1 of G_1. We consider now only the extensions of this fixed coloring. Each of these implies a proper k-coloring of G_2. Hence, by the same argument as above, G_2 must have a proper k-coloring f_2 which is restriction of an infinity of proper k-colorings. We proceed in this manner to obtain a sequence of k-colorings $f_1, f_2, \ldots$ such that $f_n (n = 1, 2, \ldots)$ is a proper k-coloring of G_n and $f_1 \prec f_2 \prec \ldots$. Now, we define a k-coloring f of G as follows: for each edge $e \in E(G)$, $f(e) = f_n(e)$ if n is the first index such that $e \in E(G_n)$. Evidently, f is well-defined and we have $\Delta_f(V') = 0$.

If G is not connected, let $\{G_a\}$ be the family of connected components of G. Let $V'_a = V' \cap V(G_a)$ for each index a. Then, $d(G_a, v) \geq k$ for each $v \in V'_a$ and since G_a is connected, by what we have just proved, each of the graphs G_a has a k-coloring f_a such that $\Delta_{f_a}(V'_a) = 0$. We define a k-coloring f of G as follows: for every $e \in E(G)$, $f(e) = f_a(e)$ if $e \in E(G_a)$. It is then clear that $\Delta_f(V') = 0$. This completes the proof of the lemma.

The theorem now follows immediately if we take $V' = V(G)$ in the above lemma. Hence, the proof of the theorem is also complete.

Remark: Let G be a finite graph which is bipartite and let $\tilde{d}(G) = k$. The proof of Lemma 1.3.1 suggests an algorithm for an actual determination of an admissible k-coloring of G or, equivalently, a decomposition of G into k covering graphs. We start with an arbitrary k-coloring f_1 of G. If f_1 is not admissible then a suitable alternating chain is found as indicated in the proof of the lemma. By interchanging the colors of all edges belonging to this chain, we obtain a new k-coloring f_2 of G such that the total deficiency of f_2 is less than the total deficiency of f_1 by atleast 1. The process is repeated with the new coloring (each time) till an admissible k-coloring of G is obtained. (Since G is finite, an admissible k-coloring of G is obtained after a finite number of repetitions of the process.)

It may also be noted that the above algorithm is quite efficient and can be programmed conveniently for use on electronic computers.

There are several important consequences of Theorem 1.3.1 which we shall state below.

We first obtain the following theorem which may be called a Decomposition Theorem for Directed Graphs.

Theorem 1.3.2. Let D be a directed graph which is locally finite and let $\tilde{d}(D) = k$. Then, there exists a decomposition

$$D = D_0 + D_1 + \dots + D_{k-1} \qquad (1.3.2)$$

such that each of the digraphs $D_i (i = 0, 1, \dots, k-1)$ is a covering graph of D. In other words, if D is a locally finite directed graph then $k(D) = \tilde{d}(D)$.

Proof: Consider the bipartite graph G_D corresponding to the given graph D as defined above. Clearly, G_D is locally finite and $\tilde{d}(G_D) = \tilde{d}(D)$. Also, it is seen that a set of arcs in D is a cover of D if and only if its corresponding set of edges in G_D is a cover of G_D so that we have evidently, $k(G_D) = k(D)$. Now, by Theorem 1.3.1, we have $k(G_D) = \tilde{d}(G_D)$ whence the theorem follows immediately.

We say that a digraph D is regular of degree k if $d^+(D, v) = d^-(D, v) = k$ for each vertex $v \in V(D)$. Any partial graph D' of D is called an r-factor of D if D' is regular of degree r. If, in Theorem 1.3.2, we take D to be regular of degree k, then evidently, each of the digraphs $D_i (i = 0, 1, \ldots, k-1)$ in the decomposition (1.3.2) will be a 1-factor of D. Hence, as a particular case of Theorem 1.3.2, we obtain the following:

Corollary 1.3.1. Let D be a directed graph which is regular of degree k. Then, there exists a decomposition (1.3.2) of D such that each of the digraphs $D_i (i = 0, 1, \ldots, k-1)$ is a 1-factor of D.

Consider an undirected graph G. The graph G is said to be regular (homogeneous, Berge [1]) of degree k if $d(G, v) = k$ for each vertex $v \in V(G)$. A partial graph G' of G is called an r-factor of G if G' is regular of degree r. It is known that if G is a regular graph of (even) degree 2k then the edges of G can be so 'directed' ('oriented'), each edge being assigned a unique direction, that the resulting digraph G_d, say, is regular of degree k. Clearly, if a partial graph G_d' of G_d is an r-factor of G_d then its corresponding partial graph G' of G is a 2r-factor of G. Hence, from Corollary 1.3.1, we obtain the following 'factorization theorem' due to Petersen (1891).

Corollary 1.3.2. (Theorem of Petersen) Let G be a regular graph of degree 2k. Then, there exists a decomposition (1.3.1) of G such that each of the graphs $G_i (i = 0, 1, \ldots, k-1)$ is a 2-factor of G.

It may also be noted that the theorem of Petersen implies, in its turn, Corollary 1. 3. 1.

The following result for regular bipartite graphs due to Konig [14], which is a special case of a more general theorem of Konig and P. Hall (1934), can also be shown to be equivalent to Corollary 1. 3. 1 or the theorem of Petersen and hence, is a consequence of Theorem 1. 3. 2.

Corollary 1. 3. 3. (Theorem of Konig) Let G be a bipartite graph which is regular of degree k. Then, there exists a decomposition (1. 3. 1) of G such that each of the graphs G_i ($i = 0, 1, \ldots, k-1$) is a 1-factor of G.

We observe that Theorem 1. 3. 1 is apparently stronger than Theorem 1. 3. 2 which implies the theorems of Petersen and Konig. It is not known if the converse implication also holds. Specifically, we may ask the following question: "Is it true that the theorem of Petersen (or Konig) implies Theorem 1. 3. 1?"

We state below another problem which seems to be of considerable interest. A digraph D is said to be 'pseudosymmetric' if $d^+(D, v) = d^-(D, v)$ for each vertex $v \in V(D)$. We believe that the following statement is true; however, the proof is not known to us.

Conjecture: Let D be a locally finite directed graph which is pseudosymmetric and let $\tilde{d}(D) = k$. Then, there exists a decomposition of D into k covering graphs each of which is also pseudosymmetric.

(For a detailed study of some of the properties of finite pseudosymmetric digraphs and other unsolved problems, see Chapter 4).

Note: The results contained in this section were announced at the "International Seminar on Graph Theory and Its Applications" held in Rome, 5 - 9 July, 1966. See [8].

1.4. The Cover Index of an s-graph

The cover index k(G) of a graph G is the maximum number k such that there exist k mutually disjoint covers of G. If the degree of a vertex $v \in V(G)$ is k then, clearly, since any cover of G must contain at least one edge incident with v, the graph G cannot have more than k mutually disjoint covers. Hence, it is obvious that $k(G) \leq \tilde{d}(G)$ where $\tilde{d}(G)$ is the greatest lower bound of the degrees of the vertices of G. In the previous section, it was proved that if G is a locally finite bipartite graph, then the equality $k(G) = \tilde{d}(G)$ holds. However, in general, we may have $k(G) < \tilde{d}(G)$, as is shown by the following examples.

For any integers $s \geq 1$ and $k = 2ms-r$ where $r \geq 0$, $m > \left[\frac{r-1}{2}\right]+s$, we give examples of s-graphs G such that $\tilde{d}(G) = k$ and $k(G) \leq \tilde{d}(G)-s$. To this end, consider a graph K with 2m+1 vertices $v_0, v_1, \ldots, v_{2m}$ in which each pair of distinct vertices are joined by exactly s edges. Clearly, K is an s-graph which is regular of degree 2ms. If $k = 2ms-r$ is even, then, by the theorem of Petersen (Corollary 1.3.2), K has a partial graph G which is regular of degree k. Obviously, G is an s-graph and $\tilde{d}(G) = k$. If $k(G) \geqslant k-s+1$, then, since any cover of G must contain at least m+1 edges, we have $|E(G)| = \frac{1}{2}k(2m+1) \geqslant (m+1)(k-s+1)$ or, after simplification, $m \leqslant \left[\frac{r-1}{2}\right]+s$, which contradicts the assumption that $m > \left[\frac{r-1}{2}\right]+s$. Hence, $k(G) \leqslant \tilde{d}(G)-s$. Also, if $k = 2ms-r$ is odd, then, it can be shown that K has a partial graph G such that $d(G, v_0) = k+1$ and $d(G, v_i) = k$ for $i = 1, 2, \ldots, 2m$. Obviously, G is an s-graph and $\tilde{d}(G) = k$. As above, it is easily proved that $k(G) \leqslant \tilde{d}(G)-s$.

The main result proved below (Theorem 1. 4. 1)is that for any locally finite s-graph G, $k(G) \geqslant \tilde{d}(G)-s$. Evidently, in view of the above examples, this bound cannot be improved in general.

Before proving Theorem 1. 4. 1, we first explain some notations and terminological conventions. Consider a graph G, and let f be any coloring of G. As earlier, for any vertex $v \in V(G)$, $c_f(v)$ or simply $c(v)$ denotes the set of colors i such that there is at least one i-edge incident with v. Further, $c_f^r(v)$ or simply $c^r(v)$, $r \geqslant 2$, denotes the set of colors i such that there are at least r i-edges incident with v, and $\bar{c}_f(v)$ or simply $\bar{c}(v)$ denotes the set of colors which do not belong to $c(v)$.

In the following, the letters g, f, f_0, f_1 etc. denote colorings of G. Let f be any coloring of G, and $\mu(v, u)$ be an (i, j)-alternating chain in G. Let g be defined as follows: for any edge $e \in E(G)$,

$g(e) = j$ if e belongs to $\mu(v, u)$ and $f(e) = i$,

$\quad = i$ if e belongs to $\mu(v, u)$ and $f(e) = j$,

$\quad = f(e)$ if e does not belong to $\mu(v, u)$.

Then, we say that g is obtained from f by interchanging the colors on $\mu(v, u)$.

For the sake of convenience, all suitable alternating chains considered below are assumed to be minimal. Any minimal, suitable (i, j)-alternating chain is referred to, for brevity, simply as an (i, j)-<u>chain</u>. (As observed earlier, if there is an i-edge incident with a vertex v, then an (i, j)-chain with v as its first vertex can always be determined.)

We are now prepared to prove the following:

<u>Theorem 1.4.1.</u> If G is an s-graph which is locally finite then $\tilde{d}(G) \geq k(G) \geq \tilde{d}(G)-s$. The bounds are best possible in the following cases: $s \geq 1$, and $\tilde{d}(G) = 2ms-r$ where $r \geq 0$, $m > \left[\frac{r-1}{2}\right] + s$.

<u>Proof:</u> Let G be a locally finite s-graph, and let $\tilde{d}(G) = k$. To complete the proof of the theorem, we have only to prove that $k(G) \geq \tilde{d}(G)-s$, or equivalently, that G possesses a (k-s)-coloring f which is admissible, the rest having been proved above. We shall prove it only for the case when G is finite. The proof can then be extended to all infinite, locally finite graphs by using the argument of Konig and Valko, as we did in the case of Lemma 1.3.1. Let, therefore, G be finite. If $k-s \leq 1$, then the assertion holds obviously. Hence, we may assume that $k-s \geq 2$. We start with an arbitrary (k-s)-coloring

f of G. If f is not admissible, i.e., if $\Delta_f(G) > 0$, then we shall show that by suitably modifying the coloring f, we can obtain another (k-s)-coloring g of G such that $\Delta_g(G) < \Delta_f(G)$. (Since G is finite, the assertion will then follow by finite induction). The operations used in modifying the coloring f are as follows:

(I) Interchange any two colors on all edges of G. (This is equivalent merely to renumbering the colors).

(II) Interchange colors on an (i, j)-chain $\mu(v, u)$ if $j \in \bar{c}(v)$, and/or there is an i-edge incident with v which does not belong to $\mu(v, u)$.

It is clear that if g is obtained from f by using (I) or (II) any number of times then $\Delta_g(G) \leq \Delta_f(G)$. Moreover, we have, evidently, the following:

(1.4.1) Let f be any coloring of G, and let $\mu(v, u)$ be an (i, j)-chain in G. Let g be obtained from f by interchanging the colors on $\mu(v, u)$. Then, $\Delta_g(G) < \Delta_f(G)$ if $j \in \bar{c}_f(v)$ and there is an i-edge incident with v which does not belong to $\mu(v, u)$.

Now, let f be any (k-s)-coloring of G which is not admissible. Then, there is a vertex $v_0 \in V(G)$ such that $\Delta_f(v_0) > 0$ or, equivalently, $\bar{c}(v_0) \neq \emptyset$. Suppose that we also have $c^3(v_0) \neq \emptyset$. Let $i \in c^3(v_0)$, and let $j \in \bar{c}(v_0)$. Consider an (i, j)-chain $\mu(v_0, u)$. Since $i \in c^3(v_0)$, it is seen that there is an i-edge incident with v_0 which does not belong to $\mu(v_0, u)$. Let g be obtained from f by interchanging the colors on $\mu(v_0, u)$. By (1. 4. 1), we then have $\Delta_g(G) < \Delta_f(G)$, as required. Hence, we may assume that $c^3(v_0) = \emptyset$.

It is observed that since $\tilde{d}(G) = k$ and the number of colors used is k-s, for any vertex v, $c^2(v) \neq \emptyset$, and if $c^3(v) = \emptyset$, then $|c^2(v)| \geq s$.

Now, let $i_1 \in c^2(v_0)$ and $e_1 = (v_0, v_1)$ be any i_1-edge. Let us assume, if possible, that (a) $c^2(v_1) \cap \bar{c}(v_0) = \emptyset$, i. e., $c^2(v_1) \subseteq c(v_0)$, (b) $c^2(v_1) \cap c^2(v_0) = \emptyset$, and (c) $c^3(v_1) = \emptyset$ hold simultaneously. Then, from (b), $i_1 \notin c^2(v_1)$ and from (c), $|c^2(v_1)| \geq s$. Since there are at most s-1 edges, other than the i_1-edge e_1, joining v_0 and v_1, it follows from (a) that for some color $i_2 \in c^2(v_1)$, there must be an i_2-edge $e_2 = (v_0, v_2)$, say, such that $v_2 \neq v_1$. Let us assume further, if possible, that $c^2(v_2) \cap \bar{c}(v_0) = \emptyset$, $c^2(v_2) \cap c^2(v_t) = \emptyset$ for $t = 0, 1$, and $c^3(v_2) = \emptyset$. Then, clearly, $i_1 \notin c^2(v_1) \cup c^2(v_2)$, $|c^2(v_1) \cup c^2(v_2)| \geq 2s$, and $c^2(v_1) \cup c^2(v_2) \subseteq c(v_0)$. Hence, as above, for some color $i_3 \in c^2(v_1) \cup c^2(v_2)$ there must be an i_3-edge $e_3 = (v_0, v_3)$, say, such that $v_3 \notin \{v_0, v_1, v_2\}$. Continuing the process, it is evident that, since $|c(v_0)| \leq k-s-1$, there must exist an index m, $1 \leq m \leq k-s-1$, such that we can find i_t-edges $e_t = (v_0, v_t)$, $t = 1, 2, \ldots, m$ satisfying the following conditions:

(i) $v_t \notin \{v_0, v_1, \ldots, v_{t-1}\}$ for $t = 1, 2, \ldots, m$,

(ii) $i_1 \in c^2(v_0)$, and for each t, $2 \leq t \leq m$, there is a (unique) index r, $1 \leq r \leq t-1$, such that $i_t \in c^2(v_r)$,

(iii) $c^2(v_t) \cap \bar{c}(v_0) = \emptyset$, i.e., $c^2(v_t) \subseteq c(v_0)$ for $t = 1, 2, \ldots, m-1$,

(iv) $c^2(v_0), c^2(v_1), \ldots, c^2(v_{m-1})$ are mutually disjoint,

(v) $c^3(v_0) = c^3(v_1) = \ldots = c^3(v_{m-1}) = \emptyset$,

and further, we have one of the following cases:

(1) $c^2(v_m) \cap \bar{c}(v_0) \neq \emptyset$, or

(2) $c^2(v_m) \cap c^2(v_0) \neq \emptyset$, or

(3) $c^2(v_m) \cap c^2(v_t) \neq \emptyset$ for some t, $1 \leq t \leq m-1$, or

(4) $c^3(v_m) \neq \emptyset$.

In each of the above cases, we shall now define a (k-s)-coloring g of G such that $\Delta_g(G) < \Delta_f(G)$.

Case (1) $c^2(v_m) \cap \bar{c}(v_0) \neq \emptyset$. Let $j \in c^2(v_m) \cap \bar{c}(v_0)$. Interchanging the colors j and 0, if necessary, we may take $j = 0$. Let this new coloring of G be denoted by f_0. (Obviously, $\Delta_{f_0}(G) = \Delta_f(G)$, and all the conditions (i) to (v) are satisfied with respect to f_0.) It is evident from conditions (ii) and (iv) that there is a (unique) sequence of vertices

(1.4.2) $\quad v_m = v_{p_{a+1}}, v_{p_a}, \ldots, v_{p_1} = v_1, v_{p_0} = v_0 (m = p_{a+1} > p_a > \cdots > p_0 = 0)$

such that $i_m = i_{p_{a+1}} \in c^2(v_{p_a}), i_{p_a} \in c^2(v_{p_{a-1}}), \ldots, i_{p_2} \in c^2(v_{p_1}), i_1 = i_{p_1} \in c^2(v_{p_0})$.
Now, considering the sequence of vertices (1.4.2), we proceed as follows:

Since $i_{p_1} \in c^2(v_0)$, there is an i_{p_1}-edge e'_{p_1}, say, incident with v_0 such that $e'_{p_1} \neq e_{p_1} = e_1$. Consider an (i_{p_1}, i_{p_2})-chain $\mu_1(v_0, u_1)$ such that the i_{p_2}-edge e_{p_2} and one of the i_{p_1}-edges e_{p_1} and e'_{p_1} do not belong to it. (Since $i_{p_2} \in c^2(v_{p_1})$, such a chain can always be found. In fact, let $\mu(v_0, u)$ be an (i_{p_1}, i_{p_2})-chain with first edge e_{p_1} and $\mu'(v_0, u')$ be an (i_{p_1}, i_{p_2})-chain with first edge e'_{p_1}. Then, it is easily checked that one of these chains must satisfy the requirement). Using (II), interchange the colors on $\mu_1(v_0, u_1)$, and let f_1 denote the coloring of G so obtained. Evidently, $\Delta_{f_1}(G) \leq \Delta_f(G)$, and with respect to f_1, we have $i_{p_2} \in c^2(v_0)$, $i_{p_j} \in c^2(v_{p_{j-1}})$ for $j = a+1, a, \ldots, 3$, and $0 \in c^2(v_m) \cap \bar{c}(v_0)$. Since $i_{p_2} \in c^2(v_0)$, there is an i_{p_2}-edge e'_{p_2}, say, incident with v_0 such that $e'_{p_2} \neq e_{p_2}$. As above, consider an (i_{p_2}, i_{p_3})-chain $\mu_2(v_0, u_2)$ such that the i_{p_3}-edge e_{p_3} and one of the i_{p_2}-edges e_{p_2} and e'_{p_2} do not belong to $\mu_2(v_0, u_2)$. Using (II), interchange the colors on $\mu_2(v_0, u_2)$ and let f_2 denote the coloring so obtained. Evidently, $\Delta_{f_2}(G) \leq \Delta_{f_1}(G) \leq \Delta_f(G)$, and with respect to f_2, we have $i_{p_3} \in c^2(v_0)$, $i_{p_j} \in c^2(v_{p_{j-1}})$ for $j = a+1, a, \ldots, 4$, and $0 \in c^2(v_m) \cap \bar{c}(v_0)$. Proceeding in this manner, it is obvious that we can obtain, using (II) a-2 times more, a coloring f_a of G such that $\Delta_{f_a}(G) \leq \Delta_f(G)$, and with respect to f_a, we have $i_{p_{a+1}} \in c^2(v_0)$, $0 \in c^2(v_{p_{a+1}}) \cap \bar{c}(v_0)$ (where $e_{p_{a+1}} = (v_0, v_{p_{a+1}})$ is an $i_{p_{a+1}}$-edge). Since $i_{p_{a+1}} \in c^2(v_0)$, there is an $i_{p_{a+1}}$-edge $e'_{p_{a+1}}$, say, incident with v_0 such that $e'_{p_{a+1}} \neq e_{p_{a+1}}$. Now, consider an $(i_{p_{a+1}}, 0)$-chain $\mu(v_0, u)$, say, with first edge $e'_{p_{a+1}}$. Since $0 \in c^2(v_{p_{a+1}})$ and $\mu(v_0, u)$ is minimal, it is seen that $e_{p_{a+1}}$ cannot belong to $\mu(v_0, u)$. Using (II), interchange the colors on $\mu(v_0, u)$ and let g be the coloring so obtained. By (1.4.1), we have, evidently, $\Delta_g(G) < \Delta_{f_a}(G) \leq \Delta_f(G)$, as required.

Case (2) $c^2(v_m) \cap c^2(v_0) \neq \emptyset$. Let $i \in c^2(v_m) \cap c^2(v_0)$. Now, by (v), $c^3(v_0) = \emptyset$, so that there are exactly two i-edges e and e', say, incident with v_0. Let $j \in \bar{c}(v_0)$ and consider an (i, j)-chain $\mu(v_0, u)$, say. Using (II), interchange the colors i and j on $\mu(v_0, u)$, and then, interchange i and 0 on all edges of G. Let f_0 denote the coloring so obtained. If one of the edges e and e' does not belong to $\mu(v_0, u)$, then we take $g = f_0$, and by (1.4.1), we have, evidently, $\Delta_g(G) < \Delta_f(G)$. If both e and e' belong to $\mu(v_0, u)$, (or, equivalently, if $u = v_0$), then it is observed that with respect to f_0, we have $0 \in c^2(v_m) \cap \bar{c}(v_0)$, and $i_{p_j} \in c^2(v_{p_{j-1}})$, where $e_{p_j} = (v_0, v_{p_j})$ are i_{p_j}-edges, for $j = a+1, a, \ldots, 1$. Hence, as in case (1), proceeding with the sequence of vertices (1.4.2), we can obtain g such that $\Delta_g(G) < \Delta_f(G)$, as required.

Case (3) $c^2(v_m) \cap c^2(v_t) \neq \emptyset$ for some t, $1 \leq t \leq m-1$. Let $i \in c^2(v_m) \cap c^2(v_t)$. From conditions (ii) and (iv), there is a (unique) sequence of vertices

$$(1.4.3) \qquad v_t = v_{q_{b+1}}, v_{q_b}, \ldots, v_{q_1} = v_1, v_{q_0} = v_0 \quad (t = q_{b+1} > q_b > \ldots > q_0 = 0)$$

such that $i_t = i_{q_{b+1}} \in c^2(v_{q_b})$, $i_{q_b} \in c^2(v_{q_{b-1}})$, ..., $i_{q_2} \in c^2(v_{q_1})$, $i_1 = i_{q_1} \in c^2(v_{q_0})$. Now, in view of the cases (1) and (2), we may assume that $i \in c(v_0)$ and $i \notin c^2(v_0)$ so that there is one and only one i-edge e, say, incident with v_0.

From (ii) and (iv), it is easily seen that $e \neq e_r$ or, equivalently, $i \neq i_r$, for any $r = 1, 2, \ldots, t$. In particular, $i \neq i_{q_r}$, for any $r = b+1, b, \ldots, 1$.

Now, let us first assume that we also have $i \neq i_{p_r}$ (or, equivalently, $e \neq e_{p_r}$) for any $r = a+1, a, \ldots, 1$. Let $j \in \bar{c}(v_0)$, and consider an (i, j)-chain $\mu(v_0, u)$, say. (Obviously, $u \neq v_0$). Using (II), interchange the colors on $\mu(v_0, u)$, and then, interchange i and 0 on all edges of G. Let f_0 denote the coloring so obtained. Evidently, $\Delta_{f_0}(G) \leq \Delta_f(G)$. Now, if $u \neq v_m = v_{p_{a+1}}$, then, it is seen that with respect to f_0, we have $0 \in c^2(v_m) \cap \bar{c}(v_0)$, and $i_{p_r} \in c^2(v_{p_{r-1}})$ where $e_{p_r} = (v_0, v_{p_r})$ are i_{p_r}-edges, for $r = a+1, a, \ldots, 1$. Hence, as in case (1), we can obtain g as required. If $u = v_m$, then, it is seen that with respect to f_0, we have $0 \in c^2(v_t) \cap \bar{c}(v_0)$, and $i_{q_j} \in c^2(v_{q_{j-1}})$, where $e_{q_j} = (v_0, v_{q_j})$ are i_{q_j}-edges, for $j = b+1, b, \ldots, 1$. Hence, evidently, as in case (1), proceeding with the sequence of vertices (1. 4. 3), we can obtain g as required.

Let us now assume that $i = i_{p_r}$ (or, equivalently, that $e = e_{p_r}$) for some $r = a+1, a, \ldots, 1$. Since, by assumption, $i \notin c^2(v_0)$, clearly, $i \neq i_{p_1}$. Let $i = i_{p_r}$ $(a+1 \geq r \geq 2)$, so that $i_{p_r} \in c^2(v_m) \cap c^2(v_{p_{r-1}})$. Let $j \in \bar{c}(v_0)$. Interchanging the colors j and 0, if necessary, we may take $j = 0$, and denote the new coloring of G by f_0. Now, it is evident that we can obtain a coloring f_{r-2} of G, proceeding as in case (1), such that $\Delta_{f_{r-2}}(G) \leq \Delta_{f_0}(G) = \Delta_f(G)$, and with respect to f_{r-2}, we have $i_{p_{r-1}} \in c^2(v_0)$, $i_{p_j} \in c^2(v_{p_{j-1}})$ for $j = a+1, a, \ldots, r$, where $e_{p_j} = (v_0, v_{p_j})$ are i_{p_j}-edges for $j = a+1, a, \ldots, r-1$, and $i_{p_r} \in c^2(v_m) \cap c^2(v_{p_{r-1}})$. Now, let $e'_{p_{r-1}}$ be an $i_{p_{r-1}}$-edge incident with v_0 such that $e'_{p_{r-1}} \neq e_{p_{r-1}}$, and let e', e'' be any two i_{p_r}-edges incident with $v_{p_{r-1}}$. Consider $(i_{p_r}, i_{p_{r-1}})$-chains $\mu'(v_{p_{r-1}}, u')$ and $\mu''(v_{p_{r-1}}, u'')$ with first edge e' and e'', respectively. Since both of these chains are minimal, it is seen that at least one of them, say $\mu'(v_{p_{r-1}}, u')$, does not contain the i_{p_r}-edge e_{p_r}. If $u' \neq v_{p_{r-1}}$ and $u' \neq v_0$, then, using (II), first interchange the colors on $\mu'(v_{p_{r-1}}, u')$, and then, change the color of $e_{p_{r-1}}$ to 0. Let g be the coloring so obtained. It is easily verified that $\Delta_g(G) < \Delta_{t_{r-2}}(G) \leq \Delta_f(G)$, as required. If $u' = v_{r-1}$, then, it is seen that $\mu_{r-1}(v_0, u') = (v_0, e_{p_{r-1}}, \mu'(v_{r-1}, u'))$ is an $(i_{p_{r-1}}, i_{p_r})$-chain which does not contain the edges $e'_{p_{r-1}}$ and e_{p_r}. Using (II), interchange the colors on $\mu_{r-1}(v_0, u')$ and let f_{r-1} denote the coloring so obtained. And, if $u' = v_0$, then $\mu'(v_{p_{r-1}}, u')$ contains the edge $e'_{p_{r-1}}$, but does not contain any one of the edges $e_{p_{r-1}}$, e_{p_r} and e''. Using (II), interchange the colors on $\mu'(v_{p_{r-1}}, u')$, and let f_{r-1} denote the coloring so obtained. In either case, it is seen that $\Delta_{f_{r-1}}(G) \leq \Delta$ and with respect to f_{r-1}, we have $i_{p_r} \in c^2(v_m) \cap c^2(v_0)$, $i_{p_j} \in c^2(v_{p_{j-1}})$ for $j = a+1, a, \ldots, r+1$, where $e_{p_j} = (v_0, v_{p_j})$ are i_{p_j}-edges for $j = a+1, a, \ldots, r$. It is evident that we have, now, a situation similar to case (2). Hence, we can obtain g satisfying $\Delta_g(G) < \Delta_f(G)$, as required.

Case (4) $c^3(v_m) \neq \emptyset$. Let $i \in c^3(v_m)$. From the above cases (1), (2) and (3), we may assume that $i \in c(v_0)$, $i \notin c^2(v_0)$ and $i \notin c^2(v_t)$ for any $t = 1, 2, \ldots, m-1$. Hence, there is one and only one i-edge incident with v_0, and $i \neq i_t$ for any $t = 1, 2, \ldots, m$. Let $j \in \bar{c}(v_0)$. Consider an (i, j)-chain $\mu(v_0, u)$. Using (II), interchange the colors on $\mu(v_0, u)$, and then, interchange i and 0 on all edges of G. Let f_0 denote the coloring of G so obtained. It is clear that $\Delta_{f_0}(G) \leq \Delta_f(G)$, and furthermore, with respect to f_0, we have $0 \in c^2(v_m) \cap c^2(v_0)$, $i_{p_j} \in c^2(v_{p_{j-1}})$, where $e_{p_j} = (v_0, v_{p_j})$ are i_{p_j}-edges for j = a+1, a, . . . , 1. Hence, as in case (1), we can obtain g as required.

Thus, we have shown that starting with any (k-s)-coloring f of G such that $\Delta_f(G) > 0$, we can obtain a (k- s)-coloring g of G satisfying $\Delta_g(G) < \Delta_f(G)$. The proof of the theorem is now completed by finite induction.

A graph is said to be <u>linear</u> if no two of its vertices are joined by more than one edge. As an important special case of Theorem 1. 4. 1, we have the following:

<u>Theorem 1. 4. 2.</u> If G is a linear graph which is locally finite, then $k(G) = \tilde{d}(G)$ or $k(G) = \tilde{d}(G)-1$. For any integer $k \geqslant 2$, there exist linear graphs G such that $\tilde{d}(G) = k$ and $k(G) = \tilde{d}(G)-1$.

It may be remarked that the proof of the inequality $k(G) \geqslant \tilde{d}(G)-s$ in Theorem 1. 4. 1 is algorithmic so that, given any finite s-graph G we can always find a partition of its set of edges into $\tilde{d}(G)-s$ sets each of which is a cover of G.

Let $s \geqslant 1$ and k be any positive integer. It was shown that if k can be expressed in the forms 2ms-r where $r \geq 0$, $m > \left\lceil \frac{r-1}{2} \right\rceil + s$, then, there exist s-graphs G for which $\tilde{d}(G) = k$ and $k(G) = \tilde{d}(G)-s$. We make the following:

Conjecture If G is any locally finite s-graph such that $\tilde{d}(G)$ cannot be expressed in the form 2ms-r where $r \geq 0$, $m > \left[\frac{r-1}{2}\right] + s$, then $k(G) > \tilde{d}(G)-s$.

The conjecture is known to be true only for a few special classes of graphs.

Note: The results contained in this section were announced in [9].

CHAPTER 2

THE CHROMATIC INDEX OF A GRAPH

2.1. Introduction and Summary

Let G be an undirected graph with set of vertices V(G) and set of edges E(G). Any nonempty subset M of E(G) is said to be a matching of G if each vertex $v \in V(G)$ is an endvertex of at most one edge in M. The chromatic index of a graph G, denoted by q(G), is the minimum number q such that there exists a decomposition of E(G) into q sets each of which is a matching of G. The upper degree $\overline{d}(G)$ of G is the least upper bound of the degrees of its vertices.

In the present chapter, we consider the problem of determining the chromatic index of a graph in terms of its upper degree and multiplicity. Evidently, for any graph G we must have $\overline{d}(G) \leq q(G)$. It is well-known (Berge [1], p. 95) that if G is a bipartite graph then the equality $q(G) = \overline{d}(G)$ holds. In general, it can be proved that $q(G) \leq 2\overline{d}(G)-1$. However, this bound is not best possible. C. E. Shannon [22] has shown that for any finite graph G, $q(G) \leq \left[\frac{3}{2}\overline{d}(G)\right]$. In this chapter, we obtain the following Theorem 2.3.1 "If G is an s-graph then $\overline{d}(G) \leq q(G) \leq \overline{d}(G)+s$. The bounds are best possible in the following cases: $s \geq 1$ and $\overline{d}(G) = 2ms-r$ where $r \geq 0$, $m > \left[\frac{r+1}{2}\right]$".

2.2. Preliminary Remarks

For definitions and terminology, see Chapter 1. Let G be any undirected graph. For convenience, we shall assume that G has at least one edge. Any nonempty set M of the edges of G, $M \subseteq E(G)$, is called a matching of G if every vertex $v \in V(G)$ is incident with at most one edge in M or, in other words, if no two edges belonging to M are adjacent. Consider an arbitrary decomposition of E(G) into q mutually disjoint sets.

$$(2.2.1) \qquad E_0, E_1, \ldots, E_{q-1}; \; E(G) = \cup E_i, \; E_i \cap E_j = \emptyset$$

The chromatic index of G, denoted by q(G) is the minimum number q such that there exists a decomposition (2.2.1) of E(G) where each of the sets $E_i (i = 0, 1, \ldots, q-1)$ is a matching of G. This terminology is suggested when we consider each set E_i as representing a color (to be denoted by the integer i). As before, we introduce a q-coloring f of G so that for every $e \in E(G)$, $f(e) = i$ if $e \in E_i$. If $f(e) = i$, we say that the edge e is colored with the color i or that e is an i-edge.

Any coloring of a graph G is called regular if no two adjacent edges of G are colored with the same color. It is readily observed that the q-coloring, corresponding to any decomposition (2.2.1), of G is regular if and only if each of the sets E_i in (2.2.1) is either empty or a matching of G. Thus, it is seen that the chromatic index of a graph G may alternatively be defined, as is popularly done, to be the minimum number $q = q(G)$ such that a regular q-coloring of G exists.

The upper degree of a graph G, denoted by $\overline{d}(G)$, is the least upper bound of the degrees of its vertices. A graph is said to be locally bounded if its upper degree is finite. Evidently, any locally bounded graph is locally finite; the converse, however, is not true.

In this chapter, we consider the problem of determining the chromatic index of a given graph in terms of its upper degree (and multiplicity). It can be shown that for any graph G if $\overline{d}(G)$ is infinite then $q(G) = \overline{d}(G)$. Hence, the problem is of interest only for those graphs which are locally bounded.

Evidently, if there are h edges incident with a vertex v of a graph G then, for any regular coloring of G, h distinct colors are required to color these edges. Hence, it is obvious that we must have $q(G) \leqslant \overline{d}(G)$. It is well-known that if G is a bipartite graph then the equality $q(G) = \overline{d}(G)$ holds. In fact, this result is known to be equivalent to the theorem of Konig (Corollary 1. 3. 3) and hence, is implied by our Theorem 1. 3. 1. It may be of interest to know if the converse implication also holds. The question, however, remains unanswered.

C. E. Shannon [22] has proved that for any finite graph G, $q(G) \leqslant \left[\frac{3}{2} \overline{d}(G) \right]$. By constructing simple examples of graphs, he also showed that the bound is best possible for any value of $\overline{d}(G) \geqslant 1$. However, the multiplicity of G (which is the smallest number s such that no two vertices of G are joined by more than s edges) is not taken into account here. Theorem 2. 3. 1, proved in the next section, yields that if the multiplicity of G is s then $q(G) \leq \bar{d}(G)+s$. Clearly, if $s \leq \left[\frac{1}{2}\bar{d}(G)\right]$ then our bound is more exact than the Shannon's bound.

2.3. The Chromatic Index of an s-graph

As observed in the previous section, for any graph G, $q(G) \geq \bar{d}(G)$. And, if G is bipartite, then $q(G) = \bar{d}(G)$. However, in general, we may have $q(G) > \bar{d}(G)$. In fact, we give below examples of s-graphs G such that $q(G) \geq \bar{d}(G)+s$.

Let $s \geq 1$ and $h = 2ms-r$, where $r \geq 0$, $m > \left[\frac{r+1}{2}\right]$, be any two integers. We shall show that there exist s-graphs G such that $\bar{d}(G) = h$ and $q(G) \geq h+s$.

Consider a graph K with 2m+1 vertices $v_0, v_1, \ldots, v_{2m}$ in which each pair of distinct vertices is joined by exactly s edges. Clearly, K is an s-graph which is regular of degree 2ms. Now, if $h = 2ms-r$ is even then, by the theorem of Petersen (Corollary 1.3.2), K has a partial graph G which is regular of degree h. If $q(G) \leq h+s-1$ then, since any matching of G can contain at most m edges, we must have $m(h+s-1) \geq |E(G)| = \frac{1}{2}h(2m+1)$ or $m \leq \left[\frac{r+1}{2}\right]$ which contradicts the assumption that $m > \left[\frac{r+1}{2}\right]$. Hence, $q(G) \geq h+s$. If $h = 2ms-r$ is odd, then it can be shown that K has a partial graph G such that $d(G, v_0) = h-1$, $d(G, v_i) = h$ for $i = 1, 2, \ldots, 2m$. As before, it is easily proved that $q(G) \geq h+s$.

It will be proved below (Theorem 2.3.1) that for any s-graph G, we have the upper bound $q(G) \leq \bar{d}(G)+s$. Obviously, in view of the above examples, this bound cannot be improved in general.

We first recall some notations and state some facts which will be used in the proof of the theorem.

Let f be any q-coloring of a graph G. For any vertex v, we denote by $c_f(v)$ or simply by c(v) the set of colors which are assigned to the edges incident with v; $\bar{c}_f(v)$ or simply $\bar{c}(v)$ denotes the set of colors not belonging to c(v). For any two fixed colors i and j, the partial graph of G defined by the set of edges colored with i or j is denoted by $G_{i,\, j}$.

Let G be a finite graph, and f be a regular q-coloring $(q \geqslant 2)$ of G. Let i and j be any two fixed colors. Then, the following statements are easily verified.

(2.3.1) Any connected component of the graph $G_{i,\, j}$ is a cycle or a chain (possibly an isolated vertex). If v is a vertex such that $i \in c(v)$ and $j \in \bar{c}(v)$, then the connected component of $G_{i,\, j}$, which contains the vertex v, is an (i, j)-alternating chain with v taken as its first vertex.

(2.3.2) If v_1, v_2, v_3 are three vertices such that one of the colors i and j does not belong to $c(v_r)$ for each r = 1, 2, 3, then all the three vertices v_1, v_2, v_3 cannot belong to the same connected component of $G_{i,\, j}$. (In fact, this follows from (2.3.1), since a chain can have only two terminal vertices.)

We shall now prove

Theorem 2. 3. 1.: If G is an s-graph which is locally bounded then $\bar{d}(G) \leq q(G) \leq \bar{d}(G)+s$. The bounds are best possible in the following cases: $s \geq 1$ and $\bar{d}(G) = 2ms-r$ where $r \geq 0$, $m > \left[\frac{r+1}{2}\right]$.

Proof: To complete the proof of the theorem, it only remains to show that $q(G) \leq \bar{d}(G)+s$, the rest having been proved above. Let $\bar{d}(G) = h$. It is clearly sufficient to prove that G possesses an (h+s)-coloring which is regular. We shall prove it first for the case when G is finite; then, assuming it to be true for all finite graphs, we shall prove it, using the familiar argument of Konig and Valko [15], for the case when G is infinite (but locally bounded).

G is finite: If G has only one or two edges then the assertion holds obviously. We can, therefore, use induction on the number of edges in the graph G. Delete any particular edge $e_0 = (v, v_0)$, say, from the graph G and let G' be the partial graph of G defined by the set of remaining edges. By hypothesis, G' has an (h+s)-coloring f which is regular. By suitably modifying the coloring f of G', we shall obtain a regular (h+s)-coloring of G so that the edge $e_0 = (v, v_0)$ will also be colored. The operations used in modifying the coloring are as follows:

(I) Interchange colors i and j on all edges of any particular connected component (or connected components) of the graph $G'_{i,j}$.

(II) Assign the color j to an edge $e = (v, u)$ if $j \in \bar{c}(v) \cap \bar{c}(u)$.

It is evident that if the initial coloring is regular, then any coloring obtained by using (I) or (II) any number of times is also regular.

Now, obviously, $\bar{d}(G') \leq h$ and $d(G', v_0) \leq h-1$, so that, since the number of colors used is $h+s$, we have $|\bar{c}(u)| \geq s$ for all vertices u, and $|\bar{c}(v_0)| \geq s+1$. If there is a color $j \in \bar{c}(v_0) \cap \bar{c}(v)$, then, using (II), we can define $g(e_0) = j$ and $g(e) = f(e)$ for all other edges $e \in E(G)$, and the assertion will be proved. Let, therefore, $\bar{c}(v_0) \cap \bar{c}(v) = \emptyset$. Clearly, then, $\bar{c}(v_0) \subseteq c(v)$, and for any chosen color $i_1 \in \bar{c}(v_0)$, there is an i_1-edge $e_1 = (v, v_1)$, say. (Obviously, $v_1 \neq v_0$). If there is a color $j \in \bar{c}(v_1) \cap \bar{c}(v)$, then, using (II) successively, we can define $g(e_1) = j$, $g(e_0) = i_1$ and $g(e) = f(e)$ for all other edges $e \in E(G)$. Assume, therefore, that $\bar{c}(v_1) \cap \bar{c}(v) = \emptyset$, and let us assume further that $\bar{c}(v_1) \cap \bar{c}(v_0) = \emptyset$. Then, clearly, $\bar{c}(v_0) \cup \bar{c}(v_1) \subseteq c(v)$, and $|\bar{c}(v_0) \cup \bar{c}(v_1)| \geq 2s+1$. Since there are at most $2s-1$ edges in G' joining v to v_0 or v_1, clearly, for some color $i_2 \in \bar{c}(v_0) \cup \bar{c}(v_1)$, there must be an i_2-edge $e_2 = (v, v_2)$, say, such that $v_2 \notin \{v_0, v_1\}$. Let us assume further that $\bar{c}(v_2)$ is disjoint from each of the sets $\bar{c}(v)$, $\bar{c}(v_0)$, $\bar{c}(v_1)$. As above, it is then seen that for some color $i_3 \in \bar{c}(v_0) \cup \bar{c}(v_1) \cup \bar{c}(v_2)$, there must be an i_3-edge $e_3 = (v, v_3)$, say, such that $v_3 \notin \{v_0, v_1, v_2\}$. Continuing the process, since $d(G', v_0) \leq h-1$, it is evident that there must exist an index k, $1 \leq k \leq h-1$, such that we can find i_t-edges $e_t = (v, v_t)$, $t = 1, 2, \ldots, k$, satisfying the following conditions:

(i) $v_t \notin \{v_0, v_1, \ldots, v_{t-1}\}$, $t = 1, 2, \ldots, k$,

(ii) for each t, $1 \leqslant t \leqslant k$, there is a (unique) index r, $0 \leqslant r \leqslant t-1$, such that $i_t \in \overline{c}(v_r)$,

(iii) $\overline{c}(v)$, $\overline{c}(v_0)$, $\overline{c}(v_1), \ldots, \overline{c}(v_{k-1})$ are mutually disjoint,

and further, we have one of the following cases:

(1) $\overline{c}(v_k) \cap \overline{c}(v) \neq \emptyset$, or

(2) $\overline{c}(v_k) \cap \overline{c}(v_r) \neq \emptyset$ for some r, $0 \leqslant r \leqslant k-1$.

In each of the above cases, we shall now define a regular $(h+s)$-coloring g of the graph G.

Case (1). $\overline{c}(v_k) \cap \overline{c}(v) \neq \emptyset$: It is clear from the conditions (ii) and (iii), that there is a unique sequence of vertices

$$(2.3.3) \qquad v_k = v_{p_{a+1}}, v_{p_a}, \ldots, v_{p_1}, v_{p_0} = v_0 \quad (p_{a+1} > p_a > \cdots > p_1 > p_0)$$

such that $i_k = i_{p_{a+1}} \in \overline{c}(v_{p_a})$, $i_{p_a} \in \overline{c}(v_{p_{a-1}}), \ldots, i_{p_2} \in \overline{c}(v_{p_1})$, $i_{p_1} \in \overline{c}(v_{p_0})$.

Now, let $j \in \overline{c}(v_k) \cap \overline{c}(v)$. Using (II) successively, we define the coloring g as follows: $g(e_k) = j$, $g(e_{p_a}) = i_{p_{a+1}}$, $g(e_{p_{a-1}}) = i_{p_a}, \ldots,$ $g(e_{p_1}) = i_{p_2}$, $g(e_0) = i_{p_1}$ and $g(e) = f(e)$ for all the remaining edges $e \in E(G)$. Evidently, g is an $(h+s)$-coloring of G which is regular.

Case (2). $\overline{c}(v_k) \cap \overline{c}(v_r) \neq \emptyset$, for some r, $0 \leqslant r \leqslant k-1$: Let $i \in \overline{c}(v_k) \cap \overline{c}(v_r)$, for some r, $0 \leqslant r \leqslant k-1$. From condition (iii), $i \in c(v)$, so that there is an i-edge $e' = (v, v')$, say, incident with v. (Obviously, $v' \neq v_r$, $v' \neq v_k$). Let $j \in \overline{c}(v)$. Consider the graph $G'_{i,j}$. From (2.3.1), the connected component of $G'_{i,j}$ which contains the vertex v, is an (i, j)-alternating chain with first vertex v and last vertex u, say. We denote this chain by $\mu(v, u)$. Since the given coloring f of G' is regular, and $j \in \overline{c}(v)$, it is evident that $u \neq v$. Also, since $i \in \overline{c}(v_k) \cap \overline{c}(v_r)$, from (2.3.2), both v_r and v_k cannot belong to $\mu(v, u)$, and if any one of them belongs to $\mu(v, u)$, then, it must coincide with u.

Now, from conditions (ii) and (iii), we can find a unique sequence of vertices,

(2.3.4) $\quad v_r = v_{q_{b+1}}, v_{q_b}, \ldots, v_{q_1}, v_{q_0} = v_0 \; (q_{b+1} > q_b > \cdots > q_1 > q_0)$

such that $i_{q_j} \in \overline{c}(v_{q_{j-1}})$ for $j = b+1, b, \ldots, 1$. Since $i \in \overline{c}(v_r)$, from conditions (ii) and (iii), it is seen that $i \neq i_t$ for any $t = 1, 2, \ldots, r$. In particular, $i \neq i_{q_j}$ for any $j = b+1, b, \ldots, 1$.

Now, let us assume first that we also have $i \neq i_{p_t}$ for any $t = a, a-1, \ldots, 1$. Now, using (I), interchange the colors on $\mu(v, u)$ and let f' denote the ooloring of G' so obtained. If $u \neq v_k$, then, it is seen that with respect to f', we have $i \in \overline{c}(v_k) \cap \overline{c}(v)$ and $i_{p_j} \in \overline{c}(v_{p_{j-1}})$ for $j = a+1, a, \ldots, 1$. Hence, as in case (1), using (II) successively, we can define g as follows: $g(e_k) = i$, $g(e_{p_a}) = i_{p_{a+1}}$, $g(e_{p_{a-1}}) = i_{p_a}, \ldots, g(e_{p_1}) = i_{p_2}$, $g(e_0) = i_{p_1}$ and $g(e) = f'(e)$ for all other edges $e \in E(G)$. If $u = v_k$, then it is seen that with respect to f', we have $i \in \overline{c}(v_r) \cap \overline{c}(v)$ and $i_{q_j} \in \overline{c}(v_{q_{j-1}})$ for

$j = b+1, b, \ldots, 1$. Hence, again, as above, we can define g as follows: $g(e_r) = i$, $g(e_{q_b}) = i_{q_{b+1}}$, $g(e_{q_{b-1}}) = i_{q_b}, \ldots$, $g(e_{q_1}) = i_{q_2}$, $g(e_0) = i_{q_1}$ and $g(e) = f'(e)$ for all other edges $e \in E(G)$.

Finally, let us assume that $i = i_{p_t}$ for some t, $1 \leq t \leq a$. Then, $i = i_{p_t} \in \bar{c}(v_{p_{t-1}}) \cap \bar{c}(v_k)$. Now, using (I), interchange the colors i_{p_t} and j on μ (v, u), and let f' denote the coloring of G' so obtained. If $u \neq v_{p_{t-1}}$, then, it is seen that with respect to f', we have $i_{p_t} \in \bar{c}(v_{p_{t-1}}) \cap \bar{c}(v)$ and $i_{p_j} \in \bar{c}(v_{p_{j-1}})$ for $j = t-1, t-2, \ldots, 1$. Hence, as above, we can define g as follows: $g(e_{p_{t-1}}) = i_{p_t}$, $g(e_{p_{t-2}}) = i_{p_{t-1}}, \ldots$, $g(e_{p_1}) = i_{p_2}$, $g(e_0) = i_{p_1}$ and $g(e) = f'(e)$ for all other edges $e \in E(G)$. Suppose, therefore, that $u = v_{p_{t-1}}$. Then, it is seen that with respect to f', we have $i_{p_t} \in \bar{c}(v_k) \cap \bar{c}(v)$ and $j \in \bar{c}(v_{p_{t-1}})$ where $e_{p_t} = (v_0, v_{p_t})$ is a j-edge. Hence, using (II), successively, we define g as follows: $g(e_k) = i_{p_t}$, $g(e_{p_a}) = i_{p_{a+1}}$, $g(e_{p_{a-1}}) = i_{p_a}, \ldots\ldots$ $\ldots$, $g(e_{p_t}) = i_{p_{t+1}}$, $g(e_{p_{t-1}}) = j$, $g(e_{p_{t-2}}) = i_{p_{t-1}}, \ldots$, $g(e_{p_1}) = i_{p_2}$, $g(e_0) = i_{p_1}$ and g is defined as f' for all the remaining edges.

Thus, starting with any regular (h+s)-coloring of G', we have obtained an (h+s)-coloring of G which is regular. This, by induction, completes the proof of the theorem for finite graphs.

<u>G is infinite.</u> If the chromatic index of each connected component of G is less than or equal to h+s then, clearly, $q(G) \leq h+s$. Hence, it is sufficient to prove the theorem with the assumption that G is connected. Since, moreover, G is locally bounded, G has an enumerable number of edges. Let

$e_1, e_2, \ldots$ be an enumeration of the edges of G in some order. We denote by $G_n (n = 1, 2, \ldots)$ the partial subgraph of G consisting of the edges $e_1, e_2, \ldots, e_n$ and the vertices which are incident with these edges. If $n < m$, G_n is a partial subgraph of G_m and each regular coloring of G_m implies a regular coloring of G_n which is called the restriction of the coloring of G_m; the coloring of G_m is said to be an extension of the coloring of G_n. Obviously, the graphs G_n are finite; and since, as proved above, the theorem holds for finite graphs, each graph G_n has regular (h+s)-colorings but only a finite number of them.

Now, the regular (h+s)-colorings of the graphs $G_2, G_3, \ldots$ imply regular (h+s)-colorings of G_1. Since G_1 has only a finite number of colorings, there must be one among them which is a restriction of an infinity of regular (h+s)-colorings. Let us fix one such coloring of G_1. We now consider only the extensions of this fixed coloring. Each of these implies a regular (h+s)-coloring of G_2. Hence, by the same argument as above, G_2 has a regular (h+s)-coloring which is a restriction of infinitely many regular colorings. We proceed in this manner to get fixed regular (h+s)-colorings of the graphs $G_1, G_2, \ldots$ such that each is an extension of its predecessor. Thus, each edge of G is assigned a well-defined color which evidently, defines a regular (h+s)-coloring of G. This completes the proof of the theorem.

As an important special case of Theorem 2.3.1, we have the following:

Theorem 2. 3. 2. If G is a linear graph which is locally bounded, then, $q(G) = \bar{d}(G)$ or $q(G) = \bar{d}(G)+1$. For any integer $h \geq 2$, there exist linear graphs G such that $\bar{d}(G) = h$ and $q(G) = h+1$.

It may be remarked that the proof of the inequality $q(G) \leq \bar{d}(G)+s$ in Theorem 2. 3. 1 is algorithmic, so that, given any finite s-graph G, we can always determine a decomposition of its set of edges into (at most) $\bar{d}(G)+s$ sets each of which is a matching of G. (We may, however, have $q(G) < \bar{d}(G)+s$.)

Let $s \geq 1$ and h be any positive integer. It was shown that if h is of the form $2ms-r$ where $r \geq 0$, $m > \left[\frac{r+1}{2}\right]$, then there exist s-graphs G for which $\bar{d}(G) = h$ and $q(G) = \bar{d}(G)+s$. We make the following

Conjecture: Let G be any s-graph such that $\bar{d}(G)$ cannot be expressed in the form $2ms-r$ where $r \geq 0$ and $m > \left[\frac{r+1}{2}\right]$. Then, $q(G) < \bar{d}(G)+s$.

The conjecture is known to be true only for a few special classes of graphs.

Note: Theorem 2. 3. 2 was announced in [10]. The upper bound $q(G) \leq \bar{d}(G)+s$, for finite s-graph G, was first obtained, independently of the author, by V. G. Vizing [24].

2. 4. Duality between Cover Index and Chromatic Index of a Graph

It was remarked earlier that there is an apparent duality between the concepts of the cover index and the chromatic index of a graph. In the light of the results obtained so far, we may, however, add the following remarks:

1. We proved (Theorem 1. 3. 1) that if G is a (locally finite) bipartite graph, then, $k(G) = \tilde{d}(G)$, and, as is well-known, we also have $q(G) = \bar{d}(G)$. It was observed that, in the case of bipartite graphs, the equality $k(G) = \tilde{d}(G)$ implies its counterpart $q(G) = \bar{d}(G)$. It is not known, however, if the converse implication also holds.

2. It may be observed that Theorem 1. 4. 1 and Theorem 2. 3. 1 are counterparts of each other, and it is tempting to ask if any one of them can be derived from the other. Very little is known in this respect.

3. It can be easily seen that if G is a regular graph, then, $k(G) = \tilde{d}(G)$ if and only if $q(G) = \bar{d}(G)$. Unfortunately, as one might expect, such a relationship does not hold in general. For instance, if G is the graph of Figure 1, then, $k(G) = \tilde{d}(G)$ but $q(G) = \bar{d}(G)+1$, and if G is the graph of Figure 2, then, $q(G) = \bar{d}(G)$ but $k(G) = \tilde{d}(G)-1$. In general, for any $s \geq 1$ and any fixed r, $1 \leq r \leq s$, one can construct examples of s-graphs G such that $k(G) = \tilde{d}(G)$ whereas $q(G) = \bar{d}(G)+r$ and also, examples of s-graphs G such that $q(G) = \bar{d}(G)$, whereas $k(G) = \tilde{d}(G)-r$. Hence, one can only ask, for some criteria under which both the equalities $k(G) = \tilde{d}(G)$ and $q(G) = \bar{d}(G)$ hold simultaneously. Also, it would be desirable to have criteria under which the equality $k(G) = \tilde{d}(G)$ or $q(G) = \bar{d}(G)$ holds.

CHAPTER 3

ON BASIS DIGRAPHS

3.1. Introduction and Summary

In a directed graph or digraph D, a vertex u is said to be reachable from a vertex v if there exists a path in D from v to u. The set of all vertices reachable from v is denoted by $R_D(v)$. A digraph D is defined to be a basis digraph if there is no partial graph D' of D (different from D) such that $R_{D'}(v) = R_D(v)$ for all vertices v.

In the present chapter, we consider the problem, proposed by O. Ore [19], of determining the largest number of arcs in any basis digraph defined on a vertex-set with n vertices. In sections 3.2 and 3.3, a few preliminary lemmas are stated and some auxiliary results concerning the structure of basis digraphs are obtained. Let D be any basis digraph with n vertices. In section 3.4, we first obtain Theorem 3.4.1 "If D is acyclic then it has at most $\left[\frac{n^2}{4}\right]$ arcs" which is a consequence of a well-known theorem of P. Turan [23] and the fact that a basis digraph having no circuits can have no triangles. We next prove Theorem 3.4.2 "If D is strongly connected then it has at most 2n-2 arcs". Combining these results, the following general theorem is then proved. Theorem 3.4.3 "If D is any basis digraph with n vertices and k strong components then it can have at most $2(n-k) + \left[\frac{k^2}{4}\right]$ arcs". It is shown (in section 3.5) that the bounds given by the above theorems are exact and the structure of extremal basis digraphs is determined.

3.2. Basic Concepts and Preliminary Lemmas

Consider a digraph D with set of vertices V(D) and set of arcs A(D) as defined in section 1.2. A sequence of the form

$$P(v_1, v_{r+1}) = (v_1, v_2)(v_2, v_3), \ldots, (v_r, v_{r+1}) \tag{3.2.1}$$

where $(v_i, v_{i+1}) \in A(D)$, $i = 1, 2, \ldots, r$ is called a path in D from v_1 to v_{r+1}. For convenience, we take the empty sequence consisting of no arc of D to be a path from any vertex to itself. The length of a path is the number of arcs appearing in it. The sequence (3.2.1) is a circuit if all the arcs appearing in it are distinct and $v_{r+1} = v_1$. A circuit (3.2.1) is said to be elementary if the vertices $v_1, v_2, \ldots, v_r$ are distinct. A digraph is said to be acyclic (or circuit-free) if it has no circuit.

In a digraph D, a vertex u is reachable from a vertex v if there exists a path from v to u. The reachability set $R_D(v)$ of a vertex v is the set of all vertices reachable from v. Two vertices are mutually reachable if there exists a path from each one to the other. It is easily proved that two vertices v and u are mutually reachable if and only if $R_D(v) = R_D(u)$. A digraph D is said to be strongly connected if every two of its vertices are mutually reachable. Equivalently, D is strongly connected if $R_D(v) = V(D)$

for every $v \in V(D)$.

Given a digraph D, there is defined a binary relation '$\sim$' in V(D) as follows: for any v, $u \in V(D)$, $v \sim u$ if and only if $R_D(v) = R_D(u)$, i. e., v and u are mutually reachable. It is obvious that $\sim$ is an equivalence relation and thus defines a partition of V(D)

(3. 2. 2) $$S_1, S_2, \ldots, S_k, \; V(D) = \cup S_i, \; S_i \cap S_j = \emptyset$$

where $S_1, S_2, \ldots, S_k$ are nonempty sets such that any two vertices v and u belong to the same set S_i if and only if $v \sim u$. The subgraph of D defined by the set of vertices S_i is denoted by D_i and the digraphs $D_1, D_2, \ldots, D_k$ are called the <u>strong components</u> of D. It may be noted that the strong components of any digraph D are uniquely determined and their number is fixed (which may be infinite if D is infinite).

Corresponding to the partition (3. 2. 2), we define a digraph D^c, called the <u>condensation digraph</u> of D, as follows: the vertices of D^c are denoted by $S_1, S_2, \ldots, S_k$ and there is an arc $(S_i, S_j) \in A(D^c)$, $i \neq j$, if and only if $(v_i, v_j) \in A(D)$ for some $v_i \in S_i$ and $v_j \in S_j$.

We state below a few lemmas which will be used subsequently.

<u>Lemma 3. 2. 1.</u> The strong components D_i, $i = 1, 2, \ldots, k$, of D are strongly connected and are maximal subgraphs of D with this property.

<u>Lemma 3. 2. 2.</u> The condensation digraph D^c of any digraph D is acyclic.

For the proof of Lemmas 3.2.1 and 3.2.2, see, for instance, Harary et al. [13], Chapter 3.

Lemma 3.2.3. A digraph D is strongly connected if and only if for every nonempty subset V' of $V(D)$, $V' \neq V(D)$, there exists a path (3.2.1) in D such that (i) $r \geq 2$, (ii) $v_1, v_{r+1} \in V'$, (iii) $v_i \in V(D)-V'$ for $i = 2, 3, \ldots, r$, and (iv) the vertices $v_1, v_2, \ldots, v_r$ are distinct.

The proof of Lemma 3.2.3 is omitted here.

3.3. Basis Digraphs

In a digraph D, an arc $(v, u) \in A(D)$ is said to be basic if it appears in every path from v to u. A digraph D is called basic or a basis digraph if every arc of D is basic. In other words, D is a basis digraph if there exists no partial graph D' of D (different from D) such that $R_{D'}(v) = R_D(v)$ for all vertices v. Evidently, if D is a basis digraph then it can have no 'arcs in parallel' (i.e. arcs which are identified with the same ordered pair of vertices) or 'loops' (i.e. arcs with coincident endvertices). A digraph having no arcs in parallel or loops is called linear. Thus, any basis digraph must be linear. (It may be remarked that there is a natural one-to-one correspondence between linear digraphs and irreflexive binary relations. Thus, all the results and properties of linear digraphs may be stated in terms of such relations and vice versa).

We prove below two theorems which throw some light on the relationship of basis digraphs and their condensation digraphs.

Theorem 3.3.1. If D is a basis digraph then there is a one-to-one correspondence between the arcs of the condensation digraph D^c of D and the arcs of D which have their endvertices in different sets S_i where the sets S_i are defined by (3.2.2).

Proof: By definition of D^c, corresponding to any arc $(S_i, S_j) \in A(D^c)$ there is an arc $(v_i, v_j) \in A(D)$ for some $v_i \in S_i$ and $v_j \in S_j$. Let, if possible, (v_i, v_j), (u_i, u_j) be two distinct arcs in A(D) where $v_i, u_i \in S_i$ and $v_j, u_j \in S_j$. Since D_i is strongly connected (Lemma 3.2.1) there is a path $P(v_i, u_i)$ in D_i from v_i to u_i. Similarly, there is a path $P(u_j, v_j)$ in D_j from u_j to v_j. Then, $P(v_i, u_i)(u_i, u_j)P(u_j, v_j)$ is a path in D from v_i to v_j in which the arc (v_i, v_j) does not appear. This shows that (v_i, v_j) is not basic, contradicting the assumption that D is a basis digraph. The proof is then completed by the definition of the digraph D^c.

Theorem 3.3.2. The condensation digraph D^c of a basis digraph D is a basis digraph.

Proof: Suppose that D^c is not a basis digraph. Then, there is an arc $(S_i, S_j) \in A(D^c)$ which is not basic. Let

$$(S_{i_0}, S_{i_1})(S_{i_1}, S_{i_2}) \ldots (S_{i_r}, S_{i_{r+1}}), \quad S_{i_0} = S_i, \; S_{i_{r+1}} = S_j$$

be a path in D^c from S_i to S_j in which the arc (S_i, S_j) does not appear. By the definition of D^c, there is an arc $(v_i, v_j) \in A(D)$ where $v_i \in S_i$, $v_j \in S_j$ and there are arcs $(u_{i_t}, v_{i_{t+1}}) \in A(D)$ where $v_{i_t}, u_{i_t} \in S_t$, $t = 0, 1, \ldots, r$. Since the digraphs D_t are strongly connected, there exist paths $P(v_i, u_{i_0})$, $P(v_{i_1}, u_{i_1}), \ldots, P(v_{i_r}, u_{i_r})$, $P(v_{i_{r+1}}, v_j)$ where $P(v, u)$ is a path in D_t if both v, u belong to $S_t = V(D_t)$. But, then

$$P(v_i, u_{i_0})\,(u_{i_0}, v_{i_1})\,P(v_{i_1}, u_{i_1}) \ldots P(v_{i_r}, u_{i_r})\,(u_{i_r}, v_{i_{r+1}})\,P(v_{i_{r+1}}, v_j)$$

is a path in D from v_i to v_j in which the arc (v_i, v_j) does not appear. This, by definition, contradicts the assumption that D is a basis digraph. This completes the proof of the theorem.

3.4. Largest Number of Arcs in a Basis Digraph

We now consider the problem, suggested by O. Ore [19], of determining the largest number of arcs in a basis digraph with a fixed number of vertices. The problem is of interest only when the number of vertices is finite. In this section, upper bounds on the number of arcs in a basis digraph with n vertices are obtained. These bounds are shown to be best possible in the next section.

We first obtain the following:

Theorem 3.4.1. Let D be any basis digraph with n vertices. If D is acyclic then $|A(D)| \leq \left\lceil \frac{n^2}{4} \right\rceil$.

Proof: Since D is acyclic, $(v, u) \in A(D)$ implies $(u, v) \notin A(D)$, so that any two vertices of D are joined by at most one arc. Also, it is easily verified that a basis digraph which is acyclic can have no triangle, i.e., a subgraph consisting of three vertices which are pairwise joined by arcs. Hence, from a well-known theorem of P. Turan [23], we must have $|A(D)| \leq \left[\frac{n^2}{4}\right]$.

We next prove the following.

Theorem 3.4.2. Let D be any basis digraph with n vertices. If D is strongly connected, then $|A(D)| \leq 2n-2$.

Proof: We may assume that $n > 1$ for, otherwise, the theorem holds trivially. Let

$$(3.4.1) \qquad (v_{01}, v_{02})(v_{02}, v_{03}) \dots (v_{0n_0}, v_{01}), \quad n_0 \geq 2,$$

be an elementary circuit in D (by Lemma 3.2.3, taking V' as consisting of a single vertex, such a circuit in D always exists). Define $V_0 = \{v_{01}, v_{02}, \dots, v_{0n_0}\}$. The partial subgraph H_0 of D consisting of, precisely the arcs appearing in (3.4.1) with $V(H_0) = V_0$ is clearly strongly connected, and since D is a basis digraph it is easily seen that H_0 is, *in fact*, a subgraph of D. If $V_0 \neq V(D)$, by Lemma 3.2.3, there exists a path

$$(3.4.2) \qquad (v, v_{11})(v_{11}, v_{12}) \dots (v_{1n_1}, u), \quad n_1 \geq 1,$$

in D such that $v, u \in V_0$, $v_{1i} \in V(D) - V_0$ for $i = 1, 2, \dots, n_1$ and the vertices $v_{11}, \dots, v_{1n_1}$ are distinct. Define $V_1 = V_0 \cup \{v_{11}, v_{12}, \dots, v_{1n_1}\}$. Clearly, the digraph H_1 consisting of the arcs of H_0 and those appearing in (3.4.2) with $V(H_1) = V_1$ is strongly connected and since D is a basis digraph, H_1 is a subgraph of D. Suppose we have already defined, proceeding in this way, the digraph H_{j-1} and $V(H_{j-1}) \neq V(D)$. Then, by Lemma 3.2.3, there is a path

(3.4.3) $$(v, v_{j1})(v_{j1}, v_{j2})\ldots(v_{jn_j}, u), \quad n_j \geqslant 1,$$

in D such that $v, u \in V(H_{j-1})$, $v_{ji} \in V(D) - V(H_{j-1})$ for $i = 1, 2, \ldots, n_j$ and the vertices $v_{j1}, v_{j2}, \ldots, v_{jn_j}$ are distinct. Define $V_j = V(H_{j-1}) \cup \{v_{j1}, \ldots, v_{jn_j}\}$. Then, by the same argument as above, the digraph H_j consisting of the arcs of H_{j-1} and those appearing in (3.4.3) with $V(H_j) = V_j$ is a subgraph of D. Since D is finite, by induction, there must exist an index $r \geqslant 0$ such that $V_r = V(D)$ and $H_r = D$. It is clear from the way in which the graphs H_j are defined, that

$$|A(H_j)| = n_0 + (n_1+1) + \ldots + (n_j+1), \qquad n = 0, 1, \ldots, r.$$

In particular, we have

(3.4.4) $$\begin{aligned} |A(D)| &= |A(H_r)| \\ &= n_0+(n_1+1)+\ldots+(n_r+1) \\ &= (2n-2) - \left[(n_0-2) + \sum_{j=1}^{r} (n_j-1)\right]. \end{aligned}$$

Since $n_0 \geqslant 2$, $n_j \geqslant 1$ for $j = 1, 2, \ldots, r$, the term in parenthesis on the right hand side of (3.4.4) is ~~positive~~ nonnegative. Hence, $|A(D)| \leqslant 2n-2$. This completes the proof of the theorem.

We now prove the following theorem for general digraphs which, as is easily seen, includes Theorems 3.4.1 and 3.4.2 as particular cases.

Theorem 3.4.3. Let D be any basis digraph with n vertices and let k be the number of strong components of D. Then,

$$|A(D)| \leqslant 2(n-k) + \left[\frac{k^2}{4}\right].$$

Proof: Consider the strong components D_i, $i = 1, 2, \ldots, k$ and the condensation digraph D^c of D defined by the partition (3.2.2) of the vertices of D. Let D_i have n_i vertices so that we have $n = n_1+n_2+\ldots+n_k$. Evidently, any subgraph of a basis digraph is a basis digraph. Also, since the digraphs D_i are strongly connected, by Theorem 3.4.2, we obtain $|A(D_i)| \leqslant 2(n_i-1)$. Further, since D^c is an acyclic basis digraph (Lemma 3.2.2 and Theorem 3.3.2) having k vertices, by Theorem 3.4.1, we obtain $|A(D^c)| \leqslant \left[\frac{k^2}{4}\right]$. Now, since every arc of D either belongs to one of the digraphs D_i or has its endvertices in different sets S_i, by Theorem 3.3.1 we obtain finally

$$(3.4.5)\qquad \begin{aligned} |A(D)| &= |A(D_1)| + |A(D_2)| +\ldots+ |A(D_k)| + |A(D^c)| \\ &\leqslant 2(n_1-1)+ 2(n_2-1) +\ldots+ 2(n_k-1) + \left[\frac{k^2}{4}\right] \\ &= 2(n-k) + \left[\frac{k^2}{4}\right]. \end{aligned}$$

Hence, the proof of the theorem is complete.

3.5. Extremal Basis Digraphs

Any basis digraph with n vertices and k strong components is called extremal if it has exactly $2(n-k) + \left[\frac{k^2}{4}\right]$ arcs. In this section, we show that the bounds on the number of arcs in a basis digraph given by the Theorems 3.4.1, 3.4.2 and 3.4.3 are exact and determine the structure of extremal basis digraphs.

Consider a digraph D with n vertices. Let V_1, V_2, V_3 be mutually disjoint subsets of V(D) such that $V(D) = V_1 \cup V_2 \cup V_3$. Let $|V_i| = n_i$ so that $n = n_1+n_2+n_3$. Suppose that D satisfies the following conditions:
(i) an arc $(v, u) \in A(D)$ if and only if $v \in V_i$ and $u \in V_{i+1}$ for i = 1, 2, and
(ii) $|A(D)| = n_2(n_1+n_3) = \left[\frac{n^2}{4}\right]$. Then, D has a specific structure. In the following, any digraph with this structure is said to be of type I. It is easily seen that for any $n \geq 1$ there exist digraphs with n vertices which are of type I.

Evidently, any digraph of type I is an acyclic, basis digraph which, by Theorem 3.4.1, is extremal. Conversely, we assert that any extremal acyclic basis digraph must be of type I. To prove this, let D be any such digraph. It is clearly sufficient to show that D has no path of length greater than 2. Now, as observed in the proof of Theotem 3.4.1, any two vertices of D are joined by at most one arc and D can have no triangles. Since, moreover, D is extremal, by the Theorem of P. Turan [23], the digraph D is then such that its set of vertices V(D) is decomposed into two sets

V' and V'' ($V' \cap V'' = \emptyset$, $V(D) = V' \cup V''$) where $(v, u) \in A(D)$ if and only if $v \in V'$, $u \in V''$ or $u \in V'$, $v \in V''$. Let, if possible, (v_1, v_2) (v_2, v_3) (v_3, v_4) be a path in D of length greater than 2. We may assume without any loss of generality that $v_1 \in V'$. Then $v_3 \in V'$ and v_2, $v_4 \in V''$. From the above observation, either $(v_1, v_4) \in A(D)$ or $(v_4, v_1) \in A(D)$. However, neither of the cases can occur since, as is easily seen, the former contradicts the assumption that D is a basis digraph and the latter that D is acyclic. This proves the assertion.

The above discussion is summarized by the following:

Theorem 3.5.1. For any $n \geq 1$, there exist extremal acyclic basis digraphs with n vertices. A digraph is an extremal acyclic basis digraph if and only if it is of type I.

A (linear) digraph D is called symmetric if for any two vertices v, u of D, $(v, u) \in A(D)$ implies $(u, v) \in A(D)$. Any digraph D with n vertices is said to be of type II if it is (i) symmetric, (ii) strongly connected, and (iii) $|A(D)| = 2n-2$. Equivalently, D is of type II if it is strongly connected and has no elementary circuits of length greater than 2. (It is easily verified that a digraph is of type II if and only if it is a 'symmetric tree'. See [13], p. 260). It is known that for any $n \geq 1$, there exist digraphs with n vertices which are of type II.

Evidently, any digraph of type II is a strongly connected basis digraph which, by Theorem 3.4.2, is extremal. Conversely, we assert that any extremal strongly connected basis digraph must be of type II. To prove this, let D be any such graph with n vertices. If $n = 1$, the assertion holds trivially. Otherwise, since $|A(D)| = 2n-2$, in the notations of the proof of Theorem 3.4.2, from (3.4.4) we must have $n_0 = 2$. Since the circuit (3.4.1) was chosen arbitrarily, it follows that D can have no circuit (3.4.1) which is of length greater than 2 which proves the assertion. Hence, we have proved the following:

<u>Theorem 3.5.2.</u> For any $n \geq 1$, there exist extremal strongly connected basis digraphs with n vertices. A digraph is an extremal strongly connected basis digraph if and only if it is of type II.

In general, we now have the following:

<u>Theorem 3.5.3.</u> For any integers n and k, $n \geq k \geq 1$, there exist extremal basis digraphs with n vertices and k strong components. A digraph is an extremal basis digraph (with a fixed number of vertices and strong components) if and only if its condensation digraph is of type I and each of its strong components is of type II.

<u>Note</u>: The results contained in this chapter are to appear in [11].

CHAPTER 4

ON PSEUDOSYMMETRIC DIGRAPHS

4.1. Introduction and Summary

All directed graphs considered in this chapter, are finite.

A digraph D is called pseudosymmetric if for every vertex $v \in V(D)$, the outward degree of v equals the inward degree of v. We first note a few characteristic properties of pseudosymmetric digraphs, in section 4.2. A well-known characterization, due essentially to Euler [2], is provided by Theorem 4.2.1 "A digraph D is pseudosymmetric if and only if each connected component of D possesses a circuit which passes through every one of its arcs once and only once".

In section 4.3, we introduce the concept of a circuitsymmetric digraph which is apparently stronger than the known concept of a path-symmetric digraph. Each one of these is shown to be weaker than the concept of a pseudosymmetric digraph. Specifically, we prove the following Theorem 4.3.1 "If a digraph D is pseudosymmetric, then D is circuit-symmetric". Also, we have Theorem 4.3.2 "If a digraph D is circuitsymmetric, then D is pathsymmetric". As an immediate corollary to Theorem 4.3.1, we then obtain Theorem 4.3.3 "If a digraph D is pseudosymmetric, then D is pathsymmetric", which was proved earlier by A. Kotzig [16] and later, in a slightly more general form, by the author [5].

A Kotzig [16] first asked the question "Is the converse of Theorem 4. 3. 3 true?" which remains unanswered. It may now be asked if the converse of Theorem 4. 3. 1 or Theorem 4. 3. 2 is true. We believe and, thereby, conjecture, that the converse statements of the above theorems also hold.

In conclusion, some remarks on the problem of analysis of digraphs are made and a few unsolved problems are pointed out.

4. 2. Pseudosymmetric Digraphs

Consider a (finite) directed graph D with set of vertices V(D) and set of arcs A(D). We recall that for any vertex $v \in V(D)$, the outward degree $d^+(D, v)$ of v is the number of arcs in A(D) with initial vertex v and the inward degree $d^-(D, v)$ of v is the number of arcs in A(D) with terminal vertex v.

Definition 4. 2. 1. A digraph D is said to be pseudosymmetric if

$$d^+(D, v) = d^-(D, v) \qquad \text{for all } v \in V(D). \tag{4. 2. 1}$$

Let G be any finite, undirected graph. A cycle in G is said to be Eulerian or unicursal if every edge of G appears in the cycle once and only once. It may be asked under what conditions a given graph G possesses a Eulerian cycle. The mathematical theory of graphs had its origin in unicursal problems, in particular, the celebrated Euler's problem of the 'Bridges of Konigsberg'. Euler [2] provided the following criterion: "A graph G possesses a Eulerian cycle if and only if (i) G is connected, and (ii) the degree of each vertex of G is even".

In a directed graph D, a circuit is called Eulerian if every arc of D appears in the circuit exactly once. It is well-known that 'a directed graph D possesses a Eulerian circuit if and only if it is (i) connected, and (ii) pseudosymmetric". Thus, we have the following.

Theorem 4.2.1. A digraph D is pseudosymmetric if and only if each connected component of D possesses a Eulerian circuit.

Consider a digraph D and let S be any subset of V(D). Extending the previous notation, let $d^+(D, S)$ denote the number of arcs in A(D) with initial vertex in S and terminal vertex in V(D)-S; similarly, let $d^-(D, S)$ denote the number of arcs in A(D) with terminal vertex in S and initial vertex in V(D)-S. Clearly, if S is empty, then $d^+(D, S) = d^-(D, S) = 0$. Now, if we add (4.2.1) over all vertices $v \in S$ and cancel out the common part from either side (i.e., the number of those arcs which have both of their endvertices in S), we get

$$d^+(D, S) = d^-(D, S) \qquad \text{for all } S \subseteq V(D). \tag{4.2.2}$$

Conversely, by taking $S = \{v\}$, it is seen that (4.2.2) implies (4.2.1) trivially.

Thus, we obtain the following:

Theorem 4.2.2. A digraph D is pseudosymmetric if and only if it satisfies the equalities (4.2.2).

An immediate consequence, which can be easily derived from Theorem 4.2.2, is that "if any subset S of vertices of a pseudosymmetric digraph is 'condensed' to a single special vertex (where all the arcs of the digraph which have exactly one endvertex in common to S are joined with this vertex), the condensed digraph so obtained will also be pseudosymmetric

We state below another elementary characterization of pseudosymmetric digraphs.

Theorem 4.2.3. A digraph D is pseudosymmetric if and only if the maximal number of mutually arc-disjoint circuits passing through each vertex v of D equals the number of arcs having v as initial (or terminal) vertex.

The proof of the above theorem, which may also be derived as a corollary to Theorem 4.3.1, is omitted here.

4.3. Circuitsymmetric and Pathsymmetric Digraphs

In the following, all paths considered in any digraph D are assumed to have at least one arc. Two paths are said to be arc-disjoint if they share no arc in common. For any two vertices v, u of a digraph D, the maximal number of mutually arc-disjoint paths from v to u is denoted by $p_D(v, u)$ and the maximal number of mutually arc-disjoint circuits each passing through both v and u is denoted by $q_D(v, u)$.

We now introduce the following:

Definition 4. 3. 1. A digraph D is defined to be circuitsymmetric if

(4. 3. 1) $$q_D(v, u) = p_D(v, u) \quad \text{for all } v, u \in V(D).$$

In other words, a digraph D is circuitsymmetric if and only if whenever there exist k mutually arc-disjoint paths from a vertex v to a vertex u, we can find k mutually arc-disjoint circuits each of which passes through both v and u.

We shall show that the concept of a circuitsymmetric digraph is weaker than the concept of a pseudosymmetric digraph. Indeed, we prove the following

Theorem 4. 3. 1. If a digraph D is pseudosymmetric then D is circuit-symmetric.

Proof: The theorem will be proved by mathematical induction. If for any $v, u \in V(D)$, $p_D(v, u) = 0$, then clearly, we must have $q_D(v, u) = 0$ and the equality $q_D(v, u) = p_D(v, u)$ holds trivially. Let us assume, as induction hypothesis, that in all pseudosymmetric digraphs D, for any $v, u \in V(D)$ if

$p_D(v, u) < k$, where k is some positive integer, then $q_D(v, u) = p_D(v, u)$. Suppose that D is pseudosymmetric and let there exist vertices $v_0, u_0 \in V(D)$ such that $p_D(v_0, u_0) = k$. We shall show $q_D(v_0, u_0) = k$.

Now, by our assumption, there are k mutually arc-disjoint paths from v_0 to u_0. Let these be denoted by $P_1(v_0, u_0)$, $P_2(v_0, u_0), \ldots, P_k(v_0, u_0$ We can assume, without any loss of generality, that no arc appears in any of the paths $P_i(v_0, u_0)$ more than once. If $v_0 = u_0$, then each of the paths $P_i(v_0, u_0)$ is a circuit and there is nothing to prove. Suppose, therefore, that $v_0 \neq u_0$. Deleting from D all arcs which appear in $P_i(v_0, u_0)$, $i = 1, 2, \ldots, k$, we obtain a partial graph of D which we denote by D'. In D', we have, evidently,

$$(4.3.2) \qquad \begin{aligned} &d^+(D', v_0) = d^-(D', v_0) - k, \; d^+(D', u_0) = d^-(D, u_0) + k \\ &d^+(D', v) = d^-(D', v) \quad \text{for all } v \neq v_0, u_0. \end{aligned}$$

We now determine a path from u_0 to v_0 in D' proceeding as follows: Begin at the vertex u_0 and choose an arc (u_0, u_1) with initial vertex u_0 (from (4.3.2), there are at least k such arcs). If $u_1 \neq v_0$, then, from (4.3.2), there is an arc (u_1, u_2). If $u_2 = v_0$, the process terminates. Otherwise, from (4.3.2), there is an arc (u_2, u_3) different from those already encountered. The process continues till we reach the vertex v_0. Since the number of arcs in D' is finite, we must reach v_0 in a finite number of steps. Let the path so determined be denoted by $P_1'(u_0, v_0)$.

The path $P_1'(u_0, v_0)$, obviously, has no arc in common with any of the paths $P_i(v_0, u_0)$, and the arcs in $P_1(v_0, u_0)$, say, and $P_1'(u_0, v_0)$, put together, give us a circuit $C_1(v_0, u_0)$ in D passing through v_0 and u_0. Now, deleting from D all arcs which appear in $C_1(v_0, u_0)$, we obtain a digraph D'' which, as is easily seen, is also pseudosymmetric. Furthermore, in D'', we have k-1 mutually arc-disjoint paths, namely, $P_2(v_0, u_0), \ldots, P_k(v_0, u_0)$ from v_0 to u_0, i.e., $p_{D''}(v_0, u_0) = k-1$. By our induction hypothesis, we can find k-1 mutually arc-disjoint circuits in D'' each passing through v_0 and u_0 which together with $C_1(v_0, u_0)$ give us k mutually arc-disjoint circuits in D each passing through v_0 and u_0.

Thus, we have proved that $q_D(v_0, u_0) \geq p_D(v_0, u_0)$. The reverse inequality holds trivially. Hence, the proof of the theorem is complete.

We now have the following:

Definition 4.3.2. A digraph D is said to be pathsymmetric if

(4.3.3) $$p_D(v, u) = p_D(u, v) \text{ for all } v, u \in V(D).$$

It is clear from the definitions that (4.3.1) implies (4.3.3) trivially. Hence, we have readily

Theorem 4.3.2. If a digraph D is circuitsymmetric, then D is pathsymmetric.

As an immediate corollary to Theorem 4.3.1 and Theorem 4.3.2, we obtain

Theorem 4.3.3. If a digraph D is pseudosymmetric then D is pathsymmetric.

It may be remarked that the proof of Theorem 4.3.1, which is apparently stronger than Theorem 4.3.3, is quite elementary. Theorem 4.3.3 was proved initially by A. Kotzig [16] and in a slightly more general, though essentially equivalent form by the author [5] by making use of the maximum-flow minimum-cut theorem of Ford and Fulkerson [3].

A Kotzig [16] asked if the converse of Theorem 4.3.3 is also true. It may now be asked if the converse of the Theorems 4.3.1 and 4.3.2 are also true. We believe that the converse of these theorems also hold and thereby make the following conjectures.

Conjecture 1: If a digraph D is pathsymmetric, then D is circuitsymmetric.

Conjecture 2: If a digraph D is circuitsymmetric, then D is pseudosymmetric.

Conjecture 3: If a digraph D is pathsymmetric, then D is pseudosymmetric.

It is observed that the Conjectures 1 and 2 are independent of each other and together they are equivalent to Conjecture 3.

4.4. The Problem of Analysis of a Digraph

For any two fixed vertices v, u of a digraph D, algorithms for determining the value of $p_D(v, u)$ are well-known, e.g., the 'flow-algorithm' of Ford and Fulkerson [3]. The problem of analysis of a digraph D consists essentially in determining the values $p_D(v, u)$ for all ordered pairs of distinct vertices $v, u \in V(D)$. Let a digraph D with n vertices be denoted by D_n. It is known [4] that if D_n is pathsymmetric, then there are at most n-1 numerically different values among the possible n(n-1) values $p_{D_n}(v, u)$, $v, u \in V(D_n)$, $v \neq u$. Let a procedure of analysis of D_n be called <u>efficient</u> if it requires the determination of the values $p_{D_n}(v, u)$ for just n-1 pairs of vertices (so that the remaining ones can be determined easily with the help of these n-1 values). An efficient procedure of analysis for any pseudosymmetric digraph D_n has been given by the author [6]. This procedure of analysis is similar to that given by Gomory and Hu [4] for symmetric digraphs (or undirected networks).

We now propose the following problems.

Problem 1: Find an efficient procedure of analysis for any directed graph which is pathsymmetric.

Problem 2: Find an efficient procedure of analysis for any directed graph which is circuitsymmetric.

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