SOME OPTIMISATION PROBLEMS

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A C K N O W L E D G E M E N T S

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CONTENTS

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GENERAL INTRODUCTION AND SUMMARY

In this thesis we have considered two optimisation problems - optimal grouping in inventory control and optimal cutting procedures for products which are produced in continuous length. The motivation for these two pieces of research work was actual plant problems encountered in industry. In part I, we have introduced the approach of Group Economic Order Quantity (EOQ) in inventory control and developed optimum grouping procedures. In Part II, we have derived a statistical distribution which can be put to use to solve a variety of industrial problems. In particular, we have used this distribution to develop optimum cutting procedures. A brief summary of the contents of Parts I and II is given below.

PART I : GROUP EOQ REPLENISHMENT POLICY

Let A be the per order cost of ordering and I the cost of ordering expressed as rate of interest. We assume that A and I are the same for all the items. The Economic Order Quantity (EOQ) in terms of money value for an individual item with money value of yearly demand equal to y is given by the well-known square-root formula

$$Q^* = \sqrt{\frac{2AY}{I}} \qquad \ldots \quad \ldots \quad \ldots (1)$$

When we use the square-root formula (1) to calculate the order quantity seperately for each item, we refer to such a situation as Individual EOQ Replenishment Policy.

Suppose now that we have a group of N items and that the average money value of yearly demand for the group is $\bar{y}$. It is desired to use a common ordering rule (either in terms of money value or of the frequency of orders) for all the items in the group instead of the Individual EOQ Replenishment Policy. For example, for every item in the group an order of Rs.5000/- worth of material may be placed, or every item in the group may be ordered 10 times a year. The optimum common group ordering rule in terms of money value is then shown to be given by

$$Q^* = \sqrt{\frac{2A\bar{y}}{I}} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

When we use a formula of type (2), to calculate the common group ordering rule, we refer to the situation as Group EOQ Replenishment Policy. Use of Group EOQ approach will always mean additional cost as compared to Individual EOQ approach.

Let $-\infty = y_0 < y_1 < y_2 < \ldots\ldots < y_{r-1} < y_r = \infty$ be suitably chosen numbers. We form r groups of items as follows: The ith group consists of all items whose money value of yearly demand lie in the range y_{i-1} to y_i. We refer to $y_1, y_2, \ldots\ldots y_{r-1}$ as group boundaries. Suppose that Group EOQ Replenishment Policy is used for each of the groups. Using the log normal distribution to describe the distribution of money value of yearly demand over the items; we derive an explicit expression for the average cost per item (over all the r groups) for any given group boundaries.

We then consider the problem of determining the optimal group boundaries which minimise the average cost per item. This optimisation problem is studied in detail and the existence and uniqueness of the optimal solution are established. It is shown that this optimisation problem is closely related to another optimisation problem encountered by Ogawa in the determination of optimal spacings for the large sample estimation of the mean of a normal distribution (when the standard deviation is known) by a selected set of sample quantiles. The relationship between these two optimisation problems is studied in detail.

Let C_r be the average cost per item when there are r groups with optimal group boundaries and when Group EOQ approach is used for each of the r groups. Let C_∞ be the average cost per item when Individual EOQ Replenishment policy is used. It is shown that the ratio C_r / C_∞ depends only on the number of groups r and on σ - one of the parameters of the log-normal distribution. This ratio gives an indication of the additional cost incurred when the Group EOQ approach is used instead of Individual EOQ approach. As such this ratio provides an objective criterion for determining the number of groups when σ is known. The values of the ratios C_r / C_∞ and optimal group boundaries have been calculated $r = 2$ to 10 for $\sigma = 0.2\ (0.2)\ 4.0$ by programming on a computer. Various useful Approximations for optimal group boundaries are also given.

The Group EOQ approach has been successfully used in many industrial organisations and super-markets. Details of various steps involved in the determination of the number of groups, calculation of group boundaries, calculation of group EOQ's, etc., are described with the help of an actual case study.

In connection with the above work, we have derived some new results about truncated normal distributions. These results are of interest on their own, and one of them is a generalisation of a result due to Sampford on single truncation.

PART II : OPTIMAL CUTTING PROCEDURES :

We consider a sequence of events which is a mixture of a completely regular sequence and a Poisson process. Let S denote a systematic event occuring at the end of every T time units and R a random event which occurs according to a Poisson process of rate λ. Let the event ξ be defined to occur when either S or R does. The equilibrium marginal frequency distribution of the interval between successive occurences of the event ξ is derived. This distribution can be used to solve a variety of industrial problemsand some specific applications are discussed.

Automatic winding machines in textile industry consist of 200 to 300 spindles; each of which is used to wind yarn from a relatively small supply bobbin to a larger cone. The winding at any spindle, stops when the yarn on the bobbin gets exhausted or when the yarn breaks. An automatic head which patrols the machine in fixed time replaces the empty bobbin by a full one or knots the broken yarn. Under reasonable assumptions we can identify the stoppage of the spindle due to exhaustion of yarn as the occurence of the systematic event S and stoppage of the spindle due to yarn break as the random event R, and use the distribution mentioned earlier for developing a suitable model. We have derived (i) the distribution of idle time of a spindle (ii) the distribution of busy time, and (iii) the distribution of the number of patrols of the automatic head between two consecutive restartings of the spindle. We have also derived an expression for machine efficiency. Howie and Shenton have also derived an expression for machine efficiency using an entirely different approach.

In textile industry, cloth is cut and taken out of the looms in some definite lengths - say L units. Weaving and processing defects occur at random. The money value realised depends on the length of defect-free cloth sold. By identifying the cut at every L units as a systematic event S and the occurence of a defect as a random event R; we have developed a suitable model for this situation.

Products which are produced in continuous length are sold to customers in defect-free pieces of some specific length which can be taken as unit length without loss of generality.

The cutting procedure is as follows: First, relatively longer pieces of some predetermined length L are cut out. These are then inspected and, depending on the positions where defects occur, are suitably cut so as to get the maximum number of defect-free pieces of unit length from each of the pieces of longer pieces of length L. Let $Y(L, \lambda)$ be the average yield when the initial out length is L and λ is the average number of defects per unit length. Using a different approach, Sibuya derived expressions for yield when $1 \leq L < 2$ and $2 \leq L < 3$. Using the distribution mentioned earlier, we have derived an expression for yield for any initial cut length L. Let n be any positive integer. It is shown that there exists a λ_n^* for a given n such that for $0 < \delta < 1$

$$Y(n+\delta, \lambda) \begin{cases} > Y(n,\lambda) & \text{when } \lambda > \lambda_n^* \\ = Y(n,\lambda) & \text{when } \lambda = \lambda_n^* \\ < Y(n,\lambda) & \text{when } \lambda < \lambda_n^* \end{cases}$$

It is also shown λ_n^* is the unique solution of

$$\sum_{i=1}^{n} (\lambda i - 1)\, e^{-\lambda(i-1)} = 0 \quad \ldots \quad \ldots \quad (3)$$

The assymptotic properties of λ_n^* are then studied and it is shown that

$$\lim_{n \to \infty} \left(n \exp(-n\lambda_n^*)\right) = \tfrac{1}{2},$$

and that $\frac{1}{n} \log 2n$ provides a good approximation for λ_n^* for $n \geq 5$. The properties of the yield function $Y(L, \lambda)$ ~~is~~ are studied in detail and it is indicated that it is not a desirable practice to use fractional initial cut lengths. The yield values $Y(n, \lambda)$ have been computed for a wide range of n and λ, and tables are given. It is felt that this table will be useful in practical applications. Finally, a suitable model for the determination of optimum n is suggested.

PART II

OPTIMAL CUTTING PROCEDURES

1. INTRODUCTION

In this part we introduce and develop the concept of Group EOQ. The use of Economic Order Quantity (EOQ) square root formula in practical inventory control in fairly wide spread. In Group EOQ concept, instead of applying EOQ formula to each and every item, a common order quantity or a common frequency of order is used for all the items in the group. Obviously this will mean additional cost as compared to calculation of order quantity seperately for each item i.e., individual EOQ approach. The items are grouped as in classical ABC analysis on the basis of money value of yearly demand into one or more groups. For each group a common order quantity (either in terms of money value or in terms of number of orders per year) is calculated according to the Group EOQ rule. Using lognormal distribution to describe the distribution of money value of yearly demand, explicit expressions are obtained for the additional cost for any number of groups and given group boundaries. The optimal group boundaries which minimise the additional cost for a given number of groups are obtained. Aggregate inventory measures like total number of orders, average total inventory investment etc., are obtained for Group EOQ approach with optimal group boundaries. Finally an illustrative example of Group EOQ approach is given with the help of a case study.

In section 2, we briefly state some of the simple and well known properties of the two-parameter lognormal distribution. In section 3, we briefly review the application of lognormal distribution due to Brown ([2] and [3]) when individual EOQ approach is used.

Sections 4, 5 and the Appendix are the work done by the author on Group EOQ approach. A brief summary of contents of these sections are as follows.

Section 4 : Group EOQ Replenishment Policy

Suppose we have some group of N items and that the average money value of yearly demand for the group is $\bar{y}$. Suppose the cost of ordering is A and cost of carrying inventory expressed as rate of interest is I. A and I are assumed to be the same for all the items in the group. Suppose we decide to have a common ordering rule for all the items in a group. The common group ordering rule is either to order the same money value for all the items in the group or to order all the items in the group with the same frequency. The best common group order is then given by the group EOQ formulae :

$$\text{Common order Quantity} = Q^* = \sqrt{\frac{2A\bar{y}}{I}} \quad \ldots \quad \ldots \quad (1\text{-}1)$$

(as money value)

$$\text{Common order Quantity} = 1^* = \sqrt{\frac{I\,\bar{y}}{2A}} \quad \ldots \quad \ldots \quad (1\text{-}2)$$

(as frequency of orders)

Use of group EOQ approach will always mean additional cost as compared to individual EOQ approach.

The distribution of money value of yearly demand over the items can often be adequately described by the lognormal distribution. Let $y_o \leq y_1 \leq \ldots\ldots \leq y_{r-1} \leq y_r$ where $y_o = 0$ and $y_r = +\infty$. Consider the grouping procedure where we have r groups and the i^{th} group consists of all the items whose money value of yearly demand lies in the range y_{i-1} to y_i. We define the weighted average cost per item (over all the r groups) $G(y_1, \ldots\ldots y_{r-1})$ when the group boundaries are $y_1, y_2, \ldots\ldots y_{r-1}$ and Group EOQ approach is used for each group. Using lognormal model we derive an explicit expression for $G(y_1, \ldots\ldots y_{r-1})$. We define C_r by

$$C_r = \text{Min } G(y_1, \ldots\ldots y_{r-1}) \quad \ldots \quad \ldots \quad (1\text{-}3)$$

where the minimum is taken over all the y's satisfying $y_o \leq y_1 \ldots\ldots y_{r-1} \leq y_r$. The values of y's where the minimum of (1-3) is attained are called <u>optimal group boundaries</u>. Define C_∞ to be the average cost per item when Individual EOQ approach is used. The ratio C_r/C_∞ then gives an indication of the additional cost one has to incur when Group EOQ approach is used with optimum group boundaries. It is shown that the ratio C_r/C_∞ is only a function of r-the number of groups and σ - one of the parameters of the lognormal distribution. The constrained non-linear minimisation problem (1-3) is then studied and solved. Extensive tables giving the values of the ratio C_r/C_∞ as well optimal group boundaries are computed and given.

It is shown that as σ (one of the parameters of the lognormal distribution) tends to 0^+, a certain simple transformation of optimal group boundaries tend to some definite values. These turn out to be the optimum spacings for large sample estimate of the mean of a normal distribution (when standard deviation is known) by a selected set of (r-1) sample quantiles. The optimum spacings for the estimation problem has been worked out by Ogawa [7]. It is also shown that the optimum spacings worked out by Ogawa for the estimation problem can be used to provide a very good approximation to the optimal group boundaries. Finally expressions for aggregate measures of systems effectiveness like total number of orders per year, total inventory average investment etc., are worked out.

Section 5 : Application

In this section we discuss the applicational aspects of the theory developed in section 4. Details of various steps involved in the choice of number of groups, calculation of optimal group boundaries, calculation of Group EOQ's etc. are described. An illustrative case study is also given.

Appendix : Section 6

Higuchi [8] proved the existence and uniqueness of the optimal solution of the optimisation problem encountered by Ogawa in large sample estimation of the mean of a normal distribution (with known standard deviation) by a selected set of sample quantiles for determination of optimum spacings. The system of equations which the optimal spacings should satisfy (obtained by Ogawa) is the limiting case of the system of equations obtained for the solution of problem (1-3). We use Higuchi's approach to prove the existence and uniqueness of the optimal solution to problem (1-3). In addition we obtain a number of interesting results about the properties of the solution to problem (1-3).

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2. TWO PARAMETER LOGNORMAL DISTRIBUTION

In this section, some of the well-known results of the lognormal distribution are given [1]. We consider an essentially positive variate y $(0 < y < \infty)$ such that $\log y$ is distributed normally with mean μ and standard deviation σ. We then say that y has a lognormal distribution. Its frequency function and distribution function are denoted by $f(y|\mu, \sigma)$ and $F(y|\mu, \sigma)$ i.e.

$$f(y|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma y} \exp\left[-\frac{1}{2\sigma^2}(\log y - \mu)^2\right] \quad ; \sigma > 0$$

$$= \frac{1}{\sigma y} \phi\left(\frac{\log y - \mu}{\sigma}\right) \quad \ldots \quad \ldots \quad (2\text{-}1)$$

$$F(y|\mu,\sigma) = \int_0^y f(x|\mu,\sigma)\, dx$$

$$= \Phi\left[\frac{\log y - \mu}{\sigma}\right] \quad \ldots \quad \ldots \quad (2\text{-}2)$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \phi(t)\, dt$ ... (2-3)

This distribution is positively skewed and, the greater the value of σ, the greater the skewness. It has also positive kurtosis and again the kurtosis increases with σ.

The distribution possesses moments of all orders and the j^{th} moment about the origin is given by

$$E(y^j) = \int_0^\infty y^j f(y|\mu,\sigma)\,dy$$

$$= e^{j\mu + \frac{1}{2} j^2 \sigma^2} \quad \ldots \quad (2\text{-}4)$$

Hence the mean M is given by

$$M = e^{\mu + \sigma^2/2} \quad \ldots \quad \ldots \quad \ldots \quad (2\text{-}5)$$

If b and c are constants, where $c > 0$ (say $c = e^a$), then $c\,y^b$ is again lognormally distributed with parameters $a + b\mu$ and $|b|\sigma$. In particular, the expected value of $\sqrt{y}$ is given by

$$E(\sqrt{y}) = e^{\frac{\mu}{2} + \frac{1}{2}\left(\frac{\sigma}{2}\right)^2} = e^{\frac{\mu}{2} + \frac{\sigma^2}{8}}$$

$$= e^{\frac{\mu}{2} + \frac{\sigma^2}{4} - \frac{\sigma^2}{8}} = \sqrt{M}\; e^{-\sigma^2/8} \quad \ldots \quad (2\text{-}6)$$

The j^{th} moment distribution of a lognormal distribution with parameters μ and σ is defined to be

$$F_j(y|\mu,\sigma) = \frac{1}{E(y^j)} \int_0^y x^j f(x|\mu,\sigma)\,dx$$

$$= F(y|\mu + j\sigma^2, \sigma) \quad \ldots \quad \ldots \quad \ldots \quad (2\text{-}7)$$

i.e., the j^{th} moment distribution is again lognormal with parameters

$\mu + j\sigma^2$ and σ. It follows from (2-7) that

$$F_1(y \mid \mu, \sigma) = F(y \mid \mu + \sigma^2, \sigma)$$

$$= \Phi\left(\frac{\log y - \mu - \sigma^2}{\sigma}\right)$$

$$= \Phi\left(\frac{\log y - \mu}{\sigma} - \sigma\right) \quad \ldots \quad \ldots \qquad (2\text{-}8)$$

The Lorenz curve is a plot of $F_1(y \mid \mu, \sigma)$ against $F(y \mid \mu, \sigma)$ for different values of y. If we put $x = (\log y - \mu)/\sigma$, then the Lorenz curve is a plot of $\Phi(x-\sigma)$ against $\Phi(x)$ for different values of x. It is well known that the Lorenz curve is strictly convex and for the lognormal distribution is symmetrical about the diagonal (0,1) and (1,0) - see figure 1.

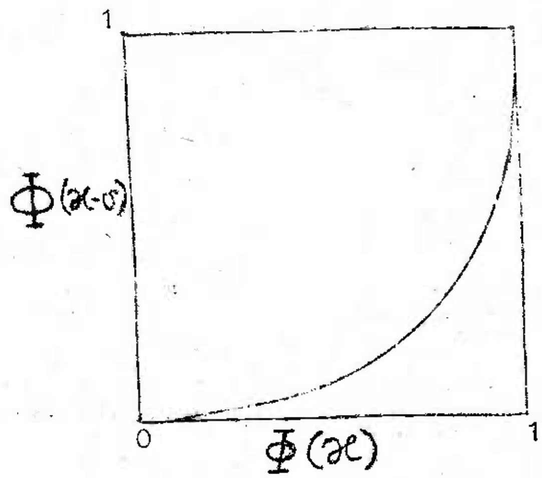

Figure 1 Lorenz curve

3. INDIVIDUAL EOQ REPLENISHMENT POLICY

In this section, we shall review the work of Brown ([2] and [3]) on the use of the lognormal distribution in getting aggregate measures like total inventory investment, total number of orders etc., when the economic order quantity (EOQ) rule is used individually in case of each item for determining the order quantity. The following notation is used.

A = cost of ordering

I = cost of carrying inventory expressed as yearly rate of interest.

D = yearly demand (for a given item)

C = unit cost of the item

y = DC = Money value of yearly demand for the item.

It is further assumed that A and I are the same for all the items in all subsequent discussions.

We consider a situation in which a fairly large number of items are carried in stock. Brown ([2] and [3]) has repeatedly pointed out that in any homogeneous inventory, the distribution of money value of yearly demand can be adequately represented by a lognormal distribution. Specifically it is assumed that the proportion of items having their money value of yearly demand in the range (y_1, y_2)

is given by $\int_{y_1}^{y_2} f(y|\mu,\sigma)\,dy$ where $f(y|\mu,\sigma)$ is the frequency function of the lognormal distribution (2-1). Under this assumption the average money value of yearly demand M over all the items is by (2-5).

$$M = e^{\mu+\sigma^2/2} \qquad \ldots \quad \ldots \quad \ldots \qquad (3\text{-}1)$$

Consider a group of items with the money value of yearly demand in the range (y_1, y_2). Then it follows from (2-2) and (2-8) that

(i) proportion of items in the group (y_1, y_2)

$$= F(y_2|\mu,\sigma) - F(y_1|\mu,\sigma)$$

$$= \Phi\left(\frac{\log y_2-\mu}{\sigma}\right) - \Phi\left(\frac{\log y_1-\mu}{\sigma}\right) \quad ..(3\text{-}2)$$

(ii) proportion of money value explained by the group (y_1, y_2)

$$= F_1(y_2|\mu,\sigma) - F_1(y_1|\mu,\sigma)$$

$$= F(y_2|\mu+\sigma^2,\sigma) - F(y_1|\mu+\sigma^2,\sigma)$$

$$= \Phi\left(\frac{\log y_2-\mu}{\sigma} - \sigma\right) - \Phi\left(\frac{\log y_1-\mu}{\sigma} - \sigma\right) \quad ..(3\text{-}3)$$

(iii) average money value of yearly demand for the group (y_1, y_2)

$$= M \frac{\Phi\left(\frac{\log y_2-\mu}{\sigma} - \sigma\right) - \Phi\left(\frac{\log y_1-\mu}{\sigma} - \sigma\right)}{\Phi\left(\frac{\log y_2-\mu}{\sigma}\right) - \Phi\left(\frac{\log y_1-\mu}{\sigma}\right)} \quad ..(3\text{-}4)$$

It follows from (3-2) and (3-3) that in ABC analysis [4] we are plotting $1 - \Phi(x-\sigma)$ against $1 - \Phi(x)$ for different values of x.

This is but a minor variation of the Lorenz curve. The extent of concentration of the total money value of yearly demand in a relatively small percentage of items depends only on σ and the concentration increases as σ increases. In practical situations σ will lie between 0.6 to 2.6 [2]. For example the percentage of total money value explained by the top 15% of the items is $100 \left[1 - \Phi(1.04 - \sigma) \right]$ and is given in the table below for different values of σ.

TABLE 3.1

σ	% money value explained by the top 15% of the items
0.6	33.0
0.8	40.5
1.0	48.4
1.2	56.4
1.4	64.1
1.6	71.2
1.8	77.6
2.0	83.2
2.2	87.7
2.4	91.3
2.6	94.1

Incidentally, table 3.1 can be used to obtain a quick and rough estimate of σ. The estimates of the parameters μ and σ can be easily obtained by considering the logarithms of money value of yearly demand for an adequate sample of items.

If y is the money value of yearly demand for a given item, then

Economic order Quantity in terms of money value $= Q^* = \sqrt{\frac{2Ay}{I}}$ ... ... (3-5)

Economic order Quantity in terms of number of orders per year $= 1^* = \sqrt{\frac{I\,y}{2A}}$ ... ... (3-6)

Minimum total cost $= \sqrt{2AIy}$ ... ... (3-7)

Even though the square-root formula ((3-5) or (3-6)) arises out of the simplest deterministic model, it has wide applicability. The order quantities calculated from this formula often provide surprisingly good approximations to the optimum order quantities of the more complicated stochastic models [5]. In practice it often suffices to calculate the order quantity using the square-root formula and use that in the calculation of safety stock [6]. If the square-root formula is used to calculate the order quantity separately for each item, we refer to such a situation as <u>Individual EOQ Replenishment Policy</u>. The word '<u>cost</u>' for any item refers to the minimum total cost for the item given by (3-7).

The average cost (per item) under the Individual EOQ Replenishment Policy is denoted by C_∞ and is given by

$$C_\infty = E(\sqrt{2AIy}) = \sqrt{2AI}\; E(\sqrt{y})$$

$$= \sqrt{2AIM}\; e^{-\sigma^2/8} \quad \ldots \quad \ldots \quad \ldots \quad (3\text{-}8)$$

We use the symbol C_∞ to denote this average cost, because order quantities are seperately calculated for each item using the square-root formula and there are a large number of items. The average money value of (working) inventory and the number of orders per year for an item with money value of yearly demand y are $\frac{1}{2}\sqrt{\frac{2Ay}{I}}$ and $\sqrt{\frac{Iy}{2A}}$ respectively. It follows from (2-6) that

(i) average (working) inventory investment per item

$$= \tfrac{1}{2}\sqrt{\frac{2A}{I}}\; E(\sqrt{y})$$

$$= \tfrac{1}{2}\sqrt{\frac{2AM}{I}}\; e^{-\sigma^2/8} \quad \ldots \quad (3\text{-}9)$$

(ii) average number of orders per year per item

$$= \sqrt{\frac{I}{2A}}\; E(\sqrt{y})$$

$$= \sqrt{\frac{IM}{2A}}\; e^{-\sigma^2/8} \quad \ldots \quad (3\text{-}10)$$

We can also get the equation for the exchange curve, a useful concept introduced by Brown. If the total number of items is N, then the total inventory investment (TI) is $\frac{N}{2}\sqrt{\frac{2AM}{I}}\; e^{-\sigma^2/8}$ and the total number of orders per year (TO) is $N\sqrt{\frac{IM}{2A}}\; e^{-\sigma^2/8}$. Hence the equation for the exchange curve for the Individual EOQ Replenishment Policy is

$$TI \times TO = \frac{N^2}{2} M e^{-\sigma^2/4} \quad \ldots \quad \ldots \quad (3\text{-}11)$$

The exchange curve indicates how one can trade off between TI and TO and helps to get an imputed value for the ratio A/I. All the results in this section are due to Brown ([2] and [3]).

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4. GROUP EOQ REPLENISHMENT POLICY

4.1 INTRODUCTION

In this section, the items are intended to be grouped as in ABC analysis [4] on the basis of money value of yearly demand. However, the number of groups small not be restricted only to three as is done in classical ABC analysis. A common group ordering rule is used for all the items in a group. This is expected to simplify the administrative aspects in purchase. The common group ordering rule considered is one of the following types.

(i) Common Money value rule :- Under this rule, we place an order for the same money value for each item in the group. For example, for ~~any~~ every item in the group an order for Rs.5000/- worth of material is always placed.

(ii) Common Frequency rule :- Under this rule, all the items in the group are ordered the same number of times a year. For example each of the item in a group is ordered 12 times a year.

The optimum group ordering rule which minimises the total of ordering cost and carrying cost over all the items in the group is determined. The optimum group ordering rule turns out to be the use of the EOQ square root formula (3-5) or (3-6) for the hypothetical item with money value of yearly demand equal to average money value of yearly demand for the group. This is referred to as <u>Group EOQ Replenishment Policy</u>. The use of Group EOQ Replenishment Policy means always more cost as compared to the use of Individual EOQ replenishment policy.

Let $y_o = 0 \leqslant y_1 \leqslant y_2 \ldots\ldots \leqslant y_{r-1} \leqslant y_r = \infty$ be arbitrarily chosen numbers in the order shown. We form r groups as follows: The i^{th} group $(i = 1 \text{ to } r)$ consists of all the items whose money values of yearly demand lie in the range y_{i-1} to y_i. The group boundaries for the i^{th} group are y_{i-1} and y_i. For each of the groups, Group EOQ replenishment policy is used. We determine the optimum group boundaries so that the total cost over all the items (over all the groups) is minimised. The percent additional cost one has to incur because of Group EOQ Replenishment Policy as against Individual EOQ Replenishment Policy is worked out for $r = 1$ to 10. It is shown that this depends only on r and σ. This provides an objective criterion for deciding on the number of groups one should have in a practical situation. It is also shown that if one desires to ensure that the additional cost due to grouping is to be less than 5% in practice, it is necessary to have at most six groups.

Let y_i $(i = 1 \text{ to } r-1)$ be the optimal group bounderies. Let

$$t_i = \frac{\log y_i - \mu}{\sigma} - \sigma/2 \quad (\text{i.e. } y_i = M e^{\sigma t_i}) \quad \text{for } i = 1 \text{ to } r-1.$$

We call t's 'transformed' optimal group bounderies. It is shown that $\lim_{\sigma \to o} t_i$ turn out to be the optimal spacings for large sample estimation of the mean of a normal distribution (with known standard deviation) by a selected set of (r-1) sample quantiles. It is further shown that these limiting values provide a good approximation for the transformed optimal group boundaries when $0 < \sigma < 3$.

Finally, aggregate inventory measures like total inventory investment, total number of orders etc., are worked out for the group EOQ Replenishment Policy.

4.2 GROUP EOQ FORMULA

Consider a group of N items and let y_j be the money value of yearly demand for the j^{th} item in the group. Let $\bar{y} = \frac{1}{N} \sum_{j=1}^{N} y_j$ be the average money value of yearly demand for the group. Suppose we use the common money value rule of ordering for this group. Let Q be the common money value of an order for this group. The total cost for an item with money value of yearly demand y_j will then be $A \frac{y_j}{Q} + \frac{1}{2} IQ$. Since Q is same for all the items in the group, the average total cost per item will be

$$\frac{1}{N} \Sigma \left(A \frac{y_j}{Q} + \frac{1}{2} IQ \right) = A \frac{\bar{y}}{Q} + \frac{1}{2} IQ \qquad \ldots \quad \ldots \qquad (4\text{-}1)$$

The minimum of (4-1) is attained when

$$Q^* = \sqrt{\frac{2A\bar{y}}{I}}, \qquad \ldots \quad \ldots \quad \ldots \qquad (4\text{-}2)$$

and its value is

$$\text{minimum average total cost per item} = \sqrt{2AI\bar{y}} \quad \ldots \quad \ldots \quad (4\text{-}3)$$

Similarly it can be shown that optimum common frequency of orders per year is given by

$$l^{*} = \sqrt{\frac{I\bar{y}}{2A}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}4)$$

and again the minimum average total cost per item is given by (4-3). Thus in either case the best common ordering rule for the group is found by calculating the EOQ in terms of money value or number of orders per year for the hypothetical item with money value of yearly demand equal to $\bar{y}$. If we use either (4-2) or (4-4) to calculate the EOQ of the group, we refer to the situation as Group EOQ Replenishment Policy. The words "average cost" per item when Group EOQ Replenishment policy is used is always taken to mean $\sqrt{2AI\bar{y}}$ (i.e. (4-3)). Group EOQ Replenishment policy will result in additional cost as compared to Individual EOQ Replenishment policy, since the inequality

$$\sqrt{2AI\bar{y}} > \frac{I}{N} \sum_{j=1}^{N} \sqrt{2AIy_j} \quad \ldots \quad \ldots \quad (4\text{-}5)$$

holds good provided that $y_i \neq y_j$ for at least one i and j.

4.3 OPTIMAL GROUP BOUNDARIES

Let $y_o = 0$ and $y_r = \infty$ and $y_1, y_2, \ldots\ldots y_{r-1}$ be such that $y_o \leq y_1 \leq y_2 \ldots\ldots \leq y_{r-1} \leq y_r$. We define the i[th] group to consist of all items which have their money value of yearly demand in the range $[y_{i-1}, y_i)$. Thus we have (r-1) group boundaries and r groups. Let p_i and m_i be the proportion of items and the average money value of yearly demand respectively for the i[th] group. If we put

$$t_i = \frac{\log y_i - \mu}{\sigma} - \sigma/2 \quad \text{or} \quad y_i = e^{\mu + \sigma^2/2} e^{\sigma t_i} = M e^{\sigma t_i}, \quad \ldots (4\text{-}6)$$

We have $-\infty = t_o \leq t_1 \ldots\ldots \leq t_{r-1} \leq t_r = +\infty$. Further, from (3-2) and (3-4) it follows that

$$p_i = \Phi(t_i + \sigma/2) - \Phi(t_{i-1} + \sigma/2) \quad \ldots \quad \ldots \quad (4\text{-}7)$$

$$m_i = M \frac{\Phi(t_i - \sigma/2) - \Phi(t_{i-1} - \sigma/2)}{\Phi(t_i + \sigma/2) - \Phi(t_{i-1} + \sigma/2)} \quad \ldots \quad (4\text{-}8)$$

If we use Group EOQ Replenishment policy for each of the r groups, then the optimum common group ordering rule is given by either (4-2) or (4-4) and the average cost per item for the i[th] group will be

$\sqrt{2AIm_i}$ (by (4-3)). For $r \geq 2$ define $G(y_1, y_2, \ldots y_{r-1})$ by

$$G(y_1, y_2, \ldots y_{r-1}) = \sqrt{2AI} \sum_{i=1}^{r} p_i \sqrt{m_i} \quad \ldots \quad (4\text{-}9)$$

Hence $G(y_1, y_2, \ldots y_{r-1})$ is the weighted average cost per item (over all the r groups) when Group EOQ Replenishment policy is used for each group and the group boundaries are specified to be $y_1, y_2, \ldots \ldots y_{r-1}$. Further define C_r by

$$C_1 = \sqrt{2AIM} \quad \text{for } r = 1, \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}10)$$

$$\text{and} \quad C_r = \text{Min } G(y_1, y_2, \ldots y_{r-1}) \quad \text{for } r \geq 2, \quad \ldots (4\text{-}11)$$

where the minimum is taken over all y's satisfying $0 = y_0 \leq y_1 \leq y_2 \ldots \ldots y_{r-1} \leq y_r = +\infty$. C_r gives the minimum average cost per item under Group EOQ Replenishment Policy when there are r groups and the group boundaries are chosen in an optimal manner. By using (4-7) and (4-8) we get

$$G(y_1, y_2, \ldots y_{r-1}) = \sqrt{2AIM} \sum_{i=1}^{r} \Big[\left\{ \Phi(t_i - \sigma/2) - \Phi(t_{i-1} - \sigma/2) \right\} \left\{ \Phi(t_i + \sigma/2) - \Phi(t_{i-1} + \sigma/2) \right\} \Big]^{\frac{1}{2}}$$

$$= \sqrt{2AIM}\ H(t_1, t_2, \ldots t_{r-1}), \text{ say}, \quad \ldots \quad (4\text{-}12)$$

where the y's and t's are related by (4-6). By making use of (3-8) we get

$$C_1/C_\infty = e^{\sigma^2/8} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}13)$$

and for $r \geq 2$

$$C_r/C_\infty = e^{\sigma^2/8} \text{ Min } H(t_1, t_2, \ldots\ldots t_{r-1}) \quad \ldots \quad \ldots \quad (4\text{-}14)$$

where the minimum is taken over all the t's satisfying

$$-\infty = t_o \leq t_1 \ldots\ldots \leq t_{r-1} \leq t_r = +\infty \quad \ldots \quad \ldots \quad (4\text{-}15)$$

It is noted that the ratio C_r/C_∞ depends only on r and σ. Further $(\frac{C_r}{C_\infty} - 1)\, 100$ gives the percent extra cost one has to incur when group EOQ Replenishment policy is used with optimal choice of group boundaries as against Individual EOQ Replenishment policy. Hence the ratio $\frac{C_r}{C_\infty}$ provides a criterion for deciding on the number of groups in a practical situation.

The problem of finding the optimal group boundaries is thus equivalent to the minimisation problem :

$$\text{Minimise } H(t_1, t_2, \ldots\ldots t_{r-1}) \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}16)$$

$$\text{subject to } -\infty = t_o \leq t_1 \ldots\ldots \leq t_{r-1} \leq t_r = +\infty$$

where $H(t_1, t_2, \ldots\ldots t_{r-1})$ is given by

$$H(t_1, t_2, \ldots t_{r-1}) = \sum_{i=1}^{r} \left[\left\{ \Phi\left(t_i - \frac{\sigma}{2}\right) - \Phi\left(t_{i-1} - \frac{\sigma}{2}\right) \right\} \left\{ \Phi\left(t_i + \frac{\sigma}{2}\right) - \Phi\left(t_{i-1} + \frac{\sigma}{2}\right) \right\} \right]^{\frac{1}{2}} \quad \ldots \quad (4\text{-}17)$$

If we put $t'_i = -t_{r-i}$ for $i = 1$ to r, we note that $-\infty = t'_0 \leq t'_1 \ldots\ldots\ldots \leq t'_{r-1} \leq t'_r = +\infty$. By making use of the fact $\Phi(-x) = 1 - \Phi(x)$, we can easily show that the function $H(t_1, t_2, \ldots\ldots t_{r-1})$ has the following property of symmetry.

$$H(t_1, t_2, \ldots\ldots t_{r-1}) = H(t'_1, t'_2, \ldots\ldots t'_{r-1})$$

$$= H(-t_{r-1}, -t_{r-2}, \ldots -t_2, -t_1) \quad \ldots \quad (4\text{-}18)$$

We refer to solution of problem (4-16) as transformed optimal group boundaries. Theorem 1 characterises the property of optimal solution.

<u>THEOREM 1</u> A necessary condition for the minimum of problem (4-16) to occur at $(t_1, t_2, \ldots\ldots t_{r-1})$ is that the t's should satisfy the system of equations

$$2\sigma t_i - \log \theta_i - \log \theta_{i+1} = 0 \quad \text{for } i = 1 \text{ to } r-1 \quad \ldots \quad (4\text{-}19)$$

$$\text{where } \theta_i = \frac{\Phi\left(t_i - \frac{\sigma}{2}\right) - \Phi\left(t_{i-1} - \frac{\sigma}{2}\right)}{\Phi\left(t_i + \frac{\sigma}{2}\right) - \Phi\left(t_{i-1} + \frac{\sigma}{2}\right)} \quad \text{for } i = 1 \text{ to } r \quad \ldots \quad (4\text{-}20)$$

<u>Proof</u> :- It is seen from lemma 9 of the Appendix that the minimum to the problem (4-16) can only occur at a point where the constraints are satisfied as strict inequalities i.e., $-\infty = t_o < t_1 < t_2 \ldots\ldots < t_{r-1} < t_r = +\infty$. As such the necessary condition is $\frac{\partial H}{\partial t_i} = 0$ for $i = 1$ to $r-1$. By noting that $\frac{d\Phi(x)}{dx} = \phi(x)$, we have

$$2\frac{\partial H}{\partial t_i} = \left(\frac{1}{\sqrt{\theta_i}} - \frac{1}{\sqrt{\theta_{i+1}}}\right)\phi\left(t_i - \frac{\sigma}{2}\right) + \left(\sqrt{\theta_i} - \sqrt{\theta_{i+1}}\right)\phi\left(t_i + \frac{\sigma}{2}\right) = 0$$

for $i = 1$ to $r-1$ ... (4-21)

Since $t_{i-1} < t_i < t_{i+1}$, it is seen from lemma 4 of the Appendix that $\theta_{i+1} > \theta_i$. Hence (4-21) reduces to

$$\frac{\phi\left(t_i - \frac{\sigma}{2}\right)}{\phi\left(t_i + \frac{\sigma}{2}\right)} = \sqrt{\theta_i\,\theta_{i+1}}$$

or $$e^{\sigma t_i} = \sqrt{\theta_i\,\theta_{i+1}} \quad \ldots \ldots \ldots (4\text{-}22)$$

or $$2\sigma t_i - \log\theta_i - \log\theta_{i+1} = 0 \quad \ldots (4\text{-}23)$$

There is a unique solution to the system of equations (4-19) and it is the global minimum of problem (4-16). The proof of this is rather involved and long and as such it is postponed to the Appendix. Only the main results are given in this section.

<u>THEOREM 2</u> The system of equations (4-19) has one and only one solution of real numbers and it satisfies the constraints of problem (4-16) with strict inequalities. Moreover this unique solution is the minimum point of problem (4-16).

<u>Proof</u> See Theorems 5 and 6 of the Appendix.

The following theorem characterises the symmetrical properties of the unique optimal solution of problem (4-16).

<u>THEOREM 3</u> Let $(t_1, t_2, \ldots\ldots t_{r-1})$ be the optimal solution of problem (4-16). Then it satisfies the following symmetrical properties

$$\text{(i)} \quad t_i = -t_{r-i} \quad \text{and} \quad \sum_{i=1}^{r-1} t_i = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(4-24)}$$

$$\text{(ii)} \quad \Phi(t_i - \tfrac{\sigma}{2}) - \Phi(t_{i-1} - \tfrac{\sigma}{2}) = \Phi(t_{r-i+1} + \tfrac{\sigma}{2}) - \Phi(t_{r-i} + \tfrac{\sigma}{2}) \quad \ldots \qquad \text{(4-25)}$$

$$\text{(iii)} \quad \Phi(t_i + \tfrac{\sigma}{2}) - \Phi(t_{i-1} + \tfrac{\sigma}{2}) = \Phi(t_{r-i+1} - \tfrac{\sigma}{2}) - \Phi(t_{r-i} - \tfrac{\sigma}{2}) \quad \ldots \qquad \text{(4-26)}$$

$$\text{(iv)} \quad \theta_i = \frac{1}{\theta_{r-i+1}}$$

for $i = 1$ to r.

<u>Proof</u> If $t_1, t_2, \ldots\ldots t_{r-1}$ is an optimal solution of problem (4-16), then $-t_{r-1}, -t_{r-2}, \ldots\ldots -t_2, -t_1$ is also an optimal solution to problem (4-16), because of (4-18). Because of uniqueness of the

optimal solution, (i) follows. If we use the fact $\Phi(-x) = 1 - \Phi(x)$, (ii) and (iii) follow from (i). It is easily seen that (iv) follows from (ii) and (iii).

We now make use of Theorems 1, 2 and 3 to characterise the properties of the optimal solution of problem (4-11) i.e., the optimal group boundaries.

<u>THEOREM 4</u> Problem (4-11) has a unique optimal solution : $(y_1, y_2, \ldots\ldots y_{r-1})$. We refer to y_i (i = 1 to r-1) as <u>optimal group boundaries</u>. Further the optimal group boundaries satisfy the following properties

(i) $0 = y_o < y_1 < y_2 \ldots\ldots < y_{r-1} < y_r = +\infty$

(ii) $y_i = \sqrt{m_i \, m_{i+1}}$

(iii) $\prod_{i=1}^{r-1} y_i = M^{r-1}$

(iv) The percentage of items in the i^{th} group is equal to the percent money value of yearly demand explained by the $(r-i+1)^{th}$ group, and the percent money value of yearly demand explained by the i^{th} group is equal to the percentage of items in the $(r-i+1)^{th}$ group.

<u>Proof</u> Let $(t_1, t_2, \ldots\ldots t_{r-1})$ be the unique optimal solution of problem (4-16). Then by (4-6) we have the unique optimal solution

to problem (4-11) to be

$$y_i = M e^{\sigma t_i} \quad \ldots \quad \ldots \quad \ldots \qquad (4\text{-}27)$$

We get (i) by the fact $-\infty = t_o < t_1 < t_2 \ldots\ldots t_{r-1} < t_r = +\infty$. We get (ii) by multiplying both the sides of (4-22) by M and making use of (4-8). We get (iii) from (4-27) and (4-24). Finally we get (iv) by using (3-2), (3-3), (4-25) and (4-26).

4.4 COMPUTATION OF THE RATIO C_r/C_∞

Instead of solving (4-19) numerically, the following approach was used. The symmetrical property $t_i = -t_{r-i}$ of the optimal solution of problem (4-16) was used and a direct search was carried out on the computer to find the minimum of the function $H(t_1, t_2, \ldots t_{r-1})$ and the corresponding optimal values of $t_1, t_2, \ldots t_{r-1}$. The following approximation due to Hastings [9] for the normal integral was used in the computer runs.

$$\text{Let } \alpha(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \ldots \quad \ldots \quad \ldots \qquad (4\text{-}28)$$

Then for $x \geq 0$

$$\alpha(x) \simeq 1 - \frac{1}{(1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6)^{16}} \quad \ldots \quad (4\text{-}29)$$

where a_1 = 0.0705, 2304, 84 ; a_2 = 0.0422, 8201, 23

a_3 = 0.0092, 7052, 72 ; a_4 = 0.0001, 5201, 43

a_5 = 0.0002, 7652, 72 ; a_6 = 0.0000, 4306, 38

The maximum possible error in this approximation is $\pm$ 0.0000003

For $r = 3$ to 10, $\sigma = 0.2\ (0.2)\ 4.0$, the search for the minimum of $H\ (t_1, t_2, \ldots\ t_{r-1})$ was carried out to the nearest 3rd place of decimal for the t's. However, in the case of $r = 2$, we know that the minimum of $H(t_1)$ occurs when $t_1 = 0$ and

$$\frac{C_2}{C_\infty} = 2\ e^{\sigma^2/8} \left[\Phi(\sigma/2)\ \Phi(-\sigma/2) \right]^{\frac{1}{2}} \quad \ldots \qquad (4\text{-}30)$$

Tables I to X at the end of this part give the results of numerical evaluation. Table I is an extract of Tables II to X and gives the values of the ratio C_r/C_∞ for $r = 1$ to 7 and for $\sigma = 0.4\ (0.2)\ 3.0$. The range of σ in practical situations is 0.6 to 2.6. Table II gives the values of C_1/C_∞ and C_2/C_∞ for $\sigma = 0.2\ (0.2)\ 4.0$. Tables III to X give the values of C_r/C_∞ and the optimal values of $t_1, t_2, \ldots\ t_{r-1}$ for the same range of σ as in Table II. The optimal group boundaries can be easily calculated by making use of transformed optimal group boundaries given in Tables III to X and the formula

$$y_i = M\ e^{t_i \sigma} \qquad i = 1 \text{ to } r-1 \quad \ldots \quad \ldots \qquad (4\text{-}31)$$

It is easily seen, that for $r = 2$, the optimal group boundary is $y_1 = M$. A careful look at tables 1 to X reveals, that

(i) We need utmost six groups in practice if we want to keep the additional cost due to grouping less than 5%.

(ii) The transformed optimal group boundaries change very slowly with σ in the range studied.

(iii) As $\sigma \longrightarrow 0$, it appears that the transformed optimal group boundaries tend to some definite values.

We shall now briefly investigate the limiting nature of transformed optimal group boundaries as $\sigma \longrightarrow 0$. A detailed study is postponed to the appendix. For $\sigma > 0$, the system of equations (4-19) is equivalent to

$$2\, t_i - \frac{1}{\sigma} \log \theta_i - \frac{1}{\sigma} \log \theta_{i+1} = 0 \quad \ldots \quad \ldots \quad (4\text{-}32)$$

for $i = 1$ to $r-1$. Further, we have,

$$\lim_{\sigma \longrightarrow 0^+} \theta_i = 1 \text{ for } i = 1 \text{ to } r \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}33)$$

$$\lim_{\sigma \longrightarrow 0^+} \frac{1}{\sigma} \log \theta_i = - \frac{\phi(t_i) - \phi(t_{i-1})}{\Phi(t_i) - \Phi(t_{i-1})} \quad \ldots \quad (4\text{-}34)$$

Hence the limiting form of (4-32) as $\sigma \longrightarrow 0^+$ is

$$2\, t_i + \frac{\phi(t_i) - \phi(t_{i-1})}{\Phi(t_i) - \Phi(t_{i-1})} + \frac{\phi(t_{i+1}) - \phi(t_i)}{\Phi(t_{i+1}) - \Phi(t_i)} = 0 \quad \ldots \quad (4\text{-}35)$$

In the determination of optimum spacings for large sample estimation of the mean of a normal population (with known standard deviation) by a selected set of k sample quantiles, Ogawa [7] obtained the same system of equations as (4-35). Of course in our notation r-1 = k. He also noted that if $-\infty < t_1 < t_2 \; \ldots\ldots < t_k < \infty$ is a solution to (4-35), then $-\infty < -t_k < -t_{k-1} \; \ldots\ldots < -t_1 < \infty$ is also a solution to (4-35). Higuchi [8] proved that the system (4-35) has one and only one solution and that this solution satisfies the constraints of problem (4-16) with strict inequalities. Ogawa [7] solved the system of equations (4-35) numerically and his results are given in Table XI at the end of this part. We refer to these as Ogawa's optimum spacings.

We shall now investigate how good an approximation the solution to (4-35) is for the solution of the system (4-19). Numerical evaluation of the function H at the optimal point and also at Ogawa's optimum spacings were done for r = 3 to 10 and $\sigma = 0.2$ to 4.0. It was observed that the approximation was very good for $\sigma \leq 3.0$. We denote by C'_r the average cost per item for the case of r groups when Group EOQ Replenishment Policy is used and the group boundaries are based on Ogawa's optimum spacings given in Table XI. For example for r = 4, we get from Table XI, $t_1 = -0.982$, $t_2 = 0$, $t_3 = 0.982$.

By (4-6) we get the nearly optimum group boundaries to be

$$y_1 = M e^{-0.982\sigma}, \quad y_2 = M \text{ and } y_3 = M e^{0.982\sigma}$$

and

$$C'_4 = C(y_1, y_2, y_3)$$

$$= \sqrt{2AIM} \quad H(-0.982, 0, 0.982)$$

$$\frac{C'_4}{C_\infty} = e^{\sigma^2/8} \cdot \quad H(-0.982, 0, 0.982)$$

The values of the ratio $\frac{C'_r}{C_\infty}$ for $r = 3$ to 10 and $\sigma = 0.2(0.2)4.0$ were computed and are given in Tables III to X. It is seen for $\sigma \leq 3.0$, that C'_r/C_∞ is practically same as C_r/C_∞. Since in practice $\sigma < 3.0$, we can safely use Ogawa's optimum spacings for calculating the nearly optimum group boundaries.

The following steps are used in obtaining the nearly optimum group boundaries based on Ogawa's optimal spacings.

(i) For $r = 2$, the optimum group boundary is $y_1 = M$ for all

(ii) For $r \geq 3$, read from table XI, the values $t_1, t_2, \ldots$ $\ldots t_{r-1}$.

(iii) The nearly optimum group boundaries are calculated as

$$y_1 = M e^{\sigma t_1}, \quad y_2 = M e^{\sigma t_2} \ldots y_{r-1} = M e^{\sigma t_{r-1}} \quad \ldots (4\text{-}36)$$

4.5 AGGREGATE MEASURES

We now derive expressions for aggregate measures of system effectiveness when Group EOQ Replenishment Policy is used with optimal group boundaries. The common money value of an order for the i^{th} group is by (4-2)

$$Q_i^* = \sqrt{\frac{2Am_i}{I}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}37)$$

Hence the average inventory investment per item for the i^{th} group will be $\frac{1}{2}\sqrt{\frac{2A}{I}} \sqrt{m_i}$. Hence the average inventory investment per item over all the groups is given by $\frac{1}{2}\sqrt{\frac{2A}{I}} \Sigma\, p_i \sqrt{m_i}$. If N is total number of items in all the groups, then the average total inventory investment (TI) is given by

$$TI = \frac{N}{2}\sqrt{\frac{2A}{I}} \sum_{i=1}^{r} p_i \sqrt{m_i} \quad \ldots \quad \ldots \quad (4\text{-}38)$$

$$= \frac{N}{2} \frac{C_r}{C_1} \sqrt{\frac{2AM}{I}} \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}39)$$

We get (4-39) from (4-38) and the definition of C_r and by recalling $C_1 = \sqrt{2AIM}$. Similarly we get the total number of orders per year (TO) over all the groups to be

$$TO = N \frac{C_r}{C_1} \sqrt{\frac{IM}{2A}} \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}40)$$

The equation to the exchange curve (3-11) under Group EOQ Replenishment Policy will be

$$TI \times TO = \frac{N^2}{2} \left(\frac{C_r}{C_1} \right)^2 M \quad \ldots \quad \ldots \quad \ldots \quad (4\text{-}41)$$

$$= \frac{N^2}{2} \left(\frac{C_r}{C_\infty} \right)^2 \left(\frac{C_\infty}{C_1} \right)^2 M$$

$$= \frac{N^2}{2} \left(\frac{C_r}{C_\infty} \right)^2 e^{-\sigma^2/4} M \quad \ldots \quad \ldots \quad (4\text{-}42)$$

since $\frac{C_1}{C_\infty} = e^{\sigma^2/8}$. We note that (4-42) is same as (3-11) except for the multiplying factor $\left(\frac{C_r}{C_\infty} \right)^2$.

5. APPLICATION

In this section we discuss the applicational aspects of the Group EOQ approach developed in section 4. This approach has been successfully used in many industrial organisations and super-markets. The validity of the lognormal model for distribution of money value of yearly demand can be easily verified by plotting the cumulative percentage of items against money value of yearly demand on a logarithmic probability paper. A straight line fit indicates the validity of the assumption. Occassionally such a plot indicates a curvature at one or both the ends. This means that more than one type of inventory is included in the group of items considered and seperation into homogeneous groups generally gives good straight line fits. The basic quantity required for the use of Group EOQ approach is σ - one of the parameters of the lognormal distribution. Brown [2] uses $\rho = e^{\sigma}$ as an index of the concentration of money value in a relatively small percentage of items and calls 'ρ' the standard ratio. However we prefer to use σ itself an index of extent of concentration. The value of σ encountered in practice is usually in the range 0.6 to 2.6. In case of retail inventories like large departmental stores etc., σ is found to be in the range of 0.6 to 1.2 and in case of industrial organisations σ is found to be in the range 1.2 to 2.6.

The average money value of yearly demand 'M' can be very easily estimated by dividing the total money value of yearly demand by the total number of items. The estimate of σ - the index of concentration can be obtained from the logarithmic normal probability plot. The best fit straight line through the points plotted is drawn by eye. Let Y_{16} and Y_{50} be the money values of yearly demand corresponding to 16% and 50% points respectively of the probability plot. Then the estimate of σ is given by

$$\sigma = \log_e \frac{Y_{50}}{Y_{16}} = 2.3026 \, (\log_{10} Y_{50} - \log_{10} Y_{16}).$$

Table I can be used to decide upon the number of groups once σ is estimated. The values of the ratio C_r/C_∞ for that particular σ give an indication the extent of additional cost one has to incur if Group EOQ approach is used with r groups as against individual EOQ approach. Actually the percent extra cost is given by $100 \, (\frac{C_r}{C_\infty} - 1)$. The number of groups 'r' is so chosen such that the additional cost due to Group EOQ approach is reasonable, say less than 5%. Once the number of groups 'r' is decided, the nearly optimal group boundaries are obtained with the help Table XI. Table XI gives Ogawa's optimal spacings. For the value of 'r' chosen we read

Ogawa's optimal spacings $t_1, t_2, \ldots\ldots\ t_{r-1}$

from Table XI. Then the nearly optimal group boundaries are given by

$$y_1 = M e^{t_1 \sigma}, \quad y_2 = M e^{t_2 \sigma}, \quad \ldots\ldots\ y_{r-1} = M e^{t_{r-1} \sigma}.$$

All the calculations needed for Group EOQ replenishment approach are now illustrated with the help of a case-study.

This case study refers to an automobile manufacturing unit and the category of items considered were 'Ready-Purchase Local Parts'. There were 2032 items and their total money value of yearly demand was about Rs.89,42,000. The cumulative percent of items (having their money value of yearly demand less than or equal to a particular value) was plotted against money value of yearly demand on a logarithmic probability paper (see figure 1). As the plotted points lie very close to a straight-line, the lognormal model could be adopted. The 16th and 50th percentile points are 70 and 500. Hence the estimate of σ was, $\sigma = \log_e 500 - \log_e 70 = 2.0$. The average money value of yearly demand M for this category was $M = \frac{89,42,000}{2032} = 4400$.

For $\sigma = 2.0$ we get from Table I, $\frac{C_4}{C_\infty} = 1.0625$, $\frac{C_5}{C_\infty} = 1.0420$ and $\frac{C_6}{C_\infty} = 1.0303$. Since it involved only 4.2% additional cost, the management felt that five groups were adequate. From Table XI we get Ogawa's optimal spacings (rounded off to two places of decimal) to be

$$t_1 = -1.24, \quad t_2 = -0.38, \quad t_3 = 0.38 \text{ and } t_4 = 1.24$$

Hence the nearly optimum group bounderies were

$$y_1 = M e^{t_1 \sigma} = 4400 \; e^{-1.24 \times 2.0} = 368$$

$$y_2 = M e^{t_2 \sigma} = 4400 \; e^{-0.38 \times 2.0} = 2058$$

$$y_3 = M e^{t_3 \sigma} = 4400 \; e^{0.38 \times 2.0} = 9408$$

$$y_4 = M e^{t_4 \sigma} = 4400 \; e^{1.24 \times 2.0} = 52542$$

The proportion of items and proportion of money value of yearly demand for the ith group ($i = 1$ to 5) is given by $\Phi(t_i + \sigma/2) - \Phi(t_{i-1} + \sigma/2)$ and $\Phi(t_{r-i+1} + \sigma/2) - \Phi(t_{r-i} + \sigma/2)$ respectively, where $t_o = -\infty$ and $t_5 = +\infty$. Hence we get from the Normal probability integral table.

Group No. (i)	Group Boundaries (ii)	%items (iii)	% value (iv)	Avg. Money value of yearly demand (m_i) (v)= $4400 \frac{(iv)}{(iii)}$	$\sqrt{m_i}$
I	$y < 368$	40.52	1.25	135	11.62
II	$368 \leq y < 2058$	33.22	7.13	944	30.72
III	$2058 \leq y < 9408$	17.88	17.88	4400	65.32
IV	$9408 \leq y < 52542$	7.13	33.22	20500	140.32
V	$y \geq 52542$	1.25	40.52	142630	377.66
		100.00	100.00		

The cost of ordering (A) was Rs.20/- per order and the cost of carrying the inventory (I) expressed as yearly rate of interest was 0.15 for this category of items. Hence the EOQ constant (K) was

$$K = \sqrt{\frac{2A}{I}} = \sqrt{266.7} = 16.33$$

The group ordering quantities were therefore as follows :

Group (i)	Average money value of yearly demand (m_i) (ii)	order Qtty.in Rupees (Q^*) (iii) = $K\sqrt{m_i}$	order Qtty. in number of orders per year (1^*) (iv) = $\sqrt{m_i}/K$
I	135	190	0.7
II	944	502	1.9
III	4400	1067	4.0
IV	20500	2291	8.6
V	142630	6167	23.2

Finally, we have for the aggregate inventory measures, (by (4-39) and (4-40)).

Total no. of orders per year (TO) $= \frac{C_5}{C_\infty} \frac{C_\infty}{C_1} N \times \frac{\sqrt{M}}{K}$

$$= \frac{1.0416}{1.6487} \times 2032 \times \frac{\sqrt{4400}}{16.33}$$

$$= \frac{83855}{16.33} = 5135$$

Total average inventory investment (TI) $= \frac{C_5}{C_\infty} \cdot \frac{C_\infty}{C_1} \times \frac{N}{2} \times K\sqrt{M}$

$$= \frac{83855 \times 16.33}{2} = 6{,}85{,}095 \text{ (Rupees)}.$$

6. APPENDIX

6.1 INTRODUCTION

In this appendix, a detailed study of some of the mathematical aspects of Group EOQ Replenishment Policy are undertaken. The main result proved here is that the minimum occurs at unique point in problem (4-16). In addition, a number of interesting results about the properties of the solution of the system of equations (4-19) are also obtained. In sections 6.3 and 6.4 we prove certain inequalities and related results. These are used in section 6.5 to study the Group EOQ Replenishment policy. A brief summary of the contents of sections 6.3, 6.4 and 6.5 is as follows.

Section 6.3

Let x and y be truncation points of the standard normal distribution. In theorem 1, we prove that the mean of the truncated distribution has the same sign as that of x+y. In theorem 2, we prove that all the odd moments about the mean of the truncated distribution has the same sign as the mean. Theorem 2 is a generalisation of a result to due to Samford [10]. In theorem 3, we show that if the normal distribution is truncated at two points which are at a fixed distance apart, the variance of the truncated distribution is maximum when the truncation points are placed symmetrically about the mean.

Section 6.4

The results of section 6.3 are used to establish some important inequalities and related results. The main result of this section is theorem 4, where we establish certain inequalities which the partial derivatives of the function $\log [(\Phi(y-\sigma/2) - \Phi(x-\sigma/2))/(\Phi(y+\sigma/2) - \Phi(x+\sigma/2))]$ satisfy. Lemmas 10 and 11 which are later used in section 6.5 are due to Higuchi [8].

Section 6.5

In this section, various theoretical aspects of the Group EOQ Replenishment Policy are studied in some detail. We call the optimal solution of problem (4-16) as transformed optimal group boundaries as they are related to optimal group boundaries (i.e. optimal solution of problem (4-11) by a simple transformation (i.e. (4-6)). In theorem 5 we show that the system of equations (4-19) has a unique solution. Theorem 6 shows that the minimum of problem (4-16) is attained at this point. In theorems 10 and 11 we characterise a certain important property of the solution of (4-19). In theorem 12, we show that transformed optimal group boundaries which are positive (negative) strictly increase (decrease) with σ. It was seen in section 4 that the limiting form of the system of equations (4-19) as $\sigma \longrightarrow 0^+$ is (4-35). The system of equations (4-35) was obtained by Ogawa [7] in determination of optimal spacings in an estimation problem. We call the solution of the system of equations (4-35) as Ogawa's optimal spacings. In theorem 13, we establish a certain relationship between transformed optimal group boundaries and Ogawa's optimal spacings. In theorem 14, we prove that transformed optimal group boundaries actually tend to Ogawa's optimal spacings as $\sigma \longrightarrow 0^+$.

6.2 NOTATION

For easy reference, we list various functions which have either been defined earlier, or will be defined later in this appendix.

(1) $$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(2) $$\Phi(x) = \int_{-\infty}^{x} \phi(t)\, dt$$

(3) $$\lambda(x, y) = -\frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)} = k_1(x, y)$$

(4) $$\mu_2(x,y) = 1 - \frac{y\,\phi(y) - x\,\phi(x)}{\Phi(y) - \Phi(x)} - \left(\frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)}\right)^2$$

$$= k_2(x, y)$$

(5) $$\mu_3(x,y) = -\frac{y^2\,\phi(y) - x^2\,\phi(x)}{\Phi(y) - \Phi(x)} + \frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)}$$

$$- 3\,\frac{y\,\phi(y) - x\,\phi(x)}{\Phi(y) - \Phi(x)}\,\frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)} - 2\left(\frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)}\right)^3$$

$$= k_3(x, y)$$

(6) $$\theta(x,y|\sigma) = \frac{\Phi(y-\sigma/2) - \Phi(x-\sigma/2)}{\Phi(y+\sigma/2) - \Phi(x+\sigma/2)}$$

(7) $$W(x,y|\sigma) = \sigma y - \log \theta(x,y|\sigma)$$

(8) $$\Omega(x,y) = y - \lambda(x,y)$$

(9) $$h(x,y|\sigma) = \begin{cases} \frac{1}{\sigma} \log \theta(x,y|\sigma) & \text{when } \sigma \neq 0 \\ \lambda(x,y) & \text{when } \sigma = 0 \end{cases}$$

(10) $$w(x,y|\sigma) = \begin{cases} \frac{1}{\sigma} W(x,y|\sigma) & \text{when } \sigma \neq 0 \\ \Omega(x,y) & \text{when } \sigma = 0 \end{cases}$$

6.3 TRUNCATED NORMAL DISTRIBUTION

Let x and y be such that $-\infty \leq x \leq y \leq \infty$ and at least one of them is finite. We consider x and y as the truncation points of the standard normal distribution. When $-\infty < x < y < \infty$ we have double truncation and when $x = -\infty$ or $y = \infty$ we have single truncation. The probability density function of the truncated distribution is

$$g(u) = \frac{\phi(u)}{\Phi(y) - \Phi(x)} \quad \text{for } x < u < y \qquad \ldots \qquad \ldots (6\text{-}1)$$

$$= 0 \quad \text{otherwise}$$

We denote the mean of the truncated distribution by $\lambda(x,y)$, the r^{th} moment about the mean by $\mu_r(x,y)$, and the r^{th} cumulant by $k_r(x,y)$. To avoid discussing separately the cases where $x = -\infty$ or $y = \infty$, we take $\Phi(\infty) = 1$, $\Phi(-\infty) = 0$ and also $u^r \phi(u) = 0$ when $u = \pm\infty$.

The moment generating function for the truncated distribution is given by

$$\frac{\int_x^y e^{tu}\,\phi(u)\,du}{\Phi(y) - \Phi(x)} = e^{t^2/2}\,\frac{\Phi(y-t) - \Phi(x-t)}{\Phi(y) - \Phi(x)} \quad \ldots \quad (6\text{-}2)$$

Hence $G(t|x,y)$, the generating function of the cumulants is given by

$$G(t|x,y) = \frac{t^2}{2} + \log\left[\Phi(y-t) - \Phi(x-t)\right] - \log\left[\Phi(y) - \Phi(x)\right] \ldots (6\text{-}3)$$

We then have

$$\frac{d\,G(t|x,y)}{dt} = t - \frac{\phi(y-t) - \phi(x-t)}{\Phi(y-t) - \Phi(x-t)} \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}4)$$

$$\frac{d^2G(t|x,y)}{dt^2} = 1 - \frac{(y-t)\,\phi(y-t) - (x-t)\,\phi(x-t)}{\Phi(y-t) - \Phi(x-t)} - \left(\frac{\phi(y-t) - \phi(x-t)}{\Phi(y-t) - \Phi(x-t)}\right)^2$$

$$\ldots \quad \ldots \quad (6\text{-}5)$$

$$\frac{d^3G(t|x,y)}{dt^3} = - \frac{(y-t)^2\,\phi(y-t) - (x-t)^2\,\phi(x-t)}{\Phi(y-t) - \Phi(x-t)} + \frac{\phi(y-t) - \phi(x-t)}{\Phi(y-t) - \Phi(x-t)}$$

$$- 3\,\frac{(y-t)\,\phi(y-t) - (x-t)\,\phi(x-t)}{\Phi(y-t) - \Phi(x-t)}\cdot\frac{\phi(y-t) - \phi(x-t)}{\Phi(y-t) - \Phi(x-t)}$$

$$- 2\left(\frac{\phi(y-t) - \phi(x-t)}{\Phi(y-t) - \Phi(x-t)}\right)^3 \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}6)$$

Hence it follows that

$$k_1(x,y) = \lambda(x,y) = - \frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)} \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}7)$$

$$\frac{d\,G(t|x,y)}{dt} = t + k_1(x-t,\ y-t) = t + \lambda(x-t,\ y-t) \quad \ldots \quad (6\text{-}8)$$

$$k_2(x,y) = \mu_2(x,y) = 1 - \frac{y\,\phi(y) - x\,\phi(x)}{\Phi(y) - \Phi(x)} - \left(\frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)}\right)^2 \ldots (6\text{-}9)$$

$$\frac{d^2\,G(t|x,y)}{dt^2} = k_2(x-t,\ y-t) = \mu_2(x-t,\ y-t) \quad \ldots \quad \ldots \quad (6\text{-}10)$$

$$\frac{d\,\lambda(x-t,\ y-t)}{dt} = \mu_2(x-t,\ y-t) - 1 \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}11)$$

$$k_3(x,y) = \mu_3(x,y) = - \frac{y^2\,\phi(y) - x^2\,\phi(x)}{\Phi(y) - \Phi(x)} + \frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)}$$

$$- 3\,\frac{y\phi(y) - x\phi(x)}{\Phi(y) - \Phi(x)}\,\frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)} - 2\left(\frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)}\right)^3 \ldots (6\text{-}12)$$

$$\frac{d^3 G(t|x,y)}{dt^3} = k_3 (x-t, y-t) = \mu_3 (x-t, y-t) \quad \cdots \quad \cdots \quad (6\text{-}13)$$

$$\frac{d}{dt} k_2(x-t, y-t) = \frac{d}{dt}\mu_2(x-t, y-t) = k_3(x-t, y-t) = \mu_3(x-t,y-t) \quad ..(6\text{-}14)$$

In general for $r \geq 2$, we have $\frac{d^r G(t|x,y)}{dt^r}$ will be some function $g_r (u, v)$ of the form, where $u = x-t$, $v = y-t$. Hence

$$k_r (x,y) = \frac{d^r G(t|x,y)}{dt^r} \Big/_{t=o} = g_r (x-t, y-t)/_{t=o} = g_r (x,y).$$

Therefore we get

$$k_r(x-t, y-t) = g_r (x-t, y-t) = \frac{d^r G(t|x,y)}{dt^r} \quad \cdots \quad \cdots \quad (6\text{-}15)$$

$$k_{r+1} (x-t, y-t) = \frac{d}{dt} k_r (x-t, y-t) \quad \cdots \quad \cdots \quad \cdots \quad (6\text{-}16)$$

The recurrence relations can be summarised as

$$\frac{d}{dt} \lambda (x+t, y+t) = 1 - \mu_2 (x+t, y+t) \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (6\text{-}17)$$

$$\frac{d}{dt} \mu_2 (x+t, y+t) = - \mu_3 (x+t, y+t) \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (6\text{-}18)$$

$$\frac{d}{dt} k_r (x+t, y+t) = - k_{r+1} (x+t, y+t) \text{ for } r \geq 2 \quad \cdots \quad (6\text{-}19)$$

The relations (6-17) and (6-18) interms of moments have been written because of their subsequent use.

Lemma 1

Let x and y be such that $-\infty \leq x < y \leq \infty$ and at least one of them is finite. Then $0 < \mu_2(x, y) < 1$.

Proof :

Since it is obvious that $\mu_2(x, y) > 0$, it is enough to show that $\mu_2(x, y) < 1$. For this purpose it is enough to show that

$$\frac{y \phi(y) - x \phi(x)}{\Phi(y) - \Phi(x)} + \left(\frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)} \right)^2 > 0 \quad \ldots \quad (6\text{-}20)$$

Noting that $x < \lambda(x, y) < y$, the left hand member of this inequality can be written as

$$= \frac{[y - \lambda(x, y)] \phi(y) + [\lambda(x,y) - x] \phi(x)}{\Phi(y) - \Phi(x)} \quad \ldots \quad (6\text{-}21)$$

$$> 0$$

THEOREM 1

Let x and y be such that $-\infty \leq x < y \leq \infty$ and at least one of them is finite. Then

(i) $\lambda(x,y)$ has the same sign as that of $x+y$;

(ii) $\lambda(x,y) = 0$ when $x+y = 0$; $\lambda(x,y) < \frac{x+y}{2}$ when $x+y > 0$ and $\lambda(x, y) > \frac{x + y}{2}$ when $x+y < 0$.

Proof

The result (i) and also the first part of (ii) are trivial. We recall that $\lambda(x,y)$ is finite. If $-\infty < x < y = \infty$, then we have $x + y > 0$ and $\lambda(x,y) < \frac{x+y}{2} = \infty$. If $-\infty = x < y < \infty$, then $x + y < 0$ and $\lambda(x,y) > \frac{x+y}{2} = -\infty$. Hence it is enough to consider the case where $-\infty < x < y < \infty$. Let $a = \frac{y-x}{2}$ and $b = \frac{x+y}{2}$. It is seen that $a > 0$ and $x = -a + b$ and $y = a + b$. Consider the function

$$\xi(t) = t - \lambda(-a + t, a + t).$$

We observe that $\xi(0) = 0$ and

$$\xi(b) = \frac{x+y}{2} - \lambda(x,y).$$

Further we have from (6-17)

$$\frac{d\,\xi(t)}{dt} = 1 - (1 - \mu_2(-a + t, a + t)) = \mu_2(-a + t, a + t) > 0.$$

Hence $\xi(b) > 0$ when $b > 0$ and $\xi(b) < 0$ when $b < 0$.

The required result follows.

THEOREM 2

Let x and y be such that $-\infty \leq x < y \leq \infty$ and at least one of them is finite. Then all the odd order moments about the mean of the truncated distribution have the same sign as that of the mean.

Proof

Let $x+y = 0$, then $\lambda(x,y) = 0$ and $\mu_r(x,y) = 0$ for all odd integers r. Hence we need consider the case where $x+y \neq 0$. Let $x+y \neq 0$ and r be an odd integer. We also recall that $x < \lambda(x,y) < y$. We have

$$[\Phi(y) - \Phi(x)]\, \mu_r(x,y) = \int_x^y (z-\lambda)^r \phi(z)\, dz \quad \ldots \quad (6\text{-}22)$$

where we write λ for $\lambda(x,y)$

$$\int_x^y (z-\lambda)^r \phi(z)dz = \int_{x-\lambda}^{y-\lambda} z^r \phi(z+\lambda)dz = \int_{x-\lambda}^{0} z^r \phi(z+\lambda)\, dz + \int_0^{y-\lambda} z^r \phi(z+\lambda)dz$$

$$\ldots \quad (6\text{-}23)$$

Since r is an odd integer, the transformation $z = -u$ gives

$$\int_{x-\lambda}^{0} z^r \phi(z+\lambda)\, dz = \int_{\lambda-x}^{0} u^r \phi(-u+\lambda)\, du = \int_{\lambda-x}^{0} u^r \phi(u-\lambda)du$$

$$= -\int_0^{\lambda-x} u^r \phi(u-\lambda)du \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}24)$$

Combining (6-23) and (6-24) we get for an odd integer r

$$\int_x^y (z-\lambda)^r \phi(z)dz = -\int_0^{\lambda-x} z^r \phi(z-\lambda)dz + \int_0^{y-\lambda} z^r \phi(z+\lambda)dz \quad \ldots \quad (6\text{-}25)$$

If we put $r = 1$ in (6-25) we get

$$\int_x^y (z-\lambda)\phi(z)dz = 0 = -\int_0^{\lambda-x} z\phi(z-\lambda)dz + \int_0^{y-\lambda} z\, \phi(z+\lambda)\, dz \quad \ldots \quad (6\text{-}26)$$

Case (i)

Let $x+y > 0$. This implies $x > -\infty$. If $y < \infty$, then we have from theorem 1, $x < \lambda < \frac{x+y}{2} < y$ and $\lambda > 0$. Since $\lambda < \frac{x+y}{2}$, we have $0 < \lambda - x < y - \lambda$. This inequality is obvious when $y = \infty$. Hence if $x+y > 0$, irrespective of whether $y = \infty$ or $y < \infty$, we have $0 < \lambda - x < y - \lambda$ and $0 < \lambda < \infty$. Since $\lambda > 0$, we observe that $\phi(z-\lambda) > \phi(z+\lambda)$ for all $z > 0$. We get from (6-25) for an odd integer $r \geq 3$

$$\int_x^y (z-\lambda)^r \phi(z)dz = -\int_0^{\lambda-x} z^r \phi(z-\lambda)dz + \int_0^{\lambda-x} z^r \phi(z+\lambda)dz + \int_{\lambda-x}^{y-\lambda} z^r \phi(z+\lambda)dz$$

$$= \int_0^{\lambda-x} z^r [\phi(z+\lambda) - \phi(z-\lambda)]dz + \int_{\lambda-x}^{y-\lambda} z^r \phi(z+\lambda)dz$$

$$> (\lambda-x)^{r-1} \int_0^{\lambda-x} z[\phi(z+\lambda) - \phi(z-\lambda)]dz + (\lambda-x)^{r-1} \int_{\lambda-x}^{y-\lambda} z\phi(z+\lambda)dz$$

$$> (\lambda-x)^{r-1} [\int_0^{\lambda-x} z\{\phi(z+\lambda) - \phi(z-\lambda)\}dz + \int_{\lambda-x}^{y-\lambda} z\phi(z+\lambda)dz]$$

$$> (\lambda-x)^{r-1} [-\int_0^{\lambda-x} z\phi(z-\lambda)dz + \int_0^{y-\lambda} z\phi(z+\lambda)dz] = 0 \quad ..(6\text{-}27)$$

Case (ii)

Let $x+y < 0$. This implies $y < \infty$. If $x > -\infty$, then we have from theorem 1, $x < \frac{x+y}{2} < \lambda < y$ and $-\infty < \lambda < 0$. Since $(x+y)/2 < \lambda$, we have $0 < y - \lambda < \lambda - x$. This inequality also is obvious when $x = -\infty$.

Hence irrespective of whether $x = -\infty$ or $x > -\infty$, we have $0 < y - \lambda < \lambda - x$ and $-\infty < \lambda < 0$. Since $\lambda < 0$, we observe that $\phi(z-\lambda) < \phi(z+\lambda)$ for all $z > 0$. Proceeding on similar lines as in case of (i), we get

$$\int_x^y (z-\lambda)^r \phi(z)\, dz < 0 \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}28)$$

In equalities (6-27) and (6-28) along with (6-22) prove the required result.

If we take $r = 3$ in theorem 2, we get the result that the sign of $\mu_3(x,y)$ i.e., of the expression in the right hand member of (6-12) is same as that of $x+y$. This is a generalisation of a result due to Sanford [10]. If we put $y = \infty$ in (6-12), we get

$$\mu_3(x,\infty) = \frac{x^2\,\phi(x)}{1-\Phi(x)} - \frac{\phi(x)}{1-\Phi(x)} - 3x\left(\frac{\phi(x)}{1-\Phi(x)}\right)^2 + 2\left(\frac{\phi(x)}{1-\Phi(x)}\right)^3 > 0 \quad \ldots (6\text{-}29)$$

for all $x \in (-\infty, \infty)$. This result for truncation of one tail was conjectured first by Birnbaum and later proved by Samford [10].

<u>Corollary to Theorem 2:</u>

For all $u \in (-\infty, \infty)$, $\mu_2(u, \infty)$ is a strictly decreasing function of u and $\mu_2(-\infty, u)$ is a strictly increasing function of u.

Proof

We have

$$\frac{d}{dt} \mu_2 (-\infty, y+t) = -\mu_3 (-\infty, y+t) > 0$$

$$\frac{d}{dt} \mu_2 (x+t, \infty) = -\mu_3 (x+t, \infty) < 0$$

and these prove the required result.

It is seen from (6-7) and (6-9) that $\lambda(x,y) = \lambda(y,x)$ and $\mu_2(x,y) = \mu_2(y,x)$ and there is no need for us to always designate the lower truncation point by x and the upper truncation point by y. We shall now state and prove a result in theorem 3 which is later frequently used in the study of Group EOQ Replenishment policy.

THEOREM 3

Consider the non-linear constrained maximisation problem

$$\text{Maximise } \mu_2(x, y) \text{ subject to}$$

$$(x,y) \in R^2 \text{ and } |x-y| = 2a \qquad \ldots \qquad \ldots \qquad (6\text{-}30)$$

where $\mu_2(x,y)$ is the function defined in (6-9) and 'a' is a given positive constant. Then (i) the maximum of $\mu_2(x,y)$ subject to the constraints is attained when $x+y = 0$ (that is at the points $(-a,a)$ and $(a, -a)$), (ii) if (x_1, y_1) and (x_2, y_2) are two points such that $|x_1 - y_1| = |x_2 - y_2| = 2a$, then $\mu_2(x_1, y_1)$ is equal to or greater than or less than $\mu_2(x_2, y_2)$ respectively, according as $|x_1 + y_1|$ is equal to or less than or greater than $|x_2 + y_2|$.

Proof

Since $\mu_2(x, y) = \mu_2(y, x)$, it is enough to consider the maximisation with the additional constraint $x < y$. Any point $(x,y) \in R^2$ and satisfying constraints of the problem (6-30) and $x < y$ can be represented as $x = -a + t$ and $y = a + t$ where $t \in R$. It can be easily verified that

$$\mu_2(-a + t, \; a + t) = \mu_2(-a - t, \; a - t)$$

Hence the function $\mu_2(-a + t, \; a + t)$ is symmetrical about $t = 0$. Further, we have from (6-18),

$$\frac{d}{dt} \mu_2(-a + t, a + t) = -\mu_3(-a + t, a + t)$$

By theorem 2, we have

$$\mu_3(-a + t, a + t) \begin{cases} > 0 & \text{when } t > 0 \\ = 0 & \text{when } t = 0 \\ < 0 & \text{when } t < 0 \end{cases}$$

This shows that $\mu_2(-a + t, a + t)$ is strictly increasing with t for $t < 0$ and strictly decreasing for $t > 0$. Therefore $\mu_2(-a+t, a+t)$ attains its maximum at $t=0$ and this proves the theorem. If we truncate a normal distribution at two points which are at a fixed distance apart, theorem 3 states that the variance of the truncated distribution is maximum when the truncation points are placed symmetrically around the mean.

Lemma 2

Let x and y be such that $-\infty \leq x \leq \infty$, $-\infty \leq y \leq \infty$, $x \neq y$ and at least one of them is finite. Then $\mu_2\,(x+\sigma/2,\ y+\sigma/2)$ is equal to or greater than or less than $\mu_2\,(x-\sigma/2,\ y-\sigma/2)$ respectively, according as $x+y=0$ or $x+y<0$ or $x+y>0$ for all $\sigma>0$.

Proof

Case (i) Let x and y be both finite. The required result then follows from theorem 3.

Case (ii) When either x or y is infinite, we know from the corollary to theorem 2 that $\mu_2\,(-\infty,\ u)$ is a strictly increasing function of u and $\mu_2\,(\infty,\ u)$ is a strictly decreasing function of u. This proves the required result.

6.4 SOME MISCELLANEOUS RESULTS

Define the function $\theta(x,y\,|\,\sigma)$ for all $(x,\ y,\ \sigma) \in R^3$ by

$$\theta(x,y\,|\,\sigma) = \frac{\Phi(y-\sigma/2) - \Phi(x-\sigma/2)}{\Phi(y+\sigma/2) - \Phi(x+\sigma/2)} \quad \text{when } x \neq y \qquad \ldots \quad (6\text{-}31)$$

$$= e^{\sigma x} \quad \text{when } x = y$$

It is easily verified that $\theta\,(x,y\,|\,\sigma) = 1$ for all σ when $x+y=0$.

The following facts are obvious

(i) $\theta(x, y \mid \sigma) = \theta(y, x / \sigma)$; (ii) $\theta(x, y \mid 0) = 1$;

(iii) $\theta(x, y \mid \sigma) \longrightarrow e^{\sigma x}$ as $y \longrightarrow x$, (iv) limits of $\theta(x, y \mid \sigma)$ when $x \longrightarrow \pm\infty$ exist and are denoted by $\theta(\pm\infty, y \mid \sigma)$ and

(v) $\theta(-\infty, y \mid \sigma) \longrightarrow 1$ as $y \longrightarrow \infty$ and $\theta(x, \infty \mid \sigma) \longrightarrow 1$ as $x \longrightarrow -\infty$.

<u>Lemma 3</u>

Let $-\infty < x < y < \infty$. Then we have for all $\sigma > 0$.

(i) $e^{\sigma x} < \theta(x, y \mid \sigma) < e^{\sigma y}$

(ii) $\theta(-\infty, x \mid \sigma) < e^{\sigma x} < \theta(x, \infty \mid \sigma)$

<u>Proof</u>

Let $f(u \mid \mu, \sigma)$ be the log-normal probability density function i.e.,

$$f(u \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma u} \exp\left(-\frac{1}{2\sigma^2}(\log u - \mu)^2\right), \quad 0 \leq u \leq \infty, \quad \sigma > 0;$$

and $u_1 = \exp(\mu + \sigma^2/2 + \sigma x)$ & $u_2 = \exp(\mu + \sigma^2/2 + \sigma y)$. Since $x < y$, we have $u_1 < u_2$. It follows from (3- 4) that

$$\frac{\int_{u_1}^{u_2} u\, f(u \mid \mu, \sigma)\, du}{\int_{u_1}^{u_2} f(u \mid \mu, \sigma)\, du} = \exp(\mu + \sigma^2/2)\, \theta(x, y \mid \sigma) \quad \ldots \quad (6\text{-}32)$$

It follows from (6-32) that

$$\exp(\mu+\sigma^2/2+\sigma x) < \exp(\mu+\sigma^2/2)\ \theta(x,y|\sigma) < \exp(\mu+\sigma^2/2+\sigma y) \quad \ldots (6\text{-}33)$$

The two results of the lemma follow from these inequalities, observing that σ - one of the parameters of the log-normal distribution can take only positive values.

Lemma 4

Let x, y and z be such that $-\infty \leq x < y < z \leq \infty$. Then we have for all $\sigma > 0$, $\theta(x, y|\sigma) < \theta(y, z|\sigma)$.

Proof

It is well known that the Lorenz curve is strictly convex. For the log-normal case, the Lorenz curve is a plot of $\Phi(u - \sigma/2)$ against $\Phi(u + \sigma/2)$ for different u. The required reslt follows.

We shall now consider the partial derivatives of $\theta(x, y|\sigma)$. For $x \neq y$ and $\sigma \neq 0$, we have

$$\frac{\partial \theta(x, y|\sigma)}{\partial x} = \frac{\phi(x-\sigma/2) - \theta(x, y|\sigma)\,\phi(x+\sigma/2)}{\Phi(x+\sigma/2) - \Phi(y+\sigma/2)} \quad \ldots (6\text{-}34)$$

$$= \phi(x+\sigma/2)\,\frac{e^{\sigma x} - \theta(x, y|\sigma)}{\Phi(x+\sigma/2) - \Phi(y+\sigma/2)} \quad \ldots (6\text{-}35)$$

$$\frac{\partial \theta(x, y|\sigma)}{\partial y} = \frac{\phi(y-\sigma/2) - \theta(x, y|\sigma)\,\phi(y+\sigma/2)}{\Phi(y+\sigma/2) - \Phi(x+\sigma/2)} \quad \ldots (6\text{-}36)$$

$$= \phi\,(y + \sigma/2)\ \frac{e^{\sigma y} - \theta\,(x,y|\sigma)}{\Phi(y+\sigma/2) - \Phi(x+\sigma/2)} \quad \ldots \quad (6\text{-}37)$$

Since $\frac{\partial \theta\,(x,\ y|0)}{\partial x} = \frac{\partial \theta\,(x,\ y|0)}{\partial y} = 0$, it is seen that the above hold true even when $\sigma = 0$. Further

$$\lim_{y \longrightarrow x} \frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial x}\Big/_{y=x} = \frac{\partial \theta}{\partial y}\Big/_{y=x} = \lim_{x \longrightarrow y} \frac{\partial \theta}{\partial y}$$

$$= \frac{\sigma}{2}\ e^{\sigma x} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}38)$$

Finally for $x \neq y$

$$\frac{\partial \theta(x,y|\sigma)}{\partial \sigma} = -\tfrac{1}{2}\left(\frac{\phi\,(y-\sigma/2)-\phi(x-\sigma/2)+\theta(x,y|\sigma)\,[\,\phi(y+\sigma/2)-\phi(x+\sigma/2)\,]}{\Phi(y+\sigma/2) - \Phi(x+\sigma/2)}\right) \quad (6\text{-}39)$$

It is easily verified that

$$\frac{\partial \theta\,(x,\ x|\sigma)}{\partial \sigma} = x\ e^{\sigma x} = \lim_{y \longrightarrow x} \frac{\partial \theta(x,\ y|\sigma)}{\partial \sigma} \quad \ldots \quad (6\text{-}40)$$

<u>Lemma 5</u>

For all $\sigma > 0$, we have

(i) $\frac{\partial}{\partial x}\ \theta\,(\pm\infty,\ x|\sigma) > 0$

(ii) $\frac{\partial}{\partial x}\ \theta\,(x,\ y|\sigma) > 0$ and $\frac{\partial}{\partial y}\ \theta\,(x,\ y|\sigma) > 0 \quad \ldots \quad (6\text{-}41)$

Proof

$$\frac{\partial \theta(-\infty, x|\sigma)}{\partial x} = \phi(x+\sigma/2)\,\frac{e^{\sigma x} - \theta(-\infty, x|\sigma)}{\Phi(x+\sigma/2)}$$

$$> 0 \text{ by lemma 3.}$$

The other inequalities of lemma are proved in a similar manner.

It is seen that $\theta(x, y|\sigma)$ and $\theta(\pm\infty, y|\sigma)$ are positive for all x and y. We now define the function $W(x, y|\sigma)$ for all $(x, y, \sigma) \in R^3$ by

$$W(x, y|\sigma) = \sigma y - \log \theta(x, y|\sigma) \qquad \ldots \qquad \ldots (6\text{-}42)$$

The following observations can be easily verified

(i) $W(x, y|0) = 0$ for all (x, y).

(ii) $W(x, y|\sigma) = 0$ for all σ when $x = y$

(iii) For $\sigma > 0$, $W(x, y|\sigma) \longrightarrow \pm\infty$ as $y \longrightarrow \pm\infty$

(iv) limits of $W(x, y|\sigma)$ when $x \longrightarrow \pm\infty$ exist and are denoted by $W(\pm\infty, y|\sigma)$

(v) For $\sigma > 0$, $W(-\infty, y|\sigma) \longrightarrow \infty$ as $y \longrightarrow \infty$ and $W(\infty, y|\sigma) \longrightarrow -\infty$ as $y \longrightarrow -\infty$

Further we have for $x \neq y$.

$$\frac{\partial W(x, y|\sigma)}{\partial x} = -\frac{1}{\theta}\,\frac{\partial \theta}{\partial x} = -\,\frac{\phi(x-\sigma/2) - \theta(x, y|\sigma)\,\phi(x+\sigma/2)}{\Phi(x-\sigma/2) \text{ [illegible] } \Phi(\text{[illegible]}/2)} \qquad \ldots \qquad (6\text{-}43)$$

$$\frac{\partial W(x,y|\sigma)}{\partial y} = \sigma - \frac{1}{\Theta}\frac{\partial \Theta}{\partial y} = \sigma - \frac{\phi(y-\sigma/2) - \Theta(x,y|\sigma)\ \phi(y+\sigma/2)}{\Phi(y-\sigma/2) - \Phi(x-\sigma/2)} \quad \ldots (6\text{-}44)$$

$$\frac{\partial W(x,y|\sigma)}{\partial x} + \frac{\partial W(x,y|\sigma)}{\partial y} = \sigma - \frac{\phi(y-\sigma/2) - \phi(x-\sigma/2)}{\Phi(y-\sigma/2) - \Phi(x-\sigma/2)} + \frac{\phi(y+\sigma/2) - \phi(x+\sigma/2)}{\Phi(y+\sigma/2) - \Phi(x+\sigma/2)}$$

$$= \sigma + \lambda(x-\sigma/2,\ y-\sigma/2) - \lambda(x+\sigma/2,\ y+\sigma/2) \quad \ldots\ldots (6\text{-}45)$$

It is easily seen that

$$\frac{\partial W(x,y|\sigma)}{\partial x} \longrightarrow -\sigma/2 \quad \text{when} \quad x \longrightarrow y \quad \text{or} \quad y \longrightarrow x$$

$$\frac{\partial W(x,y|\sigma)}{\partial y} \longrightarrow \sigma/2 \quad \text{when} \quad x \longrightarrow y \quad \text{or} \quad y \longrightarrow x$$

Finally we have

$$\frac{\partial \log \Theta(x,y|\sigma)}{\partial \sigma} = -\frac{1}{2\Theta}\left(\frac{\phi(y-\sigma/2) - \phi(x-\sigma/2) + \Theta\ (\phi(y+\sigma/2) - \phi(x+\sigma/2))}{\Phi(y+\sigma/2) - \Phi(x+\sigma/2)} \right)$$

$$= -\frac{1}{2}\left(\frac{\phi(y-\sigma/2) - \phi(x-\sigma/2)}{\Phi(y-\sigma/2) - \Phi(x-\sigma/2)} + \frac{\phi(y+\sigma/2) - \phi(x+\sigma/2)}{\Phi(y+\sigma/2) - \Phi(x+\sigma/2)} \right)$$

$$= \frac{1}{2}\left[\lambda(x-\sigma/2,\ y-\sigma/2) + \lambda(x+\sigma/2,\ y+\sigma/2) \right] \quad \ldots (6\text{-}46)$$

where we have written Θ for $\Theta(x, y|\sigma)$ for the sake of simplicity.

It is easily verified that

$$\frac{\partial \log \Theta(x,y|\sigma)}{\partial \sigma} \longrightarrow x \quad \text{when} \quad x \longrightarrow y \quad \ldots (6\text{-}47)$$

Lemma 6

For all $\sigma > 0$, we have

(i) $W(-\infty, y|\sigma) > 0$ and $W(+\infty, y|\sigma) < 0$ for all $y \in (-\infty, \infty)$.

(ii) For all $(x, y) \in R^2$

$$W(x, y|\sigma) \begin{cases} > 0 & \text{if } y > x \\ = 0 & \text{if } y = x \\ < 0 & \text{if } y < x \end{cases}$$

Proof

We prove only the second result and the proof for the first one is similar. We have from lemma 3 for any u and v such that $-\infty < u < v < \infty$

$$e^{\sigma u} < \theta(u, v|\sigma) < e^{\sigma v}$$

Further when $x = y$, we have

$$e^{\sigma x} = \theta(x, y|\sigma)$$

and this proves the lemma.

We shall now state and prove an important result in theorem 4. This result is later used frequently.

THEOREM 4

For all $\sigma > 0$, we have

(i) $\dfrac{\partial W(x, y|\sigma)}{\partial x} + \dfrac{\partial W(x, y|\sigma)}{\partial y} > 0$ when $x \neq y$ for all $(x,y) \in R^2$.

(ii) $\frac{\partial W(\pm\infty, y|\sigma)}{\partial y} > 0$ for all $y \in (-\infty, \infty)$.

Proof

We have from (6-45)

$$\frac{\partial W(x,y|\sigma)}{\partial x} + \frac{\partial W(x,y|\sigma)}{\partial y} = \sigma + \lambda(x-\sigma/2,\ y-\sigma/2) - \lambda(x+\sigma/2,\ y+\sigma/2) \quad \ldots(6\text{-}48)$$

Consider for given x and y, (6-48) as a function of σ i.e.,

$$G(\sigma) = \sigma + \lambda(x-\sigma/2,\ y-\sigma/2) - \lambda(x+\sigma/2,\ y+\sigma/2) \quad \ldots \quad (6\text{-}49)$$

It is seen that $G(0) = 0$. Further, we have from (6-11) and (6-17).

$$\frac{d\,G(\sigma)}{d\,\sigma} = 1 + \tfrac{1}{2}\left[\mu_2(x-\sigma/2,\ y-\sigma/2)-1\right] - \tfrac{1}{2}\left[1-\mu_2(x+\sigma/2,\ y+\sigma/2)\right]$$

$$= \tfrac{1}{2}\left[\mu_2(x-\sigma/2,\ y-\sigma/2) + \mu_2(x+\sigma/2,\ y+\sigma/2)\right] \quad \ldots \quad (6\text{-}50)$$

$$> 0 \text{ when } x \neq y$$

This proves the first part of the theorem. We shall now prove the second part. It is easily verified that (6-44) holds true even when $x = \pm\infty$. We have

$$\frac{\partial W(-\infty, y|\sigma)}{\partial y} = \sigma - \frac{\phi(y-\sigma/2)}{\Phi(y-\sigma/2)} + \frac{\phi(y+\sigma/2)}{\Phi(y+\sigma/2)}$$

$$= \sigma + \lambda(-\infty,\ y-\sigma/2) - \lambda(-\infty,\ y+\sigma/2)$$

$$\frac{\partial W(+\infty, y|\sigma)}{\partial y} = \sigma - \frac{\phi(y-\sigma/2)}{\Phi(y-\sigma/2)-1} + \frac{\phi(y+\sigma/2)}{\Phi(y+\sigma/2)-1}$$

$$= \sigma + \lambda(\infty, y-\sigma/2) - \lambda(\infty, y+\sigma/2)$$

The rest of the proof is same as before.

Note: This theorem can be proved directly by Cauchy Schwarz's inequality.

Corollary to Theorem 4

For all $\sigma > 0$, $x \in [-\infty, \infty]$ and $y \in (-\infty, +\infty)$, we have

$$\frac{\partial W(x, y|\sigma)}{\partial y} > 0 \quad \dots \quad \dots \quad \dots \quad (6\text{-}51)$$

Proof

It has been already shown in the theorem that

$$\frac{\partial W(\pm\infty, y|\sigma)}{\partial y} > 0 \quad \dots \quad \dots \quad (6\text{-}52)$$

Hence we need prove the result when x is finite.

It was seen in lemma 5 that $\frac{\partial \Theta(x, y|\sigma)}{\partial x} > 0$ for all $\sigma > 0$.

Hence we have

$$\frac{\partial W(x, y|\sigma)}{\partial x} = -\frac{1}{\Theta}\frac{\partial \Theta}{\partial x} < 0 \text{ for all } \sigma > 0 \quad \dots \quad (6\text{-}53)$$

Since

$$\frac{\partial W(x, y|\sigma)}{\partial x} + \frac{\partial W(x, y|\sigma)}{\partial y} > 0 \text{ when } x \neq y$$

It follows that

$$\frac{\partial W(x, y\,|\,\sigma)}{\partial y} > 0 \quad \text{for} \quad x \neq y$$

Further since

$$\frac{\partial W(x, y\,|\,\sigma)}{\partial y} \Big/_{x=y} = \sigma/2$$

we have $\frac{\partial W(x, y\,|\,\sigma)}{\partial y} > 0$ for all $\sigma > 0$ and $(x,y) \in R^2$.

Lemma 7

Let x and y be such that $-\infty \leq x \leq \infty$, $-\infty \leq y \leq \infty$, $x \neq y$ and at least one of them is finite. Then for all $\sigma > 0$ we have that

$$\log \theta(x, y\,|\,\sigma) - \frac{\sigma}{2} \left[\lambda(x-\sigma/2,\ y-\sigma/2) + \lambda(x+\sigma/2,\ y+\sigma/2) \right] \quad ..(6\text{-}54)$$

is equal to zero or greater than zero or less than zero respectively according as $x+y = 0$ or $x+y < 0$ or $x+y > 0$.

Proof

When $x+y = 0$, we have $\theta(x, y\,|\,\sigma) = 1$ and also

$\lambda(x-\sigma/2,\ y-\sigma/2) = -\lambda(x+\sigma/2, y+\sigma/2)$. This proves the required result for the case $x+y = 0$.

Let $x+y \neq 0$, and consider for given x and y, the expression (6-54) as a function of σ i.e.,

$$G(\sigma) = \log \theta(x,y\,|\,\sigma) - \frac{\sigma}{2} \left[\lambda(x-\sigma/2,\ y-\sigma/2) + \lambda(x+\sigma/2,\ y+\sigma/2) \right]$$

It is easily verified that (6-46) holds true even when $x = \pm\infty$.

Thus we have from (6-46), (6-11) and (6-17)

$$\frac{d\,G(\sigma)}{d\,\sigma} = \frac{\sigma}{4}\left[\mu_2\,(x+\sigma/2,\; y+\sigma/2) - \mu_2\,(x-\sigma/2,\; y-\sigma/2)\right].$$

By lemma 2, we have for $\sigma > 0$

$$\frac{d\,G(\sigma)}{d\sigma}\begin{cases} > 0 & \text{when } x+y < 0 \\ < 0 & \text{when } x+y > 0 \end{cases}.$$

Since $G(0) = 0$, the required result follows.

<u>Lemma 8</u>

Let x, y, z be such that $-\infty \leq x < y < z \leq \infty$ and at least one of x & z is finite. Then we have for all $\sigma > 0$, that

$$\log\Theta\,(x,y\,|\,\sigma) + \log\Theta\,(y,z\,|\,\sigma) - \frac{\sigma}{2}\left[\lambda(x-\sigma/2,\; y-\sigma/2) + \lambda(x+\sigma/2,\; y+\sigma/2)\right.$$
$$\left. + \lambda(y-\sigma/2,\; z-\sigma/2) + \lambda(y+\sigma/2,\; z+\sigma/2)\right] \quad \ldots \quad \ldots \quad (6\text{-}54)$$

is equal to or greater than or less than zero respectively according as $x+z = 0$, $y = 0$ or $y+z \leq 0$ or $x+y \geq 0$.

<u>Proof</u>

<u>Case (i)</u> Let $x+z = 0$ and $y = 0$, then we have

$$\Theta\,(x,\,y|\,\sigma) = 1/\Theta\,(y,\,z|\,\sigma)$$

$$\lambda\,(x-\sigma/2,\, y-\sigma/2) = -\lambda(y+\sigma/2,\; z+\sigma/2)$$

$$\lambda\,(x+\sigma/2,\, y+\sigma/2) = -\lambda(y-\sigma/2,\; z-\sigma/2)$$

and this proves the required result.

Case (ii) Let $x+z \neq 0$. If $y+z \leq 0$, then $x+y < 0$. Similarly if $x+y \geq 0$, then $y+z > 0$. The required result follows from lemma 7.

Lemma 9

Let x, y and z be such that $-\infty \leq x < y < z \leq \infty$. Then we have for all $\sigma \neq 0$.

$$\left[\left\{ \Phi(z-\sigma/2) - \Phi(x-\sigma/2) \right\} \left\{ \Phi(z+\sigma/2) - \Phi(x+\sigma/2) \right\} \right]^{\frac{1}{2}}$$
$$> \left[\left\{ \Phi(z-\sigma/2) - \Phi(y-\sigma/2) \right\} \left\{ \Phi(z+\sigma/2) - \Phi(y+\sigma/2) \right\} \right]^{\frac{1}{2}}$$
$$+ \left[\left\{ \Phi(y-\sigma/2) - \Phi(x-\sigma/2) \right\} \left\{ \Phi(y+\sigma/2) - \Phi(x+\sigma/2) \right\} \right]^{\frac{1}{2}}$$

Proof

The result is a direct consequence of Cauchy's inequality.

We shall state now lemma 10 and lemma 11 which are due to Higuchi [8]. The results of these two lemmas are used later.

Lemma 10

Let $f(y)$ be a one-valued and differentiable function defined over $(-\infty, \infty)$, with the properties

(i) $\lim\limits_{y \to \infty} |f(y)| < \infty$, (ii) $\lim\limits_{y \to -\infty} f(y) < \infty$, (iii) $\left| \frac{df}{dy} \right| < 1$

Let $g(x, z)$ be a one valued and continuous function (as a function of two variables) defined over the whole x, z - plane, whose partial derivatives exist every where, and which has the following properties;

(iv) $g(x,z) = g(z,x)$, (v) for any fixed z, $\lim_{x \to \pm\infty} |g(x,z)| < \infty$

(vi) $\lim_{\substack{z \to \infty \\ x \to -\infty}} |g(x,z)| < \infty$ where x and z tend to infinities

independently, (vii) $\left| \frac{dg(\pm\infty, z)}{dz} \right| < 1$ and (viii) $\left|\frac{\partial g}{\partial x}\right| + \left|\frac{\partial g}{\partial z}\right| < 1$.

Then there exists uniquely a function $h(z)$ defined over $(-\infty, \infty)$ such that the relation $y = h(z)$ implies relations $x = f(y)$ and $y = g(x, z)$ and vice versa. Moreover $h(z)$ is differentiable in the whole interval and satisfies

(ix) $\lim_{z \to \infty} |h(z)| < \infty$, (x) $\lim_{z \to -\infty} h(z) < \infty$ and (xi) $\left|\frac{dh}{dz}\right| < 1$.

<u>Note</u>: The condition (viii) can be relaxed to $\left|\frac{\partial g}{\partial x}\right| + \left|\frac{\partial g}{\partial z}\right| \leq 1$ if $\frac{\partial g}{\partial x} \neq 0$.

<u>Lemma 11</u>

A matrix $((a_{ij}))$ of real numbers having the following properties is positive definite.

(i) It is symmetric

(ii) All elements except the diagonal ones and those adjacent to any diagonal one, are zero.

(iii) Every diagonal element is greater than the row - sum of the absolute values of the elements adjacent to it; hence the diagonal elements are positive.

For proofs of lemmas 10 and 11, see Higuchi [8].

6.5 FURTHER THEORETICAL STUDY OF GROUP EOQ REPLENISHMENT POLICY

In this section we shall further study some of the theoretical aspects of Group EOQ Replenishment approach. In section 4, we encountered the system of equations (4-19) which the transformed optimal group boundaries satisfy. In the notation of this appendix, the system of equations (4-19) is the same as

$$W(t_{i-1}, t_i \mid \sigma) + W(t_{i+1}, t_i \mid \sigma) = 0 \quad \text{for} \quad i = 1 \text{ to } r-1.$$

We shall now state and prove an important theorem about the existence and uniqueness of the solution to the above system of equations.

THEOREM 5

For any given positive constant σ, the system of equations

$$W(t_{i-1}, t_i \mid \sigma) + W(t_{i+1}, t_i \mid \sigma) = 0, \quad i=1 \text{ to } r-1, \quad \ldots \quad (6\text{-}55)$$

where $t_o = -\infty$ and $t_r = \infty$, has one and only one solution of real numbers, and its constituents satisfy the order condition

$$-\infty = t_o < t_1 < t_2 \; \ldots\ldots < t_{r-1} < t_r = \infty \quad \ldots \quad (6\text{-}56)$$

Proof

Since $W(x,y \mid \sigma) \longrightarrow \pm\infty$ as $y \longrightarrow \pm\infty$ and $\frac{\partial W(x,y|\sigma)}{\partial y} > 0$ for $\sigma > 0$ it follows that for given $(x, z) \in R^2$, there exists one and only one real valued function.

$$y = g(x, z) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}57)$$

which satisfies the equation

$$W(x, y \mid \sigma) + W(z, y \mid \sigma) = 0 \qquad \ldots \quad \ldots \quad (6\text{-}58)$$

identically. This holds true also for the case when x or $z = \pm\infty$ This is seen from the additional facts that $W(-\infty, y|\sigma) \longrightarrow \infty$ as $y \longrightarrow \infty$, $W(\infty, y|\sigma) \longrightarrow -\infty$ as $y \longrightarrow -\infty$ and $\frac{\partial W(\pm\infty, y|\sigma)}{\partial y} > 0$ for all y. It is noted that $g(x, z)$ is a symmetric function of x and z i.e. $g(x, z) = g(z, x)$. Since $\frac{\partial W(x, y|\sigma)}{\partial x} < 0$ it follows that

$$\frac{\partial g}{\partial x} = - \frac{\partial W(x,y|\sigma)}{\partial x} \Big/ \left(\frac{\partial W(x,y|\sigma)}{\partial y} + \frac{\partial W(z,y|\sigma)}{\partial y} \right) > 0$$

Similarly we have $\frac{\partial g}{\partial z} > 0$. Hence $g(x, z)$ is monotone increasing with respect to each argument. Further, by theorem 4, we have

$$\left|\frac{\partial g}{\partial x}\right| + \left|\frac{\partial g}{\partial z}\right| = - \frac{\frac{\partial W(x,y|\sigma)}{\partial x} + \frac{\partial W(z,y|\sigma)}{\partial z}}{\frac{\partial W(x,y|\sigma)}{\partial y} + \frac{\partial W(z,y|\sigma)}{\partial y}} \leq 1$$

Actually it can be shown that $\left|\frac{\partial g}{\partial x}\right| + \left|\frac{\partial g}{\partial z}\right|$ is equal to unity when $x = z$ and strictly less than unity when $x \neq z$. It is easily seen that $g(-\infty, \infty) = 0$, $\lim_{z \to -\infty} g(-\infty, z) = -\infty$ and $0 < \frac{dg(-\infty, z)}{dz} < 1$. Since any x, y, z which donot satisfy (6-58) evidently donot satisfy (6-57), the system of equations

$$t_1 = g(-\infty, t_2), \quad t_2 = g(t_1, t_3) \; \ldots\ldots \; t_{r-1} = g(t_{r-2}, \infty) \quad \ldots (6\text{-}59)$$

and the system of equations (6-55) are one and the same. Consequently, we can prove the existence and uniqueness of the solution of (6-59). Further $g(-\infty, y)$ satisfies all the conditions imposed on $f(y)$ of lemma 10 and $g(x, z)$ satisfies all the conditions imposed on $g(x,z)$ of the same lemma.

Now consider the system of equations

$$t_1 = g(-\infty, t_2), \quad t_2 = g(t_1, t_3), \; \ldots\ldots \; t_{r-1} = g(t_{r-2}, t_r) \quad \ldots (6\text{-}60)$$

where we take t_r to be a variable instead of the usual convention $t_r = \infty$. We obtain by applying lemma 10 successively, a class of one - valued continuous functions

$$t_1 = h_1(t_2), \quad t_2 = h_2(t_3), \; \ldots\ldots \; t_{r-1} = h_{r-1}(t_r) \quad \ldots \quad \ldots (6\text{-}61)$$

which satisfy the system (6-60) simultaneously. As no other t_{r-1}, t_r than those which satisfy $t_{r-1} = h_{r-1}(t_r)$ satisfy the system (6-60) and $\lim_{t_r \to \infty} h_{r-1}(t_r)$ exists, the continuity properties of h_{r-1} and $g(t_{r-2}, t_r)$ assert the existence and uniqueness of the solution of (6-59).

That the unique solution of (6-59) satisfies the order condition (6-56) is proved as follows. From lemma 6, we know for all $\sigma > 0$.

$$\text{(i)} \quad W(-\infty, y\,|\,\sigma) > 0 \text{ and } W(+\infty, y) < 0 \quad \ldots \quad \ldots \quad (6\text{-}62)$$

$$\text{(ii)} \quad W(x, y\,|\,\sigma) \begin{cases} > 0 & \text{if } y > x \\ = 0 & \text{if } y = x \\ < 0 & \text{if } y < x \end{cases} \quad \text{for all } (x,y) \in R^2 \ \ldots (6\text{-}63)$$

For any y and z which satisfy $y = g(-\infty, z)$, the relation $-\infty < y < z$ holds. Suppose $y \geq z$, then we have from (6-62) and (6-63)

$$W(-\infty, y\,|\,\sigma) + W(z, y\,|\,\sigma) > 0 \quad \ldots \quad \ldots \quad (6\text{-}64)$$

which contradicts that y and z satisfy (6-64) with equality. Similarly it can be shown that for any x, y, z, which satisfy $y = g(x, z)$ and $x < y$, then $x < y < z$. Since in the system of equations $t_1 = g(-\infty, t_2)$, we have $-\infty < t_1 < t_2$. Again since t_1, t_2, t_3 are such that $t_2 = g(t_1, t_3)$ and $t_1 < t_2$, we have $t_1 < t_2 < t_3$ and so on.

<u>THEOREM 6</u>

The minimum of problem (4-16) is attained at the unique solution of the system of equations (4-19).

Proof

We recall that the unique solution of (4-19) satisfies the order condition (6-56) and also satisfies (see (4-22)).

$$\frac{\phi(t_i - \sigma/2)}{\phi(t_i + \sigma/2)} = \sqrt{\theta(t_{i-1}, t_i | \sigma)\, \theta(t_i, t_{i+1} | \sigma)}$$

$$= \sqrt{\theta_i\, \theta_{i+1}} \quad \text{for } i = 1 \text{ to } r-1 \quad \ldots \quad (6\text{-}65)$$

where we write θ_i for $\theta(t_{i-1}, t_i | \sigma)$. We have from (4-21)

$$2\frac{\partial H}{\partial t_i} = \left(\frac{1}{\sqrt{\theta_i}} - \frac{1}{\sqrt{\theta_{i+1}}}\right)\phi(t_i - \sigma/2) + \left(\sqrt{\theta_i} - \sqrt{\theta_{i+1}}\right)\phi(t_i + \sigma/2)$$

for $i = 1$ to $r-1$. It is enough to prove that the Hessian matrix $\left(\left(\frac{\partial^2 H}{\partial t_i \partial t_j}\right)\right)$ is positive definite at the unique solution of (4-19).

We note

$$\frac{\partial^2 H}{\partial t_i\, \partial t_j} = 0 \quad \text{for } |i-j| \geq 2$$

and the matrix is symmetric. Further we have

$$2\frac{\partial^2 H}{\partial t_i^2} = \frac{\sqrt{\theta_{i+1}} - \sqrt{\theta_i}}{\sqrt{\theta_i\, \theta_{i+1}}}(-t_i + \sigma/2)\,\phi(t_i - \sigma/2) + \left(\sqrt{\theta_i} - \sqrt{\theta_{i+1}}\right)(-t_i - \sigma/2) \times$$

$$\phi(t_i + \sigma/2) + \tfrac{1}{2}\frac{\partial \theta_i}{\partial t_i}\left(-\theta_i^{-3/2}\,\phi(t_i - \sigma/2) + \theta_i^{-\frac{1}{2}}\,\phi(t_i + \sigma/2)\right)$$

$$+ \tfrac{1}{2}\frac{\partial \theta_{i+1}}{\partial t_i}\left(\theta_{i+1}^{-3/2}\,\phi(t_i - \sigma/2) - \theta_{i+1}^{-\frac{1}{2}}\,\phi(t_i + \sigma/2)\right) \quad \ldots \quad \ldots (6\text{-}66)$$

$$= (\sqrt{\theta_{i+1}} - \sqrt{\theta_i}) \, \emptyset(t_i+\sigma/2) \, [-t_i+\sigma/2 + t_i +\sigma/2 - \tfrac{1}{2} \frac{1}{\theta_i} \frac{\partial \theta_i}{\partial t_i}$$

$$- \tfrac{1}{2} \frac{1}{\theta_{i+1}} \frac{\partial \theta_{i+1}}{\partial t_i}] \quad .... \quad (6\text{-}67)$$

$$= (\sqrt{\theta_{i+1}} - \sqrt{\theta_i}) \, \emptyset(t_i+\sigma/2) \, [\sigma - \tfrac{1}{2} \frac{\partial \log \theta_i}{\partial t_i} - \tfrac{1}{2} \frac{\partial \log \theta_{i+1}}{\partial t_i}] \quad ..(6\text{-}68)$$

We get (6-67) from (6-66) by making use of (6-65).

$$2 \frac{\partial^2 H}{\partial t_{i+1} \, \partial t_i} = \tfrac{1}{2} \theta_{i+1}^{-3/2} \frac{\partial \theta_{i+1}}{\partial t_{i+1}} \emptyset(t_i-\sigma/2) - \tfrac{1}{2} \theta_{i+1}^{-\frac{1}{2}} \frac{\partial \theta_{i+1}}{\partial t_{i+1}} \emptyset(t_i+\sigma/2)$$

$$... \quad (6\text{-}69)$$

$$= -\tfrac{1}{2} (\sqrt{\theta_{i+1}} - \sqrt{\theta_i}) \, \emptyset(t_i+\sigma/2) \frac{1}{\theta_{i+1}} \frac{\partial \theta_{i+1}}{\partial t_{i+1}} \quad ... \quad (6\text{-}70)$$

$$= -\tfrac{1}{2} (\sqrt{\theta_{i+1}} - \sqrt{\theta_i}) \, \emptyset(t_i+\sigma/2) \frac{\partial \log \theta_{i+1}}{\partial t_{i+1}} < 0 \quad ... \quad (6\text{-}71)$$

$$2 \frac{\partial^2 H}{\partial t_{i-1} \, \partial t_i} = -\tfrac{1}{2} \theta_i^{-3/2} \frac{\partial \theta_i}{\partial t_{i-1}} \emptyset(t_i-\sigma/2) + \tfrac{1}{2} \theta_i^{-\frac{1}{2}} \frac{\partial \theta_i}{\partial t_{i-1}} \emptyset(t_i+\sigma/2)$$

$$... \quad (6\text{-}72)$$

$$= -\tfrac{1}{2} (\sqrt{\theta_{i+1}} - \sqrt{\theta_i}) \, \emptyset(t_i+\sigma/2) \frac{1}{\theta_i} \frac{\partial \theta_i}{\partial t_{i-1}} \quad ... \quad (6\text{-}73)$$

$$= -\tfrac{1}{2} (\sqrt{\theta_{i+1}} - \sqrt{\theta_i}) \, \emptyset(t_i+\sigma/2) \frac{\partial \log \theta_i}{\partial t_{i-1}} < 0 \quad ... \quad (6\text{-}74)$$

We get (6-70) and (6-73) from (6-69) and (6-72) respectively after making use of (6-65). Since $\sigma > 0$, we have $\theta_{i+1} > \theta_i$ and (6-71) and (6-74) follow from lemma 5. Hence

$$2\left[\frac{\partial^2 H}{\partial t_i^2} - \left|\frac{\partial^2 H}{\partial t_{i-1}\,\partial t_i}\right| - \left|\frac{\partial^2 H}{\partial t_{i+1}\,\partial t_i}\right|\right]$$

$$= \left(\sqrt{\theta_{i+1}} - \sqrt{\theta_i}\right) \phi(t_i+\sigma/2) \left\{\sigma - \frac{1}{2}\frac{\partial \log \theta_i}{\partial t_i} - \frac{1}{2}\frac{\partial \log \theta_{i+1}}{\partial t_i} - \frac{1}{2}\frac{\partial \log \theta_{i+1}}{\partial t_{i+1}} - \frac{1}{2}\frac{\partial \log \theta_i}{\partial t_{i-1}}\right\} \quad \ldots \quad (6\text{-}75)$$

$$= \frac{1}{2}\left(\sqrt{\theta_{i+1}} - \sqrt{\theta_i}\right) \phi(t_i+\sigma/2) \left\{\frac{\partial W(t_{i-1}, t_i|\sigma)}{\partial t_{i-1}} + \frac{\partial W(t_{i-1}, t_i|\sigma)}{\partial t_i} + \frac{\partial W(t_{i+1}, t_i|\sigma)}{\partial t_i} + \frac{\partial W(t_{i+1}, t_i|\sigma)}{\partial t_{i+1}}\right\} \quad \ldots \quad (6\text{-}76)$$

We obtain (6-76) from (6-75) by noting that

$$W(t_{i-1}, t_i|\sigma) = \sigma t_i - \log \theta_i$$

$$W(t_{i+1}, t_i|\sigma) = \sigma t_i - \log \theta_{i+1}$$

We get from (6-76) and theorem 4, that for all $\sigma > 0$

$$\frac{\partial^2 H}{\partial t_i^2} - \left|\frac{\partial^2 H}{\partial t_{i-1}\, t_i}\right| - \left|\frac{\partial^2 H}{\partial t_{i+1}\,\partial t_i}\right| > 0$$

Hence the elements of the Hessian matrix evaluated at the unique solution of (4-19) satisfy all the conditions imposed in lemma 11. The Hessian is positive definite at that point. This proves the theorem.

Define the function $\Omega(x, y)$ for all $(x, y) \in R^2$ by

$$\Omega(x, y) = y + \frac{\phi(y) - \phi(x)}{\Phi(y) - \Phi(x)}$$

$$= y - \lambda(x, y) \quad \dots \quad \dots \quad \dots \quad (6\text{-}77)$$

The limits of $\Omega(x, y)$ as $x \longrightarrow \pm\infty$ exist and they are denoted by $\Omega(\pm\infty, y)$. Further in section 4, we saw ((4-33) and (4-34)).

$$\lim_{\sigma \to 0} \frac{1}{\sigma} W(x,y|\sigma) = \lim_{\sigma \to 0} \left\{ y - \frac{\log \Theta(x,y|\sigma)}{\sigma} \right\}$$

$$= \Omega(x, y) \quad \dots \quad \dots \quad \dots \quad (6\text{-}78)$$

If we put $k = (r-1)$ in (6-55), we get the equivalent system of equations for $\sigma > 0$.

$$\frac{1}{\sigma} W(t_{i-1}, t_i|\sigma) + \frac{1}{\sigma} W(t_{i+1}, t_i|\sigma) = 0 \quad i=1 \text{ to } k \quad \dots \quad (6\text{-}79)$$

Hence the system of equations

$$\Omega(t_{i-1}, t_i) + \Omega(t_{i+1}, t_i) = 0, \quad i=1 \text{ to } k \quad \dots \quad \dots \quad (6\text{-}80)$$

is the limiting form of the system (6-79) as $\sigma \longrightarrow 0^+$. In both(6-79) and (6-80) we adopt the convention that $t_o = -\infty$ and $t_{k+1} = \infty$. In the determination of optimum spacings for large sample estimation of the mean of a normal population (with known standard deviation) by a selected set of k sample quantiles, Ogawa [7] obtained the system of equations (6-79). He also noted that if $-\infty < t_1 < t_2 \cdots\cdots < t_k < \infty$ is a solution of (6-80), then $-\infty < -t_k < -t_{k-1} \cdots\cdots < -t_1 < \infty$ is also a solution of (6-80). Later Higuchi [8] showed that (6-80) has a unique solution which satisfies the order condition (6-56). We shall now study in some detail the equations of type (6-79) and their relationship with equations of type (6-80).

THEOREM 7

Let x and z be such that $-\infty \leq x \leq \infty$ and $-\infty \leq z \leq \infty$. Further for a given $\sigma > 0$, let y be such that

$$W(x, y \mid \sigma) + W(z, y \mid \sigma) = 0 \quad \cdots \quad \cdots \quad (6\text{-}81)$$

Then we have, $y=x$ when $x=z$ and $x < y < z$ when $x < z$. Further if at least one of x and z is finite and $x \neq z$, then $y=0$ when $x+z=0$, $y < 0$ when $x+z < 0$ and $y > 0$ when $x+z > 0$.

Proof

We saw in theorem 5, that for given x and z, there exists a unique y which satisfies (6-81). Actually the first part of the present theorem has already been proved in theorem 5. Hence we prove the second part only.

Case (i) Let $x+z = 0$, it is easily seen that

$$\theta(x,y|\sigma) = 1/\theta(z,y|\sigma) \text{ when } x+z=0 \text{ and}$$

$$W(x, 0|\sigma) + W(z, 0|\sigma) = 0 \text{ for all } \sigma$$

Case (ii) Let $x+z \neq 0$. In this case, we can take without loss of generality that $x < z$. We recall that

$$\frac{\partial W(x, y|\sigma)}{\partial x} < 0 \text{ and } \frac{\partial W(x, y|\sigma)}{\partial y} > 0$$

for all $\sigma > 0$.

Let $x = -\infty$. It can be easily verified that for all σ

$$W(-\infty, 0|\sigma) + W(\infty, 0|\sigma) = 0 \qquad \ldots \quad \ldots \quad (6\text{-}82)$$

For $\sigma > 0$, we have $W(z, 0|\sigma) > W(\infty, 0|\sigma)$ for all $z \in R$.

Hence (6-82) implies that for $\sigma > 0$

$$W(-\infty, 0|\sigma) + W(z, 0|\sigma) > 0 \qquad \ldots \quad \ldots \quad (6\text{-}83)$$

which in turn implies $y < 0$. Similarly we can prove $y > 0$ when $z = \infty$.

Let $-\infty < x < z < \infty$. If $x+z < 0$, we have $x < 0$ and $-x > z$. It is easily verified that

$$W(x, 0|\sigma) + W(-x, 0|\sigma) = 0 \qquad \ldots \quad \ldots \quad (6\text{-}84)$$

Since $\sigma > 0$ and $-x > z$, we have $W(z, 0|\sigma) > W(-x, 0|\sigma)$.

Hence (6-84) implies

$$W(x, 0 \mid \sigma) + W(z, 0 \mid \sigma) > 0 \quad \ldots \quad \ldots \quad (6\text{-}85)$$

which in turn implies $y < 0$. The required result for the case $x+z > 0$ can be proved in a similar manner.

THEOREM 8

Let $\sigma_o > 0$ and x, z be such that $-\infty \leq x < z \leq \infty$ and at least one of them is finite. Further, let $y = y_o$ be the unique solution of

$$W(x, y \mid \sigma_o) + W(z, y \mid \sigma_o) = 0 \quad \ldots \quad (6\text{-}86)$$

Then we have

$$W(x, y_o \mid \sigma) + W(z, y_o \mid \sigma) \begin{cases} > 0 & \text{for all } \sigma > \sigma_o \text{ when } y_o+z \leq 0 \\ = 0 & \text{for all } \sigma \text{ when } x+z = 0 \\ < 0 & \text{for all } \sigma > \sigma_o \text{ when } x+y_o \geq 0 \end{cases}$$

Proof

The required result for the case when $x+z = 0$ is trivial. Hence we consider the case when $x+z \neq 0$. We know that $x < y_o < z$. For fixed x, y_o, and z, let us denote by $G(\sigma)$ the function of σ

$$G(\sigma) = 2\sigma y_o - \log \Theta (x, y_o \mid \sigma) - \log \Theta (y_o, z \mid \sigma) \quad \ldots \quad (6\text{-}87)$$

We have by (6-46), (6-11) and (6-17)

$$\frac{dG(\sigma)}{d\sigma} = 2y_o - \tfrac{1}{2} \left[\lambda(x-\sigma/2, y_o-\sigma/2) + \lambda(x+\sigma/2, y_o+\sigma/2) + \lambda(y_o-\sigma/2, z-\sigma/2) + \lambda(y_o+\sigma/2, z+\sigma/2) \right] \quad \ldots \quad (6\text{-}88)$$

$$\frac{d^2G(\sigma)}{d\sigma^2} = \tfrac{1}{4}\left[\mu_2(x+\sigma/2,\ y_o+\sigma/2) - \mu_2(x-\sigma/2,\ y_o-\sigma/2) + \mu_2(y_o+\sigma/2,\ z+\sigma/2) - \mu_2(y_o-\sigma/2,\ z-\sigma/2)\right] \quad \ldots \quad (6\text{-}89)$$

Since $y_o+z \leq 0$ implies $x+y_o < 0$ and $x+y_o \geq 0$ implies $y_o+z > 0$,

We have for all $\sigma > 0$ by lemma 2, that

$$\frac{d^2\, G(\sigma)}{d\,\sigma^2} \begin{cases} > 0 & \text{when } y_o + z \leq 0 \\ < 0 & \text{when } x + y_o \geq 0 \end{cases}$$

Since y_o satisfies (6-86) we have

$$2\,\sigma_o\, y_o = \log \theta\,(x,\ y_o|\sigma_o) + \log \theta\,(y_o,\ z|\sigma_o)$$

Therefore it follows that

$$\frac{dG}{d\sigma}\Big/_{\sigma=\sigma_o} = \frac{1}{\sigma_o}\left[\log \theta\,(x,\ y_o|\sigma_o) + \log \theta\,(y_o,\ z|\sigma_o)\right]$$

$$- \tfrac{1}{2}\left[\lambda(x-\sigma_o/2,\ y_o-\sigma_o/2) + \lambda\,(x+\sigma_o/2,\ y_o+\sigma_o/2)\right.$$

$$\left. + \lambda(y_o-\sigma_o/2,\ z-\sigma_o/2) + \lambda(y_o+\sigma_o/2,\ z+\sigma_o/2)\right]$$

Since $x < y_o < z$, we have by lemma 8, for all $\sigma_o > 0$

$$\frac{dG}{d\sigma}\Big|\,\sigma=\sigma_o \begin{cases} > 0 & \text{when } y_o + z \leq 0 \\ < 0 & \text{when } y_o + x \geq 0 \end{cases}$$

Since $G(\sigma_o) = 0$, the required result follows.

THEOREM 9

Let x and y be such that $-\infty \leq x \leq \infty$, and $-\infty \leq y \leq \infty$, $x \neq y$ and at least one of them is finite. Then we have for all $\sigma > 0$.

$$W(x, y|\sigma) - \sigma\Omega(x, y) \begin{cases} > 0 & \text{if } x+y < 0 \\ = 0 & \text{if } x+y = 0 \\ < 0 & \text{if } x+y > 0 \end{cases}$$

Proof

It is easily verified that $W(x, y|\sigma) = \sigma\Omega(x, y)$ when $x+y = 0$. Hence we need consider the case $x+y \neq 0$.

We note that $W(x, y|0) = 0$ and

$$\frac{\partial W(x,y|\sigma)}{\partial \sigma} = y - \tfrac{1}{2}\left[\lambda(x-\sigma/2, y-\sigma/2) + \lambda(x+\sigma/2, y+\sigma/2)\right]$$

$$\frac{\partial^2 W(x,y|\sigma)}{d\sigma^2} = \tfrac{1}{4}\left[\mu_2(x+\sigma/2, y+\sigma/2) - \mu_2(x-\sigma/2, y-\sigma/2)\right]$$

For fixed x and y, we can consider $W(x, y|\sigma)$ as a function of σ and expanding $W(x, y|\sigma)$ as a Taylor's series about $\sigma = 0$, we get

$$W(x,y|\sigma) = \sigma\left[y - \lambda(x,y)\right] + \frac{\sigma^2}{8}\left[\mu_2(x+\alpha\sigma/2, y+\alpha\sigma/2) - \mu_2(x-\alpha\sigma/2, y-\alpha\sigma/2)\right] \quad \ldots \quad (6\text{-}90)$$

where $0 < \alpha < 1$. The required result follows immediately from lemma 2.

THEOREM 10

For any given positive constant σ, consider the system of equations

$$W(t_{i-1}, t_i | \sigma) + W(t_{i+1}, t_i | \sigma) = 0, \quad i=1 \text{ to } 2n \qquad \ldots \quad (6\text{-}91)$$

where $t_o = -\infty$, $t_{2n+1} = \infty$ and n is some given positive integer.

Let $g_i(\sigma)$, $i = 1$ to $2n$ be the unique solution of (6-91). Let t_i', $i = 1$ to $2n$ be such that $-\infty < t_1' < t_2' \ldots\ldots < t_{2n}' < \infty$, $t_{2n+1-i}' = -t_i'$ for $i = 1$ to $2n$ and

$$W(t_{i-1}', t_i' | \sigma) + W(t_{i+1}', t_i' | \sigma) > 0 \quad \text{for } i = 1 \text{ to } n$$
$$W(t_{i-1}', t_i' | \sigma) + W(t_{i+1}', t_i' | \sigma) < 0 \quad \text{for } i = n+1 \text{ to } 2n \qquad \ldots \quad (6\text{-}92)$$

where for notational convenience we again take $t_o' = -\infty$ and $t_{2n+1}' = \infty$. Then we have $g_i(\sigma) < t_i'$ for $i=1$ to n and $g_i(\sigma) > t_i'$ for $i = n+1$ to $2n$.

Proof

For simplifying the notation, we shall write g_i for $g_i(\sigma)$. We recall that g_i are such that $-\infty < g_1 < \ldots < g_n < 0$ and $g_{2n+1-i} = -g_i$ for $i = 1$ to $2n$. Let $g_i = t_i' + \delta_i$ for $i = 1$ to $2n$. We have $g_{n+1} = -g_n = -t_n' - \delta_n = t_{n+1}' - \delta_n$, i.e., $\delta_{n+1} = -\delta_n$. In fact we have $\delta_{2n+1-i} = -\delta_i$. Hence it is enough to show that $\delta_i < 0$ for $i = 1$ to n. We recall from theorem 4 that for $\sigma > 0$.

$$\frac{\partial W(x, y|\sigma)}{\partial x} < 0, \quad \frac{\partial W(x, y|\sigma)}{\partial y} > 0$$

$$\frac{\partial W(x, y|\sigma)}{\partial x} + \frac{\partial W(x, y|\sigma)}{\partial y} > 0$$

We shall now prove a simple result which is later used to prove the theorem, that is,

$$\delta_{i+1} \geq 0, \ \delta_{i+1} > \delta_i \ \text{ implies } \ \delta_{i+2} > \delta_{i+1} \quad \text{for } i=1 \text{ to } n-1 \quad \ldots(6\text{-}93)$$

To prove (6-93) consider the $(i+1)^{th}$ inequality of (6-92)

$$W(t_i', t_{i+1}' | \sigma) + W(t_{i+2}', t_{i+1}' | \sigma) > 0 \quad \ldots \quad \ldots(6\text{-}94)$$

Since $\delta_{i+1} \geq 0$ and $\delta_{i+1} > \delta_i$, (6-94) implies

$$W(t_i' + \delta_i, t_{i+1}' + \delta_{i+1}|\sigma) + W(t_{i+2}' + \delta_{i+1}, t_{i+1}' + \delta_{i+1}|\sigma) > 0$$

or $W(g_i, g_{i+1}|\sigma) + W(t_{i+2}' + \delta_{i+1}, g_{i+1} | \sigma) > 0 \quad \ldots \quad \ldots (6\text{-}95)$

It is seen that (6-95) implies $g_{i+2} > t_{i+2}' + \delta_{i+1}$ or $\delta_{i+2} > \delta_{i+1}$ and this proves (6-93).

We shall now show that $\delta_1 < 0$. Suppose $\delta_1 \geq 0$, and consider the first inequality of (6-92)

$$W(-\infty, t_1' |\sigma) + W(t_2', t_1' |\sigma) > 0 \quad \ldots \quad \ldots (6\text{-}96)$$

Since $\delta_1 \geq 0$, (6-96) implies

$$W(-\infty, t_1' + \delta_1 | \sigma) + W(t_2' + \delta_1, t_1' + \delta_1 | \sigma) > 0$$

or $$W(-\infty, g_1 | \sigma) + W(t_2' + \delta_1, g_1 | \sigma) > 0 \quad \ldots \ldots \quad (6\text{-}97)$$

We see that (6-97) implies $g_2 > t_2' + \delta_1$ or $\delta_2 > \delta_1$. Now using the implication result (6-93) successively, we see that

$$\delta_1 \geq 0 \text{ implies } \delta_{n+1} > \delta_n > \ldots\ldots \delta_1 \geq 0 \quad \ldots \ldots \quad (6\text{-}98)$$

We note that $\delta_{n+1} = -\delta_n$ and as such it is impossible to have $\delta_n \geq 0$ and $\delta_{n+1} > \delta_n$. Hence the assumption $\delta_1 \geq 0$ leads to a contradiction. Therefore we must have $\delta_1 < 0$.

We shall now show that $\delta_2 < 0$. Suppose $\delta_2 \geq 0$, then since $\delta_1 < 0$, we have $\delta_2 \geq 0$ and $\delta_2 > \delta_1$. Again using the implication result (6-93) successively, we get

$$\delta_1 < 0, \quad \delta_2 \geq 0 \text{ implies } \delta_{n+1} > \delta_n > \ldots\ldots \delta_2 \geq 0$$

which leads to a contradiction. Hence we must have $\delta_2 < 0$. Similarly we can prove $\delta_3 < 0$, ... and $\delta_n < 0$.

<u>THEOREM 11</u>

For a given positive constant σ , consider the system of equations

$$W(t_{i-1}, t_i | \sigma) + W(t_{i+1}, t_i | \sigma) = 0 \text{ for } i = 1 \text{ to } 2n+1 \quad \ldots \quad (6\text{-}99)$$

where $t_o = -\infty$, $t_{2n+2} = \infty$ and n is some given positive integer. Let $g_i(\sigma)$, $i = 1$ to $2n+1$ be the unique solution of (6-99). Let t_i', $i = 1$ to $2n+1$ be such that $-\infty < t_1' < t_2' \ldots < t_{2n+1}' < \infty$; $t_{2n+2-i}' = -t_i'$ and

$$W(t_{i-1}', t_i' | \sigma) + W(t_{i+1}', t_i' | \sigma) > 0 \quad \text{for } i = 1 \text{ to } n$$

$$W(t_n', t_{n+1}' | \sigma) + W(t_{n+2}', t_{n+1}' | \sigma) = 0$$

$$W(t_{i-1}', t_i' | \sigma) + W(t_{i+1}', t_i' | \sigma) < 0 \quad \text{for } i = n+2 \text{ to } 2n+1$$

where for notational convenience, we again take $t_o' = -\infty$ and $t_{2n+2}' = \infty$. Then we have $g_i(\sigma) < t_i'$ for $i = 1$ to n; $g_{n+1}(\sigma) = t_{n+1}' = 0$ and $g_i(\sigma) > t_i'$ for $i = n+2$ to $2n+1$.

Proof : Similar to that of theorem 10.

THEOREM 12

Consider the system of equations

$$W(t_{i-1}, t_i | \sigma) + W(t_{i+1}, t_i | \sigma) = 0, \quad i = 1 \text{ to } k \quad \ldots \quad (6\text{-}100)$$

Let $g_i(\sigma)$, $i = 1$ to k be the unique solution of (6-100) for any given $\sigma > 0$. Then it follows that

(i) $g_i(\sigma) = 0$ when $i = \frac{k+1}{2}$ for all $\sigma > 0$.

(ii) $g_i(\sigma)$ strictly decreases (increases) with σ when $i-(k+1)/2$ is negative (positive) for all $\sigma > 0$.

Proof

We shall prove the theorem only for the case where k is an even integer, the proof the for the case where k is odd being similar. Let $k = 2n$ and $g_i(\sigma_o)$ $i = 1$ to $2n$ be the unique solution of (6-100) for some given $\sigma_o > 0$. It is known that

$$-\infty < g_1(\sigma_o) < g_2(\sigma_o) \quad \ldots\ldots < g_n(\sigma_o) < 0$$

and $g_{2n+1-i}(\sigma_o) = -g_i(\sigma_o)$. Hence it follows from theorem 8 that for all $\sigma > \sigma_o$

$$W(g_{i-1}(\sigma_o), g_i(\sigma_o) | \sigma) + W(g_{i+1}(\sigma_o), g_i(\sigma_o) | \sigma) > 0 \quad \text{for } i=1 \text{ to } n$$

$$W(g_{i-1}(\sigma_o), g_i(\sigma_o) | \sigma) + W(g_{i+1}(\sigma_o), g_i(\sigma_o) | \sigma) < 0 \quad \text{for } i=n+1 \text{ to } 2n.$$

From theorem 10, we get $g_i(\sigma) < g_i(\sigma_o)$ for $i = 1$ to n and $g_i(\sigma) > g_i(\sigma_o)$ for $i = n+1$ to $2n$ for all $\sigma > \sigma_o$

THEOREM 13

Let $g_i(\sigma)$, $i = 1$ to k be the unique solution of

$$W(t_{i-1}, t_i | \sigma) + W(t_{i+1}, t_i | \sigma) = 0, \quad i = 1 \text{ to } k \quad \ldots \quad \ldots(6\text{-}101)$$

for any given $\sigma > 0$. Let t_i^*, $i = 1$ to k be the unique solution of

$$\Omega(t_{i-1}, t_i) + \Omega(t_{i+1}, t_i) = 0, \quad i = 1 \text{ to } k \quad \ldots \quad (6-102)$$

In both (6-101) and (6-102), we adopt the convention that $t_0 = -\infty$ and $t_{k+1} = \infty$. Then it follows that

(i) $g_i(\sigma) = t_i^*$ when $i - \frac{k+1}{2} = 0$

(ii) $g_i(\sigma)$ is less (greater) than t_i^* when $i - \frac{k+1}{2}$ is less (greater) than zero.

<u>Proof</u>

We shall prove the theorem only for the case where k is an even positive integer and the proof for the case where k is odd is similar. Let $k = 2n$ and in this case we know (Ogawa [7] and Higuchi [8]), that $-\infty < t_1^* < t_2^* < \ldots < t_n^* < 0$ and $t_{2n+1-i}^* = -t_i^*$. By theorem 9, we have for all $\sigma > 0$

$$W(t_{i-1}^*, t_i^* | \sigma) + W(t_{i+1}^*, t_i^* | \sigma) > \sigma [\Omega(t_{i-1}^*, t_i^*) + \Omega(t_{i+1}^*, t_i^*)] = 0$$

for $i = 1$ to n.

$$W(t_{i-1}^*, t_i^* | \sigma) + W(t_{i+1}^*, t_i^* | \sigma) < \sigma [\Omega(t_{i-1}^*, t_i^*) + \Omega(t_{i+1}^*, t_i^*)] = 0$$

for $i = n+1$ to $2n$.

The required result then follows from theorem 10.

We shall now slightly generalise lemma 11 for subsequent use :

Lemma 12

A square matrix $((a_{ij}))$ of real numbers having the following properties is positive definite.

(i) All elements except diagonal ones and those adjacent to any diagonal one, are zero.

(ii) Every diagonal element is greater than the row sum of the absolute values of the elements adjacent to it.

Proof

Let n be the order of the matrix and A_m be the determinant of the submatrix $((a_{ij}))$; $i = 1$ to m and $j = 1$ to m. We have from the hypothesis of the lemma

$$a_{11} > |a_{12}|$$

$$a_{ii} > |a_{i,i-1}| + |a_{i,i+1}| \quad \text{for } i = 2 \text{ to } n-1 \qquad \ldots (6\text{-}103)$$

$$a_{nn} > |a_{n,n-1}|$$

It is easily verified that

$$A_m = a_{mm} A_{m-1} - a_{m,m-1} \, a_{m-1,m} \, A_{m-2} \qquad \ldots \quad \ldots (6\text{-}104)$$

for $3 \leq m \leq n$. We shall now show that for $2 \leq m \leq n-2$, $A_{m-1} > 0$

and $A_m > |a_{m,m+1}| A_{m-1}$ implies that $A_{m+1} > |a_{m+1,m+2}| A_m$.

Suppose $A_{m-1} > 0$ and $A_m > |a_{m,m+1}| A_{m-1}$, then we get from (6-104) and (6-103)

$$A_{m+1} = a_{m+1,m+1} A_m - a_{m+1,m} a_{m,m+1} A_{m-1}$$

$$\geq a_{m+1,m+1} A_m - |a_{m+1,m}| |a_{m,m+1}| A_{m-1}$$

$$> a_{m+1,m+1} A_m - |a_{m+1,m}| A_m = A_m(a_{m+1,m+1} - |a_{m+1,m}|)$$

$$> A_m |a_{m+1,m+2}| \geq 0 \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}105)$$

We note that

$$A_1 = a_{11} > |a_{12}| \geq 0$$

$$A_2 = a_{11} a_{22} - a_{21} a_{12}$$

$$\geq a_{11} a_{22} - |a_{21}| |a_{12}|$$

$$> a_{11} a_{22} - a_{11} |a_{21}| = a_{11} (a_{22} - |a_{21}|) > 0$$

$$> a_{11} |a_{23}|$$

Hence we see that the result $A_{m-1} > 0$ and $A_m > |a_{m,m+1}| A_{m-1}$ holds true for $m=2$. Hence we get by mathematical induction

$$A_1 > 0, \; A_2 > 0, \ldots . A_{n-1} > 0 \text{ and } A_{n-1} > |a_{n-1,n}| A_{n-2} \quad \ldots (6\text{-}106)$$

Finally we get by

$$A_n = a_{n,n} A_{n-1} - a_{n,n-1} a_{n-1,n} A_{n-2}$$

$$\geq a_{n,n} A_{n-1} - |a_{n,n-1}| \; |a_{n-1,n}| \; A_{n-2}$$

$$> a_{n,n} A_{n-1} - |a_{n,n-1}| A_{n-1} = A_{n-1}(a_{n,n} - |a_{n,n-1}|) > 0$$

and this proves the required result.

We recall from theorem 5, that there exists a unique solution of the system of equations (6-55) or, equivalently of the system (6-79). For a given $\sigma > 0$, let $(g_1(\sigma), g_2(\sigma), \ldots\ldots g_k(\sigma))$ be the unique solution of (6-79). We shall now show that $g_i(\sigma)$, $i = 1$ to k are differentiable functions of σ and also $\lim_{\sigma \longrightarrow 0+} g_i(\sigma) = t_i^*$ where $(t_1^*, t_2^*, \ldots\ldots t_k^*)$ is the unique solution of (6-30). This is equivalent to the statement that the transformed optimal group boundaries (see section 4.4) tend to Ogawa's optimal spacings as $\sigma \longrightarrow 0^+$.

For this purpose define the function $h(x, y|\sigma)$ by

$$h(x, y|\sigma) = \begin{cases} \frac{1}{\sigma} \log \theta(x, y|\sigma) & \text{when } \sigma \neq 0 \\ \lambda(x, y) & \text{when } \sigma = 0 \end{cases}$$

It is easily verified that $h(x, y|\sigma) \longrightarrow \lambda(x, y)$ as $\sigma \longrightarrow 0$.

It can also be verified that

$$\frac{\partial}{\partial x} h(x,y|\sigma) \longrightarrow \frac{\partial}{\partial x} \lambda(x,y) \text{ as } \sigma \longrightarrow 0$$

$$\frac{\partial}{\partial \sigma} h(x,y|\sigma) \longrightarrow \frac{\partial}{\partial \sigma} h(x,y|\sigma) /_{\sigma=0} \text{ as } \sigma \longrightarrow 0.$$

Hence it is noted that the function $h(x,y|\sigma)$ has continuous partial derivatives. Define $w(x,y|\sigma)$ by

$$w(x,y|\sigma) = y - h(x,y|\sigma) \qquad \dots \quad \dots \quad \dots \quad (6\text{-}107)$$

It is easily seen that

$$w(x,y|\sigma) = \begin{cases} \frac{1}{\sigma} W(x,y|\sigma) & \text{when } \sigma \neq 0 \\ \Omega(x,y) & \text{when } \sigma = 0. \end{cases}$$

By theorem 4, we have for $\sigma > 0$

$$\frac{\partial W(x,y|\sigma)}{\partial x} < 0, \quad \frac{\partial W(x,y|\sigma)}{\partial y} > 0$$

$$\frac{\partial W(x,y|\sigma)}{\partial x} + \frac{\partial W(x,y|\sigma)}{\partial y} > 0 \quad \text{for } x \neq y$$

Higuchi [8] has proved similar results for $\Omega(x,y)$ i.e.,

$$\frac{\partial \Omega(x, y)}{\partial x} < 0, \quad \frac{\partial \Omega(x,y)}{\partial y} > 0$$

$$\frac{\partial \Omega(x,y)}{\partial x} + \frac{\partial \Omega(x,y)}{\partial y} > 0 \quad \text{for} \quad x \neq y$$

Hence it is seen that for $\sigma \geq 0$, we have

$$\frac{\partial w(x,y \mid \sigma)}{\partial x} < 0 \, , \quad \frac{\partial w(x,y \mid \sigma)}{\partial y} > 0 \qquad \ldots \quad \ldots (6\text{-}108)$$

$$\frac{\partial w(x,y \mid \sigma)}{\partial x} + \frac{\partial w(x,y \mid \sigma)}{\partial y} > 0 \quad \text{for} \quad x \neq y$$

we can now represent (6-79) and (6-80) together as

$$w(t_{i-1}, t_i \mid \sigma) + w(t_{i+1}, t_i \mid \sigma) = 0, \quad i = 1 \text{ to } k \quad \ldots (6\text{-}109)$$

where the functions $w(x,y \mid \sigma)$ posess continuous partial derivatives.

THEOREM 14

Let $g_i(\sigma)$, $i = 1$ to k be the unique solution of the system of equations

$$w(t_{i-1}, t_i \mid \sigma) + w(t_{i+1}, t_i \mid \sigma) = 0, \quad i = 1 \text{ to } k \quad \ldots (6\text{-}110)$$

for $\sigma \geq 0$. *Then* $g_i(\sigma)$, $i = 1$ *to* k *are differentiable functions of* σ *for* $\sigma \geq 0$.

Proof

We know that for any $\sigma \geq 0$, there exists a unique solution $g_i(\sigma)$, $i = 1$ to k to (6-110). Further we have

$$-\infty < g_1(\sigma) < g_2(\sigma) \; \ldots\ldots < g_k(\sigma) < \infty \qquad \ldots \quad \ldots (6\text{-}111)$$

Let $F_i(t_1, t_2, \ldots\ldots t_k | \sigma) = w(t_{i-1}, t_i | \sigma) + w(t_{i+1}, t_i | \sigma)$. Then it is enough to show that the Jacobian

$$\frac{\partial(F_1, F_2, \ldots\ldots F_k)}{\partial(t_1, t_2, \ldots\ldots t_k)} \neq 0$$

at the solution of (6-110) for any $\sigma \geq 0$. The required result then follows from the well known theorem of Implicit Functions (Goursat [11]). By the definition of the functions F_i, it follows.

$$\frac{\partial F_1}{\partial t_1} = \frac{\partial}{\partial t_1} w(-\infty, t_1 | \sigma) + \frac{\partial}{\partial t_1} w(t_2, t_1 | \sigma)$$

$$\frac{\partial F_1}{\partial t_2} = \frac{\partial}{\partial t_2} w(t_2, t_1 | \sigma) ; \quad \frac{\partial F_1}{\partial t_j} = 0 \quad \text{for} \quad j > 2$$

$$\left.\begin{aligned}
\frac{\partial F_i}{\partial t_{i-1}} &= \frac{\partial}{\partial t_{i-1}} w(t_{i-1}, t_i | \sigma) \\
\frac{\partial F_i}{\partial t_i} &= \frac{\partial}{\partial t_i} w(t_{i-1}, t_i | \sigma) + \frac{\partial}{\partial t_i} w(t_{i+1}, t_i | \sigma) \\
\frac{\partial F_i}{\partial t_{i+1}} &= \frac{\partial}{\partial t_{i+1}} w(t_{i+1}, t_i | \sigma) \\
\frac{\partial F_i}{\partial t_j} &= 0 \quad \text{for} \quad |i - j| \geq 2
\end{aligned}\right\} \quad \text{for } i = 2 \text{ to } k-1$$

$$\frac{\partial F_k}{\partial t_j} = 0 \quad \text{for} \quad j \leq k-2$$

$$\frac{\partial F_k}{\partial t_{k-1}} = \frac{\partial}{\partial t_{k-1}} \cdot w(t_{k-1}, t_k|\sigma)$$

$$\frac{\partial F_k}{\partial t_k} = \frac{\partial}{\partial t_k} w(t_{k-1}, t_k|\sigma) + \frac{\partial}{\partial t_k} w(t_{k+1}, t_k|\sigma)$$

Further it follows from (6-108) that at any point satisfying

$-\infty < t_1 < t_2 \cdots\cdots < t_k < \infty$, that

$$\frac{\partial F_1}{\partial t_1} - \left|\frac{\partial F_1}{\partial t_2}\right| = \frac{\partial}{\partial t_1} w(-\infty, t_1|\sigma) + \frac{\partial w(t_2, t_1|\sigma)}{\partial t_1} + \frac{\partial w(t_2, t_1|\sigma)}{\partial t_2} > 0$$

$$\frac{\partial F_i}{\partial t_i} - \left|\frac{\partial F_i}{\partial t_{i-1}}\right| - \left|\frac{\partial F_i}{\partial t_{i+1}}\right| =$$

$$= \frac{\partial}{\partial t_i} w(t_{i-1}, t_i|\sigma) + \frac{\partial}{\partial t_i} w(t_{i+1}, t_i|\sigma) + \frac{\partial}{\partial t_{i-1}} w(t_{i-1}, t_i|\sigma)$$

$$+ \frac{\partial}{\partial t_{i+1}} w(t_{i+1}, t_i|\sigma) > 0 \quad \text{for } i = 2 \text{ to } k-1$$

Similarly it can be verified that

$$\frac{\partial F_k}{\partial t_k} - \left|\frac{\partial F_k}{\partial t_{k-1}}\right| > 0$$

Therefore it follows from lemma 12, that the Jacobian $\frac{\partial(F_1, F_2, \ldots F_k)}{\partial(t_1, t_2, \ldots t_k)}$ evaluated at any point such that $-\infty < t_1 < t_2 \; \ldots\ldots < t_k < \infty$ is non-zero. It therefore follows from (6-111) that the Jacobian evaluated at the solution of (6-110) for any $\sigma \geq 0$ is non-zero.

By definition $g_i(0) = t_i^*$ for $i = 1$ to k and theorem 14 tells that $g_i(\sigma)$ are continuous functions of σ for $\sigma \geq 0$. (Actually $g_i(\sigma)$ is continuous in some open interval $(-\delta, \infty)$ where $\delta > 0$). It follows therefore that $\lim_{\sigma \to 0^+} g_i(\sigma) = g_i(0) = t_i^*$ for $i = 1$ to k. That is the transformed optimal group boundaries tend to Ogawas optimal spacings as $\sigma \longrightarrow 0^+$.

We recall that 'r' is used to denote the number of groups in Group EOQ approach; k (=r-1) is used to denote the number of group boundaries and 'i' is a subscript used for identifying a particular group boundary for a given r. The values of $g_i(\sigma)$ (i.e. transformed optimal group boundaries) are given in tables II to X for i=1 to r-1; r = 2 to 10 and for $\sigma = 0.2$ to 4.0 in steps of 0.2. The values of $t_i^* = g_i(0)$ (i.e. Ogawa's optimal spacings) as computed by Ogawa are given in table XI. A look at the rate of change of $g_i(\sigma)$ with σ in tables III to X indicates that t_i^* must be very nearly same as $g_i(0.2)$. A comparison of values of t_i^* as computed by Ogawa with g_i (0.2) is given in the table below.

COMPARISON OF t_i^* AS COMPUTED

BY OGAWA WITH $g_i(0.2)$

i \ r	3	4	5	6	7	8	9	10
1	-0.613	-0.982	-1.243	-1.447	-1.598	-1.751	-1.866	-1.977
	-0.612	-0.982	-1.244	-1.447	-1.611	-1.748	-1.865	-1.969
2			-0.383	-0.659	-0.860	-1.049	-1.200	-1.329
			-0.382	-0.659	-0.875	-1.050	-1.197	-1.325
3					-0.266	-0.502	-0.684	-0.834
					-0.281	-0.500	-0.681	-0.834
4							-0.222	-0.404
							-0.222	-0.405

Note: The first figure is Ogawa's computation of t_i^* and the second figure is g_i(0.2). Because of symmetry only negative values are given.

It is seen from the above table that $g_i(0.2)$ differs slightly from Ogawa's computation of t_i^* in some cases. Initially it was thought that these differences may be due to the particular form of approximation used in this thesis to evaluate the normal probability integral in the computation of $g_i(\sigma)$. However later calculations indicated that Ogawa's computations are slightly in error in these cases and $g_i(0.2)$'s are nearer the correct values. Since these differences are of no practical significance, Ogawa's values for t_i^* are given in table XI as they are published figures readily available.

We saw in section 4.4 that t_i^*'s are fairly good approximations for $g_i(\sigma)$ when $\sigma \leq 3$. We shall now derive a much better approximation to $g_i(\sigma)$ which give almost the exact values when $\sigma \leq 4$. The arguement used in this derivation is mostly heuristic. For a given k and i, a plot of $g_i(\sigma)$ against σ^2 indicated a surprisingly good linear relationship with intercept t_i^* (see for example figure 1). Hence it was felt that $g_i(\sigma)$ can be well represented by $t_i^* + b_i \sigma^2$ where b_i is some suitable constant. If we put $t_i = t_i^* + b_i \sigma^2$ in (6-55) we get

$$2\sigma(t_i^* + b_i \sigma^2) - \log \theta(t_{i-1}^* + b_{i-1}\sigma^2, t_i^* + b_i \sigma^2 | \sigma)$$

$$- \log \theta(t_i^* + b_i \sigma^2, t_{i+1}^* + b_{i+1}\sigma^2 | \sigma) = 0 \qquad \ldots \qquad \ldots (6\text{-}112)$$

for $i = 1$ to k

We shall now try to find b_i's which satisfy (6-112) approximately for all $0 \leq \sigma \leq 4$. Since t_i^*'s and b_i's are constants, $\theta(t_{i-1}^* + b_{i-1}\sigma^2, t_i^* + b_i\sigma^2 | \sigma)$ can be considered as a function of σ. Expanding $\log \theta(t_{i-1}^* + b_{i-1}\sigma^2, t_i^* + b_i\sigma^2 | \sigma)$ as Taylor series about $\sigma = 0$, we get after omitting terms of fourth order or more

$$\log \theta(t_{i-1}^* + b_{i-1}\sigma^2, t_i^* + b_i\sigma^2 | \sigma) = \log \frac{\Phi(t_i^* + b_i\sigma^2 - \sigma/2) - \Phi(t_{i-1}^* + b_{i-1}\sigma^2 - \sigma/2)}{\Phi(t_i^* + b_i\sigma^2 + \sigma/2) - \Phi(t_{i-1}^* + b_{i-1}\sigma^2 - \sigma/2)}$$

$$= -\sigma \frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} + \frac{\sigma^3}{6}\Bigg[6\, \frac{b_i t_i^* \phi(t_i^*) - b_{i-1} t_{i-1}^* \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)}$$

$$+ 6\, \frac{b_i \phi(t_i^*) - b_{i-1}\phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} \; \frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)}$$

$$+ \frac{2}{8}\Bigg\{ - \frac{t_i^{*2}\phi(t_i^*) - t_{i-1}^{*2}\phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} + \frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)}$$

$$- 3\, \frac{t_i^*\phi(t_i^*) - t_{i-1}^*\phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} \; \frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)}$$

$$- 2 \left(\frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} \right)^3 \Bigg\} \Bigg]$$

$$= -\sigma \frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} + \sigma^3 \left\{ \frac{b_i\, t_i^*\, \phi(t_i^*) - b_{i-1}\, t_{i-1}^*\, \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} \right.$$

$$\left. + \frac{b_i\, \phi(t_i^*) - b_{i-1}\, \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} \; \frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} + \frac{1}{24}\, \mu_3\, (t_{i-1}^*,\, t_i^*) \right\}$$

... ...(6-113)

Substituting (6-113) in (6-112) and noting that t_i^* satisfy (6-80), we get for $\sigma > 0$, the system of equations.

$$2b_i - \frac{b_i\, t_i^*\, \phi(t_i^*) - b_{i-1}\, t_{i-1}^*\, \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} - \frac{b_{i+1}\, t_{i+1}^*\, \phi(t_{i+1}^*) - b_i\, t_i^*\, \phi(t_i^*)}{\Phi(t_{i+1}^*) - \Phi(t_i^*)}$$

$$- \frac{b_i \phi(t_i^*) - b_{i-1}\phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} \; \frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)}$$

$$- \frac{b_{i+1}\, \phi(t_{i+1}^*) - b_i\, \phi(t_i^*)}{\Phi(t_{i+1}^*) - \Phi(t_i^*)} \; \frac{\phi(t_{i+1}^*) - \phi(t_i^*)}{\Phi(t_{i+1}^*) - \Phi(t_i^*)}$$

$$= \frac{1}{24} \left[\mu_3\, (t_{i-1}^*,\, t_i^*) + \mu_3\, (t_i^*\, ,\, t_{i+1}^*) \right] \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}114)$$

for $i = 1$ to k. We can rewrite the system (6-114) as

$$2b_i - b_i \left[1 - \mu_2(t_{i-1}^*,\, t_i^*) + 1 - \mu_2\, (t_i^*\, ,\, t_{i+1}^*) \right]$$

$$- (b_i - b_{i-1}) \; \frac{\phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} \left[t_{i-1}^* + \frac{\phi(t_i^*) - \phi(t_{i-1}^*)}{\Phi(t_i^*) - \Phi(t_{i-1}^*)} \right]$$

$$- (b_{i+1} - b_i) \frac{\phi(t^*_{i+1})}{\Phi(t^*_{i+1}) - \Phi(t^*_i)} \left[t^*_{i+1} + \frac{\phi(t^*_{i+1}) - \phi(t^*_i)}{\Phi(t^*_{i+1}) - \Phi(t^*_i)} \right]$$

$$= \frac{1}{24} \left[\mu_3 (t^*_{i-1}, t^*_i) + \mu_3 (t^*_i, t^*_{i+1}) \right] \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}115)$$

for $i = 1$ to k. In (6-115), we have $b_o = 0$ and $b_{k+1} = 0$. Because of the symmetrical properties of $g_i(\sigma)$, we have $b_{k+1-i} = -b_i$. Solving of (6-115) for b_i's, we get the required approximation. We now solve (6-115) in particular cases.

(i) Three groups

In this case we have $g_i(\sigma) = t^*_i + b_i \sigma^2$ for $i = 1$ to 2; $t^*_2 = - t^*_1$ and $b_2 = -b_1$ where b_1 is the solution of

$$2 b_1 - b_1 \left[1 - \mu_2 (-\infty, t_1) + 1 - \mu_2 (t^*_1, t^*_2) \right]$$

$$+ 2b_1 \frac{\phi(t^*_2)\, t^*_2}{\Phi(t^*_2) - \Phi(t^*_1)} = \frac{1}{24} \mu_3 (-\infty, t^*_1) \quad \ldots \quad (6\text{-}116)$$

or equivalently

$$2b_1 - b_1 (1 - \mu_2(-\infty, t^*_1)) = \frac{1}{24} \mu_3 (-\infty, t^*_1) \quad \ldots \quad \ldots (6\text{-}117)$$

Further $t^*_1 = -0.612$, $t^*_2 = -t^*_1$, $1 - \mu_2 (-\infty, t^*_1) = 0.749093$, $\mu_3 (-\infty, t^*_1) = -0.151336$ and substituting these in (6-117) and solving for b_1 we get $b_1 = -0.005041$. Thus the required approximations are

$$g_1(\sigma) = -0.612 - 0.00504\,\sigma^2$$

$$g_2(\sigma) = 0.612 + 0.00504\,\sigma^2$$

(ii) Four Groups

In this case we have $g_1(\sigma) = t_1^* + b_1\sigma^2$, $g_2(\sigma) = t_2^* = 0$, $g_3(\sigma) = t_3^* + b_3\sigma^2$ where $t_3^* = -t_1^*$, $b_3 = -b_1$ and b_1 is the solution of

$$2b_1 - b_1 \left[1 - \mu_2(-\infty, t_1^*) + 1 - \mu_2(t_1^*, t_2^*) \right]$$

$$+ \; b_1 \frac{\phi(t_2^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \; \frac{\phi(t_2^*) - (t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)}$$

$$= \frac{1}{24} \left[\mu_3(-\infty, t_1^*) + \mu_3(t_1^*, t_2^*) \right] \quad \ldots \quad \ldots \quad \ldots \quad (6\text{-}118)$$

Further we have $t_1^* = -0.982$, $t_2^* = 0$, $\mu_3(-\infty, t_1^*) = -0.1183645$, $\mu_3(t_1^*, t_2^*) = -0.0034083$, $1 - \mu_2(-\infty, t_1^*) = 0.7987849$, $1 - \mu_2(t_1^*, t_2^*) = 0.9230366$ and

$$\frac{\phi(t_2^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \; \frac{\phi(t_2^*) - \phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} = 0.5362665$$

Substituting these in (6-118) and solving we get $b_1 = -0.0062298$. Hence the required approximations are

$g_1(\sigma) = -0.982 - 0.00623\,\sigma^2$, $g_2(\sigma) = 0$ and $g_3(\sigma) = 0.982 + 0.00623\,\sigma^2$

(iii) Five Groups

In this case we have $g_i(\sigma) = t_i^* + b_i \sigma^2$ for $i = 1$ to 4,

$t_{5-i}^* = - t_i^*$, $b_{5-i} = - b_i$ and where (b_1, b_2) is the solution of

$$2b_1 - b_1 \left[1 - \mu_2(-\infty, t_1^*) + 1 - \mu_2(t_1^*, t_2^*) \right]$$

$$- (b_2 - b_1) \frac{\phi(t_2^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \left[t_2^* + \frac{\phi(t_2^*) - \phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \right]$$

$$= \frac{1}{24} \left[\mu_3(-\infty, t_1^*) + \mu_3(t_1^*, t_2^*) \right] \quad \ldots \quad \ldots \quad (6\text{-}119)$$

$$2b_2 - b_2 \left[1 - \mu_2(t_1^*, t_2^*) + 1 - \mu_2(t_2^*, t_3^*) \right]$$

$$- (b_2 - b_1) \frac{\phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \left[t_1^* + \frac{\phi(t_2^*) - \phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \right]$$

$$+ 2b_2 \frac{\phi(t_3^*)\, t_3}{\Phi(t_3^*) - \Phi(t_2^*)} = \frac{1}{24} \mu_3(t_1^*, t_2^*) \quad \ldots \quad (6\text{-}120)$$

Since $t_3^* = - t_2^*$, (6-120) is equivalent to

$$2b_2 - b_2 \left[1 - \mu_2(t_1^*, t_2^*) \right] - (b_2 - b_1) \frac{\phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \left\{ t_1^* + \frac{\phi(t_2^*) - \phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \right\}$$

$$= \frac{1}{24} \mu_3(t_1^*, t_2^*) \quad \ldots \quad \ldots \quad (6\text{-}121)$$

Further we have $t_1^* = -1.244$, $t_2^* = -0.382$, $t_3^* = -t_2^*$,

$\mu_3(-\infty, t_1^*) = -0.0990353$, $\mu_3(t_1^*, t_2^*) = -0.0033564$,

$1-\mu_2(-\infty, t_1^*) = 0.8271927$, $1-\mu_2(t_1^*, t_2^*) = 0.9409895$ and

$$\frac{\phi(t_2^*)}{\Phi(t_2^*)-\Phi(t_1^*)}\left[t_2^* + \frac{\phi(t_2^*)-\phi(t_1^*)}{\Phi(t_2^*)-\Phi(t_1^*)}\right] = 0.5798997$$

$$\frac{\phi(t_1)}{\Phi(t_2^*)-\Phi(t_1^*)}\left[t_1^* + \frac{\phi(t_2^*)-\phi(t_1^*)}{\Phi(t_2^*)-\Phi(t_1^*)}\right] = -0.3610928$$

Substituting these in (6-119) and (6-121) and solving for b_1 and b_2 we get $b_1 = -0.0065084$ and $b_2 = -0.0017556$.

Hence the required approximations are

$g_1(\sigma) = -1.244 - 0.00651\,\sigma^2$, $g_2(\sigma) = -0.382 - 0.00176\,\sigma^2$

$g_3(\sigma) = 0.382 + 0.00176\,\sigma^2$, $g_4(\sigma) = -1.244 + 0.00651\,\sigma^2$

(iv) <u>Six Groups</u>

In this case we have $g_i(\sigma) = t_i^* + b_i\sigma^2$ for $i = 1$ to 5,

$t_{6-i}^* = -t_i^*$, $b_{6-i} = -b_i$, $t_3^* = 0$, $b_3 = 0$, and where (b_1, b_2) is the solution of

$$2b_1 - b_1 \left[1 - \mu_2(-\infty, t_1^*) + 1 - \mu_2(t_1^*, t_2^*) \right]$$

$$- (b_2 - b_1) \frac{\phi(t_2^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \left[t_2^* + \frac{\phi(t_2^*) - \phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \right]$$

$$= \frac{1}{24} \left[\mu_3(-\infty, t_1^*) + \mu_3(t_1^*, t_2^*) \right] \quad \dots \quad \dots \quad (6\text{-}122)$$

$$2b_2 - b_2 \left[1 - \mu_2(t_1^*, t_2^*) + 1 - \mu_2(t_2^*, t_3^*) \right]$$

$$- (b_2 - b_1) \frac{\phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \left[t_1 + \frac{\phi(t_2^*) - \phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \right]$$

$$+ b_2 \frac{\phi(t_3^*)}{\Phi(t_3^*) - \Phi(t_2^*)} \frac{\phi(t_3^*) - \phi(t_2^*)}{\Phi(t_3^*) - \Phi(t_2^*)}$$

$$= \frac{1}{24} \left[\mu_3(t_1^*, t_2^*) + \mu_3(t_2^*, t_3^*) \right] \quad \dots \quad \dots \quad (6\text{-}123)$$

Further we have $t_1^* = -1.447$, $t_2^* = -0.659$, $t_3^* = 0$,

$\mu_3(-\infty, t_1^*) = -0.0862092$, $\mu_3(t_1^*, t_2^*) = -0.0030251$,

$\mu_3(t_2^*, t_3^*) = -0.0004952$, $1 - \mu_2(-\infty, t_1) = 0.8459641$,

$1 - \mu_2(t_1^*, t_2^*) = 0.9509529$, $1 - \mu_2(t_2^*, t_3^*) = 0.9644130$ and

$$\frac{\phi(t_2^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \left[t_2^* + \frac{\phi(t_2^*) - \phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \right] = 0.6052765$$

$$\frac{\phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \left[t_1^* + \frac{\phi(t_2^*) - \phi(t_1^*)}{\Phi(t_2^*) - \Phi(t_1^*)} \right] = -0.3456766$$

$$\frac{\phi(t_3^*)}{\Phi(t_3^*) - \Phi(t_2^*)} \quad \frac{\phi(t_3^*) - \phi(t_2^*)}{\Phi(t_3^*) - \Phi(t_2^*)} = 0.5173037$$

Substituting these in (6-122) and (6-123) and solving for b_1, b_2 we get,

$b_1 = -0.0064869$ and $b_2 = -0.0025209$. Hence the required approximations are

$$g_1(\sigma) = -1.447 - 0.00649\,\sigma^2, \quad g_2(\sigma) = -0.659 - 0.00252\,\sigma^2$$

$$g_4(\sigma) = 0.659 + 0.00252\,\sigma^2, \quad g_5(\sigma) = 1.447 + 0.00649\,\sigma^2$$

$$g_3(\sigma) = 0$$

Here we note that $g_3(\sigma) = 0$ is the exact value.

These approximations are also given as foot-notes in tables III to VI. Similar approximations can also be obtained for $r = 7$ to 10. We shall now compare the exact values of $g_i(\sigma)$ with those given by these approximations for the case $r = 6$ in the table below for defferent σ.

EXACT AND APPROXIMATE VALUES OF $g_i(\sigma)$

FOR SIX GROUPS

σ	$g_1(\sigma)$ Exact	$g_1(\sigma)$ Approximation	$g_2(\sigma)$ Exact	$g_2(\sigma)$ Approximation
0.4	-1.448	-1.448	-0.659	-0.659
1.0	-1.453	-1.453	-0.661	-0.662
1.4	-1.460	-1.460	-0.664	-0.664
2.0	-1.473	-1.473	-0.669	-0.669
2.4	-1.485	-1.484	-0.674	-0.674
3.0	-1.506	-1.505	-0.682	-0.682
3.4	-1.523	-1.522	-0.688	-0.688
4.0	-1.551	-1.551	-0.699	-0.699

It is seen from the above table that the approximation developed is really very good. Similar results were obtained for $r = 3$ to 5. Since the approximations derived on partly heuristic basis agrees very well with the numerical results, the author feels that a more rigorous justification of the approximation should be possible.

TABLE - I

VALUES OF THE RATIO C_r/C_∞

σ	C_1/C_∞	C_2/C_∞	C_3/C_∞	C_4/C_∞	C_5/C_∞	C_6/C_∞	C_7/C_∞
0.4	1.0202	1.0073	1.0038	1.0024	1.0016	1.0012	1.0009
0.6	1.0460	1.0165	1.0086	1.0053	1.0036	1.0026	1.0020
0.8	1.0833	1.0296	1.0154	1.0095	1.0064	1.0047	1.0035
1.0	1.1331	1.0468	1.0242	1.0149	1.0101	1.0073	1.0055
1.2	1.1972	1.0682	1.0351	1.0215	1.0146	1.0106	1.0080
1.4	1.2776	1.0943	1.0483	1.0295	1.0200	1.0144	1.0109
1.6	1.3771	1.1255	1.0638	1.0389	1.0262	1.0189	1.0143
1.8	1.4993	1.1621	1.0818	1.0496	1.0334	1.0241	1.0182
2.0	1.6487	1.2047	1.1024	1.0619	1.0416	1.0299	1.0226
2.2	1.8313	1.2542	1.1259	1.0757	1.0508	1.0365	1.0275
2.4	2.0544	1.3112	1.1524	1.0912	1.0610	1.0437	1.0329
2.6	2.3280	1.3767	1.1823	1.1085	1.0723	1.0517	1.0389
2.8	2.6645	1.4519	1.2158	1.1277	1.0847	1.0605	1.0454
3.0	3.0802	1.5382	1.2533	1.1489	1.0984	1.0701	1.0525

TABLE - II

VALUES OF C_1/C_∞ AND C_2/C_∞

σ	C_1/C_∞	C_2/C_∞
0.2	1.005013	1.001819
0.4	1.020201	1.007302
0.6	1.046028	1.016526
0.8	1.083287	1.029622
1.0	1.133148	1.046779
1.2	1.197217	1.068246
1.4	1.277621	1.094340
1.6	1.377128	1.125452
1.8	1.499303	1.162059
2.0	1.648721	1.204735
2.2	1.831252	1.254165
2.4	2.054433	1.311164
2.6	2.327978	1.376700
2.8	2.664456	1.451920
3.0	3.080217	1.538184
3.2	3.596640	1.637104
3.4	4.241852	1.750593
3.6	5.053090	1.880928
3.8	6.079971	2.030821
4.0	7.389056	2.203510

Note : The transformed optimal group boundary for the case of two groups is always $t_1 = 0.000$

TABLE - III

VALUES OF TRANSFORMED OPTIMAL GROUP BOUNDARIES AND C_3/C_∞ AND C_3^1/C_∞ FOR THREE GROUPS.

σ	t_1	C_3/C_∞	C_3^1/C_∞
0.2	-0.612	1.000952	1.000952
0.4	-0.613	1.003815	1.003815
0.6	-0.614	1.008615	1.008615
0.8	-0.615	1.015394	1.015394
1.0	-0.617	1.024212	1.024213
1.2	-0.619	1.035149	1.035153
1.4	-0.622	1.048304	1.048314
1.6	-0.625	1.063797	1.063821
1.8	-0.628	1.081771	1.081824
2.0	-0.632	1.102396	1.102499
2.2	-0.636	1.125868	1.126058
2.4	-0.641	1.152416	1.152748
2.6	-0.646	1.182301	1.182857
2.8	-0.652	1.215824	1.216723
3.0	-0.658	1.253330	1.254741
3.2	-0.664	1.295213	1.297370
3.4	-0.670	1.341922	1.345148
3.6	-0.678	1.393970	1.398701
3.8	-0.685	1.451945	1.458766
4.0	-0.693	1.516516	1.526205

Note : (i) $t_2 = - t_1$

(ii) $t_1 \simeq -0.612 - 0.00504\ \sigma^2$

TABLE - IV

VALUES OF TRANSFORMED OPTIMAL GROUP BOUNDARIES AND C_4/C_∞ AND C_4'/C_∞ FOR FOUR GROUPS.

σ	t_1	C_4/C_∞	C_4'/C_∞
0.2	-0.982	1.000588	1.000588
0.4	-0.983	1.002354	1.002354
0.6	-0.984	1.005311	1.005311
0.8	-0.986	1.009477	1.009477
1.0	-0.988	1.014877	1.014878
1.2	-0.991	1.021547	1.021550
1.4	-0.994	1.029528	1.029536
1.6	-0.998	1.038872	1.038889
1.8	-1.002	1.049639	1.049675
2.0	-1.007	1.061899	1.061969
2.2	-1.012	1.075734	1.075860
2.4	-1.018	1.091235	1.091453
2.6	-1.024	1.108508	1.108868
2.8	-1.031	1.127670	1.128248
3.0	-1.038	1.148856	1.149754
3.2	-1.046	1.172215	1.173575
3.4	-1.054	1.197915	1.199930
3.6	-1.063	1.226145	1.229073
3.8	-1.073	1.257113	1.261297
4.0	-1.082	1.291057	1.296946

Note : (i) $t_2 = 0.000$ and $t_3 = -t_1$

(ii) $t_1 \simeq -0.982 - 0.00623\,\sigma^2$

TABLE - V

VALUES OF TRANSFORMED OPTIMAL GROUP BOUNDARIES AND C_5/C_∞ AND C_5'/C_∞ FOR FIVE GROUPS

σ	t_1	t_2	C_5/C_∞	C_5'/C_∞
0.2	-1.244	-0.382	1.000400	1.000400
0.4	-1.246	-0.383	1.001601	1.001601
0.6	-1.247	-0.383	1.003610	1.003610
0.8	-1.248	-0.383	1.006436	1.006436
1.0	-1.251	-0.384	1.010092	1.010093
1.2	-1.254	-0.385	1.014597	1.014599
1.4	-1.257	-0.386	1.019971	1.019977
1.6	-1.261	-0.387	1.026241	1.026254
1.8	-1.266	-0.388	1.033438	1.033463
2.0	-1.270	-0.389	1.041596	1.041644
2.2	-1.276	-0.391	1.050757	1.050841
2.4	-1.282	-0.392	1.060967	1.061109
2.6	-1.289	-0.394	1.072276	1.072508
2.8	-1.296	-0.396	1.084744	1.085109
3.0	-1.303	-0.398	1.098433	1.098994
3.2	-1.312	-0.400	1.113417	1.114258
3.4	-1.321	-0.403	1.129775	1.131009
3.6	-1.330	-0.405	1.147596	1.149374
3.8	-1.340	-0.408	1.166979	1.169498
4.0	-1.350	-0.411	1.188031	1.191549

Note : (i) $t_3 = -t_2$ and $t_4 = -t_1$

(ii) $t_1 \simeq -1.244 - 0.00651\,\sigma^2$ and $t_2 \simeq -0.382 - 0.00176\,\sigma^2$.

TABLE - VI

VALUES OF TRANSFORMED OPTIMAL GROUP BOUNDARIES AND C_6/C_∞ AND C'_6/C_∞ FOR SIX GROUPS.

σ	t_1	t_2	C_6/C_∞	C'_6/C_∞
0.2	-1.447	-0.659	1.000290	1.000290
0.4	-1.448	-0.659	1.001161	1.011161
0.6	-1.449	-0.660	1.002617	1.002617
0.8	-1.451	-0.661	1.004662	1.004662
1.0	-1.453	-0.661	1.007305	1.007305
1.2	-1.457	-0.663	1.010556	1.010557
1.4	-1.460	-0.664	1.014428	1.014430
1.6	-1.463	-0.665	1.018934	1.018941
1.8	-1.468	-0.667	1.024094	1.024108
2.0	-1.473	-0.669	1.029927	1.029953
2.2	-1.478	-0.671	1.036455	1.036502
2.4	-1.485	-0.674	1.043706	1.043787
2.6	-1.491	-0.676	1.051707	1.051840
2.8	-1.498	-0.679	1.060492	1.060704
3.0	-1.506	-0.682	1.070096	1.070423
3.2	-1.514	-0.685	1.080559	1.081052
3.4	-1.523	-0.688	1.091926	1.092650
3.6	-1.532	-0.692	1.104246	1.105290
3.8	-1.542	-0.696	1.117571	1.119052
4.0	-1.551	-0.699	1.131963	1.134032

Note: (i) $t_4 = -t_2$, $t_3 = 0.000$ and $t_5 = -t_1$

(ii) $t_1 \simeq -1.447 - 0.00649\,\sigma^2$ and $t_2 \simeq -0.659 - 0.00252\,\sigma^2$.

TABLE - VII

VALUES OF TRANSFORMED OPTIMAL GROUP BOUNDARIES AND C_7/C_∞ AND C_7'/C_∞ FOR SEVEN GROUPS.

σ	t_1	t_2	t_3	C_7/C_∞	C_7'/C_∞
0.2	-1.611	-0.875	-0.281	1.000220	1.000220
0.4	-1.612	-0.875	-0.281	1.000881	1.000882
0.6	-1.614	-0.876	-0.281	1.001985	1.001987
0.8	-1.615	-0.876	-0.281	1.003535	1.003539
1.0	-1.617	-0.877	-0.281	1.005536	1.005544
1.2	-1.620	-0.879	-0.282	1.007995	1.008007
1.4	-1.623	-0.880	-0.282	1.010920	1.010938
1.6	-1.627	-0.882	-0.283	1.014319	1.014347
1.8	-1.632	-0.884	-0.283	1.018203	1.018245
2.0	-1.636	-0.886	-0.284	1.022586	1.022647
2.2	-1.642	-0.889	-0.285	1.027481	1.027570
2.4	-1.648	-0.891	-0.285	1.032904	1.033034
2.6	-1.654	-0.894	-0.286	1.038874	1.039059
2.8	-1.661	-0.897	-0.287	1.045409	1.045673
3.0	-1.668	-0.900	-0.288	1.052533	1.052905
3.2	-1.677	-0.904	-0.289	1.060270	1.060789
3.4	-1.685	-0.908	-0.290	1.068647	1.069364
3.6	-1.694	-0.912	-0.292	1.077694	1.078675
3.8	-1.705	-0.917	-0.293	1.087444	1.088775
4.0	-1.715	-0.921	-0.294	1.097933	1.099724

Note: $t_4 = -t_3$; $t_5 = -t_2$; $t_6 = -t_1$

TABLE - VIII

VALUES OF TRANSFORMED OPTIMAL GROUP BOUNDARIES AND C_8/C_∞ AND C'_8/C_∞ FOR EIGHT GROUPS.

σ	t_1	t_2	t_3	C_8/C_∞	C'_8/C_∞
0.2	-1.748	-1.050	-0.500	1.000173	1.000173
0.4	-1.749	-1.051	-0.501	1.000692	1.000692
0.6	-1.750	-1.051	-0.501	1.001558	1.001558
0.8	-1.752	-1.052	-0.501	1.002774	1.002774
1.0	-1.754	-1.053	-0.502	1.004343	1.004343
1.2	-1.757	-1.055	-0.503	1.006269	1.006269
1.4	-1.760	-1.056	-0.503	1.008557	1.008558
1.6	-1.764	-1.058	-0.504	1.011214	1.011216
1.8	-1.768	-1.060	-0.505	1.014246	1.014251
2.0	-1.773	-1.062	-0.506	1.017662	1.017672
2.2	-1.778	-1.065	-0.507	1.021472	1.021490
2.4	-1.784	-1.068	-0.508	1.025685	1.025716
2.6	-1.790	-1.071	-0.510	1.030315	1.030366
2.8	-1.797	-1.074	-0.511	1.035373	1.035455
3.0	-1.805	-1.078	-0.513	1.040874	1.041003
3.2	-1.811	-1.081	-0.514	1.046836	1.047031
3.4	-1.821	-1.086	-0.516	1.053275	1.053563
3.6	-1.829	-1.090	-0.518	1.060212	1.060630
3.8	-1.838	-1.094	-0.520	1.067670	1.068264
4.0	-1.849	-1.099	-0.522	1.075671	1.076503

Note: $t_4 = 0$, $t_5 = -t_3$, $t_6 = -t_2$ and $t_7 = -t_1$

TABLE - IX

VALUES OF TRANSFORMED OPTIMAL GROUP BOUNDARIES AND C_9/C_∞ AND C'_9/C_∞ FOR NINE GROUPS.

σ	t_1	t_2	t_3	t_4	C_9/C_∞	C'_9/C_∞
0.2	-1.865	-1.197	-0.681	-0.222	1.000139	1.000139
0.4	-1.866	-1.198	-0.681	-0.222	1.000558	1.000558
0.6	-1.868	-1.199	-0.682	-0.222	1.001256	1.001256
0.8	-1.869	-1.199	-0.682	-0.222	1.002235	1.002235
1.0	-1.872	-1.201	-0.683	-0.222	1.003498	1.003499
1.2	-1.875	-1.203	-0.684	-0.223	1.005048	1.005049
1.4	-1.878	-1.204	-0.685	-0.223	1.006888	1.006889
1.6	-1.880	-1.205	-0.685	-0.223	1.009023	1.009025
1.8	-1.885	-1.208	-0.687	-0.224	1.011457	1.011461
2.0	-1.890	-1.210	-0.688	-0.224	1.014196	1.014204
2.2	-1.895	-1.213	-0.689	-0.224	1.017247	1.017262
2.4	-1.901	-1.216	-0.691	-0.225	1.020617	1.020642
2.6	-1.907	-1.219	-0.692	-0.225	1.024315	1.024355
2.8	-1.913	-1.222	-0.694	-0.226	1.028349	1.028412
3.0	-1.920	-1.226	-0.696	-0.226	1.032730	1.032827
3.2	-1.928	-1.230	-0.698	-0.227	1.037469	1.037615
3.4	-1.936	-1.234	-0.700	-0.228	1.042580	1.042793
3.6	-1.946	-1.239	-0.703	-0.229	1.048075	1.048382
3.8	-1.952	-1.243	-0.705	-0.229	1.053971	1.054405
4.0	-1.962	-1.248	-0.707	-0.230	1.060286	1.060890

Note : $t_5 = -t_4$, $t_6 = -t_3$, $t_7 = -t_2$ and $t_8 = -t_1$

TABLE - X

VALUES OF TRANSFORMED OPTIMAL GROUP BOUNDARIES AND C_{10}/C_{∞} AND C'_{10}/C_{∞} FOR TEN GROUPS.

σ	t_1	t_2	t_3	t_4	C_{10}/C_{∞}	C'_{10}/C_{∞}
0.2	-1.969	-1.325	-0.834	-0.405	1.000115	1.000115
0.4	-1.969	-1.325	-0.834	-0.405	1.000459	1.000459
0.6	-1.970	-1.325	-0.834	-0.405	1.001034	1.001034
0.8	-1.972	-1.327	-0.835	-0.405	1.001840	1.001840
1.0	-1.974	-1.328	-0.836	-0.406	1.002879	1.002879
1.2	-1.977	-1.329	-0.836	-0.406	1.004154	1.004154
1.4	-1.980	-1.331	-0.837	-0.406	1.005666	1.005666
1.6	-1.984	-1.333	-0.839	-0.407	1.007419	1.007419
1.8	-1.986	-1.334	-0.839	-0.407	1.009417	1.009418
2.0	-1.992	-1.337	-0.841	-0.408	1.011663	1.011666
2.2	-1.997	-1.340	-0.843	-0.409	1.014162	1.014168
2.4	-2.002	-1.343	-0.844	-0.409	1.016921	1.016931
2.6	-2.008	-1.346	-0.846	-0.410	1.019944	1.019962
2.8	-2.014	-1.349	-0.848	-0.411	1.023238	1.023269
3.0	-2.021	-1.353	-0.850	-0.412	1.026812	1.026862
3.2	-2.029	-1.357	-0.852	-0.413	1.030672	1.030750
3.4	-2.037	-1.362	-0.855	-0.414	1.034830	1.034948
3.6	-2.045	-1.366	-0.857	-0.415	1.039295	1.039470

Note : $t_5 = 0.000$, $t_6 = -t_4$, $t_7 = -t_3$, $t_8 = -t_2$ and $t_9 = -t_1$.

TABLE - XI

OGAWA'S OPTIMUM SPACINGS

t_i \ r	2	3	4	5	6	7	8	9	10
t_1	0.000	-0.613	-0.982	-1.243	-1.447	-1.598	-1.751	-1.866	-1.997
t_2		0.613	0.000	-0.383	-0.659	-0.860	-1.049	-1.200	-1.329
t_3			0.982	0.383	0.000	-0.266	-0.502	-0.684	-0.834
t_4				1.243	0.659	0.266	0.000	-0.222	-0.404
t_5					1.447	0.860	0.502	0.222	0.000
t_6						1.598	1.049	0.684	0.404
t_7							1.751	1.200	0.834
t_8								1.866	1.329
t_9									1.977

(i) r = no. of groups.

(ii) Ogawa's computations are slightly in error in some cases (see appendix).

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1. INTRODUCTION

In this part, we derive a certain statistical distribution which can be put to use to solve a variety of industrial problems. For purpose of illustration, we indicate the use of this distribution in textile winding, cloth cutting and also cutting procedures for products produced in a continuous length. The motivation for this piece of research was an actual cloth-cutting problem encountered by the author in his consultancy work in a cotton textile mill.

For the purpose of developing a suitable model, we consider a sequence of events which is a mixture of a completely regular sequence and a Poisson's process. Let S denote a systematic event occuring at the end of every T time units and R a random event which occurs according to a Poisson process of rate λ. Let the event $\mathcal{E}$ be defined to occur when either S or R does. The equilibrium marginal frequency distribution of the time interval between successive occurences of the event $\mathcal{E}$ is derived in section 2.1. This distribution can be used under variety of situations, for example, in the case of the products which are produced in a continuous length (like cloth, extrusion products etc.) and are cut in definite lengths and taken out of the machine, we can take the point of the initial cut as an occurence of S and the existence of a defect , which occurs in a random fashion, as an occurence of R. Here we are interested in the distribution of defect free lengths.

In section 2.2 we consider the application of this distribution to textile winding. Automatic winding machines used in textile industry consist of 200 to 300 spindles; each of which is used to wind yarn from a relatively small supply bobbin on to a larger cone. The winding at any spindle stops when the yarn on the bobbin gets exhausted or when the yarn breaks. An automatic head patrols the machine in a fixed time and spends fixed time in servicing each spindle, i.e. replacing an exhausted bobbin by a full one and starting it, or knotting an yarn which has broken and starting the spindle again. Under reasonable assumptions, we can identify the stoppage of a spindle due to exhaustion of yarn as the occurence of the systematic event S and the stoppage of the spindle due to yarn break as the occurence of the random event R. We derive the distributions of i) idle time of a spindle, ii) busy time of a spindle and iii) the number of patrols of the automatic head between two consecutive restartings of a spindle. We also derive an expression for Machine Efficiency. Howie and Shenton [2] have also derived an expression for Machine Efficiency and their approach was entirely different.

In section 2.3; we consider the application of the basic distribution of section 2.1 for problems of cutting of cloth in textile industry. To analyse this problem, it is enough to consider the cutting of the cloth in definite lengths while taking it out of a loom as a systematic event S and the occurence of weaving or processing defects as the random event. A suitable model is then developed.

In section 2.4, we generalise some of Sibuya's [3] work on cutting procedures by using the basic distribution of section 2.1. Products which are produced in continuous length are generally sold to final customers in defect free pieces of some given specific length, say unit length. The cutting procedure followed is that first, relatively longer pieces of some definite length L are cut out. These are then inspected and, depending on the positions where defects occur, are suitably cut into pieces of unit length so as to get the maximum number of defect free pieces of unit length from each of the longer pieces of length L. We call the procedure when $L = 1$ as simple cutting and when $L = \infty$ as sequential cutting. We denote by $Y(L,\lambda)$ the yield when the initial cut length is L and λ is the average number of defects per unit length. Using a different approach Sibuya [3] derived expressions for yield when $1 \leq L < 2$ and $2 \leq L < 3$. Using the basic distribution of section 2.1, we derive an expression for any initial cut length L. Let n be any positive integer. It is then shown that for $0 < \delta < 1$ that

$$Y(n+\delta,\lambda) \begin{cases} > Y(n,\lambda) & \text{when } \lambda > \lambda_n^* \\ = Y(n,\lambda) & \text{when } \lambda = \lambda_n^* \\ < Y(n,\lambda) & \text{when } \lambda < \lambda_n^* \end{cases}$$

where λ_n^* is called the critical value of λ for a given n and is the solution of

$$\sum_{i=1}^{n} (\lambda i - 1)\, e^{-\lambda(i-1)} = 0$$

It is also shown that $\lim_{n \to \infty} (n\, e^{-n\lambda_n^*}) = \frac{1}{2}$ and $\frac{1}{n}$ log 2n provides a very good approximation for λ_n^*. It is also indicated that it is not a desirable practice to have fractional values for initial cut length. The yield values $Y(n, \lambda)$ have been computed for a wide range of n and λ and given in Table III (at the end of this part). It is felt that this table will be useful in practical applications. Finally a suitable model for determination of optimum n is suggested.

2. OPTIMAL CUTTING PROCEDURES

Suppose that a sequence of events is a mixture of a completely regular sequence and a Poisson process. The distribution of the interval between successive events in the combined process is obtained. The use of this distribution in various industrial situations is discussed.

2.1 DERIVATION OF THE MODEL

Let S be a systematic event occuring every T time units and R a random event occuring in a Poisson process of rate λ. Let ξ be defined to occur whenever either S or R does. Let f(x) be the equilibrium marginal frequency distribution of the intervals between successive occurences of $\mathcal{E}$. The rigorous definition of this distribution is that it is the limit as $n \longrightarrow \infty$ of the frequency distribution $f_n(x)$ of the interval between the n^{th} and (n+1)st occurence of $\mathcal{E}$; given, say, that the first event is an S (or an R). The argument in this thesis is, however, heuristic.

Let X be the interval between two successive occurences of $\mathcal{E}$. Let $P(x) = Pr\{X \geq x\}$ and $F(x) = Pr\{X \leq x\}$. We define P and F seperately in this manner for reasons which will become clear later. It is obvious that the maximum value which X can take is T. Define the recurrence time y of $\mathcal{E}$ as the time interval from a randomly chosen point of time to the next occurence of $\mathcal{E}$. The maximum value possible for recurrence time is T. Let $\phi(y)$ be its probability density function and $\Phi(y)$ its distribution function. The probability that the recurrence time is greater than or equal to y is given by the probability that neither S nor R occur in a randomly chosen interval of length y i.e.

$$\int_y^T \phi(t)\,dt = 1 - \Phi(y) = e^{-\lambda y}\left(1 - \frac{y}{T}\right) \quad \ldots \quad (1)$$

$$\text{or} \quad \phi(y) = \frac{\lambda T + 1}{T}\left(1 - \frac{\lambda y}{\lambda T+1}\right) e^{-\lambda y} \quad \ldots \quad (2)$$

It is well known (Cox [1]) that ~~ø and f are related~~ by

$$\phi(y) = \frac{\int_y^\infty f(x)\,dx}{\int_0^\infty x f(x)\,dx} = \frac{P(y)}{E(X)} \quad \ldots \quad \ldots \quad (3)$$

$$\frac{Pr\{X \geq y\}}{E(x)}$$

where E (X) is the expected value of X. Since $\mathcal{E}$ occurs on an average $(\lambda T + 1)$ times in an interval of time T, it follows that

$$E(X) = \frac{T}{\lambda T + 1} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (4)$$

Hence we get by (2), (3) and (4)

$$P(x) = \left(1 - \frac{\lambda x}{\lambda T + 1}\right) e^{-\lambda x} \quad \text{for } 0 \leq x \leq T \quad \cdots \quad \cdots \quad (5)$$

$$= 1 \quad \text{when } x < 0$$

$$= 0 \quad \text{when } x > T$$

If we put $x = T$ in (5) we get

$$P(T) = \frac{1}{\lambda T + 1} e^{-\lambda T} > 0 \quad \cdots \quad \cdots \quad \cdots \quad (6)$$

This is because the probability of an interval between two successives $\mathcal{E}$'s being exactly equal to T is

$$Pr(X = T) = \frac{1}{\lambda T + 1} e^{-\lambda T} \quad \cdots \quad \cdots \quad (7)$$

and since the maximum possible length of an interval is T, we have

$$Pr(X \geq T) = P(T) = Pr(X = T) = \frac{1}{\lambda T + 1} e^{-\lambda T} \quad \cdots \quad (8)$$

Further

$$F(x) = 1 - \left(1 - \frac{\lambda x}{\lambda T + 1}\right) e^{-\lambda x} \quad \text{for } 0 \leq x < T \quad \cdots \quad (9)$$

$$= 1 \quad \text{when } x \geq T$$

$$= 0 \quad \text{when } x < 0$$

It follows from (5) that the frequency function of x can be obtained by differentiating $-P(x)$ for $x < T$ i.e.

$$f(x) = \frac{\lambda T}{\lambda T+1} \left\{ \frac{2}{T} + \lambda \left(1 - \frac{x}{T}\right) \right\} e^{-\lambda x} \quad \text{for } 0 \leq x < T \quad \ldots (10)$$

$$f(T) = \Pr \left\{ X = T \right\} = \frac{1}{1+\lambda T} e^{-\lambda T} \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$f(x) = 0 \quad \text{when } x < 0 \quad \text{or} \quad x > T \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

It can be easily verified that

$$\int_0^T x\, f(x)\, dx + T\, f(T) = \frac{T}{\lambda T+1} \quad \ldots \quad \ldots \quad (13)$$

In sections 2.2, 2.3 & 2.4 we shall discuss the application of this distribution in industry.

2.2 TEXTILE WINDING

Automatic winding machines used in the Textile industry consist of 200 to 300 spindles, each of which is used to wind yarn from a relatively small supply bobbin on to a larger cone. An automatic head patrols the machine in a fixed time and spends a fixed time servicing each spindle, i.e. replacing an exhausted supply bobbin by a full one and starting it or knotting a yarn which has broken and re-starting it. We **define** Machine Efficiency (ME) as the ratio of time spent on actual unwinding to total running time of the machine.

The period of time for which the unwinding goes on un-interupted at a spindle is called a busy period of that spindle. The time period for which a spindle waits for either replacing an empty bobbin or for knotting a broken yarn by the automatic patrolling head is called an idle period for that spindle. The probability distributions of i) busy period, ii) idle period and iii) the number of patrols between two consecutive restartings of a spindle are derived. The patrolling time, i.e. the time between consecutive arrivals of the automatic head at a particular spindle, can be varied within certain limits, and is set at a value which gives maximum efficiency. This problem has been considered by Howie & Shenton [2] and they have also derived an expression for Machine Efficiency using a different approach. It is shown here that the expression for Machine Efficiency can be obtained in a very simple manner by the use of the distribution derived in section 2.1.

Let T be the constant time required to unwind a bobbin completely in the absence of any breaks, λ the average number of breaks per unit spindle running time and d be constant patrolling time. Let m be the smallest non-negative integer, such that

$$m < \frac{T}{d} \leq m + 1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

It is seen that $m = 0$ when $d \geq T$. Further let

$$h = (m+1)d - T \quad \text{or} \quad d = \frac{T + h}{m + 1} \quad \ldots \quad \ldots \quad (15)$$

and it is seen that $0 \leq h < d$. The physical interpretation of (14) is that in the absence of breaks (m+1) patrols are required to unwind a bobbin and start a new one.

We consider the stoppage of a spindle for the exhaustion of yarn on a bobbin as a systematic event and the stoppage due to yarn break as a random event. Hence the distribution derived in section 2.1 can be used for an analysis of this problem. The following notation is used.

$f(x)$ = frequency function of the busy period distribution

$g(y)$ = " " " idle period "

$P(x)$ = Probability $\{$busy period $\geq x\}$

$G(y)$ = Probability $\{$idle period $\leq y\}$

μ_B = Average duration of a busy period

μ_I = Average duration of an idle period

It is obvious that the probability P(x) that a busy period is greater than or equal to x is given by (5) i.e.

$$P(x) = \left(1 - \frac{\lambda x}{\lambda T + 1}\right) e^{-\lambda x} \quad \text{for } 0 \leq x \leq T \qquad \ldots (16)$$

$$= 0 \quad \text{for } x > T$$

$$= 1 \quad \text{for } x < 0$$

Thus the frequency function for the busy period distribution **is** given by

$$f(x) = \frac{\lambda T}{\lambda T+1} \left\{ \frac{2}{T} + \lambda \left(1 - \frac{x}{T}\right) \right\} e^{-\lambda x} \quad \text{for} \quad 0 \leq x < T$$

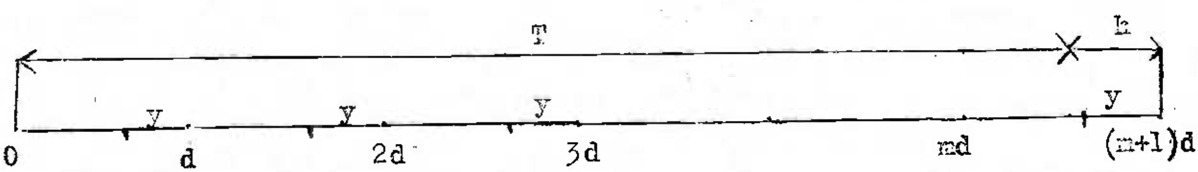
$$f(T) = \text{Prob} \left\{ \text{busy period} = T \right\} = P(T) = \frac{e^{-\lambda T}}{\lambda T+1} \qquad \ldots \quad (17)$$

$$f(x) = 0 \quad \text{for} \quad x < 0 \quad \text{or} \quad x > T$$

The average duration μ_B of busy period is given by (13) i.e.

$$\mu_B = \int_0^T x\, f(x)\, dx + T\, P(T) = \frac{T}{\lambda T+1} \qquad \ldots \quad \ldots \quad (18)$$

Let us consider the idle period and note that busy periods and idle periods alternate. It is seen that a busy period for a particular spindle can start only at the instant of time when the automatic head visits that particular spindle. For a given spindle, consider a busy period which is taken for convenience to start at time 0. It is observed that the automatic head visits the spindle again at instants of time d, 2d, 3d, and so on, and the maximum value for the busy period is T. It is easy to visualise the situation with the help of figure 1.



It is also noted that the maximum value idle period can take is d and the idle period immediately after the busy period under consideration has to end either at d or 2d .... or at (m+1) d. The probability G(y) that the idle period is less than or equal to y is given by

$$G(y) = P(d-y)-P(d)+P(2d-y)-P(d)+ \dots +P(md-y)-P(md)+P(\overline{m+1}\, d-y)$$

$$= \sum_{i=1}^{m+1} P(id-y) - \sum_{i=1}^{m} P(id) \quad \text{for} \quad 0 \leq y < d$$

$$= 0 \quad \text{for} \quad y < 0 \qquad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (19)$$

$$= 1 \quad \text{for} \quad y \geq d$$

We note that G(y) is discontinuous at y=h because P(x) is so at x=T. In fact we have, since $P(\overline{m+1}\, d-y) = 0$ for $y < h$

$$G(y) = \sum_{i=1}^{m} P(id-y) - \sum_{i=1}^{m} P(id) \quad \text{for} \quad y < h$$

$$= \sum_{i=1}^{m+1} P(id-y) - \sum_{i=1}^{m} P(id) \quad \text{for} \quad y \geq h$$

Hence the frequency function of the idle time distribution is given by

$$g(y) = \sum_{i=1}^{m} f(id-y) \quad \text{for} \quad 0 \leq y < h \qquad \dots \quad \dots \qquad (20)$$

$$g(h) = \text{Prob}\left\{ \text{idle period} = h \right\} = P(\overline{m+1}\, d - h)$$

$$= P(T) = \frac{1}{\lambda T+1} e^{-\lambda T} \qquad \dots \quad \dots \quad \dots \qquad (21)$$

$$g(y) = \sum_{i=1}^{m+1} f(id-y) \quad \text{for} \quad h < y < d \qquad \dots \qquad \dots \qquad (22)$$

$$g(y) = 0 \quad \text{for} \quad y < 0 \quad \text{and} \quad y \geq d$$

The average duration μ_I of an idle period is given by

$$\mu_I = \int_0^h y\, g(y)dy + h\, g(h) + \int_h^d y\, g(y)\, dy$$

$$= \int_0^h y\left(\sum_{i=1}^{m} f(id-y)\right)dy + \frac{h}{\lambda T+1} e^{-\lambda T} + \int_h^d y\left(\sum_{i=1}^{m+1} f(id-y)\right)dy$$

$$= \sum_{i=1}^{m}\left(\int_0^h y\, f(id-y)dy + \int_h^d y\, f(id-y)dy\right) + \int_h^d y\, f(\overline{m+1}\, d-y)dy$$

$$+ \frac{h}{\lambda T+1} e^{-\lambda T}$$

$$= \sum_{i=1}^{m} \int_0^d y\, f(id-y)dy + \int_h^d y\, f(\overline{m+1}\, d-y)dy + h\, e^{-\lambda T}/(\lambda T+1)$$

$$= \sum_{i=1}^{m} \int_{(i-1)d}^{id} (id-z)\, f(z)dz + \int_{md}^{(m+1)d-h} (\overline{m+1}\, d-z)f(z)dz + h\, e^{-\lambda T}/(\lambda T+1).$$

$$= d \sum_{i=1}^{m} i\left[P(\overline{i-1}\, d) - P(id)\right] + (m+1)d\left[P(md) - P(\overline{m+1}\, d-h)\right]$$

$$- \int_0^T z\, f(z)dz + h\, e^{-\lambda T}/(\lambda T+1).$$

$$= d \sum_{i=0}^{m} P(id) - (m+1)d\, P(T) + h\, e^{-\lambda T}/(\lambda T+1) - \int_0^T z\, f(z)dz$$

$$- TP(T) + TP(T).$$

$$= d \sum_{i=0}^{m} P(id) - \frac{(m+1)d}{\lambda T+1} e^{-\lambda T} + \frac{h e^{-\lambda T}}{(\lambda T+1)} - \frac{T}{\lambda T+1} + \frac{T e^{-\lambda T}}{\lambda T + 1}$$

$$= d \sum_{i=o}^{m} P(id) + \frac{e^{-\lambda T}}{\lambda T+1} (-(m+1)d + h + T) - \frac{T}{\lambda T+1} \quad \ldots \quad \ldots \quad (23)$$

$$= d \sum_{i=0}^{m} P(id) - \frac{T}{\lambda T+1} = - \mu_B + d \sum_{i=o}^{m} P(id) \quad \ldots \quad \ldots \quad (24)$$

In arriving at (24), we make use of (18), (17) and (15)

Hence we have Machine Efficiency (ME) to be

$$M E = \frac{\mu_P}{\mu_B + \mu_I}$$

$$= \frac{T}{(\lambda T+1) d \sum_{i=o}^{m} P(id)} \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Howie and Shenton [2] obtained essentially the same expression as (25) for M.E. Their approach was entirely different and they used integro-difference equations and generating functions. The proof that the expression (25) for Machine efficiency is same as that obtained by Howie and Shanton [2] is given in the appendix.

We now consider the probability distribution of the number of patrols between two consecutive restartings of a spindle. Let D(n) denote the probability that the number of patrols between two consecutive restartings of a spindle is n, then

$$D(n) = P(\overline{n-1}\, d) - P(nd) \quad \text{for} \quad 0 < n \leq m$$

$$D(m+1) = P(m\, d) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (26)$$

$$E(n) = \sum_{n=1}^{m+1} n\, D(n) = \sum_{n=0}^{m} P(n\, d) \qquad \ldots \quad \ldots \qquad (27)$$

$$= 1 + \sum_{n=1}^{m} P(n\, d)$$

Let $n_1, n_2, \ldots\ldots n_k$ be the number of patrols between k successive re-startings of a spindle and let $t_1, t_2, \ldots\ldots t_k$ be the corresponding busy periods. Then the Average Number of Patrols (ANP) required to unwind a bobbin completely and start the next one is given by

$$\text{ANP} = \lim_{k \to \infty} \frac{n_1 + n_2 + \ldots\ldots + n_k}{\frac{1}{T}(t_1 + \ldots\ldots + t_k)}$$

$$= \lim_{k \to \infty} \frac{T\,(n_1 + \ldots\ldots + n_k)/k}{(t_1 + \ldots\ldots + t_k)/k}$$

$$= \frac{T\, E(n)}{T/(\lambda T + 1)} = (\lambda T + 1)\, E(n) \qquad \ldots \quad \ldots \qquad (28)$$

Hence the Average Number of patrols required to unwind a bobbin completely is equal to the product of average number of stoppages per bobbin $(\lambda T + 1)$ and the average number of patrols between two consecutive restartings ($E(n)$). Since the average machine running time in d x ANP units of time is T, the machine efficiency is given by

$$ME = \frac{T}{d \times ANP} \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

$$= \frac{T}{d\ (\lambda T + 1)\ E(n)}$$

$$= \frac{T}{d\ (\lambda T + 1) \sum_{n=0}^{m} P(nd)}$$

and this is same as (25).

2.3 CLOTH CUTTING

Suppose that cloth is cut and taken out of the looms in definite lengths - say L yards. Suppose also that the weaving and processing defects occur at random at a rate λ per unit length, and the cloth is cut again wherever a defect occurs to get defect free lengths. We can consider a cut at every L units of length as a systematic event and processing and weaving defects as a random event. Then the frequency function of the distribution of length of defect free cloth is given by (10) and (11) i.e.,

$$f(x) = \frac{\lambda L}{\lambda L + 1} \left\{ \frac{2}{L} + \lambda \left(1 - \frac{x}{L}\right) \right\} e^{-\lambda x} \qquad 0 \leq x < L$$

$$f(L) = \text{Prob (defect free length} = L) = P(L) = \frac{1}{\lambda L + 1} e^{-\lambda L} \quad ..(30)$$

$$f(x) = 0 \quad \text{otherwise}$$

The value realised by selling the cloth depends on the length of defect free cloth sold. Let $r(x)$ denote the monetary realisation from selling a continuous length x of defect free cloth. Step functions for realisations are quite common in textile industry. Let

$$\pi = \max_{x > 0} \left\{ \frac{r(x)}{x} \right\}$$

i.e. π gives the maximum realisation per unit length. We define yield as the ratio of actual realisation to the maximum possible realisation. Then the yield $Y(L, \lambda)$ which is a function of λ and L is given by

$$Y(L, \lambda) = \frac{\frac{\lambda L}{\lambda L+1} \int_0^L r(x)\, e^{-\lambda x} \left\{ \frac{2}{L} + \lambda \left(1 - \frac{x}{L}\right) \right\} dx + r(L) \cdot \frac{e^{-\lambda L}}{\lambda L+1}}{\pi \frac{\lambda L}{\lambda L+1} \int_0^L x \left\{ \frac{2}{L} + \lambda \left(1 - \frac{x}{L}\right) \right\} e^{-\lambda x}\, dx + \pi L \frac{e^{-\lambda L}}{\lambda L+1}}$$

$$= \frac{\frac{\lambda L}{\lambda L+1} \int_0^L r(x) \left\{ \frac{2}{L} + \lambda \left(1 - \frac{x}{L}\right) \right\} e^{-\lambda x}\, dx + r(L) \frac{e^{-\lambda L}}{\lambda L+1}}{\pi L / (\lambda L + 1)} \quad \ldots(31)$$

Expression (31), can be used to study the increase in yield as L increases. The cost of inspection at final inspection increases as the initial cut length L increases and the optimum initial cut length can be found using (31).

2.4 CUTTING PROCEDURES FOR MATERIAL WITH POISSON DEFECTS

Products like wire, extrusions etc., produced in a continuous length are eventually cut into pieces of some specified length for shipment to consumers. We consider the situation where a final piece is acceptable if and only if it is defect free. Without loss of generality we can assume that the final piece is of unit length. It is further assumed that defects occur at random with rate λ. We define yield as the acceptable proportion of the total throughput. We can visualise two extreme procedures for cutting the product into final pieces of unit length.

Simple cutting : As the product comes out of the machine, it is straight away cut into pieces of unit length. Since the proportion of defect free pieces will be $e^{-\lambda}$, we have the yield, α , for this procedure to be

$$\alpha = e^{-\lambda} \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

Sequential Cutting : From the starting point we measure an interval of unit length and if it does not contain a defect, we cut out an interval of unit length, otherwise we move the origin to the defect and repeat the procedure. Theoretically maximum yield, β can be obtained from this procedure. The frequency function of the defect free length of the material (interval between two consecutive defects) is $\lambda e^{-\lambda x}$ i.e. negative exponential. We get i acceptable pieces from a defect free length of x when $i \leq x < i+1$. Hence the yield, β for this procedure is given by

$$\beta = \frac{\sum_{i=1}^{\infty} i \int_{i}^{i+1} e^{-\lambda x} \, dx}{\lambda \int_{0}^{\infty} x \, e^{-\lambda x} \, dx}$$

$$= \frac{\sum_{i=1}^{\infty} i \left\{ e^{-\lambda i} - e^{-\lambda(i+1)} \right\}}{1/\lambda}$$

$$= \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}} = \frac{\lambda}{e^{\lambda} - 1} = \frac{\lambda \alpha}{1 - \alpha} \qquad \ldots \qquad (32)$$

In actual practice the sequential cutting procedure is often impracticable to adopt, for example in case of high speed automatic process. On the other hand the yield will be low for simple cutting procedure for relatively **high** λ. The values α and β are given in table 1 below for selected values of λ.

Table - 1

Yields for simple and sequential cutting procedures

λ	Percent yield for Simple cutting ($100\,\alpha$)	Sequential cutting ($100\,\beta$)
0.05	95.1299	97.5208
0.1	90.4837	95.0833
0.2	81.8731	90.3331
0.3	74.0818	85.7489
0.4	67.0320	81.3298
0.5	60.6531	77.0747
0.6	54.8812	72.9822
0.7	49.6585	69.0504
0.8	44.9329	65.2733
0.9	40.6570	61.6606
1.0	36.7879	58.1977

It is seen from the above table that considerable scope for potential improvement in yield exists as a result of sophistification in cutting procedure for $0.1 \leq \lambda \leq 1.0$.

In practice, we can follow a cutting procedure which may be in-between the above two extremes. First, relatively longer pieces of some definite length L are cut out. These are inspected and depending on the positions where defects occur, are suitably cut into pieces of unit length so as to get maximum number of acceptable pieces of unit length from each of the longer pieces of length L. It is noted that $L = 1$ corresponds to simple cutting and $L = \infty$ corresponds to sequential cutting. Under some conditions, the yield increases as L increases and at the same time the cost of the cutting procedure also increases. The problem is then one of finding an optimum value of L.

Sibuya [3] (using a different approach) derived expressions for yield when $L = 1 + \delta$ and $L = 2 + \delta$ where $0 \leq \delta < 1$. We shall now derive an expression for yield for any cut-length L and also generalise some of Sibuya's results. For this purpose it is enough to substitute in (31) $\pi = 1$ and $r(x) = i$ when $i \leq x < i+1$. If we denote by $[L]$, the integral part of L, then we get the yield $Y(L, \lambda)$ to be

$$Y(L, \lambda) = \frac{\lambda L+1}{L} \sum_{i=1}^{[L]-1} i\,[P(i) - P(i+1)] + [L]\,P([L])$$

$$= \frac{\lambda L+1}{L} \sum_{i=1}^{[L]} P(i) = \frac{\lambda L+1}{L} \sum_{i=1}^{[L]} \left(1 - \frac{\lambda i}{\lambda L+1}\right) e^{-\lambda i}$$

$$= \sum_{i=1}^{[L]} \left\{ \lambda + \frac{1 - \lambda i}{L} \right\} e^{-\lambda i} \qquad \ldots \quad \ldots \quad (33)$$

If we restrict the value of L in the range $1 \leq L < 2$ and $2 \leq L < 3$ in (33) we get

$$Y(L,\lambda) = \begin{cases} \left(\lambda + \frac{1-\lambda}{L}\right) e^{-\lambda} & \text{for } 1 \leq L < 2 \\ \left(\lambda + \frac{1-\lambda}{L}\right) e^{-\lambda} + \left(\lambda + \frac{1-2\lambda}{L}\right) e^{-2\lambda} & \text{for } 2 \leq L < 3 \end{cases} \qquad \ldots \quad (34)$$

which are due to Sibuya. If we carryout the summation in (33) and put $\alpha = e^{-\lambda}$ and $\beta = \frac{\lambda\alpha}{1-\alpha}$, we get

$$Y(L,\lambda) = \lambda\alpha \frac{1-\alpha^{[L]}}{1-\alpha} + \frac{\alpha}{L} \cdot \frac{1-\alpha^{[L]}}{1-\alpha} - \frac{\lambda\alpha}{L}\left(\frac{1-\alpha^{[L]}}{(1-\alpha)^2} - \frac{[L]\,\alpha^{[L]}}{1-\alpha}\right)$$

$$= \beta\left(1-\alpha^{[L]}\right) + \frac{\alpha}{L}\,\frac{1-\alpha^{[L]}}{1-\alpha} - \frac{\beta}{L}\,\frac{1-\alpha^{[L]}}{1-\alpha} + \frac{\beta[L]\,\alpha^{[L]}}{L}$$

$$= \beta - \frac{L-[L]}{L}\,\beta\,\alpha^{[L]} - \frac{\beta-\alpha}{1-\alpha}\,\frac{1-\alpha^{[L]}}{L} \qquad \dots \quad \dots \qquad (35)$$

If L takes an integral value, say n, then $L = [L] = n$ and

$$Y(n,\lambda) = \beta - \frac{\beta-\alpha}{1-\alpha}\,\frac{1-\alpha^{n}}{n} \qquad \dots \quad \dots \qquad (36)$$

We shall now study the properties of the yield function $Y(L,\lambda)$ in some detail. For this purpose, we digress for the moment and establish some inequalities and related results. For all $0 < \lambda < \infty$ the following inequalities hold good.

$$\lambda - 1 + e^{-\lambda} > 0 \qquad \dots \quad \dots \qquad (37)$$

$$1 - 2\lambda\, e^{-\lambda} - e^{-2\lambda} > 0 \qquad \dots \qquad (38)$$

The proofs for these are quite simple. It is well known that $e^{x} > 1+x$ for $x \neq 0$. If we put $x = -\lambda$ we get (37). To establish (38) it is enough to show that $e^{2\lambda} - 2\lambda e^{\lambda} - 1 > 0$ for all $\lambda > 0$. Let $\theta(\lambda) = e^{2\lambda} - 2\lambda e^{\lambda} - 1$, and we have $\theta(0) = 0$ and $\frac{d\theta}{d\lambda} = 2e^{\lambda}(e^{\lambda} - 1 - \lambda) > 0$ for $\lambda > 0$. Hence $e^{2\lambda} - 2\lambda e^{\lambda} - 1 > 0$ for $\lambda > 0$. The in equality (37) can be used to establish the intuitively obvious fact that β - the yield for the sequential cutting procedure is greater than α - the yield for the simple cutting procedure. In fact

$$\beta - \alpha = \frac{\lambda\alpha}{1-\alpha} - \alpha = \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}} - e^{-\lambda}$$

$$= \frac{\lambda e^{-\lambda} - e^{-\lambda} + e^{-2\lambda}}{(1 - e^{-\lambda})} > 0 \quad \text{for } \lambda > 0$$

Further we have for $\lambda > 0$, from the inequality (37)

$$\sum_{i=1}^{\infty} (\lambda i - 1) e^{-\lambda(i-1)} = \frac{\lambda}{(1-e^{-\lambda})^2} - \frac{1}{(1-e^{-\lambda})}$$

$$= \frac{\lambda - 1 + e^{-\lambda}}{(1-e^{-\lambda})^2} > 0 \quad \text{for } \lambda > 0 \quad \ldots \quad (39)$$

Similarly for $\lambda > 0$, we have by inequality (38) that

$$\sum_{i=1}^{\infty} \left\{ 2i - \lambda i(i-1) - 1 \right\} e^{-\lambda(i-1)} = \frac{2}{(1-e^{-\lambda})^2} - \frac{2\lambda e^{-\lambda}}{(1-e^{-\lambda})^3} - \frac{1}{(1-e^{-\lambda})}$$

$$= \frac{2(1-e^{-\lambda}) - 2\lambda e^{-\lambda} - (1-e^{-\lambda})^2}{(1-e^{-\lambda})^3}$$

$$= \frac{1 - 2\lambda e^{-\lambda} - e^{-2\lambda}}{(1 - e^{-\lambda})^3} > 0 \quad \text{for } \lambda > 0 \qquad \ldots (40)$$

Finally, for any positive integer n, we define the function $\xi(n,x)$ for $0 < x < 1$ by

$$\xi(n, x) = \frac{1 - x^n}{n(1 - x)} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (41)$$

If we denote the forward difference operator with respect to n by Δ then

$$\Delta \xi(n, x) = \xi(n+1, x) - \xi(n, x)$$

$$= \frac{1 - x^{n+1}}{(n+1)(1-x)} - \frac{1 - x^n}{n(1-x)}$$

$$= \frac{1}{(1-x)} \frac{n - nx^{n+1} - (n+1) + (n+1)x^n}{n(n+1)}$$

$$= -\frac{(1-x)}{n} \frac{1 - (n+1)x^n + n x^{n+1}}{(n+1)(1-x)^2}$$

$$= -\frac{(1-x)}{n} \frac{d\xi(n+1, x)}{dx} \qquad \ldots \quad \ldots \quad \ldots (42)$$

or equivalently, we have

$$n\xi(n,x) = n\xi(n+1,x) + (1-x)\frac{d\xi(n+1, x)}{dx} \qquad \ldots \quad \ldots (43)$$

Defferentiating both sides of (43) with respect to x and replacing n by (n+1) we get

$$(n+1)\frac{d\xi(n+1,x)}{dx} = n\frac{d\xi(n+2,x)}{dx} + (1-x)\frac{d^2\xi(n+2,x)}{dx} \quad \ldots \quad (44)$$

Further, we have from (42) and (44)

$$\Delta^2\xi(n,x) = \Delta\xi(n+1,x) - \Delta\xi(n,x)$$

$$= -(1-x)\left\{\frac{1}{n+1}\frac{d\xi(n+2,x)}{dx} - \frac{1}{n}\frac{d\xi(n+1,x)}{dx}\right\}$$

$$= \frac{-(1-x)}{n(n+1)}\left\{n\frac{d\xi(n+2,x)}{dx} - (n+1)\frac{d\xi(n+1,x)}{dx}\right\}$$

$$= (-1)^2\frac{(1-x)^2}{n(n+1)}\frac{d^2\xi(n+2,x)}{dx^2}$$

In general we have

$$(n+r)\frac{d^r\xi(n+r,x)}{dx} = n\frac{d^r\xi(n+r+1,x)}{dx} + (1-x)\frac{d^{r+1}\xi(n+r+1,x)}{dx} \quad \ldots \quad (45)$$

$$\Delta^r\xi(n,x) = (-1)^r\frac{(1-x)^r}{n(n+1)\ldots(n+r-1)}\frac{d^r\xi(n+r,x)}{dx^r} \quad \ldots \quad (46)$$

For example, we have

$$\Delta\xi(n,x) = (-1)\frac{1-x}{n}\frac{d\xi(n+1,x)}{dx}$$

$$= (-1)\frac{1-x}{n}\frac{d}{dx}\left(\frac{1-x^{n+1}}{(n+1)(1-x)}\right) = (-1)\frac{1-x}{n(n+1)}\frac{d}{dx}\left(\sum_{i=1}^{n+1}x^{i-1}\right)$$

$$= (-1) \frac{1-x}{n(n+1)} \sum_{i=2}^{n+1} (i-1)\, x^{i-2} < 0 \quad \text{when } 0 < x < 1 \quad \ldots\ldots \quad (47)$$

$$\Delta^2 \xi (n,x) = (-1)^2 \frac{1-x}{n(n+1)} \frac{d^2}{dx^2} \left(\frac{1-x^{n+2}}{(n+2)(1-x)} \right)$$

$$= (-1)^2 \frac{1-x}{n(n+1)(n+2)} \sum_{i=3}^{n+2} (i-1)(i-2)\, x^{i-3} > 0 \quad \text{when } 0 < x < 1 \ldots (48)$$

We shall now make use of the above results to study the properties of the yield function. If $L = n + \delta$ where $0 \leq \delta < 1$, and n is a positive integer, then we have $[L] = n$ and get from (33)

$$Y(n+\delta, \lambda) = \sum_{i=1}^{n} \left(\lambda + \frac{1-i\lambda}{n+\delta} \right) e^{-\lambda i}$$

$$Y(n, \lambda) = \sum_{i=1}^{n} \left(\lambda + \frac{1-i\lambda}{n} \right) e^{-\lambda i}$$

$$Y(n+\delta, \lambda) - Y(n, \lambda) = \frac{\delta}{n+\delta} \sum_{i=1}^{n} (\lambda i - 1)\, e^{-\lambda i} \quad \ldots \quad \ldots (49)$$

Thus when we increase the initial cut length L from n to $n+\delta$ (where $0 < \delta < 1$), the yield increases or remains constant or decreases depending on whether $\sum_{i=1}^{n} (\lambda i - 1)\, e^{-\lambda(i-1)}$ is positive or zero or negative. The value of λ for which $Y(n+\delta, \lambda) = Y(n, \lambda)$ for all δ such that $0 < \delta < 1$ is called the critical value of λ and is denoted by λ_n^*. It is seen that λ_n^* is the solution of the equation

$$\sum_{i=1}^{n} (\lambda i - 1)\, e^{-\lambda(i-1)} = 0 \quad \ldots \quad \ldots \quad (50)$$

Sibuya [3] got the particular cases of (50) for $n = 1$ & 2 and also λ_1^* and λ_2^*. We shall now analytically establish for any n what Sibuya [3] numerically verified for $n = 2$. This is done in the following theorem.

THEOREM 1 : For any given positive integer n, there exists a λ_n^* greater than zero such that

$$Y(n+\delta, \lambda) \begin{cases} > Y(n,\lambda) & \text{for } \lambda > \lambda_n^* \\ = Y(n,\lambda) & \text{for } \lambda = \lambda_n^* \\ < Y(n,\lambda) & \text{for } 0 < \lambda < \lambda_n^* \end{cases} \quad \ldots \quad \ldots \quad (51)$$

for all $\delta \in (0, 1)$. Further (i) $\lambda_1^* = 1$, (ii) $\lambda_{n+1}^* < \lambda_n^*$ and $\lim_{n \to \infty} \lambda_n^* = 0$.

Proof : Let $\phi(n,\lambda) = \sum_{i=1}^{n} (\lambda i - 1)\, e^{-\lambda(i-1)}$. To prove (51) it is enough to show the existence of λ_n^* such that $\phi(n, \lambda) \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0$ when $\lambda \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} \lambda_n^*$. Since $\phi(1,\lambda) = \lambda - 1$, this is trivially true for the case $n = 1$. Hence we consider the case where $n \geq 2$. Since

$\emptyset(n, 0) = -n$ and $\emptyset(n, 1) = \sum_{i=1}^{n} (i-1)\, e^{-(i-1)} > 0$, it is enough if we can show that $\frac{d\,\emptyset(n, \lambda)}{d\,\lambda} > 0$ for all $\lambda \in (0, \infty)$. We have

$$\frac{d\,\emptyset(n, \lambda)}{d\,\lambda} = \sum_{i=1}^{n} \left\{ 2\,i - \lambda i\,(i-1) - 1 \right\} e^{-\lambda(i-1)}$$

$$= 1 + \sum_{i=2}^{n} \left\{ \frac{2i - 1}{i(i-1)} - \lambda \right\} i(i-1)\, e^{-\lambda(i-1)}$$

$$= 1 + \sum_{i=2}^{n} \left(\frac{1}{i} + \frac{1}{i-1} - \lambda \right) i\,(i-1)\, e^{-\lambda(i-1)} \quad \ldots \quad (52)$$

We shall consider the case where $0 < \lambda < \frac{1}{n} + \frac{1}{n-1}$ first and then the case where $\lambda \geq \frac{1}{n} + \frac{1}{n-1}$. When $0 < \lambda < \frac{1}{n} + \frac{1}{n-1}$, we have $\frac{1}{i} + \frac{1}{i-1} > \lambda$ for $i = 2$ to n and hence $\frac{d\,\emptyset(n, \lambda)}{d\,\lambda} > 0$. When $\lambda \geq \frac{1}{n} + \frac{1}{n-1}$, we have $\lambda > \frac{1}{i} + \frac{1}{i-1}$ for $i \geq n+1$. Hence

$$\frac{d\,\emptyset(n, \lambda)}{d\,\lambda} = 1 + \sum_{i=2}^{n} \left(\frac{1}{i} + \frac{1}{i-1} - \lambda \right) i\,(i-1)\, e^{-\lambda(i-1)}$$

$$> 1 + \sum_{i=2}^{\infty} \left\{ \frac{1}{i} + \frac{1}{i-1} - \lambda \right\} i(i-1)\, e^{-\lambda(i-1)}$$

$$> \sum_{i=1}^{\infty} \left\{ 2i - \lambda i(i-1) - 1 \right\} e^{-\lambda(i-1)}$$

$$> \frac{1 - 2.\lambda e^{-\lambda} - e^{-2\lambda}}{(1 - e^{-\lambda})^3} > 0 \quad \ldots \quad \ldots \quad (53)$$

where (53) follows from the inequality (38). Hence for all $\lambda \in (0,\infty)$, we have $\frac{d\,\emptyset(n,\lambda)}{d\lambda} > 0$. This proves the existence of λ_n^*. Since $\emptyset(n, \frac{1}{n}) = \sum_{i=1}^{n} (\frac{i}{n} - 1)\, e^{-\frac{1}{n}(i-1)} < 0$ for $n \geq 2$, it follows $\lambda_n^* > \frac{1}{n}$ for $n \geq 2$. Further

$$\emptyset(n+1, \lambda_n^*) = \emptyset(n, \lambda_n^*) + [(n+1)\lambda_n^* - 1]\, e^{-n\lambda_n^*}$$

$$= [(n+1)\lambda_n^* - 1]\, e^{-n\lambda_n^*} > 0 \quad \text{since } \lambda_n^* > \frac{1}{n}$$

This implies $\lambda_{n+1}^* < \lambda_n^*$. Finally consider an arbitrarily small positive λ_o. Let n_o be the smallest integer such that $n_o > \frac{1}{\lambda_o}$. Then $\emptyset(n, \lambda_o) < 0$ for $n < n_o$ and is strictly increasing with n for $n \geq n_o$. By (39) we have since $\lambda_o > 0$

$$\lim_{n \to \infty} \emptyset(n, \lambda_o) = \frac{\lambda_o - 1 + e^{-\lambda_o}}{(1-e^{-\lambda_o})^2} > 0 \quad \ldots \quad \ldots \quad (54)$$

Hence there should exist an $n_1 > n_o$ such that

$$\emptyset(n, \lambda_o) > 0 \quad \text{for all } n \geq n_1$$

This implies that $\lambda_n^* < \lambda_o$ for all $n \geq n_1$ and this proves the required result that $\lim_{n \to \infty} \lambda_n^* = 0$.

We shall now investigate the assymptotic behaviour of λ_n^* in Theorem 2.

<u>THEOREM 2</u> Let λ_n^* be the unique positive solution of

$$\sum_{i=1}^{n} (\lambda i-1)\, e^{-\lambda(i-1)} = 0$$

for any given positive integer n. Then $n\lambda_n^* \longrightarrow \infty$ and $n\, e^{-n\lambda_n^*} \longrightarrow \frac{1}{2}$ as $n \longrightarrow \infty$.

<u>Proof</u> : As before let $\emptyset(n,\lambda) = \sum_{i=1}^{n} (\lambda i-1)\, e^{-\lambda(i-1)}$. Carrying out the summation we get

$$\emptyset(n,\lambda) = \frac{1}{(1-e^{-\lambda})^2}\left[\lambda\left\{1-e^{-n\lambda} - n\, e^{-n\lambda}(1-e^{-\lambda})\right\} - (1-e^{-\lambda})(1-e^{-n\lambda})\right] \quad ..(55)$$

Since the function $\log x - x + 1$ has one maximum at $x = 1$, we have $\log x < x - 1$ for $x > 0$ and $x \neq 1$. If we put $x = n^{1/n}$, then we get

$$\frac{\log n}{n} < n^{\frac{1}{n}} - 1 \quad \text{for} \quad n > 1 \quad \ldots \quad \ldots \quad (56)$$

If we now put $\lambda = \frac{1}{n} \log n$ or equivalently $e^{-n\lambda} = \frac{1}{n}$ and $e^{-\lambda} = n^{-\frac{1}{n}}$ in (55), we get, by using (56), for $n > 1$

$$(1-n^{-\frac{1}{n}})^2\, \emptyset\left(n, \frac{\log n}{n}\right) = \frac{\log n}{n}\left(n^{-\frac{1}{n}} - \frac{1}{n}\right) - (1-n^{-\frac{1}{n}})(1-\frac{1}{n})$$

$$< (n^{\frac{1}{n}} - 1)(n^{-\frac{1}{n}} - \frac{1}{n}) - (1-n^{-\frac{1}{n}})(1-\frac{1}{n}) = -\frac{n^{\frac{1}{n}}}{n}(1-n^{-\frac{1}{n}})^2 < 0 \quad ..(57)$$

Since we know by theorem 1, that $\frac{d\,\emptyset(n,\lambda)}{d\lambda} > 0$ for $\lambda > 0$; (57) implies that $\lambda_n^* > \frac{1}{n} \log n$.

Hence we get

$$n\lambda_n^* \longrightarrow \infty \text{ as } n \longrightarrow \infty \qquad \ldots \quad \ldots \quad \ldots \qquad (58)$$

We know that for any given n, λ_n^* is such that

$$\lambda_n^* \left\{ 1 - e^{-n\lambda_n^*} - n e^{-n\lambda_n^*}(1 - e^{-\lambda_n^*}) \right\} - (1 - e^{-\lambda_n^*})(1 - e^{-n\lambda_n^*}) = 0$$

$$\text{or} \quad n\lambda_n^* e^{-n\lambda_n^*}(1 - e^{-\lambda_n^*}) = (1 - e^{-n\lambda_n^*}) \left\{ \lambda_n^* - (1 - e^{-\lambda_n^*}) \right\}$$

$$\text{or} \quad n e^{-n\lambda_n^*} = (1 - e^{-n\lambda_n^*}) \left\{ \frac{\lambda_n^* - (1 - e^{-\lambda_n^*})}{\lambda_n^* (1 - e^{-\lambda_n^*})} \right\} \qquad \ldots \quad \ldots \qquad (59)$$

We know by theorem 1, that $\lim_{n \to \infty} \lambda_n^* = 0$, and by (58) we have

$n\lambda_n^* \longrightarrow \infty$ as $n \longrightarrow \infty$. Hence

$$\lim_{n \to \infty} n e^{-n\lambda_n^*} = \lim_{x \to \infty} (1 - e^{-x}) \lim_{\lambda \to 0} \frac{\lambda - 1 + e^{-\lambda}}{\lambda(1 - e^{-\lambda})}$$

$$= \tfrac{1}{2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (60)$$

We can use (60) to get an approximation for λ_n^* when n is large, i.e.,

$$\lambda_n^* \approx \frac{1}{n} \log 2n \qquad \ldots \quad \ldots \quad \ldots \qquad (61)$$

Equation (50) was solved numerically for n=1 to 200 in steps of 1 and n = 200 to 1000 in steps of 50. The procedure used was to find λ_n^* such that

$$\phi(n, \lambda_n^*) \geq 0 \quad \text{and} \quad \phi(n, \lambda_n^* - 0.001) < 0 \quad \ldots \quad \ldots (62)$$

and as such these values may be slightly higher than the correct values. To study the extent to which the approximation (61) for is good, the values of λ_n^* and $\frac{1}{n}$ log 2n are given in Table 2 for some selected n.

Table - 2

Values of λ_n^* and $\frac{1}{n}$ log 2n

n	λ_n^*	$\frac{1}{n}\log 2n$	n	λ_n^*	$\frac{1}{n}\log 2n$	n	λ_n^*	$\frac{1}{n}\log 2n$
1	1.000	0.6931	6	0.418	0.4142	20	0.185	0.1844
2	0.759	0.6931	7	0.379	0.3770	30	0.137	0.1365
3	0.621	0.5973	8	0.348	0.3466	40	0.110	0.1096
4	0.531	0.5199	9	0.322	0.3212			
5	0.467	0.4005	10	0.300	0.2996			

Considering the possibility that the values of λ_n^* given in Table 2 may be larger than the correct values by 0.001, we can conclude that $\frac{1}{n}$ log 2n gives a very good approximation of λ_n^* for $n \geq 5$. Even though (50) was solved numerically for a very wide range of n, these are not given because of this.

We see by theorem 1, that, when we increase the initial cut length from n to n+ δ (where $0 < \delta < 1$), the yield may increase or remain same or decrease depending on whether the defect rate λ is greater than or equal to or less than the corresponding critical value λ_n^*. We shall now show that $Y(n+1, \lambda) > Y(n, \lambda)$ for all $\lambda > 0$ and also study the properties of the yield function for integral values of L. We have from (36), (41), (47), (48) & (46) for $\lambda \in (0, \infty)$

$$Y(n, \lambda) = \beta - \frac{\beta - \alpha}{1 - \alpha} \frac{1 - \alpha^n}{n} = \beta - (\beta - \alpha) \xi (n, \alpha)$$

$$\Delta Y(n, \lambda) = - (\beta - \alpha) \Delta \xi (n, \alpha)$$

$$= (\beta - \alpha) \frac{1 - \alpha}{n (n+1)} \sum_{i=2}^{n+1} (i-1) \alpha^{i-2} > 0 \quad \dots \quad \dots (63)$$

$$\Delta^2 Y(n, \lambda) = - (\beta - \alpha) \Delta^2 \xi (n, \alpha)$$

$$= - (\beta - \alpha) \frac{(1 - \alpha)^2}{n(n+1)(n+2)} \sum_{i=3}^{n+2} (i-1)(i-2) \alpha^{i-3} < 0 \dots (64)$$

$$\Delta^r Y(n, \lambda) = - (\beta - \alpha) \Delta^r \xi (n, \alpha)$$

$$= (-1)^{r+1} \frac{(\beta - \alpha)(1 - \alpha)^r}{n(n+1)\dots(n+r)} \sum_{i=r+1}^{n+r} (i-1)\dots(i-r) \alpha^{i-r-1} \quad ..(65)$$

In (63), (64) and (65) we recall that $\beta > \alpha > 0$ for $\lambda \in (0, \infty)$. Hence for any finite defect rate λ, the yield increases at a decreasing rate with n. It is also seen that

$$\lim_{n \longrightarrow \infty} \Delta^{r} Y(n, \lambda) = 0 \quad \text{for all } r \geq 1 \quad \ldots \quad \ldots (66)$$

Finally, we have from (33) for $\lambda \in (0, \infty)$

$$Y(n+1, \lambda) = \sum_{i=1}^{n+1} \left(\lambda + \frac{1 - \lambda i}{n+1}\right) e^{-\lambda i}$$

$$Y(n+\delta, \lambda) = \sum_{i=1}^{n} \left(\lambda + \frac{1 - \lambda i}{n+\delta}\right) e^{-\lambda i} \quad \text{for } 0 \leq \delta < 1$$

$$Y(n+1, \lambda) - \lim_{\delta \longrightarrow 1^{-}} Y(n+\delta, \lambda) = \frac{1}{n+1} e^{-(n+1)\lambda} > 0 \quad \ldots \quad (67)$$

We shall use the above results to show that generally it is not desirable to have fractional values for L, especially so when λ is small. For example, when $\lambda = 0$, the yield decreases from 100% to 66.7% when L is changed from 1 to 1.5. We have from (51) that $Y(n+\delta, \lambda) >$ or $=$ or $< Y(n, \lambda)$ according to whether $\lambda >$ or $=$ or $< \lambda_n^*$ for $0 < \delta < 1$. We have seen in theorem 1, that λ_n^* decreases with n. Hence when we change the initial cut length from n to $n+\delta$, there is a risk of decrease in yield for small λ, unless n is sufficiently large. We have from table 2, $\lambda_{10}^* = 0.3$. The defect rates (λ) of magnitude more than 0.3 for the final product length are uneconomical in an industrial process. Similarly initial cut lengths above 10 are impracticable in practice. Hence exact multiples of the final product length are sensible values for the initial cut length in practical situations. An added reason for using integral values of L is (67) where it is shown that $Y(n+1, \lambda) > \lim_{\delta \to 1^-} Y(n+\delta, \lambda)$ for all λ.

The yield values $Y(n, \lambda)$ when the initial cut length L is an integer n, has been computed for a wide range of n and λ by programming on a computer. Table III at the end of this part gives the value of $Y(n, \lambda)$ for $n = 1(1)\ 10(5)\ 50(10)\ 100$ and ∞, $\lambda = 0.02\ (0.02)\ 0.10(0.05)\ 1.0$. It is felt that this table will be useful in practical application of the model.

The cost of inspection and the cutting procedure increases as the initial cut length n increases. For convenience we can denote this cost by C (n) for some definite volume of production, say a lot of N pieces. In general C (n) and C (n+1) - C(n) will be increasing with n. If we get more number of acceptable pieces from a lot through sophistication in cutting procedure there will be direct savings by way of profit, reduction in material consumed, reduction in rework and the corresponding reduction in production costs etc. The indirect benefit will be greater production rate. Hence in general it will be possible either to estimate (or impute) in terms of money value of the above. Let A (r) denote the monetory benefit if r more good pieces are obtained from a lot of N pieces. Hence the optimal value of n satisfies.

$$A\ (NY(n,\lambda)) - A(NY(n-1,\lambda)) \geq C(n) - C(n-1)$$

$$\qquad\qquad \ldots \qquad (68)$$

$$A\ (NY(n+1,\lambda)) - A(NY(n,\lambda)) < C(n+1) - C(n)$$

This can be found easily with the help of Table III once the savings and the costs are estimated.

APPENDIX

In this appendix, we show that the expression (25) for Machine Efficiency is essentially the same as the one obtained by Howie and Shenton [2]. If we put

$$k = e^{-\lambda d}, \quad c = \lambda h \text{ and } b = \lambda d$$

then by (15), we have

$$(\lambda T+1) = \lambda \left\{ (m+1)d - h \right\} + 1 = (m+1)b - c + 1 \quad \ldots (A1)$$

$$\sum_{n=o}^{m} P(nd) = \sum_{n=o}^{m} \left(1 - \frac{\lambda n d}{\lambda T+1}\right) e^{-\lambda nd} \quad \ldots \ldots \ldots (A2)$$

$$= \sum_{n=o}^{m} k^n - \frac{\lambda d}{\lambda T+1} \sum_{n=1}^{m} n k^n \quad \ldots \ldots (A3)$$

$$\sum_{n=0}^{m} k^n = \frac{1 - k^{m+1}}{1 - k} \quad \ldots \ldots \ldots \ldots (A4)$$

$$\sum_{n=1}^{m} n k^n = k \frac{-(1-k)(m+1)k^m + 1-k^{m+1}}{(1-k)^2} \quad \ldots \ldots (A5)$$

$$= \frac{k - (m+1)k^{m+1} + m k^{m+2}}{(1-k)^2} \quad \ldots \ldots (A6)$$

$$(\lambda T+1)\sum_{n=o}^{m} P(nd) = (\lambda T+1)\left[\frac{1-k^{m+1}}{1-k} - \frac{\lambda d}{\lambda T+1}\;\frac{k-(m+1)k^{m+1}+m\,k^{m+2}}{(1-k)^2}\right]$$

$$= \frac{(m+1)b-c+1}{1-k} - \frac{[(m+1)b-c+1]k^{m+1}}{1-k} - b\,\frac{k-(m+1)k^{m+1}+m\,k^{m+2}}{(1-k)^2}$$

$$= \frac{(m+2)b-c+1}{1-k} - \frac{b}{(1-k)^2} - \frac{[(m+1)b-c+1]\,k^{m+1}}{(1-k)} - b\,\frac{m\,k^{m+2}-(m+1)\,k^{m+1}}{(1-k)^2}$$

$$= \frac{(m+2)b-c+1}{(1-k)} - \frac{b}{(1-k)^2} - k^m\left[\frac{\{(m+1)b-c+1\}k}{(1-k)} + \frac{mbk^2-(m+1)bk}{(1-k)^2}\right]$$

$$= \frac{(m+2)b-c+1}{(1-k)} - \frac{b}{(1-k)^2} - k^m\left[-(b-c+1) + \frac{mbk+b-c+1}{(1-k)} + \frac{mbk^2-(m+1)bk}{(1-k)^2}\right]$$

$$= \frac{(m+2)b-c+1}{(1-k)} - \frac{b}{(1-k)^2} - k^m\left[-(b-c+1) + \frac{2b-c+1}{(1-k)} + \frac{mbk-b}{(1-k)} + \frac{mbk^2-(m+1)bk}{(1-k)^2}\right]$$

$$= \frac{(m+2)b-c+1}{(1-k)} - \frac{b}{(1-k)^2} - k^m\left[-(b-c+1) + \frac{2b-c+1}{(1-k)} + \frac{mbk-b}{(1-k)} + \frac{mbk^2-mbk-bk+b-b}{(1-k)^2}\right]$$

$$= \frac{(m+2)b-c+1}{(1-k)} - \frac{b}{(1-k)^2} - k^m\left[-(b-c+1) + \frac{2b-c+1}{(1-k)} - \frac{b}{(1-k)^2}\right]$$

$$= \frac{(m+2)b-c+1}{(1-k)} - \frac{b}{(1-k)^2} + k^m\left[(b-c+1) - \frac{2b-c+1}{(1-k)} + \frac{b}{(1-k)^2}\right] \quad \ldots \quad (A7)$$

If we substitute (A7) for $(\lambda T+1)\sum_{n=o}^{m} P(nd)$ in (25) we get the expression of Howie & Shanton [2] for Machine efficiency.

TABLE - III

VALUES OF $Y(n, \lambda)$

n \ λ	0.02	0.04	0.06	0.08	0.10
1	0.980199	0.960789	0.941765	0.923116	0.904837
2	.980296	.961169	.942595	.924555	.907026
3	.980392	.961538	.943394	.925919	.909076
4	.980487	.961898	.944162	.927215	.910996
5	.980581	.962248	.944900	.928444	.912797
6	.980673	.962589	.945610	.929612	.914487
7	.980764	.962982	.946293	.930722	.916073
8	.980854	.963246	.946950	.931777	.917563
9	.980943	.963562	.947583	.932780	.918963
10	.981030	.963869	.948192	.933734	.920280
15	.981452	.965294	.950915	.937861	.925800
20	.981846	.966550	.953179	.941113	.929937
25	.982216	.967660	.955073	.943701	.933087
30	.982564	.968642	.956667	.945783	.935524
35	.982890	.969514	.958014	.947474	.937441
40	.983196	.970290	.959161	.948863	.938971
45	.983484	.970983	.960143	.950014	.940212
50	.983754	.971602	.960988	.950978	.941232
60	.984249	.972657	.962357	.952489	.942798
70	.984688	.973514	.963405	.953607	.943935
80	.985079	.974218	.964225	.954460	.944794
90	.985427	.974802	.964880	.955130	.945463
100	.985739	.975290	.965412	.955668	.946000
∞	.990034	.980134	.970300	.960533	.950833

(Contd. Table III)

λ / n	0.15	0.20	0.25	0.30	0.35
1	0.860708	0.818731	0.778801	0.740818	0.704688
2	.865316	.826398	.790016	.755938	.723957
3	.869497	.833140	.799577	.768445	.739433
4	.873294	.839080	.807759	.778847	.751955
5	.876747	.844328	.814787	.787547	.762165
6	.879893	.848974	.820847	.794865	.770556
7	.882761	.853099	.826095	.801057	.777506
8	.885381	.856771	.830655	.806325	.783309
9	.887777	.860046	.834636	.810834	.788191
10	.889972	.862976	.838124	.814715	.792332
15	.898544	.873766	.850360	.827812	.805882
20	.904346	.880423	.857436	.835037	.813113
25	.908320	.884788	.861902	.839493	.817514
30	.911216	.887813	.864931	.842486	.820458
35	.913374	.890009	.867107	.844628	.822562
40	.915028	.891667	.868743	.846235	.824140
45	.916331	.892961	.870016	.847485	.825367
50	.917379	.893997	.871035	.848486	.826349
60	.918958	.895553	.872563	.849986	.827823
70	.920089	.896664	.873654	.851058	.828875
80	.920937	.897497	.874473	.851862	.829664
90	.921596	.898145	.875109	.852487	.830278
100	.922124	.898664	.875619	.852987	.830769
∞	.926874	.903331	.880203	.857489	.835188

(TableIII Contd.)

n \ λ	0.40	0.45	0.50	0.60	0.70
1	0.670320	0.637628	0.606531	0.548812	0.496585
2	.698889	.665565	.638838	.589646	.545396
3	.712277	.686753	.662670	.618198	.577825
4	.726766	.703026	.680529	.638624	.600058
5	.738299	.715687	.694128	.653974	.615789
6	.747574	.725667	.704651	.664784	.627265
7	.755110	.733637	.712925	.673369	.635884
8	.761297	.740084	.719533	.680086	.642531
9	.766427	.745364	.724889	.685447	.647782
10	.770724	.749741	.729293	.689803	.652018
15	.784457	.763485	.742939	.703079	.664824
20	.791621	.770546	.749880	.709762	.671243
25	.795951	.774798	.754053	.713774	.675095
30	.798842	.777635	.756835	.716449	.677664
35	.800907	.779661	.758823	.718359	.679498
40	.802456	.781181	.760313	.719792	.680874
45	.803660	.782363	.761472	.720906	.681944
50	.804624	.783308	.762400	.721798	.682800
60	.806070	.784727	.763791	.723135	.684084
70	.807102	.785740	.764785	.724090	.685001
80	.807877	.786500	.765530	.724807	.685689
90	.808479	.787091	.766110	.725364	.686224
100	.808961	.787563	.766570	.725810	.686652
∞	.813298	.791818	.770747	.729822	.690504

(Table III Contd.)

n \ λ	0.80	0.90	1.00
1	0.449329	0.406570	0.367879
2	.505344	.468891	.435547
3	.540796	.506556	.474699
4	.564176	.530540	.498853
5	.580237	.546605	.514694
6	.591705	.557883	.525667
7	.600190	.566137	.533636
8	.606669	.572397	.539654
9	.611754	.577292	.544348
10	.615841	.581217	.548109
15	.628143	.593010	.559397
20	.634301	.598909	.565042
25	.637995	.602449	.568429
30	.640458	.604808	.570687
35	.642217	.606494	.572300
40	.643537	.607758	.573509
45	.644563	.608741	.574450
50	.645384	.609527	.575203
60	.646616	.610707	.576332
70	.647495	.611550	.577138
80	.648155	.612182	.577743
90	.648668	.612673	.578213
100	.649079	.613067	.578590
∞	.652773	.616606	.581977

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