MAXIMAL AND MINIMAL TOPOLOGIES

ASIT BARAN RAHA

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INTRODUCTION

Given a topological property π and a non-void set X, let $\pi(X)$ denote the set of topologies on X with property π. $\pi(X)$ is obviously partially ordered by inclusion. A topological space $(X,\underline{\underline{T}})$ is minimal π if $\underline{\underline{T}}$ is a minimal element in $\pi(X)$. $(X,\underline{\underline{T}})$ is said to be maximal π if $\underline{\underline{T}}$ is a maximal element in $\pi(X)$. The study of maximal and minimal topological space is, as a matter of fact, a study of maximal π and minimal π spaces. Topological spaces closely related to minimal π spaces are π-closed and Katětov π spaces. $(X,\underline{\underline{T}})$ is π-closed if $\underline{\underline{T}}$ has property π and X is a closed subspace of every π space in which it can be embedded. A π space $(X,\underline{\underline{T}})$ is Katětov π provided $\underline{\underline{T}}$ is stronger than some minimal π topology on X.

The study of maximal topologies owes its origin to the remarkable fact that a compact Hausdorff space is maximal compact. This was first observed by R. Vaidyanathaswamy in his book [Va] published in the year 1947. The first substantial work in maximal topologies appeared in 1963 when N. Smythe and C.A.Wilkins characterised maximal compact spaces [SW]. In this brilliant paper they also produced an example of a maximal compact space which is not Hausdorff. It was followed up by J.P.Thomas in 1968 when he studied maximal connected spaces [Th]. In Chapter I of this thesis, as promised by the title of the chapter, we proceed to investigate maximal π spaces for some further topological properties like H-closed, connected, lightly compact, pseudocompact, Lindelöf and countably compact. The paucity of published literature on maximal topologies is largely due to the simple reason that for most topological properties the maximal π spaces turn out to be discrete.

The impetus to study minimal topological space came from the well-known topological result, first observed by A.S. Parkhomenko [Pa] in 1939, that a compact Hausdorff space is minimal Hausdorff. Earlier in 1924 P.S. Alexandroff and P. Urysohn [AU] investigated 'H-closed' spaces and proved that a regular space is H-closed if and only if it is compact. In considering the H-closedness of a Hausdorff space they were guided by the important observation that if a compact Hausdorff space X is embedded in a Hausdorff space Y, the image of X is always closed in Y. Parkhomenko found out the relationship between minimal Hausdorff and H-closed spaces by demonstrating that a minimal Hausdorff space is always H-closed. It was left to Katětov [K] to obtain the characterisations of minimal Hausdorff spaces. In 1941 E. Cartan [Bo1] could obtain characterisations of both minimal Hausdorff and H-closed spaces in terms of filters. He is the first to produce a non-compact minimal Hausdorff space. Since then minimal π and π-closed spaces have been studied for a wide spectrum of topological properties including various separation properties. In chapter II of this thesis we carry out further studies in minimal π and π-closed spaces for π = realcompact, first countable realcompact, locally H-closed, E_1 space, P-space, Hausdorff P-space, analytic and borelian (Relevant definitions are supplied later in this thesis).

CHAPTER 0

DEFINITIONS OF BASIC CONCEPTS AND NOTATIONS

We shall denote a topological space by a pair of the form $(X, \underline{\underline{T}})$ where X is any non-void abstract set and $\underline{\underline{T}}$ is the class of open subsets of X. If a fixed topology $\underline{\underline{T}}$ is under consideration, $\underline{\underline{T}}$ will be, usually, suppressed. Suppose $(X, \underline{\underline{T}})$ is any topological space. For a subset A of X, $\overline{A}$ and A^{o} respectively stand for closure of A and interior of A in $(X, \underline{\underline{T}})$. If $\underline{\underline{T}}$ and $\underline{\underline{S}}$ are two topologies on the same set X, $\underline{\underline{T}}$-clA denotes closure of A in $(X, \underline{\underline{T}})$ and $\underline{\underline{T}}$-intA denotes interior of A in $(X, \underline{\underline{T}})$. Similarly $\underline{\underline{S}}$-clA and $\underline{\underline{S}}$-intA stand respectively for closure and interior of A in $(X, \underline{\underline{S}})$.

An open subset U of a topological space $(X, \underline{\underline{T}})$ is said to be <u>regular-open</u> if $U = (\overline{U})^{o}$. The regular-open subsets of X are easily seen to be closed under finite intersections and thus form a base for a unique topology $\underline{\underline{T}}_o$ on X. $\underline{\underline{T}}_o$ is called the <u>semi-regular topology</u> on X associated with the topology $\underline{\underline{T}}$, and $(X, \underline{\underline{T}}_o)$ the <u>semi-regular space</u>. We shall make the convention to denote the semi-regular topology associated with a space $(X, \underline{\underline{T}})$ by $\underline{\underline{T}}_o$. A (topological) space $(X, \underline{\underline{T}})$ is called <u>semi-regular</u> if $\underline{\underline{T}} = \underline{\underline{T}}_o$. Here we list some important properties of the semi-regular space $(X, \underline{\underline{T}}_o)$ associated with the space $(X, \underline{\underline{T}})$.

(i) $(X, \underline{\underline{T}})$ is Hausdorff if and only if $(X, \underline{\underline{T}}_0)$ is Hausdorff.

(ii) $U \varepsilon \underline{\underline{T}} \Rightarrow \underline{\underline{T}}\text{-cl } U = \underline{\underline{T}}_0\text{-cl } U$ (Hence, for any $U \varepsilon \underline{\underline{T}}$ we shall use $\overline{U}$ to denote either of the two).

(iii) $\underline{\underline{T}}$ is regular $\Rightarrow \underline{\underline{T}} = \underline{\underline{T}}_0$.

Suppose $(X, \underline{\underline{T}}_0)$ is a semi-regular space. Put $E(\underline{\underline{T}}_0) = \{ \underline{\underline{S}} : \underline{\underline{S}}$ a topology on X such that $\underline{\underline{S}}_0 = \underline{\underline{T}}_0 \}$. $E(\underline{\underline{T}}_0)$ is partially ordered by the relation "$\underline{\underline{S}}'$ is weaker than $\underline{\underline{S}}$". It is known that $E(\underline{\underline{T}}_0)$ has a maximal element. A maximal element of $E(\underline{\underline{T}}_0)$ is referred to as a <u>submaximal topology</u> and X, endowed with a submaximal topology is called a <u>submaximal space</u>. The above facts are to be found in Bourbaki [Bo2]. An interesting property of a submaximal space will be frequently called for in later chapters and will be mentioned in the appropriate place.

By a <u>G-delta</u> we shall mean a countable intersection of open sets while a countable union of closed sets will be called an <u>F-sigma</u>. N, R and I will be used to denote the set of natural numbers, the set of real numbers and the unit interval [0,1] respectively (unless specifically stated otherwise) all of them endowed with their usual topologies.

An <u>open filter</u> $\underline{\underline{F}}$ is a non-empty collection of open sets satisfying

(i) $\emptyset \notin \underline{\underline{F}}$

(ii) if $U, V \in \underline{\underline{F}}$ and G open $\supseteq U \cap V$ then $G \in \underline{\underline{F}}$.

An <u>open ultrafilter</u> is an open filter which is maximal in the collection of open filters.

Let $\underline{\underline{F}}$ be any open filter in a topological space $(X, \underline{\underline{T}})$. Then $\bigcap \{\overline{F} : F \in \underline{\underline{F}}\}$ is called the <u>adherence</u> of $\underline{\underline{F}}$ or ad $(\underline{\underline{F}})$ in short. Further, an element of ad $(\underline{\underline{F}})$ is called an <u>adherent point</u> of $\underline{\underline{F}}$.

If X and Y are two topological spaces $C(X,Y)$ stands for the set of all continuous function from X into Y. $C(X)$ is the abbreviation for $C(X,R)$ and $F(X)$ for $C(X,I)$. The set $f^{-1}\{0\}$ is called the zero-set of f. We shall follow [GJ] in order to denote this set by $Z(f)$ i.e., for $f \in C(X)$ $Z(f) = \{x \in X : f(x) = 0\} = f^{-1}\{0\}$. Any set that is a zero-set of some function in $C(X)$ is called a zero-set in X. The family of all zero-sets of X i.e., $\{Z(f) : f \in C(X)\}$ will be denoted by $Z(X)$. It is a matter of elementary calculation to see that $Z(X) = \{Z(f) : f \in F(X)\}$.

A non-void subfamily $\underline{\underline{F}}$ of $Z(X)$ is called a <u>z-filter</u> on X provided that

(i) $\emptyset \notin \underline{\underline{F}}$

(ii) if $Z_1, Z_2 \in \underline{\underline{F}}$ and $Z_3 \in Z(X)$ such that $Z_3 \supseteq Z_1 \cap Z_2$ then $Z_3 \in \underline{\underline{F}}$.

By a <u>z-ultrafilter</u> on X is meant a maximal z-filter. It is known that every subfamily of $Z(X)$ with finite intersection property is contained in some z-ultrafilter. A z-filter $\underline{\underline{F}}$ is free or fixed according as $\bigcap \{Z : Z \in \underline{\underline{F}}\} = \emptyset$ or not.

βX, as usual, is the Stone-Čech compactification of a Tychonoff (i.e., completely regular Hausdorff) space X. Call a Tychonoff space X <u>realcompact</u> if every z-ultrafilter with the countable intersection property is fixed.

Let us denote the irrationals with usual topology by Σ. f is said to be a <u>compact correspondence</u> on Σ to a Tychonoff space X if for each $\sigma \in \Sigma$, $f(\sigma)$ is a compact subset of X. A compact correspondence f is called upper <u>semi-continuous</u> if, for every open subset U of X, the set $\{\sigma : f(\sigma) \subseteq U\}$ is open in Σ.

A <u>pseudocompact</u> space is a topological space in which every real valued continuous function is bounded. Throughout the term 'space' will indicate a topological space. A T_2 space will, as usual, mean a Hausdorff space. The separation properties like regularity, complete regularity, normality will include Hausdorff separation axiom.

CHAPTER I

MAXIMAL TOPOLOGIES

Summary: In this chapter we shall investigate some maximal topological spaces. The topological properties under consideration are compact, Lindelöf, countably compact, H-closed, connected, lightly compact or pseudocompact. The results can be classified into two types. The first type of results consists of characterisations of maximal π spaces where π = compact, Lindelöf, countably compact or H-closed, while the other type contains only necessary conditions to be satisfied by maximal π spaces when π = connected, lightly compact or pseudocompact. We start out with requisite definitions in section 1. Our results on maximal compact, maximal Lindelöf, maximal countably compact and maximal H-closed spaces are presented respectively in sections 2,3,4 and 5. Sections 6,7 and 8 are devoted to results on maximal π spaces where π = connected, lightly compact and pseudocompact. Products and subspaces of maximal π spaces and questions regarding embedding π spaces into maximal π spaces are dealt with at the end of this chapter.

1. Definitions

1.1 Given a topological property π a space $(X, \underline{\underline{T}})$ is said to be <u>maximal π</u> if $(X, \underline{\underline{T}})$ is a π space and $\underline{\underline{S}}$ is a π-topology on X with $\underline{\underline{T}} \subseteq \underline{\underline{S}}$ then $\underline{\underline{T}} = \underline{\underline{S}}$.

1.2 A space (X, $\underline{\underline{T}}$) is termed a <u>P-space</u> if every G-delta subset of X is open in $\underline{\underline{T}}$.

1.3 Call a space (X, $\underline{\underline{T}}$) <u>countably compact</u> if every countable open cover of X admits a finite open subcover.

1.4 A space (X, $\underline{\underline{T}}$) is <u>lightly compact</u> if every locally finite system of open sets of X is finite.

1.5 A space X is called an <u>E_1 space</u> if every point is the intersection of a countable number of closed neighbourhoods.

1.6 Call a Hausdorff space <u>H-closed</u> if every homeomorphic image of the space in a Hausdorff space is a closed subspace therein.

1.7 A space X is said to be <u>embedded</u> in a space Y if there exists a homeomorphism of X into a subspace of Y. X is said to be <u>densely embedded</u> in Y if X is embedded in Y and the subspace of Y which is homeomorphic to X is dense in Y.

2. <u>Compact spaces</u>

Let us recall that any 1-1 continuous map from a compact space onto a Hausdorff space (i.e., a continuous bijection from a compact space to a T_2 space) is a homeomorphism. In the above fact the Hausdorff property of the range space is a sufficient condition and a close examination of the proof of it will reveal that the closedness of every compact subset of the range space is precisely what is called for. The following theorem while characterising maximal compact spaces brings out an intimate relationship existing between maximal compact spaces and those spaces onto which any continuous bijection of a compact space is a homeomorphism.

2.1 Theorem: The following are equivalent :

(a) X is maximal compact

(b) The set of all closed subsets of X = the set of all compact subsets of X.

(c) Any continuous bijection f from a compact space Y onto X is a homeomorphism.

Proof. The equivalence of (a) and (b) is due to Smythe and Wilkins [SW]. We shall only prove (b) ⇒ (c) ⇒ (a).

(b) ⇒ (c) : Let f be a continuous bijection of a compact space Y onto X. If A is a closed subset of Y, A is compact. But $(f^{-1})^{-1}(A) = f(A)$ is compact and so closed in X by (b). Thus f^{-1} is also continuous i.e., f is a homeomorphism.

(c) ⇒ (a). Let $\underline{\underline{T}}$ be the topology on X. Let $\underline{\underline{S}}$ be a topology on X which is finer than $\underline{\underline{T}}$ and also compact. Now the identity map $i : (X, \underline{\underline{S}}) \to (X,\underline{\underline{T}})$ is a continuous bijection from the compact space $(X,\underline{\underline{S}})$ onto $(X,\underline{\underline{T}})$. By (c) i is a homeomorphism i.e., $\underline{\underline{T}} = \underline{\underline{S}}$. Naturally $(X,\underline{\underline{T}})$ is maximal compact. (Q.E.D.)

From theorem 2.1 it is easy to see that a maximal compact space is T_1. Since any compact T_2 space satisfies condition (b) of theorem 2.1, it is maximal compact. Smythe and Wilkins have explicitly constructed an example of a maximal compact space which is not Hausdorff [SW]. The example is reproduced here :

Let R be the set of real numbers. Let us choose two points $a, b \notin R$ and let $E = R \cup \{a, b\}$ with topology $\underline{\underline{T}}_1$ defined by neighbourhoods (to be, henceforth, abbreviated as nbhd)

$$\underline{\underline{W}}_1(x) = \{ V \subseteq E : V \supseteq (x-d,\ x+d) \text{ for some } d > 0 \} \text{ if } x \in R$$

$$\underline{\underline{W}}_1(b) = \{ V \subseteq E : V \supseteq \{b\} \cup \bigcup_{|n|=N}^{\infty} (2n-1, 2n) \text{ for some integer } N \}$$

$$\underline{\underline{W}}_1(a) = \{ V \subseteq E : V \supseteq \{a\} \cup \bigcup_{|n|=N}^{\infty} (2n-d_n,\ 2n+1+d_n) \text{ where } d_n > 0 \text{ are real numbers for } |n| = N, N+1, \ldots, \text{ and } N \text{ some integer} \}.$$

3. Lindelöf Spaces

Theorem 3.1 is clearly motivated by theorem 2.1 and offers a complete characterisation of maximal Lindelöf spaces.

3.1 Theorem. The following are equivalent :

(i) $(X, \underline{\underline{T}})$ is maximal Lindelöf

(ii) The set of all closed subsets of X coincides with the set of all Lindelöf subspaces of X.

(iii) If Y is a Lindelöf space and f is any continuous bijection from Y onto X, then f is a homeomorphism.

<u>Proof</u>: (i) $\Rightarrow$ (ii) Suppose there exists a Lindelöf subspace A of $(X, \underline{\underline{T}})$ which is not closed. Obviously, $A^c(=X-A) \notin \underline{\underline{T}}$. Let $\underline{\underline{S}}$ be the topology generated by $\underline{\underline{T}} \cup \{A^c\}$. Then $\underline{\underline{S}} = \{(A^c \cap U) \cup V : U, V \in \underline{\underline{T}}\}$ and is strictly finer than $\underline{\underline{T}}$. We shall now show that $(X, \underline{\underline{S}})$ is Lindelöf. Let $\{W_i : i \in I\}$ be an open cover of $(X, \underline{\underline{S}})$. Let $W_i = (A^c \cap U_i) \cup V_i$. Obviously $\bigcup \{V_i : i \in I\} \supseteq A$ and A is Lindelöf in the topology $\underline{\underline{T}}$. So there exists a countable subset I_1 of I such that $\bigcup \{V_i : i \in I_1\} \supseteq A$. Put $V = \bigcup \{V_i : i \in I_1\}$. Then $V \in \underline{\underline{T}}$. Consequently V^c is closed in $\underline{\underline{T}}$ and is, therefore, Lindelöf. Again $V^c \subseteq A^c$. Consider $W_i \cap V^c = (A^c \cap U_i \cap V^c) \cup (V^c \cap V_i) = V^c \cap (U_i \cup V_i)$. Thus $\underline{\underline{T}}|V^c = \underline{\underline{S}}|V^c$. Inasmuch as V^c is Lindelöf when $\underline{\underline{T}}$ is relativised to it, there exists a countable subset $I_2 \subseteq I$ such that $\bigcup \{W_i : i \in I_2\} \supseteq V^c$ and thence $\bigcup \{W_i : i \in I_1 \cup I_2\} \supseteq V^c \cup V = X$. But $I_1 \cup I_2$ is a countable subset of I. Hence, $(X, \underline{\underline{S}})$ is Lindelöf. A contradiction to the fact that $(X, \underline{\underline{T}})$ is maximal Lindelöf. Thus (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Since f is a continuous bijection onto X, the inverse f^{-1} is well-defined from X onto Y. Only we need to show f^{-1} is continuous. Suffices to show that for each closed subset F of Y, $(f^{-1})^{-1}(F) = f(F)$ is closed in X. Now F is closed in $Y \Rightarrow F$ is a Lindelöf subset of $Y \Rightarrow f(F)$ is a Lindelöf subset of X i.e., $f(F)$ is a closed subset of X.

(iii) $\Rightarrow$ (i) : If $\underline{\underline{T}}'$ is any Lindelöf topology on X such that $\underline{\underline{T}}$ is contained in $\underline{\underline{T}}'$, the identity map $i : (X, \underline{\underline{T}}') \rightarrow (X, \underline{\underline{T}})$ satisfies the conditions of (iii). So $\underline{\underline{T}}' = \underline{\underline{T}}$, i.e., $(X, \underline{\underline{T}})$ is maximal Lindelöf.

(Q.E.D.)

<u>Fact</u> : It follows from above theorem that maximal Lindelöf spaces are T_1. If X is a countable set, the maximal Lindelöf space is nothing but the discrete space. Through the next theorem we shall presently obtain a connection between maximal Lindelöf Hausdorff spaces and Lindelöf P-spaces which are Hausdorff. Moreover, in course of the proof it will be discovered that every maximal Lindelöf space is a P-space.

3.2 <u>Theorem</u>. The following are equivalent :

(i) X is maximal Lindelöf and Hausdorff.

(ii) X is a Lindelöf Hausdorff P-space.

<u>Proof</u>: (i) $\Rightarrow$ (ii) : By theorem 3.1 we know that a subset of X is Lindelöf if and only if it is closed. Let $G = \bigcap_{n=1}^{\infty} G_n$ be a G-delta subset of X where each G_n is open in X.

Now $X-G = \bigcup_{n=1}^{\infty} G_n^c$ and G_n^c $(=X-G_n)$ is closed and, a fortiori, Lindelöf for each n. Naturally, X-G is a Lindelöf subspace of X and so it is closed. Consequently G is open in X. X is thus a P-space. Since X is assumed to be Hausdorff, X becomes a Lindelöf T_2 P-space. (We have, in fact, proved that any maximal Lindelöf space is a P-space).

(ii)⇒(i) : Since X is Lindelöf, every closed subset of X is Lindelöf. We need only to show that every Lindelöf subspace is closed. Invoking theorem 3.1 we then conclude that X is maximal Lindelöf. Suppose, A is a Lindelöf subspace of X. Let $x \in \bar{A}$, the closure of A in X. Suffices to prove that $\bar{A} = A$. Suppose $x \notin A$. Let $\underline{\underline{N}}(x)$ denote the filter base of open nbhds of x. Since $x \in \bar{A}$, $\underline{\underline{F}} = \{V \cap A : V \in \underline{\underline{N}}(x)\}$ is a filter base of open subsets of A. If $\{V_n \cap A\}$ is a countable collection from $\underline{\underline{F}}$, $\bigcap_{n=1}^{\infty} (V_n \cap A) \neq \emptyset$ because $\bigcap_{n=1}^{\infty}(V_n \cap A) = (\bigcap V_n) \cap A = V \cap A \neq \emptyset$ as $V \in \underline{\underline{N}}(x)$ by (ii). Since X is T_2, $\{x\} = \bigcap\{\bar{V} : V \in \underline{\underline{N}}(x)\}$ and $x \notin A \Rightarrow \{x\}^c \supseteq A$ i.e., $\bigcup\{(\bar{V})^c : V \in \underline{\underline{N}}(x)\} \supseteq A$. But A is Lindelöf; therefore there exist $V_n \in \underline{\underline{N}}(x), n \geq 1$, such that $\bigcup_{n=1}^{\infty} (\bar{V}_n)^c \supseteq A$ i.e., $\bigcap_{n=1}^{\infty} V_n \subseteq \bigcap_{n=1}^{\infty} \bar{V}_n \subseteq A^c$. But $\bigcap_{n=1}^{\infty} V_n \in \underline{\underline{N}}(x)$ giving $A \cap (\bigcap V_n) = \emptyset$. A contradiction. So A must be closed.

(Q.E.D.)

3.3. <u>Corollary</u> : Every Hausdorff maximal Lindelöf space is regular and, hence, normal and paracompact.

Proof: Let X be a Hausdorff maximal Lindelöf space and let x be a point in X and F a closed subset of X such that $x \notin F$. Since X is T_2, for each $y \in F$ there exists an open set V_y in X such that $x \notin \overline{V}_y$. Now $\{V_y : y \in F\}$ is an open cover of F and, as F is Lindelöf, we can extract a countable subcover $\{V_n : n \in N\}$ from $\{V_y\}$. Now $F \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq \bigcup_{n=1}^{\infty} \overline{V}_n$ and $x \notin \bigcup_{n=1}^{\infty} \overline{V}_n$. Put $V = \bigcup V_n$. Then $\overline{V} = \overline{\bigcup V_n} \supseteq \bigcup \overline{V}_n \supseteq V$. But X is a P-space implies that $\bigcup \overline{V}_n$ is a closed set and contains V. So $\bigcup \overline{V}_n = \overline{V}$. Put $U = (\overline{V})^c$. Then $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. So X is regular. It is well-known fact that a regular Lindelöf space is normal and paracompact. So X is normal and paracompact.

(Q.E.D.)

Remark: We recall that any Lindelöf Tychonoff (= completely regular T_2) space is realcompact and any normal T_2 space is Tychonoff. Corollary 3.3 then shows that any Hausdorff maximal Lindelöf space is realcompact. As is already mentioned that a maximal Lindelöf space is always T_1 a non-Hausdorff maximal Lindelöf space can never be regular. Thus we arrive at the stronger conclusion that a maximal Lindelöf space is regular if and only if it is Hausdorff.

4. Countably compact spaces

Before formulating the theorem which states necessary and sufficient conditions for a countably compact space to be maximal countably compact let us list some well-known properties of countably compact spaces without proof (cf. Bourbaki [Bo2]).

4.1 (a) Every closed subspace of a countably compact space is countably compact.

(b) Let f be a continuous mapping of a countably compact space X into a topological space Y. Then f(X) is a countably compact subspace of Y.

(c) If X is Hausdorff and first countable, then every countably compact subspace of X is closed in X.

(d) Any first countable, countably compact T_2 space is regular (i.e., T_3).

Theorem 4.2 which follows is also motivated by theorem 2.1. The proof is omitted as it is similar to that of theorem 3.1.

4.2 Theorem : For a topological space $(X,\underline{\underline{T}})$ the following are equivalent :

(i) $(X,\underline{\underline{T}})$ is maximal countably compact.

(ii) The set of all closed subsets of X = The set of all countably compact subspaces of X.

(iii) Any continuous bijection f from a countably compact space Y onto X is a homeomorphism.

4.3 Corollary : Any first countable, countably compact Hausdorff space is maximal countably compact.

Proof: Suppose X satisfies the given conditions. By 4.1(a) all closed subspaces of X are countably compact. Since X is first countable too and T_2, property 4.1(c) yields that every countably compact subset of X is closed. Consequently a subset of X is closed when and only when it is countably compact. An application of theorem 4.2 helps us to conclude that X is maximal countably compact. (Q.E.D.)

4.4 Corollary (Aull [Au]): Every countably compact E_1 space is maximal countably compact.

Proof: Let X be a countably compact E_1 space. We shall establish that X is first countable. Then the conclusion will follow from previous corollary. Let $x \in X$. By definition there exist open sets U_n, $n = 1,2,\ldots$, such that $\{x\} = \bigcap \overline{U}_n$. We claim that given any open set V containing x there exists U_{n_0} such that $x \in U_{n_0} \subseteq V$. If not, $U_n \cap V^c \neq \emptyset$ for all n. We can easily assume that $U_{n+1} \subseteq U_n$ for $n \geqq 1$. Then $\{U_n \cap V^c : n \geqq 1\}$ is a countable filter base on X which is countably compact. The filter base must have non-empty adherence i.e. $\emptyset \neq \bigcap_{n=1}^{\infty} \overline{U_n \cap V^c} \subseteq \bigcap_{n=1}^{\infty} \overline{U}_n \cap V^c = V^c \cap (\bigcap_{n=1}^{\infty} \overline{U}_n) = \emptyset$ as $x \notin V^c$. A contradiction. Thus $\{U_n : n \in N\}$ forms a countable base at the point x. X is thus seen to be 1st countable, X is already T_2 as every E_1 space is. (Q.E.D.)

Some more interesting results on maximal countably compact spaces will be noticed in the next chapter. Theorem 4.2, incidentally, shows that every maximal countably compact space is T_1, and, therefore, T_0. We shall now present three examples of maximal countably compact spaces such that the first one is first countable but non-Hausdorff, the second one is Hausdorff but fails to be first countable (or E_1) and the last one is neither first countable nor Hausdorff.

Examples :

(1) The example due to Smythe and Wilkins, occurring at the end of section 2 furnishes, at the same time, an illustration of a maximal countably compact space which is non-Hausdorff.

(2) Let X be any uncountable set. Let p be a point not belonging to X. Let $Y = X \cup \{p\}$ denote the one-point compactification of X where X is endowed with the discrete topology. Now Y is easily seen to be a Hausdorff countably compact space which is not first countable (it is not even E_1). If A is any countably compact subset of Y then case (i) $p \notin A$ or case (ii) $p \in A$. In the former case $A \subseteq X$ and as it is countably compact A is finite and so closed. In case

(ii), since $p \in A$, $Y-A$ is a subset of X. Clearly $Y-A$ is a open making A closed. Thus any countably compact subset of Y is closed. By theorem 4.2, Y is maximal countably compact.

(3) Let E be the space considered in Example 1 and we can without loss of generality assume that E and Y are disjoint where Y is the space considered in Example 2. If $D = E \cup Y$ is endowed with the union topology, D will turn out to be compact and hence countably compact. Using Examples 1 and 2 it can be easily seen that D is non-Hausdorff and non-first-countable. But D is maximal countably compact.

5. H-closed spaces

H-closedness (or absolute closedness according to Bourbaki [Bo2]) is a slight weakening of compactness and was first introduced by Alexandroff and Urysohn [AU]. Let us call a Hausdorff space minimal Hausdorff if any Hausdorff topology weaker than the given topology necessarily equals it. H-closed and minimal Hausdorff spaces are intimately connected as the following fact testifies {Katětov [K]}:

5.1 A space is minimal Hausdorff if and only if it is H-closed and semiregular.

The following statement contains a very useful characterisation of H-closed spaces in terms of minimal Hausdorff spaces and can be found in Bourbaki [Bo2, page 146] :

5.2 A Hausdorff space is H-closed if and only if its associated semiregular topology is minimal Hausdorff.

Our first theorem asserts that every H-closed space admits a maximal H-closed topology.

5.3 <u>Theorem</u>: Suppose $(X, \underline{\underline{T}})$ is an H-closed space. Then there exists a maximal H-closed topology $\underline{\underline{S}}$ on X such that $\underline{\underline{T}}$ is weaker than $\underline{\underline{S}}$. Also every $\underline{\underline{S}}$-dense subset of X is open in $\underline{\underline{S}}$.

<u>Proof</u>: Since $\underline{\underline{T}}_o$, the associated semiregular topology of $\underline{\underline{T}}$, is weaker than $\underline{\underline{T}}$ and $\underline{\underline{T}}$ is closed, $\underline{\underline{T}}_o$ is also H-closed. By 5.2 $\underline{\underline{T}}_o$ is, in fact, minimal Hausdorff. As $\underline{\underline{T}}_o$ is semiregular, let us consider $E(\underline{\underline{T}}_o) = \{ \underline{\underline{U}}$ a topology on $X : \underline{\underline{U}}_o = \underline{\underline{T}}_o \}$ ($\underline{\underline{U}}_o$ is the semiregular topology associated with $\underline{\underline{U}}$). Let $\underline{\underline{S}}$ be a maximal element of $E(\underline{\underline{T}}_o)$ such that $\underline{\underline{T}} \subseteq \underline{\underline{S}}$. Then $\underline{\underline{S}}$ is a submaximal topology (cf. Chapter 0). Since $\underline{\underline{S}}_o = \underline{\underline{T}}_o$, by 5.2 $\underline{\underline{S}}$ is H-closed. To show the maximality of $\underline{\underline{S}}$ let us assume : $\underline{\underline{U}}$ is an H-closed topology on X such that $\underline{\underline{S}} \subseteq \underline{\underline{U}}$. We shall now show that for any $\underline{\underline{U}}$-open set V, $\underline{\underline{U}}\text{-cl } V = \underline{\underline{T}}\text{-cl } V$. As $\underline{\underline{U}}$ is H-closed $\underline{\underline{U}}\text{-cl } V$ is H-closed [Bo2]. H-closedness being preserved under continuous maps into Hausdorff spaces and the identity map of $(X,\underline{\underline{U}}) \rightarrow (X,\underline{\underline{T}})$ being continuous, $\underline{\underline{U}}\text{-cl } V$ is

H-closed in $\underline{\underline{T}}$ and, in particular, closed in $\underline{\underline{T}}$. Since $\underline{\underline{T}} \subseteq \underline{\underline{U}}$ and $\underline{\underline{U}}$-cl V $\supseteq$ V we can conclude that $\underline{\underline{U}}$-cl V = $\underline{\underline{T}}$-cl V = $\bar{V}$(say).

Now let V be a regular open set in $\underline{\underline{T}}$. As $\underline{\underline{T}} \subseteq \underline{\underline{U}}$, V ε $\underline{\underline{U}}$. Because of the last paragraph $\underline{\underline{U}}$-cl V = $\underline{\underline{T}}$-cl V. So $(\underline{\underline{U}}\text{-cl } V)^c = (\underline{\underline{T}}\text{-cl } V)^c$. Again, by the observation made in the last paragraph, $(\underline{\underline{U}}\text{-cl } (\underline{\underline{U}}\text{-cl } V)^c)^c = (\underline{\underline{T}}\text{-cl}(\underline{\underline{T}}\text{-cl } V)^c)^c =$ $\underline{\underline{T}}$-int($\underline{\underline{T}}$-cl V) = V since V is regular open in $\underline{\underline{T}}$. This shows that V is a regular open set in $\underline{\underline{U}}$. Hence $\underline{\underline{T}}_o \subseteq \underline{\underline{U}}_o$. Since $\underline{\underline{U}}$ is H-closed, $\underline{\underline{U}}_o$ is minimal T_2 and this forces $\underline{\underline{T}}_o = \underline{\underline{U}}_o$. Immediately $\underline{\underline{U}}$ ε E($\underline{\underline{T}}_o$) and $\underline{\underline{S}}$ being a maximal element of E($\underline{\underline{T}}_o$) we can conclude that $\underline{\underline{S}} = \underline{\underline{U}}$. It is thus established that $\underline{\underline{S}}$ is a maximal H-closed topology stronger than $\underline{\underline{T}}$. We have already noticed that $\underline{\underline{S}}$ is submaximal and hence every $\underline{\underline{S}}$-dense subset of X must be open in $\underline{\underline{S}}$ [Bo2, p. 139]. (Q.E.D.)

The main theorem of this section is the following theorem which succeeds in obtaining necessary and sufficient conditions for an H-closed space to be come maximal H-closed. The proof depends heavily on theorem 5.3

5.4 <u>Theorem</u> : An H-closed space (X,$\underline{\underline{T}}$) is maximal H-closed if and only if every $\underline{\underline{T}}$-dense subset of X is open in $\underline{\underline{T}}$.

<u>Proof</u>: We shall use the characterisation of submaximal spaces given in Bourbaki [Bo2, p.139] : (X,$\underline{\underline{T}}$) is submaximal iff every

$\underline{\underline{T}}$-dense subset of X is open in $\underline{\underline{T}}$. So suffices to show that, for an H-closed space $(X,\underline{\underline{T}})$,

$\underline{\underline{T}}$ is maximal iff $\underline{\underline{T}}$ is submaximal. Let us assume $\underline{\underline{T}}$ to be maximal H-closed and $\underline{\underline{T}}_o$, as usual, to be its associated semi-regular topology. Then $\underline{\underline{T}} \varepsilon E(\underline{\underline{T}}_o)$. Since $\underline{\underline{T}}_o$ is minimal Hausdorff (by 5.1) we can say that every member of $E(\underline{\underline{T}}_o)$ is H-closed (by 5.2). Naturally, $\underline{\underline{T}}$ is a maximal element of $E(\underline{\underline{T}}_o)$ i.e., $\underline{\underline{T}}$ is submaximal.

Conversely, let $\underline{\underline{T}}$ be submaximal. **Inasmuch** as $\underline{\underline{T}}$ is H-closed, by invoking theorem 5.3, there exists a maximal H-closed topology $\underline{\underline{S}}$ on X stronger than $\underline{\underline{T}}$. It was observed in course of the proof of theorem 5.3 that $\underline{\underline{S}} \varepsilon E(\underline{\underline{T}}_o)$ and as $\underline{\underline{T}}$, by assumption, is a maximal element of $E(\underline{\underline{T}}_o)$ $\underline{\underline{S}} = \underline{\underline{T}}$ i.e., $\underline{\underline{T}}$ is maximal H-closed.

(Q.E.D.)

5.5 <u>Examples of maximal H-closed spaces</u>

(1) Katětov [K] proved that for any Hausdorff space X, there is an H-closed space kX in which X is densely imbedded as an open subset and which has the characteristic property : If Y is an H-closed space containing X as a dense subset and if $i : X \to Y$ is the identity map on X, then there exists a continuous onto map $f : kX \to Y$ such that $f|X = i$(cf. C.T.Liu [L]). Let X be, in particular, the discrete space of natural numbers N. It is known that βN and kN are identical as sets.

But the topology of kN is much larger; in fact every point in kN-N is isolated in kN-N. Now the relative topology of N as a subspace of kN is discrete. If kN can support any larger H-closed topology, the relative topology of N induced by this topology is, of course, discrete. The characteristic property of kN alongwith the fact that N is dense open in kX will show that kN is a maximal H-closed space. Consequently every dense subset of kN is open.

(2) We shall now present an example of an infinite compact Hausdorff space which is maximal H-closed.

Let N_1 denote the one-point compactification of the set of natural numbers N endowed with discrete topology. N_1 is compact T_2 and so H-closed. N_1 is, in fact, maximal H-closed. We shall return to this example sometime later in this chapter.

6. Connected spaces

We shall only present a necessary condition for a connected space to be maximal connected. The theorem runs as follows :

6.1 Theorem: Suppose $(X, \underline{T})$ is a maximal connected space. Then every dense subset of X is open in $\underline{T}$.

The proof will be accomplished with the aid of the following string of lemmas.

6.2 <u>Lemma</u> : A space $(X, \underline{\underline{T}})$ is connected if and only if $(X, \underline{\underline{T}}_0)$ is connected, where $\underline{\underline{T}}_0$ (as per our convention) denotes the associated semiregular topology [Bo2, p.155].

<u>Proof</u>. As $\underline{\underline{T}}_0$ is weaker than $\underline{\underline{T}}$, the connectedness of $\underline{\underline{T}} \Rightarrow$ the same for $\underline{\underline{T}}_0$. Conversely, if $\underline{\underline{T}}$ is not connected, there should exist nonempty disjoint open sets G and V in $\underline{\underline{T}}$ such that $G \cup V = X$. As G and V are both open and closed sets in $\underline{\underline{T}}$, G and V are definitely regular-open i.e., they are in $\underline{\underline{T}}_0$. So $\underline{\underline{T}}_0$ is also not connected.

(Q.E.D.)

6.3 <u>Lemma</u>: A maximal connected space $(X, \underline{\underline{T}})$ is submaximal.

<u>Proof</u>: $\underline{\underline{T}}$ is a member of $E(\underline{\underline{T}}_0)$. Lemma 6.2 together with maximal connectedness of $\underline{\underline{T}}$ implies that $\underline{\underline{T}}$ is submaximal.

(Q.E.D.)

6.4 <u>Lemma</u>: A topology $\underline{\underline{T}}$ on X is submaximal if and only if every subset of X which is dense in the topology $\underline{\underline{T}}$ is open in $\underline{\underline{T}}$. (cf. Bourbaki [Bo2]).

Now the proof of theorem 6.1 immediately follows from lemmas 6.3 and 6.4.

<u>Remark</u>. J.P.Thomas [Th] has proved that any maximal connected space is $\underline{\underline{T}}_0$. But we can arrive at the same conclusion from the following proposition and lemma 6.3.

<u>Proposition</u> : Every submaximal topology is T_0.

Proof: Suppose $(X,\underline{\underline{T}})$ is a submaximal space. Let x,y be two distinct points in X. To show that there is a nbhd. of one which excludes the other. If any one of them is isolated we are done. Suppose none of them is isolated. Then $X - \{x\}$ is dense in $\underline{\underline{T}}$ and, in virtue of lemma 6.4, is open in $\underline{\underline{T}}$. Now $y \in X - \{x\}$. So $(X,\underline{\underline{T}})$ is T_o.

Theorem 6.1 states that a maximal connected space is necessarily submaximal. But submaximality of a connected space is not a sufficient condition of its being maximal connected. We shall substantiate this by means of the following example.

6.5 Example

$$X = \{1,2,3,4\}$$

$$\underline{\underline{T}} = \{X, \emptyset, \{1\}, \{2\}, \{1,2\} \{1,2,3\}, \{1,2,4\}\}$$

$(X,\underline{\underline{T}})$ is easily seen to be connected and submaximal (dense sets are sets containing both 1 and 2 and they are all open in $\underline{\underline{T}}$). Let us look at the topology

$$\underline{\underline{T}}_1 = \{X, \emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,4\}\}$$

$\underline{\underline{T}}_1$ is connected and is strictly bigger than $\underline{\underline{T}}$. So $\underline{\underline{T}}$ is not maximal connected.

Remark. It is still an open question whether there exists a maximal connected Hausdorff space.

7. Lightly compact spaces

To start out with we shall present few characterisations of a lightly compact space.

7.1 Proposition : On a space X the following are equivalent.

(a) X is lightly compact

(b) If $\underline{U}$ is a countable open cover of X, then there is a finite sub-collection of $\underline{U}$ whose closures cover X.

(c) Every countable open filter base on X has an adherent point.

For proof of the above proposition we refer to Stephenson [Ste2, page 439, theorem 2.6].

It is evident from the above position that compact, H-closed, countably compact spaces are all lightly compact. The purpose of lemma 7.2 is to show that a lightly compact space is pseudocompact.

7.2 Lemma: If X is a lightly compact space then X is pseudo-compact.

Proof: Suppose f is any real-valued continuous function on X. To show that f is bounded. Put $V_n = f^{-1}\{(-n,n)\}$. Each V_n is open and $\bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} f^{-1}(-n,n) = X$. Thus $\{V_n\}$ is an open cover of X which is countable. By 7.1(b) there exists a finite collection $V_{n_1}, V_{n_2}, \ldots, V_{n_k}$ such that $X = \bigcup_{i=1}^{k} \overline{V}_{n_i}$.

Now $\bigcup_{i=1}^{k} \overline{V}_{n_i} = \overline{(\bigcup_{i=1}^{k} V_{n_i})} = \overline{f^{-1}(-n,n)}$ where $n = \max_{1 \leq i \leq k} n_i$.

Again $\overline{f^{-1}(-n,n)} \subseteq f^{-1}[-n,n]$ $\therefore X = f^{-1}[-n,n]$ i.e., $|f| \leq n$. So f is bounded. (Q.E.D.)

7.3 <u>Lemma</u> : A space $(X,\underline{\underline{T}})$ is lightly compact if and only if $(X, \underline{\underline{T}}_o)$ is lightly compact. (Cf. Stepenson [Ste 3, page 116]).

<u>Proof</u>: Since $\underline{\underline{T}}_o \subseteq \underline{\underline{T}}$, from the definition it follows that $(X,\underline{\underline{T}}_o)$ is lightly compact whenever $(X,\underline{\underline{T}})$ is lightly compact. Suppose, conversely, $(X,\underline{\underline{T}}_o)$ is lightly compact. Let $\underline{\underline{U}}$ be a countable cover of X consisting of sets from $\underline{\underline{T}}$. If $\underline{\underline{U}} = \{V_n : n \in N\}$, put $G_n = (\overline{V}_n)^o = \underline{\underline{T}}\text{-int}(\underline{\underline{T}}\text{-cl}(V_n))$. Then $G_n \in \underline{\underline{T}}_o$, and $V_n \subseteq G_n$, so that $\{G_n : n \in N\}$ is an open cover of $(X,\underline{\underline{T}}_o)$ and hence admits a finite subfamily $\{G_{n_i} : 1 \leq i \leq k\}$ such that $\bigcup \{\underline{\underline{T}}_o\text{-cl}(G_{n_i}) : i = 1,2,\ldots,k\} = X$. Obviously $G_{n_i} \in \underline{\underline{T}}$ for $1 \leq i \leq k$ and since $\underline{U \in \underline{\underline{T}}} \Rightarrow \underline{\underline{T}}_o\text{-cl}(U) = \underline{\underline{T}}\text{-cl}(U)$ we have $\bigcup_{i=1}^{k} \overline{G}_{n_i} = X$ i.e., $\bigcup_{i=1}^{k} (\overline{V}_{n_i})^o = X$ i.e., $\bigcup_{i=1}^{k} \overline{V}_{n_i} = X$.

This shows that $(X, \underline{\underline{T}})$ is lightly compact. (Q.E.D.)

Our next lemma states that every maximal lightly compact space is submaximal. It is analogous to lemma 6.3. We shall omit the proof since it resembles the proof of lemma 6.3.

7.4 Lemma : Suppose $(X, \underline{\underline{T}})$ is maximal lightly compact. Then $(X, \underline{\underline{T}})$ is submaximal.

The following theorem is the natural consequence of lemmas 6.4, 7.3 and 7.4.

7.5 Theorem : If $(X, \underline{\underline{T}})$ is a maximal lightly compact space, then every dense subset of X is open in $\underline{\underline{T}}$.

Next we shall show that theorem 7.5 provides us only a necessary condition, That submaximality does not guarantee maximality of a lightly compact space is brought out by the following example.

7.6 Example

X = Any infinite set.

Let x_o be a fixed point in X.

Let $\underline{\underline{T}} = \{V \subset X : x_o \in V\} \cup \{\emptyset\}$.

Then $(X, \underline{\underline{T}})$ is a submaximal topological space. We shall first see that $(X, \underline{\underline{T}})$ is lightly compact. If $\{G_n\}$ is any countable open cover of X, then $x_o \in G_n$ for each n. So $\overline{G}_n = X$ for each n so that by proposition 7.1(b) X is lightly compact. Let us now fix $x_1 \in X$ such that $x_1 \neq x_o$. Look at the topology $\underline{\underline{S}}$ on X determined as follows : base for the nbhd filter at x consists of the set $\{x, x_o\}$ if $x \neq x_o, x_1$.

base for the nbhd filter at x_0 consists of $\{x_0\}$ and the base at x_1 consists of $\{x_1\}$ alone.

$\underline{\underline{S}}$ is strictly stronger than $\underline{\underline{T}}$. We shall prove now that $(X, \underline{\underline{S}})$ is lightly compact. Let $\{G_n\}$ be a countable open cover of X. There exist n_1 and n_2 such that $x_0 \varepsilon G_{n_1}, x_1 \varepsilon G_{n_2}$ and obviously $\underline{\underline{S}}\text{-cl}(G_{n_1} \cup G_{n_2}) = X$. Then $(X, \underline{\underline{T}})$ is submaximal and lightly compact but not maximal lightly compact.

(Q.E.D.)

8. Pseudocompact spaces

Our main result regarding maximal pseudocompact spaces is as follows :

8.1 Theorem: Every maximal pseudocompact space is submaximal.

The first step in order to prove the theorem is the next lemma.

8.2 Lemma : A space $(X, \underline{\underline{T}})$ is pseudocompact if and only if $(X, \underline{\underline{T}}_0)$ is pseudocompact.

Proof. The only nontrivial part is to show the 'if' part of the assertion. Denote by $C(X;\underline{\underline{T}}_0)$ the space of all continuous real-valued functions on $(X, \underline{\underline{T}}_0)$ and by $C(X;\underline{\underline{T}})$ the space of all continuous real-valued function on $(X, \underline{\underline{T}})$. We claim that $C(X;\underline{\underline{T}}_0) = C(X;\underline{\underline{T}})$. For this we need only to prove $C(X;\underline{\underline{T}}) \subset C(X;\underline{\underline{T}}_0)$. Let $f \varepsilon C(X;\underline{\underline{T}})$, and U be any open subset of the real line. Then $f^{-1}(U) \varepsilon \underline{\underline{T}}$. Let $x \varepsilon f^{-1}(U)$. Then $f(x) \varepsilon U$. By using regularity of the real line we can catch hold of a $V \varepsilon \underline{\underline{T}}$ such that $x \varepsilon V \subseteq \overline{V} \subseteq f^{-1}(U)$ and so $x \varepsilon V \subseteq (\overline{V})^{o} \subseteq \overline{V} \subseteq f^{-1}(U)$. But $(\overline{V})^{o} \varepsilon \underline{\underline{T}}_o$. So $f^{-1}(U) \varepsilon \underline{\underline{T}}_o$. f thus belongs to $C(X;\underline{\underline{T}}_o)$.

(Q.E.D.)

Proof of theorem 8.1: As usual let $E(\underline{\underline{T}}_o) = \{ \underline{\underline{S}} : \underline{\underline{S}}$ a topology on X with $\underline{\underline{S}}_o = \underline{\underline{T}}_o \}$. By maximality of $(X,\underline{\underline{T}})$ and lemma 8.2 and since $\underline{\underline{T}} \varepsilon E(\underline{\underline{T}}_o)$, we conclude that $\underline{\underline{T}}$ is submaximal.

(Q.E.D.)

Fact: Since by lemma 7.2 a lightly compact space is pseudocompact, Example 7.6 offers even an example of a submaximal pseudocompact space which is not maximal pseudocompact.

We are going to close this section by mentioning an example of a compact Hausdorff space which acts as an omnibus example for a maximal compact, maximal countably compact, maximal H-closed, maximal lightly compact or maximal pseudocompact space.

8.3 Example

Let N_1 denote, as in Example 2 of 5.5, the one-point compactification of the discrete space N of natural numbers. Let $N_1 = N \cup \{\omega\}$. N_1 is compact T_2 and so is maximal compact. It has been already mentioned as an example of a maximal H-closed space. Everything will be shown in one stroke if we show that

N_1 is maximal pseudocompact (N_1 is obviously pseudocompact). Suppose not. Then there must exist a strictly bigger topology on N_1 which is still pseudocompact. In that topology there should be an open set V containing ω such that $N_1 - V$ is infinite. It is easy to see that $N_1 - V$ is open in the latter topology. Define the function $f : N_1 \to R$ as follows :

$$f(n) = n \text{ if } n \in N_1 - V$$
$$= -1 \text{ if } n \in V$$

f is a continuous real-valued function on N_1, no doubt, but f is not bounded. So a contradiction arises.

9. Products and subspaces of maximal spaces

In this section we intend to study the behaviour of maximal topological spaces under products and make an attempt to determine those subspaces of maximal spaces which retain the maximality property. Let us first try to determine which subspaces of maximal π spaces are maximal π where π = compact, Lindelöf, H-closed, countably compact, connected, lightly compact or pseudocompact.

In the cases when π = compact, Lindelöf or countably compact we really possess complete characterisations of maximal π subspaces. Theorems 2.1, 3.1 and 4.2 imply that if π = compact, Lindelöf or countably compact, closed subspaces of maximal π

spaces are precisely the maximal π subspaces.

We could not ascertain which subspaces of maximal lightly compact or maximal pseudocompact spaces are of the same type. In fact there is no characterisation available regarding which subspaces of a pseudocompact space are pseudocompact. The case of lightly compact spaces is not so bad. In [BaCM] it was proved that a space X is lightly compact if and only if the closure of each open subset of X is lightly compact.

Turning our attention to connected spaces we find that J.P.Thomas [Th] has established that connected open subsets of a maximal connected space are maximal connected. But a maximal connected subspace of a maximal connected space need not be open. We shall demonstrate this by an illustration.

Let $X = \{1,2,3\}$ and

$$\underline{\underline{T}} = \{X, \emptyset, \{1\}, \{2\}, \{1,2\}\}$$

$(X,\underline{\underline{T}})$ is obviously connected ; in fact it is maximal connected. Look at the subset $\{1,3\}$. It is closed in $\underline{\underline{T}}$ and not open. The topology $\underline{\underline{T}}$ restricted to $\{1,3\}$ is as follows

$\underline{\underline{T}} \mid \{1,3\} = [\{1,3\}, \emptyset, \{1\}]$. Evidently $\{1,3\}$ is a maximal connected subspace of X.

A theorem of Thomas [Th, theorem 5, page 701] helps us to assert that any connected subspace of a finite maximal connected space is maximal connected. Recently it has been proved by Guthrie and Stone [GS, theorem 1] that every connected subspace of a maximal connected space is maximal connected. However, no access to the above paper could be made. A connected space need not be maximal connected despite the fact that all proper connected subspaces are maximal connected. An example is here incorporated to substantiate this statement. The connected space X of Example 6.5 is not maximal connected. But all of its proper connected subspaces are maximal connected.

In order to determine the maximal H-closed subspaces in a maximal H-closed space let us first observe that any H-closed subspace of an H-closed space is closed. On the other hand a closed subset of an H-closed space need not be H-closed. The following is an example :

Let $Z = \{a_{ij}, b_{ij}, c_i, \alpha, \beta : i = 1,2,\ldots, j = 1,2,\ldots\}$ where all these elements are assumed to be distinct.

Define the following neighbourhood system on Z : each a_{ij} is isolated and each b_{ij} is isolated. We define the basic nbhds of c_i, α, β as follows :

$$\underline{\underline{B}}(c_i) = \{V_n(c_i) = \bigcup_{j=n}^{\infty} \{a_{ij}, b_{ij}, c_i\} : n = 1,2,\ldots\},$$

$$\underline{\underline{B}}(\alpha) = \{V_n(\alpha) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{a_{ij}, \alpha\} : n = 1,2,\ldots\},$$

$$\underline{\underline{B}}(\beta) = \{V_n(\beta) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{b_{ij}, \beta\} : n = 1,2,\ldots\}.$$

It is well-known that Z is H-closed (in fact, minimal Hausdorff). The subset $\{c_i : i = 1,2,\ldots\}$ is a closed discrete infinite subset and hence cannot be H-closed.

Submaximality has been characterised in lemma 6.4. The next proposition shows that submaximality is a hereditary property.

9.1 Proposition: If X is a submaximal space, then every subspace of X is submaximal. [Bo2]

Proof: Let A be a non-void subset of a submaximal space X. To show A is submaximal in its relative topology it suffices to show that every dense subset of A is open in A (lemma 6.4). Let B be a dense subset of A. Consider A^c and choose a dense subset D of A^c. It is easy to see that $B \cup D$ is dense in X and hence open in X. Since $B = (B \cup D) \cap A$, B is open in A.

(Q.E.D.)

The following proposition characterises maximal H-closed subspaces of a maximal H-closed space in terms of H-closed subspaces.

9.2 Proposition : In a maximal H-closed space, the class of all maximal H-closed subspaces and the class of H-closed subspaces are identical.

Proof: Let X be a maximal H-closed space. We are done if we show that every H-closed subspace of X is maximal H-closed. Theorem 5.4 asserts that X is submaximal. If A is any H-closed subspace of X, A is also submaximal due to proposition 9.1. An application of theorem 5.4 now yields that A is maximal H-closed.

(Q.E.D.)

Inasmuch as a compact Hausdorff space is maximal compact, Hausdorff maximal compact spaces are just the compact T_2 spaces. Consequently, Hausdorff maximal compact spaces are indeed closed under product (in fact, arbitrary product). In the case of non-Hausdorff maximal compact space we intend to prove the following.

9.3 Proposition: If X is a non-Hausdorff maximal compact space, $X \times X$ endowed with the product topology is never going to be maximal compact.

Proof: Let $D = \{(x,x) ; x \in X\}$ denote the diagonal of $X \times X$. D is easily seen to be compact. Since X is not T_2, D cannot be closed. The existence of a non-closed compact set, viz. D, shows that the compact space $X \times X$ cannot be maximal compact.

(Q.E.D.)

Let us have a proposition regarding products of submaximal spaces.

9.4 <u>Proposition</u> : If X is submaximal and X has a non-isolated point (or the topology of X is not discrete), then $X \times X$ is not submaximal.

<u>Proof</u>: Let p be a non-isolated point of X. Put $Y = X - \{p\}$. Y is then dense in X and as X is submaximal Y is open in X. Consider the subset $B = Y \times Y \bigcup \{(p,p)\}$ of $X \times X$. B is dense in $X \times X$ as $Y \times Y$ is already dense in $X \times X$. It is easy to see that B is not open in $X \times X$ since it does not contain any nbhd of (p,p). This shows that $X \times X$ cannot be submaximal (lemma 6.4).

(Q.E.D.)

An interesting application of the last proposition is the next theorem :

9.5 <u>Theorem</u>: If X is a maximal H-closed space with at least one non-isolated point, $X \times X$ with the product topology is H-closed but never maximal H-closed.

<u>Proof</u>: Since H-closed spaces are known to be closed under products [CF], $X \times X$ is definitely H-closed. X is assumed to maximal H-closed and, hence, submaximal (theorem 5.4). Since the conditions of proposition 9.4 are satisfied, $X \times X$ is not submaximal. So $X \times X$ cannot become maximal H-closed. (Q.E.D.)

Products of maximal connected spaces, like maximal H-closed spaces, are not in general maximal connected. As an illustration we mention the following :

$X = \{a, b\}$ with a, b distinct.

$\underline{\underline{T}} = \{X, \emptyset, \{a\}\}$, $(X, \underline{\underline{T}})$ is maximal connected.

The product topology $\underline{\underline{T}} \times \underline{\underline{T}}$ on $X \times X$ is given by

$[X \times X, \emptyset, \{(a,a)\}, X \times \{a\}, \{a\} \times X, (X \times \{a\}) \cup (\{a\} \times X)]$

which is definitely not maximal connected as we can produce a strictly larger connected topology as follows

$$[X \times X, \emptyset, \{(a,a)\}, \{(a,b)\}, \{a\} \times X, X \times \{a\}, (X \times \{a\}) \cup (\{a\} \times X)].$$

It is worthwhile to mention that the study of products of maximal π spaces where π = Lindelöf, countably compact, lightly compact or pseudocompact is rendered uninteresting by the fact that these topological properties are not, in general, productive. Still we shall examine some special cases and see what we are able to establish. Let us take up the case of maximal Lindelöf spaces. If we happen to consider a non-Hausdorff maximal Lindelöf space X such that $X \times X$ is Lindelöf when endowed with the product topology, $X \times X$ cannot be maximal Lindelöf due to the fact that diagonal of $X \times X$ is not closed but Lindelöf (the argument is analogous to one used in Proposition 9.3). In the Hausdorff case, however, we can prove the following assertion.

9.6 Theorem: If X is a maximal Lindelöf space and if $X \times X$ is Hausdorff and Lindelöf, then $X \times X$ is a maximal Lindelöf space.

Proof: By invoking theorem 3.2 we can assert that X is a P-space. By hypothesis $X \times X$ is a Lindelöf Hausdorff space. If we can show that $X \times X$ is a P-space, theorem 3.2 will yield that it is maximal Lindelöf. Let $A \subseteq X \times X$ be any G-delta i.e., $A = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open in $X \times X$. If $(x,y) \in A$, $(x,y) \in G_n$ for each n. Then for each n, there exists U_n and V_n open in X such that $(x,y) \in U_n \times V_n \subseteq G_n$, so $(x,y) \in (\bigcap_n U_n) \times (\bigcap_n V_n) \subseteq A$. By hypothesis $\bigcap U_n$ and $\bigcap V_n$ are open subsets of X. Thus A is open in $X \times X$ i.e. $X \times X$ is a P-space. (Q.E.D.)

A trivial consequence of corollary 4.3 is that every first countable compact T_2 space is maximal countably compact. Thus if we start with a first countable compact Hausdorff space X, the product space $X \times X$ is also a space of the same type and, a fortiori, maximal countably compact. But arguments similar to those of Proposition 9.3 show that if X is a non-T_2 maximal countably compact space $X \times X$ is not going to be maximal countably compact when it is given that $X \times X$ is countably compact.

With respect to products maximal lightly compact or maximal pseudocompact spaces behave like maximal H-closed spaces. If X is a maximal lightly compact (or resp. pseudocompact) space and if $X \times X$ is known to be lightly compact (or resp. pseudocompant), the product space may fail to be maximal lightly compact (or resp. pseudocompact) mainly because of Proposition 9.4.

10. Conditions under which π spaces can be imbedded into maximal π spaces

In this section we make an attempt to find out conditions under which a π space admits an embedding or a dense embedding into a maximal π space where π = compact, Lindelöf, countably compact, H-closed, connected, lightly compact or pseudocompact. In some of the cases necessary and sufficient conditions are lacking.

If π is compact, Lindelöf or countably compact, then in order that a π space X be embedded in a maximal π space, it is necessary and sufficient that X be maximal π. In these cases the problem of dense embedding is trivially solved.

For π = H-closed, connected, lightly compact or pseudocompact we know that maximal π spaces are submaximal. We also note that submaximality is hereditary. So we can conclude :

A necessary condition for a π space X to be embeddable into a maximal π space is that X is submaximal. Whether this condition is also sufficient while dealing with lightly compact or pseudocompact spaces is not known to us.

If π stands for H-closedness, we recall that such a space is maximal as soon as it is submaximal (Theorem 5.4). So submaximality is a necessary and sufficient condition for an H-closed space to be embedded in a maximal H-closed space.

Finally when we come to connected spaces we recall that a connected subspace of maximal connected space is maximal connected [GS]. We can immediately infer that a necessary and sufficient condition for a connected space to be imbedded in a maximal connected space is that the space is maximal connected.

11. Locally Lindelöf Hausdorff P-spaces and imbedding in a maximal Lindelöf space

In the last section it has been established that a Lindelöf space allows an imbedding into a maximal Lindelöf space if it is already maximal Lindelöf. The aim of this section is to introduce a certain type of spaces which admit an embedding in maximal Lindelöf spaces. We admit that our results in this direction are naturally guided by the simple observation that the one-point compactification of a locally compact T_2 space is nothing but an embedding of the space in a maximal compact (Hausdorff) space (a compact T_2 space being maximal compact). The study of locally H-closed spaces, introduced by Obreanu [O], also supplied the motivation for locally Lindelöf Hausdorff P-spaces. (Cf. Section 4 of Chapter II).

11.1 A locally Lindelöf space is a topological space in which every point has a Lindelöf neighbourhood.

11.2 Call $(Y,\underline{\underline{S}})$ a <u>one-point maximal Lindelöf extension</u> of a topological space $(X,\underline{\underline{T}})$ if $X \subset Y$, $\underline{\underline{T}} = \underline{\underline{S}}|X$ (i.e., $\underline{\underline{T}}$ is the topology obtained by relativising $\underline{\underline{S}}$ to X), $(Y,\underline{\underline{S}})$ is maximal Lindelöf and $Y-X$ is a singleton.

11.3 A topological space $(Y,\underline{\underline{S}})$ is said to be a <u>maximal Lindelöf extension</u> of a space $(X,\underline{\underline{T}})$ if $(Y,\underline{\underline{S}})$ is a maximal Lindelöf space and X is embedded as a <u>dense subspace</u> of Y.

11.4 <u>The one-point Lindelöf extension</u> of a topological space $(X,\underline{\underline{T}})$ is the set $X' = X \cup \{p\}$ (where $p \notin X$) with the topology $\underline{\underline{T}}'$ where

$\underline{\underline{T}}' = \underline{\underline{T}} \cup [\{p\} \cup V : V \subset X$ and $X-V$ is a closed Lindelöf subset of $X]$.

The one-point Lindelöf extension $(X',\underline{\underline{T}}')$ of a topological space $(X,\underline{\underline{T}})$ is Lindelöf and X is an open subspace. If X is Lindelöf to start with, then $\{p\}$ is an isolated point of the one-point Lindelöf extension. Conversely, if p is an isolated point of X', then X is closed in X' and is therefore Lindelöf.

Let us call a topological space $(X,\underline{\underline{T}})$ a locally Lindelöf Hausdorff P-space (to be abbreviated as llh P-space) if $(X,\underline{\underline{T}})$ is a locally Lindelöf space and at the same time a Hausdorff P-space also. The reason behind introducing this concept will be clear from the following theorem.

11.5 Theorem: If $(X,\underline{\underline{T}})$ is a locally Lindelöf Hausdorff P-space, then the one-point Lindelöf extension $(X',\underline{\underline{T}}')$ of the space $(X,\underline{\underline{T}})$ is Hausdorff.

Proof: The Hausdorffness of $(X',\underline{\underline{T}}')$ will be established if we separate a point $x \in X$ from p by means of disjoint open sets. By hypothesis X is a Hausdorff P-space. So every Lindelöf subspace of X is going to be closed in X (cf. theorem 3.2). Moveover, $(X,\underline{\underline{T}})$ is a locally Lindelöf space, so x has a Lindelöf neighbourhood in X, say F. Then F is a closed and Lindelöf subspace of X containing x, By definition of $\underline{\underline{T}}'$, now $X-F \cup \{p\}$ is an open nbhd of p in $(X',\underline{\underline{T}}')$. Thus x and p are separated by disjoint nbhds.

(Q.E.D.)

11.6 Proposition: Let $(Y,\underline{\underline{S}})$ be a one-point maximal Lindelöf Hausdorff extension of $(X,\underline{\underline{T}})$. Then

(a) $(X, \underline{\underline{T}})$ is a llh P-space

(b) $X \in \underline{\underline{S}}$.

Proof: (a) Let $\{r\} = Y - X$. Since $(Y, \underline{\underline{S}})$ is maximal Lindelöf T_2, $(Y, \underline{\underline{S}})$ is a P-space and X, being a subspace of Y, is a Hausdorff P-space. In order to show that $(X, \underline{\underline{T}})$ is locally Lindelöf let $x \in X$. Obviously $x \neq r$. There exists $U \in \underline{\underline{S}}$ such that $r \notin$ closure of U in Y and $x \in U$. Clearly closure of U in Y is a subset of X and being closed in Y is Lindelöf. Thus x has a Lindelöf nbhd in $(X, \underline{\underline{T}})$. So $(X, \underline{\underline{T}})$ is a llhP-space.

(b) Y is $T_2 \Rightarrow \{r\}$ is closed in Y $\Rightarrow$ X is open in Y i.e., $X \in \underline{\underline{S}}$. (Q.E.D.)

The above proposition tells us that a necessary condition for a topological space to possess a one-point maximal Lindelöf extension is that it should be a llh P-space. Naturally one may want to know whether every locally Lindelöf Hausdorff P-space always admits a one-point maximal Lindelöf extension. The following proposition provides an affirmative answer.

11.7 Proposition: The one-point Lindelöf extension $(X', \underline{\underline{T}}')$ of a llh P-space $(X, \underline{\underline{T}})$ is, in fact, a one-point T_2 maximal Lindelöf extension of $(X, \underline{\underline{T}})$.

Proof: We need to show $(X', \underline{\underline{T}}')$ is a maximal Lindelöf T_2 space. By theorem 11.5, $(X', \underline{\underline{T}}')$ is Hausdorff. It has been proved

(Theorem 3.2) that a Hausdorff space is maximal Lindelöf if and only if it is a Lindelöf P-space. Since $(X', \underline{\underline{T}}')$ is already Lindelöf we are done if we can show that $(X', \underline{\underline{T}}')$ is a P-space. In view of the well-known fact that a space is a P-space iff every point is a P-point (a point is a P-point if countable intersections of nbhds of the point are again nbhds of that point, cf [GJ]) we are required to show that every point of X' is a P-point. Let $x \in X'$. Case (i) $x \neq p$. Then $x \in X$ and X is a P-space and every open set in X is also open in X'. So x is a P-point in X'. Case (ii) $x=p$. Let $\{G_n : n \in N\}$ be a countable collection of open nbhds of p. $X'-G_n = X-(G_n-\{p\})$ is a Lindelöf subspace of X (by definition) for each n. $X' - G = X' - \bigcap_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (X'-G_n)$ is then a Lindelöf subspace of X and hence closed. Consequently, by definition, G is an open nbhd of p in X'. p is then a P-point. (Q.E.D.)

It is well-known that every locally compact T_2 space has a unique (i.e., unique upto a homeomorphism) one-point Hausdorff compactification. The above statement can be reformulated as : every locally compact T_2 space has a unique one-point maximal compact Hausdorff extension. Similarly, it is quite a pertinent question to enquire whether a locally Lindelöf Hausdorff P-space has a unique one-point maximal Lindelöf extension. The answer is contained in the following proposition.

11.8 <u>Proposition</u> : Suppose $(X,\underline{\underline{T}})$ is a llh P-space. Let $(Y,\underline{\underline{S}})$ be a one-point maximal Lindelöf extension of $(X,\underline{\underline{T}})$. Then $(X',\underline{\underline{T}}')$ and $(Y,\underline{\underline{S}})$ are homeomorphic where $(X',\underline{\underline{T}}')$ is the one-point Lindelöf extension of $(X,\underline{\underline{T}})$.

<u>Proof</u>: Let $Y = X \cup \{r\}$. Define $f : X' \to Y$ as follows : $f(x) = x$ if $x \in X$ and $f(p) = r$. f is obviously a bijection. In order that f be continuous we should show that f is continuous at each point of X'. For each $x \in X$, it is trivial for f is identity on X. Let U be an open set containing r. Since $p \in f^{-1}(U)$, to show $f^{-1}(U)$ is open we shall prove that $X'-f^{-1}(U)$ is a Lindelöf subspace of X(then it will be automatically closed as X is a Hausdorff P-space). Now $X'-f^{-1}(U) = f^{-1}(Y-U) = Y-U$ as $Y-U$ is a subset of X and f is identity on X. Again $Y-U$ is a closed subset of Y and therefore Lindelöf. Thus $X'-f^{-1}(U)$ is a Lindelöf subspace of X. So f becomes a continuous bijection from the Lindelöf space $(X',\underline{\underline{T}}')$ onto the maximal Lindelöf space $(Y,\underline{\underline{S}})$. From theorem 3.1 it follows that f is a homeomorphism. (Q.E.D.)

<u>Remark</u>: The proof of Proposition 11.8 shows that we need not assume that the space Y is Hausdorff. The last proposition shows that any non-Lindelöf locally Lindelöf Hausdorff P-space X always admits an embedding (as a matter of fact, a dense imbedding) into a maximal Lindelöf space, viz, in its one-point

Lindelöf extension $(X',\underline{\underline{T}}')$. We continue our study of llh P-spaces in the next section.

12. *More on locally Lindelöf Hausdorff P-spaces*

We recall that in a locally compact T_2 space, at every point the compact nbhds form a base for the neighbourhood filter. Analogously we prove :

12.1 Proposition : In a locally Lindelöf Hausdorff P-space Lindelöf nbhds of a point form a base for the nbhd system at that point.

Proof: Let $(X,\underline{\underline{T}})$ be a llh P-space. Let $x \in X$ and U be any nbhd of x. Consider the one-point Lindelöf extension $(X',\underline{\underline{T}}')$ of $(X,\underline{\underline{T}})$. By Proposition 11.7, $(X',\underline{\underline{T}}')$ is a maximal Lindelöf Hausdorff space. So X' is regular. Now U is also a nbhd of x in $\underline{\underline{T}}'$. By regularity there exists $V \in \underline{\underline{T}}'$ such that $x \in V \subseteq$ closure of V in $X' \subseteq U$. Since $U \subseteq X$, $V \in \underline{\underline{T}}$ and closure of V in X' = closure of V in $X = \bar{V}$(say). So $x \in V \subseteq \bar{V} \subseteq U$. As $\bar{V}$ is a closed subset of X', it is also Lindelöf. Thus $\bar{V}$ is a Lindelöf nbhd of x contained in U. (Q.E.D.)

12.2 Corollary. Any open subset of a locally Lindelöf Hausdorff P-space is also locally Lindelöf.

12.3 Proposition: Suppose X is a dense locally Lindelöf subset of a Hausdorff P-space Y. Then X is open in Y. (Compare : any locally compact dense subspace of a Hausdorff space is open).

Proof: Suffices to show that each $x \in X$ has a nbhd contained in X. As X is locally Lindelöf x has a Lindelöf nbhd V contained in X. Since V is a Lindelöf subset of X it is so in Y and Y being a Hausdorff P-space V is closed in Y. Let U be a nbhd of x in Y such that $U \cap X = V$. Now $V = \text{cl } V$ in $Y = \text{cl}(U \cap X)$ in $Y = \text{cl}(U)$ in Y(as X is dense in Y). We conclude $V \subseteq U \subseteq \text{cl } U$ in $Y = V$. So $U = V$. X is then open in Y.

(Q.E.D.)

It is clear from proposition 12.1 that a maximal Lindelöf T_2 space becomes a llh P-space as soon as one point is removed. Conversely, proposition 11.7 has already established that any llh P-space can be transformed into a maximal Lindelöf Hausdorff space just by adjoining one point. In fact we can assert the following.

12.4 Proposition: Let $(X,\underline{\underline{T}})$ be a non-Lindelöf Hausdorff P-space. Then the following are equivalent :

(1) $(X,\underline{\underline{T}})$ is locally Lindelöf.

(2) X is open in some Hausdorff maximal Lindelöf extension of $(X, \underline{\underline{T}})$.

(3) X is open in every Hausdorff, maximal Lindelöf extension of $(X, \underline{\underline{T}})$.

Proof: (1) $\Rightarrow$ (2). Since X is non-Lindelöf, the one-point Lindelöf extension $(X',\underline{\underline{T}}')$ of $(X,\underline{\underline{T}})$ is a Hausdorff maximal

Lindelöf space containing X as an open dense subset, that is $(X', \underline{\underline{T}}')$ is a maximal Lindelöf extension in which X is open.

(2) ⇒ (1) Follows from corollary 12.2.

(1) ⇒ (3). X becomes a dense locally Lindelöf subspace in a Hausdorff maximal Lindelöf extension and is therefore open in virtue of proposition 12.3.

(3) ⇒ (1). Follows from corollary 12.2.

(Q.E.D.)

Finally, we shall establish that any non-Lindelöf llh P-space contains a strictly weaker maximal Lindelöf Hausdorff topology. To start with let us take a llh P-space $(X, \underline{\underline{T}})$ which is not already Lindelöf. Let $(X', \underline{\underline{T}}')$ be, as usual, the one-point Lindelöf extension of $(X, \underline{\underline{T}})$. Suppose $\underline{\underline{B}}(x)$ denotes the nbhd filter of x in $(X, \underline{\underline{T}})$. If $\underline{\underline{G}} = \{V \cap X : V \text{ a nbhd of } p \text{ in } X'\}$, $\underline{\underline{G}}$ is a filter on X. Fix $a \in X$. Let $\underline{\underline{S}}$ be the topology on X generated by the following neighbourhood filters :

$$\underline{\underline{B}}'(x) = \underline{\underline{B}}(x) \text{ if } x \neq a$$

$$\underline{\underline{B}}'(a) = \{N \cup G : N \in \underline{\underline{B}}(a),\ G \in \underline{\underline{G}}\}$$

(i) <u>$(X, \underline{\underline{S}})$ is a P-space</u> : Let us first observe that a space is a P-space if and only if every point is a P-point. If $x \neq a$, $\underline{\underline{B}}(x) = \underline{\underline{B}}'(x) \Rightarrow x$ is a P-point in the topology $\underline{\underline{S}}$. We have to show now that a is a P-point in $\underline{\underline{S}}$ i.e., every countable intersection of nbhds of a is again a nbhd of a. We look at $\bigcap_{i=1}^{\infty}(N_i \cup G_i)$ where $N_i \cup G_i \in \mathcal{B}'(a)$. Clearly, $\bigcap_{i=1}^{\infty}(N_i \cup G_i) \supset (\bigcap_{i=1}^{\infty} N_i) \cup (\bigcap_{i=1}^{\infty} G_i)$. By hypothesis $N = \bigcap_{i=1}^{\infty} N_i$ and $G = \bigcap_{i=1}^{\infty} G_i$ belong to $\mathcal{B}(a)$ and $\mathcal{G}$ respectively. Then $\bigcap_{i=1}^{\infty}(N_i \cup G_i) \supset N \cup G \in \mathcal{B}'(a)$ and hence $\bigcap_{i=1}^{\infty}(N_i \cup G_i) \in \mathcal{B}'(a)$.

(ii) <u>$\mathcal{S}$ is Hausdorff</u> : Let x and y be two distinct points in X : If both are different from a, they can be separated by means of $\mathcal{T}$-open sets and, hence, by $\mathcal{S}$-open sets. Let $x = a$. As $\mathcal{T}'$ and $\mathcal{T}$ are T_2 topologies we can get $y \in V \in \mathcal{T}$, $N \in \mathcal{B}(a)$ and $G \in \mathcal{G}$ such that $V \cap (N \cup G) = \emptyset$. But $V \in \mathcal{B}(y) = \mathcal{B}'(y)$ and $N \cup G \in \mathcal{B}'(a)$. So $\mathcal{S}$ is T_2.

(iii) <u>$(X,\mathcal{S})$ is Lindelöf</u> : Let $\mathcal{V}$ be an open cover of $(X,\mathcal{S})$. Since $\mathcal{S}$ is weaker than $\mathcal{T}$ and every $\mathcal{T}$-open set is also open in $\mathcal{T}'$, $\mathcal{V}$ is an open cover of X in $(X',\mathcal{T}')$. Let $V_o \in \mathcal{V}$ such that $a \in V_o$. Then by definition of $\mathcal{S}$, $V_o = N \cup G$ where $N \in \mathcal{B}(a)$ and $G \in \mathcal{G}$. From the definition of $\mathcal{G}$ it follows that $G \cup \{p\}$ is a nbhd of p for each $G \in \mathcal{G}$. So $V_o \cup \{p\}$ is an open nbhd of p in X'. Let $\mathcal{W} = \mathcal{V} - \{V_o\}$. Then $\mathcal{W} \cup \{V_o \cup \{p\}\}$ is an open cover of $(X',\mathcal{T}')$ which is Lindelöf and, consequently, it has a countable subcover. The countable subcover must contain the set $V_o \cup \{p\}$. Suppose $\{W_1, W_2, \ldots\}$ together with the set $V_o \cup \{p\}$ denotes the countable subcover. Then clearly $\{V_o, W_1, \ldots, W_n \ldots\}$ is a countable subfamily of $\underline{\underline{V}}$ which covers X. Thus $(X, \underline{\underline{S}})$ is Lindelöf.

(iv) <u>$\underline{\underline{S}}$ is strictly weaker than $\underline{\underline{T}}$</u> : We note that $(X, \underline{\underline{T}})$ is not Lindelöf $\Longleftrightarrow$ $\{p\}$ is not isolated in $(X', \underline{\underline{T}}')$ $\Longleftrightarrow$ $\underline{\underline{G}}$ is a filter on X. By definition we have $\underline{\underline{B}}'(a) \subseteq \underline{\underline{B}}(a)$. We shall show that $\underline{\underline{B}}'(a) \neq B(a)$ when $\underline{\underline{G}}$ is a filter on X thereby establishing that $\underline{\underline{S}}$ is strictly weaker than $\underline{\underline{T}}$. Suppose not i.e., $\underline{\underline{B}}(a) = \underline{\underline{B}}'(a)$. Note that $\underline{\underline{T}}'$ is T_2 i.e., a and p have disjoint open neighbourhoods in $(X', \underline{\underline{T}}')$ i.e., there exist N and V belonging to $\underline{\underline{T}}'$ such that $a \in N$, $p \in V$ and $N \cap V = \emptyset$. Since $N \subseteq X$, $N = N \cap X \in \underline{\underline{B}}(a)$ and $G = V \cap X \in \underline{\underline{G}}$. Now $N \in \underline{\underline{B}}'(a)$ by hypothesis. So, $N = N_1 \cup G_1$ where $N_1 \in \underline{\underline{B}}(a)$ and $G_1 \in \underline{\underline{G}}$. But $\emptyset = G \cap N = (G \cap N_1) \cup (G \cap G_1) \Rightarrow G \cap G_1 = \emptyset$. A contradiction as $\underline{\underline{G}}$ is a filter on X. We are then forced to conclude that $\underline{\underline{S}}$ is strictly weaker than $\underline{\underline{T}}$.

So far we have succeeded in showing that $(X, \underline{\underline{S}})$ is a Lindelöf Hausdorff P-space strictly contained in the non-Lindelöf llh P-space $(X, \underline{\underline{T}})$. Since a Lindelöf Hausdorff P-space is maximal Lindelöf (Theorem 3.2) we arrive at the following conclusion ;

12.6 Theorem : If $(X,\underline{\underline{T}})$ is a non-Lindelöf locally Lindelöf Hausdorff P-space, $\underline{\underline{T}}$ contains a strictly weaker maximal Lindelöf Hausdorff topology on X.

12.7 Examples of llh P-spaces. Any discrete topological space is an example of a llh P-space. Let us look at the following example :

Let Y be the set of reals with the topology in which every $x \neq 0$ is isolated and a typical nbhd of 0 is a cocountable set (i.e. complement of a countable set). Y is an example of a non-discrete P-space. It is, in fact, a Lindelöf Hausdorff P-space and hence maximal Lindelöf. We shall look at the space Y from another point of view. Let $X = Y - \{0\}$. If $\underline{\underline{T}}$ denotes the relative topology of X, then $\underline{\underline{T}}$ is discrete. According to a remark above $(X,\underline{\underline{T}})$ is a llh P-space. Let $(X', \underline{\underline{T}}')$ be the one-point Lindelöf extension of $(X,\underline{\underline{T}})$. As $(X,\underline{\underline{T}})$ is a llh P-space, $(X',\underline{\underline{T}}')$ is a Hausdorff maximal Lindelöf space. If $X' = X \cup \{p\}$, an open nbhd V of p in X' is such that $X'-V$ is a closed Lindelöf subset of X. But X is discrete, so $X'-V$ is a Lindelöf subset of X and hence must be countable. Thus cocountable subsets of X' containing p are the nbhds of p. Thus we can identify Y as the one-point Lindelöf extension of $(X,\underline{\underline{T}})$.

An easy observation based on theorem 12.6 is that a non-Lindelöf llh P-space does not admit any finer Lindelöf topology. Every Hausdorff maximal Lindelöf space being a llh P-space, an important implication of theorem 12.6 is that every non-Lindelöf llh P-space admits a strictly weaker llh P-space.

13. In this penultimate section we shall settle the following question ; ''Is a given Lindelöf topology always weaker than some maximal Lindelöf topology? ''

The answer to the above question is, in general, no. The fact that a maximal Lindelöf space must be a P-space (Theorem 3.2) will be exploited to produce an example corroborating our claim.

13.1 Example.

X = Real line

$\underline{\underline{T}}$ = Usual topology on X.

$(X,\underline{\underline{T}})$ is a Lindelöf T_2 space which is not maximal Lindelöf (because X is not a P-space in the usual topology). Suppose $\underline{\underline{S}}$ is a maximal Lindelöf topology on X finer than $\underline{\underline{T}}$. Let $x \in X$. Then $\{x\}$ is a G-delta in $\underline{\underline{T}}$ and so in $\underline{\underline{S}}$ also. As $(X, \underline{\underline{S}})$ is a P-space $\{x\}$ must be open in $\underline{\underline{S}}$ i.e., $\underline{\underline{S}}$ is discrete. Then $(X, \underline{\underline{S}})$ cannot be Lindelöf. A contradiction. We, therefore, conclude that the <u>usual topology on the reals is not weaker than any maximal Lindelöf topology on the real numbers</u>. Of course, any topology on a countable set is Lindelöf and is weaker than the discrete topology which is, obviously, maximal Lindelöf.

Remark. In the above question if 'Lindelöf' is replaced by 'H-closed' the answer is, always, yes as theorem 5.3 shows. For the other properties π considered in this chapter we do not know definite answers.

14. Historical Notes

In 1947 Vaidyanathaswamy [Va] showed that every compact T_2 space is maximal compact. Maximal compact spaces have been characterised in an excellent paper of N. Smythe and C. A. Wilkins [SW] published in 1963. These characterisations have been incorporated in theorem 2.1 and were chiefly instrumental in our obtaining similar characterisations for maximal Lindelöf and maximal countably compact spaces that appear in sections 3 and 4. Necessary and sufficient conditions for an H-closed space to become maximal H-closed have been obtained by us in 1970. During the preparation of this thesis and after we had obtained these results we came to know that these results had been independently obtained by Mioduszewski and Rudolf [MR] in 1969. They are included as theorems 5.3 and 5.4. J.P.Thomas [Th] has studied maximal connected spaces (1968) and has shown that a maximal connected space is T_0 (Cf. the Remark following lemma 6.4), a connected open subset is maximal connected and other related results. He also showed that maximal connectedness is not, in general, productive by means of an example. Our example is just a variant of his. He has raised the question about the existence of maximal connected Hausdorff spaces. The contents of Sections 9,10,11 and 13 are inspired by some interesting questions raised by Prof. M.P.Berri through a personal communication to the author. As a concluding remark it may be mentioned that this chapter is the result of further addition to and elaboration of the contents of the paper of Raha [R2].

CHAPTER II

MINIMAL TOPOLOGIES

1. Summary

Given a topological property π a space $(X,\underline{\underline{T}})$ is called minimal π provided $(X,\underline{\underline{T}})$ is a π-space and there is no π topology defined on X which is properly contained in $\underline{\underline{T}}$ (i.e., strictly weaker than $\underline{\underline{T}}$). Closely associated with minimal π spaces are π-closed (or π-complete) and Katětov π spaces. A π space $(X,\underline{\underline{T}})$ is π-closed if X is a closed subset in every π space into which X can be homeomorphically imbedded. A π space $(X,\underline{\underline{T}})$ is said to be Katětov π in case $\underline{\underline{T}}$ is stronger than some minimal π topology on X. We devote this chapter to investigations of minimal π and other related spaces, stated above, for π = realcompact, first countable realcompact, locally H-closed, P-space, locally Lindelöf Hausdorff P-space, E_1, analytic or borelian. Necessary definitions follow.

2. Definitions

2.1 A Hausdorff space is locally H-closed if every point has a neighbourhood which is H-closed (H-closed spaces have been defined in Chapter I).

2.2 $(Y,\underline{\underline{S}})$ is a one-point H-closed extension of $(X,\underline{\underline{T}})$ if X is a subset of Y, $\underline{\underline{T}} = \underline{\underline{S}}|X$, $(Y,\underline{\underline{S}})$ is H-closed, Y-X is a singleton and X is dense in Y.

2.3 A space X is called an $\underline{E_o \text{ space}}$ if every singleton point of X is a G-delta.

2.4 Call a completely regular Hausdorff space (i.e. a Tychonoff) space <u>analytic</u> if X is an USCO-compact image of irrationals Σ (i.e., there exists a correspondence f defined on Σ onto X satisfying (i) for $\sigma \varepsilon \Sigma$, $f(\sigma)$ is a compact subset of X and $X = \bigcup \{f(\sigma) : \sigma \varepsilon \Sigma\}$, (ii) For every non-void open subset U of X, $\{\sigma : f(\sigma) \subseteq U\}$ is open in Σ).

2.5 By a completely regular filter base $\underline{\underline{B}}$ in a space X we mean an open filter base such that for each $C \varepsilon \underline{\underline{B}}$ there exist $D \varepsilon \underline{\underline{B}}$ and a continuous function f from X into [0,1] such that $D \subseteq C$, $f = 0$ on D and $f = 1$ on X-C. For such a filter base, $\bigcap \underline{\underline{B}} = \bigcap \{C : C \varepsilon \underline{\underline{B}}\} = \bigcap \{\bar{C} : C \varepsilon \underline{\underline{B}}\}$ where $\bar{C}$ denotes the closure of C in X. A completely regular filter base $\underline{\underline{B}}$ is fixed (free) if $\bigcap \underline{\underline{B}} \neq \emptyset (\bigcap \underline{\underline{B}} = \emptyset)$.

3. Realcompact spaces

In this section our main result is that both minimal realcompact and minimal first countable realcompact spaces are compact. Although either of spaces under consideration enjoys the same property, separate proofs are needed for their establishment. The following results will play important roles in the proofs of our theorems.

3.1 A completely regular Hausdorff space X is realcompact if and only if for every $y \in \beta X - X$ there exists a continuous function $h : \beta X \to [0,1]$ such that $f(y) = 0$ and $f > 0$ on X, where βX is the Stone-Čech compactification of X. (Cf. Engelking [E])

3.2 A realcompact space is compact if and only if it is pseudocompact. (Cf. Engelking [E])

3.3 A space Y is pseudocompact if and only if every countable completely regular filter base on Y is fixed. (Stephenson [Ste 2, page 438])

Let us recall that by a first countable realcompact space we mean a realcompact space which is, at the same time, first countable. Now we are in a position to state our theorems. These theorems are to be found in Raha [R1].

3.4 <u>Theorem</u> : Let $(X,\underline{T})$ be a topological space. The following are equivalent :

(i) X is minimal realcompact.

(ii) X is realcompact-closed.

(iii) X is compact Hausdorff.

<u>Proof</u>: (i) $\Rightarrow$ (ii) : We assume that $(X,\underline{T})$ is minimal realcompact. In order to show that $(X,\underline{T})$ is realcompact-closed let Y be a realcompact space containing X as a subset. We wish to show that X is closed in Y. Let $q \in$ closure of X in Y. Let

$\underline{\underline{N}}$ denote the open filter of open neighbourhoods in Y of q. Then $\underline{\underline{C}} = \underline{\underline{N}}|X = \{N \cap X : N \in \underline{\underline{N}}\}$ is a completely regular filter on X. Suppose $q \notin X$ i.e., X is not closed in Y, then $\bigcap \underline{\underline{C}} = \emptyset$ i.e., $\underline{\underline{C}}$ is free. Fix $x_o \in X$. Consider the topology $\underline{\underline{T}}'$ on X generated by the following neighbourhoods :

$$\underline{\underline{B}}'(x) = \underline{\underline{B}}(x) \quad \text{if} \quad x \neq x_o$$

$$\underline{\underline{B}}'(x_o) = \{U \subseteq X : U \supseteq V \cup C \text{ for some } V \in \underline{\underline{B}}(x_o) \text{ and } C \in \underline{\underline{C}}\}$$

where $\underline{\underline{B}}(x)$ stands for the neighbourhood filter of x in $(X,\underline{\underline{T}})$ for each $x \in X$.

Before we proceed further let us introduce some notations to be followed throughout this proof. F(X) = The space of all cont. functions on $(X,\underline{\underline{T}}) \to [0,1]$. βX denote the Stone-Čech compactification of $(X,\underline{\underline{T}})$ and $F(\beta X)$ = The space of all cont. functions from $\beta X \to [0,1]$.

(a) $\underline{\underline{T}}' \subseteq \underline{\underline{T}}$ but $\underline{\underline{T}}' \neq \underline{\underline{T}}$: Obviously $\underline{\underline{B}}'(x_o) \subseteq \underline{\underline{B}}(x_o)$. So suffices to show that $\underline{\underline{B}}'(x_o) \neq \underline{\underline{B}}(x_o)$. We have $\emptyset = \bigcap \underline{\underline{C}} = \bigcap \{\bar{C} : C \in \underline{\underline{C}}\}$. So there exists $C \in \underline{\underline{C}}$ such that $x_o \notin \bar{C}$ i.e., there exists $V_o \in \underline{\underline{B}}(x_o)$ such that $V_o \cap C = \emptyset$ (here $\bar{C}$ = closure of C in $\underline{\underline{T}}$). If $\underline{\underline{B}}'(x_o) = \underline{\underline{B}}(x_o)$, there exist $V \in \underline{\underline{B}}(x_o)$ and $D \in \underline{\underline{C}}$ such that $V \cup D \subseteq V_o$. Then $D \subseteq V_o$ and $V_o \cap C = \emptyset$, which implies that $D \cap C = \emptyset$. A contradiction. Thus $\underline{\underline{B}}'(x_o) \neq \underline{\underline{B}}(x_o)$.

(b) $\underline{\underline{T}}'$ is Hausdorff : Suffices to separate a point $x \neq x_o$ from x_o by means of $\underline{\underline{T}}'$-open sets. As $\underline{\underline{T}}$ is Hausdorff there exist $U \varepsilon \underline{\underline{B}}(x)$ and $V \varepsilon \underline{\underline{B}}(x_o)$ such that $U \cap V = \emptyset$. Now $x \notin \cap \underline{\underline{C}} \Rightarrow$ there exist $U' \varepsilon \underline{\underline{B}}(x)$ and $C \varepsilon \underline{\underline{C}}$ such that $U' \cap C = \emptyset$. Let $W = U \cap U'$ and then $W \cap (V \cup C) = \emptyset$. But $W \varepsilon \underline{\underline{B}}(x) = \underline{\underline{B}}'(x)$ and $V \cup C \varepsilon \underline{\underline{B}}'(x_o)$.

(c) $\underline{\underline{T}}'$ is completely regular : Let $x \varepsilon X$ and U an open neighbourhood (to be abbreviated as nbhd.) of x in $\underline{\underline{T}}'$. Case (i) $x = x_o$. The U can be taken to be of the form $V \cup C$ for some $V \varepsilon \underline{\underline{B}}(x_o)$ and $C \varepsilon \underline{\underline{C}}$. By complete regularity of $\underline{\underline{T}}$ and definition of $\underline{\underline{C}}$ there exist $f, g \varepsilon F(X)$ and $D \varepsilon \underline{\underline{C}}$ with $D \subset C$ such that $f(x_o) = 0$, $f(X-V) = 1$; $g(D) = 0$ and $g(X-C) = 1$. Set $h = \min(f,g)$. Then $h(x_o) = 0$ and $h = 1$ on $X - (V \cup C) = X-U$. Obviously $h \varepsilon F(X)$. If we put $F'(X)$ = The space of all continuous functions on $(X,\underline{\underline{T}}') \to [0,1]$, h belongs, in fact, to $F'(X)$. Case (ii) $x \neq x_o$. By Hausdorff property of $\underline{\underline{T}}'$ we can get $U' \varepsilon \underline{\underline{B}}'(x)$ and $V \cup C \varepsilon \underline{\underline{B}}'(x_o)$ such that $U' \cap (V \cup C) = \emptyset$. We can, without loss of generality, assume that $U = U'$. Now, by complete regularity of $\underline{\underline{T}}$ we can find $f \varepsilon F(X)$ such that $f(x) = 0$ and $f = 1$ on $X-U$ which incidentally implies that $f(V \cup C) = 1$. So $f \varepsilon F'(X)$.

(d) $(X,\underline{\underline{T}}')$ is realcompact: First, let $D_o'(X)$ = The set of all zero-sets of $(X,\underline{\underline{T}}')$. Suppose $\underline{\underline{G}} \subset D_o'(X)$ be a maximal subfamily with respect to countable intersection property. To show that $\bigcap \underline{\underline{G}} = \bigcap \{G : G \in \underline{\underline{G}}\} \neq \emptyset$. Suppose not i.e., $\bigcap \underline{\underline{G}} = \emptyset$. We set $A = \{g \in F'(X) : g^{-1}\{0\} \in \underline{\underline{G}}\}$ (here we note that there can exist more than one function in A corresponding to one $G \in \underline{\underline{G}}$). Let $A' = \{g' \in F(\beta X) : g \in A\}$. (Note that $A \subseteq F'(X) \subseteq F(X)$ and thus every $g \in A$ has a unique continuous extension g' to βX). Clearly there is a 1-1 correspondence between A and A'. Let $\bar{G}$ stand for closure of G in βX for each $G \in \underline{\underline{G}}$. Now $G \in \underline{\underline{G}} \Rightarrow G = g^{-1}\{0\}$ for some $g \in A$ and also $\bar{G} \subseteq g'^{-1}\{0\}$ where $g' \in A'$ corresponds to $g \in A$. Again $G = X \cap \bar{G} = X \cap g'^{-1}\{0\}$ so that $\bigcap \underline{\underline{G}} = \bigcap(\bar{G} \cap X) = \bigcap \{X \cap g'^{-1}\{0\} : g' \in A'\}$ i.e., $\bigcap \underline{\underline{G}} = X \cap (\bigcap \bar{G}) = X \cap (\bigcap \{g'^{-1}\{0\} : g' \in A'\})$. By hypothesis $\bigcap \underline{\underline{G}} = \emptyset$. So $\bigcap \{g'^{-1}\{0\} : g' \in A'\}$ is disjoint from X and $\bigcap \bar{G} \subseteq \bigcap \{g'^{-1}\{0\} : g' \in A'\}$. (Here $\bigcap \bar{G} = \bigcap \{\bar{G} : G \in \underline{\underline{G}}\}$). As $\underline{\underline{G}}$ has countable intersection property, the set $\{\bar{G} : G \in \underline{\underline{G}}\}$ will have countable intersection property and, a fortiori, finite intersection property. By compactness of βX we have $\bigcap \bar{G} \neq \emptyset$. Since $\bigcap \bar{G} \subseteq \bigcap \{g'^{-1}\{0\} : g' \in A'\} \subseteq \beta X - X$, $\bigcap \bar{G}$ is a nonempty subset of $\beta X - X$. Let us fix $y \in \bigcap \bar{G}$. Since $x_0 \in X$, $x_0 \notin \bigcap \{g'^{-1}\{0\} : g' \in A'\}$ and, hence, there exists $g_1' \in A'$ with property that $g_1'(x_0) > 0$. Put $\alpha = g_1'(x_0)$. Clearly $g_1'(y) = 0$. We can get a continuous function $f : [0,1] \to [0,1]$ such $f(0) = 0$ and $f = 1$ on $[\alpha/2, 1]$. Then

$h = f \circ g_1' \varepsilon F(\beta X)$ such that $h(x_0) = 1$ and $h(y) = 0$. Then $f \circ g_1$ becomes the restriction of h to X and belongs to $F'(X)$ as $g_1 \varepsilon F'(X)$. On the other hand X is realcompact in the topology $\underline{\underline{T}}$ and $y \varepsilon \beta X - X$. So there exists $p \varepsilon F(\beta X)$ such that $p(y) = 0$ and $p > 0$ on X (By 3.1). Let $r = \max(p,h)$ $\varepsilon F(\beta X)$. Then $r > 0$ on X, $r(y) = 0$ and $r(x_0) = 1$. Let $u = r$ restricted to X. Then $u \varepsilon F(X)$. Note that $x_0 \varepsilon u^{-1}\{1\}$. Now $X \cap r^{-1}\{1\} \supseteq X \cap h^{-1}\{1\} = X \cap g_1'^{-1}[f^{-1}\{1\}] \supseteq g_1'^{-1}(\alpha/2,1] \cap X = g_1^{-1}(\alpha/2,1]$ = a $\underline{\underline{T}}'$-open nbhd of x_0 because $g_1 \varepsilon F'(x)$. So $u \varepsilon F'(X)$ and has the properties that $u(x_0) = 1$ and $u > 0$ on X. Now $r(y) = 0$ and so $r^{-1}[0,1/n)$ is an open nbhd of y in βX for each $n \geq 1$. Again $y \varepsilon \bigcap \bar{G} \Rightarrow y \varepsilon \bar{G}$ for all $G \varepsilon \underline{\underline{G}} \Rightarrow r^{-1}[0, 1/n) \cap G \neq \emptyset$ for all $G \varepsilon \underline{\underline{G}}$ and $n = 1,2,\ldots$. This implies $r^{-1}[0,1/n]$ intersects every $G \varepsilon \underline{\underline{G}}$, for each n. Thus $\emptyset \neq r^{-1}[0,1/n] \cap G = u^{-1}[0,1/n] \cap G$ (since $G \subseteq X$) $= Z_n \cap G$ (say) for each $G \varepsilon \underline{\underline{G}}$, $n = 1,2,\ldots$. As $u \varepsilon F'(X)$, $Z_n \varepsilon D_0'(X)$ for $n = 1,2,\ldots$. Now $\underline{\underline{G}}$ is a maximal subfamily of $D_0'(X)$ with respect to countable intersection property and $Z_n \cap G \neq \emptyset$ for all $G \varepsilon \underline{\underline{G}}$ and $n = 1,2,\ldots$, We, then, must conclude that $Z_n \varepsilon \underline{\underline{G}}$ for $n = 1,2,\ldots$ and, as a result, $\emptyset \neq \bigcap_{i=1}^{\infty} Z_i = \bigcap_{n=1}^{\infty} u^{-1}[0,1/n] = u^{-1}\{0\} = \emptyset$ as $u > 0$ on X. A contradiction. $(X, \underline{\underline{T}}')$ is thus realcompact. But $\underline{\underline{T}}'$ is strictly weaker than $\underline{\underline{T}}$ and is just shown to be a realcompact topology on X. This is a contradiction to the hypothesis that $(X,\underline{\underline{T}})$ is minimal realcompact. As a result X is closed in Y.

(ii) ⟹ (iii): We assume that $(X,\underline{\underline{T}})$ is realcompact-closed. We are to show the compactness of X. Consider βX, the Stone-Čech compactification of $(X,\underline{\underline{T}})$. Now X is a dense subset of βX which is trivially realcompact. As X is realcompact-closed X is also a closed subset of βX. So $X = \beta X$ i.e., X is a compact T_2 space.

(iii) ⟹ (i) : If $(X,\underline{\underline{T}})$ is a compact T_2 space, it is realcompact. Now the well-known fact that a compact Hausdorff space is minimal Hausdorff clearly implies the minimal realcompactness of $(X,\underline{\underline{T}})$.

(Q.E.D.)

Let us recall that a first countable realcompact space $(X,\underline{\underline{T}})$ is referred to as a <u>minimal first countable realcompact space</u> if any topology on X which is weaker than $\underline{\underline{T}}$ and is first countable realcompact necessarily coincides with $\underline{\underline{T}}$, and X is called <u>first countable realcompact-closed</u> if it becomes a closed subset in every embedding of it into some realcompact and first countable space.

3.5 The following statements are equivalent for a topological space :

(i) X is minimal first countable realcompact.

(ii) X is first countable realcompact-closed.

(iii) X is first countable compact Hausdorff.

Proof: (i) ⇒ (ii) : We assume that $(X,\underline{\underline{T}})$ is minimal first countable realcompact. To show that $(X,\underline{\underline{T}})$ is first countable realcompact-closed let Y be a first countable realcompact space containing X as a subset. We intend to show that X is closed in Y. Let $q \in$ closure of X in Y. Let $\underline{\underline{N}}$ denote a countable fundamental system of open nbhds of q in Y. Then $\underline{\underline{C}} = \underline{\underline{N}}|X = \{N \cap X : N \in \underline{\underline{N}}\}$ is a countable completely regular filter base on X. Suppose $q \notin X$ i.e., X is not closed in Y, then $\bigcap \underline{\underline{C}} = \emptyset$ i.e., $\underline{\underline{C}}$ is free. Let $\underline{\underline{C}} = \{C_n : n \geq 1\}$. Fix x_o in X. Consider the topology $\underline{\underline{T}}'$ on X generated by the following neighbourhoods :

$$\underline{\underline{B}}'(x) = \underline{\underline{B}}(x) \text{ if } x \neq x_o$$

$$\underline{\underline{B}}'(x_o) = \text{filter generated by } \{V \cup C_n : n \geq 1, V \in \underline{\underline{B}}(x_o)\}$$

Arguments very similar to those which appear in the proof of theorem 3.4 obtain that $(X,\underline{\underline{T}}')$ is a first countable completely regular Hausdorff space, and $\underline{\underline{T}}'$ is strictly weaker than $\underline{\underline{T}}$ (cf. [Ste 3, lemma 2.8]).

$(X,\underline{\underline{T}}')$ is realcompact : First, let $D_o'(X) = \{f^{-1}\{0\} : f \in F'(X)\}$ = the set of all zero-sets of $(X,\underline{\underline{T}}')$; where $F'(X)$ = the space of all continuous functions on $(X, \underline{\underline{T}}') \to [0,1]$. Let us consider a z-ultrafilter $\underline{\underline{G}}'$ on $(X, \underline{\underline{T}}')$ which has the countable intersection property (i.e., a maximal subfamily $\underline{\underline{G}}'$ of $D_o'(X)$ with respect to countable intersection property). We eish to prove that $\underline{\underline{G}}'$ is fixed (i.e., $\bigcap \underline{\underline{G}}' \neq \emptyset$). If $x_o \in \bigcap \underline{\underline{G}}'$ there is nothing to prove. Let us suppose then that there exists a set $G' \in \underline{\underline{G}}'$ with $x_o \notin G'$. Since $\underline{\underline{T}}'$ is weaker than $\underline{\underline{T}}$, if follows from Zorn's lemma that there is a z-ultrafilter $\underline{\underline{G}}$ on $(X, \underline{\underline{T}})$ such that $\underline{\underline{G}}' \subseteq \underline{\underline{G}}$. If $\underline{\underline{G}}$ has the countable intersection property, then, by the realcompactness of $(X, \underline{\underline{T}})$, $\underline{\underline{G}}$ and, a fortiori, $\underline{\underline{G}}'$ are fixed. So, the proof will be complete as soon as we show that $\underline{\underline{G}}$ has the countable intersection property.

Let $\{Z_i : i \geq 1\} \subseteq \underline{\underline{G}}$. We may assume that $Z_i = Z(f_i) = f_i^{-1}\{0\}$ where $f_i \in F(X)$. Also, let $G' = g^{-1}\{0\} = Z(g)$ for some $g \in F'(X)$. Since $x_o \notin G'$, $g(x_o) > 0$. Put $\alpha = g(x_o)$. There exists a continuous function $r : [0,1] \to [0,1]$ such that $r^{-1}\{0\} = \{0\}$ and $r = 1$ on $[\alpha/2, 1]$. If $f = r \circ g$, then $f \in F'(X)$, f equals 1 on some $\underline{\underline{T}}'$-nbhd of x_o and $G' = Z(f) \in D_o'(X)$. We set $h_i = \max(f, f_i)$, then $h_i \in F'(X)$ and $h_i^{-1}\{0\} = Z(h_i) = Z(f) \cap Z(f_i) = G' \cap Z(f_i) = G' \cap Z_i$. So each $G' \cap Z_i$, being equal to $Z(h_i)$, belongs to $D_o'(X)$. Now for each $i \geq 1$, $G' \cap Z_i$ has non-empty intersection with every member of the z-ultrafilter $\underline{\underline{G}}'$ and, hence $G' \cap Z_i \in \underline{\underline{G}}'$.

Inasmuch as $\underline{\underline{G}}'$ has the countable intersection property $\bigcap_{i=1}^{\infty} (G' \cap Z_i) = G' \cap (\bigcap_{i=1}^{\infty} Z_i) \in \underline{\underline{G}}' \subset \underline{\underline{G}}$. Since $\underline{\underline{G}}$ is a z-ultrafilter $\bigcap_{i=1}^{\infty} Z_i \in \underline{\underline{G}}$. Thus the realcompactness of $(X, \underline{\underline{T}}')$ has been established.

Now, we see that $(X, \underline{\underline{T}}')$ is a strictly weaker first countable realcompact space than $(X, \underline{\underline{T}})$. This contradicts the hypothesis about $(X, \underline{\underline{T}})$. We are then forced to conclude that $q \in X$ i.e., X is closed in Y.

(ii) $\Rightarrow$ (iii) : We assume that $(X, \underline{\underline{T}})$ is first countable realcompact-closed. We have to show that $(X, \underline{\underline{T}})$ is pseudocompact in order to establish the compactness of X (3.2). In view of 3.3 we only need to show that every countable completely regular filter base is fixed. Let $\underline{\underline{C}} = \{C_m : m \geq 1\}$ be a countable completely regular filter base. Suppose it is free i.e., $\bigcap \underline{\underline{C}} = \bigcap \{C_m : m \geq 1\} = \emptyset$. Take $p \notin X$, let $Y = X \cup \{p\}$. If $\underline{\underline{T}}$ denotes the topology of X we define a topology $\underline{\underline{S}}$ on Y as follows :

$$\underline{\underline{S}} = \underline{\underline{T}} \cup \{V \subset Y : V \supset \{p\} \cup C_m \text{ for some } m \geq 1\}.$$

<u>Claim</u> : Y is a first countable realcompact space in which X is imbedded as a proper dense subset.

<u>Proof of the claim</u> : It is fairly easy to see that Y is a first countable Tychonoff space in which X is dense. Let us note that $\{p\}$ is a compact subset of Y and X is a realcompact subspace of Y. Invoking the following theorem of Gillman and Jerison [G, J, page 121] the realcompactness of Y follows :

In any Tychonoff space, the union of a compact set with a realcompact set is realcompact.

The claim is thus justified. But this contradicts the first countable realcompact-closedness of $(X, \underline{T})$. Therefore we conclude that $\underline{C}$ is fixed.

(iii) $\Rightarrow$ (i) : The proof of (iii) $\Rightarrow$ (i) in theorem 3.4 is applicable mutatis mutandis.

(Q.E.D.)

It is well-known that a compact Hausdorff space is of the second category. The following corollary is immediate from theorems 3.4 and 3.5.

3.6 <u>Corollary</u> : Any minimal (first countable) realcompact space is of the second category.

Since for π = realcompact or first countable realcompact, minimal π and π-closed spaces are identical, any π-closed space is, trivially, Katětov π. That this is not the case for general π spaces is the contention of our next example.

3.7 Example

Let Q denote the set of rational numbers with usual topology. Q is a realcompact space; in fact it is a first countable realcompact space. Q is not Katětov first countable realcompact as Q cannot support any compact Hausdorff topology weaker than the usual topology (Baire category theorem implies that any countable, compact Hausdorff space has isolated points).

4. Locally H-closed spaces

Parallel to the concept of locally compact Hausdorff spaces, locally H-closed spaces were introduced. The natural analogue of one-point compactification has also been deduced, the nomenclature being one-point H-closed extension of a locally H-closed space. Before we mention the results on minimal locally H-closed spaces we quote below a theorem, due to Obreanu [O], regarding one-point H-closed extensions from Porter [Po2, theorem 2.2, p.195]. This will be needed in the sequel.

4.1 Theorem : Let $(X, \underline{\underline{T}})$ be a locally H-closed space which is not H-closed and let $X' = X \cup \{p\}$ where p is not a point of X.

(a) $\underline{\underline{T}}' = \underline{\underline{T}} \cup \{\{p\} \cup V : V \in \underline{\underline{G}}\}$ is a topology on X' where $\underline{\underline{G}}$ is the open filter which is the intersection of all free open filters in $(X,\underline{\underline{T}})$.

(b) $(X',\underline{\underline{T}}')$ is a one-point H-closed extension of $(X,\underline{\underline{T}})$.

(c) $\underline{\underline{G}}$ is the open filter generated by $\{V \in \underline{\underline{T}} ; X-V$ is H-closed$\}$.

M.P.Berri ([Bel], theorem 5.2, p.104) has shown that if $(X,\underline{\underline{T}})$ is a locally compact non-compact Hausdorff space, $\underline{\underline{T}}$ contains a strictly weaker compact Hausdorff topology on X. Motivated by this result we prove the locally H-closed analogue of it in the following lemma.

4.2 <u>Lemma</u> : If $(X,\underline{\underline{T}})$ is a non H-closed locally H-closed space, there exists an H-closed topology $\underline{\underline{S}}$ on X, strictly weaker than $\underline{\underline{T}}$.

<u>Proof</u>: By theorem 4.1 there exists a one-point H-closed extension $(X',\underline{\underline{T}}')$ of $(X,\underline{\underline{T}})$. Obviously X is an open dense subset of X'. If $\underline{\underline{B}}(p)$ denotes the nbhd filter of p in X', $\underline{\underline{G}} \subseteq \underline{\underline{B}}(p) \cap X$ and $\underline{\underline{G}}$ is an open filter on X. Obviously $\underline{\underline{G}}$ has no adherent point in X. For any $x \in X$, let $\underline{\underline{N}}(x)$ denote the filter of $\underline{\underline{T}}$-nbhds of x. Fix a point $a \in X$. We shall define the topology $\underline{\underline{S}}$ on X by the following neighbourhood systems :

$$\underline{\underline{N}}'(x) = \underline{\underline{N}}(x) \quad \text{if} \quad x \neq a$$

$$\underline{\underline{N}}'(a) = \{V \cup F : V \in \underline{\underline{N}}(a), \quad F \in \underline{\underline{B}}(p) \cap X\}$$

(i) $\underline{\underline{S}}$ is Hausdorff : Suffices to separate a point $x \neq a$ from a by means of $\underline{\underline{S}}$-open sets. Since $(X', \underline{\underline{T}}')$ is Hausdorff it is possible to find $U \in \underline{\underline{N}}(x)$, $V \in \underline{\underline{N}}(a)$ and $G \in \underline{\underline{G}}$ such that $U \cap (V \cup G) = \emptyset$. Now $U \in \underline{\underline{N}}'(x)$ and $V \cup G \in \underline{\underline{N}}'(a)$ as $G \in \underline{\underline{G}} \Rightarrow G \in \underline{\underline{B}}(p) \cap X$.

(ii) $\underline{\underline{S}}$ is strictly weaker than $\underline{\underline{T}}$: By definition $\underline{\underline{S}}$ is weaker than $\underline{\underline{T}}$ as $\underline{\underline{N}}'(x) \subseteq \underline{\underline{N}}(x)$ for each $x \in X$. We shall show that $\underline{\underline{N}}'(a) \neq \underline{\underline{N}}(a)$ which proves that $\underline{\underline{S}} \neq \underline{\underline{T}}$. If not, let $\underline{\underline{N}}'(a) = \underline{\underline{N}}(a)$. Now $a \neq p$ and $(X', \underline{\underline{T}}')$ is T_2, so we can find $\underline{\underline{T}}'$-open sets V and W such that $a \in V$ and $p \in W$ and $V \cap W = \emptyset$. Since $V \cap W = \emptyset$, V, in fact, belongs to $\underline{\underline{N}}(a)$. Again $W \in \underline{\underline{B}}(p)$ and $W \in \underline{\underline{T}}'$. So $W \cap X \in \underline{\underline{G}}$. Put $G = W \cap X$. Since $V \in \underline{\underline{N}}(a) = \underline{\underline{N}}'(a)$, we, definitely, can find $V_1 \in \underline{\underline{N}}(a)$, $F \in \underline{\underline{B}}(p) \cap X$ such that $V = V_1 \cup F$. But this will mean $F \cap G = \emptyset$ which is false as they belong to $\underline{\underline{B}}(p) \cap X$. Thus $\underline{\underline{N}}'(a) \neq \underline{\underline{N}}(a)$.

(iii) $(X, \underline{\underline{S}})$ is H-closed : Let $\underline{\underline{F}}$ be any open filter base on $(X, \underline{\underline{S}})$. Suffices to show that $\underline{\underline{F}}$ has an adherent point. If $\underline{\underline{F}}$ has an adherent point in $(X, \underline{\underline{T}})$, $\underline{\underline{F}}$ also has an adherent point in $(X, \underline{\underline{S}})$ since $\underline{\underline{S}}$ is weaker than $\underline{\underline{T}}$. Suppose $\underline{\underline{F}}$ has no adherent point in $(X, \underline{\underline{S}})$. Obviously, $\underline{\underline{F}}$ cannot have any in $(X, \underline{\underline{T}})$ then. Since any $\underline{\underline{T}}$-open set is also $\underline{\underline{T}}'$-open set, $\underline{\underline{F}}$ is an open filter base on $(X', \underline{\underline{T}}')$ and the latter being

H-closed $\underline{\underline{F}}$ will have non-empty adherence in $\underline{\underline{T}}'$. Then by hypothesis, p will be an adherent point of $\underline{\underline{F}}$ in $(X', \underline{\underline{T}}')$. This means that every member of $\underline{\underline{B}}(p)$ intersects every member of $\underline{\underline{F}}$, which, in view of the fact that every member of $\underline{\underline{F}}$ is a subset of X, implies that for each $F \varepsilon \underline{\underline{F}}$ and $G \varepsilon \underline{\underline{B}}(p) \cap X$, $F \cap G \neq \emptyset$. As a result, every $\underline{\underline{S}}$-nbhd of a intersects F for every $F \varepsilon \underline{\underline{F}}$ i.e., a is an adherent point of $\underline{\underline{F}}$ in $(X, \underline{\underline{S}})$ A contradiction. So $(X, \underline{\underline{S}})$ is H-closed.

(Q.E.D.)

From the definition it is evident that an H-closed space is always locally H-closed. Lemma 4.2, then, implies that any non H-closed locally H-closed space always admits of a strictly weaker locally H-closed topology. It has been mentioned in Chapter I (Section 5) that any H-closed topology contains a minimal Hausdorff topology (i.e., an H-closed space is Katětov Hausdorff) and further a minimal Hausdorff space is H-closed [Bo2]. We can immediately conclude the following from the previous lines :

(i) A minimal locally H-closed space is H-closed.

(ii) A minimal Hausdorff space is minimal locally H-closed.

These observations help us to arrive at our main theorem :

4.3 <u>Theorem</u> : A space is minimal locally H-closed iff it is minimal H-closed iff it is minimal Hausdorff.

Let us recall that a minimal T_2 space is semiregular. In virtue of the theorem 4.3 we can state : Every minimal loçally H-closed space is semiregular. Let us observe that if X is any non H-closed locally H-closed space, theorem 4.1 asserts that X has a one-point H-closed extension X'. We see that X is an open dense subset of X' which is H-closed. Here, X is trivially embedded into an H-closed space, viz. X', as a dense open subset. We recall that a locally H-closed space is called locally H-closed-complete if it is a closed subspace in any locally H-closed space in which it is embedded. By definition X cannot then be locally H-closed-complete. Obviously we infer the following :

4.4 Proposition : Every locally H-closed-complete space is H-closed.

An important conclusion from lemma 4.2 and theorem 4.3 is that every locally H-closed space contains a minimal locally H-closed topology i.e., a locally H-closed space is Katetov locally H-closed. In this context it may be mentioned that J.R.Porter [Po1] has established the following in his dissertation.

4.5 Proposition : A Hausdorff space which is locally H-closed except at most one point is Katětov Hausdorff.

Remarks: Lemma 4.2, the key result of this section, is slightly generalised by proposition 4.5. It was not possible to contact the aforesaid dissertation in original. We have been apprised of the proposition 4.5 by Porter through a personal communication.

5. E_1 spaces

Let us recall that a topological space is an E_1-space if every point is the intersection of a countable number of closed nbhds. Clearly every E_1 space is Hausdorff. In fact, a first countable T_2 space is E_1 as is shown by the following lemma, though there exist regular E_1 spaces which are not 1st countable (c.f. Kelley [Ke, p.77]).

5.1 Lemma : If X is a first countable T_2 space, then X is an E_1 space.

5.2 Lemma : Any countably compact subspace of an E_1 space is closed.

Proof: Let A be a countably compact subspace of an E_1 space X. If A is not closed let $x \in \bar{A} - A$. As X is an E_1 space there exists a sequence $\{V_n\}$ of open sets such that $\{x\} = \bigcap_{n=1}^{\infty} \bar{V}_n$. Put $G_n = (\bar{V}_n)^c$. Then $\bigcup_{n=1}^{\infty} G_n = X - \{x\} \supseteq A$. By countable compactness of A there exist $G_1, G_2, \ldots, G_k$, say, such that $\bigcup_{i=1}^{k} G_i \supseteq A$ i.e. $x \in \bigcap_{i=1}^{k} \bar{V}_i \subseteq A^c$. This implies that the open nbhd $\bigcap_{i=1}^{k} V_i$ of x does not intersect A — a contradiction to the fact that $x \in \bar{A}$. Hence A is closed. (Q.E.D.)

Lemma 5.2 together with theorem 4.2 of Chapter I implies that a countably compact E_1 space is maximal countably compact (Cf. Corollary 4.4 of Chapter I). As a matter of fact, we can arrive at the following proposition.

5.3 Proposition : (C.E. Aull [Au, p.50]) Every countably compact E_1 space is maximal countably compact and minimal E_1.

Proof: Let $(X, \underline{\underline{T}})$ be a countably compact E_1 space. Then, as we have already mentioned, X is maximal countably compact. If $(X, \underline{\underline{T}})$ fails to be minimal E_1 let $\underline{\underline{S}}$ be a strictly smaller E_1 topology on X. But $(X, \underline{\underline{S}})$ still continues to be countably compact and, ipso facto, is maximal countably compact. This contradicts the hypothesis that $(X, \underline{\underline{T}})$ is countably compact.

(Q.E.D.)

5.4 Lemma : A countably compact E_1 space is regular.

Proof : Let A be a closed subset of countably compact E_1 space X and $x \notin A$. We have open sets $\{U_n\}$ such that $\{x\} = \bigcap_{n=1}^{\infty} \bar{U}_n$. So, $A \subseteq \bigcup_{n=1}^{\infty} V_n$ where $V_n = (\bar{U}_n)^c$, an open set in X for $n \geqq 1$. Being a closed subset of X, A is countably compact and, hence, there should exist a finite subcover, say, $V_1, \ldots, V_m$. Then $x \in \bigcap_{i=1}^{m} U_i \subseteq \bigcap_{i=1}^{m} \bar{U}_i \subseteq A^c$. Now $U = \bigcap_{i=1}^{m} U_i$ and $V = \bigcup_{i=1}^{m} V_i$ are open sets with the property that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$. Thus X is regular.

(Q.E.D.)

Now, if a countably compact E_1 space X is embedded into an E_1 space Y then (identifying X as a subset of Y), by dint of lemma 5.2, X is a closed subset of Y as X is countably compact in $\dot{Y}$ (c.f. Chapter I, 4.2). This observation, naturally, has led us to study those E_1 spaces which when embedded into E_1 spaces become closed subspaces, that is to say, E_1 closed spaces. Needless to say, a countably compact E_1 space is E_1-closed. Historically such a consideration gave rise to H-closed spaces from compact T_2 spaces [AU]. The last point needs some elaboration in order to adumbrate or, at least, anticipate our characterisations of E_1-closed spaces. We know that for a compact T_2 space every open cover admits of a finite open subcover and such a space is H-closed too. An H-closed space has been proved [Bo2] to be a T_2 space for which every open cover of the space has a finite subfamily whose closures cover X (such a finite subfamily is called a finite proximate subcover). Similarly a countably compact E_1 space possesses the property that every countable open cover of the space admits a finite subcover. Following the instance of H-closed spaces we are naturally led to consider spaces which satisfy the condition : every countable open cover of the space has a finite proximate subcover. These spaces are nothing but the lightly compact spaces introduced in Chapter I, Sec. 7 (c.f. [Ste 2]). That our guess is accurate will be testified by the next theorem.

5.5 Theorem : Let X be an E_1 space. The following are equivalent ;

(i) X is E_1-closed.

(ii) Every countable open filter base on X has an adherent point.

(iii) Every countable open cover of X has a finite proximate open subcover.

Proof: (i) $\Rightarrow$ (ii) : Let $\underline{F} = \{F_n : n \geq 1\}$ be a countable open filter base on X. Let $\bigcap_{n=1}^{\infty} \bar{F}_n$ (= adherence of the filter base) be, if possible, empty. Let p be a point not belonging to X. Set $Y = X \cup \{p\}$. We define a topology $\underline{S}$ on X as follows : A subset $U \subseteq Y$ is open if (i) $U \cap X$ is open in X and (ii) if $p \in U$, $U \supseteq F_n$ for some n. $(Y, \underline{S})$ is an E_1-space: Let $y \in Y$. If $y = p$, put $G_n = F_n \cup \{p\}$. Then G_n is open in Y. If $Cl(A)$, $A \subseteq Y$, stands for closure of A in Y and $\bar{B}$, $B \subseteq X$, stands for closure of B in X, $Cl(G_n) = Cl(F_n) \cup \{p\} = \bar{F}_n \cup \{p\}$ so that $\bigcap_{n=1}^{\infty} Cl(G_n) = (\bigcap_{n=1}^{\infty} \bar{F}_n) \cup \{p\} = \{p\}$. If $y \in X$, as $\bigcap_n \bar{F}_n = \emptyset$, there exists U open in X containing y and $F_n \in \underline{F}$ such that $U \cap F_n = \emptyset$. Then $p \notin Cl(U)$. We know there exists a countable sequence of open sets $\{V_n\}$ in X such that $\{y\} = \bigcap_{n=1}^{\infty} \bar{V}_n$. Without loss of generality we can assume that $V_n \subseteq U$ for each n, so that $Cl(V_n) \subseteq Cl(U)$ for each n. Then $p \notin Cl(V_n)$ and, hence, $Cl(V_n) = \bar{V}_n$ for each n. Then we have $\{y\} = \bigcap_{n=1}^{\infty} \bar{V}_n = \bigcap_{n=1}^{\infty} Cl(V_n)$. Thus $(Y, \underline{\underline{S}})$ is an E_1 space containing X as a dense open subspace. So X cannot be E_1-closed. Since X is assumed to be E_1-closed, $\bigcap_{n=1}^{\infty} \bar{F}_n \neq \emptyset$.

(ii) $\Rightarrow$ (iii) : Let $\{V_n : n \geq 1\}$ be a countable open cover of an E_1 space X. Let us suppose this open cover has no finite proximate open subcover. Suppose $\{G_n\}$ is the countable collection of open sets which are finite unions of the sets V_n. Let $\underline{\underline{F}} = \{F_n : F_n = (\bar{G}_n)^c\}$. By hypothesis $\underline{\underline{F}}$ is an open filter base on X. Since $\underline{\underline{F}}$ is also countable, adherence of $\underline{\underline{F}} = \bigcap_{n=1}^{\infty} \bar{F}_n \neq \emptyset$. Then $X \neq \bigcup_{n=1}^{\infty} (\bar{F}_n)^c = \bigcup_n (F_n^c)^o = \bigcup_n ((\bar{G}_n)^o) \supseteq \bigcup_n G_n = X$. A contradiction. So (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) : Suppose an E_1 space X is a subset of another E_1 space Y. To show that (iii) $\Rightarrow$ X is closed in Y. Suppose X is not closed in Y, then let $y \in \bar{X} - X$ where $\bar{X}$ = closure of X in Y. Since Y is an E_1 space there exist open sets U_n, $n \geq 1$ in Y such that $\{y\} = \bigcap_{n=1}^{\infty} \bar{U}_n$. So, $X \subseteq Y - \{y\} = \bigcup_{n=1}^{\infty} V_n$ where $V_n^c = \bar{U}_n$, $n \geq 1$. Since every countable open cover of X has a finite proximate open subcover, we should get open sets $V_1, V_2, \ldots, V_m$ (say) such that $X \subseteq \bigcup_{i=1}^{m} \bar{V}_i$. Now since $\bigcap_{i=1}^{m} (\bar{V}_i)^c = \bigcap_{i=1}^{m} (V_i^c)^\circ = \bigcap_{i=1}^{m} (\bar{U}_i)^\circ \supseteq \bigcap_{i=1}^{m} U_i = U$ (say) and $X \cap (\bigcap_{i=1}^{m} (\bar{V}_i)^c) = \emptyset$, $U \cap X = \emptyset$. But U is an open nbhd of y in Y and $y \in \bar{X}$ so that $U \cap X \neq \emptyset$. The contradiction proves our contention.

(Q.E.D.)

What is the relationship between E_1-closed and minimal E_1 spaces ? Is a minimal E_1 space always E_1-closed? (Compare with the fact that a minimal T_2 space is H-closed). The answer is yes and we prove the following.

5.6 <u>Proposition</u> : A minimal E_1 space is E_1-closed and semiregular.

<u>Proof</u>: Let $(X, \underline{\underline{T}})$ be minimal E_1. Consider the semiregular topology $\underline{\underline{T}}_o$ on X. We intend to establish that $\underline{\underline{T}}_o$ is E_1 and then it will follow that $\underline{\underline{T}} = \underline{\underline{T}}_o$, i.e., $\underline{\underline{T}}$ is semiregular. Let $x \in X$. By definition there exists a sequence U_n of open sets in $\underline{\underline{T}}$ such that $\{x\} = \bigcap_{n=1}^{\infty} \bar{U}_n$. Let $V_n = (\bar{U}_n)^\circ$. Then $V_n \in \underline{\underline{T}}_o$ and $\bar{V}_n = \bar{U}_n$ and $x \in V_n$ for all n. Obviously $\{x\} = \bigcap_{n=1}^{\infty} \bar{U}_n = \bigcap_{n=1}^{\infty} \bar{V}_n$. Thus $\underline{\underline{T}}_o$ is a topology of E_1 space.

If $(X,\underline{\underline{T}})$ is not E_1-closed, suppose there exists an E_1 space Y containing X as a dense subspace. Let $p \in Y-X$. Let $\underline{\underline{N}} = \{N : N = G \cap X$ for some open nbhd G of p in $Y\}$. For $A \subseteq X$, $Cl(A)$ denotes closure of A in X, $\bar{A}$, the closure of A in Y. Since $\underline{\underline{N}}$ converges to p, adherence of $\underline{\underline{N}}$ in $X = \bigcap \{Cl(N) : N \in \underline{\underline{N}}\} = \emptyset$. Fix $a \in X$. We define a topology $\underline{\underline{S}}$ on X as follows : If $x \neq a$ the open nbhds of x in $\underline{\underline{S}}$ = the open nbhds of x in $\underline{\underline{T}}$ where $\underline{\underline{T}}$ is the original topology of X. If $x = a$, the open nbhds of $a = \{V \cup N : N \in \underline{\underline{N}}$ and V an open nbhd of a in $\underline{\underline{T}}\}$.

Obviously the topology $\underline{\underline{S}}$ is weaker than $\underline{\underline{T}}$. Since $\underline{\underline{N}}$ is a filter base on X, the topology $\underline{\underline{S}}$ is strictly weaker than $\underline{\underline{T}}$. We shall show that $(X,\underline{\underline{S}})$ is an E_1 space. Suppose $x \in X$. Case (i) $x \neq a$. Since $(X,\underline{\underline{T}})$ is E_1 and adherence of $\underline{\underline{N}}$ is $\emptyset$, we can get $U \in \underline{\underline{T}}$, $V \in \underline{\underline{T}}$ and $N \in \underline{\underline{N}}$ such that $x \in U$, $a \in V$ and $(V \cup N) \cap U = \emptyset$ and $V \cup N$ is an open nbhd of a in $\underline{\underline{S}}$. So $a \notin$ closure of U in $\underline{\underline{S}}$ i.e., closure of U in $\underline{\underline{S}} = Cl(U)$. Again there exist open sets $U_n \in \underline{\underline{T}}$, $n \geq 1$ such that $U_n \subseteq U$ and $\{x\} = \bigcap_{n=1}^{\infty} Cl(U_n)$. Since $U_n \subseteq U$, $Cl(U_n)$ = closure of U_n in $\underline{\underline{S}}$. So $\{x\} = \bigcap_{n=1}^{\infty} \underline{\underline{S}}\text{-}Cl(U_n)$. Case (ii) $x = a$. First there exists a sequence of open sets W_n in Y

such that $\bigcap_{n=1}^{\infty} \bar{W}_n = \{a\}$, similarly there exists a sequence $\{G_n\}$ of open sets in Y such that $\{p\} = \bigcap_{n=1}^{\infty} \bar{G}_n$. Then, for each n, $(W_n \cup G_n) \cap X \in \underline{\underline{S}}$ and contains a. We first observe that for $n \geq 1$, $\underline{\underline{S}}\text{-Cl}(Z_n) = \text{Cl}(Z_n)$ where $Z_n = X \cap (W_n \cup G_n)$. It is not difficult to see that $\bigcap_{n=1}^{\infty} Z_n = \bigcap_{n=1}^{\infty} \text{Cl}(Z_n) = \{a\}$. So $(X, \underline{\underline{S}})$ is an E_1 space. By hypothesis $(X, \underline{\underline{T}})$ is minimal E_1. So we arrive at a contradiction. Consequently $(X, \underline{\underline{T}})$ is E_1-closed.

(Q.E.D.)

The earlier proposition brings out a necessary condition for a minimal E_1 space. In fact, this condition is sufficient too.

5.7 Theorem: An E_1 space $(X, \underline{\underline{T}})$ is minimal E_1 if and only if it is E_1-closed and semiregular.

Proof: The 'only if' part is the proposition 5.6. We need only to prove the 'if' part. Let $(X, \underline{\underline{T}})$ be an E_1-closed and semi-regular space. Let $(X, \underline{\underline{S}})$ be a strictly weaker E_1 space. As $\underline{\underline{T}}$ is semiregular and $\underline{\underline{S}} \neq \underline{\underline{T}}$ there exists a regular open set $U \in \underline{\underline{T}} - \underline{\underline{S}}$. Obviously, $\emptyset \neq U \subseteq \bar{U} \neq X$ (where $\bar{A}$ refers to closure of A in $(X, \underline{\underline{T}})$ for $A \subseteq X$). Since $U \notin \underline{\underline{S}}$, we should get a point $x \in U$ such that U does not contain any $\underline{\underline{S}}$-nbhd. of x. As $\underline{\underline{S}}$ is an E_1 topology a sequence $\{V_n\} \subseteq \underline{\underline{S}}$ is available such that

$\{x\} = \bigcap_{n=1}^{\infty} \underline{\underline{S}}\text{-Cl}(V_n)$ and $V_n \supseteq V_{n+1}$. Now $(\bar{U})^c \cap V_n \neq \emptyset$ for each n. For, if not there should exist $m \geq 1$ with the property $V_m \subseteq \bar{U}$ and since U is regular open and $V_m \in \underline{\underline{T}}$, $V_m = V_m^o \subseteq (\bar{U})^o = U$ (here A^o = interior of A in the topology $\underline{\underline{T}}$, $A \subseteq X$). But U cannot contain any V_n as it is an open $\underline{\underline{S}}$-nbhd of x. Naturally $\underline{\underline{B}} = \{(\bar{U})^c \cap V_n : n = 1,2,\ldots\}$ is a countable open filter base on $(X,\underline{\underline{T}})$ which is E_1-closed by hypothesis. Consequently $\underline{\underline{B}}$ has non-void adherence i.e.,

$$\emptyset \neq \text{adherence of } \underline{\underline{B}} = \bigcap\{\overline{V_n \cap (\bar{U})^c} : n \geq 1\} \subseteq$$
$$\bigcap\{\bar{V}_n \cap (\bar{U})^c : n \geq 1\} \subseteq (\bigcap_{n=1}^{\infty} \bar{V}_n) \cap (\bar{U}^o)^c = \{x\} \cap U^c = \emptyset.$$

A contradiction. Therefore, $(X,\underline{\underline{S}})$ cannot be an E_1 space and this establishes that $(X,\underline{\underline{T}})$ is minimal E_1.

(Q.E.D.)

5.8 <u>Lemma</u> : Suppose X is an E_1-closed space and f is a continuous function from X into an E_1 space Y. $f(X)$ is then E_1-closed.

<u>Proof</u>: The proof is easy and is omitted. (Q.E.D.)

5.9 <u>Corollary</u> : Let $(X,\underline{\underline{T}})$ be E_1-closed and $(X,\underline{\underline{S}})$ be an E_1 space. $\underline{\underline{S}}$ is weaker than $\underline{\underline{T}}$ implies $(X,\underline{\underline{S}})$ is also E_1-closed.

Now we are in a position to characterise E_1-closed spaces in terms of minimal E_1 spaces.

5.10 Proposition: An E_1 space is E_1-closed if and only if its associated semiregular space is a minimal E_1 space.

Proof: Suppose $(X,\underline{\underline{T}})$ is an E_1 space which is E_1-closed. Let $(X,\underline{\underline{T}}_o)$ be the associated semiregular space. $(X,\underline{\underline{T}}_o)$ is an E_1 space and by Corollary 5.9 it is E_1-closed. Thereom 5.7 then declares $(X,\underline{\underline{T}}_o)$ to be minimal E_1.

Conversely, suppose we are told that the associated semiregular space $(X,\underline{\underline{T}}_o)$ of an E_1 space $(X,\underline{\underline{T}})$ is minimal E_1. Let us take any countable open cover $\{V_n\}$ of $(X,\underline{\underline{T}})$. Put $W_n = (\bar{V}_n)^o$ (the closures and interiors being taken with respect to $\underline{\underline{T}}$). As $V_n \subseteq W_n$ and $W_n \in \underline{\underline{T}}_o$, $\{W_n : n \geq 1\}$ is an open cover of $(X,\underline{\underline{T}}_o)$. $(X,\underline{\underline{T}}_o)$ is minimal E_1 and, as a result, there exist $W_{n_1},\ldots,W_{n_p}$ such that $\bigcup_{i=1}^{p} \bar{W}_{n_i} = X$ (because for any $A \in \underline{\underline{T}}$, $\underline{\underline{T}}\text{-Cl}(A) = \underline{\underline{T}}_o\text{-Cl}(A)$). Now note that $\bar{W}_n = \bar{V}_n$ for each n so that $X = \bigcup_{i=1}^{p} \bar{W}_{n_i} = \bigcup_{i=1}^{p} \bar{V}_{n_i}$. By theorem 5.5 we see that $(X,\underline{\underline{T}})$ is E_1-closed. (Q.E.D.)

It has been already proved that every countably compact E_1 space is minimal E_1. The existence of a minimal E_1 space which is not countably compact is demonstrated by the next example.

5.11 Example of a non-countably compact E_1 space

Let $X = \{ a_{ij}, b_{ij}, c_i, a, b : i = 1,2,\ldots, j = 1,2,\ldots\}$ where all these elements are distinct.

Define the following neighbourhood system on X :

Each a_{ij} and each b_{ij} are isolated. The basic neighbourhoods of c_i, a, b are as follows :

$$\underline{\underline{B}}(c_i) = \{V^n(c_i) = \bigcup_{j=n}^{\infty} \{a_{ij}, b_{ij}, c_i\} : n = 1,2,\ldots\}$$

$$\underline{\underline{B}}(a) = \{V^n(a) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{a_{ij}, a\} : n = 1,2,\ldots\}$$

$$\underline{\underline{B}}(b) = \{V^n(b) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{b_{ij}, b\} : n = 1,2,\ldots\}$$

X is minimal Hausdorff, non-compact and satisfies 1st axiom of countability. Thus X is an E_1 space (lemma 5.1) and so minimal E_1. Since countable compactness for a countable space is identical with compactness, X is obviously non-countably compact. Other examples will be given after a short while. The example, just presented, is due to Urysohn [U] and was cited as the first example of a non-compact minimal Hausdorff space by E.Cartan [Bo1] in 1941.

We shall now present another characterisation of minimal E_1 spaces using open filters.

5.12 Theorem: A necessary and sufficient condition that an E_1 space $(X, \underline{\underline{T}})$ be minimal E_1 is that $\underline{\underline{T}}$ satisfies properties :

E_1(i) Every countable open filter base has an adherent point.

E_1(ii) If an open countable filter base has a unique adherent point, it converges to this point.

Proof: Necessity : Suppose $(X, \underline{\underline{T}})$ is minimal E_1. Then, due to Proposition 5.6, $(X, \underline{\underline{T}})$ is E_1-closed i.e., it satisfies E_1(i). In order to see that it satisfies E_1(ii) let us take a countable open filter base $\underline{\underline{F}} = \{F_n : n \geq 1\}$ on $(X, \underline{\underline{T}})$ which has a unique adherent point, say, x i.e., $\{x\} = \bigcap_{n=1}^{\infty} \bar{F}_n$. Set $\underline{\underline{G}} = \{F_n \cup V : n \geq 1$ and V an open nbhd of $x\}$. $\underline{\underline{G}}$ is a sub-family of $\underline{\underline{T}}$. Consider the topology $\underline{\underline{S}}$ on X generated by the following neighbourhood systems :

If $y \neq x$, the neighbourhoods are unchanged

$\underline{\underline{G}}$ is the base for the neighbourhoods at x.

$\underline{\underline{S}}$ is evidently weaker than $\underline{\underline{T}}$. The interesting fact is that $(X, \underline{\underline{S}})$ is an E_1 space. As a result $\underline{\underline{T}} = \underline{\underline{S}}$. Therefore, $\underline{\underline{G}}$ is a base for the nbhds of x in $\underline{\underline{T}}$ and clearly the filter base converges to the point x.

<u>Sufficiency</u> : Let $(X,\underline{\underline{T}})$ be an E_1 space satisfying E_1(i) and E_1(ii). Since $\underline{\underline{T}}$ satisfies E_1(i), it is E_1-closed. If we show that $(X,\underline{\underline{T}})$ is semiregular, it will be minimal E_1 by theorem 5.7. If $\underline{\underline{T}}$ is not semiregular let $\underline{\underline{T}}_o$ indicate the associated semiregular topology on X. Obviously there exists $\emptyset \neq U \varepsilon \underline{\underline{T}}-\underline{\underline{T}}_o$. Then $U \neq X$. There should exist $x \varepsilon U$ such that $x \varepsilon W \varepsilon \underline{\underline{T}}_o \Rightarrow W$ is not contained in U. Inasmuch as $(X,\underline{\underline{T}}_o)$ is an E_1 space, $\{x\} = \bigcap_{n=1}^{\infty} \bar{V}_n$ where $V_n \varepsilon \underline{\underline{T}}_o$ and $x \varepsilon V_n$ for each $n \geq 1$. We can easily assume $V_n \supseteq V_{n+1}$ for all n. Clearly $\{V_n\}$ is an open filter base on $(X,\underline{\underline{T}})$ which is countable also. x is the unique adherent point of this countable open filter base and so by E_1(ii) it converges to x in $\underline{\underline{T}}$. Since $U \varepsilon \underline{\underline{T}}$ there should exist a V_n such that $x \varepsilon V_n \subseteq U$. A contradiction. We, therefore, conclude that $(X,\underline{\underline{T}})$ is semiregular i.e. $(X,\underline{\underline{T}})$ is minimal E_1. (Q.E.D.)

5.13 <u>Corollary</u> : Every minimal E_1 space satisfies first axiom of countability.

<u>Proof</u> : Suppose x is any arbitrary point in a minimal E_1 space X. We know $\{x\} = \bigcap_{n=1}^{\infty} \bar{U}_n$ where each U_n is an open nbhd of x in X. No loss of generality in assuming $U_n \supseteq U_{n+1}$ for $n = 1,2,\ldots$, so that $\{U_n\}$ is a countable open filter base with a unique adherent point x. By property

E_1(ii) of theorem 5.12 the filter base $\{U_n\}$ must converge to x i.e., U_n's form a base for the neighbourhood filter of x. Consequently X is first countable.

(Q.E.D.)

Corollary 5.13 at once gives the following in view of the fact that any 1st countable T_2 space is E_1.

5.14 Corollary : Every minimal E_1 space is minimal first countable Hausdorff.

5.15 Proposition : A countably compact E_1 space is minimal first countable Hausdorff.

Proof: Immediately follows from proposition 5.3 and corollary 5.14.

(Q.E.D.)

Remark : Let us note that in Chap. I, Section 4 (Corollary 4.4) it has been established that a countably compact E_1 space is first countable and then using Corollary 4.3 of the same chapter it can be readily seen that a countably compact E_1 space is minimal 1st countable Hausdorff. In fact the following statement is true.

5.16 Proposition : Every countably compact, first countable Hausdorff space is minimal first countable Hausdorff (cf. Stephenson [Ste 3], lemma 2.12, p.119).

Proof: Follows easily from lemma 5.1 and proposition 5.15

(Q.E.D.)

Proposition 5.6 can, as well, be a starting point for a probe into first countable T_2-closed spaces exactly in the same manner as has been initiated by proposition 5.3 in the case of E_1-closed spaces. The paragraph just preceding theorem 5.5 contains the hint that the characterisations of 1st countable Hausdorff-closed and minimal 1st countable Hausdorff spaces are likely to resemble those of E_1-closed and minimal E_1 spaces respectively. As a matter of fact, they do and we can get them simply by replacing by 'first countable Hausdorff' the term 'E_1' in theorems 5.5-5.7. But R. M. Stephenson Jr. has obtained the same results on first countable Hausdorff-closed and minimal first countable T_2 spaces while he has been studying various minimal first countable topologies. [Ste 3, theorems 2.4 and 2.5]. Thus his starting point has been different from ours.

Let us suppose that X is a minimal 1st countable Hausdorff space. In the light of the characterisations mentioned in the last paragraph X is lightly compact and semiregular. Application of lemma 5.1 together with theorems 5.5 and 5.7 yields X as a minimal E_1 space. Use of Corollary 5.14 then finally leads to :

5.17 <u>Proposition</u> : X is minimal first countable T_2 $\Longleftrightarrow$ X is minimal E_1.

Despite the observation that minimal first countable Hausdorff spaces are identical with minimal E_1 spaces, first countable T_2-closedness and E_1-closedness are distinct concepts. From the properties of 1st countable T_2-closed spaces, mentioned earlier, it, however, follows that the former spaces are always E_1-closed. We are going to cite an example of an E_1-closed space that fails to be first countable T_2-closed. The example has been taken from Bourbaki [Bo2] where, of course, it appears in a different context.

5.18 Example.

$$X = [0, 1]$$

$$\underline{\underline{T}} = \text{Usual topology on } X$$

$$\underline{\underline{C}} = \{A \subseteq X : X - A \text{ is countable}\}$$

Let $\underline{\underline{S}}$ be the topology on X generated by $\underline{\underline{C}}$ and $\underline{\underline{T}}$. Obviously, $(X, \underline{\underline{S}})$ is an E_1 space. We shall show that $(X,\underline{\underline{S}})$ is lightly compact. It is easy to see that $A \varepsilon \underline{\underline{C}} \Rightarrow A$ is dense in $(X,\underline{\underline{T}})$. Again $\underline{\underline{C}}$ is a filter base. Since $(X,\underline{\underline{T}})$ is compact T_2, it is evidently H-closed and semiregular. It now follows from an exercise of Bourbaki [Bo2, page 138, Ex. 20] that $(X, \underline{\underline{S}})$ is H-closed and from the definition of lightly compact spaces it is clear that an H-closed space is lightly compact. $(X,\underline{\underline{S}})$ is thus both lightly compact and E_1. By theorem 5.5, $(X,\underline{\underline{S}})$ is

E_1-closed. As soon as it is shown that $(X,\underline{\underline{S}})$ fails to be 1st countable $(X,\underline{\underline{S}})$ will be an example of an E_1-closed space which is not first countable T_2-closed. Let $x \in X$. Suppose $\{U_n ; n \geq 1\}$ is a countable base at x. We further assume that U_n's are decreasing. Choose $x_n \in U_n - U_{n+1}$, $n \geq 1$. If G is the set consisting only of x_n's, G is countable so $G^c \in \underline{\underline{C}}$. Obviously, $G^c \in \underline{\underline{S}}$ and is a nbhd of x. There should exist U_{n_o} such that $x \in U_{n_o} \subseteq G^c$. But by choice $x_{n_o} \in U_{n_o}$ and $x_{n_o} \notin G^c$. Hence this is impossible. No point of $(X,\underline{\underline{S}})$ can have a countable base for its neighbourhood filter.

5.19 At the end of example 5.11 we promised to produce further examples of minimal E_1 spaces which are not countably compact. According to proposition 5.17 it suffices for us to produce such examples of minimal first countable Hausdorff spaces. Here we refer to two minimal first countable Hausdorff spaces, due to Stephenson [Ste 4], neither of which is regular (and hence neither is countably compact by lemma 5.4) or minimal Hausdorff.

5.20 <u>Category of minimal E_1 spaces</u>

Minimal E_1 spaces, like minimal Hausdorff spaces, can be of either category. One of the two examples referred to in 5.19 offers an illustration of a second category minimal E_1 space while the other is an evidence of a minimal E_1 space which is of the first category. Let us look at the following result which has been taken from Bourbaki [Bo2, page 147].

5.21 Proposition: Let X be a countable H-closed space. Then isolated points of X are dense in the topology of X.

5.22 Corollary : Any countable H-closed space is of 2nd category.

5.23 Proposition : Let X be a countable set and be endowed with a first countable Hausdorff-closed (resp. E_1-closed) topology. Then X is H-closed.

Proof: Let us note that a Lindelöf space is H-closed iff it is lightly compact and T_2. Since any countable topological space is Lindelöf, X is Lindelöf. Again by our characterisations, X is lightly compact so that X must be H-closed. (Q.E.D.)

Combining Corollary 5.22 with proposition 5.23 we can state the following :

5.24 Theorem: Every countable E_1-closed or first countable Hausdorff-closed space is of the second category.

5.25 Remarks: Let us note that every E_1-closed or 1st countable T_2-closed space contains a minimal E_1 (which is same as minimal first countable T_2) space. Thus an E_1-closed space is Katětov E_1 and a first countable T_2-closed space Katětov first countable

T_2. Further it is easy to see that a countable T_2 space is lightly compact if and only if it is H-closed. Using this fact we are able to demonstrate that an E_1 space need not be Katětov E_1. The example is as follows :

Example : Let Q be the set of rationals with usual topology. Q is a first countable T_2 space, so is E_1. If Q has to support a minimal E_1 topology weaker than the usual topology, the topology ought to be lightly compact (theorems 5.5 and 5.7) as Q is countable it would be H-closed. Now appealing to Proposition 5.21 we know that Q then should have isolated points. But the usual topology of Q has no isolated points and hence no weaker topology can have isolated points thereby rendering the existence of any minimal E_1 topology on Q weaker than the usual topology impossible. The same argument also establishes that Q is neither Katětov E_1 nor Katětov first countable T_2.

We shall conclude this section by demonstrating an application of Proposition 5.16 to metric spaces. We shall exhibit an alternative proof of the following well-known fact.

5.26 Theorem : A countably compact metric space is compact.

Proof: If X is a countably compact metric space it is definitely a 1st countable countably compact T_2 space. By appealing to theorem 5.16 we know that X is minimal first countable T_2. As any metric space is a 1st countable Hausdorff space, X has to be a minimal metric space. Scarborough and Stephenson [SSe] have shown that a minimal metric space is compact.

(Q.E.D.)

6. P-spaces

In Chapter I we have already come across P-spaces which are defined to be spaces in which every G-delta subset is open. The nomenclature 'P-space' has been taken from Gillman and Jerison [GJ, 4J page 62]. To start with let us list some important properties of P-spaces. Most of them can be found in [GJ].

6.1 (i) Every discrete and every indiscrete space are P-spaces.

(ii) Every countable subset of a P-space is closed and discrete. Hence every countable P-space is discrete.

(iii) Every subspace of a P-space is a P-space.

(iv) Finite products of P-spaces are P-spaces.

(v) X is a P-space if and only if every point is a P-point (i.e., every G-delta containing the point is a nbhd of the point).

(vi) Every Lindelöf subspace of a Hausdorff P-space is closed (cf. Chap. I, Section 3.2 for the proof).

The implication of 6.1(i) is that both the smallest and the largest topologies on any set are P-spaces. Thus the questions regarding minimal as well as maximal P-spaces are, rather, trivially answered. We need to consider more restrictive set-up, as for example, P-spaces which are T_1. Here we can state an interesting result regarding minimal T_1 P-spaces. It is as follows :

6.2 Theorem : On any set X the collection of T_1 P-space topologies has a unique minimum.

Proof: Case (i) X is countable. In this case the discrete topology is the only T_1 P-space topology (6.1 (ii)) and is, naturally, the minimum.

Case (ii) X is uncountable. The minimum T_1 P-space topology is given by the topology whose non-void open sets are complements of countable subsets. Let this topology be denoted by $\underline{\underline{T}}$. Suffices to show that any T_1 P-space topology $\underline{\underline{S}}$ on X is stronger than $\underline{\underline{T}}$. Let $\emptyset \neq V \in \underline{\underline{T}}$. Then X-V is countable. X-V is an F-sigma as $\underline{\underline{T}}$ is $T_1 \Rightarrow$ V is a G-delta in $\underline{\underline{T}} \Rightarrow$ V is a G-delta in $\underline{\underline{S}} \Rightarrow$ V is open in $\underline{\underline{S}}$ (since $\underline{\underline{S}}$ is a P-space). So $\underline{\underline{T}}$ is weaker than $\underline{\underline{S}}$. (Q.E.D.)

6.3 Hausdorff P-spaces : From now onwards we shall only consider Hausdorff P-spaces (to be abbreviated as HP-spaces). The following theorem is meant for showing how naturally the concept of minimal HP-spaces enters our discussion.

6.4 Theorem : Let $(X,\underline{\underline{T}})$ be a maximal Lindelöf Hausdorff space. Then $(X,\underline{\underline{T}})$ is a minimal Hausdorff P-space.

Proof: Suppose not. Then there exists a strictly smaller Hausdorff P-space topology $\underline{\underline{S}}$ on X. Consider the identity mapping $i : (X, \underline{\underline{T}}) \rightarrow (X,\underline{\underline{S}})$. Obviously i is a continuous bijection and hence $\underline{\underline{S}}$ is a Lindelöf Hausdorff P-topology. So $(X,\underline{\underline{S}})$ is maximal Lindelöf due to theorem 3.2 of Chapter I. By theorem 3.1 of Chapter I i must now be a homeomorphism i.e., $\underline{\underline{S}} = \underline{\underline{T}}$. So $(X,\underline{\underline{T}})$ is a minimal Hausdorff P-space.

(Q.E.D.)

Now we wish to study HP-closed spaces. First let us observe that if a maximal Lindelöf T_2 space is homeomorphically imbedded in a Hausdorff P-space, the homeomorph, being a Lindelöf subspace of an HP-space, necessarily turns out to be a closed subset (6.1 (vi)). Thus maximal Lindelöf Hausdorff spaces provide examples of HP-closed spaces. Necessary and sufficient conditions for HP-closed spaces are obtained in the following theorem.

6.5 Theorem : The following are equivalent for a Hausdorff P-space $(X,\underline{\underline{T}})$.

(a) $(X,\underline{\underline{T}})$ is an HP-closed space.

(b) Every open filter on X such that every countable intersection of sets in the filter is also in the filter (i.e., every open filter that is closed under countable intersections) has non-empty adherence.

(c) Every open cover $\underline{\underline{V}}$ of X has a countable subfamily $\{V_n\}$ such that $\bigcup_{n=1}^{\infty} \bar{V}_n = X$ ($\bar{V}_n$ being the closure of V_n in X).

<u>Proof</u>: (a) $\Rightarrow$ (b) : Let $\underline{\underline{F}}$ be an open filter on X which is closed under countable intersections. To show that $\bigcap\{\bar{F} : F \in \underline{\underline{F}}\}$ = adherence of $\underline{\underline{F}} \neq \emptyset$. Suppose not. Put $Y = X \bigcup \{p\}$ where $p \notin X$. A subset U of Y is open if (i) $U \bigcap X$ is open in X and (ii) if $p \in U$ then $U = \{p\} \bigcup F$ for some $F \in \underline{\underline{F}}$. To check that Y is a P-space we are to show that p is a P-point and that is trivially done by using the property of $\underline{\underline{F}}$. Y is Hausdorff : let $x \in X$ and the other point be p. Since $x \notin \bigcap \{\bar{F} : F \in \underline{\underline{F}}\}$, $\bar{F}$ = the closure of F in $(X, \underline{\underline{T}})$, there exist $\underline{\underline{T}}$-open set V containing x and $F_o \in \underline{\underline{F}}$ such that $V \bigcap F_o = \emptyset$. As $V \subseteq X$, $V \bigcap (F_o \bigcup \{p\}) = \emptyset$ and V and $F_o \bigcup \{p\}$ are disjoint open neighbourhoods in Y of x and p respectively. Consequently, X is a dense open subset of a Hausdorff P-space Y - a contradiction to the fact that X is HP-closed.

(b) $\Rightarrow$ (c) : Let $\underline{\underline{V}}$ be any open cover of X. Suppose, also, that for every countable subfamily $\underline{\underline{C}}$ of $\underline{\underline{V}}$, $\bigcup\{\bar{V} : V \in \underline{\underline{C}}\} \neq X$.

Without loss of generality we can assume that $\underline{\underline{V}}$ is also closed under countable unions. Put $\underline{\underline{F}}$ = open filter generated by $\{(\bar{V})^c : V \in \underline{\underline{V}}\}$. Easy to see that $\underline{\underline{F}}$ is an open filter closed under countable intersections. By hypothesis $\bigcap\{(\bar{V})^c : V \in \underline{\underline{V}}\} = \bigcap\{\bar{F} : F \in \underline{\underline{F}}\} \neq \emptyset$ i.e., $\bigcup\{(\bar{F})^c : F \in \underline{\underline{F}}\} \neq X$. Since $((\bar{V})^c)^c = (\bar{V})^o \supseteq V$, $X \neq \bigcup\{(\bar{F})^c : F \in \underline{\underline{F}}\} = \bigcup\{(\bar{V})^o : V \in \underline{\underline{V}}\} \supseteq \bigcup\{V : V \in \underline{\underline{V}}\} = X$. A contradiction.

(c) $\Rightarrow$ (a) : Let X be a dense subset of a Hausdorff P-space Y such that $\underline{\underline{T}}$ is the relative topology of X. Take $p \in Y-X$. Suppose $\underline{\underline{N}}$ = the family of open nbhds of p in Y. Set $\underline{\underline{G}} = \{X \cap N : N \in \underline{\underline{N}}\}$. As X is dense in Y, $\emptyset \notin \underline{\underline{G}}$. Since Y is Hausdorff $\{p\} = \bigcap\{Cl_Y N : N \in \underline{\underline{N}}\}$ and so $\emptyset = X \cap \{p\} = \bigcap\{(Cl_Y N) \cap X\}$. Now $Cl_X(N \cap X) = X \cap Cl_Y(N \cap X) = (Cl_Y N) \cap X$ (since X is dense in Y) $\Rightarrow \emptyset = \bigcap\{Cl_X(N \cap X) : N \in \underline{\underline{N}}\} = \bigcap\{Cl_X G : G \in \underline{\underline{G}}\}$ giving $X = \bigcup\{X - Cl_X(G) : G \in \underline{\underline{G}}\}$. By hypothesis there exists a countable subfamily $\{G_n\}$ of $\underline{\underline{G}}$ such that $X = \bigcup_{n=1}^{\infty} Cl_X(X - Cl_X(G_n))$ which yields $\bigcap_{n=1}^{\infty} Cl_X(G_n)$ has empty interior in X. As X is a P-space, $\bigcap_{n=1}^{\infty} G_n$ is open in X and $\bigcap_{n=1}^{\infty} G_n \subseteq \bigcap_{n=1}^{\infty} Cl_X(G_n)$. So $\bigcap_{n=1}^{\infty} G_n = \emptyset$. Now $G_n \in \underline{\underline{G}} \Rightarrow G_n = X \cap N_n$ for some $N_n \in \underline{\underline{N}}$. Therefore, $\bigcap_{n=1}^{\infty}(X \cap N_n) = \bigcap_{n=1}^{\infty} G_n = \emptyset$ i.e., $X \cap (\bigcap_{n=1}^{\infty} N_n) = \emptyset$. But Y is a P-space $\Rightarrow \bigcap_{n=1}^{\infty} N_n = N \in \underline{\underline{N}}$

so that $X \cap N = \emptyset$. A contradiction to the fact that $\emptyset \notin \underline{\underline{G}}$.

(Q.E.D.)

6.6 <u>Lemma</u> : Suppose $(X, \underline{\underline{T}})$ is a P-space. If $(X, \underline{\underline{T}}_o)$ is the semi-regular topology associated with $(X, \underline{\underline{T}})$, then $(X, \underline{\underline{T}}_o)$ is also a P-space.

<u>Proof</u>: Suffices to show that every $x \in X$ is a P-point. Take any $x \in X$ and suppose $\{V_n : n \geq 1\}$ is a countable family of regular-open nbhds. of x. Since $(X, \underline{\underline{T}})$ is a P-space, $V = \bigcap_{n=1}^{\infty} V_n$ is an open set in $\underline{\underline{T}}$ containing x. In order to show that $V \in \underline{\underline{T}}_o$ we shall, in fact, show that V is regular-open. Since $\bar{V} \subseteq \bigcap \bar{V}_n$ (closure with respect to $(X, \underline{\underline{T}})$), $(\bar{V})^o \subseteq (\bigcap \bar{V}_n)^o \subseteq \bigcap (\bar{V}_n)^o = \bigcap V_n$ (as each V_n is regular-open by choice). Since $V \in \underline{\underline{T}}$, $V \subseteq (\bar{V})^o \subseteq V$ i.e., V is regular-open.

(Q.E.D.)

6.7 <u>Theorem</u> : A minimal HP-space is semiregular and HP-closed.

<u>Proof</u>: Let us recall that a topological space is Hausdorff iff its associated semiregular topology is Hausdorff [Bo2]. Now with the help of lemma 6.6 it is easy to see that a minimal HP-space should necessarily be semiregular. Suppose a minimal HP-space $(X, \underline{\underline{T}})$ is not HP-closed i.e., there exists an open filter $\underline{\underline{F}}$ on X closed under countable intersections such that $\underline{\underline{F}}$ has no adherent point. Fix $x_o \in X$. Consider the topology $\underline{\underline{S}}$ on X generated by the following neighbourhood filters.

$$\underline{\underline{B}}'(x) = \underline{\underline{B}}(x) \quad \text{if} \quad x \neq x_o$$

$$\underline{\underline{B}}'(x_o) = \{G \subseteq X : G \supseteq V \cup F \text{ for some } V \in \underline{\underline{B}}(x_o) \text{ and } F \in \underline{\underline{F}}\}$$

where $\underline{\underline{B}}(x)$ denotes the neighbourhood filter of x in $(X,\underline{\underline{T}})$.

Since $\underline{\underline{B}}'(x_o) \subseteq \underline{\underline{B}}(x_o)$ but $\underline{\underline{B}}'(x_o) \neq \underline{\underline{B}}(x_o)$, $\underline{\underline{S}}$ is strictly weaker than $\underline{\underline{T}}$. $\underline{\underline{S}}$ is easily seen to be T_2. To show that $(X,\underline{\underline{S}})$ is a P-space we shall show that x_o is a P-point in the topology $\underline{\underline{S}}$. Let $\{G_n = V_n \cup F_n, n = 1,2,\ldots\}$ be a countable family of $\underline{\underline{S}}$-nbhds of x_o. Now $\bigcap_{n=1}^{\infty} G_n \supseteq (\bigcap V_n) \cup (\bigcap F_n) = V \cup F$ where $V = \bigcap V_n \in \underline{\underline{B}}(x_o)$ and $F = \bigcap F_n \in \underline{\underline{F}}$. So $\bigcap G_n \in \underline{\underline{B}}'(x_o)$. Thus $(X, \underline{\underline{S}})$ is a Hausdorff P-space strictly weaker than $(X,\underline{\underline{T}})$. The minimality of $(X,\underline{\underline{T}})$ is then contradicted. (Q.E.D.)

Theorem 6.7 furnishes necessary conditions for a Hausdorff P-space to be minimal HP. The next theorem is called for to establish the sufficiency of these conditions.

6.8 Theorem : Any semiregular and HP-closed space is a minimal HP-space.

Proof: Suppose $(X,\underline{\underline{T}})$ is a semiregular and HP-closed space. Let $(X,\underline{\underline{S}})$ be a strictly weaker P-space. Since $\underline{\underline{S}} \neq \underline{\underline{T}}$ and $\underline{\underline{T}}$ has a base consisting of regular-open sets, we can get a regular-open set $U \in \underline{\underline{T}}-\underline{\underline{S}}$. Obviously, $\emptyset \neq U \subseteq \bar{U} \neq X$ ($\bar{A}$ refers to closure of

A in $(\underline{\underline{X}},\underline{\underline{T}})$). As $U \not\in \underline{\underline{S}}$, there exists $x \in U$ such that U does not contain any $\underline{\underline{S}}$-nbhd of x. Put $\underline{\underline{B}} = \underline{\underline{B}}(x,\underline{\underline{S}}) = \{V \subseteq X : V$ is an $\underline{\underline{S}}$-nbhd of x $\}$. Naturally $U \not\in \underline{\underline{B}}$ and so $U^c \cap V \neq \emptyset$ for all $V \in \underline{\underline{B}}$. We can claim that $(\bar{U})^c \cap V \neq \emptyset$ for all $V \in \underline{\underline{B}}$; for, if otherwise, we should be able to find $V \in \underline{\underline{B}}$ with $V \subseteq \bar{U}$ i.e., $V^o \subseteq (\bar{U})^o = U$ (where, of course, A^o denotes the interior operation with respect to $\underline{\underline{T}}$). Now $V \in \underline{\underline{B}} \Rightarrow V^o \in \underline{\underline{B}}$ and this means $U \in \underline{\underline{B}}$ as $U \supseteq V^o$. If $\underline{\underline{F}} = \{\bar{U}^c \cap V : V \in \underline{\underline{B}} \cap \underline{\underline{S}}\}$, $\underline{\underline{F}}$ is an open filter base which is also closed under countable intersection in virtue of $(X,\underline{\underline{S}})$ being a P-space. If $\underline{\underline{G}}$ is the open filter generated by $\underline{\underline{F}}$, it will, of course, be closed under countable intersections and by hypothesis it will have nonempty adherence in $\underline{\underline{T}}$ i.e., $\emptyset \neq$ adherence $(\underline{\underline{G}})$ = adherence$(\underline{\underline{F}}) = \bigcap \overline{\{\bar{U}^c \cap V : V \in \underline{\underline{B}} \cap \underline{\underline{S}}\}} \subseteq \bigcap \{\overline{(\bar{U}^c)} \cap \bar{V} : V \in \underline{\underline{B}} \cap \underline{\underline{S}}\} = (\bigcap \bar{V}) \cap (\bar{U}^o)^c$
$= (\bigcap \bar{V}) \cap U^c$. Clearly, $\bigcap \bar{V} = \bigcap \{\bar{V} : V \in \underline{\underline{B}} \cap \underline{\underline{S}}\} \neq \{x_o\}$ since $\{x_o\} \cap U^c = \emptyset$. We at once conclude that $\underline{\underline{S}}$ is not Hausdorff, for in a Hausdorff space the intersection of the closures of all the open nbhds of a point must equal the singleton point. We have thus established that X cannot support any Hausdorff P-space topology strictly smaller than $\underline{\underline{T}}$. Naturally $(X, \underline{\underline{T}})$ is minimal HP.

(Q.E.D.)

It follows from theorem 3.2 of Chapter I that maximal Lindelöf T_2 spaces are nothing but the Lindelöf HP-spaces. Consequently a Lindelöf HP-space is HP-closed. We recall that in a Lindelöf space every open cover of the space has a countable open subcover. Weakening the condition of Lindelöfness by demanding that every open cover of the space should possess a countable proximate open subcover (i.e., a countable subfamily whose closures cover the space) is just to follow the genesis of H-closed spaces from compact T_2 spaces and lightly compact spaces from countably compact spaces, dwelt on in some detail in Sec. 5. Theorem 6.5 vindicates our emulation. Any Lindëlof Hausdorff P-space, according to theorem 6.4, is minimal HP. Does there exist a minimal HP-space which fails to be Lindelöf The following example affords us to offer an affirmative answer to this question.

6.9 Example of a non-Lindelöf minimal HP-space

Suppose ω_1 denotes the first uncountable ordinal. Let X be the set $\{a_{\alpha\beta}, b_{\alpha\beta}, c_\alpha : 1 \leq \alpha, \beta < \omega_1\} \cup \{a, b\}$, where $a_{\alpha\beta}, b_{\alpha\beta}, c_\alpha, a, b$ are all distinct.

The topology $\underline{T}$ on X is determined as follows :

$a_{\alpha\beta}$ and $b_{\alpha\beta}$ are isolated points, $1 \leq \alpha, \beta < \omega_1$.

Basic open neighbourhoods of c_α are of the form

$V_{\beta_0}(c_\alpha) = \{c_\alpha, a_{\alpha\beta}, b_{\alpha\beta} : 1 \leqq \beta_0 < \beta\}$, $1 \leqq \beta_0 < \omega_1$.

Basic open neighbourhoods of a are of the type

$V_{\alpha_0}(a) = \{a, a_{\alpha\beta} : \alpha_0 \leqq \alpha, 1 \leqq \beta < \omega_1\}$, $1 \leqq \alpha_0 < \omega_1$.

Basic open nbhds of b are as follows

$V_{\alpha_0}(b) = \{b, b_{\alpha\beta} : \alpha_0 \leqq \alpha, 1 \leqq \beta < \omega_1\}$, $1 \leqq \alpha_0 < \omega_1$.

$(X, \underline{\underline{T}})$ is a Hausdorff P-space which is not Lindelöf. With the assistance of theorems 6.5 and 6.8 it can be shown that $(X, \underline{\underline{T}})$ is a minimal HP-space.

6.10 <u>Lemma</u> : The continuous image of an HP-closed space into an HP-space is HP-closed.

<u>Proof</u>: Let f be a continuous function on an HP-closed space X into a Hausdorff P-space Y. In order to show that f(X) is HP-closed, let us consider an open cover $\{V_\alpha\}$ of f(X) i.e., $\bigcup V_\alpha \supseteq f(X)$ and V_α is open in Y. Consider the open cover of X formed by $\{f^{-1}(V_\alpha)\}$. According to theorem 6.5 there exists a countable subfamily $\{f^{-1}(V_n) : n \in N\}$ such that $\bigcup_{n=1}^{\infty} Cl_X f^{-1}(V_n) = X$. Now continuity of f implies that $f(Cl_X f^{-1}(V_n)) \subseteq Cl_Y V_n$ and, therefore, $f(X) \subseteq \bigcup_{n=1}^{\infty} Cl_Y V_n$. As $\{V_n\}$ is a countable

subfamily of $\{V_\alpha\}$, in view of theorem 6.5 again, $f(X)$ is HP-closed. (Q.E.D.)

6.11 Corollary: If $(X,\underline{\underline{T}})$ is HP-closed and $(X,\underline{\underline{S}})$ is a Hausdorff P-space such that $\underline{\underline{S}}$ is weaker than $\underline{\underline{T}}$, then $(X,\underline{\underline{S}})$ is HP-closed.

6.12 Proposition: A Hausdorff P-space is HP-closed iff its associated semiregular space is a minimal HP-space.

Proof: If a Hausdorff P-space $(X,\underline{\underline{T}})$ is HP-closed, the associated semiregular topology $(X,\underline{\underline{T}}_o)$ is, due to lemma 6.6 and Corollary 6.11, HP-closed and, no doubt, semiregular. By appealing to theorem 6.8 it is immediate that $(X,\underline{\underline{T}}_o)$ is a minimal HP-space. Conversely, if $(X,\underline{\underline{T}}_o)$ is minimal HP, $(X,\underline{\underline{T}}_o)$ is HP-closed and semiregular because of theorem 6.7. If $\underline{\underline{V}}$ is any open cover of $(X,\underline{\underline{T}})$ let $W = (\bar{V})^o$ for $V \varepsilon \underline{\underline{V}}$ [the closures and the interiors are in the topology of $(X,\underline{\underline{T}})$]. Now $W \varepsilon \underline{\underline{T}}_o$ and $V \subset W$ implying thereby that $\underline{\underline{W}} = \{W\}$ forms an open cover of $(X,\underline{\underline{T}}_o)$ and now we can extract a countable subfamily $\{W_n\}$ of $\underline{\underline{W}}$ with the property that $\bigcup_{n=1}^{\infty} \bar{W}_n = X$ (since closures in $\underline{\underline{T}}$ and $\underline{\underline{T}}_o$ are identical). Note that $W_n = (\bar{V}_n)^o$ for some $V_n \varepsilon \underline{\underline{V}}$ and, ipso facto, $\bar{W}_n = \bar{V}_n$. Then $\bigcup \bar{V}_n = \bigcup \bar{W}_n = X$. Applying theorem 6.5 we infer that $(X, \underline{\underline{T}})$ is HP-closed. (Q.E.D.)

Another characterisation of minimal HP-spaces will be described now.

6.13 <u>Theorem</u> : A necessary and sufficient condition that a Hausdorff P-space $(X, \underline{\underline{T}})$ be minimal HP is that $\underline{\underline{T}}$ should satisfy properties

HP(i) Every open filter closed under countable intersections has an adherent point.

HP(ii) If an open filter which is closed under countable intersections has a unique adherent point, then it converges to this point.

6.14 <u>Lemma</u> : An HP-space which satisfies HP(ii) also satisfies HP(i). Hence such a space is minimal HP.

<u>Proof</u>: Assume that there exists an open filter $\underline{\underline{G}}$ which is closed under countable intersections having no adherent point. Fix $p \varepsilon X$. Let $\underline{\underline{N}}$ be the filter of open neighbourhoods of p. Set $\underline{\underline{F}} = \{N \cup G : N \varepsilon \underline{\underline{N}}, G \varepsilon \underline{\underline{G}}\}$. $\underline{\underline{F}}$ is an open filter which is closed under countable intersections. p is the unique adherent point of $\underline{\underline{F}}$. By HP(ii) $\underline{\underline{F}}$ converges to p. By construction $\underline{\underline{F}}$ is weaker than $\underline{\underline{G}}$ so $\underline{\underline{G}}$ converges to p. Hence p is an adherent point of $\underline{\underline{G}}$. A contradiction. So HP(ii) implies HP(i) for an HP-space and hence, by theorem 6.13, an HP-space satisfying HP(ii) is minimal HP.

(Q.E.D.)

Proof of theorem 6.13: Necessity : Let $(X,\underline{\underline{T}})$ be minimal HP. Then $(X,\underline{\underline{T}})$ is HP-closed and so HP(i) must be satisfied (Theorem 6.5). To show that $\underline{\underline{T}}$ satisfies HP(ii), let $\underline{\underline{F}}$ be an open filter closed under countable intersections having a unique adherent point x(say). Then $\bigcap\{\bar{F} : F \in \underline{\underline{F}}\} = \{x\}$. Let $\underline{\underline{B}}$ denote the filter of $\underline{\underline{T}}$-open nbhds of x. We shall show that given $V \in \underline{\underline{B}}$ there exists $F \in \underline{\underline{F}}$ such that $F \subseteq V$. Obviously $F \cup V \in \underline{\underline{B}}$ for all $F \in \underline{\underline{F}}$. Suppose $\underline{\underline{F}}$ does not converge to x i.e., there exists $V_0 \in \underline{\underline{B}}$ such that $F \subseteq V_0$ for no $F \in \underline{\underline{F}}$. Consider $\underline{\underline{G}} = \{F \cup V : F \in \underline{\underline{F}}, V \in \underline{\underline{B}}\}$ which is contained in $\underline{\underline{B}}$. Consider the topology $\underline{\underline{S}}$ on X generated by the following neighbourhood systems :

If $y \neq x$, the neighbourhoods are unaltered and $\underline{\underline{G}}$ is the base for the neighbourhoods of x.

Since $\underline{\underline{G}} \subseteq \underline{\underline{B}}$, $\underline{\underline{S}}$ is weaker than $\underline{\underline{T}}$. In fact, $\underline{\underline{S}}$ is strictly weaker than $\underline{\underline{T}}$ as $V_0 \in \underline{\underline{T}}$ but $V_0 \notin \underline{\underline{S}}$. It is easily checked that x is a P-point in $(X,\underline{\underline{S}})$ so that $(X,\underline{\underline{S}})$ becomes a P-space. If we can separate any point y different from x by means of $\underline{\underline{S}}$-open sets it follows from the nature of the topology $\underline{\underline{S}}$ that it is Hausdorff. Let $y \in X$ such that $y \neq x$. Since $\underline{\underline{T}}$ is, originally, Hausdorff there are open sets U,V in $\underline{\underline{T}}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Now $V \in \underline{\underline{T}}$ and $x \notin V \Rightarrow V \in \underline{\underline{S}}$.

Again $y \neq x \Rightarrow$ there exists $F \in \underline{\underline{F}}$ such that $y \notin \bar{F}$ (the closure is with respect to $\underline{\underline{T}}$) i.e., $y \in (\bar{F})^c$. Again $x \notin (\bar{F})^c$ gives us the following : $(\bar{F})^c \in \underline{\underline{S}}$ so that $y \in V \cap (\bar{F})^c$ which belongs to $\underline{\underline{S}}$. Moreover, $V \cap (\bar{F})^c \subseteq V \cap F^c \subseteq U^c \cap F^c = (U \cup F)^c$. Now $x \in U \cup F$ which is an $\underline{\underline{S}}$-open nbhd of x. Consequently $(X,\underline{\underline{S}})$ is a Hausdorff space. So the minimal HP-space property of $(X,\underline{\underline{T}})$ is contradicted. Hence $\underline{\underline{F}}$ converges to x, i.e., $\underline{\underline{T}}$ satisfies HP(ii).

<u>Sufficiency</u> : Let $(X,\underline{\underline{T}})$ be a Hausdorff P-space satisfying HP(i) and HP(ii). Since $\underline{\underline{T}}$ satisfies HP(i), $(X,\underline{\underline{T}})$ is HP-closed. If $(X,\underline{\underline{T}})$ is semiregular, then it will be minimal HP(theorem 6.8). Let $x \in X$. Let $\underline{\underline{F}} = \{U \subseteq X : U$ is a regular-open set containing $x\}$. For every $V \in \underline{\underline{T}}$, $(\bar{V})^o$ is regular-open and $(\bar{V})^o = \bar{V}$. So $x \in \bigcap\{\bar{U} : U \in \underline{\underline{F}}\} = \bigcap\{\bar{V} : V \in \underline{\underline{T}}$ and $x \in V\} = \{x\}$ as $(X,\underline{\underline{T}})$ is Hausdorff. Thus x is the only cluster point of the open filter base $\underline{\underline{F}}$ which is closed under countable intersections. By HP(ii), $\underline{\underline{F}}$ must converge to x. So $\underline{\underline{F}}$ is a base for the neighbourhoods of x i.e., $\underline{\underline{T}}$ is semiregular as x is entirely arbitrary.

(Q.E.D.)

6.15 <u>Locally Lindelöf Hausdorff P-spaces</u> : These spaces, to be, henceforth, abbreviated as llh P-spaces, have been introduced in the previous Chapter in Sec.11 in connection with one-point

Lindelöf extensions of topological spaces. An important observation supplied by theorem 12.6 of Chapter I reads as follows :

6.16 Proposition : If $(X,\underline{\underline{T}})$ is a non-Lindelöf locally Lindelöf Hausdorff P-space, $\underline{\underline{T}}$ contains a strictly weaker maximal Lindelöf Hausdorff topology on X.

As mentioned earlier a maximal Lindelöf T_2 is a Lindelöf HP-space and, a fortiori, is a llh P-space as well. 6.16 at once yields that every non-Lindelöf llh P-space admits a strictly weaker llh P-space. This immediately gives rise to the following theorem.

6.17 Theorem : $(X,\underline{\underline{T}})$ is a minimal llh P-space only if $(X,\underline{\underline{T}})$ is a Lindelöf HP-space.

But Lindelöf HP-spaces, being identical with maximal Lindelöf T_2 spaces, have been earlier found to be minimal HP. As a corollary we therefore get :

6.18 Corollary : A minimal llh P-space is a minimal Hausdorff P-space.

Minimal HP-spaces need not, in general, be minimal llh P-spaces. Example 6.9 offers a ready illustration. The space $(X,\underline{\underline{T}})$ is minimal HP but not a locally Lindelöf Hausdorff P-space because if it were so, in virtue of theorem 6.4, proposition 6.16 and $(X,\underline{\underline{T}})$ being non-Lindelöf, $(X,\underline{\underline{T}})$ would turn out to be strictly stronger than a minimal HP-space which is impossible.

It has been established (Chap.I, Sec.11) that every non-Lindelöf locally Lindelöf Hausdorff P-space X has a unique one-point maximal Lindelöf extension (which turns out to be a Lindelöf HP-space) in which X is a dense open subset. Thus such a space always admits of a dense embedding into a llh P-space (since any Lindelöf HP-space is, trivially, a llh P-space). Consequently a llh P-space which is a closed subspace in every llh P-space in which it is embedded i.e., a <u>llh P-closed</u> space must be Lindelöf and, ipso facto, minimal HP. Summing up we get the following.

6.19 <u>Theorem</u> : The following are equivalent for a llh P-space X.

(i) X is minimal llh P-space.

(ii) X is llh P-closed.

(iii) X is minimal HP.

<u>Proof</u>: (i) ⇒ (ii): Suppose X is a minimal llh P-space. By theorem 6.17 X is a Lindelöf HP-space. Inasmuch as every Lindelöf subspace of a Hausdorff P-space is closed, X must be llh P-closed.

(ii) ⇒ (iii) : By the remark preceding this theorem, the llh P-closed space X is Lindelöf, so that X is a Lindelöf HP-space and is thus minimal HP.

(iii) $\Rightarrow$ (i) : Obvious. (Q.E.D.)

From proposition 6.16, theorem 6.19 and the fact that any Lindelöf Hausdorff P-space is minimal HP, it is evident that any locally Lindelöf Hausdorff P-space is a Katětov llh P-space.

7. Analytic and borelian spaces

Let X be any topological space (no separation axiom is assumed) whic is an USCO-compact image of irrationals Σ, i.e., there exists a correspondence f defined on Σ onto X satisfying

(i) for $\sigma \in \Sigma$, $f(\sigma)$ is a compact set in X and $X = \bigcup\{f(\sigma) : \sigma \in \Sigma\}$,

(ii) for any non-void open set U of X, $\{\sigma : f(\sigma) \subseteq U\}$ is an open set of Σ.

Let the topology of X be denoted by $\underline{\underline{T}}$. If $\underline{\underline{S}}$ is any topology on X weaker than $\underline{\underline{T}}$, then we prove the following useful lemma.

7.1 Lemma : $(X,\underline{\underline{S}})$ is an USCO-compact image of irrationals.

Proof: Let f be the USCO-compact correspondence under which $(X,\underline{\underline{T}})$ becomes an USCO-compact image of irrationals. As $\underline{\underline{S}}$ is weaker than $\underline{\underline{T}}$, $f(\sigma)$ is still compact in the topology $\underline{\underline{S}}$. Let $U \in \underline{\underline{S}}$, then $\{\sigma : f(\sigma) \subseteq U\}$ is open in Σ as U also belongs to $\underline{\underline{T}}$. Naturally, $(X,\underline{\underline{S}})$ is an USCO-compact image of Σ under the correspondence f. (Q.E.D.)

If, in the previous paragraph, the space $(X,\underline{\underline{T}})$ satisfies the following condition in addition to (i) and (ii) :

(iii) σ_1, σ_2 are distinct points of $\Sigma \Rightarrow f(\sigma_1) \cap f(\sigma_2) = \emptyset$.

We then call $(X,\underline{\underline{T}})$ a DUSCO-compact image of Σ. The proof of lemma 7.1 will immediately lead us to infer.

7.2 Lemma : If $(X,\underline{\underline{T}})$ is a DUSCO-compact image of Σ then so also $(X,\underline{\underline{S}})$ where $\underline{\underline{S}}$ is any topology weaker than $\underline{\underline{T}}$.

Z. Frolik [Fr] defines an analytic space X as a Tychonoff space which is an USCO-compact image of Σ. According to him a borelian space X is a Tychonoff space that is a DUSCO-compact image of Σ.

Our main results about minimal analytic spaces and minimal borelian spaces are recorded in the following theorems.

7.3 Theorem : A minimal analytic space is compact.

7.4 Theorem : A minimal borelian space is compact.

Our proofs for both the theorems are almost identical and rely on the fact, first proved by Banaschewski [Ba1], that every minimal Tychonoff space is compact.

Proof of theorem 7.3 : Let $(X,\underline{\underline{T}})$ be a minimal analytic space. If it is not compact, as X is a Tychonoff space it cannot be a minimal Tychonoff space. So there exists a strictly weaker

Tychonoff (or equivalently completely regular T_2) topology $\underline{\underline{S}}$ on X. Since $(X,\underline{\underline{S}})$ is a Tychonoff space, an application of lemma 7.1. yields that it is also an analytic space. $\underline{\underline{S}}$ is strictly weaker than $\underline{\underline{T}}$ now leads to a contradiction. So $(X,\underline{\underline{T}})$ is a compact space.

(Q.E.D.)

<u>Proof of theorem 7.4</u>: Same as the proof of theorem 7.3, the only change being the use of Lemma 7.2 in lieu of lemma 7.1.

(Q.E.D.)

Let us make the following observation [Fr].

7.5 <u>Lemma</u> : Any analytic space is Lindelöf and hence realcompact.

Alternative proofs of theorems 7.3 and 7.4 can be constructed based on the previous lemma and theorem 3.4. The proof of theorem 7.4 follows. The same proof works for theorem 7.3 mututis mutandis.

7.6 <u>Alternative proof of theorem 7.4</u> : Let us note that any borelian space is analytic and as a result realcompact (lemma 7.5). If $(X,\underline{\underline{T}})$ is any minimal borelian space which is not compact, it is definitely not minimal realcompact (theorem 3.4). Naturally, X admits a strictly weaker realcompact topology, say, $\underline{\underline{S}}$. As any realcompact space, by definition, is Tychonoff lemma 7.2 implies that $(X,\underline{\underline{S}})$ is a borelian space. A contradiction to the assumption that $(X,\underline{\underline{T}})$ is minimal borelian.

(Q.E.D.)

Suppose $(X,\underline{\underline{T}})$ is a compact Hausdorff space. Then the correspondence which associates X to every $\sigma \varepsilon \Sigma$ is obviously as USCO-compact correspondence. Consequently every compact Hausdorff space is analytic. This observation together with theorem 7.3 characterises minimal analytic spaces as mentioned below.

7.7 Theorem : The class of minimal analytic spaces is identical with that of compact T_2 spaces.

Proof: The only fact that remains to be shown is that every compact T_2 space is minimal analytic. A compact T_2 space is analytic and minimal T_2 at the same time. Naturally it is minimal analytic. (Q.E.D.)

Any non-compact analytic space X, being Tychonoff, can be embedded into its Stone-Čech compactification βX which is analytic being compact. As a result such a space always admits a dense embedding into another analytic space. This clearly indicates that an analytic-closed space must be compact. We have proved:

7.8 Theorem : The following are equivalent for an analytic space X

(i) X is minimal analytic

(ii) X is analytic-closed

(iii) X is compact T_2

We shall presently derive the analogue of theorem 7.7 for borelian spaces. The main fact needed is that every compact T_2 space is borelian. We shall sketch the proof below.

Let X be a compact T_2 space. Fix $\sigma_o \varepsilon \Sigma$. Define the correspondence f from Σ into X as follows :

$$f(\sigma_o) = X$$

$$f(\sigma) = \emptyset \quad \text{if} \quad \sigma \varepsilon \Sigma - \{\sigma_o\}$$

Now $\{\sigma : f(\sigma) \subset U\} = \Sigma \qquad \text{if} \quad U = X$

$$= \Sigma - \{\sigma_o\} \text{ if } \quad U \text{ is any open set} \neq X.$$

$\therefore$ f is, obviously a DUSCO-compact correspondence and X is the immage of Σ under f. So X is a borelian space. In fact X is minimal borelian.

Combining this fact with theorem 7.4 we conclude the following :

7.9 <u>Theorem</u> : A borelian space is compact if and only if it is minimal borelian.

As any compact T_2 space is shown to be borelian, the arguments showing the compactness of any analytic-closed space also show that any borelian-closed space is compact. We can assert the following.

7.10 Theorem : If X is any borelian space, the following are equivalent

(i) X is minimal borelian

(ii) X is borelian-closed

(iii) X is compact T_2.

7.11 Connection between classical analytic and Borel spaces and analytic and borelian spaces of Frolik : Z. Frolik [Fr] has clearly established that restricted to separable metric spaces his analytic and borelian spaces coincide with analytic spaces and Borel spaces respectively of Kuratowski [Ku]. Using this simple connection we shall cite an example of a borelian space which is not Katětov borelian.

7.12 Example

Let Q denote the set of rationals with usual topology. Q is known to be a classical Borel space and so, by remarks in 7.11, is a borelian space. As mentioned earlier that the topology of Q is not stronger than any compact Hausdorff topology, theorem 7.4 indicates that Q cannot be Katětov borelian. As any borelian space is analytic and any minimal analytic space is also compact this example automatically shows that Q is not Katětov analytic too.

Needless to say that both minimal analytic and minimal borelian spaces are of second category.

8. Subspaces:

If π is any one of the properties Realcompact, first countable realcompact, analytic or borelian, a closed subspace of a π space X is also a π space. Next theorem describes the minimal π subspaces of minimal π spaces.

8.1 Theorem : Let π denote any one of realcompact, first countable realcompact, analytic or borelian and let X be a minimal π space. The collection of minimal π subspaces of X coincides with that of closed subspaces of X.

Proof: By theorem 3.4, 3.5, 7.7 and 7.9, a π space X is minimal π iff X is compact Hausdorff. The conclusion follows now from the well-known fact that a subset of a compact T_2 space is compact iff it is closed.

(Q.E.D.)

8.2 In virtue of theorem 4.3 minimal locally H-closed spaces and minimal Hausdorff spaces are identical. So the collection of minimal locally H-closed subspaces of a minimal locally H-closed space X coincides with class of all H-closed subspaces of X which are semiregular.

Let us note the following characterisation of lightly compact spaces [BaCM].

8.3 Lemma : A space X is lightly compact iff the closure of every open subset is lightly compact.

8.4 Theorem : A space X is E_1-closed if and only if the closure of every open subset of X is E_1-closed.

Proof: Suppose a space X is E_1-closed. Then by theorem 5.5 X is lightly compact. By application of lemma 8.3 the closure of every open subset of X is lightly compact. As any subspace of an E_1-space is E_1, the closure of every open subset is lightly compact and E_1 implying thereby that it is E_1-closed (Theorem 5.5). The sufficiency part is trivial for X is the closure of an open set, namely X. (Q.E.D.)

The following theorem is an immediate consequence of theorem 8.4 and theorem 5.7.

8.5 Theorem : Let X be a minimal E_1 space and let A be an open subset of X. Then $\bar{A}$ is minimal E_1 if and only if it is semi-regular (c.f. theorem 5.2 of Stephenson [Ste 3]).

8.6 Remark : In the above theorem the assumption of semiregularity is needed. For, an example can be constructed such that closure of an open set A need not be semiregular. The example follows (due to Stephenson [Ste 3]).

8.7 Example:

Let R be the set of real numbers. Let us choose two points $a, b \notin R$ and let $E = R \cup \{a, b\}$ with topology $\underline{T}$ defined by neighbourhoods :

$$\underline{\underline{W}}(x) = \{V \subset E : V \supset (x-d, x+d) \text{ for some } d > 0\} \text{ if } x \in R.$$

$$\underline{\underline{W}}(a) = \{V \subset E : V \supset \{a\} \cup \bigcup_{n=N}^{\infty} (2n, 2n+1) \cup \bigcup_{n=-N}^{-\infty} (2n, 2n+1) \text{ for some integer } N\}$$

$$\underline{\underline{W}}(b) = \{V \subset E : V \supset \{b\} \cup \bigcup_{n=N}^{\infty} (2n-1, 2n) \cup \bigcup_{n=-N}^{-\infty} (2n-1, 2n) \text{ for some integer } N\}$$

$(E, \underline{\underline{T}})$ is a first countable minimal Hausdorff space and so is minimal E_1. Let A be the open subset $\bigcup_{n=1}^{\infty} (2n, 2n+1)$. Then $\bar{A} = \bigcup_{n=1}^{\infty} [2n, 2n+1] \cup \{a\}$. A basic nbhd of a in $\bar{A}$ is of the form $\{a\} \cup \bigcup_{n=N}^{\infty} (2n, 2n+1)$ for some positive integer N. Let V be one such of the form $\{a\} \cup \bigcup_{n=N_0}^{\infty} (2n, 2n+1)$. Then, Cl V in $\bar{A}$ = $\bar{A} \cap \bar{V}$ ('$-$' denoting the closure in E) $= \bar{A} \cap (\{a\} \cup \bigcup_{n=N_0}^{\infty} [2n, 2n+1])$ $= \{a\} \cup \bigcup_{N_0}^{\infty} [2n, 2n+1]$. For any $n > 0$, $(2n-\frac{1}{2}, 2n+1+\frac{1}{2})$ is open in E and, therefore, $\bar{A} \cap (2n-\frac{1}{2}, 2n+1+\frac{1}{2})$ is open in $\bar{A}$ i.e., $[2n, 2n+1]$ is open in $\bar{A}$. Consequently $\{a\} \cup \bigcup_{n=N_0}^{\infty} [2n, 2n+1]$ is open in $\bar{A}$. This implies that V is not a regular open subset of $\bar{A}$. Then $\bar{A}$ cannot be semiregular.

We know that every closed subset of a countably compact space is countably compact. Further if every closed subset is countably compact then so is the space. But in the case of T_1 spaces we can state a much stronger version as follows (which, incidentally, strengthens theorem 5.6 of Stephenson [Ste 3]).

8.8 <u>Theorem</u> : A T_1 space X is countably compact if and only if every proper closed subset of X is pseudocompact.

<u>Proof</u>: Necessity is obvious. For the sufficiency part we take a countably infinite proper subset C of X. C has a cluster point in X. For, if not no x in X is an accumulation point of C i.e. $x \in X \Rightarrow x \in \overline{C - \{x\}}$ i.e., C is a closed discrete subset of X. Since C is infinite, C cannot be pseudocompact. A contradiction to the hypothesis. If X itself is countably infinite we have to show that X has accumulation points. Suppose not, then X is a discrete space. Let $x \in X$. The $C = \{x\}^c$ is a proper countably infinite subset of X which is, in fact, closed in X. But C is not pseudocompact. A contradiction. So X is countably compact.

(Q.E.D.)

<u>Remark</u>: The proof of theorem 8.8 shows that we can as well assume 'every countable closed subset of X is pseudocompact' instead of 'every proper closed subset of X is pseudocompact'. So it strengthens theorem 5.6 of Stephenson as stated earlier.

8.9 Theorem : Every closed subset of a minimal E_1 (or equivalently minimal first countable Hausdorff) space X is minimal E_1 (or equivalently minimal first countable Hausdorff) if and only if X is countably compact.

Proof: Sufficiency follows from Propositions 5.3 and 5.17. For necessity, we observe that every closed subset of X is minimal E_1 $\Rightarrow$ every closed subset of X is E_1-closed $\Rightarrow$ every closed subset of X is lightly compact $\Rightarrow$ every subset of X is pseudocompact. By theorem 8.8 it immediately follows that X is countably compact.

(Q.E.D.)

Let X be any E_1 space such that every closed subset of X is E_1-closed. Then every closed subset is lightly compact and, a fortiori, pseudocompact. So X must be countably compact by means of theorem 8.8. Theorem 5.3 then implies that X is minimal E_1. So we have come to the conclusion.

8.10 Theorem : An E_1 space X in which every proper closed subset is E_1-closed is countably compact and, ipso facto, minimal E_1.

8.11 Corollary: A first countable Hausdorff space in which every proper closed subset is first countable T_2-closed is countably compact and, therefore, minimal first countable T_2.

Remark: Corollary 8.11 reminds us of the following result due to M.H.Stone [Sto] : A Hausdorff space in which every proper closed subset is H-closed is compact and, a fortiori, minimal Hausdorff.

The following theorem is an important step in our attempt to deterime the minimal HP subspaces of a minimal HP-space.

8.12 Theorem : A space X is HP-closed if and only if the closure of every nonvoid open subset of X is HP-closed.

Proof: Suppose X is HP-closed. Let A be any non-void open set. To show that $\bar{A}$ (= the closure of A in X) is HP-closed. Let $\underline{\underline{F}}$ be an open filter on $\bar{A}$ such that $\underline{\underline{F}}$ is closed under countable intersections. By theorem 6.5(b) we are required to show that $\underline{\underline{F}}$ has nonvoid adherence in $\bar{A}$. Now adherence of $\underline{\underline{F}}$ in $\bar{A}$ = $\bigcap\{Cl_{\bar{A}}(F) : F \in \underline{\underline{F}}\} = \bigcap\{\bar{F} : F \in \underline{\underline{F}}\}$. Let $\underline{\underline{G}} = \{F \cap A : F \in \underline{\underline{F}}\}$. $\underline{\underline{G}}$ is open filter base on X. Further if $\{G_n = F_n \cap A ; n \geq 1\}$ forms a countable collection of sets from $\underline{\underline{G}}$, $\bigcap_{n=1}^{\infty} G_n = (\bigcap_{n=1}^{\infty} F_n) \cap A$ = $F \cap A$ where $F = \bigcap_{n=1}^{\infty} F_n \in \underline{\underline{F}}$ (by the property of $\underline{\underline{F}}$) and, hence, $F \cap A \in \underline{\underline{G}}$. So $\bigcap_{n=1}^{\infty} G_n \in \underline{\underline{G}}$. Thus $\underline{\underline{G}}$ is the filter base for an open filter $\underline{\underline{H}}$ on X which is closed under countable intersections. As X is HP-closed, adherence of $\underline{\underline{G}}$ = adherence of $\underline{\underline{H}} \neq \emptyset$. But adherence of $\underline{\underline{G}} = \bigcap\{\overline{F \cap A} : F \in \underline{\underline{F}}\} \subseteq \bigcap\{\bar{F} : F \in \underline{\underline{F}}\}$ = adherence of $\underline{\underline{F}}$ in $\bar{A}$. So $\underline{\underline{F}}$ has nonvoid adherence in $\bar{A}$. For the converse X is trivially a nonvoid open subset of X and so X is HP-closed.

(Q.E.D.)

We immediately arrive at, using the above theorem and theorem 6.8 :

8.13 Theorem : Let X be a minimal HP-space and let A be an open subset of X. $\bar{A}$ is minimal HP if and only if it is semiregular.

It would have been pleasant to prove the following assertion which resembles Corollary 8.11. But we have yet to know whether it is valid or not.

Assertion : If every proper closed subset of an HP-closed space is HP-closed, the space is Lindelöf.

9. Products

9.1 Theorem : Let $\{X_i : i \in I\}$ be a collection of Hausdorff spaces and let $X = \prod_{i \in I} X_i$. If π stands for any one of realcompact, analytic or borelian, X is minimal π if and only if each X_i is minimal π.

Proof: By dint of theorems 3.4, 7.7 and 7.9, X is minimal π $\iff$ X is compact T_2 $\iff$ X_i is compact T_2 for each $i \in I$ $\iff$ X_i is minimal π for each $i \in I$.

(Q.E.D.)

In the following theorem M denotes a subset of N, the natural numbers.

9.2 Theorem: Let $\{X_n : n \in M\}$ be a collection of topological spaces and let $X = \prod_{n \in M} X_n$. Then X is minimal first countable realcompact if and only if each X_n is minimal first countable realcompact.

Proof: In virtue of Theorem 3.5, X is minimal first countable realcompact $\iff$ X is compact first countable Hausdorff $\iff$ X_n is compact first countable T_2 for each $n \in M \iff X_n$ is minimal first countable realcompact for each $n \in M$. (Q.E.D.)

9.3 Proposition : Let $\{X_n : n \in N\}$ be a collection of topological spaces and let $X = \prod_{n \in N} X_n$. A necessary and sufficient condition for X to be E_1 is that each X_n be E_1.

Proof: Suppose X is E_1. Fix $n \in N$. Let $x_n \in X_n$. Choose a point $x \in X$ such that $pr_n(x) = x_n$ (here pr_m denotes the projection map of X onto X_m, $m \in N$). Since X is E_1 there exist open sets G_i $(i \in N)$ such that $\{x\} = \bigcap_{i=1}^{\infty} \bar{G}_i$ (where $\bar{G}_i$ = closure of G_i in X). Without loss of generality we can assume that G_i's are basic open sets. Put $U_i(n) = pr_n(G_i)$. We claim $\{x_n\} = \bigcap_{i=1}^{\infty}$ (Cl $U_i(n)$ in X_n). Since G_i's are basic open sets, Cl $U_i(n)$ in $X_n = pr_n(\bar{G}_i)$. So we need only to show that $\{x_n\} = \bigcap_{i=1}^{\infty} pr_n(\bar{G}_i)$. If possible, let $y_n \neq x_n$ be such that $y_n \in \bigcap_{i=1}^{\infty} pr_n(\bar{G}_i)$. Consider the point $z \in X$ such that $pr_m(z) = pr_m(x)$ for $m \neq n$ and $pr_n(z) = y_n$. Then $z \in \bar{G}_i$ for each i.

So $z \in \bigcap_{i=1}^{\infty} \bar{G}_i$ i.e., $z = x$. A contradiction. Thus we conclude that $\{x_n\} = \bigcap_{i=1}^{\infty} (\text{Cl } U_i(n))$ which shows that X_n is E_1. Hence, X is $E_1 \Rightarrow X_n$ is E_1 for each n. Conversely if each X_n is E_1 we would like to show that X is E_1. Let $x = (x_n) \in X$. Since each X_n is E_1 we can suppose that $\{x_n\} = \bigcap_{i=1}^{\infty} (\text{Cl } U_i(n)$ in $X_n)$ where each $U_i(n)$ is open in X_n. Define $G_{n_1 n_2 \ldots n_k} = U_{n_1}(1) \times U_{n_2}(2) \times \ldots \times U_{n_k}(k) \times X_{k+1} \times \ldots$ for $n_1, n_2, \ldots, n_k \in N$, $k \in N$. Then $\overline{G}_{n_1 n_2 \ldots n_k} = \text{Cl}_1\, U_{n_1}(1) \times \text{Cl}_2\, U_{n_2}(2) \times \ldots \times \text{Cl}_k\, U_{n_k}(k) \times X_{k+1} \times \ldots$ where $\text{Cl}_i\, U_{n_i}(i)$ stands for $\text{Cl } U_{n_i}(i)$ in X_i, $i \in N$. Now $\{x\} = \bigcap_{k=1}^{\infty} \bigcap_{\substack{n_i=1 \\ 1 \leq i \leq k}}^{\infty} \overline{G}_{n_1 n_2 \ldots n_k}$.

So X is an E_1-space. (Q.E.D.)

9.4 Corollary : Statement of the previous proposition remains valid even if N is replaced by any non-empty subset M of N.

9.5 Theorem : Let $\{X_n : n \in M\}$ be a collection of topological spaces and $X = \prod X_n$. Then X is E_1-closed if each X_n is E_1-closed, and if (except perhaps for one value of n) each non-P point of X_n has a countable base of neighbourhoods in X_n.

Proof: Let us recall that a point x of a space Y is said to be a P-point in Y if every intersection of countably many nbhds of x is again a nbhd of x. Now theorem 4.4 of Scarborough and Stone [SSo, p 141] states : If $X = \prod X_a$, where X_a is lightly compact, and if (except perhaps for one value of a) each non-P-point of X_a has a countable base of nbhds in X_a, then X is lightly compact. Since each X_n is E_1-closed, X_n is lightly compact by theorem 5.5. According to our hypothesis X is lightly compact by invoking the theorem mentioned above. By Corollary 9.4, X is also E_1. So X is a lightly compact E_1 space and hence E_1-closed in view of theorem 5.5 again. (Q.E.D.)

9.6 <u>Theorem</u> : In order that the product $X = \prod X_n$ of a collection of topological spaces $\{X_n : n \in M\}$ be E_1-closed it is necessary that each X_n be E_1-closed.

<u>Proof</u>: Corollary 9.4 implies that each X_n is E_1 when X is E_1. Since X is E_1-closed $\Rightarrow$ X is lightly compact (theorem 5.5), by invoking theorem 4.2 in [SSo, p 141] we conclude that each X_n is lightly compact so that each X_n is E_1-closed. (Q.E.D.)

9.7 <u>Theorem</u> : Let $\{X_n : n \in M\}$ be a family of topological spaces and let $X = \prod X_n$. Then X is minimal E_1 if and only if each X_n is minimal E_1.

<u>Proof</u>: This theorem is identical with the theorem 4.3 of Stephenson [Ste 3, p 123] because of our proposition 5.17

which says that minimal E_1 spaces are nothing but the minimal first countable T_2 spaces. (Q.E.D.)

We recall that P-spaces are closed under finite products. Naturally we would want to find out whether finite products of HP-closed spaces are HP-closed. We do not know whether the product of even two HP-closed spaces is HP-closed or not. We shall discuss this point now in detail. It is well-known that any product of H-closed spaces is H-closed. But the proof, even in the case of product of two such spaces, depends on the fact that every open filter on a topological space is contained in a maximal open filter (obviously with the aid of Zorn's lemma) [CF]. Since the characterisations of HP-closed spaces depend not merely on open filters but open filters which are closed under countable intersections, and Zornification is not applicable to open filters closed under countable intersections in order to obtain maximal such open filters, we cannot imitate the proof for the product of H-closed spaces in the case of HP-closed spaces. Nevertheless we can establish theorem 9.9. The following proposition is important for the proof of theorem 9.9.

9.8 Proposition : Let X_1 be a Lindelöf space and X_2 a P-space. Then the projection map $pr_2 : X_1 \times X_2 \to X_2$ is a closed map.

Proof : Let $F \subseteq X_1 \times X_2$ be a closed subset. We are required to show that $pr_2(F)$ is a closed set in X_2. Let $y \notin pr_2(F)$.

Then for every $x \in X_1$, $(x,y) \in F^c$ and F^c is open in $X_1 \times X_2$. There exist open sets U_x and V_x such that $x \in U_x \subseteq X_1$ and $y \in V_x \subseteq X_2$ and $(x,y) \in U_x \times V_x \subseteq F^c$. Consider $\{U_x : x \in X_1\}$. It is an open cover for X_1 and has a countable subcover $\{U_{x_n} : n \geq 1\}$. Consider the sets $\{V_{x_n}\}$. Each is a nbhd of y and X_2 is a P-space; so $V = \bigcap_{n=1}^{\infty} V_{x_n}$ is also a nbhd of y. Easy to see that $V \cap pr_2(F) = \emptyset$. Naturally, $pr_2(F)$ must be closed in X_2.

(Q.E.D.)

9.9 <u>Theorem</u> : If X_1 is a Lindelöf Hausdorff P-space and X_2 an HP-closed space, the product $X_1 \times X_2$ is HP-closed.

<u>Proof</u>: Let $\underline{\underline{F}}$ be an open filter on $X_1 \times X_2$ such that every countable intersection of sets in $\underline{\underline{F}}$ is also in $\underline{\underline{F}}$. By theorem 6.5 we are required to show that $\underline{\underline{F}}$ has non-empty adherence. Let pr_i $(i = 1,2)$ stand for the projection of $X_1 \times X_2$ onto X_i. Let $\underline{\underline{E}} = \{pr_2(F) : F \in \underline{\underline{F}}\}$. Every member of $\underline{\underline{E}}$ is a non-void open subset of X_2. Let A be an open subset of X_2 such that $A \supseteq E = pr_2(F)$ for some $F \in \underline{\underline{F}}$. Then easy to see that $X_1 \times A \supseteq F$ and, hence, $X_1 \times A \in \underline{\underline{F}}$ so that $A = pr_2(X_1 \times A)$ $\in \underline{\underline{E}}$. Thus $\underline{\underline{E}}$ is an open filter on X_2. Let $E_n = pr_2(F_n) \in \underline{\underline{E}}$, $n \geq 1$. Then $\bigcap_{n=1}^{\infty} E_n \supseteq pr_2 \left(\bigcap_{n=1}^{\infty} F_n \right) = pr_2(F)$ where $F \in \underline{\underline{F}}$. If $E = pr_2(F)$ then $E \neq \emptyset$ $E \in \underline{\underline{E}}$ and $\bigcap_{n=1}^{\infty} E_n \supseteq E$. So $\bigcap_{n=1}^{\infty} E_n \in \underline{\underline{E}}$ as $\underline{\underline{E}}$ is an open filter and $\bigcap E_n$ is an open set as X_2 is a P-space. Since X_2 is HP-closed, by theorem 6.5

$\underline{\underline{E}}$ has non-empty adherence. Let $x_2 \in$ adherence $(\underline{\underline{E}})$. Then $x_2 \in$ closure of $pr_2(F)$ in X_2 for each $F \in \underline{\underline{F}}$. Now since X_1 is a Lindelöf HP-space and X_2 is a P-space pr_2 is a closed map (Proposition 9.8). So $pr_2(\bar{F})$ equals the closure of $pr_2(F)$ in X_2 for each $F \in \underline{\underline{F}}$(where $\bar{F}$ denotes the closure of F in $X_1 \times X_2$). Therefore $\bar{F} \cap (X_1 \times \{x_2\}) \neq \emptyset$ for each $F \in \underline{\underline{F}}$. Put $\underline{\underline{G}} = \{\bar{F} \cap (X_1 \times \{x_2\}) : F \in \underline{\underline{F}}\}$. Let $\{G_n : n \geq 1\} \subset \underline{\underline{G}}$. Then $\bigcap_{n=1}^{\infty} G_n = (\bigcap \bar{F}_n) \cap (X_1 \times \{x_2\}) \supset \overline{\bigcap F_n} \cap (X_1 \times \{x_2\})$ $= G$ (say) so that $G \in \underline{\underline{G}}$. So $\underline{\underline{G}}$ is a filter base on $X_1 \times X_2$. Consider $pr_1(\underline{\underline{G}})$. Obviously it is a filter base on X_1. Moreover, if $\{pr_1(G_n) : n \geq 1\}$ is a countable subfamily of $pr_1(\underline{\underline{G}})$ then $\bigcap_{n=1}^{\infty} pr_1(G_n) \supset pr_1(\bigcap_{n=1}^{\infty} G_n) \supset pr_1(G)$ where $G \in \underline{\underline{G}}$ (as shown earlier) i.e., every countable intersection of sets in $pr_1(\underline{\underline{G}})$ contains a set of $pr_1(\underline{\underline{G}})$. Let $\underline{\underline{D}}$ be the filter on X_1 generated by $pr_1(\underline{\underline{G}})$. Then $\underline{\underline{D}}$ is closed under countable intersections. Again X_1 is Lindelöf, so $\emptyset \neq$ adherence of $\underline{\underline{D}}$ = adherence of $pr_1(\underline{\underline{G}})$. Let $x_1 \in$ adherence of $pr_1(\underline{\underline{G}})$. Let U be any open set $\subset X_1$ and let $x_1 \in U$. Then $U \cap pr_1(G) \neq \emptyset$ for each $G \in \underline{\underline{G}}$ i.e., $(U \times X_2) \cap G \neq \emptyset$ for each $G \in \underline{\underline{G}}$ i.e., for each $F \in \underline{\underline{F}}$, $\emptyset \neq (\bar{F} \cap (X_1 \times \{x_2\})) \cap (U \times X_2) = \bar{F} \cap (X_1 \times \{x_2\}) \cap (U \times X_2) = \bar{F} \cap (U \times \{x_2\})$. So, $(x_1\ x_2) \in \bar{F}$ for each $F \in \underline{\underline{F}}$ i.e., $(x_1, x_2) \in \bigcap \{\bar{F} : F \in \underline{\underline{F}}\}$ = adherence of $\underline{\underline{F}}$. Consequently $\underline{\underline{F}}$ has a non-void adherence. (Q.E.D.)

<u>Remark</u> : We have seen earlier that every Lindelöf Hausdorff P-space is HP-closed (in fact, minimal HP). It has been also observed that if X_1 and X_2 are two Lindelöf HP-spaces and if the product space $X_1 \times X_2$ is Lindelöf then $X_1 \times X_2$ is again a Lindelöf Hausdorff P-space and, a fortiori, HP-closed. But theorem 9.9 provides us with the information that even if $X_1 \times X_2$ fails to be Lindelöf, $X_1 \times X_2$ retains the property of being HP-closed.

9.10 <u>Theorem</u> : Let $\{X_i : 1 \leqq i \leqq n\}$ be Hausdorff P-spaces and let $X = \prod_{i=1}^{n} X_i$. If X is HP-closed then so also is each X_i $(1 \leqq i \leqq n)$.

<u>Proof</u>: Let us fix i. Let us take an open filter $\underline{\underline{F}}$ on X_i such that $\underline{\underline{F}}$ is closed under countable intersections. Consider the filter base of open sets $pr_i^{-1}(\underline{\underline{F}})$ on X. Let $\{pr_i^{-1}(F_n) : n \geqq 1\}$ be a countable subfamily of $pr_i^{-1}(\underline{\underline{F}})$. $\bigcap_{n=1}^{\infty} pr_i^{-1}(F_n) = pr_i^{-1}(\bigcap_{n=1}^{\infty} F_n) = pr_i^{-1}(F) \in pr_i^{-1}(\underline{\underline{F}})$ where $F = \bigcap_{n=1}^{\infty} F_n \in \underline{\underline{F}}$ by assumptions on $\underline{\underline{F}}$. The open filter $\underline{\underline{G}}$ on X generated by $pr_i^{-1}(\underline{\underline{F}})$ is then closed under countable intersection. As X is HP-closed, $\emptyset \neq$ adherence $(\underline{\underline{G}})$ = adherence $(pr_i^{-1}(\underline{\underline{F}}))$. Let $x \in$ adherence $(pr_i^{-1}(\underline{\underline{F}}))$, then $x_i = pr_i(x) \in$ closure of F in X_i for each $F \in \underline{\underline{F}}$, i.e., x_i is an adherent point of $\underline{\underline{F}}$. So X_i is HP-closed.

(Q.E.D.)

9.11 Theorem : Let X_1 be a Lindelöf HP-space and X_2 a minimal HP-space. Then the product space $X_1 \times X_2$ is minimal HP. If $\{X_i : 1 \leq i \leq n\}$ is a finite collection of Hausdorff P-spaces and their product space $X = \prod_{i=1}^{n} X_i$ is minimal HP then each factor X_i is also minimal HP.

Proof: Since both X_1 and X_2 are semi-regular (Theorem 6.7), $X_1 \times X_2$ is also semi-regular. As X_2 is HP-closed, $X_1 \times X_2$ is HP-closed by theorem 9.9 and, in virtue of theorem 6.8, $X_1 \times X_2$ becomes minimal HP. For the second assertion, as X is minimal HP, it is HP-closed and semi-regular. Then each factor X_i becomes a semiregular space and by theorem 9.10 each X_i is HP-closed. Consequently, invoking theorem 6.8 we conclude that each X_i is minimal HP. (Q.E.D.)

10. Embedding of π spaces into π-closed and minimal π spaces

It is well-known that every Tychonoff space X can be densely embedded into a compact Hausdorff space, namely in its Čech-Stone compactification βX. We have observed that a minimal realcompact space or a realcompact-closed space is a compact T_2 space and vice versa (Theorem 3.4). Thus each realcompact space X can be embedded densely in a realcompact-closed or a minimal realcompact space, because in each case βX serves our purpose. So we arrive at the following theorem.

10.1 Theorem : Any realcompact space admits of a dense embedding into a minimal realcompact space (or equivalently, into a realcompact-closed space).

If we start out with a first countable realcompact space X, the Stone-Čech compactification βX is not goint to be first countable and fails to act as a first countable compactification of X. In order that X can be densely embedded in a minimal first countable realcompact space, X should possess a first countable compactification. As a matter offact the following theorem is true.

10.2 Theorem : In order that a first countable realcompact space X be densely embeddable in a minimal first countable realcompact (or first countable realcompact-closed) space it is necessary and sufficient that X have a first countable Hausdorff compactification.

Results concerning the existence of first countable compactifications of first countable Tychonoff spaces are, in general, absent. We can mention one condition under which a first countable Tychonoff space cannot admit any first countable Hausdorff compactification. Arhangel'skii [A] has shown that no first countable compact T_2 space can have cardinality greater than c. Thus if we begin with a first countable Tychonoff space X with cardinality exceeding c, X cannot have any first countable Hausdorff compactification.

In view of the facts that a locally H-closed space X always admits a one-point H-closed extension X' such that X is embedded densely in X' and an H-closed space is locally H-closed-complete we can state the following :

10.3 Theorem : Suppose X is a locally H-closed (non H-closed) space. Then X can be embedded densely in a locally H-closed-complete space.

Proof: Since X is non H-closed, X is densely embedded in its one-point H-closed extension X' (Theorem 4.1). Since X' is H-closed, X is locally H-closed-complete. So the theorem is proved.

(Q.E.D.)

Banaschewski [Ba2] obtained the following theorem :

10.4 Theorem: A Hausdorff space X can be densely embedded in a minimal Hausdorff space if and only if it is semiregular.

It has been earlier observed that minimal Hausdorff spaces are precisely the minimal locally H-closed spaces. Now we can surely conclude the following (a locally H-closed space being always T_2) :

10.5 <u>Theorem</u> : A locally H-closed space X can be embedded densely in a minimal locally H-closed space if and only if X is semi-regular.

In the next few theorems we shall study embedding problems for E_1-spaces.

10.6 <u>Theorem</u> : Let X be an E_1-space, and let $\underline{N} = \{\underline{F} : \underline{F}$ is a countable open filter base on X which has no adherent points $\}$, and let $\underline{M}$ be a subset of $\underline{N}$ which is maximal with respect to having the property that whenever $\underline{F}, \underline{G} \in \underline{M}$, $\underline{F} \neq \underline{G}$, then there exist sets $F \in \underline{F}$ and $G \in \underline{G}$ such that $F \cap G = \emptyset$. Let $Y = X \cup \underline{M}$, topologised as follows : a set $U \subset Y$ is open if and only if (i) $U \cap X$ is open in X, and (ii) if $\underline{F} \in U$, then $U \cap X$ contains a set belonging to $\underline{F}$. Then Y is an E_1-closed space in which X is embedded as a dense open subspace.

<u>Proof</u> : Let us first show that Y is an E_1-space. Let $y \in Y$. If $y = x$ for some $x \in X$, we can get open subsets $U_n \subset X$ such that $\{x\} = \bigcap_{n=1}^{\infty} Cl_X(U_n)$. It is quite easy to see that $\{x\} = \bigcap_{n=1}^{\infty} \bar{U}_n$ where, for each n, $\bar{U}_n$ = closure of U_n in Y and $Cl_X(U_n)$ = closure of U_n in the topology of X. If $y = \underline{F} \in \underline{M}$, we first note that, if $\underline{F} = \{F_n : n \geq 1\}$, then

$H_n = F_n \cup \{\underline{\underline{F}}\}$ is a basic nbhd of $\underline{\underline{F}}$. We claim that $\{\underline{\underline{F}}\} = \bigcap_{n=1}^{\infty} \bar{H}_n$. Firstly, $X \cap (\bigcap_{n=1}^{\infty} \bar{H}_n) = \bigcap_{n=1}^{\infty} (X \cap \bar{H}_n) = \bigcap_{n=1}^{\infty} (X \cap \overline{X \cap H_n})$

(as X is dense in Y and H_n is open in Y for each n) $=$

$\bigcap_{n=1}^{\infty} Cl_X(E_n) = \emptyset$ since $\underline{\underline{F}} \in \underline{\underline{M}} \subseteq \underline{\underline{N}}$. So $\bigcap_{n=1}^{\infty} \bar{H}_n \subseteq \underline{\underline{M}}$. Again from the definition of $\underline{\underline{M}}$ it follows that $\underline{\underline{G}} \in \underline{\underline{M}}$, $\underline{\underline{G}} \neq \underline{\underline{F}} \Rightarrow$ $\underline{\underline{G}} \notin \bigcap_{n=1}^{\infty} \bar{H}_n$. So $\bigcap_{n=1}^{\infty} \bar{H}_n = \{\underline{\underline{F}}\}$. So Y is an E_1-space. In order to prove that Y is E_1-closed, let us take any countable open filter base $\underline{\underline{P}} = \{P_n : n \geq 1\}$ on Y. To show that adherence of $\underline{\underline{P}} = \bigcap_{n=1}^{\infty} \bar{P}_n \neq \emptyset$. Suppose $\emptyset = \bigcap_{n=1}^{\infty} \bar{P}_n$. Since X is dense in Y, $\underline{\underline{P}} \cap X = \{P_n \cap X : n \geq 1\}$ is a countable open filter base on X. Since $\bar{P}_n = \overline{P_n \cap X}$ (as P_n is open in Y), $\bar{P}_n \supseteq Cl_X(P_n \cap X)$.

So $\bigcap_{n=1}^{\infty} Cl_X(P_n \cap X) \subseteq \bigcap_{n=1}^{\infty} \bar{P}_n = \emptyset$ i.e., $\underline{\underline{P}} \cap X$ is a countable open filter base on X with empty adherence i.e., $\underline{\underline{P}} \cap X \in \underline{\underline{N}}$. If $\underline{\underline{G}} \in \underline{\underline{M}}$, $\underline{\underline{G}} \notin \bigcap_{n=1}^{\infty} \bar{P}_n$ i.e., $\exists\ n_o$ such that $\underline{\underline{G}} \notin \bar{P}_{n_o}$ i.e., there exists a nbhd $G_m \cup \{\underline{\underline{G}}\}$ of $\underline{\underline{G}}$ (where $G_m \in \underline{\underline{G}}$) such that $(G_m \cup \{\underline{\underline{G}}\}) \cap P_{n_o} = \emptyset$. So, $G_m \cap (X \cap P_{n_o}) = \emptyset$. Now $X \cap P_{n_o} \in \underline{\underline{P}} \cap X$. Since $\underline{\underline{G}}$ is arbitrary, by maximality of the subset $\underline{\underline{M}}$ of $\underline{\underline{N}}$, we infer that $\underline{\underline{P}} \cap X \in \underline{\underline{M}}$. A basic nbhd of $\underline{\underline{P}} \cap X$ is of the form $\{\underline{\underline{P}} \cap X\} \cup (P_n \cap X)$ and $P_n \cap X$ meets every member of $\underline{\underline{P}}$. So $\underline{\underline{P}} \cap X \in \bigcap_{n=1}^{\infty} \bar{P}_n$. A contradiction to our assumption $\bigcap_{n=1}^{\infty} \bar{P}_n = \emptyset$. Thus Y is E_1-closed.

(Q.E.D.)

10.7 Theorem : Let X be an E_1-space, and let $(Y,\underline{\underline{T}})$ be the E_1-closed space given by theorem 10.6. The following are equivalent.

(1) X is semiregular.

(2) $(Y,\underline{\underline{T}}_o)$ is a minimal E_1-space in which X is imbedded as a dense subspace.

Proof: (1) $\Rightarrow$ (2) : Since $(Y, \underline{\underline{T}})$ is an E_1-closed space, the corresponding semiregular space $(Y, \underline{\underline{T}}_o)$ is a minimal E_1 space by theorem 5.10. We want to show that X is imbedded as a dense subspace of $(Y, \underline{\underline{T}}_o)$. Since X is dense in $(Y, \underline{\underline{T}})$, it is dense in $(Y, \underline{\underline{T}}_o)$. We shall be done as soon as we show that X inherits the same topology from both the topologies $\underline{\underline{T}}$ and $\underline{\underline{T}}_o$ on Y. Let U be any non-empty open subset of X. So U is an open set in $\underline{\underline{T}}$. Let $x \in U$. Then, as X is semiregular, there exists V, open in X, such that $x \in V \subseteq U$ and V is regular-open i.e., $V = \mathrm{Int}_X(\mathrm{Cl}_X(V)) = \underline{\underline{T}} - \mathrm{Int}(\mathrm{Cl}_X(V)) = \mathrm{Int}(X \cap \bar{V})$ [where $\bar{V} = \underline{\underline{T}}\text{-}\mathrm{Cl}(V)$ and Int=$\underline{\underline{T}}$-Int]

$= X \cap \text{Int}(\bar{V})$ (as $\text{Int}(X) = X$). Now $\text{Int}(\bar{V}) \varepsilon \underline{\underline{T}}_o$ so that $X \cap \text{Int}(\bar{V})$ belongs to the relative topology of X w.r.t. $\underline{\underline{T}}_o$. Hence $U \varepsilon X \cap \underline{\underline{T}}_o$ = the relative topology of X w.r.t. $\underline{\underline{T}}_o$. But U is any arbitrary non-empty open set from $X \cap \underline{\underline{T}}$. So $X \cap \underline{\underline{T}}_o = X \cap \underline{\underline{T}}$. So X is, in fact, embedded as a dense subspace of $(Y, \underline{\underline{T}}_o)$.

(2) $\Rightarrow$ (1) : Since $(Y, \underline{\underline{T}}_o)$ is a minimal E_1 space, it is semi-regular. (Proposition 5.6). Now X is a dense subspace of Y. We shall show that X is also semiregular. Since $(Y, \underline{\underline{T}}_o)$ is semiregular, regular-open sets form a base for $\underline{\underline{T}}_o$. In order to show that $(X, X \cap \underline{\underline{T}}_o)$ is a semiregular space it suffices to show that if V is a regular open subset of Y, $X \cap V$ is regular-open in the relative topology $X \cap \underline{\underline{T}}_o$. Now, $Cl_X(V \cap X)$ $= X \cap \overline{V \cap X}$ (The closure is with respect to the topology $\underline{\underline{T}}_o$) $= X \cap \bar{V}$ (as V is open in Y and X is dense). Obviously, $X \cap \bar{V} \supseteq X \cap V$. Again, if for any $W \varepsilon \underline{\underline{T}}_o$, $X \cap W \subseteq X \cap \bar{V}$ we should necessarily have $W \subseteq \bar{V}$ because, if not, $W - \bar{V}$ is an open set such that $W - \bar{V} \neq \emptyset$ and $X \cap (W - \bar{V}) = X \cap W - X \cap \bar{V} = \emptyset$. As X is dense this is impossible. Since V is regular-open, $\bar{V} \supseteq W \Rightarrow W \subseteq V$ where W is any set belonging to $\underline{\underline{T}}_o$. Thus $X \cap V$ is the largest open subset of X contained in $X \cap \bar{V}$. So, $X \cap V = \text{Int}_X [Cl_X(V \cap X)]$ i.e., $X \cap V$ is regular-open in the topology of X.

(Q.E.D.)

10.8 <u>Remarks</u> : Theorems 10.6 and 10.7 are motivated by the marked resemblance between the results on E_1-closed and minimal E_1-spaces and those of first countable H-closed and minimal first countable T_2 spaces (observed in Sec.5) and are the precise analogues of theorems 5.7 and 5.9 of Stephenson [Ste 3, p 125-126]. The rest of this section will be devoted to the investigations of the problem of imbedding a Hausdorff P-space in an HP-closed or a minimal HP-space.

10.9 <u>Proposition</u>: Let X be any Hausdorff P-space. Let $\underline{\underline{M}} = \{ \underline{\underline{F}} : \underline{\underline{F}}$ is an open filter on X closed under countable intersections and has no adherent points$\}$. Then there exists a subset $\underline{\underline{D}}$ of $\underline{\underline{M}}$ which is maximal with respect to having the property that whenever $\underline{\underline{F}}, \underline{\underline{G}} \in \underline{\underline{D}}$, $\underline{\underline{F}} \neq \underline{\underline{G}}$, then there exists sets $F \in \underline{\underline{F}}$ and $G \in \underline{\underline{G}}$ such that $F \cap G = \emptyset$.

<u>Proof</u>: Let $\underline{\underline{C}} = \{ \underline{\underline{A}} : \underline{\underline{A}} \subseteq \underline{\underline{M}}$ such that $\underline{\underline{F}}, \underline{\underline{G}} \in \underline{\underline{A}}, \underline{\underline{F}} \neq \underline{\underline{G}}$ $\Rightarrow$ there exist $F \in \underline{\underline{F}}$ and $G \in \underline{\underline{G}}$ such that $F \cap G = \emptyset\}$. $\underline{\underline{C}}$ is non-empty. Partially order $\underline{\underline{C}}$ by inclusion '$\subseteq$'. Let $(\underline{\underline{A}}_i)_{i \in I}$ be a linearly ordered subfamily of $\underline{\underline{C}}$. Put $\underline{\underline{A}} = \bigcup_{i \in I} \underline{\underline{A}}_i$. Then $\underline{\underline{A}} \subseteq M$. Let $\underline{\underline{F}}, \underline{\underline{G}} \in \underline{\underline{A}} \ni \underline{\underline{F}} \neq \underline{\underline{G}}$. Then $\underline{\underline{F}}, \underline{\underline{G}} \in \underline{\underline{A}}_i$ for some $i \in I$. So, $\exists$ $F \in \underline{\underline{F}}$ and $G \in \underline{\underline{G}}$ such that $F \cap G = \emptyset$. So $\underline{\underline{A}} \in \underline{\underline{C}}$. By Zorn's lemma, now, there exists a subset $\underline{\underline{D}}$ of $\underline{\underline{M}}$ such that $\underline{\underline{D}}$ is a maximal element of $\underline{\underline{C}}$. So the assertion is proved. (Q.E.D.)

10.10 <u>Definition</u> : Let X and Y be Hausdorff P-spaces such that

(a) X is dense in Y

(b) Y is HP-closed.

We call Y an HP-closure of X.

10.11 <u>Theorem</u> : Let X be any Hausdorff P-space. There exists a HP-closure Y of X.

<u>Proof</u>: Let us first outline the construction of Y. Let $\underline{\underline{M}}$ be the set defined in Proposition 10.9. Choose a set $\underline{\underline{D}} \subseteq \underline{\underline{M}}$ which is maximal with respect to having the property that whenever $\underline{\underline{F}}, \underline{\underline{G}} \in \underline{\underline{D}}$ $(\underline{\underline{F}} \neq \underline{\underline{G}})$, then $\exists$ $F \in \underline{\underline{F}}$ and $G \in \underline{\underline{G}}$ such that $F \cap G = \emptyset$. Existence of such a subset $\underline{\underline{D}}$ is guaranteed by Proposition 10.9. Set $Y = X \cup \underline{\underline{D}}$. We define $\underline{\underline{B}}$, a base for the topology of Y, as follows : $B \in \underline{\underline{B}} \iff B = G \cup \{\underline{\underline{F}}\}$ where $\underline{\underline{F}} \in \underline{\underline{D}}$ and $G \in \underline{\underline{F}}$, or $B = G$ where G is an open subset of X. So X is obviously a dense open set in Y.

1°) Y is a T_2 space : Let $y_1, y_2 \in Y \ni y_1 \neq y_2$. If $y_1, y_2 \in X$ y_1, y_2 can be separated by open sets as X is Hausdorff. If $y_1 \in X$ and $y_2 = \underline{\underline{F}}$ for $\underline{\underline{F}} \in \underline{\underline{D}}$, then as $\underline{\underline{F}} \in \underline{\underline{M}}$, $\underline{\underline{F}}$ has empty adherence. So there exist open set $G \subseteq X$ and $F \in \underline{\underline{F}}$ such that $y_1 \in G$ and $G \cap F = \emptyset$.

So consider the disjoint open sets G and $F \cup \{\mathcal{F}\}$ in Y. If $y_1 = \mathcal{F}$ and $y_2 = \mathcal{G}$ where $\mathcal{F}, \mathcal{G} \in \mathcal{D}$. By the property of $\mathcal{D}$ there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \emptyset$. So the required open sets can be taken to be $F \cup \{\mathcal{F}\}$ and $G \cup \{\mathcal{G}\}$.

2°) Y is a P-space : Suffices to show that $y \in Y \Rightarrow y$ is a P-point. We need to consider only the **points** coming from $\mathcal{D}$. Let $\mathcal{F} \in \mathcal{D}$. Let $\{U_n\}$ be a countable family of nbhds of $\mathcal{F}$. Then, for each n, there exists a basic open set B_n containing $\mathcal{F}$ such that $B_n \subset U_n$. So $\bigcap_{n=1}^{\infty} U_n \supset \bigcap_{n=1}^{\infty} B_n$. But $B_n = G_n \cup \{\mathcal{F}\}$ where $G_n \in \mathcal{F}$ for $n \geq 1$. As $\mathcal{F}$ is closed under countable interesections, $G = \bigcap_{n=1}^{\infty} G_n \in \mathcal{F}$ so that $G \cup \{\mathcal{F}\}$ is again a basic open nbhd. of $\mathcal{F}$. Naturally, $\bigcap U_n$ is a nbhd. of $\mathcal{F}$ in Y.

3°) Y is HP-closed : Let $\mathcal{U}$ be an open filter on Y such that countable intersections of members of $\mathcal{U}$ are again in $\mathcal{U}$. We want to claim that $\mathcal{U}$ has non-empty adherence. Suppose not, then $\bigcap \{\bar{U} : U \in \mathcal{U}\} = \emptyset$. [$\bar{U}$ = closure in Y of U]. Let $\mathcal{V} = \{U \cap X : U \in \mathcal{U}\}$. $\mathcal{V}$ is an open filter on X closed under countable intersection. Easy to see that $\mathcal{V}$ has empty adherence in X. So $\mathcal{V} \in \mathcal{M}$. If $\mathcal{F} \in \mathcal{D}$, $\mathcal{F} \notin \bigcap \{\bar{U} : U \in \mathcal{U}\}$

i.e., there exist $U_0 \in \underline{\underline{U}}$ and a basic open nbhd B_0 of $\underline{\underline{F}}$ such that $B_0 \cap U_0 = \emptyset$. But $B_0 = G_0 \cup \{\underline{\underline{F}}\}$ for some $G_0 \in \underline{\underline{F}}$ so that $G_0 = B_0 \cap X$ and $B_0 \cap U_0 = \emptyset \Rightarrow B_0 \cap X \cap U_0 = \emptyset$ $\Rightarrow G_0 \cap (X \cap U_0) = \emptyset$. As $V = X \cap U_0 \in \underline{\underline{V}}$ we get that there exist $V \in \underline{\underline{V}}$ and $G_0 \in \underline{\underline{F}}$ with the property that $V \cap G_0 = \emptyset$. But $\underline{\underline{F}} \in \underline{\underline{D}}$ is arbitrary. This forces $\underline{\underline{V}} \in \underline{\underline{D}}$ by the maximality property of $\underline{\underline{D}}$. A basic nbhd of $\underline{\underline{V}}$ is of the form $V \cup \{\underline{\underline{V}}\}$ where $V \in \underline{\underline{V}}$. So $\underline{\underline{V}}$ is an adherent point of $\underline{\underline{U}}$. A contradiction. So $\underline{\underline{U}}$ has non-empty adherence i.e., Y is HP-closed.

(Q.E.D.)

10.12 Theorem : Let X be an HP-space and let $(Y,\underline{\underline{T}})$ be the HP-closure given by theorem 10.11. The following are then equivalent.

(1) X is semiregular

(2) $(Y,\underline{\underline{T}}_0)$ is a minimal HP-space in which X is imbedded as a dense subspace.

Proof : We first note that $(Y, \underline{\underline{T}}_0)$ is a minimal HP-space as $(Y, \underline{\underline{T}})$ is HP-closed. The line of proof adopted in theorem 10.7 is applicable here and the assertions are proved. (Q.E.D.)

If we focus our attention on locally Lindelöf Hausdorff P-spaces (i.e., llh P-spaces) we know from previous chapter (Chap. I, Sec. 11) that every non-Lindelöf llh P-space X has a unique one-point maximal Lindelöf extension (which turns out to be a Lindelöf HP-space and, a fortiori, a minimal HP-space) in which X is a dense open subset. According to theorem 6.19 every minimal HP-space, which is simultaneously a llh P-space, is a minimal llh P-space. Since the aforesaid one-point maximal Lindelöf extension is a Lindelöf HP-space it is consequently a llh P-space as well. So this extension is a minimal llh P-space as it is already minimal HP. So we can come to following conclusion :

10.13 Theorem : The one-point maximal Lindelöf extension of a non-Lindelöf locally Lindelöf Hausdorff P-space X is a minimal llh P-space in which X is a dense (open) subspace.

Theorem 10.13 completely solves the problem of embedding a llh P-space as a dense subspace of a minimal llh P-space. The question of embedding an analytic (or a borelian) space densely in a minimal analytic (or borelian) space is rather trivially answered. It is as follows :

10.14 Theorem : Every analytic (or borelian) space can be densely imbedded in a minimal analytic (or borelian) space.

Proof: If X is an analytic (or borelian) space, consider the Stone-Čech compactification βX. As a compact T_2 space is both a minimal analytic and a minimal borelian space (Theorems 7.8 and 7.10), the analytic (or borelian) space X gets densely imbedded in the minimal analytic (or borelian) space βX.

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