STUDIES IN PROBABILITY THEORY

RAMANATHAN SUBRAMANIAN

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R. SUBRAMANIAN

CONTENTS

INTRODUCTION

This thesis consists of four chapters.

The impetus for the work in Chapter 1 comes from the concept of 'conditional atom' introduced by Neveu [19]. Here, using conditional atoms we generalize the concept of nonatomicity of measures.(We confine ourselves to probability measures). We obtain generalizations of results on nonatomic measures in [1], [3] and of Liapounoff's theorem.

The results in Chapter 2 have their origins in a paper by Boylan [7]. To study 'equiconvergence of martingales' Boylan introduced in [7] a metric on the space of complete sub σ-algebras of a probability space. A little later, Neveu showed in [19a], this metric space of complete sub σ-algebras is 'tight' ; that is, if two sub σ-algebras are 'close' under this metric with one contained in the other, then there is a set (a conditional atom), with 'high' probability on which the traces of these σ-algebras coincide. In Chapter 2 we investigate what else this metric space is besides being 'tight'. We prove a host of results concerning the topological properties of this metric space ; we also study an isomorphism problem.

Chapters 3 and 4 are devoted to problems in martingales. In Chapter 3 we prove a convergence theorem for 'fairer with time processes' (a generalization due to Blake of martingales in [5]). In Chapter 4 we disprove with the help of an example a conjecture on singular martingales made by Luis Baez-Duarte in [14].

Some of the results in this thesis have already appeared in print. See [23] and [24].

CHAPTER 1

A GENERALIZATION OF NONATOMICITY OF MEASURES

1. Introduction

Let $(\Omega, \underline{\underline{A}}, \mu)$ be a probability space. Let $\underline{\underline{N}}_\mu$ denote the class of all μ-null sets. In [19], Neveu gives the following definition that generalizes the classical concept of an atom of a probability space.

Definition : Let $\underline{\underline{B}}$ be a sub σ-algebra of $\underline{\underline{A}}$. A set $A_o \in \underline{\underline{A}}$ is called a conditional atom of $\underline{\underline{A}}$ with respect to $\underline{\underline{B}}$ if $\mu(A_o) > 0$ and the traces $A_o \cap \underline{\underline{A}} = A_o \cap \underline{\underline{B}}$ (modulo $\underline{\underline{N}}_\mu$) ; (that is, if for every $A \in \underline{\underline{A}}$ one can find $B \in \underline{\underline{B}}$ such that $\mu(A \cap A_o \,\Delta\, B \cap A_o) = 0$). In symbols, A_o is a $(\underline{\underline{B}}, \underline{\underline{A}})(\mu)$ atom.

It is easily verified that if $\underline{\underline{B}} = \{\emptyset, \Omega\}$, the conditional atom as defined above coincides with the classical concept of an atom. This generalization leads, in a natural way, to that of the concept of nonatomicity of a probability.

Definition : μ is said to be $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic if there is no $(\underline{\underline{B}}, \underline{\underline{A}})(\mu)$ atom.

Clearly when $\underline{\underline{B}} = \{\emptyset, \Omega\}$, this concept coincides with the classical concept of nonatomicity ; so when μ is $(\underline{\underline{B}}, \underline{\underline{A}})$

nonatomic for $\underline{\underline{B}} = \{ \emptyset, \Omega \}$, we shall simply write μ is nonatomic on $\underline{\underline{A}}$.

In this chapter in section 2 we give some characterisations of $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomicity, in section 3 we consider the product problem and in section 4 we obtain a generalization of Liapounoff's theorem. Section 5 is devoted to a generalization of a result of Halmos. In section 6 we study this nonatomicity on Polish (complete separable metric) spaces.

In the sequel $(\Omega, \underline{\underline{A}})$ will stand for a measurable space where $\underline{\underline{A}}$ is a σ-algebra. $\underline{\underline{B}}, \underline{\underline{B}}_1, \underline{\underline{B}}_2, \ldots$, etc., will denote sub σ-algebras of $\underline{\underline{A}}$. $\mu, \lambda, \ldots$ etc., will stand for probability measures on $(\Omega, \underline{\underline{A}})$. Let $A \in \underline{\underline{A}}$ and f be a $\underline{\underline{A}}$-measurable function. The conditional expectation under μ of I_A (respectively f) given $\underline{\underline{B}}$ will be denoted by $P_\mu(A|\underline{\underline{B}})$ (respectively $P_\mu(f|\underline{\underline{B}})$). A statement like '$P_\mu(f|\underline{\underline{B}}) = g$' will mean that g is a $\underline{\underline{B}}$ measurable function and $P_\mu(f|\underline{\underline{B}}) = g$ a.s. $[\mu]$.

Following Boylan [7], we can introduce a pseudo-metric d_μ on $\underline{\underline{S}}$, the space of all sub σ-algebras of the probability space $(\Omega, \underline{\underline{A}}, \mu)$, by setting

$$d_\mu(\underline{\underline{B}}_1, \underline{\underline{B}}_2) = \sup_{B_1 \in \underline{\underline{B}}_1} \inf_{B_2 \in \underline{\underline{B}}_2} \mu(B_1 \Delta B_2) + \sup_{B_2 \in \underline{\underline{B}}_2} \inf_{B_1 \in \underline{\underline{B}}_1} \mu(B_1 \Delta B_2)$$

for $\underset{=}{B}_1, \underset{=}{B}_2 \in \underset{=}{S}$. This distance gives us an useful characterisation of nonatomicity. For our study we need the following result, a proof of which can be found in [19].

<u>Theorem 1.1</u> Let μ be $(\underset{=}{B}, \underset{=}{A})$ nonatomic. Given any $\underset{=}{B}$-measurable function f with $0 \leq f \leq 1$, we can find $A \in \underset{=}{A}$ such that $P_{\mu}(A|\underset{=}{B}) = f$.

The following well-known results will be used without mention. Every countably generated σ-algebra is atomic. A measure on a countably generated σ-algebra is nonatomic if and only if the measure gives mass zero to every atom of the σ-algebra.

2. <u>Some characterisations of nonatomicity</u>.

<u>Theorem 2.1</u> Let μ be $(\underset{=}{B}, \underset{=}{A})$ nonatomic and $\lambda << \mu$. Then λ is also $(\underset{=}{B}, \underset{=}{A})$ nonatomic.

<u>Proof</u>: Let g denote a version of the Radon-Nikodym derivative of λ with respect to μ. Let $D = \{g > 0\}$. Observe that (i) $\lambda(D) = 1$ and (ii) for any $A \in \underset{=}{A}$, $A \subseteq D$ we have $\lambda(A) = 0$ if and only if $\mu(A) = 0$. Now, let if possible A_0 be a $(\underset{=}{B}, \underset{=}{A})$ (λ) atom. By (i) $\lambda(A_0 \cap D) = \lambda(A_0) > 0$. Also, since any subset of positive measure of a conditional atom is itself a conditional atom, it follows that $A_0 \cap D$ is a $(\underset{=}{B}, \underset{=}{A})$ (λ) atom.

That is, $(A_o \cap D) \cap \mathcal{A} = (A_o \cap D) \cap \mathcal{B}$ (modulo $\mathcal{N}_\lambda$). This implies, in view of (ii), $(A_o \cap D) \cap \mathcal{A} = (A_o \cap D) \cap \mathcal{B}$ (modulo $\mathcal{N}_\mu$). Again, by (ii), $\mu(A_o \cap D) > 0$. So $A_o \cap D$ is a $(\mathcal{B}, \mathcal{A})$ (μ) atom. A contradiction. Hence the theorem.

Before proceeding with the next result we would like to observe the following. Let $\mathcal{B}_1 \subset \mathcal{B}_2$. Then for any μ, $d_\mu(\mathcal{B}_1, \mathcal{B}_2) \leq 1/2$. This is because, for any $B_1 \in \mathcal{B}_1$,

$\inf_{B_2 \in \mathcal{B}_2} \mu(B_1 \Delta B_2) = 0$; and for any $B_2 \in \mathcal{B}_2$, $\inf_{B_1 \in \mathcal{B}_1} \mu(B_1 \Delta B_2) \leq$

$\min \{\mu(B_2 \Delta \emptyset), \mu(B_2 \Delta \Omega)\} \leq 1/2.$

<u>Theorem 2.2</u> μ is $(\mathcal{B}, \mathcal{A})$ nonatomic if and only if for every $\lambda << \mu$, we have $d_\lambda(\mathcal{B}, \mathcal{A}) = 1/2$.

<u>Proof</u>: Let μ be $(\mathcal{B}, \mathcal{A})$ nonatomic. Let $\lambda << \mu$. Then by Theorem 2.1 λ is $(\mathcal{B}, \mathcal{A})$ nonatomic. By Theorem 1.1 there exists $A_o \in \mathcal{A}$ such that $P_\lambda(A_o | \mathcal{B}) = 1/2$. Now for any $B \in \mathcal{B}$, $\lambda(A_o \Delta B) = \lambda(A_o - B) + \lambda(B - A_o)$

$$= \int_{B^c} I_{A_o} \, d\lambda + \int_B I_{A_o^c} \, d\lambda$$

$$= \int_{B^c} P_\lambda(A_o | \mathcal{B}) \, d\lambda + \int_B P_\lambda(A_o^c | \mathcal{B}) \, d\lambda$$

$$= (1/2)\lambda(B^c) + (1/2)\lambda(B) = 1/2$$

$\therefore \inf_{B \in \mathcal{B}} \lambda(A_o \Delta B) = 1/2$. So $\frac{1}{2} \geq \sup_{A \in \mathcal{A}} \inf_{B \in \mathcal{B}} \lambda(A \Delta B) \geq 1/2$.

Hence $d_\lambda(\underline{\underline{B}}, \underline{\underline{A}}) = 1/2$ and this completes the 'only if' part.

Let A_o be a $(\underline{\underline{B}}, \underline{\underline{A}})$ (μ) atom. Define λ on $\underline{\underline{A}}$ by,

for $A \varepsilon \underline{\underline{A}}$, $\lambda(A) = \dfrac{\mu(A \cap A_o)}{\mu(A_o)}$. Clearly $\lambda << \mu$. Since, for every $A \varepsilon \underline{\underline{A}}$ there is a $B \varepsilon \underline{\underline{B}}$ with $\mu[(A \Delta B) \cap A_o] = 0$, we have $\lambda(A \Delta B) = 0$; hence $d_\lambda(\underline{\underline{B}}, \underline{\underline{A}}) = 0$ and this completes the 'if' part.

<u>Remarks</u> : With the same proof, therem 2.2 can be recast as 'μ is $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic if and only if for every $\lambda << \mu$, we have $d_\lambda(\underline{\underline{B}}, \underline{\underline{A}}) \neq 0$'. The following stronger version of Theorem 2.2 can also be proved. μ is $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic if and only if for every $\lambda \equiv \mu$ we have $d_\lambda(\underline{\underline{B}}, \underline{\underline{A}}) = 1/2$.

<u>Theorem 2.3</u> Let $\underline{\underline{B}}_1 \subset \underline{\underline{B}}_2 \subset \underline{\underline{B}}_3$. If μ is $(\underline{\underline{B}}_1, \underline{\underline{B}}_2)$ nonatomic then it is $(\underline{\underline{B}}_1, \underline{\underline{B}}_3)$ nonatomic as well.

<u>Proof</u>: For any measure λ, from the definition of d_λ it is clear that $d_\lambda(\underline{\underline{B}}_1, \underline{\underline{B}}_2) \leq d_\lambda(\underline{\underline{B}}_1, \underline{\underline{B}}_3) \leq \frac{1}{2}$. Now, an application of theorem 2.2 gives us the result.

The next theorem helps us to generalize the result that μ is nonatomic on $\underline{\underline{A}}$ if and only if it is nonatomic on a countably generated sub σ-algebra $\underline{\underline{C}}$ of $\underline{\underline{A}}$. (See [1]).

<u>Theorem 2.4</u> Let $\underline{\underline{B}}$ and $\underline{\underline{C}}$ be sub σ-algebras of $\underline{\underline{A}}$. Let μ be such that (i) μ is nonatomic on $\underline{\underline{C}}$ and (ii) under μ, the sub σ-algebras $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are independent. Then μ is $(\underline{\underline{B}}, \underline{\underline{B}} \vee \underline{\underline{C}})$ nonatomic where $\underline{\underline{B}} \vee \underline{\underline{C}}$ stands for the σ-algebra generated by $\underline{\underline{B}}$ and $\underline{\underline{C}}$.

<u>Proof</u> : To prove this theorem we assume that μ is not $(\underline{\underline{B}}, \underline{\underline{B}} \vee \underline{\underline{C}})$ nonatomic and arrive at a contradiction.

Observe that the collection

$$\underline{\underline{D}} = \left\{ \begin{array}{l} \bigcup_{j=1}^{m} B_j C_j : B_j \in \underline{\underline{B}}, \quad C_j \in \underline{\underline{C}}, \quad 1 \leq j \leq m \\ \qquad B_j C_j \cap B_{j'} C_{j'} = \emptyset \text{ if } 1 \leq j \neq j' \leq m \\ \qquad \text{and} \quad m \geq 1 \end{array} \right\}$$

is an algebra generating $\underline{\underline{B}} \vee \underline{\underline{C}}$. Let A_o be a $(\underline{\underline{B}}, \underline{\underline{B}} \vee \underline{\underline{C}})(\mu)$ atom. Let $0 < \varepsilon = P(A_o)$. Choose and fix $\bigcup_{j=1}^{m_o} B_j C_j \in \underline{\underline{D}}$ such that $\mu(A_o \,\Delta\, \bigcup_{j=1}^{m_o} B_j C_j) < \varepsilon/4$. For easy understanding the remainder of the proof has been divided into a few steps numbered (S1) through (S4).

(S_1) For any $A \in \underline{\underline{B}} \vee \underline{\underline{C}}$, there is $B \in \underline{\underline{B}}$ such that

$$\mu[(A \,\Delta\, B) \cap \bigcup_{j=1}^{m_o} B_j C_j] \leq \varepsilon/4.$$

Since A_o is a $(\underline{\underline{B}}, \underline{\underline{B}} \vee \underline{\underline{C}})(\mu)$ atom, given A there is $B \in \underline{\underline{B}}$ such that $\mu[(A \,\Delta\, B) \cap A_o] = 0$. Therefore,

$$\mu[(A \,\Delta\, B) \cap \bigcup_{j=1}^{m_o} B_j C_j] \leq \mu[(A \,\Delta\, B) \cap A_o] +$$
$$\mu[(A \,\Delta\, B) \cap (\bigcup_{j=1}^{m_o} B_j C_j - A_o)]$$
$$\leq 0 + \mu[(\bigcup_{j=1}^{m_o} B_j C_j) \,\Delta\, A_o] \leq \varepsilon/4.$$

Hence (S1)

(S2) For $1 \leq j \leq m_o$, choose $C_j^o \subset C_j$ such that $\mu(C_j^o) = \frac{\mu(C_j)}{2}$. Such a choice is possible as μ is nonatomic on $\underline{\underline{C}}$. Take $D_o = \bigcup_{j=1}^{m_o} B_j \, C_j^o$. By (S1) there is $B_o \in \underline{\underline{B}}$ with $\mu[(D_o \,\Delta\, B_o) \cap \bigcup_{j=1}^{m_o} B_j C_j] \leq \varepsilon/4$.

(S3) $\mu[(D_o \Delta B_o) \cap (\bigcup_{j=1}^{m_o} B_j C_j)] = \dfrac{\mu(\bigcup_{j=1}^{m_o} B_j C_j)}{2}$. We present the proof below.

$$(D_o \Delta B_o) \cap (\bigcup_{j=1}^{m_o} B_j C_j) = [((\bigcup_{j=1}^{m_o} B_j C_j^o) \,\Delta\, B_o) \cap (\bigcup_{j=1}^{m_o} B_j C_j)]$$
$$= [(\bigcup_{j=1}^{m_o} B_j C_j^o) - B_o] \cup [B_o \cap (\bigcup_{j=1}^{m_o} B_j (C_j - C_j^o))]$$

(using the fact that $B_j C_j \cap B_{j'} C_{j'} = \emptyset$ if $j \neq j'$).

So, $\mu[(D_o \Delta B_o) \cap (\bigcup_{j=1}^{m_o} B_j C_j)] = \mu[(\bigcup_{j=1}^{m_o} B_j C_j^o) - B_o] +$

$$\mu[B_o \cap (\bigcup_{j=1}^{m_o} B_j (C_j - C_j^o))]$$

$$= \sum_{j=1}^{m_o} \mu(C_j^o B_j B_o^c) + \sum_{j=1}^{m_o} \mu[(C_j - C_j^o) B_j B_o]$$

(using the fact that $B_j C_j \cap B_{j'} C_{j'} = \emptyset$ if $j \neq j'$)

$$= \sum_{j=1}^{m_o} \mu(C_j^o) . \mu(B_j B_o^c) + \sum_{j=1}^{m_o} \mu(C_j - C_j^o) . \mu(B_j B_o)$$

(because of independence of $\underline{\underline{B}}$ and $\underline{C}$)

$$= \frac{1}{2}[\sum_{j=1}^{m_o} \mu(C_j) \mu(B_j B_o^c) + \sum_{j=1}^{m_o} \mu(C_j) \mu(B_j B_o)]$$

(by the choice of C_j^o, $1 \leq j \leq m_o$)

$$= \frac{1}{2} \sum_{j=1}^{m_o} \mu(C_j) \mu(B_j) = \frac{1}{2} \sum_{j=1}^{m_o} \mu(C_j B_j)$$

$$= \frac{1}{2} \mu(\bigcup_{j=1}^{m_o} B_j C_j).$$

Hence (S3).

(S4) By (S3) and (S2) $\dfrac{\mu(\bigcup_{j=1}^{m_o} B_j C_j)}{2} \leq \varepsilon/4$. So,

$$\mu(\bigcup_{j=1}^{m_o} B_j C_j) \leq \varepsilon/2.$$

On the other hand $\mu((\bigcup_{j=1}^{m_o} B_j C_j) \Delta A_o) \leq \varepsilon/4$. This implies that

$\mu(\bigcup_{j=1}^{m_o} B_j C_j) \geq \varepsilon - \varepsilon/4 = \frac{3\varepsilon}{4}$. A contradiction! Hence the theorem.

<u>Lemma 2.5</u> Let μ be $(\underline{\underline{B}}, \underline{A})$ nonatomic. Let $A_o \varepsilon \underline{A}$ be such that $P_\mu(A_o|\underline{\underline{B}}) = \varepsilon > 0$. Then there is $A_1 \varepsilon \underline{A}$ $A_1 \subset A_o$ with $P_\mu(A_1|\underline{\underline{B}}) = \varepsilon/2$.

<u>Proof</u> : Let μ_{A_o} be defined on $\underline{A}$ by $\mu_{A_o}(A) = \frac{\mu(A \cap A_o)}{\mu(A_o)}$, $A \varepsilon \underline{A}$. Since $\mu_{A_o} \ll \mu$, we have μ_{A_o} is $(\underline{\underline{B}}, \underline{A})$ nonatomic. So, by theorem 1.1, there is $A_1 \varepsilon \underline{A}$ with $P_{\mu_{A_o}}(A_1|\underline{\underline{B}}) = \frac{1}{2}$. As $\mu_{A_o}(A_o) = 1$, we have $\mu_{A_o}[(A_1 \cap A_o) \Delta A_o] = 0$; so we can take $A_1 \subset A_o$ and $P_{\mu_{A_o}}(A_1|\underline{\underline{B}}) = \frac{1}{2}$. Now, for any $B \varepsilon \underline{\underline{B}}$

$$\mu_{A_o}(A_1 \cap B) = \frac{1}{2}\mu_{A_o}(B) \Rightarrow \frac{\mu(A_1 \cap B)}{\mu(A_o)} = \frac{1}{2}\frac{\mu(A_o \cap B)}{\mu(A_o)} \text{(as } A_1 \subset A_o\text{)}$$

$$\Rightarrow \frac{\mu(A_1 \cap B)}{\mu(A_o)} = \frac{1}{2} \cdot \frac{\mu(B)}{\mu(A_o)} \cdot \varepsilon$$

$$\Rightarrow \mu(A_1 \cap B) = \frac{\varepsilon}{2}\mu(B).$$

Therefore, $P_\mu(A_1|\underline{\underline{B}}) = \varepsilon/2$. Hence the lemma.

<u>Theorem 2.6</u> μ is $(\underline{\underline{B}}, \underline{A})$ nonatomic if and only if there is a countably generated sub σ-algebra $\underline{C}$ of $\underline{A}$ such that, under μ $\underline{\underline{B}}$ and $\underline{C}$ are independent and μ is nonatomic on $\underline{C}$.

Proof: Let $\underline{\underline{C}}$ be a sub σ-algebra of $\underline{\underline{A}}$ such that μ is nonatomic on $\underline{\underline{C}}$ and under μ, $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are independent. By theorem 2.4 μ is $(\underline{\underline{B}}, \underline{\underline{B}} \vee \underline{\underline{C}})$ nonatomic. Since $\underline{\underline{B}} \vee \underline{\underline{C}} \subset \underline{\underline{A}}$, by theorem 2.3 μ is $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic. This completes the proof of 'if' part.

For the proof of 'only if' part, we obtain a collection

$$\begin{array}{llllll} A_{11} & A_{12} & & & & \\ A_{21} & A_{22} & A_{23} & A_{24} & & \\ \cdots & \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & & \\ A_{n1} & A_{n2} & A_{n3} & \cdots & \cdots & A_{n2^n} \end{array} \qquad (*)$$

of elements of $\underline{\underline{A}}$ as follows $A_{11} \in \underline{\underline{A}}$ with $P_\mu(A_{11}|\underline{\underline{B}}) = \frac{1}{2}$ and $A_{12} = \Omega - A_{11}$; note that $P_\mu(A_{12}|\underline{\underline{B}}) = \frac{1}{2}$. After having obtained the sets in the nth row of (*) satisfying the conditions that $\{A_{nk}, 1 \le k \le 2^n\}$ is a partition of Ω by $\underline{\underline{A}}$-sets and for any $1 \le k \le 2^n$, $P_\mu(A_{nk}|\underline{\underline{B}}) = 1/2^n$, we obtain the sets in the (n+1)th row as follows. $A_{n+1,1} \subset A_{n1}$ with $P_\mu(A_{n+1,1}|\underline{\underline{B}}) = \frac{1}{2^{n+1}}$ and $A_{n+1,2} = A_{n1} - A_{n+1,1}$. More generally for $1 \le k \le 2^n$, $A_{n+1,2k-1} \subset A_{nk}$ with $P_\mu(A_{n+1,2k-1}|\underline{\underline{B}}) = \frac{1}{2^{n+1}}$ and $A_{n+1,2k} = A_{nk} - A_{n+1,2k-1}$. Such a construction is possible by Lemma 2.5. It is clear that $\{A_{n+1,k}, 1 \le k \le 2^{n+1}\}$ is a partition of Ω by $\underline{\underline{A}}$-sets and $P_\mu(A_{n+1,k} \mid \underline{\underline{B}}) = \frac{1}{2^{n+1}}$.

Define $\underline{\underline{C}}$ to be the σ-algebra generated by $\{A_{nk}, 1 \leq k \leq 2^n$ and $n \geq 1\}$. Since for $1 \leq k \leq 2^n$, $\mu(A_{nk}) = \frac{1}{2^n}$ it is clear that μ-measure of any atom of $\underline{\underline{C}}$ is zero; also $\underline{\underline{C}}$ is countably generated. Hence μ is nonatomic on $\underline{\underline{C}}$. Observe that the collection of sets $\{D : D$ is a union of some sets in the nth row of (*) for some $n \geq 1\}$ is an algebra generating $\underline{\underline{C}}$. So, to check the independence of $\underline{\underline{C}}$ and $\underline{\underline{B}}$ under μ it is enough to check that $\mu(D \cap B) = \mu(D).\,\mu(B)$ for any $B \in \underline{\underline{B}}$ and D from the above collection. Let $D_o = \bigcup_{k \in J} A_{n_o k}$ where J is a subset of $1, 2, \ldots, 2^{n_o}$.

For any $B \in \underline{\underline{B}}$, $\mu(D_o \cap B) = \sum_{k \in J} \mu(A_{n_o k} \cap B)$

$$= \sum_{k \in J} \int_B I_{A_{n_o k}} \, d\mu$$

$$= \sum_{k \in J} \int_B P_\mu (A_{n_o k} | \underline{\underline{B}}) \, d\mu$$

$$= \sum_{k \in J} \frac{1}{2^{n_o}} \mu(B)$$

$$= \mu(B).\,\mu(D_o).$$

Thus $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are independent under μ. This completes the proof of 'only if' part and hence the theorem.

3. The Product Problem

Let $(\Omega_1, \underline{\underline{A}}_1, \mu_1)$ and $(\Omega_2, \underline{\underline{A}}_2, \mu_2)$ be probability spaces. Let $\underline{\underline{B}}_i \subset \underline{\underline{A}}_i$ $i = 1,2$. In this section we investigate the problem of $\mu = \mu_1 \times \mu_2$ being $(\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2, \underline{\underline{A}}_1 \otimes \underline{\underline{A}}_2)$ nonatomic.

Theorem 3.1 μ is $(\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2, \underline{\underline{A}}_1 \otimes \underline{\underline{A}}_2)$ nonatomic if and only if either μ_1 is $(\underline{\underline{B}}_1, \underline{\underline{A}}_1)$ nonatomic or μ_2 is $(\underline{\underline{B}}_2, \underline{\underline{A}}_2)$ nonatomic.

Proof: If there are sets A_1 and A_2 such that A_i is a $(\underline{\underline{B}}_i, \underline{\underline{A}}_i)$ (μ_i) atom, $i = 1,2$, then clearly $A_1 \times A_2$ is a $(\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2, \underline{\underline{A}}_1 \otimes \underline{\underline{A}}_2)$ $(\mu_1 \times \mu_2)$ atom. This proves the 'only if' part.

Let μ_1 be $(\underline{\underline{B}}_1, \underline{\underline{A}}_1)$ nonatomic. We shall show that $\mu_1 \times \mu_2$ is $(\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2, \underline{\underline{A}}_1 \otimes \underline{\underline{A}}_2)$ nonatomic. By Theorem 2.6 there is $\underline{\underline{C}}_1 \subset \underline{\underline{A}}_1$ such that μ_1 is nonatomic on $\underline{\underline{C}}_1$ and under μ_1 the σ-algebras $\underline{\underline{C}}_1$ and $\underline{\underline{B}}_1$ are independent. It is easy to see that $\mu_1 \times \mu_2$ is nonatomic on $\underline{\underline{C}}_1 \otimes \{(\emptyset, \Omega_2)\}$ and under $\mu_1 \times \mu_2$, $\underline{\underline{C}}_1 \otimes \{\emptyset, \Omega_2\}$ and $\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2$ are independent. Hence by Theorem 2.4 the measure $\mu_1 \times \mu_2$ is $(\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2, \underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2 \vee \underline{\underline{C}}_1 \otimes \{\emptyset, \Omega_2\})$ nonatomic. Now, an application of Theorem 2.3 gives us that $\mu_1 \times \mu_2$ is $(\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2, \underline{\underline{A}}_1 \otimes \underline{\underline{A}}_2)$ nonatomic. This completes the proof of 'if' part and hence the theorem.

Remark: The above theorem generalizes the theorem of [1] for product measures.

Let λ be a measure on $(\Omega_1 \times \Omega_2, \underline{\underline{A}}_1 \otimes \underline{\underline{A}}_2)$. Let λ_i be the marginal on $(\Omega_i, \underline{\underline{A}}_i)$, $i = 1, 2$. Then it is known that if one of the λ_i's is nonatomic then so is λ (See [1]). This however does not generalize as shown by the example given below.

Example : For $i = 1, 2$ let $\Omega_i = [0, 1]$ and $\underline{\underline{A}}_i$ = the Borel σ-algebra on $[0,1]$. Take $\underline{\underline{B}}_1 = \{\emptyset, \Omega_1\}$ and $\underline{\underline{B}}_2 = \underline{\underline{A}}_2$. Let λ be the uniform measure on the diagonal of $\Omega_1 \times \Omega_2$. Then it is clear that $\lambda_1 = \lambda_2$ = the Lebesgue measure on $[0,1]$. So λ_1 is $(\underline{\underline{B}}_1, \underline{\underline{A}}_1)$ nonatomic. However λ is not $(\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2, \underline{\underline{A}}_1 \otimes \underline{\underline{A}}_2)$ nonatomic since the diagonal of $\Omega_1 \times \Omega_2$ is a $(\underline{\underline{B}}_1 \otimes \underline{\underline{B}}_2, \underline{\underline{A}}_1 \otimes \underline{\underline{A}}_2)(\lambda)$ atom.

4. A generalization of Liapounoff's theorem

In this section we obtain a generalization of the classical Liapounoff's theorem. Our proof is essentially that of Lindenstrauss' of the classical version [13] with suitable modifications.

Theorem 4.1 (Liapounoff's Theorem) Let $\mu_1, \mu_2, \ldots, \mu_n$ be $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic measures. Equip $L_\infty(\Omega, \underline{\underline{B}}, \mu_i)$ with the

w^* topology (that is, the weak topology induced by $L_1(\Omega, \underline{\underline{B}}, \mu_1)$); let $L_\infty(\mu_i)$ stand for this space. The set $\{(P_{\mu_1}(A|\underline{\underline{B}}), P_{\mu_2}(A|\underline{\underline{B}}), \ldots, P_{\mu_n}(A|\underline{\underline{B}}))/A \in \underline{\underline{A}}\}$ is a closed convex subset of $L_\infty(\mu_1) \times L_\infty(\mu_2) \times \cdots \times L_\infty(\mu_n)$.

Proof: Let $n = 1$. By theorem 1.1 it is clear that $\{P_{\mu_1}(A|\underline{\underline{B}})/ A \in \underline{\underline{A}}\} = \{f : 0 \leq f \leq 1 \text{ and } f \in L_\infty(\mu_1)\}$ and hence the result. We complete the proof by induction.

Let $\mu = \mu_1 + \mu_2 + \cdots + \mu_n$. Let $W = \{g : 0 \leq g \leq 1,\ g \in L_\infty(\Omega, \underline{\underline{A}}, \mu)\}$. Equip $L_\infty(\Omega, \underline{\underline{A}}, \mu)$ with the w^*-topology. Then it is known that W is a compact set. Clearly, W is convex as well. Define a mapping T from W to $L_\infty(\mu_1) \times L_\infty(\mu_2) \times \cdots \times L_\infty(\mu_n)$ by setting, for g in W, $T(g) = (P_{\mu_1}(g|\underline{\underline{B}}), P_{\mu_2}(g|\underline{\underline{B}}), \ldots, P_{\mu_n}(g|\underline{\underline{B}}))$. Presently we show that T is continuous. Let $\{g_\alpha\}$ be a net in W converging to g_0 in W. Consider $h \in L_1(\Omega, \underline{\underline{B}}, \mu_i)$ for some i. Observing that $h.\frac{d\mu_i}{d\mu}$ is in $L_1(\Omega, \underline{\underline{A}}, \mu)$ we have

$$\int P_{\mu_i}(g_\alpha|\underline{\underline{B}})h\, d\mu_i = \int g_\alpha . h\, d\mu_i$$

$$= \int g_\alpha . h . \frac{d\mu_i}{d\mu} d\mu \longrightarrow \int g_0\, h \frac{d\mu_i}{d\mu}\, d\mu$$

$$= \int g_0\, h\, d\mu_i = \int P_{\mu_i}(g_0|\underline{\underline{B}})\, h\, d\mu_i .$$

Since h and i are arbitrary it follows that T is continuous. So, $T(W)$ is a compact (and hence closed) and convex (since T is affine) subset of $L_\infty(\mu_1) \times L_\infty(\mu_2) \times \dots \times L_\infty(\mu_n)$. The proof would be complete if we could show $T(W) = \{T(I_D) \mid D \in \underline{A}\}$.

Let $(h_1, h_2, \dots, h_n) \in T(W)$. We have to exhibit $D \in \underline{A}$ with $T(I_D) = (h_1, h_2, \dots, h_n)$. Let $W_o = T^{-1}(h_1, h_2, \dots, h_n)$. W_o is a compact convex subset of W. By Krein-Milman theorem W_o has extreme points. Let g be an extreme point of W_o. We shall prove that $g = I_D$ a.s. (μ) for some $D \in \underline{A}$.

Suppose not. Then for some $\varepsilon > 0$, $\mu(\{\varepsilon \leq g \leq 1 - \varepsilon\}) > 0$. This implies that for some i, say $i = 1$, $\mu_i(\{\varepsilon \leq g \leq 1 - \varepsilon\}) > 0$; that is $\mu_1(\{\varepsilon \leq g \leq 1 - \varepsilon\}) > 0$. Denote $\{\varepsilon \leq g \leq 1 - \varepsilon\}$ by Z. Choose an $\underline{A}$ measurable subset A of Z such that for every $B \in \underline{\underline{B}}$, $\mu_1[A \,\Delta\, (B \cap Z)] > 0$ and $\mu_1[(Z - A) \,\Delta\, (B \cap Z)] > 0$. (Such a choice is possible since Z is not a $(\underline{\underline{B}}, \underline{A})(\mu_1)$ atom). By induction hypothesis, we can find $\underline{A}$-measurable sets B and C satisfying (i) $B \subset A$, $C \subset Z - A$ and (ii) for $2 \leq i \leq n$,

$$2P_{\mu_i}(B|\underline{\underline{B}}) = P_{\mu_i}(A|\underline{\underline{B}}) \text{ and } 2P_{\mu_i}(C|\underline{\underline{B}}) = P_{\mu_i}(Z - A|\underline{\underline{B}}).$$

Choose and fix versions of $P_{\mu_1}(B|\underline{\underline{B}})$, $P_{\mu_1}(A|\underline{\underline{B}})$, $P_{\mu_1}(C|\underline{\underline{B}})$ and $P_{\mu_1}(Z - A|\underline{\underline{B}})$. Denote by x and y the functions $2P_{\mu_1}(B|\underline{\underline{B}}) - P_{\mu_1}(A|\underline{\underline{B}})$ and $2P_{\mu_1}(C|\underline{\underline{B}}) - P_{\mu_1}(Z - A|\underline{\underline{B}})$ respectively.

Define two $\underline{B}$-measurable functions s and t as follows.

$s = \varepsilon$ and $t = \varepsilon$ on $\{x = 0, y = 0\}$,

$s = \varepsilon$ and $t = 0$ on $\{x = 0, y \neq 0\}$,

$s = 0$ and $t = \varepsilon$ on $\{x \neq 0, y = 0\}$,

$s = t.\frac{y}{x}$ and $t = \varepsilon$ on $\{0 < |\frac{y}{x}| \leq 1\}$ and

$s = \varepsilon$ and $t = s.\frac{x}{y}$ on $\{|\frac{y}{x}| > 1\}$.

Clearly $|s(w)| \leq \varepsilon$ and $|t(w)| \leq \varepsilon$ for all $w \varepsilon \Omega$ and $sx = ty$. Define a function h by $h = s(2I_B - I_A) + t(I_{Z-A} - 2I_C)$.

1°. For $1 \leq i \leq n$, $P_{\mu_i}(h|\underline{B}) = 0$.

If $2 \leq i \leq n$, this is clear from the choice of sets B and C; for $i = 1$, this is by the choice of functions s and t.

2°. $g + h$ and $g - h$ are in W.

This is because $|h| \leq g \leq 1 - |h|$ on Ω; so $0 \leq g + h \leq 1$ and $0 \leq g - h \leq 1$.

3°. $g + h$ and $g - h$ are in W_o.

This is immediate from 1° and 2°.

4°. $\mu_1(h \neq 0) > 0$.

Suppose not. Then $h = 0$ on Z a.s. (μ_1). Note that

$h = s$ or $-s$ on A and $h = t$ or $-t$ on $Z - A$. So, $s = 0$ on A a.s (μ_1) and $t = 0$ on $Z - A$ a.s. (μ_1). Then, from the definition of s and t we have

$A \subset \{s = 0\} = \{x \neq 0,\ y = 0\}$ a.s. (μ_1) and

$Z - A \subset \{t = 0\} = \{x = 0,\ y \neq 0\}$ a.s. (μ_1).

Now, $\mu_1(A \triangle [\{s = 0\} \cap Z]) = \mu_1[(A \triangle \{s = 0\}) \cap Z]$

$$= \mu_1(\{s = 0\} \cap Z - A) = 0.$$

A contradiction to the choice of A since $\{s = 0\}$ is a $\underline{\underline{B}}$ measurable set. Thus 4°.

Now, from 4° it is clear that $\mu(h \neq 0) > 0$. In view of 3°, therefore, it follows that g is not an extreme point of W_o. This contradiction concludes the proof of the theorem.

Remark: In the above we have shown more than what we set out to prove. We have shown that the set

$\{(P_{\mu_1}(A|\underline{\underline{B}}),\ P_{\mu_2}(A|\underline{\underline{B}}),\ \ldots,\ P_{\mu_n}(A|\underline{\underline{B}}))/A \in \underline{\underline{A}}\}$ is convex even when the 'weights' are $\underline{\underline{B}}$-measurable functions. The paragraph below is intended to amplify this.

Let $\mu_1, \mu_2, \ldots, \mu_n$ be $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic measures. Let W and T be as in theorem 4.1. Let A_1, A_2 be elements of $\underline{\underline{A}}$. Let g_1, g_2 be two $\underline{\underline{B}}$-measurable functions with $0 \leq g_i \leq 1$, $i = 1, 2$ and $g_1 + g_2 \leq 1$. Clearly $g_1 I_{A_1} + g_2 I_{A_2} \in W$. Now

$$P_{\mu_j}(g_1 I_{A_1} + g_2 I_{A_2} \mid \underline{\underline{B}}) = g_1 P_{\mu_j}(I_{A_1} \mid \underline{\underline{B}}) + g_2 P_{\mu_j}(I_{A_2} \mid \underline{\underline{B}}),$$

(as g_1 and g_2 are $\underline{\underline{B}}$ measurable),

$j = 1, 2, \ldots n.$

So, $T(g_1 I_{A_1} + g_2 I_{A_2}) = g_1 T(I_{A_1}) + g_2 T(I_{A_2})$. Since $T(W) = \{T(I_D) / D \varepsilon \underline{\underline{A}}\}$ we have that $g_1 T(I_{A_1}) + g_2 T(I_{A_2}) \varepsilon \{T(I_D) / D \varepsilon \underline{\underline{D}}\}$.

As a consequence of the above theorem we have the following generalization of theorem 2.6.

<u>Theorem 4.2</u> Let $\mu_1, \mu_2, \ldots, \mu_n$ be $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic measures. Then there is a countably generated sub σ-algebra $\underline{\underline{C}}$ of $\underline{\underline{A}}$ such that each μ_i is nonatomic on $\underline{\underline{C}}$ and under each μ_i, $\underline{\underline{B}}$ and $\underline{\underline{C}}$ and independent.

<u>Proof</u> : Use theorem 4.1 and imitate the construction of $\underline{\underline{C}}$ in theorem 2.6. We omit the details.

5. <u>A generalization of a result of Halmos</u>

Let μ be a nonatomic measure on $(\Omega, \underline{\underline{A}})$. It is known that μ is uniquely determined by the class of $\underline{\underline{A}}$ sets on which it takes a constant value r $(0 < r < 1)$. More precisely :

<u>Theorem 5.1</u>. Let μ_1 and μ_2 be two nonatomic measures on $(\Omega, \underline{\underline{A}})$. Let for some r with $0 < r < 1$,

$\{A \in \underline{A} / \mu_1(A) = r\} = \{A \in \underline{A} / \mu_2(A) = r\}$. Then $\mu_1 = \mu_2$

This result is implicit in Halmos's proof of Liapounoff's theorem [8], as has been pointed out by Bolkar. For a proof of this result see corallary 1.23 of [6] or theorem 1.5.1 of [3]. Below, we obtain a generalization of this result.

Theorem 5.2 Let μ_1 and μ_2 be two $(\underline{\underline{B}}, \underline{A})$ nonatomic measures. Let f_o be a $\underline{\underline{B}}$ measurable function with the property $\mu_1(\{0 < f_o < 1\}) = 1$. Further, let

$$\{A \in \underline{A} / P_{\mu_1}(A|\underline{\underline{B}}) = f_o\} = \{A \in \underline{A} / P_{\mu_2}(A|\underline{\underline{B}}) = f_o\}.$$

Then $\mu_1 = \mu_2$.

The proof is completed with the help of two lemmas.

Lemma 5.3 Assume the hypothesis of theorem 5.2. Then $\mu_1 \equiv \mu_2$ on $\underline{A}$. Also for any $B_o \in \underline{\underline{B}}$,

$$\{A \in \underline{A} / P_{\mu_1}(A|\underline{\underline{B}}) = f_o \cdot I_{B_o}\} = \{A \in \underline{A} / P_{\mu_2}(A|\underline{\underline{B}}) = f_o \cdot I_{B_o}\}.$$

Proof: Let $A \in \underline{A}$ with $\mu_1(A) = 0$. Choose $C \in \underline{A}$ with $P_{\mu_1}(C|\underline{\underline{B}}) = f_o$. (Such a choice is possible in view of theorem 1.1). Then $P_{\mu_1}(C \cup A|\underline{\underline{B}}) = f_o$ and $P_{\mu_1}(C - A|\underline{\underline{B}}) = f_o$. So, $P_{\mu_2}(C \cup A \mid \underline{\underline{B}}) = f_o$ and $P_{\mu_2}(C - A|\underline{\underline{B}}) = f_o$. Hence,

$$P_{\mu_2}[(C \cup A) - (C - A) \mid \underline{\underline{B}}] = 0$$

i.e. $P_{\mu_2}(A|\underline{\underline{B}}) = 0$. This implies $\mu_2(A) = 0$. Thus $\mu_2 << \mu_1$. Interchanging the roles of μ_1 and μ_2 in the above we conclude $\mu_1 << \mu_2$. Thus $\mu_1 \equiv \mu_2$ on $\underline{\underline{A}}$.

Fix $B_o \in \underline{\underline{B}}$. Let $A_o \in \underline{\underline{A}}$ be such that $P_{\mu_1}(A_o|\underline{\underline{B}}) = f_o\, I_{B_o}$. To show $P_{\mu_2}(A_o|\underline{\underline{B}}) = f_o\, I_{B_o}$

$$\mu_1(A_o \cap B_o^c) = \int_{B_o^c} I_{A_o}\, d\mu_1 = \int_{B_o^c} P_{\mu_1}(A_o|\underline{\underline{B}})\, d\mu_1$$

$$= \int_{B_o^c} f_o . I_{B_o}\, d\mu_1 = 0.$$

So, $\mu_2(A_o \cap B_o^c) = 0$ (as $\mu_1 \equiv \mu_2$). Choose $C \in \underline{\underline{A}}$ with $P_{\mu_1}(C|\underline{\underline{B}}) = f_o$ and $P_{\mu_2}(C|\underline{\underline{B}}) = f_o$. (Such a choice is possible in view of the remark following theorem 4.1). Let $D = C \cap B_o^c$. Then $P_{\mu_1}(D|\underline{\underline{B}}) = f_o . I_{B_o^c}$ and $P_{\mu_2}(D|\underline{\underline{B}}) = f_o\, I_{B_o^c}$. Now,

$$P_{\mu_1}(A_o \cup D|\underline{\underline{B}}) = P_{\mu_1}(A_o|\underline{\underline{B}}) + P_{\mu_1}(D|\underline{\underline{B}})$$

$$(\because \mu_1(A_o \cap D) \leq \mu_1(A_o \cap B_o^c) = 0)$$

$$= f_o\, I_{B_o} + f_o\, I_{B_o^c} = f_o.$$

$$\therefore P_{\mu_2}(A_o \cup D|\underline{\underline{B}}) = f_o.$$

i.e. $P_{\mu_2}(A_o|\underline{\underline{B}}) + P_{\mu_2}(D|\underline{\underline{B}}) = f_o$ $(\because \mu_2(A_o \cap D) \leq \mu_2(A_o \cap B_o^c) = 0)$

i.e. $P_{\mu_2}(A_o|\underline{\underline{B}}) + f_o I_{B_o^c} = f_o$.

Hence $P_{\mu_2}(A_o|\underline{\underline{B}}) = f_o I_{B_o}$. Interchanging the roles of μ_1 and μ_2 the proof of the lemma can be completed.

Lemma 5.4 Assume the hypothesis of theorem 5.2. Let g be a $\underline{\underline{B}}$ measurable function with $0 \leq g \leq 1$. Then

(i) $\{A \varepsilon \underline{\underline{A}} / P_{\mu_1}(A|\underline{\underline{B}}) = g\, I_{\{g=f_o\}}\} =$

$$\{A \varepsilon \underline{\underline{A}} / P_{\mu_2}(A|\underline{\underline{B}}) = g\, I_{\{g=f_o\}}\},$$

(ii) $\{A \varepsilon \underline{\underline{A}} / P_{\mu_1}(A|\underline{\underline{B}}) = g\, I_{\{g > f_o\}}\} =$

$$\{A \varepsilon \underline{\underline{A}} / P_{\mu_2}(A|\underline{\underline{B}}) = g\, I_{\{g > f_o\}}\} \text{ and}$$

(iii) $\{A \varepsilon \underline{\underline{A}} / P_{\mu_1}(A|\underline{\underline{B}}) = g\, I_{\{g < f_o\}}\} =$

$$\{A \varepsilon \underline{\underline{A}} / P_{\mu_2}(A|\underline{\underline{B}}) = g\, I_{\{g < f_o\}}\}.$$

Proof: Since $g\, I_{\{g = f_o\}} = f_o \cdot I_{\{g = f_o\}}$ and $\{g = f_o\}$ is a $\underline{\underline{B}}$ measurable set an application of lemma 5.3 proves (i).

Let $A_o \varepsilon \underline{\underline{A}}$ be such that $P_{\mu_1}(A_o|\underline{\underline{B}}) = g\, I_{\{g > f_o\}}$. To show $P_{\mu_2}(A_o|\underline{\underline{B}}) = g\, I_{\{g > f_o\}}$. Since $\mu_1(\{f_o > 0\}) = 1$, the function $\frac{f_o}{g}$ can be meaningfully talked of. Take $h = \frac{f_o}{g} \wedge 1$. By the remark following theorem 4.1, there is a $D \varepsilon \underline{\underline{A}}$ with

$$(P_{\mu_1}(D|\underline{\underline{B}}),\ P_{\mu_2}(D|\underline{\underline{B}})) = (h\ P_{\mu_1}(A_o|\underline{\underline{B}}),\ h\ P_{\mu_2}(A_o|\underline{\underline{B}})).$$

Now, $P_{\mu_1}(D|\underline{\underline{B}}) = h\ P_{\mu_1}(A_o|\underline{\underline{B}})$

$$= h \quad g\ I_{\{g > f_o\}}$$

$$= \frac{f_o}{g} \quad g\ I_{\{g > f_o\}} \quad \left(\text{as } h = \frac{f_o}{g} \text{ on the set } \{g > f_o\}\right)$$

$$= f_o\ I_{\{g > f_o\}}$$

$\therefore$ By lemma 5.3, $P_{\mu_2}(D|\underline{\underline{B}}) = f_o\ I_{\{g > f_o\}}$

i.e. $h\ P_{\mu_2}(A_o|\underline{\underline{B}}) = f_o\ I_{\{g > f_o\}}$ (*)

Observe that since $\mu_2(\{f_o > 0\}) = 1$, we have $\mu_2(\{h = 0\}) = 0$. Now, r.h.s. of (*), and consequently $h\ P_{\mu_2}(A_o|\underline{\underline{B}})$, is 0 on $\{g \leq f_o\}$. So, $P_{\mu_2}(A_o|\underline{\underline{B}}) = 0$ a.s. (μ_2) on $\{g \leq f_o\}$.

Again, r.h.s. of (*), and consequently $h\ P_{\mu_2}(A_o|\underline{\underline{B}})$, is f_o on $\{g > f_o\}$

i.e. $\frac{f_o}{g} P_{\mu_2}(A_o|\underline{\underline{B}}) = f_o$ on $\{g > f_o\}$ (as $h = \frac{f_o}{g}$ on $\{g > f_o\}$)

i.e. $P_{\mu_2}(A_o|\underline{\underline{B}}) = g$ a.s. (μ_2) on $\{g > f_o\}$, since $\mu_2(\{f_o > 0\})=1$.

Thus $P_{\mu_2}(A_o|\underline{\underline{B}}) = g\, I_{\{g > f_o\}}$.

The other part of (ii) is similarly proved . Hence (ii).

Let $A_o \varepsilon \underline{\underline{A}}$ be such that $P_{\mu_1}(A_o|\underline{\underline{B}}) = g\, I_{\{g < f_o\}}$.

Choose $D \varepsilon \underline{\underline{A}}$ with

$$(P_{\mu_1}(D|\underline{\underline{B}}),\ P_{\mu_2}(D|\underline{\underline{B}})) = (\frac{1-f_o}{1-g} \wedge 1)\ (P_{\mu_1}(A_o|\underline{\underline{B}}), P_{\mu_2}(A_o|\underline{\underline{B}})) + I_{\{g < f_o\}} \frac{f_o - g}{1-g}(1,\ 1).$$

Verifying that there is such a D is routine (using the remark following theorem 4.1) and is omitted.

$$\text{Now,}\quad P_{\mu_1}(D|\underline{\underline{B}}) = (\frac{1-f_o}{1-g} \wedge 1)\ P_{\mu_1}(A_o|\underline{\underline{B}}) + I_{\{g<f_o\}} \cdot \frac{f_o - g}{1-g}$$

$$= (\frac{1-f_o}{1-g} \wedge 1)\ g\, I_{\{g < f_o\}} + I_{\{g < f_o\}} \cdot \frac{f_o - g}{1-g}$$

$$= f_o\, I_{\{g < f_o\}}.$$

So, by lemma 5.3, $P_{\mu_2}(D|\underline{\underline{B}}) = f_o\, I_{\{g < f_o\}}$

$$\text{i.e.}\quad (\frac{1-f_o}{1-g} \wedge 1)\ P_{\mu_2}(A_o|\underline{\underline{B}}) + I_{\{g<f_o\}} \frac{f_o - g}{1-g} = f_o\, I_{\{g<f_o\}} \qquad (**)$$

On $\{g < f_o\}$ (**) reduces to

$$\frac{1-f_o}{1-g}\, P_{\mu_2}(A_o \mid \underline{\underline{B}}) + \frac{f_o - g}{1-g} = f_o$$

or $(1-f_o)\, P_{\mu_2}(A_o|\underline{\underline{B}}) + f_o - g = f_o(1-g)$ (note that since $f_o < 1$, $g < f_o \Rightarrow g < 1$)

or $(1-f_o)\, P_{\mu_2}(A_o|\underline{\underline{B}}) = g(1-f_o)$

or $$P_{\mu_2}(A_o|\underline{\underline{B}}) = g$$

i.e. $P_{\mu_2}(A_o|\underline{\underline{B}}) = g$ a.s. (μ_2) on $\{g < f_o\}$.

On $\{g \geq f_o\}$ (**) reduces to

$$\left(\frac{1-f_o}{1-g} \wedge 1\right) P_{\mu_2}(A_o|\underline{\underline{B}}) = 0$$

or $$P_{\mu_2}(A_o|\underline{\underline{B}}) = 0$$

i.e. $P_{\mu_2}(A_o|\underline{\underline{B}}) = 0$ a.s. (μ_2) on $\{g \geq f_o\}$.

Thus $P_{\mu_2}(A_o|\underline{\underline{B}}) = g\, I_{\{g < f_o\}}$.

Interchanging the roles of μ_1 and μ_2 the proof of (iii) is completed.

Hence the lemma.

<u>Proof of theorem 5.2</u>: Let $A_o \in \underline{A}$. Let

$g = P_{\mu_1}(A_o|\underline{\underline{B}})$. Then

$$A_0 = (A_0 \cap \{g<f_0\}) \cup (A_0 \cap \{g=f_0\}) \cup (A_0 \cap \{g>f_0\}).$$

Now,

$$P_{\mu_1}(A_0 \cap \{g < f_0\}|\underline{\underline{B}}) = g.I_{\{g < f_0\}},$$

$$P_{\mu_1}(A_0 \cap \{g = f_0\}|\underline{\underline{B}}) = g.I_{\{g = f_0\}} \quad \text{and}$$

$$P_{\mu_1}(A_0 \cap \{g > f_0\}|\underline{\underline{B}}) = g.I_{\{g > f_0\}}.$$

So, by lemma 5.4,

$$P_{\mu_2}(A_0 \cap \{g < f_0\}|\underline{\underline{B}}) = g.I_{\{g < f_0\}},$$

$$P_{\mu_2}(A_0 \cap \{g = f_0\}|\underline{\underline{B}}) = g.I_{\{g = f_0\}} \quad \text{and}$$

$$P_{\mu_2}(A_0 \cap \{g > f_0\}|\underline{\underline{B}}) = g.I_{\{g > f_0\}}.$$

Hence $P_{\mu_2}(A_0|\underline{\underline{B}}) = g$.

$\therefore$ $P_{\mu_1}(A_0|\underline{\underline{B}}) = P_{\mu_2}(A_0|\underline{\underline{B}})$. This implies $\mu_1(A_0) = \mu_2(A_0)$.

Hence the theorem.

In the above theorem the condition that $\mu_1(\{0 < f_0 < 1\})=1$ (which is analogous to the condition that $0 < r < 1$ in theorem 5.1) cannot be relaxed. For this, consider the following example.

Let $\Omega = [0, 1]$, $\underline{A}$ its Borel σ-algebra. Let μ be the Lebesgue measure on $(\Omega, \underline{A})$. Let α_1 and α_2 be numbers such that $\alpha_1 \neq \alpha_2$, $0 < \alpha_i < 1$, $i = 1,2$. Define measures μ_1

and μ_2 by setting for $A \varepsilon \underline{\underline{A}}$,

$$\mu_1(A) = \frac{\alpha_1}{2} \frac{\mu(A \cap [0, \frac{1}{4}])}{\mu([0, \frac{1}{4}])} + \frac{(1-\alpha_1)}{2} \frac{\mu(A \cap [\frac{1}{4}, \frac{1}{2}])}{\mu([\frac{1}{4}, \frac{1}{2}])} + \mu(A \cap [\frac{1}{2}, 1])$$

$$\mu_2(A) = \frac{\alpha_2}{2} \frac{\mu(A \cap [0, \frac{1}{4}])}{\mu([0, \frac{1}{4}])} + \frac{(1-\alpha_2)}{2} \frac{\mu(A \cap [\frac{1}{4}, \frac{1}{2}])}{\mu([\frac{1}{4}, \frac{1}{2}])} + \mu(A \cap [\frac{1}{2}, 1]).$$

Let $\underline{\underline{B}} = \{ \phi, [0, \frac{1}{2}), [\frac{1}{2}, 1], \Omega \}$. Clearly μ_1 and μ_2 are $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic measures. Now, for any A with

$\mu(A \cap [0, \frac{1}{2}]) = 0$,

$$\begin{aligned} P_{\mu_1}(A \mid \underline{\underline{B}}) &= 0 \quad \text{on } [0, \tfrac{1}{2}) \\ &= 2\,\mu(A \cap [\tfrac{1}{2}, 1]) \text{ on } [\tfrac{1}{2}, 1], \text{ and} \end{aligned}$$

$$\begin{aligned} P_{\mu_2}(A \mid \underline{\underline{B}}) &= 0 \quad \text{on } [0, \tfrac{1}{2}) \\ &= 2\,\mu(A \cap [\tfrac{1}{2}, 1]) \text{ on } [\tfrac{1}{2}, 1]; \text{ that is} \end{aligned}$$

$P_{\mu_1}(A|\underline{\underline{B}}) = P_{\mu_2}(A|\underline{\underline{B}})$.

Let $f_0 = I_{[\frac{1}{2}, 1]}$.

$$P_{\mu_1}(A|\underline{\underline{B}}) = f_0 \Rightarrow \mu(A \,(\overline{\ })\, [0, \tfrac{1}{2})) = 0$$

$$\Rightarrow P_{\mu_2}(A|\underline{\underline{B}}) = P_{\mu_1}(A|\underline{\underline{B}})$$

$$\Rightarrow P_{\mu_2}(A|\underline{\underline{B}}) = f_0 .$$

Likewise $P_{\mu_2}(A|\underline{\underline{B}}) = f_0 \Rightarrow P_{\mu_1}(A|\underline{\underline{B}}) = f_0$.

However $\mu_1 \neq \mu_2$ as $\mu_1([0, \frac{1}{4}]) = \frac{\alpha_1}{2}$ and $\mu_2([0, \frac{1}{4}]) = \frac{\alpha_2}{2}$.

6. Nonatomicity in Polish Spaces

Let $\underline{(\overline{\ })}$ be a Polish space and $\underline{A}$ its Borel σ-algebra. In this section we consider the problem of characterizing those sub σ-algebras $\underline{\underline{B}}$ of $\underline{A}$ which admit a $(\underline{B}, \underline{A})$ non-atomic measure ; also, we give an equivalent characterization of a measure λ being $(\underline{B}, \underline{A})$ nonatomic in terms of the atoms of $\underline{\underline{B}}$. For our study, we need the following two results.

Theorem 6.1 (Blackwell). Let $A \in \underline{A}$. Let $\underline{\underline{B}}$ be a countably generated sub σ-algebra of $\underline{A}$ such that $A \,(\overline{\ })\, \underline{\underline{B}}$ has singleton atoms. Then $A \,(\overline{\ })\, \underline{\underline{B}} = A \,(\overline{\ })\, \underline{A}$.

This follows as an easy consequence of Corollary 1 to theorem 3 of [4].

Theorem 6.2 (Lusin). Let $\underline{\underline{B}}$ be a countably generated sub σ-algebra of $\underline{A}$ with every atom countable. Then we can get a sequence $\{A_n\}_{n\geq 1}$ of Borel sets such that

(1) $A_i \cap A_j = \phi$, $i \neq j$ and $\bigcup_{n=1}^{\infty} A_n = \Omega$ and

(2) each A_n contains at most one point from every atom of $\underline{\underline{B}}$.

A proof of this is available in page 335 of [10].

The next theorem gives a complete solution to the problem mentioned in the beginning of this section when the sub σ-algebra considered is countably generated.

<u>Theorem 6.3</u> Let $\underline{\underline{B}}$ be a countably generated sub σ-algebra of $\underline{\underline{A}}$. Then there exists a $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic measure if and only if $\underline{\underline{B}}$ has at least one uncountable atom.

<u>Proof</u>. Let $\underline{\underline{B}}$ have an uncountable atom, say B_o. By Borel isomorphism theorem B_o supports a nonatomic measure, say λ. Let λ_1 be the extension of λ to $\underline{\underline{A}}$ defined by $\lambda_1(A) = \lambda(A \cap B_o)$, $A \in \underline{\underline{A}}$. It is clear that λ_1 is a non-atomic measure. We shall show that λ_1 is $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic. Let $A \in \underline{\underline{A}}$ with $\lambda_1(A) > 0$; so $\lambda_1(A \cap B_o) > 0$. Choose $A_1 \subseteq A \cap B_o$ with $\lambda_1(A_1) = \frac{1}{2}\lambda_1(A \cap B_o)$.

$$(A \cap B_o) \cap \underline{\underline{B}} = A \cap (B_o \cap \underline{\underline{B}})$$

$$= A \cap \{\phi, \Omega\} \quad \text{(since } B_o \text{ is a } \underline{\underline{B}}\text{-atom)}$$

$$= \{\phi, A\}.$$

On the other hand $A_1 \in A \cap B_0 \cap \underline{\underline{A}}$ and $\lambda_1(A_1 \Delta \phi) > 0$ and $\lambda_1(A_1 \Delta A) > 0$. So, the traces of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ do not coincide (upto λ_1-null sets) on $A \cap B_0$; hence they do not on A. Thus, A is not a $(\underline{\underline{B}}, \underline{\underline{A}})$ (λ_1) atom. Since A is arbitrary it follows that λ_1 is $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic. Hence the 'if' part.

Let now every atom of $\underline{\underline{B}}$ be countable. By theorem 6.2 there are Borel sets $A_1, A_2, \ldots, A_n, \ldots$ such that

(1) $A_i \cap A_j = \phi$ if $i \neq j$ and $\bigcup_{n=1}^{\infty} A_n = \Omega$ and

(2) each A_n contains at most one point from every atom of $\underline{\underline{B}}$. We now claim that $A_n \cap \underline{\underline{B}} = A_n \cap \underline{\underline{A}}$, for every n. (Note that there is no measure involved). For, $A_n \cap \underline{\underline{B}}$ has singleton atoms (in view of property (2) mentioned above); hence by theorem 6.1, $A_n \cap \underline{\underline{B}} = A_n \cap \underline{\underline{A}}$. Let λ be any measure on $\underline{\underline{A}}$. Let n be such that $\lambda(A_n) > 0$. Then clearly A_n is a $(\underline{\underline{B}}, \underline{\underline{A}})$ (λ) atom. Thus no measure on $\underline{\underline{A}}$ is a $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic measure. Hence the 'only if' part and the theorem.

Regarding the case of not-countably generated $\underline{\underline{B}}$ we have the following two examples.

Example 1: Let Ω be uncountable. Let

$\underline{\underline{B}} = \{ B \,/\, B$ is Borel, either B or B^c is a countable set $\}$

Clearly $\underline{\underline{B}}$ is not countably generated. Let λ be a nonatomic measure on $(\Omega, \underline{\underline{A}})$. (There exist many such). Since $\{\phi, \Omega\} = \underline{\underline{B}}$ (modulo $\underline{\underline{N}}_\lambda$) it is clear that λ is $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic. Thus there are not-countably generated sub σ-algebras $\underline{\underline{B}}$ admitting $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic measures.

<u>Example 2</u>: Let $\Omega = [0, 1]$. Let A be a non-Borel universal null set. (The existence of such sets assumes the continuum hypothesis and the axim of choice. For details see pages 525 and 532 of [11]).

Define $\underline{\underline{B}} = \{B \,/\, B$ is Borel, and either $B \subset A^c$ or $B^c \subset A^c\}$.

That $\underline{\underline{B}}$ is a σ-algebra is evident. Since $\underline{\underline{A}}$ is a non-Borel set, it follows that $\underline{\underline{B}}$ is not countably generated. Let λ be any nonatomic measure on $\underline{\underline{A}}$. By the choice of A, we have A is λ-measurable and $\lambda(A^c) = 1$. So, $\underline{\underline{B}} = \underline{\underline{A}}$ (modulo $\underline{\underline{N}}_\lambda$). Hence λ is not $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic. Thus $\underline{\underline{B}}$ does not admit any $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic measure. So, there are not-countably generated sub σ-algebras $\underline{\underline{B}}$ not admitting any $(\underline{\underline{B}}, \underline{\underline{A}})$ non-atomic measure.

Before stating our next result we need a definition.

<u>Definition</u>: Let $\underline{\underline{B}}$ be a countably generated sub σ-algebra of $\underline{\underline{A}}$. $A \in \underline{\underline{A}}$ is said to be one-sheeted with respect to $\underline{\underline{B}}$ if it contains at most one point from each atom of $\underline{\underline{B}}$.

<u>Theorem 6.4</u> Let λ be a measure on $(\Omega, \underline{\underline{A}})$. Let $\underline{\underline{B}}$ be a countably generated sub σ-algebra of $\underline{\underline{A}}$. Then λ is $(\underline{\underline{B}}, \underline{\underline{A}})$ nonatomic if and only if there is no $A \in \underline{\underline{A}}$ which is one-sheeted with respect to $\underline{\underline{B}}$ and has positive λ-measure.

<u>Proof</u>: Let D be a $(\underline{\underline{B}}, \underline{\underline{A}})$ (λ) atom. Let $\{A_n\}_{n\geq 1}$ be a sequence of sets generating $\underline{\underline{A}}$. Since D is a $(\underline{\underline{B}}, \underline{\underline{A}})$ (λ) atom, for each n we can get a $\underline{\underline{B}}$-set B_n with $\lambda((B_n \Delta A_n) \cap D) = 0$. Let $A = D - \bigcup_{n=1}^{\infty} (B_n \Delta C_n)$.
Clearly $\lambda(A) = \lambda(D) > 0$. Presently we show that A is one-sheeted with respect to $\underline{\underline{B}}$. It is easily verified that $A \cap B_n = A \cap A_n$ for all n. So,

$$A \cap \underline{\underline{A}} = \text{the } \sigma\text{-algebra generated by } \{A \cap A_1, A \cap A_2, \ldots, A \cap A_n, \ldots\}$$
$$= \text{the } \sigma\text{-algebra generated by } \{A \cap B_1, A \cap B_2, \ldots, A \cap B_n, \ldots\}$$
$$\subset A \cap \underline{\underline{B}}$$

$\therefore$ $A \cap \underline{\underline{A}} = A \cap \underline{\underline{B}}$. Since $A \cap \underline{\underline{A}}$ has singleton atoms, $A \cap \underline{\underline{B}}$ has singleton atoms. Hence A contains at most one point from each atom of $\underline{\underline{B}}$. This completes the proof of 'if' part.

Let now A be one-sheeted with respect to $\underline{\underline{B}}$ and have positive λ-measure. Then $A \cap \underline{\underline{B}}$ is a sub σ-algebra of $A \cap \underline{\underline{A}}$ and has singleton atoms. Therefore, by theorem 6.1,

$A \cap \mathcal{B} = A \cap \mathcal{A}$; in particular, A is a $(\mathcal{B}, \mathcal{A})$ (λ) atom. This completes the proof of 'only if' part and hence the theorem.

Remarks :

(1) The concept of 'one-sheeted' subsets has been introduced by Rohlin in [22], for Lebesgue spaces.

(2) Theorem 6.4 is not the best possible. A careful perusal of the proof of the theorem reveals that the 'if' part remains valid so long $\mathcal{A}$ is a countably generated σ-algebra with singleton atoms ; no assumption of topological nature need be made on Ω. However the same thing cannot be said of the 'only if' part. Below we present an example to elucidate this.

Example: Let Ω be an arbitrary set, $\mathcal{A}$ a σ-algebra of its subsets. Let $\mathcal{B} \subset \mathcal{A}$ and λ be a $(\mathcal{B}, \mathcal{A})$ nonatomic measure. Let λ^* denote the λ-outer-measure. Let D be a subset of Ω with $\lambda^*(D) = 1$. Define λ_1 on $D \cap \mathcal{A}$ by $\lambda_1 (D \cap A) = \lambda(A)$, $A \in \mathcal{A}$. It is routine to verify that λ_1 is unambiguously defined. The fact that λ is $(\mathcal{B}, \mathcal{A})$ nonatomic implies that λ_1 is $(D \cap \mathcal{B}, D \cap \mathcal{A})$ nonatomic. We sketch the proof of this. Let $\lambda_1(D \cap A) > 0$. So, $\lambda(A) > 0$. Since A is not a $(\mathcal{B}, \mathcal{A})$ (λ) atom, there is $A_1 \subset A$ with $\lambda((A_1 \Delta B) \cap A) > 0$ for all $B \in \mathcal{B}$.

So, $\lambda_1((A_1 \Delta B) \cap (A \cap D)) > 0$ for all $B \in \mathcal{B}$.

i.e. $\lambda_1((A_1 D \Delta B D) \cap AD) > 0$ for all $B \in \mathcal{B}$.

Hence the traces of $D \cap \mathcal{A}$ and $D \cap \mathcal{B}$ do not coincide (upto $\mathcal{N}_{\lambda_1}$ sets) on $D \cap A$. So, $D \cap A$ is not a $(D \cap \mathcal{B}, D \cap \mathcal{A})$ (λ_1) atom. Thus λ_1 is $(D \cap \mathcal{B}, D \cap \mathcal{A})$ nonatomic.

Now let us specialize. Take $\Omega = [0, 1] \times [0, 1]$, $\mathcal{A}$ = the Borel σ-algebra of Ω. Let $\mathcal{B} = \{B \times [0, 1] / B$ is a Borel subset of $[0, 1]\}$. Let λ be the product Lebesgue measure on $(\Omega, \mathcal{A})$. Let λ^* denote the outer Lebesgue measure. Clearly λ is $(\mathcal{B}, \mathcal{A})$ nonatomic. Let D be a set with $\lambda^*(D) = 1$; further let for every $x \in [0, 1]$, $D^x = \{y : (x, y) \in \Omega\}$ be at most a singleton. (A construction of such a set D is available, for instance, in Lemma 2 of [21]). Define λ_1 as in above. Then λ_1 is $(D \cap \mathcal{B}, D \cap \mathcal{A})$ nonatomic. Evidently $D \cap \mathcal{A}$ is countably generated with singleton atoms, $D \cap \mathcal{B}$ is countably generated and D is one-sheeted with respect to $D \cap \mathcal{B}$ (as D^x is at most a singleton for each $x \in [0, 1]$) with $\lambda_1(D) = 1$.

CHAPTER 2

THE METRIC SPACE OF SUB σ-ALGEBRAS OF A PROBABILITY SPACE

1. Introduction

Let $(\Omega, \underset{=}{A}, P)$ be a probability space. A sub σ-algebra $\underset{=}{B}$ of $\underset{=}{A}$ is said to be complete if every $\underset{=}{A}$ measurable set of P-measure zero is in $\underset{=}{B}$. By the completion of a sub σ-algebra $\underset{=}{B}$ of $\underset{=}{A}$ we mean the smallest complete sub σ-algebra containing $\underset{=}{B}$. Let $\underset{=}{S}(\Omega, \underset{=}{A}, P)$ be the class of all complete sub σ-algebras of $\underset{=}{A}$. In [7], Boylan introduced a mertic d on $\underset{=}{S}(\Omega, \underset{=}{A}, P)$ by letting,

$$d(\underset{=}{B}_1, \underset{=}{B}_2) = \sup_{B_1 \varepsilon \underset{=}{B}_1} \inf_{B_2 \varepsilon \underset{=}{B}_2} P(B_1 \Delta B_2) + \sup_{B_2 \varepsilon \underset{=}{B}_2} \inf_{B_1 \varepsilon \underset{=}{B}_1} P(B_1 \Delta B_2),$$

where $\underset{=}{B}_1$ and $\underset{=}{B}_2$ are elements of $\underset{=}{S}(\Omega, \underset{=}{A}, P)$. In that paper, this metric is used to obtain an elegant sufficient condition for 'equiconvergence of martingales'. (More about it in section 3). When there is no room for confusion we will just write $\underset{=}{S}$ or $(\underset{=}{S}, d)$ in place of $(S(\Omega, \underset{=}{A}, P), d)$.

In this chapter we study the topological properties like completeness, compactness, local compactness, separability, connectedness, total disconnectedness, perfectness and dimension of $(\underset{=}{S}, d)$. Also we consider an isomorphism problem.

Let X be any metric space with a bounded metric ρ. On 2^X (that is, on the class of all closed subsets of X) one introduces what is known as the Hausdorff metric, ρ_1, by setting

$$\rho_1(C,D) = \sup_{x\varepsilon C} \inf_{y\varepsilon D} \rho(x,y) + \sup_{y\varepsilon D} \inf_{x\varepsilon C} \rho(x,y), \quad C,D \varepsilon 2^X.$$

Some topological properties of $(2^X, \rho_1)$ can be characterised using similar properties of (X, ρ). (See sections dealing with $(2^X)_m$ of [11] and [12]).

For any $A \varepsilon \underline{\underline{A}}$, denote by $[A]$ the class of all measurable sets that are equivalent to A (i.e. those sets which differ from A by a set of measure zero). Let $X = \{[A]/A \varepsilon \underline{\underline{A}}\}$ and define $\rho([A], [B]) = P(A \Delta B)$, where $[A], [B] \varepsilon X$. (X, ρ) is a metric space. X is usually referred to as the measure algebra $\underline{\underline{A}}(P)$ [9, P.167]. Define a map $(+)$ from $\underline{\underline{S}}$ to 2^X as follows.

$(+)(\underline{\underline{B}}) = \{[B]/B \varepsilon \underline{\underline{B}}\}$, $\underline{\underline{B}} \varepsilon \underline{\underline{S}}$. $(+)(\underline{\underline{B}})$ is a closed subset of (X, ρ) and hence an element of 2^X.

For any $\underline{\underline{B}}_1, \underline{\underline{B}}_2 \varepsilon \underline{\underline{S}}$,

$d(\underline{\underline{B}}_1, \underline{\underline{B}}_2) = \rho_1((+)(\underline{\underline{B}}_1), (+)(\underline{\underline{B}}_2))$, where ρ_1 is the Hausdorff distance as introduced above. Denoting by $\widehat{\underline{\underline{S}}}$ the range of $(+)$ in 2^X, we, therefore, have that $(+)$ is an isometry between $\underline{\underline{S}}$ and $\widehat{\underline{\underline{S}}}$. We will use this fact to prove some of our results.

We recall a few definitions. Let $(\Omega, \mathcal{A}, P)$ be a probability space. $A \in \mathcal{A}$ is said to be an atom if (i) $P(A) > 0$ and (ii) $B \in \mathcal{A}$, $B \subset A \Rightarrow P(B) = 0$ or $P(B) = P(A)$. The probability P is said to be nonatomic on $\mathcal{A}$ if there are no atoms ; it is said to be completely atomic on $\mathcal{A}$ if Ω is the union of atoms. If P is completely atomic on $\mathcal{A}$ then Ω is a disjoint union of countably many atoms. For $A_0 \in \mathcal{A}$, P is said to be nonatomic on A_0 if (i) $P(A_0) > 0$ and (ii) no subset of A_0 is an atom.

For our study, we need the following facts about conditional atoms (as introduced in chapter 1). Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be elements of $\mathcal{S}$ with $\mathcal{B}_1 \subset \mathcal{B}_2$. Let A be a conditional atom for $(\mathcal{B}_1, \mathcal{B}_2)$. Since for any $B_2^0 \in \mathcal{B}_2$, there exists $B_1^0 \in \mathcal{B}_1$ with $B_2^0 \cap A = B_1^0 \cap A$, we have

$$P(B_2^0 \Delta B_1^0) = P[(B_2^0 \Delta B_1^0) \cap A] + P[(B_2^0 \Delta B_1^0) \cap A^c]$$
$$\leq P(A^c).$$

Therefore, for any $B_2^0 \in \mathcal{B}_2$ $\inf_{B_1 \in \mathcal{B}_1} P(B_2^0 \Delta B_1) \leq P(A^c)$.

Hence $d(\mathcal{B}_1, \mathcal{B}_2) \leq P(A^c)$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be elements of $\mathcal{S}$ with $\mathcal{B}_1 \subset \mathcal{B}_2$. Let $\{A_\alpha\}$ be a collection of measurable sets generating a sub σ-algebra whose completion is $\mathcal{B}_2$. Then, it is not hard to see that $A \in \mathcal{B}_2$ is a conditional atom for $(\mathcal{B}_1, \mathcal{B}_2)$ if and only if for every A_α one can find $B_\alpha \in \mathcal{B}_1$ such that $P[(A \cap A_\alpha) \Delta (A \cap B_\alpha)] = 0$. The proof of 'if' part uses the fact that the class of $B \varepsilon \underline{\underline{B}}_2$ with the property that there is $B_1 \varepsilon \underline{\underline{B}}_1$ such that $P[(A \cap B) \Delta (A \cap B_1)] = 0$ is a complete sub σ-algebra containing all A_α's. The proof of 'only if' part is trivial.

In the sequel, $(\Omega, \underline{A}, P)$ will stand for a fixed probability space and X for the measure algebra $\underline{A}(P)$. $\widehat{\underline{\underline{S}}}$ will stand for the image in 2^X of $\underline{\underline{S}}$ under the mapping considered above. Given any $A_0 \varepsilon \underline{A}$ with $P(A_0) > 0$, the symbol P_{A_0} will stand for the probability measure on $A_0 \cap \underline{A}$, defined by $P_{A_0}(B \cap A_0) = \dfrac{P(B \cap A_0)}{P(A_0)}$, $B \varepsilon \underline{A}$. Given a collection $\{A_\alpha, \alpha \varepsilon \Gamma\}$ of $\underline{A}$-measurable sets, the symbol $\sigma\{A_\alpha, \alpha \varepsilon \Gamma\}$ will denote the smallest σ-algebra containing all A_α's. A word about the symbol $(\dagger)$. This has not been reserved exclusively for the mapping considered above. In the latter sections of this chapter we have occasions to use $(\dagger)$ to denote other mappings as well ; however, it will be clear from the context which mapping is being referred to.

2. Completeness of $(\underline{\underline{S}}, d)$

It is known that X is complete and hence 2^X is complete [11, the theorem in P.407]. So, to prove that $(\underline{\underline{S}}, d)$ is complete, it is enough to prove that $\widehat{\underline{\underline{S}}}$ is a closed subset of 2^X.

Lemma 2.1 $\underset{\approx}{\widetilde{S}}$ is a closed subset of 2^X.

Proof : Let $\underset{=}{B}_n$ be a sequence of elements of $\underset{\approx}{\widetilde{S}}$ converging to an element $\underset{=}{F}$ of 2^X. Since $\underset{=}{F}$ is a closed subset of X, $[F] \varepsilon \underset{=}{F}$ if and only if $\inf\limits_{[G] \varepsilon \underset{=}{F}} P(F \Delta G) = 0$. We have to show that $\underset{=}{F} \varepsilon \underset{\approx}{\widetilde{S}}$. For this, in the light of the above observation, enough to show that

(i) $[F] \varepsilon \underset{=}{F} \Rightarrow \inf\limits_{[G] \varepsilon \underset{=}{F}} P(F^c \Delta G) = 0$ and

(ii) $[F_1], [F_2], \ldots, [F_n], \ldots, \varepsilon \underset{=}{F} \Rightarrow \inf\limits_{[G] \varepsilon \underset{=}{F}} P[((\underset{n}{\cup}) F_n) \Delta G] = 0.$

Let $\varepsilon > 0$ be given. Let $[F] \varepsilon \underset{=}{F}$. Since $\rho_1(\underset{=}{B}_n, \underset{=}{F}) \to 0$, there exists $\underset{=}{B}_{n_o}$, for some sufficiently large n_o, such that $\rho_1(\underset{=}{B}_{n_o}, \underset{=}{F}) < \frac{\varepsilon}{2}$. Choose $[B_{n_o}] \varepsilon \underset{=}{B}_{n_o}$, for which $P(F \Delta B_{n_o}) < \varepsilon/2$. Then

$$\inf_{[G] \varepsilon \underset{=}{F}} P(F^c \Delta G) \leq P(F^c \Delta B^c_{n_o}) + \inf_{[G] \varepsilon \underset{=}{F}} P(B^c_{n_o} \Delta G)$$

$$= P(F \Delta B_{n_o}) + \inf_{[G] \varepsilon \underset{=}{F}} P(B^c_{n_o} \Delta G)$$

$$< \frac{\varepsilon}{2} + \rho_1(\underset{=}{F}, \underset{=}{B}_{n_o}) < \varepsilon .$$

Now, ε being arbitrary, proof of (i) follows.

Let $[F_1]$ and $[F_2]$ belong to $\underset{=}{F}$. Choose n_o large enough that $\rho_1(\underset{=}{F}, \underset{=}{B}_{n_o}) < \frac{\varepsilon}{4}$. This implies that there exist $[B_1]$ and $[B_2]$ in $\underline{\underline{B}}_{n_o}$ such that $P(F_j \Delta B_j) < \frac{\varepsilon}{4}$, for $j = 1,2$.

$$\inf_{[G]\varepsilon \underline{\underline{F}}} P[(F_1 \cup F_2) \Delta G] \leq P[(F_1 \cup F_2) \Delta (B_1 \cup B_2)] + \inf_{[G]\varepsilon \underline{\underline{F}}} P[(B_1 \cup B_2) \Delta G]$$

$$\leq P(F_1 \Delta B_1) + P(F_2 \Delta B_2) + \rho_1(\underline{\underline{F}}, \underline{\underline{B}}_{n_o})$$

$$< 3\frac{\varepsilon}{4} < \varepsilon .$$

ε being arbitrary, we have $\inf_{[G]\varepsilon \underline{\underline{F}}} P[(F_1 \cup F_2) \Delta G] = 0$;

i.e. $[F_1 \cup F_2] \varepsilon \underline{\underline{F}}$. So, by induction

$$[F_1], [F_2], \ldots, [F_n] \varepsilon \underline{\underline{F}} \Rightarrow [\bigcup_{m=1}^{n} F_m] \varepsilon \underline{\underline{F}} .$$

Now, let $\{[F_n], n \geq 1\} \subset \underline{\underline{F}}$. Choose n_o so large that $P(\bigcup_{m=1}^{\infty} F_m - \bigcup_{m=1}^{n_o} F_m) < \varepsilon$. Since $[\bigcup_{m=1}^{n_o} F_m] \varepsilon \underline{\underline{F}}$. We have,

$\inf_{[G]\varepsilon \underline{\underline{F}}} P[(\bigcup_{m=1}^{\infty} F_m) \Delta G] < \varepsilon$. Again, ε being arbitrary, the proof of (ii) follows.

<u>Theorem 2.2</u> $(\underline{\underline{S}}, d)$ is a complete metric space.

<u>Proof</u>. The proof follows from lemma 2.1 and the observation made in the beginning of this section.

Remark: Boylan gives a different proof of this theorem in [7].

3. Compactness and local compactness of $(\underline{\underline{S}}, d)$

In this section we show that the following are equivalent.

(i) P is completely atomic on $\underline{\underline{A}}$.

(ii) $(\underline{\underline{S}}, d)$ is compact.

(iii) $(\underline{\underline{S}}, d)$ is locally compact.

Let $B_o \in \underline{\underline{A}}$ be such that P is nonatomic on B_o. Let $\underline{\underline{B}}_o$ be the complete sub σ-algebra of $\underline{\underline{A}}$ defined by

$$\underline{\underline{B}}_o = \{A \in \underline{\underline{A}} / P(A \cap B_o) = P(A) \text{ or } P(A \cap B_o^c) = P(B_o^c)\}.$$

Then we have the following lemma.

Lemma 3.1 Let ε be such that $0 < \varepsilon < P(B_o)$. A sequence $\{\underline{\underline{B}}_n\}_{n \geq 1}$, from $\underline{\underline{S}}$, can be found satisfying

(i) $d(\underline{\underline{B}}_n, \underline{\underline{B}}_o) < \varepsilon$, $n \geq 1$ and

(ii) $d(\underline{\underline{B}}_n, \underline{\underline{B}}_m) \geq \frac{\varepsilon}{4}$, $n, m \geq 1, n \neq m$.

Proof : Since P is nonatomic on B_o given any measurable subset B of B_o with $0 < P(B)$ and given any $\gamma < P(B)$, one can find a measurable subset B' of B with $P(B') = \gamma$. We construct a family

$$\begin{array}{llll} B_{11} & & & \\ B_{21} & B_{22} & & \\ \vdots & \vdots & & \\ B_{n1} & B_{n2} & \cdots & B_{n2^{n-1}} \\ \vdots & \vdots & & \vdots \end{array} \qquad (*)$$

of meaurable subsets of B_o as given below.

B_{11} is a subset of B_o with $P(B_{11}) = \frac{\varepsilon}{2}$. Having obtained the sets in the first n rows, the sets in the $(n+1)^{th}$ row are obtained as follows.

$B_{n+1,1}$ is a subset of B_{n1} with $P(B_{n+1,1}) = \frac{P(B_{n1})}{2}$ and $B_{n+1,2} = B_{n1} - B_{n+1,1}$. In general, for $1 \leq k \leq 2^{n-1}$, the set $B_{n+1,2k-1}$ is a subset of B_{nk} with $P(B_{n+1,2k-1}) = \frac{P(B_{nk})}{2}$ and $B_{n+1,2k} = B_{nk} - B_{n+1,2k-1}$. The rows of (*) form successively finer partitions of B_{11} with $P(B_{n,k}) = \frac{\varepsilon}{2^n}$, $1 \leq k \leq 2^{n-1}$ and $n \geq 1$.

Now, define

$\underline{\underline{B}}_1 = \{A \in \underline{\underline{B}}_o \,/\, P(A \cap B_{11}) = P(B_{11}) \text{ or } 0\}$. More generally, for $n \geq 1$

$\underline{\underline{B}}_n = \{A \in \underline{\underline{B}}_o \,/\, P(A \cap B_{nk}) = P(B_{nk}) \text{ or } 0, \text{ for all } 1 \leq k \leq 2^{n-1}\}$.

Clearly (α) $\{\underline{\underline{B}}_n\}_{n \geq 1} \subset \underline{\underline{S}}$, ($\beta$) $\underline{\underline{B}}_n \subset \underline{\underline{B}}_o$ for all $n \geq 1$, (γ) $\underline{\underline{B}}_n$'s are increasing with n and (δ) for every n, the set B_{11}^c is a conditional atom for $(\underline{\underline{B}}_n, \underline{\underline{B}}_o)$. Hence $d(\underline{\underline{B}}_n, \underline{\underline{B}}_o) \leq P(B_{11}) = \frac{\varepsilon}{2}$. This proves (i).

Let D be the $\underline{\underline{B}}_{n+1}$ set defined by $D = \bigcup_{k=1}^{2^{n-1}} B_{n+1,2k-1}$. Observe that for any $B \in \underline{\underline{B}}_n$, we have $D \Delta B \supset D \Delta (B \cap B_{11})$; and that $(B \cap B_{11})$ is the union of some of B_{nk}'s. Therefore, it follows that $\inf_{B \in \underline{\underline{B}}_n} P(D \Delta B) = \frac{\varepsilon}{4}$. Hence $d(\underline{\underline{B}}_n, \underline{\underline{B}}_{n+1}) \geq \frac{\varepsilon}{4}$. As $\underline{\underline{B}}_n$'s increase with n, for $m > n$ we have $d(\underline{\underline{B}}_n, \underline{\underline{B}}_m) \geq d(\underline{\underline{B}}_n, \underline{\underline{B}}_{n+1}) \geq \frac{\varepsilon}{4}$. This proves (ii).

<u>Theorem 3.2</u> The following are equivalent.

(i) P is completely atomic on $\underline{\underline{A}}$.

(ii) $(\underline{\underline{S}}, d)$ is compact.

(iii) $(\underline{\underline{S}}, d)$ is locally compact.

<u>Proof.</u> Let P be completely atomic on $\underline{\underline{A}}$. Then X is compact([15, Section 1, p.94] or [2]). So, 2^X is compact ([by the theorem on p.47 and theorem 1, on p.45 of (12)]). $\widetilde{\underline{\underline{S}}}$ being a closed subset of 2^X, is, therefore compact. Hence $\underline{\underline{S}}$ is compact.

The fact that (ii) implies (iii) is immediate.

Suppose P is not completely atomic. So, there exists $B_o \varepsilon \underline{\underline{A}}$ with $P(B_o) > 0$ and P is nonatomic on B_o. Then it is clear that, by lemma 3.1, no closed sphere around $\underline{\underline{B}}_o$ (as defined before that lemma) is compact. Thus (iii) $\Rightarrow$ (i).

<u>A digression</u>: Theorem 3.2 can be used to make an observation on equiconvergence of martingales. First, the necessary details.

Let $\{\underline{\underline{F}}_n\}_{n \geq 1}$ be a sequence of elements from $\underline{\underline{S}}$, increasing to $\underline{\underline{F}}_\infty$; i.e. $\underline{\underline{F}}_n$'s are increasing and $\underline{\underline{F}}_\infty$ is the completion of $\bigcup_n \underline{\underline{F}}_n$. Let $Z \varepsilon L_1(\Omega, \underline{\underline{A}}, P)$. For any $\underline{\underline{B}} \varepsilon \underline{\underline{S}}$, let $P(Z|\underline{\underline{B}})$ stand for the conditional expectation of Z given $\underline{\underline{B}}$. It is part of the folklore that $\{P(Z|\underline{\underline{F}}_n)\}_{n \geq 1}$ is a martingale and $P(Z|\underline{\underline{F}}_n) \xrightarrow{L_1} P(Z|\underline{\underline{F}}_\infty)$. (See, theorem V T 18 of [16]). Let J be a subset of $L_1(\Omega, \underline{\underline{A}}, P)$.

<u>Definition</u> (Boylan, [7]): The martingales

$\{P(Z|\underline{\underline{F}}_1), P(Z|\underline{\underline{F}}_2), \ldots, P(Z|\underline{\underline{F}}_n), \ldots / Z \varepsilon J\}$ are strongly equiconvergent (over J) if $P(Z|\underline{\underline{F}}_n) \xrightarrow{L_1} P(Z|\underline{\underline{F}}_\infty)$ uniformly over J. That is, given $\varepsilon > 0$ one can find n_o such that for all $n \geq n_o$ and <u>for all $Z \varepsilon J$</u>,

$$\int |P(Z|\underline{\underline{F}}_n) - P(Z|\underline{\underline{F}}_\infty)| \, dP \leq \varepsilon .$$

The metric d gives an elegant sufficient condition for strong equiconvergence of martingales.

Theorem (Page 554, [7]) Let $J = \{Z / \|Z\|_\infty \leq 1\}$. Let $d(\underline{\underline{F}}_n, \underline{\underline{F}}_\infty) \to 0$. Then the martingales $\{P(Z|\underline{\underline{F}}_1), P(Z|\underline{\underline{F}}_2), \ldots, P(Z|\underline{\underline{F}}_n), \ldots, / Z \varepsilon J\}$ are strongly equiconvergent.

As pointed out by Boylan, it is not true that $\underline{\underline{F}}_n$'s increase to $\underline{\underline{F}}_\infty$ implies $d(\underline{\underline{F}}_n, \underline{\underline{F}}_\infty) \to 0$. Also, in this theorem, J can be taken to be any uniformly integrable subset of $L_1(\Omega, \underline{\underline{A}}, P)$.

We want to assert that when P is completely atomic on on $\underline{\underline{A}}$, this sufficient condition (viz. $d(\underline{\underline{F}}_n, \underline{\underline{F}}_\infty) \to 0$) is satisfied and hence strong equiconvergence of martingales obtains. For this we need a lemma.

Lemma Let $\underline{\underline{S}}$ be compact. Let $\{\underline{\underline{F}}_n\}_{n \geq 1}$ be a sequence from $\underline{\underline{S}}$ increasing to $\underline{\underline{F}}_\infty$. Then $d(\underline{\underline{F}}_n, \underline{\underline{F}}_\infty) \to 0$.

Proof: Since $\underline{\underline{S}}$ is compact, there is a subsequence $\{n_k\}$ and $\underline{\underline{G}} \varepsilon \underline{\underline{S}}$ such that $d(\underline{\underline{F}}_{n_k}, \underline{\underline{G}}) \to 0$. We will first show $\underline{\underline{G}} = \underline{\underline{F}}_\infty$. Let $G \varepsilon \underline{\underline{G}}$. Let $\varepsilon > 0$ be given. Since $d(\underline{\underline{F}}_{n_k}, \underline{\underline{G}}) \to 0$, there exists $F \varepsilon \underline{\underline{F}}_{n_k}$ (for some sufficiently large n_k) such that $P(G \Delta F) < \varepsilon$. So, $\inf_{F \varepsilon \underline{\underline{F}}_\infty} P(G \Delta F) < \varepsilon$. Since ε is arbitrary, we have, $\inf_{F \varepsilon \underline{\underline{F}}_\infty} P(G \Delta F) = 0$. Using the fact that $\underline{\underline{F}}_\infty$ is complete, we conclude $G \varepsilon \underline{\underline{F}}_\infty$. Thus $\underline{\underline{G}} \subset \underline{\underline{F}}_\infty$. To show $\underline{\underline{F}}_\infty \subset \underline{\underline{G}}$, it is enough to show $\bigcup_n \underline{\underline{F}}_n \subset \underline{\underline{G}}$. Let $F \varepsilon \underline{\underline{F}}_n$ for some n. Since $\underline{\underline{F}}_n \uparrow$, $F \varepsilon \underline{\underline{F}}_{n_k}$ for all sufficiently large n_k. Now let $\varepsilon > 0$ be given. Since $d(\underline{\underline{F}}_{n_k}, \underline{\underline{G}}) \rightarrow 0$, there is $G \varepsilon \underline{\underline{G}}$ such that $P(F \Delta G) < \varepsilon$. Therefore $\inf_{G \varepsilon \underline{\underline{G}}} P(F \Delta G) < \varepsilon$. Again, the fact that ε is arbitrary and $\underline{\underline{G}}$ is complete implies $F \varepsilon \underline{\underline{G}}$. Thus $\bigcup_n \underline{\underline{F}}_n$ and consequently $\underline{\underline{F}}_\infty$ are contained in $\underline{\underline{G}}$.

$\therefore d(\underline{\underline{F}}_{n_k}, \underline{\underline{F}}_\infty) \longrightarrow 0$.

Now, let $\varepsilon > 0$ be given. Choose n_{k_o} such that $d(\underline{\underline{F}}_{n_{k_o}}, \underline{\underline{F}}_\infty) < \varepsilon$. Let $n \geq n_{k_o}$. Then

$d(\underline{\underline{F}}_n, \underline{\underline{F}}_\infty) \leq d(\underline{\underline{F}}_{n_{k_o}}, \underline{\underline{F}}_\infty)$ (as $\underline{\underline{F}}_n \uparrow$) $< \varepsilon$.

Thus $d(\underline{\underline{F}}_n, \underline{\underline{F}}_\infty) \rightarrow 0$ and hence the lemma.

Let P be completely atomic on $\underline{\underline{A}}$. By theorem 3.2 $\underline{\underline{S}}$ is compact. So, by the lemma above $d(\underline{\underline{F}}_n, \underline{\underline{F}}_\infty) \longrightarrow 0$. Hence strong equiconvergence of martingales obtains.

4. <u>Separability of ($\underline{\underline{S}}$, d)</u>

We show that $\underline{\underline{S}}$ is separable if and only if P is completely atomic on $\underline{\underline{A}}$.

Consider the two-point space $\{0, 1\}$, equipped with the discrete σ-algebra $\underline{\underline{D}}$ and the probability measure Q which gives mass $\frac{1}{2}$ to each point. Let N be the set of all natural numbers. Let $\Omega_1 = \{0, 1\}^N$, Q^N the product measure on $\underline{\underline{D}}^N$ and $\underline{\underline{C}}$ the Q^N-completion of $\underline{\underline{D}}^N$.

Lemma 4.1 The space $\underline{\underline{S}}(\Omega_1, \underline{\underline{C}}, Q^N)$ is not separable.

Proof : Let d_1 denote the distance in $\underline{\underline{S}}(\Omega_1, \underline{\underline{C}}, Q^N)$. We shall exhibit an uncountable family $\{\underline{\underline{C}}_\alpha\}_{\alpha \in \Gamma}$ of elements of $\underline{\underline{S}}(\Omega_1, \underline{\underline{C}}, Q^N)$ with $d_1(\underline{\underline{C}}_\alpha, \underline{\underline{C}}_{\alpha'}) \geq \frac{1}{2}$, $\alpha \neq \alpha'$ and $\alpha, \alpha' \in \Gamma$.

Let

$\Gamma = \{\alpha \,/\, \alpha$ is a subsequence (finite or infinite) of the sequence $1, 2, \ldots, n, \ldots\}$.

For $m \in N$, let π_m denote the m^{th} coordinate mapping from Ω_1 to $\{0, 1\}$. Set $\underline{\underline{C}}_\alpha$ to be the Q^N-completion of $\sigma\{\pi_m^{-1}(\underline{\underline{D}})/m$ is an element of $\alpha\}$. Let $\alpha \neq \alpha'$. Then either α has an element m which is not in α' or vice versa; say, m belongs to α and does not belong to α'. Since the coordinate mappings are independent under Q^N, it is clear that $\pi_m^{-1}(\underline{\underline{D}})$ is independent of $\underline{\underline{C}}_{\alpha'}$. Let $A_0 = \pi_m^{-1}(\{0\})$. Then for any B in $\underline{\underline{C}}_{\alpha'}$,

$$
\begin{aligned}
Q^N(A_o \,\Delta\, B) &= Q^N(A_o - B) + Q^N(B - A_o) \\
&= Q^N(A_o)\, Q^N(B^c) + Q^N(B).Q^N(A_o^c) \\
&= \tfrac{1}{2}[Q^N(B^c) + Q^N(B)] \quad (\text{since } Q^N(A_o) = \tfrac{1}{2}) \\
&= \tfrac{1}{2}.
\end{aligned}
$$

So, $\sup\limits_{A \in \underline{\underline{C}}_\alpha} \inf\limits_{B \in \underline{\underline{C}}_{\alpha'}} Q^N(A \,\Delta\, B) \geq \frac{1}{2}$. This implies that

$d_1(\underline{\underline{C}}_\alpha, \underline{\underline{C}}_{\alpha'}) \geq \frac{1}{2}$ if $\alpha \neq \alpha'$.

Evidently Γ is an uncountable set.

<u>Lemma 4.2</u> Let P be nonatomic on $\underline{\underline{A}}$. Then $\underline{\underline{S}}$ is not separable.

<u>Proof</u>: Since P is nonatomic on $\underline{\underline{A}}$, we can find a countably generated sub σ-algebra $\underline{\underline{A}}_o$ of $\underline{\underline{A}}$ such that P on $\underline{\underline{A}}_o$ is nonatomic. Enough to show that $\underline{\underline{S}}(\Omega$, completion of $\underline{\underline{A}}_o$, P) is not separable.

Now, by Halmos-Von Neumann theorem [9, theorem C p.173], the measure algebra $\underline{\underline{A}}_o(P)$ is isomorphic to the measure algebra $\underline{\underline{D}}^N(Q^N)$ (as introduced in the beginning of this section). This establishes an isometry between $\underline{\underline{S}}(\Omega$, completion of $\underline{\underline{A}}_o$, P) and $\underline{\underline{S}}(\Omega_1, \underline{\underline{C}}, Q^N)$. (For details see section 9). Since $\underline{\underline{S}}(\Omega_1, \underline{\underline{C}}, Q^N)$ is not separable by lemma 4.1, the desired conclusion follows.

<u>Theorem 4.3</u> $\underline{\underline{S}}$ is separable if and only if P is completely atomic on $\underline{\underline{A}}$.

<u>Proof</u>: Let P be completely atomic on $\underline{\underline{A}}$. Then by theorem 3.2 $(\underline{\underline{S}}, d)$ is compact and hence separable.

Let now there exist $B_0 \varepsilon \underline{\underline{A}}$ such that P is nonatomic on B_0. From lemma 4.2 it follows that $\underline{\underline{S}}(B_0, B_0 \cap \underline{\underline{A}}, P_{B_0})$ is not separable. Define a mapping $\oplus$ from $\underline{\underline{S}}(B_0, B_0 \cap \underline{\underline{A}}, P_{B_0})$ to $\underline{\underline{S}}$ by setting for $\underline{\underline{D}} \varepsilon \underline{\underline{S}}(B_0, B_0 \cap \underline{\underline{A}}, P_{B_0})$

$$\oplus(\underline{\underline{D}}) = \text{P-completion of } \{A/A \varepsilon \underline{\underline{D}} \text{ or } \Omega - A \varepsilon \underline{\underline{D}}\}.$$

It is not difficult to check that $\oplus$ is a homeomorphism between $\underline{\underline{S}}(B_0, B_0 \cap \underline{\underline{A}}, P_{B_0})$ and $\oplus[\underline{\underline{S}}(B_0, B_0 \cap \underline{\underline{A}}, P_{B_0})]$. This implies that a subset of $\underline{\underline{S}}$ is not separable. Hence $\underline{\underline{S}}$ is not separable.

5. <u>Connectedness and total disconnectedness of $(\underline{\underline{S}}, d)$.</u>

<u>Lemma 5.1</u> Let P be such that it has at most one atom. Let A_0 stand for the atom, if there is one ; for the empty set, otherwise. Then, given any $\underline{\underline{B}} \varepsilon \underline{\underline{S}}$ a continuous function f can be defined from the interval $[0, 1 - P(A_0)]$ to $\underline{\underline{S}}$ satisfying

(i) $f(0) = \underline{\underline{B}}$ and

(ii) $f(1 - P(A_0)) = \underline{\underline{A}}$.

<u>Proof</u>: From the hypothesis of the lemma we have that P is nonatomic on A_0^c. Then, it is well-known that we can find a collection $\{B_t\}_{t \in [0,\, 1 - P(A_0)]}$ of measurable sets satisfying

(α) $\quad B_{t_1} \subset B_{t_2}$ if $t_1 < t_2$,

(β) $\quad P(B_t) = t$ and

(γ) $\quad B_{1-P(A_0)} = A_0^c$

where t, t_1, t_2 are from $[0,\, 1 - P(A_0)]$. Define complete sub σ-algebras $\underline{\underline{B}}_t$ of $\underline{\underline{A}}$ by

$$\underline{\underline{B}}_t = \{A \in \underline{\underline{A}} \,/\, P(A \cap B_t^c) = P(B_t^c) \text{ or } 0\}, \quad t \in [0,\, 1 - P(A_0)].$$

If $t_1 < t_2$, we have $B_{t_1} \subset B_{t_2}$ and so $\underline{\underline{B}}_{t_1} \subset \underline{\underline{B}}_{t_2}$. Now, $B_{t_1} \cup B_{t_2}^c$ is a conditional atom for $(\underline{\underline{B}}_{t_1}, \underline{\underline{B}}_{t_2})$. To see this, let $A \in \underline{\underline{B}}_{t_2}$. If $P(A \cap B_{t_2}^c) = 0$, consider the $\underline{\underline{B}}_{t_1}$ set A_1 defined by $A_1 = A \cap B_{t_1}$. Then

$P[A \cap (B_{t_1} \cup B_{t_2}^c) \,\Delta\, A_1 \cap (B_{t_1} \cup B_{t_2}^c)] = 0$. If $P(A \cap B_{t_2}^c) = P(B_{t_2}^c)$, consider the $\underline{\underline{B}}_{t_1}$ set A_1 defined by $A_1 = A \cup (B_{t_1}^c - B_{t_2}^c)$. Then

$P[A \cap (B_{t_1} \cup B_{t_2}^c) \,\Delta\, A_1 \cap (B_{t_1} \cup B_{t_2}^c)] = 0$.

Now define complete sub σ-algebras $\underline{\underline{C}}_t$ of $\underline{\underline{A}}$ by setting

$\underline{\underline{C}}_t$ = the completion of $\sigma\{\underline{\underline{B}} \cup \underline{\underline{B}}_t\}$, $t \varepsilon [0, 1 - P(A_0)]$.

For $t_1 < t_2$, the set $B_{t_1} \cup B^c_{t_2}$ is a conditional atom for $(\underline{\underline{C}}_{t_1}, \underline{\underline{C}}_{t_2})$. To verify this, in view of an observation made in section 1, we have to only check that for any A in $\underline{\underline{B}}_{t_2}$ or $\underline{\underline{B}}$ there exists A_1 in $\underline{\underline{C}}_{t_1}$ such that $P[A \cap (B_{t_1} \cup B^c_{t_2}) \Delta A_1 \cap (B_{t_1} \cup B^c_{t_2})] = 0$. If $A \varepsilon \underline{\underline{B}}$ take $A_1 = A$. If $A \varepsilon \underline{\underline{B}}_{t_2}$ take A_1 to be as defined in the earlier part of the proof. Therefore, if $t_1 < t_2$ we have that

$$d(\underline{\underline{C}}_{t_1}, \underline{\underline{C}}_{t_2}) \leq P[(B_{t_1} \cup B^c_{t_2})^c] = P(B_{t_2} - B_{t_1}) = t_2 - t_1.$$

Hence the function f defined on $[0, 1 - P(A_0)]$ by $f(t) = \underline{\underline{C}}_t$ is continuous.

$$\underline{\underline{B}}_0 = \{A \varepsilon \underline{\underline{A}} \,/\, P(A \cap B^c_0) = P(B^c_0) \text{ or } 0\}$$

$$= \{A \varepsilon \underline{\underline{A}} \,/\, P(A) = 1 \text{ or } 0\} \quad \text{and}$$

$$\underline{\underline{B}}_{1-P(A_0)} = \{A \varepsilon \underline{\underline{A}} \,/\, P(A \cap B^c_{1-P(A_0)}) = P(B^c_{1-P(A_0)}) \text{ or } 0\}$$

$$= \{A \varepsilon \underline{\underline{A}} \,/\, P(A \cap A_0) = P(A_0) \text{ or } 0\}$$

$= \underline{\underline{A}}$, since A_0 is <u>the</u> <u>only</u> <u>atom</u>.

So, $f(0) = \underline{\underline{B}}$ and $f(1 - P(A_0)) = \underline{\underline{A}}$.

<u>Theorem 5.2</u> The following are equivalent

(i) P has at most one atom.

(ii) $(\underline{\underline{S}}, d)$ is arcwise connected.

(iii) $(\underline{\underline{S}}, d)$ is connected.

<u>Proof</u>: Lemma 5.1 gives the proof of (i) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (iii) is trivial.

Let P have at least two atoms, say A_1 and A_2. Consider the probability space $(A_1 \cup A_2, (A_1 \cup A_2) \cap \underline{A}, P_{A_1 \cup A_2})$.

Clearly, $\underline{\underline{S}}_1 = \underline{\underline{S}}(A_1 \cup A_2, (A_1 \cup A_2) \cap \underline{A}, P_{A_1 \cup A_2})$ contains exactly two elements. Define the map ϕ from $\underline{\underline{S}}$ to $\underline{\underline{S}}_1$ by setting, for $\underline{B} \in \underline{\underline{S}}$, $\phi(\underline{B}) = (A_1 \cup A_2) \cap \underline{B}$. ϕ is continuous and onto $\underline{\underline{S}}_1$. Since $\underline{\underline{S}}_1$ is not connected, $\underline{\underline{S}}$ is not connected. Thus (iii) $\Rightarrow$ (i). Hence the theorem.

Regarding total disconnectedness we have

<u>Theorem 5.3</u> $(\underline{\underline{S}}, d)$ is totally disconnected if and only if P is completely atomic.

<u>Proof</u>: P is completely atomic implies that X is compact and totally disconnected ([2], theorem 6.1). This implies that 2^X is totally disconnected (by the theorem on p.47 of [12] and proposition 4.13.2 of [17]). So, $\widehat{\underline{\underline{S}}}$ and hence $\underline{\underline{S}}$ is totally disconnected.

Now, let P be nonatomic on $B_o \in \underline{\underline{A}}$. We shall show that there is a connected subset of $(\underline{\underline{S}}, d)$ containing at least two elements. This would then show that $(\underline{\underline{S}}, d)$ cannot be totally disconnected.

Consider the probability space $(B_o, B_o \cap \underline{\underline{A}}, P_{B_o})$. By our assumption P_{B_o} is a nonatomic probability. Let $\underline{\underline{S}}_1 = \underline{\underline{S}}(B_o, B_o \cap \underline{\underline{A}}, P_{B_o})$. By theorem 5.2, $\underline{\underline{S}}_1$ is a connected metric space. Define a function φ from $\underline{\underline{S}}_1$ to $\underline{\underline{S}}$ by setting, for every $\underline{\underline{D}} \in \underline{\underline{S}}_1$,

$$\varphi(\underline{\underline{D}}) = P\text{-completion of } \{A \in \underline{\underline{A}} \,/\, A \in \underline{\underline{D}} \text{ or } \Omega - A \in \underline{\underline{D}}\}$$

The mapping φ can be verified to be a continuous one. Therefore $\varphi(\underline{\underline{S}}_1)$ is a connected subset of $\underline{\underline{S}}$, being the continuous image of a connected set.

$$\varphi(\{\emptyset, B_o\}) = P\text{-completion of } \{\emptyset, B_o, B_o^c, \Omega\} \text{ and}$$

$$\varphi(B_o \cap \underline{\underline{A}}) = \{A \in \underline{\underline{A}} \,/\, P(A \cap B_o) = P(A) \text{ or } P(A \cap B_o^c) = P(B_o^c)\}.$$

The two σ-algebras above are obviously different. Thus $\varphi(\underline{\underline{S}}_1)$ contains at least two points. Hence the theorem.

In the remainder of this section we study the components of $(\underline{\underline{S}}, d)$.

Let A_o be the union of all atoms. (A_o may be empty). Consider $(A_o, A_o \cap \mathbf{A}, P_{A_o})$. (In case A_o is empty take $(A_o, A_o \cap \mathbf{A}, P_{A_o})$ to be a set with one element). Define a mapping ϕ from $\mathbf{S}$ to $\mathbf{S}_1 = \mathbf{S}(A_o, A_o \cap \mathbf{A}, P_{A_o})$ by setting, for $\mathbf{B} \varepsilon \mathbf{S}$, $\phi(\mathbf{B}) = A_o \cap \mathbf{B}$.

<u>Theorem 5.4</u> The components of $\mathbf{S}$ are precisely the collection $\{ \phi^{-1}(\mathbf{D}) : \mathbf{D} \varepsilon \mathbf{S}_1 \}$, where ϕ is the mapping as defined above.

<u>Proof</u>. We merely sketch the proof, omitting the details to the reader.

Since ϕ is continuous and $\mathbf{S}_1$ is totally disconnected, the image under ϕ of any component of $\mathbf{S}$ is a singleton. So, we have to show only that for every $\mathbf{D} \varepsilon \mathbf{S}_1$, the set $\phi^{-1}(\mathbf{D})$ is connected in $\mathbf{S}$. Fix $\mathbf{D}_o \varepsilon \mathbf{S}_1$. We shall show that $\phi^{-1}(\mathbf{D}_o)$ is arcwise connected. Let $\mathbf{F}_1 = \{A \varepsilon \mathbf{A} / A \cap A_o \varepsilon \mathbf{D}_o\}$ and let $\mathbf{F}_o$ be an arbitrary element of $\phi^{-1}(\mathbf{D}_o)$. Consider $\{\mathbf{B}_t\}_{t \varepsilon [0, 1-P(A_o)]}$ as defined in lemma 5.1 and set $\mathbf{C}_t$ = completion of $\sigma\{\mathbf{B}_t \cup \mathbf{F}_o\}$. It can be verified that for each t, $\mathbf{C}_t \varepsilon \phi^{-1}(\mathbf{D}_o)$. Define f from $[0, 1-P(A_o)]$ to $\phi^{-1}(\mathbf{D}_o)$ by $f(t) = \mathbf{C}_t$. Then,

f is a continuous map with $f(0) = \underline{\underline{F}}_o$ and $f(1-P(A_o)) = \underline{\underline{F}}_1$. Since $\underline{\underline{F}}_o$ is arbitrary, the result follows.

6. Perfectness of $(\underline{\underline{S}}, d)$

We show that $(\underline{\underline{S}}, d)$ is perfect if and only if the range of P, i.e. the set $\{P(A)/A \in \underline{\underline{A}}\}$, is an infinite set.

Lemma 6.1 Let $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ be elements of $\underline{\underline{S}}$ such that for some $A \in \underline{\underline{A}}$, $A^c \,(\bar{\,})\, \underline{\underline{B}}_1 = A^c \,(\bar{\,})\, \underline{\underline{B}}_2$. Then $d(\underline{\underline{B}}_1, \underline{\underline{B}}_2) \leq 2P(A)$.

Proof: The proof of this is similar to the proof of a result in the introduction.

Lemma 6.2 Let the range of P be an infinite set. Then given $\varepsilon > 0$, one can find $A_o \in \underline{\underline{A}}$ with $0 < P(A_o) < \frac{\varepsilon}{2}$.

Proof: Suppose there is $A \in \underline{\underline{A}}$ such that P restricted to A is nonatomic; then the proof is clear. Let now P be completely atomic. Since the range of P is an infinite set we can find $\{A_n\}_{n \geq 1}$ of atoms which are pairwise disjoint. The fact that $\sum_{n=1}^{\infty} P(A_n) = 1$ gives the desired conclusion.

Theorem 6.3 $(\underline{\underline{S}}, d)$ is perfect if and only if the range of P is an infinite set.

Proof: Let the range of P be a finite set. Then $\underline{\underline{S}}$ contains only finitely many elements and hence is not perfect.

Let the range of P be an infinite set. Let $\underline{\underline{B}}_o \varepsilon \underline{\underline{S}}$ and $\varepsilon > 0$ be given. We have to exhibit a $\underline{\underline{B}}_1 \varepsilon \underline{\underline{S}}$ such that $0 < d(\underline{\underline{B}}_o, \underline{\underline{B}}_1) < \varepsilon$.

Case (i) P on $\underline{\underline{B}}_o$ has a nonatomic part. Then choose $A_o \varepsilon \underline{\underline{B}}_o$ such that $0 < P(A_o) < \frac{\varepsilon}{2}$ and no $\underline{\underline{B}}_o$ measurable subset of A_o is atom for P. Let $\underline{\underline{B}}_1$ be the completion of $\sigma\{A_o, A_o^c \cap \underline{\underline{B}}_o\}$. Because of nonatomicity of P on A_o we have $0 < d(\underline{\underline{B}}_o, \underline{\underline{B}}_1)$ and by lemma 6.1, $d(\underline{\underline{B}}_o, \underline{\underline{B}}_1) < \varepsilon$.

Case (ii) P on $\underline{\underline{B}}_o$ is completely atomic and has only finitely many atoms.

Let $A_1, A_2, \ldots, A_m$ be all the atoms. By lemma 6.2, we can get $A_o \varepsilon \underline{A}$ with $0 < P(A_o) < \min\{P(A_1), P(A_2), \ldots, P(A_m), \frac{\varepsilon}{2}\}$. Take the required $\underline{\underline{B}}_1$ to be the completion of $\sigma\{A_o, A_o^c \cap \underline{\underline{B}}_o\}$. By the choice of A_o, $d(\underline{\underline{B}}_o, \underline{\underline{B}}_1) > 0$ and by lemma 6.1, $d(\underline{\underline{B}}_o, \underline{\underline{B}}_1) < \varepsilon$.

Case (iii) P on $\underline{\underline{B}}_o$ is completely atomic and has infinitely many atoms.

Choose two atoms A_1 and A_2 with $0 < P(A_1)+P(A_2) < \frac{\varepsilon}{2}$. (This can be done since there are infinitely many atoms). Let $A_o = A_1 \cup A_2$ and define $\underline{\underline{B}}_1$ by $\underline{\underline{B}}_1$ = Completion of $\sigma\{A_o, A_o^c \cap \underline{\underline{B}}_o\}$. Again by the choice of A_o, $d(\underline{\underline{B}}_o, \underline{\underline{B}}_1) > 0$ and by lemma 6.1 $d(\underline{\underline{B}}_o, \underline{\underline{B}}_1) < \varepsilon$.

This completes the proof of 'if' part. Hence the theorem.

7. Dimension of ($\underline{\underline{S}}$, d)

For notions in dimension theory we refer to Nagata [18]. By dimension we mean covering dimension in the sense of Nagata [See 18, p.9]. We need the following theorem from [2].

Theorem 7.1 Let P be nonatomic on $\underline{\underline{A}}$. Then dimension of $\underline{\underline{A}}$(P) is infinity.

For a proof see theorem 9.2 of [2].

Regarding the dimension of ($\underline{\underline{S}}$, d) we have the following theorem.

Theorem 7.2 The dimension of ($\underline{\underline{S}}$, d) is either zero or infinity. If P is completely atomic on $\underline{\underline{A}}$ the dimension is zero ; otherwise it is infinity.

Proof. Let P be completely atomic on $\underline{\underline{A}}$. Then, by theorem 5.3, ($\underline{\underline{S}}$, d) is totally disconnected ; hence the dimension of ($\underline{\underline{S}}$, d) is zero.

Let now there exist $B_o \in \underline{\underline{A}}$, such that P is nonatomic on B_o. Without loss in generality we can take that $0 < P(B_o) < \frac{1}{2}$. Consider the measure algebra $(B_o \bar{\cap} \underline{\underline{A}})(P_{B_o})$. By theorem 7.1 the dimension of $(B_o \bar{\cap} \underline{\underline{A}})(P_{b_o})$ is infinity. We shall establish a homeomorphism ψ from $(B_o \cap \underline{A})(P_{B_o})$ into $\underline{S}$ such that $\psi[(B_o \cap \underline{A})(P_{B_o})]$ is a closed subset of $\underline{S}$. Then it would follow that the dimension of $\underline{S}$, being greater than or equal to that of $\psi[(B_o \cap \underline{A})(P_{B_o})]$, is infinity.

Define the map ψ from $(B_o \cap \underline{A})(P_{B_o})$ to $\underline{S}$ by setting, for $[B_1] \varepsilon (B_o \cap \underline{A})(P_{B_o})$

$\psi([B_1])$ = the P-completion of $\{ \emptyset, B_1, \Omega - B_1, \Omega \}$.

The fact that ψ is well defined is easily verified. For two elements B_1 and B_2 of $\underline{A} \cap B_o$ we have (α) $P(B_1) < \frac{1}{2}$ and $P(B_2) < \frac{1}{2}$ and (β) $P[B_1 \Delta (\Omega - B_2)] > \frac{1}{2}$. (To prove (α) and (β) it is enough to observe that $P(B_o) < \frac{1}{2}$ by choice and the sets B_1 and B_2 are contained in B_o). So,

$$\min[P(B_1), P(\Omega - B_1), P(B_1 \Delta B_2), P(B_1 \Delta (\Omega - B_2))]$$
$$= \min[P(B_1), P(B_1 \Delta B_2)].$$

Thus $d(\psi(B_1), \psi(B_2)) = \min[P(B_1), P(B_1 \Delta B_2)] +$
$\min[P(B_2), P(B_1 \Delta B_2)]$.

Hence $P(B_1 \Delta B_2) \leq d(\psi(B_1), \psi(B_2)) \leq 2P(B_1 \Delta B_2)$ —(**)

Evidently, ψ is a 1-1 map. Using (**) we conclude that ψ is a homeomorphism between $(B_o \cap \underline{A})(P_{B_o})$ and $\psi[(B_o \cap \underline{A})(P_{B_o})]$ and that $\psi[(B_o \cap \underline{A})(P_{B_o})]$ is a complete subset of $\underline{\underline{S}}$. Since $\underline{\underline{S}}$ is complete, we have that $(+)[(B_o \cap \underline{A})(P_{B_o})]$ is closed.

In view of the observation made in the earlier part of the proof, we have that if P is not completely atomic the dimension of $(\underline{\underline{S}}, d)$ is infinity.

Hence the theorem.

8. <u>Some additional results on $(\underline{\underline{S}}, d)$</u>

In this section we show that the following two subsets of $\underline{\underline{S}}$, namely

$\underline{\underline{S}}_{c.a.} = \{ \underline{\underline{B}} \in \underline{\underline{S}} / P$ is completely atomic on $\underline{\underline{B}} \}$ and

$\underline{\underline{S}}_{s} = \{ \underline{\underline{B}} \in \underline{\underline{S}} / \underline{\underline{B}}$ is the completion of a countably generated sub σ-algebra $\}$

are closed.

Let P be completely atomic on $\underline{\underline{B}}_o$ with infinitely many atoms. Let $\{A_n\}_{n \geq 1}$ be all the atoms. Let $\varepsilon > 0$ be given. Choose n_o such that $\sum_{m > n_o} P(A_m) < \varepsilon$. Define $\underline{\underline{B}}_1$ = the completion of $\sigma\{A_1, A_2, \ldots, A_{n_o}\}$. It is clear that $\bigcup_{m=1}^{n_o} A_m$ is a conditional atom for $(\underline{\underline{B}}_1, \underline{\underline{B}}_o)$ and so $d(\underline{\underline{B}}_1, \underline{\underline{B}}_o) < \sum_{m > n_o} P(A_m) < \varepsilon$. This observation gives us that

$\underline{\underline{S}}'_{c\ a} = \{\underline{\underline{B}} / \underline{\underline{B}} \in \underline{\underline{S}}_{c.a.}$ and P on $\underline{\underline{B}}$ has only finitely many atoms$\}$

is dense in $\underline{\underline{S}}_{c.a.}$.

<u>Lemma 8.1</u> Let $\underline{\underline{B}}_o \varepsilon \underline{\underline{S}}$. Let $\{\underline{\underline{B}}_n\}_{n \geq 1} \subset \underline{\underline{S}}'_{c.a.}$ be such that for each n (i) $\underline{\underline{B}}_n \subset \underline{\underline{B}}_o$ and (ii) $d(\underline{\underline{B}}_n, \underline{\underline{B}}_o) \to 0$ as $n \to \infty$. Then $\underline{\underline{B}}_o \varepsilon \underline{\underline{S}}_{c.a.}$.

<u>Proof</u>: Suppose there exists $B_o \varepsilon \underline{\underline{B}}_o$ such that $P(B_o) > 0$ and no $\underline{\underline{B}}_o$ measurable subset of B_o is an atom.

Fix $n(\geq 1)$ and consider $\underline{\underline{B}}_n$. Let $A_1, A_2, \ldots, A_k$ be all the atoms of $\underline{\underline{B}}_n$. For each $j(1 \leq j \leq k)$, choose $B_j \varepsilon \underline{\underline{B}}$ such that $B_j = \emptyset$ if $P(A_j \cap B_o) = 0$; $B_j \subset A_j \cap B_o$ and $P(B_j) = \frac{P(A_j \cap B_o)}{2}$ if $P(A_j \cap B_o) > 0$. (Such a choice is possible since $\underline{\underline{B}}_n \subset \underline{\underline{B}}_o$). Let $B_n = \bigcup_{j=1}^{k} B_j$.

Clearly for every $D \varepsilon \underline{\underline{B}}_n$, $P(B_n \Delta D) \geq \frac{P(B_o)}{2}$. So, $d(\underline{\underline{B}}_n, \underline{\underline{B}}_o) \geq \frac{P(B_o)}{2}$. Since n was arbitrary, this would imply that $d(\underline{\underline{B}}_n, \underline{\underline{B}}_o) \geq \frac{P(B_o)}{2}$ for all n. A contradiction, since $d(\underline{\underline{B}}_n, \underline{\underline{B}}_o) \to 0$ as $n \to \infty$, by hypothesis. Hence the lemma.

<u>Theorem 8.2</u> $\underline{\underline{S}}_{c.a.}$ is a closed subset of $\underline{\underline{S}}$.

<u>Proof</u>: Let $\{\underline{\underline{C}}_n\}_{n \geq 1} \subset \underline{\underline{S}}_{c.a.}$ be such that for some $\underline{\underline{B}}_o \varepsilon \underline{\underline{S}}$, $d(\underline{\underline{C}}_n, \underline{\underline{B}}_o) \to 0$ as $n \to \infty$. We have to show that $\underline{\underline{B}}_o \varepsilon \underline{\underline{S}}_{c.a.}$. Since $\underline{\underline{S}}'_{c.a.}$ is dense in $\underline{\underline{S}}_{c.a.}$ we can

assume without loss in generality that $\{\underline{\underline{C}}_n\}_{n \geq 1} \subset \underline{\underline{S}}'_{c.a.}$.

Fix $n (\geq 1)$. Let $A_1, A_2, \ldots, A_k$ be all the atoms of $\underline{\underline{C}}_n$. For each subset J of $\{1, 2, \ldots, k\}$ choose a set B^n_J of $\underline{\underline{B}}_o$ such that $P(B^n_J \Delta \bigcup_{j \varepsilon J} A_j) < 2d(\underline{\underline{C}}_n, \underline{\underline{B}}_o)$. Let $\underline{\underline{B}}_n$ = P-completion of $\sigma\{B^n_J : J$ is a subset of $\{1, 2, \ldots, k\}\}$. Now, for any B in $\underline{\underline{B}}_o$, there exists some subset J of $\{1, 2, \ldots, k\}$ such that $P(B \Delta \bigcup_{j \varepsilon J} A_j) < 2d(\underline{\underline{C}}_n, \underline{\underline{B}}_o)$. So, for any B in $\underline{\underline{B}}_o$, there exists B^n_J in $\underline{\underline{B}}_n$ such that $P(B \Delta B^n_J) < 4d(\underline{\underline{C}}_n, \underline{\underline{B}}_o)$. Hence $d(\underline{\underline{B}}_n, \underline{\underline{B}}_o) < 4d(\underline{\underline{C}}_n, \underline{\underline{B}}_o)$. Also, we have $\underline{\underline{B}}_n \subset \underline{\underline{B}}_o$. Since n was arbitrary, we get $\{\underline{\underline{B}}_n\}_{n \geq 1} \subset \underline{\underline{S}}'_{c.a.}$ and $\underline{\underline{B}}_n \subset \underline{\underline{B}}_o$ for every n. From Lemma 8.1 the result now follows.

<u>Theorem 8.3</u> $\underline{\underline{S}}_s$ is a closed set.

<u>Proof</u>. (α) Let $\underline{\underline{D}} \varepsilon \underline{\underline{S}}_s$. Then the metric space $\underline{\underline{D}}(P)$ is separable. Let $\underline{\underline{C}} \subset \underline{\underline{D}}$. Then the metric space $\underline{\underline{C}}(P)$, being a subset of a separable metric space, is separable. So $\underline{\underline{C}} \varepsilon \underline{\underline{S}}_s$.

(β) Let $\{\underline{\underline{C}}_n\} \subset \underline{\underline{S}}_s$. Denote by $\bigvee_n \underline{\underline{C}}_n$ the completion of $\sigma\{\bigcup_n \underline{\underline{C}}_n\}$. It is easily verified that $\bigvee_n \underline{\underline{C}}_n \varepsilon \underline{\underline{S}}_s$.

(γ) Let $\{\underline{\underline{C}}_n\} \subset \underline{\underline{S}}_s$ be such that for some $\underline{\underline{B}}_o \in \underline{\underline{S}}$ $d(\underline{\underline{C}}_n, \underline{\underline{B}}_o) \to 0$ as $n \to \infty$. Let $B_o \in \underline{\underline{B}}_o$. Then

$$\inf_{D \in \vee \underline{\underline{C}}_n} P(B_o \Delta D) \leq \inf_{C_n \in \underline{\underline{C}}_n} P(B_o \Delta C_n) \text{ for every } n. \text{ So,}$$

$$\inf_{D \in \vee \underline{\underline{C}}_n} P(B_o \Delta D) = 0.$$ This gives that $\underline{\underline{B}}_o \subset \bigvee_n \underline{\underline{C}}_n$.

Now from (α), (β) and (γ) the result follows.

9. Isomorphism Problem

Let us start with an observation. Consider two metric spaces Y_1 and Y_2 equipped with bounded metrics. Let T be an isometry from Y_1 onto Y_2. Let the mapping $\widetilde{\varphi}$ be defined from 2^{Y_1} to 2^{Y_2} by

$\widetilde{\varphi}(C) = \{T(y_1) : y_1 \in Y_1\}$, $C \in 2^{Y_1}$. Then, it is easily seen that $\widetilde{\varphi}$ is onto 2^{Y_2} and is an isometry.

Let $(\Omega_1, \underline{\underline{A}}_1, P_1)$ and $(\Omega_2, \underline{\underline{A}}_2, P_2)$ be two probability spaces. Let T be an isomorphism between the measure algebras $\underline{\underline{A}}_1(P_1)$ and $\underline{\underline{A}}_2(P_2)$ (see page 167 of [9]). i.e. T is a one to one transformation from $\underline{\underline{A}}_1(P_1)$ onto $\underline{\underline{A}}_2(P_2)$ such that

$T([B] - [C]) = T([B]) - T([C])$ and

$T(\bigcup_{n=1}^{\infty} [A_{n1}]) = \bigcup_{n=1}^{\infty} T([A_{n1}])$ whenever B, C and A_{n1}'s are elements of $\underline{\underline{A}}_1$; moreover T 'preserves measure' in the sense whenever $A_{12} \in T[A_{11}]$ we have $P_1(A_{11}) = P_2(A_{12})$. Evidently, T is an isometry from $\underline{\underline{A}}_1(P_1)$ onto $\underline{\underline{A}}_2(P_2)$. Use the observation in the first paragraph of this section to define an isometry $\widetilde{\oplus}$ from $2^{\underline{\underline{A}}_1(P_1)}$ onto $2^{\underline{\underline{A}}_2(P_2)}$. Restricting $\widetilde{\oplus}$ to $\widetilde{\underline{\underline{S}}}(\Omega_1, \underline{\underline{A}}_1, P_1)$ we find it is mapped onto $\widetilde{\underline{\underline{S}}}(\Omega_2, \underline{\underline{A}}_2, P_2)$. Using the natural mapping between $\underline{\underline{S}}$ and $\widetilde{\underline{\underline{S}}}$ mentioned in the introduction (page 35), we can look upon $\widetilde{\oplus}$ as an isometry from $\underline{\underline{S}}(\Omega_1, \underline{\underline{A}}_1, P_1)$ onto $\underline{\underline{S}}(\Omega_2, \underline{\underline{A}}_2, P_2)$ and we do so. It is routine to verify that $\widetilde{\oplus}$ is a complete lattice isomorphism as well.

i.e. for $\{\underline{\underline{B}}_\alpha\} \subset \underline{\underline{S}}(\Omega_1, \underline{\underline{A}}_1, P_1)$,

$$\widetilde{\oplus}(\bigvee_\alpha \underline{\underline{B}}_\alpha) = \bigvee_\alpha \widetilde{\oplus}(\underline{\underline{B}}_\alpha) \quad \text{and} \quad \widetilde{\oplus}(\bigwedge_\alpha \underline{\underline{B}}_\alpha) = \bigwedge_\alpha \widetilde{\oplus}(\underline{\underline{B}}_\alpha).$$

(Here, 'V' of a family of sub σ-algebras denotes the smallest completed sub σ-algebra containing them and '$\wedge$' of a family denotes their intersection). We call this mapping $\widetilde{\oplus}$ as being induced by T. Now, we ask ''Is every isometry $\oplus$ from $\underline{\underline{S}}(\Omega_1, \underline{\underline{A}}_1, P_1)$ onto $\underline{\underline{S}}(\Omega_2, \underline{\underline{A}}_2, P_2)$ which preserves lattice operations induced by an isomorphism T between the measure algebras $\underline{\underline{A}}_1(P_1)$ and $\underline{\underline{A}}_2(P_2)$?''. The remainder of this section is intended to provide an affirmative answer to this question.

Since the proof will be a long one we first give an outline of it. Let Φ be an isometry from $\mathbf{S}(\Omega_1, \mathbf{A}_1, P_1)$ onto $\mathbf{S}(\Omega_2, \mathbf{A}_2, P_2)$ preserving lattice operations. (*In the rest of this section Φ will refer to this fixed mapping*). To start with, we will show how to associate with every $[A_1] \varepsilon \mathbf{A}_1(P_1)$, *satisfying* $P_1(A_1) < \frac{1}{2}$ an element $[A_2] \varepsilon \mathbf{A}_2(P_2)$ satisfying $P_2(A_2) = P_1(A_1)$. We will check such an association is well behaved. Then we will get a finite partition of Ω_1 by $\mathbf{A}_1$ -sets, consider the association on each element of the partition and piece them together to get an isomorphism T between $\mathbf{A}_1(P_1)$ and $\mathbf{A}_2(P_2)$. Finally we will check that T induces Φ.

Let $\{A_\alpha, \alpha \varepsilon \Gamma\}$ be a collection of measurable sets of a probability space $(\Omega, \mathbf{A}, P)$. In this section, for ease in presentation, we will let $\sigma\{A_\alpha, \alpha \varepsilon \Gamma\}$ stand for *the P-completion of the σ-algebra generated by $\{A_\alpha, \alpha \varepsilon \Gamma\}$*.

Before proceeding with the proof two remarks are in order.

In any $\mathbf{S}(\Omega, \mathbf{A}, P)$ the distance between any $\mathbf{B} \varepsilon \mathbf{S}$ and $\sigma\{\emptyset, \Omega\}$ is attained; that is, there is $B_o \varepsilon \mathbf{B}$ with $P(B_o) = d(\mathbf{B}, \sigma\{\emptyset, \Omega\})$. (This follows because (*)
$d(\mathbf{B}, \sigma\{\emptyset, \Omega\}) = \sup\{P(B) / B \varepsilon \mathbf{B} \text{ and } P(B) \leq \frac{1}{2}\}$ and $\{P(B) / B \varepsilon \mathbf{B} \text{ and } P(B) \leq \frac{1}{2}\}$ is a closed set).

Also if $\mathbf{B} = \sigma\{A\}$ with $P(A) < \frac{1}{2}$ then

$$d(\mathbf{B}, \sigma\{\phi, \Omega\}) = P(A). \tag{**}$$

In the sequel $(\mathbf{S}_i, d_i)$ (or simply $\mathbf{S}_i$), $i = 1, 2$ will stand respectively for $S(\Omega_i, \mathbf{A}_i, P_i)$, $i = 1, 2$. $A_{01}, A_{11}, A_{21}, \dots, A_{n1}, \dots$ will be sets from $\mathbf{A}_1$ and $A_{02}, A_{12}, A_{22}, \dots, A_{n2}, \dots$ will be sets from $\mathbf{A}_2$.

<u>Lemma 9.1</u> Let $\{\mathbf{B}_\alpha\} \subset \mathbf{S}_1$. Then

$$\Phi(\bigvee_\alpha \mathbf{B}_\alpha) = \bigvee_\alpha \Phi(\mathbf{B}_\alpha) \text{ and } \Phi(\bigwedge_\alpha \mathbf{B}_\alpha) = \bigwedge_\alpha \Phi(\mathbf{B}_\alpha).$$

<u>Proof:</u> Since Φ preserves lattice operations and since $\mathbf{B}_\alpha \subset \bigvee_\alpha \mathbf{B}_\alpha$, we have $\Phi(\mathbf{B}_\alpha) \subset \Phi(\bigvee_\alpha \mathbf{B}_\alpha)$ for each α.

So, $\bigvee_\alpha \Phi(\mathbf{B}_\alpha) \subset \Phi(\bigvee_\alpha \mathbf{B}_\alpha)$.

Let $\{\mathbf{C}_\alpha\} \subset \mathbf{S}_2$. Working with Φ^{-1} instead of Φ we have $\bigvee_\alpha \Phi^{-1}(\mathbf{C}_\alpha) \subset \Phi^{-1}(\bigvee_\alpha \mathbf{C}_\alpha)$. Take $\mathbf{C}_\alpha = \Phi(\mathbf{B}_\alpha)$.

Then $\bigvee_\alpha \Phi^{-1}(\Phi(\mathbf{B}_\alpha)) \subset \Phi^{-1}(\bigvee_\alpha \Phi(\mathbf{B}_\alpha))$

i.e. $\bigvee_\alpha \mathbf{B}_\alpha \subset \Phi^{-1}(\bigvee_\alpha \Phi(\mathbf{B}_\alpha))$.

$\Rightarrow \Phi(\bigvee_\alpha \mathbf{B}_\alpha) \subset \bigvee_\alpha \Phi(\mathbf{B}_\alpha)$.

$\therefore \Phi(\bigvee_\alpha \mathbf{B}_\alpha) = \bigvee_\alpha \Phi(\mathbf{B}_\alpha)$.

The other part is proved similarly. Hence the lemma.

<u>Lemma 9.2</u> (i) $\bigoplus(\sigma\{\Omega\}) = \sigma\{\Omega_2\}$.

(ii) For any $A_{11} \in \underline{A}_1$ with $P_1(A_{11}) < \frac{1}{2}$ there is $A_{12} \in \underline{A}_2$ with $P_1(A_{11}) = P_2(A_{12})$ such that $\bigoplus(\sigma\{A_{11}\}) = \sigma\{A_{12}\}$.

<u>Proof</u> : Observe that $\bigoplus^{-1}(\sigma\{\Omega_2\}) \supseteq \sigma\{\Omega_1\}$. So, $\bigoplus(\sigma\{\Omega_1\}) \subseteq \sigma\{\Omega_2\}$ (as $\bigoplus$ preserves lattice operations) ; hence $\bigoplus(\sigma\{\Omega_1\}) = \sigma\{\Omega_2\}$. This proves (i).

Let $A_{11} \in \underline{A}_1$ with $P_1(A_{11}) < \frac{1}{2}$. Then $d_1(\sigma\{\Omega_1\}, \sigma\{A_{11}\}) = P_1(A_{11})$ by (**). So, $d_2(\bigoplus(\sigma\{\Omega_1\}), \bigoplus(\sigma\{A_{11}\})) = P_1(A_{11})$ (as $\bigoplus$ preserves metric).

i.e. $P_1(A_{11}) = d_2(\sigma\{\Omega_2\}, \bigoplus(\sigma\{A_{11}\}))$. Using (*) we get $A_{12} \in \bigoplus(\sigma\{A_{11}\})$ with $P_2(A_{12}) = d_2(\sigma\{\Omega_2\}, \bigoplus(\sigma\{A_{11}\}))$. Then we have $P_2(A_{12}) = P_1(A_{11})$. It remains to prove $\bigoplus(\sigma\{A_{11}\}) = \sigma\{A_{12}\}$. Clearly, $\bigoplus(\sigma\{A_{11}\}) \supseteq \sigma\{A_{12}\}$. If the equality is not true, $\bigoplus(\sigma\{A_{11}\})$ will contain at least two distinct elements of $\underline{S}_2$, both of them different from $\sigma\{\Omega_2\}$ whereas $\sigma\{A_{11}\}$ contains only one such (i.e. itself); this will contradict the lattice preserving nature of $\bigoplus$. Hence (ii) and the lemma.

Let A_{11} and A_{21} be elements of $\underline{\underline{A}}_1$ with $P_1(A_{11} \Delta A_{21}) = 0$. Since $\sigma\{A_{11}\} = \sigma\{A_{21}\}$, it is obvious that the association mentioned in (ii) of lemma 9.2 preserves equivalence classes.

i.e. if for A_{11} we have A_{12} and A_{21} we have A_{22} through the association mentioned in (ii) of lemma 9.2, then $P_2(A_{12} \Delta A_{22}) = 0$. Thus we can unambiguously define a mapping T_1 from $\{[A_{11}]/ A_{11} \varepsilon \underline{\underline{A}}_1, P_1(A_{11}) < \frac{1}{2}\}$ to $\{[A_{12}]/ A_{12} \varepsilon \underline{\underline{A}}_2, P_2(A_{12}) < \frac{1}{2}\}$ using (ii) of lemma 9.2. T_1 is one to one is easy to check (using the fact that $(+)$ preserves the metric). Working with $(+)^{-1}$ in the place of $(+)$ we can conclude that T_1 is <u>onto</u> $\{[A_{12}]/ A_{12} \varepsilon \underline{\underline{A}}_2, P_2(A_{12}) < \frac{1}{2}\}$. We note for future use that T_1 'preserves measure' in the sense if $T_1([A_{11}]) = [A_{12}]$ then $P_1(A_{11}) = P_2(A_{12})$. The next lemma is meant to show that T_1 is well behaved.

<u>Lemma 9.3</u> (i) Let $A_{11} \varepsilon \underline{\underline{A}}_1$ be such that $P_1(A_{11}) < \frac{1}{2}$. Let A_{21} and A_{31} be elements of $\underline{\underline{A}}_1$ satisfying $P_1(A_{21} \cap A_{31}) = 0$, $P_1((A_{21} \cup A_{31}) \Delta A_{11}) = 0$. Let A_{12}, A_{22} and A_{32} be elements of $\underline{\underline{A}}_2$ such that $T_1([A_{j1}]) = [A_{j2}]$, $j = 1,2,3$. Then $P_2(A_{22} \cap A_{32}) = 0$ and

$P_2((A_{22} \bigcup A_{32}) \Delta A_{12}) = 0$

(ii) Let $A_{01} \varepsilon \mathcal{A}_1$ with $P_1(A_{01}) < \frac{1}{2}$. Let

$\{A_{n1}\}_{n \geq 1} \subset \mathcal{A}_1$ satisfying

$P_1(A_{n1} - A_{01}) = 0$ and $P_1(A_{n1} - A_{n+1,1}) = 0, \quad n \geq 1.$

Let A_{02} and $\{A_{n2}\}_{n \geq 1}$ be such that $T_1([A_{n1}]) = [A_{n2}], \; n \geq 0.$

Then, $P_2(A_{n2} - A_{02}) = 0, \quad P_2(A_{n2} - A_{n+1,2}) = 0$ and

$T_1([\bigcup_n A_{n1}]) = [\bigcup_n A_{n2}].$

<u>Proof</u>: (For (i)). $A_{11} \varepsilon \sigma \{A_{21}\} \vee \sigma \{A_{31}\}$.

So, $A_{12} \varepsilon \sigma\{A_{22}\} \vee \sigma \{A_{32}\}$ (recall that $(+)$ preserves lattice operations and

$T_1([A_{21}]) = [A_{22}] \Rightarrow (+)(\sigma\{A_{21}\}) = \sigma\{A_{22}\}$ etc.)

i.e. $A_{12} \varepsilon \sigma\{$the partition $A_{22} A_{32}, A_{22} A_{32}^c, A_{22}^c A_{32}, A_{22}^c A_{32}^c\}$.

Now, $P_2(A_{22} \bigcup A_{32}) \leq P_2(A_{22}) + P_2(A_{32})$

$$= P_1(A_{21}) + P_1(A_{31}) \quad (\because T_1 \text{ 'preserves measure'})$$

$$= P_1(A_{11}) < \frac{1}{2}.$$

$\therefore P_2(A_{22}^c \bigcap A_{32}^c) > \frac{1}{2}$. On the other hand,

$$P_2(A_{12}) = P_1(A_{11}) < \frac{1}{2}. \text{ Hence}$$

$P_2(A_{12} - (A_{22}\, A_{32} \,(\_)\, A_{22}\, A_{32}^c \,(\_)\, A_{22}^c\, A_{32})) = 0$

i.e. $P_2(A_{12} - (A_{22} \,(\_)\, A_{32})) = 0$

$$\Rightarrow P_1(A_{11}) = P_2(A_{12}) \leq P_2(A_{22} \,(\_)\, A_{32})$$

$$= P_2(A_{22}) + P_2(A_{32}) - P_2(A_{22} (\bar{\ }) A_{32})$$

$$= P_1(A_{21}) + P_1(A_{31}) - P_2(A_{22} (\bar{\ }) A_{32})$$

$$= P_1(A_{11}) - P_2(A_{22} (\bar{\ }) A_{32}) \leq P_1(A_{11})$$

$\Rightarrow P_2\, (A_{22}\, (\bar{\ })\, A_{32}) = 0$. Also the fact $P_2(A_{12}) = P_2(A_{22} \,(\_)\, A_{32})$ (which follows from the above string of inequalities) ensures $P_2\, (A_{12}\, \Delta\, (A_{22} \,(\_)\, A_{32})) = 0$. Hence (i).

Let now A_{11}, A_{21}, A_{12} and A_{22} be such that $P_1(A_{21}) < \frac{1}{2}$, $P_1(A_{11} - A_{21}) = 0$ and $T_1([A_{j1}]) = [A_{j2}]$, $j=1,2$. From (i) we have $P_2(A_{12}-A_{22}) = 0$. Now the proof of (ii) is trivial. Hence the lemma.

Loosely speaking Lemma 9.3 asserts that T_1 is 'monotone' and 'additive' (- part (i) -) and T_1 preserves 'increasing limits' and hence 'countably additive' (- part (ii) -). That is, T_1 is an isomorphism from $\underline{\underline{A}}_1(P_1)\, (\bar{\ })\, [A_{11}]$ onto $\underline{\underline{A}}_2\, (P_2)\, (\bar{\ })\, T_1([A_{11}])$ whenever $P_1(A_{11}) < \frac{1}{2}$.

Now we would like to dispense with two trivial cases. Let P_1 be completely atomic on $\underline{\underline{A}}_1$ with exactly one atom. Then $\underline{\underline{S}}_1$ contains only one element. So, $\underline{\underline{S}}_2$ will contain only one element. This will imply that P_2 is completely atomic on $\underset{=}{A}_2$ with only one atom. Then the answer to our question is immediate. Next, let $\underset{=}{P}_1$ be completely atomic on $\underset{=}{A}_1$ with exactly two atoms (say A_{11}, A_{21}). Then $\underset{=}{S}_1$ has two elements. This will imply that $\underset{=}{S}_2$ has only two elements. So, P_2 will be completely atomic on $\underset{=}{A}_2$ with exactly two atoms (say A_{12}, A_{22}). If $P_1(A_{11}) < \frac{1}{2}$, using T_1 it is seen that either A_{12} or A_{22} has P_2 measure equal to $P_1(A_{11})$; then the affirmative answer to our question follows immediately. The case is same if $P_1(A_{21}) < \frac{1}{2}$. If $P_1(A_{11}) = P_1(A_{21}) = \frac{1}{2}$ then using the metric preserving nature of $(\dagger)$, it is evident that $P_2(A_{12}) = P_2(A_{22}) = \frac{1}{2}$. Again, the answer is immediate. So, hereafter we will exclude these two cases from our consideration.

We are set to define the mapping T from $\underset{=}{A}_1(P_1)$ onto $\underset{=}{A}_2(P_2)$. Get a partition $\{A_{11}, A_{21}, \ldots, A_{m1}\}$ of Ω by $\underset{=}{A}_1$ sets as follows. A_{11} is the atom with the largest P_1-measure; if there are no atoms take $A_{11} = \phi$. The remaining sets $A_{21}, A_{31}, \ldots, A_{m1}$ are so obtained that $P_1(A_{j1}) < \frac{1}{2}$, $j=2,3,\ldots,m$. This can be done since we have excluded the case when P_1 is completely atomic with only one or two atoms. Let $A_{22}, A_{32}, \ldots, A_{m2}$ be $\underset{=}{A}_2$ sets such that $A_{j2} \in T_1[A_{j1}]$, $j = 2,3,\ldots,m$.

Now we claim that $P_2(A_{j2} \cap A_{j'2}) = 0$, $2 \leq j \neq j' \leq m$. For if $D \in T_1^{-1}([A_{j2} \cap A_{j'2}])$ then $P_1(D - A_{j1}) = 0$ and $P_1(D - A_{j'1}) = 0$. (This is a consequence of (i) of lemma 9.3) But $P_1(A_{j1} \cap A_{j'1}) = 0$. So, $P_1(D) = 0$. Thus $P_2(A_{j2} \cap A_{j'2}) = 0$, $2 \leq j \neq j' \leq m$. Thus $A_{j2}, A_{j3}, \ldots, A_{jm}$ are essentially disjoint. Let $A_{12} = \Omega_2 - \bigcup_{j=2}^{m} A_{j2}$. We presently show that $P_2(A_{12}) = P_1(A_{11})$ and if A_{11} is nonempty then A_{12} is also an atom. That $P_2(A_{12}) = P_1(A_{11})$ follows as $P_2(A_{j2}) = P_1(A_{j1})$, $j = 2,3,\ldots,m$. Let if possible A_{11} be nonempty and A_{12} not an atom. Choose $C \subset A_{12}$ with $0 < P_2(C) < \min\{\frac{1}{2}, P_2(A_{12})\}$. Let $D \in T_1^{-1}[C]$. Since $P_1(A_{11}) = P_2(A_{12}) > P_2(C) = P_1(D)$ and A_{11} is an atom $P_1(D \cap A_{11}) = 0$. So, $P_1(D \cap A_{j1}) > 0$ for some $j = 2,3,\ldots,m$. Let $E \in T_1[D \cap A_{j1}]$. Then $P_2(E - A_{j2}) = 0$ and $P_2(E - C) = 0$. ($\because$ $T[A_{1j}] = [A_{2j}]$ and $T[D] = [C]$). But $P_2(A_{2j} \cap C) = 0$. So, $P_2(E) = 0$ - a contradiction to the fact $P_1(D \cap A_{j1}) > 0$ and consequently $P_2(E) > 0$. Hence A_{12} is an atom.

Define T by

$$T([D]) = \bigcup_{j=1}^{m} T_1([D \cap A_{j1}]), \quad D \in \underline{A}_1$$

(Here interpret $T_1([A_{11}]) = [A_{12}]$. This interpretation is consistent if $P_1(A_{11}) < \frac{1}{2}$).

Lemma 9.3 and the observations we have made just above ensure that T is an isomorphism. In order not to lengthen an already long proof we omit the details, which are in any case easy to check.

Lemma 9.4 Let $\tilde{\Phi}$ be the map induced by T. Then for any $D \in \mathcal{A}_1$,

$$\Phi(\sigma\{D\}) = \tilde{\Phi}(\sigma\{D\}).$$

Proof: 1°. Let $P_1(D) < \frac{1}{2}$. Then

$$T([D]) = \bigcup_{j=1}^{m} T_1([D \cap A_{j1}])$$

$$= T_1\Big(\bigcup_{j=1}^{m} [D \cap A_{j1}]\Big) = T_1([D]).$$

Let $E \in T_1([D])$. By the definition of T_1 we have $\Phi(\sigma\{D\}) = \sigma\{E\}$. On the other hand $E \in T([D])$ and by the definition of $\tilde{\Phi}$ we have $\tilde{\Phi}(\sigma\{D\}) = \sigma\{E\}$.
So, $\Phi(\sigma\{D\}) = \tilde{\Phi}(\sigma\{D\})$.

2°. Let E_1, E_2 be two elements of $\mathcal{A}_2$ satisfying (i) $P_2(E_1 \cap E_2) = 0$, (ii) $0 < P_2(E_1) < \frac{1}{2}$ and (iii) $P_2(E_1 \cup E_2) = \frac{1}{2}$. Consider $\sigma\{E_1, E_2\}$. It is clear that it contains only five elements of $\mathcal{S}_2$ viz $\sigma\{\Omega_2\}$, $\sigma\{E_1\}$, $\sigma\{E_2\}$, $\sigma\{E_1 \cup E_2\}$ and $\sigma\{E_1, E_2\}$.

Now,

$d_2(\sigma\{\Omega_2\}, \sigma\{\Omega_2\}) = 0,$

$d_2(\sigma\{\Omega_2\}, \sigma\{E_1\}) = P_2(E_1) < \frac{1}{2},$

$d_2(\sigma\{\Omega_2\}, \sigma\{E_2\}) = P_2(E_2) < \frac{1}{2},$

$d_2(\sigma\{\Omega_2\}, \sigma\{E_1 \cup E_2\}) = P_2(E_1 \cup E_2) = \frac{1}{2}$ and

$d_2(\sigma\{\Omega_2\}, \sigma\{E_1, E_2\}) = \min\{P_2(E_1), P_2(E_2)\} < \frac{1}{2}.$

So, the <u>only</u> element of $\underline{\underline{S}}_2$ contained in $\sigma\{E_1, E_2\}$ and is at a distance $\frac{1}{2}$ from $\sigma\{\Omega_2\}$ is $\sigma\{E_1 \cup E_2\}$.

3^o. Let $P_1(D) = \frac{1}{2}$. Let B be an $\underline{\underline{A}}_1$ set with $B \subset D$ and $0 < P_1(B) < P_1(D)$.

$\sigma\{D\} \subset \sigma\{B\} \vee \sigma\{D - B\}$. So,

$\oplus(\sigma\{D\}) \subset \oplus(\sigma\{B\}) \vee \oplus(\sigma\{D - B\})$ and

$\widetilde{\oplus}(\sigma\{D\}) \subset \widetilde{\oplus}(\sigma\{B\}) \vee \widetilde{\oplus}(\sigma\{D - B\})$

$= \oplus(\sigma\{B\}) \vee \oplus(\sigma\{D - B\})$

(- note that $P_1(B) < \frac{1}{2}$, $P_1(D - B) < \frac{1}{2}$ and use 1^o -).

So, $\oplus(\sigma\{D\})$ and $\widetilde{\oplus}(\sigma\{D\})$ are elements of $\underline{\underline{S}}_2$ contained in $\oplus(\sigma\{B\}) \vee \oplus(\sigma\{D - B\})$.

Let $E_1 \in T_1[B]$ and $E_2 \in T_1[D - B]$. Then

(i) $P_2(E_1 \,(\overline{\ \ })\, E_2) = 0$, (ii) $0 < P_2(E_1) < \frac{1}{2}$, $0 < P_2(E_2) < \frac{1}{2}$ and (iii) $P_2(E_1 \,(\underline{\ \ })\, E_2) = \frac{1}{2}$. Also, by the definition of T_1, we have

$$\oplus(\sigma\{B\}) \vee \oplus(\sigma\{D - B\}) = \sigma\{E_1\} \vee \sigma\{E_2\} .$$

Observe also that $d_1(\sigma\{\Omega_1\}, \sigma\{D\}) = \frac{1}{2}$; the fact that $\oplus$ and $\widetilde{\oplus}$ are isometries implies,

$d_2(\sigma\{\Omega_2\}, \oplus(\sigma\{D\})) = \frac{1}{2}$ and $d_2(\sigma\{\Omega_2\}, \widetilde{\oplus}(\sigma\{D\})) = \frac{1}{2}$.

Thus $\oplus(\sigma\{D\})$ and $\widetilde{\oplus}(\sigma\{D\})$ are two elements of $\mathfrak{S}_2$, contained in $\sigma\{E_1\} \vee \sigma\{E_2\}$ and each is at a distance $\frac{1}{2}$ from $\sigma\{\Omega_2\}$. So, from 2°, $\oplus(\sigma\{D\}) = \widetilde{\oplus}(\sigma\{D\})$.

4°. Let $D \in \underline{\Delta}_1$. If $P_1(D) \neq \frac{1}{2}$, either $P_1(D) < \frac{1}{2}$ or $P_1(D^c) < \frac{1}{2}$. Then an application of 1° gives us $\oplus(\sigma\{D\}) = \widetilde{\oplus}(\sigma\{D\})$. Now, let $P_1(D) = \frac{1}{2}$. Since we have excluded the case of exactly two atoms either D or D^c is not an atom ; say D is not. Then D contains an $\underline{\Delta}_1$ set B with $P < P_1(B) < P_1(D)$. Now an application of 3° gives us

$\oplus(\sigma\{D\}) = \widetilde{\oplus}(\sigma\{D\})$.

Hence the lemma.

<u>Lemma 9.5</u> For any $\underline{B} \in \mathfrak{S}_1$, $\oplus(\underline{B}) = \widetilde{\oplus}(\underline{B})$.

Proof: Observe that $\underline{\underline{B}} = \bigvee_{D \in \underline{\underline{B}}} \sigma\{D\}$. Recall that (ϕ) and $(\widetilde{\phi})$ are complete lattice isomorphisms. So,

$$(\phi)(\underline{\underline{B}}) = (\phi)\Big(\bigvee_{D \in \underline{\underline{B}}} \sigma\{D\}\Big)$$

$$= \bigvee_{D \in \underline{\underline{B}}} (\phi)(\sigma\{D\})$$

$$= \bigvee_{D \in \underline{\underline{B}}} (\widetilde{\phi})(\sigma\{D\}) \quad \text{(by lemma 9.4)}$$

$$= (\widetilde{\phi})\Big(\bigvee_{D \in \underline{\underline{B}}} \sigma\{D\}\Big) = (\widetilde{\phi})(\underline{\underline{B}}).$$

Hence the lemma.

Thus we have

Theorem 9.6 Every isometry (ϕ) from $\underline{\underline{S}}(\Omega_1, \underline{\underline{A}}_1, P_1)$ onto $\underline{\underline{S}}(\Omega_2, \underline{\underline{A}}_2, P_2)$ preserving the lattice operations is induced by an isomorphism between $\underline{\underline{A}}_1(P_1)$ and $\underline{\underline{A}}_2(P_2)$.

CHAPTER 3

ON A GENERALIZATION, DUE TO BLAKE, OF MARTINGALES

1. Introduction

Let $(\Omega, \underline{A}, P)$ be a probability space and $\{\underline{F}_n\}_{n\geq 1}$ an increasing family of sub σ-algebras of $\underline{A}$. Let $\{X_n\}_{n\geq 1}$ be a stochastic process adapted to $\{\underline{F}_n\}_{n\geq 1}$; i.e. each X_n is $\underline{F}_n$ measurable. Following Blake [5], we refer to $\{X_n\}_{n\geq 1}$ as a game and define

Definition: The game $\{X_n\}_{n\geq 1}$ will be said to become fairer with time if for every $\varepsilon > 0$,

$$P[|E(X_n/\underline{F}_m) - X_m| > \varepsilon] \to 0 \quad \text{as} \quad n, m \to \infty \quad \text{with} \quad n \geq m.$$

(In the above definition the symbol $E(X_n/\underline{F}_m)$ stands for the conditional expectation of X_n given $\underline{F}_m$. We have used this notation for conditional expectation, rather than the one used in Chapter 1, to confirm with the usage in [5]). Any martingale is, trivially, a fairer with time game and thus this concept generalizes that of martingales.

Blake in [5] gives two convergence theorems for uniformly integrable fairer with time processes. Below we quote those results. Let $\{\alpha_n\}_{n \geq 1}$ be a monotonic sequence decreasing to zero with finite sum. The game $\{X_n\}_{n\geq 1}$ may be decomposed with respect to $\{\alpha_n\}_{n \geq 1}$ as

$$X_n = Y_n - Z_n \qquad (1.1)$$

where $\{Y_n\}_{n \geq 1}$ and $\{Z_n\}_{n \geq 1}$ are defined inductively by :

$$Y_1 = X_1$$

$$\vdots \qquad (1.2)$$

$$Y_n = Y_{n-1} + [X_n - E(X_n/\underline{F}_{n-1})] + \alpha_{n-1}$$

$$Z_n = Z_{n-1} + [X_{n-1} - E(X_n/\underline{F}_{n-1})] + \alpha_{n-1} \qquad (1.3)$$

The decomposition of $\{X_n\}_{n \geq 1}$ according to (1.1) - (1.3) will be called a Doob-like decomposition. It can be verified that $\{Y_n\}_{n \geq 1}$ of the decomposition, is a submartingale with respect to $\{\underline{F}_n\}_{n \geq 1}$. Define the collection of sets $\{B^{\alpha}_{n,m}\}$ for $m = 1, 2, \ldots$ and $n \geq m$ by

$B^{\alpha}_{n,m} = \{|E(X_n/\underline{F}_m) - X_m| > \alpha_m\}$. Now we state the convergence theorems in [5].

<u>Theorem 1.</u> Let $\{X_n\}_{n \geq 1}$ be a uniformly integrable game and $\{Y_n\}_{n \geq 1}$, the submartingale associated with its Doob-like decomposition, be uniformly dominated in absolute value by an element of $L_1(\Omega, \underline{A}, P)$. Suppose for every $\delta > 0$ there exists an integer $N(\delta)$ such that

$P[B^{\alpha}_{n,m}] < \delta$ whenever $n \geq m \geq N(\delta)$ (1.4)

and $\Omega - B^{\alpha}_{n,m} \subset \Omega - B^{\alpha}_{k,k-1}$ whenever (1.5)

$$n \geq k \geq k-1 \geq m \geq N(\delta)$$

Then, there exists a function X in $L_1(\Omega, \underline{A}, P)$ such that $\lim_{n \to \infty} \int_{\Omega} |X_n - X| \, dP = 0.$

<u>Theorem</u> 2. Let $\{X_n\}_{n \geq 1}$ be a uniformly integrable game satisfying (1.4) and (1.5). Then, there exists some constant C such that $\lim_{n \to \infty} \int_{\Omega} X_n \, dP = C.$

The condition (1.4) implies the game $\{X_n\}_{n \geq 1}$ is fairer with time. To see this fix $\varepsilon > 0$. Let $\delta > 0$ be given. Choose m_o so large that $\alpha_{m_o} < \varepsilon$ (this can be done as $\alpha_m \downarrow 0$) and $m_o \geq N(\delta)$. Now,

$$\{|E(X_n/\underline{F}_m) - X_m| > \varepsilon\} \subset \{|E(X_n/\underline{F}_m) - X_m| > \alpha_m\}$$

for all $n \geq m \geq m_o$.

So, $P[|E(X_n/\underline{F}_m) - X_m| > \varepsilon] \leq P[|E(X_n/\underline{F}_m) - X_m| > \alpha_m]$

$= P[B^{\alpha}_{n,m}] < \delta$ whenever

$n \geq m \geq m_o$.

Since δ is arbitrary it follows

$P[|E(X_n/\underline{F}_m) - X_m| > \varepsilon] \to 0$ as $n, m \to \infty$ with $n \geq m$.

Thus the above theorems give a set of sufficient conditions for a uniformly integrable fairer with time game to converge. In this chapter we show that these sufficient conditions are not needed ; in fact, we show that any uniformly integrable, fairer with time game converges in L_1.

2. Main theorems

Theorem 2.1 Any uniformly integrable fairer with time game $\{X_n\}_{n \geq 1}$ converges in L_1.

Proof : To facilitate understanding, we break up the proof into a few important steps numbered (S1) through (S5). For every m and $n \geq m$ define $Y_{m,n} = E(X_n / \underline{F}_m)$. Let Γ stand for the family $\{Y_{m,n},$ for all m and $n \geq m\}$.

(S1) Γ is uniformly integrable.

Since $\{X_n\}_{n \geq 1}$ is uniformly integrable there exists a function f defined on the non-negative real axis which is positive, increasing, and convex, such that

$$\lim_{t \to \infty} \frac{f(t)}{t} = +\infty$$

and $\sup_n E[f \circ |X_n|] < \infty$. (see [16, II T 22]).

Now, $E[f \circ |Y_{m,n}|] = E[f \circ |E(X_n/\underline{\underline{F}}_m)|]$

$$\leq E[f \circ E(|X_n|/\underline{\underline{F}}_m)] \text{ (since } f \text{ is increasing)}$$

$$\leq E[E(f \circ |X_n|/\underline{\underline{F}}_m)]$$

$$= E[f \circ |X_n|].$$

Therefore $\sup_{Y_{m,n} \in \Gamma} E[f \circ |Y_{m,n}|] \leq \sup_n E[f \circ |X_n|] < \infty$.

Another application of II T 22 of [16] ensures that Γ is uniformly integrable. Hence (S1).

(S2) Given $\varepsilon > 0$, there exists M such that for all $m \geq M$, one has $E(|X_m - Y_{m,n}|) \leq 2\varepsilon$ for all $n \geq m$.

Since Γ is uniformly integrable given $\varepsilon > 0$ there exists $\delta > 0$ such that $P(A) < \delta$ implies

$\int_A |Y_{m,n}| \, dP \leq \frac{\varepsilon}{2}$, for all $Y_{m,n} \in \Gamma$. Choose M so large that $m \geq M$ and $n \geq m$ implies $P[|X_m - E(X_n/\underline{\underline{F}}_m)| > \varepsilon] < \delta$. Then, it is not difficult to see that

$E[|X_m - Y_{m,n}|] \leq 2\varepsilon$ for all $m \geq M$ and $n \geq m$. Hence (S2).

(S3) For every fixed m, the sequence $\{Y_{m,n}\}$ converges in L_1 to an $\underline{\underline{F}}_m$ measurable random variable Z_m.

Let $m \leq n < n'$.

$$\begin{aligned}E[|Y_{m,n} - Y_{m,n'}|] &= E[|E(X_n/\underline{\underline{F}}_m) - E(X_{n'}/\underline{\underline{F}}_m)|] \\ &= E[|E(X_n - X_{n'}/\underline{\underline{F}}_m)|] \\ &= E[|E(\{X_n - X_{n'}/\underline{\underline{F}}_n)\}/ \underline{\underline{F}}_m)|] \\ &\leq E[E(\{|E(X_n - X_{n'}/\underline{\underline{F}}_n)|\}/\underline{\underline{F}}_m)] \\ &= E[|E(X_n - X_{n'}/\underline{\underline{F}}_n)|] \\ &= E[|X_n - Y_{n,n'}|]\end{aligned}$$

Now from (S2) it follows that given $\varepsilon > 0$ for all sufficiently large n and n'

$$E[|Y_{m,n} - Y_{m,n'}|] \leq E[|(X_n - Y_{n,n'})|] \leq 2\varepsilon .$$

Hence, for m fixed, the sequence $\{Y_{m,n}\}$ is Cauchy in the L_1-norm. So, there exists, an integrable random variable Z_m, such that, $Y_{m,n} \underset{n \to \infty}{\overset{L_1}{\to}} Z_m$. Without loss in generality we can take Z_m to be $\underline{\underline{F}}_m$ measurable. (Note that each $Y_{m,n}$ is $\underline{\underline{F}}_m$ measurable and there is a subsequence $\{Y_{m,n'}\}$ converging almost surely to Z_m). Hence (S3).

(S4) $\{Z_m, \underline{\underline{F}}_m\}_{m \geq 1}$ is an uniformly integrable martingale.

The fact that $\{Z_m\}_{m \geq 1}$ is uniformly integrable follows trivially because the closure in L_1 of a uniformly integrable collection is uniformly integrable. (See [16, II T 20]). To show $\{Z_m, \underline{\underline{F}}_m\}$ is a martingale it is enough to show that for every m,

$E(Z_{m+1}/\underline{\underline{F}}_m) = Z_m$ a.s. Since

$$E[|E(Y_{m+1,n}/\underline{\underline{F}}_m) - E(Z_{m+1}/\underline{\underline{F}}_m)|]$$

$$= E[|E\{(Y_{m+1,n} - Z_{m+1})/\underline{\underline{F}}_m\}|]$$

$$\leq E[E\{|Y_{m+1,n} - Z_{m+1})|/\underline{\underline{F}}_m\}]$$

$$= E[|Y_{m+1,n} - Z_{m+1}|] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

there exists a subsequence n' of $\{n : n \geq m\}$ such that

$$E(Y_{m+1,n'}/\underline{\underline{F}}_m) \xrightarrow{a.s.} E(Z_{m+1}/\underline{\underline{F}}_m).$$

We can assume (- if necessary, by choosing a further subsequence,-) that $Y_{m,n'} \xrightarrow{a.s.} Z_m$.

Now, $E(Z_{m+1}/\underline{\underline{F}}_m) = \lim_{n' \to \infty} E(Y_{m+1,n'}/\underline{\underline{F}}_m)$ a.s.

$$= \lim_{n' \to \infty} E(\{E(X_{n'}/\underline{\underline{F}}_{m+1})\}/\underline{\underline{F}}_m) \quad \text{a.s.}$$

$$= \lim_{n' \to \infty} E(X_{n'}/\underline{\underline{F}}_m) \quad \text{a.s.}$$

$$= \lim_{n' \to \infty} Y_{m,n'} \quad \text{a.s.}$$

$$= Z_m \quad \text{a.s.}$$

Hence (S4).

(S5) $\{X_n\}_{n \geq 1}$ converges in L_1.

Since $\{Z_n, \underline{\underline{F}}_n\}_{n \geq 1}$ is an uniformly integrable martingale, there exists an integrable random variable Z_∞ such that

$Z_n \xrightarrow[n\to\infty]{L_1} Z_\infty$. We shall show that $X_n \xrightarrow{L_1} Z_\infty$. From (S3) and (S2) it is easy to check that given $\varepsilon > 0$ there exists M such that for all $m \geq M$

$\int |X_m - Z_m|\, dP \leq 2\varepsilon$. Therefore, for sufficiently large m,

$\int |X_m - Z|\, dP \leq \int |X_m - Z_m| dP + \int |Z_m - Z_\infty| dP \leq 3\varepsilon$, say.

Hence (S5) and the theorem.

Since any game (stochastic process) $\{X_n\}_{n \geq 1}$ converging in L_1 can be taken to be a game fairer with time, by setting $\underline{\underline{F}}_n = \underline{\underline{A}}$ for all n, we get the following corollary.

<u>Corollary 2.1</u> Let $\{X_n\}_{n \geq 1}$ be a game. It converges in L_1 if and only if it is uniformly integrable and fairer with time with respect to some increasing family of sub σ-algebras $\{\underline{\underline{F}}_n\}_{n \geq 1}$ to which it is adapted.

Let $p > 1$.

<u>Theorem 2.2</u> Let $\{X_n\}_{n \geq 1}$ be a fairer with time game with $\{|X_n|^p\}_{n \geq 1}$ uniformly integrable. Then $\{X_n\}_{n \geq 1}$ converges in L_p.

<u>Proof</u> : Noting that the function f defined on the non-negative real axis by $f(t) = t^p$ is positive, increasing and convex and $\lim_{t \to \infty} \frac{f(t)}{t} = +\infty$, in view of II T 22 of [16], it is clear

that $\{X_n\}_{n \geq 1}$ is uniformly integrable. Hence by theorem 2.1 it converges in L_1; in particular, $\{X_n\}_{n \geq 1}$ converges in probability. Therefore $\{X_n\}_{n \geq 1}$ converges in L_p. (See proposition II 6.1 of [20]).

<u>Corollary 2.2</u> The game $\{X_n\}_{n \geq 1}$ converges in L_p if and only if $\{|X_n|^p\}_{n \geq 1}$ is uniformly integrable and $\{X_n\}_{n \geq 1}$ is fairer with time with respect to some increasing family of sub σ-algebras $\{\underline{F}_n\}_{n \geq 1}$ to which it is adapted.

CHAPTER 4

ON A CONJECTURE ON SINGULAR MARTINGALES

1. Introduction

Let $(\Omega, \underline{A}, P)$ be a probability space and $\{\underline{F}_n\}_{n \geq 1}$ an increasing sequence of sub σ-algebras of $\underline{A}$. Let $\underline{F}_\infty$ be the smallest σ-algebra which contains all the $\underline{F}_n$'s. In what follows, there is no essential loss of generality if we assume that $\underline{A} = \underline{F}_\infty$; accordingly we assume so. Let $\{X_n\}_{n \geq 1}$ be a sequence of integrable random variables on $(\Omega, \underline{A}, P)$ forming a martingale with respect to $\{\underline{F}_n\}_{n \geq 1}$ (in the sequel, we will abbreviate this by '$\{X_n, \underline{F}_n\}_{n \geq 1}$ is a martingale'). For each n, define a measure μ_n on $\underline{F}_n$ by

$$\mu_n(A) = \int_A X_n \, dP, \quad A \varepsilon \underline{F}_n . \tag{1.1}$$

In [14], Luis Baez-Duarte introduced the following definitions.

Definition 1.2 The martingale $\{X_n, \underline{F}_n\}_{n \geq 1}$ is measure-dominated if there is a finite measure μ on $\underline{B}$ whose restriction to each $\underline{F}_n$ coincides with the measure μ_n defined in (1.1). In such a case μ is said to dominate the martingale.

In view of our assumption $\underline{F}_\infty = \underline{A}$, if there is a measure μ dominating the martingale $\{X_n, \underline{F}_n\}_{n \geq 1}$ then it is unique.

<u>Definition 1.3</u> Let $\{X_n, \underline{\underline{F}}_n\}_{n \geq 1}$ be a measure-dominated martingale. It is said to be a singular martingale if the dominating measure μ is singular with respect to P.

Let for each n, S_n stand for the extended real line $[-\infty, \infty]$ and $\underline{\underline{B}}_n$ for the Borel σ-algebra of S_n. Denote by S the cartesian product $\prod_{n=1}^{\infty} S_n$ and by Y_n the projection from S to $(S_n, \underline{\underline{B}}_n)$. Let the σ-algebras generated by $\{Y_1, Y_2, \ldots, Y_n\}$ and $\{Y_n, n \geq 1\}$ be denoted respectively by $\underline{\underline{B}}^n$ and $\underline{\underline{B}}$. In the remainder of this chapter, the symbols S, S_n, $\underline{\underline{B}}^n$, $\underline{\underline{B}}$ and Y_n have the same meaning as given in this paragraph.

Given any sequence $\{X_n\}_{n \geq 1}$ of integrable random variables on $(\Omega, \underline{\underline{A}}, P)$ by canonical mapping we mean the measurable mapping T from $(\Omega, \underline{\underline{A}})$ to $(S, \underline{\underline{B}})$ given by

$$T(\omega) = (X_1(\omega), X_2(\omega), \ldots, X_n(\omega), \ldots), \forall \omega \in \Omega .$$

It is easy to see that if $\{X_n, \underline{\underline{F}}_n\}_{n \geq 1}$ is a martingale on $(\Omega, \underline{\underline{A}}, P)$ then $\{Y_n, \underline{\underline{B}}^n\}_{n \geq 1}$ is a martingale on $(S, \underline{\underline{B}}, P \circ T^{-1})$ (- where T is, of course, the canonical mapping -) with the property that $\{X_n\}_{n \geq 1}$ and $\{Y_n\}_{n \geq 1}$ are equivalent. We call $\{Y_n, \underline{\underline{B}}^n\}_{n \geq 1}$ the canonical martingale associated with $\{X_n, \underline{\underline{F}}_n\}_{n \geq 1}$. Note that for various $\{X_n\}$'s what varies is the measure $P \circ T^{-1}$ and not $\{Y_n\}_{n \geq 1}$.

It is not difficult to prove (see, for instance, theorem 3.1 of [14]) that, if $\{X_n, \underline{F}_n\}_{n \geq 1}$ is a measure-dominated martingale then the canonical martingale associated with it is also measure-dominated. In [14], Baez-Duarte conjectured that if $\{X_n, F_n\}_{n \geq 1}$ is a singular martingale on $(\Omega, \underline{A}, P)$ then the corresponding dominating measure for the canonical martingale is concentrated on the boundary of $T\,\Omega$ We disprove this with the help of an example.

2. An example

Before proceeding with the construction of the example we would like to observe the following.

Let $\Omega = S$, $\underline{A} = \underline{B}$ and $X_n = Y_n$ for all n. Then the canonical mapping from $(\Omega, \underline{A})$ to $(S, \underline{B})$ is just the identity mapping. So $T\,\Omega = S$ and hence the boundary of $T\,\Omega$ is empty. Therefore if we could find a measure Q on (S, B) such that $\{Y_n, \underline{B}_n\}_{n \geq 1}$ becomes a singular martingale on $(S, \underline{B}, Q)$ we would have disproved the conjecture.

We achieve this in two stages.

2.1 Let $\Omega = \{1, 2, 3, \ldots, n, \ldots, \infty\}$. Let $\underline{F}_1 = \{\emptyset, \Omega\}$ and, for each $n \geq 2$, $\underline{F}_n$ be the σ-algebra generated by the partition $\{1\}, \{2\}, \ldots, \{n-1\}, \{n, n+1, \ldots, \infty\}$. The symbol $[n, \infty]$ will denote $\{n, n+1, \ldots, \infty\}$. As in section 1, the σ-algebra generated by $\bigcup_n \underline{\underline{F}}_n$ is denoted by $\underline{\underline{A}}$. Clearly, $\underline{\underline{A}}$ is just the class of all subsets of Ω. Let P be the probability measure defined on $\underline{\underline{A}}$ by $P(\{n\}) = \frac{1}{2^n}$, $n \geq 1$ and $P(\{\infty\}) = 0$. Define a sequence $\{X_n\}_{n \geq 1}$ of integrable random variables on $(\Omega, \underline{\underline{A}}, P)$ by setting,

$$X_n(\omega) = \frac{1}{P[n, \infty]} \quad \text{if} \quad \omega \geq n$$

$$= 0 \quad \text{if} \quad \omega < n.$$

The fact that $\{X_n, \underline{\underline{F}}_n\}_{n \geq 1}$ is a martingale is easily verified. The measures μ_n's given by the relation (1.1) are nothing but the restriction to $\underline{\underline{F}}_n$ of the probability measure ε_∞, concentrated at $\{\infty\}$. Hence $\{X_n, \underline{\underline{F}}_n\}_{n \geq 1}$ is a singular martingale.

2.2 Consider the canonical martingale associated with the singular martingale constructed in 2.1. We show that it (the canonical martingale) is singular.

The following two facts are easily observed.

(i) Since the set Ω, defined in 2.1 is countable, T(A) is $\underline{\underline{B}}$ measurable for every subset A of Ω.

(ii) Since $T(\omega) = (X_1(\omega), X_2(\omega), \ldots, X_n(\omega), \ldots)$

$$= \left(1, \frac{1}{P[2,\infty]}, \frac{1}{P[3,\infty]}, \ldots, \frac{1}{P[\omega,\infty]}, 0, 0, \ldots\right)$$

T is one to one.

Now, denoting the measures, corresponding to the canonical martingale, defined in (1.1) by $\{\mu_n'\}_{n \geq 1}$, we have

$$\mu_n'(A) = \int_A Y_n \, d\, P \circ T^{-1} = \int_{T^{-1}(A)} X_n \, dP = \mu_n(T^{-1}(A)), \ \forall A \in \underline{\underline{B}}^n.$$

That is $\mu_n' = \mu_n \circ T^{-1}$. Let μ' be the probability measure concentrated at the point $T(\infty)$.

For any $A \in \underline{\underline{B}}^n$, $\mu'(A) = 1 \iff T(\infty) \in A$

$$\iff \mu_n(T^{-1}(A)) = 1$$

$$\iff \mu_n'(A) = 1$$

So, for every n, the restriction of μ' to $\underline{\underline{B}}^n$ coincides with μ_n'. That is, μ' is the dominating measure for the canonical martingale. Since, $P \circ T^{-1}(T(\Omega - \{\infty\})) = P(\Omega - \{\infty\}) = 1$ and $\mu'(T(\infty)) = 1$, the measures $P \circ T^{-1}$ and μ' are singular.

Thus $\{Y_n, \underline{\underline{B}}^n\}_{n \geq 1}$ is a singular martingale on $(S, \underline{\underline{B}}, P \circ T^{-1})$. Therefore, in view of the observation made at the beginning of this section, the conjecture made by Baez-Duarte is false.

3. <u>Remark</u>

A slight modification of the above example will show that even if boundary of $T\,\bar{\Omega}$ is <u>non-empty</u> the conjecture is not true. We indicate below the modification to be effected without giving detailed proofs.

Let $\bar{\Omega}$ and P be as in 2.1 and $s \in S - T\,\bar{\Omega}$. Define $S' = S - \{s\}$ and take $(\underline{\underline{B}}^n)'$ and $\underline{\underline{B}}'$ respectively to be the trace σ-algebras of $\underline{\underline{B}}^n$ and $\underline{\underline{B}}$ on S'. Let, for each n, Y_n' be the restriction of Y_n to S'. $\{Y_n', (\underline{\underline{B}}^n)'\}_{n \geq 1}$ is a singular martingale on $(S', \underline{\underline{B}}', P \circ T^{-1})$ with μ' (defined in 2.2) as the dominating measure. Now, since $T(S') = S - \{s\}$, the boundary of $T(S') = s$. But, the dominating measure corresponding to the canonical martingale associated with $\{Y_n', (\underline{\underline{B}}^n)'\}_{n \geq 1}$ is concentrated at the point

$$\left(1, \frac{1}{P[2,\infty]}, \ldots, \frac{1}{P[n,\infty]}, \ldots\right);$$ and this point is obviously different from s.

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