ON SOME NON-UNIFORM RATES OF CONVERGENCE TO NORMALITY WITH APPLICATIONS

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C O N T E N T S

Notations and Abbreviations

Abbreviations

r.v.	random variable
i.i.d.	independent and identically distributed
i.i.d. r.v.'s	independent and identically distributed random variables
a.e.	almost everywhere
d.f.	distribution function
w.r.t.	with respect to
m.g.f.	moment generating function
nbhd	neighbourhood

Notations

$\Phi(x)$	$\int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-x^2/2)\, dx$
$X_n \xrightarrow{L} X$, $X_n \xrightarrow{D} X$	The sequence of r.v.'s X_n converging in distribution to the r.v. X.
$a_n = O(b_n)$	$\limsup_{n \to \infty} a_n/b_n < \infty$
$a_n = o(b_n)$	$a_n/b_n \to 0$ as $n \to \infty$
$a_n = O_e(b_n)$	$0 < \liminf_n a_n/b_n \leq \limsup_n a_n/b_n < \infty$
I(A)	Indicator function of the set A
$X_n = O_p(a_n)$	For every $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that $P(\lvert X_i/a_i \rvert > K(\epsilon)) < \epsilon$ for all $i \geq 1$
$a_n \ll b_n$	$a_n = O(b_n)$
$a \vee b$	max(a, b)
$a \wedge b$	min(a, b)

CHAPTER 1.

1.1. INTRODUCTION

The central role of the normal distribution in statistical theory and methodology is beyond question. Apart from other attractive features, one important reason why this distribution has been found to be so useful is that it turns out to be the limiting distribution of many well-known statistics (for example the sample mean, sample moments etc.) after suitable standardizations under usually very moderate assumptions.

A result of pivotal importance in this respect is the so called Central Limit Theorem (CLT) which says that if $X_1, X_2, \ldots$ is a sequence of iidrv's with common mean μ and common variance σ^2 $(0 < \sigma^2 < \infty)$, then defining $S_n = \Sigma_1^n X_i$ $(n \geq 1)$ and $F_n(t) = P((S_n - n\mu)/(\sqrt{n}\,\sigma) \leq t)$, one has

(1.1.1) $$\sup_{-\infty < t < \infty} | F_n(t) - \Phi(t) | = 0 ,$$

where $\Phi(t)$ is the distribution function of a $N(0,1)$ variable.

One major limitation of the above result is that it does not say anything regarding the rate of convergence of F_n to Φ. The independent work of Berry and Esseen show that if in addition

(1.1.2) $$E\,|X_1|^3 < \infty ,$$

then

(1.1.3) $$\sup_t | F_n(t) - \Phi(t) | \leq C\, n^{-\frac{1}{2}}\, E(|X_1 - \mu|^3/\sigma^3)$$

then

(1.1.3) $$\sup_t |F_n(t) - \Phi(t)| \leq C\, n^{-\frac{1}{2}} E(\ |X_1 - \mu|^3 / \sigma^3\ )$$

where C is a universal constant. (For numerical value of C, see Zolotarev (1967).)

The condition (1.1.2) was later weakened by Katz (1963) to

(1.1.4) $$E\,[X^2\, g(x)] < \infty\ ,$$

where $g(x)$ is an even, nonnegative, real valued, nondecreasing function with $|x|/g(x)$ nondecreasing on $[0, \infty)$. He could then prove that

(1.1.5) $$\sup_t |F_n(t) - \Phi(t)| \leq C\, g^{-1}(\sqrt{n})\, E[\{(X-\mu)/\sigma\}^2\, g((X-\mu)/\sigma)]$$

These uniform rates of convergence, though very useful are non adequate for many purposes. For instance, if it is known that g is a function such that $E[g(|T|)] < \infty$, where T is $N(0,1)$, then in view of the fact that $(S_n - n)/(\sqrt{n}s) \xrightarrow{L} T$, one might intuitively expect that $E\, g(|(S_n - n)/\sqrt{n}s|) \longrightarrow E\, g(T)$ as $n \to \infty$. Or, one might be interested in knowing whether a L_p-version of the Berry Esseen Theorem holds, i.e., whether

(1.1.6) $$||\, F_n(t) - \Phi(t)\, ||_p = [\int_{-\infty}^{\infty} |F_n(t) - \Phi(t)|^p\, dt]^{\frac{1}{p}} \to 0$$

as $n \to \infty$. Questions of this type cannot be answered from the uniform rates of convergence given in (1.1.3) or (1.1.5).

The main reason why bounds of the type (1.1.3) or (1.1.5) are inadequate for the above purposes is that they do not reflect the role of t in the rate of convergence. The following result of Nagaev (1965) gives a nonuniform rate of convergence of $F_n(t)$ to $\Phi(t)$.

Theorem 1.1.1 *Let* $X_1, X_2, \ldots$ *be iid with* $E X_1 = 0$, $E X_1^2 = 1$ *and* $E|X_1^3| < \infty$. *Then,*

$$(1.1.7) \qquad |F_n(t) - \Phi(t)| \leq C n^{-1/2} (1 + |t|^3)^{-1},$$

where C *is a universal constant.*

Recently Michel (1976) has provided an interesting approach in the study of nonuniform rates of convergence of $F_n(t)$ to $\Phi(t)$. His way to tackle the problem is to break up the positive axis into two regions, and obtain two different bounds for the difference $|F_n(t) - \Phi(t)|$ depending on the region where t^2 belongs. This idea was possibly implict in Esseen (1945), but was explored very effectively by Michel (1976).

Cramer (1938) had an important result in the theory of deviations. He was interested in the question of values of t (might depend on n) for which $1 - F_n(t) \sim \Phi(-t)$ as $n \to \infty$. Cramer's main theorem is as follows:

Theorem 1.1.2 *If the X_i's are iid with a finite mgf, then*,

(1.1.8) $1 - F_n(t) \sim \bar{\Phi}(-t)$ *for* $t = o(n^{1/6})$ *and*

$$1 - F_n(t_n) = \exp\left[-\frac{1}{2} t_n^2(1+o(1))\right] \text{ for } 1 < t_n = o(n^{\frac{1}{2}}).$$

Later, in connection with the study of Bayes risk efficiency Rubin and Sethuraman (1965 a, 1965 b) considered the case $\lambda_n = c(\log n)^{\frac{1}{2}}$ $(c > 0)$. They showed that if $E|X|^{c+2+\delta} < \infty$ for some $\delta > 0$, then, $1 - F_n(\lambda_n) \sim \bar{\Phi}(-\lambda_n)$. The result was later proved under the weaker condition $E|X|^{c+2} < \infty$ by Michel (1974).

Michel (1976) strengthened his 1974 results further to obtain rates of convergence to normality depending on t and n, and it is the later work which is the genesis of the present thesis.

1.2 SOME WEAK DEPENDENCE STRUCTURE

In this section, we define some weakly dependent processes to be considered in Chapters 4 and 5 of this thesis. Let $\{X_i, -\infty < i < \infty\}$ denote a sequence of random variables, and let $\mathbb{B}_a^b = \sigma(X_i, a \leq i \leq b)$ denote the σ-field generated by $X_a, X_{a+1}, \ldots, X_b$.

<u>Definition</u> X_i is said to form an m-dependent sequence if for all k,

$$\sup_{A \in \mathbb{B}_{-\infty}^{k}} \sup_{B \in \mathbb{B}_{k+n}^{\infty}} | P(AB) - P(A) P(B) | = 0 ,$$

whenever $n \geq m$. In other words, the sequence $\{X_i\}$ is m-dependent if $(\ldots, X_{r-1}, X_r)$ is distributed independently of $(X_{r+m}, X_{r+m+1}, \ldots)$ for all r.

m-dependence is one of the simplest types of dependence. An important example is the moving average process generated by finite linear combinations of independent random variables.

A more general kind of dependence is the so-called ϕ-mixing. The sequence $\{X_i\}$ of rv's is said to be ϕ-mixing if for all k,

$$\sup_{A \in \mathbb{B}_{-\infty}^{k}} \sup_{B \in \mathbb{B}_{k+n}^{\infty}} | P(B|A) - P(B) | \leq \phi(n) ,$$

where $P(A) > 0$, and $\{\phi(n), n \geq 0\}$ is a nonnegative sequence of real numbers satisfying $1 = \phi(0) \geq \phi(1) \geq \phi(2) \geq \ldots$, with $\lim_{n \to \infty} \phi(n) = 0$. In a ϕ-mixing process, distant future is virtually independent of the present. The ϕ-mixing process includes as special cases m-dependent processes, some Markov processes and certain infinite order chains.

1.3 SURVEY OF RELATED WORK

As mentioned already, this research originates with the work of Michel (1976). One of the pleasant features of Michel's results is that these yield as byproducts several important results earlier proved independently by several authors. One important corollary are the moderate deviation results of Rubin and Sethuraman (1965 a). There have been several extensions of these PMD results in recent years. Ghosh (1974, 1975) obtained the PMD results for stationary and nonstationary m-dependent sequences. PMD for ϕ-mixing processes were obtained by Ghosh and Babu (1977), later extended to strong mixing sequences with exponential decay by Babu and Singh (1978). Also, nonuniform rates of convergence to normality for ϕ-mixing processes were obtained by Babu, Ghosh and Singh (1978) quite in the spirit of Michel's work.

Regarding the so-called large deviations results, i.e., deviations of the type $a\sqrt{n}$, Chernoff's (1952) classical theorem asserts that if the moment generating function is finite in a neighbourhood of the origin, then,

$$n^{-1} \log (1 - F_n(a\sqrt{n})) \to \log \rho ,$$

where $\rho = \inf_{h \geq 0} E[\exp(h(X_1 - a))]$. Several extensions were made by Sievers (1969), Plachky (1971), Plachky and Steinbach (1975) when the rv's are independent but not necessarily iid or when the rv's are dependent, but the cumulant generating functions satisfy certain stability assumptions.

This thesis is also concerned with deviations of certain nonlinear statistics, i.e., statistics which are not linear functions of the observations. Such statistics can in general be split into two components, the first one being a sum of independent random variables, the second one being a negligible remainder. To be precise, let $T_n = s_n^{-1} S_n + R_n$, where $S_n = \sum_1^n X_{ni}$ $(n \geq 1)$, $s_n^2 = \sum_1^n V(X_{ni})$, $(n \geq 1)$, where X_{ni} $(i = 1, \ldots, n)$ are independent random variables. Suppose, $\liminf_{n \to \infty} n^{-1} s_n^2 > 0$ and $R_n \xrightarrow{P} 0$ as $n \to \infty$. Then, under very mild additional conditions, $T_n \longrightarrow N(0, 1)$. Representation of the above type is fairly common, and is obtainable, say, via Hájek's projection lemma. Examples of such statistics are the U-statistics or L-statistics.

Central limit theorems for U-statistics were first obtained by Hoeffding (1948) for iid and independent rv's. Recently, interest has been focussed on the rate of convergence to normality for U-statistics. The order bound $O(n^{-\frac{1}{2}+\delta})$ $(\delta > 0)$ for U-statistics were obtained by Grams and Serfling (1973) assuming finiteness of all the moments, while the order bound $O(n^{-1/2})$ for U-statistics with bounded kernels were obtained by Bickel (1974). This condition was later weakened by Chan and Wierman (1977) who obtained the order $O(n^{-1/2})$ assuming finiteness of the fourth moment of the kernel and the order $O(n^{-1/2}(\log n)^{1/3})$ assuming finiteness of the third moment of the kernel. Finally, Callaert and Janssen (1978) obtained the sharpest order bound $O(n^{-1/2})$ assuming only the finiteness of the third moment of the kernel.

The asymptotic normality of L-statistics was first obtained by Jung (1955). Later, under various assumptions on the score functions, and the moments of the distributions, such results were obtained by Chernoff et al (1967), Bickel (1967), Moore (1968), Shorack (1972, 1973, 1974), Stigler (1972, 1973, 1974) and others. Rosenkrantz and O'Reilly (1972) obtained the rate $n^{-1/4}$ of convergence to normality for L-statistics using a Skorohod representation. Bjerve (1977) obtained the sharpest order $n^{-1/2}$ for trimmed L-statistics combining the techniques of Chernoff et al (1967) and Bickel (1974). Finally, in a recent paper Helmers (1978) obtains the Berry Esseen bound $n^{-1/2}$ for untrimmed L-statistics with smooth weight functions not allowing too much weight on extreme observations.

1.4 A BRIEF SUMMARY OF CHAPTERS 2-5

In Chapter 2 we obtain non-uniform rates of convergence to normality of the partial sums in a triangular array of random variables, where variables in each array are independently distributed. Section 2 of this chapter generalises the results of Michel (1976) mainly in the direction of considering a triangular array of random variables. A slight generality in the moment assumptions is also made. The later extension is quite in spirit with Katz's (1963) extension of the classical Berry-Esseen theorem. Since by Tomkin's theorem (see Tomkins (1971) or Stout (1974)) the laws of the iterated logarithm are directly related to the zone where $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$ as a corollary of our theorem (2.2.6) (see in particular (2.2.52)) we are able to show that laws of iterated logarithm for S_n holds if

$$\sup_{n \geq 1} \max_{1 \leq i \leq n} E X_{ni}^2 (\log (1 + |X_{ni}|))^{1+\varepsilon} < \infty$$

for some $\varepsilon > 0$, which is incidentlly best known result for independent case (see Stout again P 275). As other applications of these non-uniform rates we prove moment type convergences and a non-uniform L_p version of Berry-Esseen theorem extending the results of Erickson (1973).

In section 3 of Chapter 2 we consider the case when all the finite moments of the underlying random variables exist but the m.g.f need not necessarily exist. Consideration of this situation helps us weaken the assumptions of the known results of the last four decades. As a consequence of non-uniform bounds in this case we prove Cramér's(1938) results on normal approximation zone and on large deviations under milder conditions. We also prove Bahadur (1960) type upper class results on excessively large deviation (i.e., deviation of the form $\sqrt{n}\, a$, $a > 0$) for random variables in a triangular array, sharpening his results with an estimate of $\delta(n, \varepsilon)$ (see Bahadur, 1960 paper) in i.i.d case, for both upper and lower class estimates. Other applications of these nonuniform rates includes moment type convergence and a stronger non-uniform L_p version of the Berry-Esseen theorem.

In section 4 of Chapter 2 we partially cover the extreme case viz. when m.g.f necessarily exists but the random variables are not necessarily bounded. Since it is known (see Feller 1969) that so far normal approximation zone and large deviation zones are concerned boundedness of the random variables is not of much help compared to the milder condition of existance of moment generating function, we content ourselves with a relatively stronger form compared to section 3, of L_p versions of the Berry-Esseen theorem and moment type convergences.

Chapter 3 generalises the results of Chapter 2 for general non-linear statistics. As examples, we include L-statistics and U-statistics also show that the technique can very well be adopted to cover sampling without replacement from finite populations.

Chapter 4 of this thesis considers m-dependent process. Results of Chapter 2 are extended to this process.

Finally in Chapter 5 we consider ϕ-mixing process. The results of section 3 of Chapter 2 are extended to some (non-stationary) ϕ-mixing process.

CHAPTER 2.

NON UNIFORM RATES OF CONVERGENCE FOR STANDARDISED ROW SUMS OF RANDOM VARIABLES IN A TRIANGULAR ARRAY

In this chapter we study the non uniform rates of convergence to normality for random variables in a triangular array. Consider a double sequence $[X_{ni} : 1 \le i \le n , \ n \ge 1]$ of random variables, where variables within each row are independently distributed and satisfy

(2.0.1) $$E X_{ni} = 0 \ ; \quad \sup_{n \ge 1} \max_{1 \le i \le n} E X_{ni}^2 g(X_{ni}) < \infty$$

with g, to be specified later,

and

(2.0.2) $$\inf_{n \ge 1} n^{-1} s_n^2 > 0 \quad \text{where} \quad s_n^2 = \sum_{i=1}^{n} E X_{ni}^2 .$$

Nonuniform rates of convergence of S_n/s_n are studied under different conditions on g.

NON UNIFORM RATES WHEN SOME FINITE MOMENTS OF X_{ni}'s EXIST.

2.1 Introduction. In this section we first generalise and extend Michels (1976) results to row sum of independent random variables in a triangular array. Michel (1976) considered $g(x) = |x|^c$ for some $c > 0$. In the following we allow $g(x)$ to be slightly more general than Michel's ,let

$$(2.1.1) \qquad E X_{ni} = 0 . \qquad \sup_{n \geq 1} \max_{1 \leq i \leq n} E|X_{ni}|^{2+c} u(X_{ni}) < \infty$$

where $c \geq 0$ and $u(x)$ is non-negative, even, nondecreasing function on $[0, \infty)$ with $u(x) < |x|^{\varepsilon} + L(\varepsilon)$ for all $\varepsilon > 0$ with some $L > 0$.

Examples of $u(x)$ satisfying the above assumptions are $u(x) \equiv 1$ $(c > 0)$, $u(x) = \log (1 + |x|)$, $u(x) = \log \log(e + |x|)$ etc.

In the special case when the X_i's are i.i.d random variables, our assumptions are identical with those of Katz (1963) when $0 < c < 1$ or $c = 0$ and $\lim_{|x| \to \infty} u(x) = \infty$ or $c = 1$ and $u(x) = 1$. We further assume (2.0.2) to hold.

We now define $S_n = \sum_{i=1}^{n} X_{ni}$, $F_n(t) = P(s_n^{-1} S_n \leq t)$. In section 2 we derive some non-uniform bounds for $|F_n(t) - \Phi(t)|$ for different values of t, and use these to study the speed of convergence of $1 - F_n(t)$ to zero as $t \to \infty$, the speed of convergence of the moments of $|s_n^{-1} S_n|$ to those of $|N(0, 1)|$ variable and in finding certain L_p version of the Berry-Esseen theorem.

2.2 THE RESULTS ON ROW SUMS OF RANDOM VARIABLES IN A TRIANGULAR ARRAY

First we prove two theorems giving rates of convergence of $F_n(t)$ to $\Phi(t)$ depending on both n and t. In the special case of sums of i.i.d random variables, these include more general versions of theorems 1 and 2 of Michel (1976). For sums of i.i.d random variables our theorems are quite in the spirit of Katz's (1963) extension of the classical Berry-Esseen theorem.

Theorem 2.2.1. Let (2.1.1) and (2.0.2) hold. Then for $t^2 \leq K[(c/2)\log n + \log u(\sqrt{n})]$, $K > 0$, there exists positive constants b and r (depending on u, c and K) such that

$$(2.2.1) \qquad |F_n(t)-\Phi(t)| \leq b\, w \exp\left[-\tfrac{1}{2}t^2(1-3r)\right] + \sum_{i=1}^{n} P(|X_{ni}|>rs_n|t|),$$

where $w = w(n, |t|, c) = (n^{1/2}(|t| v 1))^{-c}\,(u(r\, s_n(|t|\ v 1))^{-1}$ or $n^{-1/2}$ according as $0 \leq c < 1$ or $c \geq 1$.

Theorem 2.2.2. Let (2.1.1) and (2.0.2) hold. Then for $t^2 \geq K[(c/2)\log n + \log u(\sqrt{n})]$ there exists b (> 0), r (> 0) depending on u, c and K such that

$$(2.2.2) \qquad |F_n(t)-\Phi(t)| \leq b\left[n^{c/2}\, u(\sqrt{n})\right]^{-(K-1)/2} |t|^{-2(K+1)} + \sum_{i=1}^{n} P(|X_{ni}|)\, r\, s_n|t|)$$

Proof of theorem 2.2.1. Throughout the proof $b_1, b_2, \ldots$ denote positive constants which might depend on u and c but not on n and t. The theorem is obvious for $t = 0$. We prove the theorem only for $t > 0$, as the proof when $t < 0$ is analogous. For $0 < t \leq 1$ the theorem follows immediately from Katz's (1963) theorem. For $t > 1$, let

(2.2.3) $$Y_i = Y_{ni} = X_{ni}\, I\,(|X_{ni}| \leq r\, s_n t)\ , \quad i = 1,2 \ldots n.$$

I being the usual indicator function. Define $S_n' = \sum_{i=1}^{n} Y_i \quad (n \geq 1)$ Then

(2.2.4) $$|P(s_n^{-1} S_n' \leq t) - F_n(t)| \leq \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n t)$$

Next define

(2.2.5) $$f_i(t) = f_{n,i}(t) = E \exp(t\, Y_i/s_n)\ , \quad i = 1,\ldots, n.$$

(2.2.6) $$m_i(t) = f_i(t)\, E\, [Y_i \exp(t\, Y_i/s_n)]\ , \quad i = 1,\ldots, n,$$

$$\bar{m}_n(t) = n^{-1} \sum_{i=1}^{n} m_i(t)$$

(2.2.7) $$m_i^2(t) + \sigma_i^2(t) = f_i^{-1}(t)\, E\, [Y_i^2 \exp(t\, Y_i/s_n)]\ , \quad i = 1,\ldots n,$$

$$\bar{\sigma}_n^2(t) = n^{-1} \sum_{i=1}^{n} \sigma_i^2(t).$$

(2.2.8) $$H_n(z) = G_n(\bar{\sigma}_n(t) \sqrt{n}\, z + n\, \bar{m}_n(t))$$

where

(2.2.9) $$dG_n(z) = \{E(\exp(t\, S_n'/s_n))\}^{-1} \exp(z/s_n)\, dP(S_n' \leq z)$$

Then standard methods (see e.g. Cramér (1938) or Bahadur and Ranga Rao (1960)) yield

$$(2.2.10) \quad P(s_n^{-1} S_n' > t) = A_n(t) \int_{B_n(t)}^{\infty} \exp(-t\, s_n^{-1} n^{\frac{1}{2}} \bar{\sigma}\, z)\, dH_n(z)$$

where

$$(2.2.11) \quad A_n(t) = \prod_{i=1}^{n} f_i(t) \exp(-t\, s_n^{-1}\, n\, \bar{m}_n(t))$$

$$(2.2.12) \quad B_n(t) = (t\, s_n - n\, \bar{m}_n(t)) / (\sqrt{n}\, \bar{\sigma}_n(t))$$

using (2.0.2) and (2.1.1) one has the estimates

$$(2.2.13) \quad |E\, Y_i| = o((r\, s_n t)^{-(c+1)}\, u^{-1}(r\, s_n t)), \quad 1 \leq i \leq n,$$

$$(2.2.14) \quad 0 \leq E\, X_{ni}^2 - E\, Y_i^2 = o((r\, s_n t)^{-c}\, u^{-1}(r\, s_n t)), \ 1 \leq i \leq n$$

$$(2.2.15) \quad E|Y_i|^3 = O(1) \quad \text{if} \quad c \geq 1$$

$$= O((r\, s_n t)^{1-c}(u(r\, s_n t))^{-1}) \quad \text{if} \quad 0 \leq c < 1$$

Now using (2.2.13) - (2.2.15),

$$(2.2.16) \quad \left|f_i(t) - 1 - \frac{t^2}{2s_n^2} E\, X_{ni}^2\right| \leq b\, w\, n^{-1} \exp(\tfrac{5}{4} r\, t^2)$$

Next we show that $w \exp(\frac{5}{4} r\, t^2) = O(1)$ by proper choice of $r >$ 0.

For $0 \leq c < 1$

$$(2.2.17) \quad w \exp(\tfrac{5}{4} rt^2) \leq (n^{1/2}\, t)^{-c}\, u^{-1}(r\, s_n t)(n^{c/2} u\sqrt{n}))^{\frac{5}{4}rK} = O(1)$$

$$\text{if} \quad r < \min(\tfrac{4}{5} K^{-1}, 1)$$

Again for $c \geq 1$, since $u(x) < |x|^{\varepsilon} + L$ for all $\varepsilon > 0$, one gets

(2.2.18) $$w \exp(\tfrac{5}{4} rt^2) = n^{-1/2}(n^{c/2}\, u\sqrt{n}\,))^{\frac{5}{4}Kr} = O(1)$$

if $r < 4/(5Kc)$.

Therefore, choose $0 < r < \min(1, (5K)^{-1}(c \vee 1)^{-1})$ so that both (2.2.17) and (2.2.18) hold. Now from (2.2.16) - (2.2.18),

(2.2.19) $$\sum_{i=1}^{n} \log f_i(t) = \frac{1}{2} t^2 + O(w \exp(\tfrac{5}{4} rt^2)) .$$

Next note that

(2.2.20) $$E[\, Y_i \exp(t\, s_n^{-1} Y_i)] = ts_n^{-1} EX_{ni}^2 + O(n^{-1/2} w \exp(\tfrac{5}{4} rt^2))$$

(2.2.21) $$E[\, Y_i^2 \exp(t\, s_n^{-1} Y_i)] = E\, X_{ni}^2 + O(n^{-1/2} w \exp(\tfrac{5}{4} rt^2))$$

Hence, from (2.2.6), (2.2.7), (2.2.16), (2.2.20) and (2.2.21) one gets

(2.2.22) $$m_i(t) = t\, s_n^{-1} EX_{ni}^2 + O(n^{-1/2} w \exp(\tfrac{5}{4} rt^2)) ,$$

(2.2.23) $$m_i^2(t) + \sigma_i^2(t) = EX_{ni}^2 + O(n^{-1/2} w \exp(\tfrac{5}{4} rt^2)) ,$$

Thus

(2.2.24) $$\bar{m}_n(t) = t\, n^{-1} s_n + O(n^{-1/2} w \exp(\tfrac{5}{4} rt^2)),$$

(2.2.25) $$\bar{\sigma}_n^2(t) = n^{-1} s_n^2 + O(n^{-1/2} w \exp(\tfrac{5}{4} rt^2))$$

Hence from (2.2.11), (2.2.19) and (2.2.24),

(2.2.26) $$A_n(t) = \exp(t^2/2)\,[1 + O(w \exp(\tfrac{5}{4} rt^2))] \exp(-t^2) \times [1 + O(w \exp(\tfrac{3}{2} rt^2))]$$

$$= \exp(-\tfrac{1}{2} t^2)\,[1 + O(w \exp(\tfrac{3}{2} rt^2))]$$

where $w \exp(\frac{3}{2} rt^2) = O(1)$ by choosing $r\ (> 0)$ appropiately small.

Also from (2.2.12), (2.2.24) and (2.2.25) one gets

(2.2.27) $$B_n(t) = O(w \exp(\tfrac{3}{2} rt^2)).$$

Finally from (2.2.10) one gets

(2.2.28) $$|P(s_n^{-1} S_n' \leq t) - \Phi(t)| = |P(s_n^{-1} S_n' > t) - \Phi(-t)|$$

$$= |A_n(t) \int_{B_n(t)}^{\infty} \exp(-t\, s_n^{-1} n^{1/2} \bar{\sigma}_n z) dH_n(z) - \Phi(-t)|$$

$$\leq I_1 + I_2 + I_3$$

where

(2.2.29) $$I_1 = |A_n(t) \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1} n^{1/2} \bar{\sigma}_n z)\, d(H_n(z) - \Phi(z))|$$

(2.2.30) $$I_2 = |A_n(t) - \exp(-t^2/2)| \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1} n^{1/2} \bar{\sigma}_n z) d\Phi(z),$$

(2.2.31) $$I_3 = \left|\exp(-\tfrac{1}{2}t^2) \int_{B_n(t)}^{\infty} \exp(-t\, s_n^{-1}\, n^{1/2}\, \bar{\sigma}_n z)\, d\Phi(z) - \Phi(-t)\right|.$$

Using (2.2.25) - (2.2.27), the Berry-Esseen theorem, the C_r inequality with $r = 3$ and (2.2.15) one gets,

(2.2.32) $$I_1 \leq \exp(-\tfrac{1}{2}t^2)\left[1 + O(w \exp(\tfrac{3}{2}rt^2))\right]$$

$$\times \exp\left[-t(1 + O(w \exp(\tfrac{5}{4}rt^2)))(O(w \exp(\tfrac{3}{2}rt^2)))\right]$$

$$\times \sup_z |H_n(z) - \Phi(z)|$$

$$\leq b_1 \exp(-t^2/2)\left[1+O(w \exp(\tfrac{3}{2}rt^2))\right] \frac{\sum_{i=1}^{n} E_{G_n}|Y_i - m_i(t)|^3}{\left\{\sum_{i=1}^{n} E_{G_n}(Y_i-m_i(t))^2\right\}^{3/2}}$$

$$\leq b_2 \exp(-t^2/2).\ \frac{8\sum_{i=1}^{n} E_{G_n}|Y_i|^3}{n^{3/2}\ \bar{\sigma}_n^3}$$

$$\leq b_3 \exp(-t^2/2).\ n^{-3/2} \sum_{i=1}^{n} E\,|Y_i|^3 \exp(t|Y_i|/s_n)$$

$$\leq b_4 \exp(-t^2/2).\ n^{-3/2}\ \exp(rt^2) \sum_{i=1}^{n} E\,|Y_i|^3$$

$$\leq b_5\, w \exp\left[-\tfrac{t^2}{2}(1-2r)\right].$$

(2.2.33) $$I_2 = |A_n(t)-\exp(-t^2/2)|\ \exp(\tfrac{1}{2}t^2\ s_n^{-2} n\ \bar{\sigma}_n^2) \times \Phi(-B_n(t)\ -t\ s_n^{-1}\ n^{1/2}\ \bar{\sigma}_n)$$

$$\leq \exp(-t^2/2).O(w\ \exp(\tfrac{3}{2}rt^2)).\exp(\tfrac{1}{2}t^2\ s_n^{-2}\ n\ \bar{\sigma}_n^2)$$

$$\times \exp(-\tfrac{1}{2}(B_n(t)\ +t\ s_n^{-1}\ n^{1/2}\ \bar{\sigma}_n)^2)|B_n(t)\ +t\ s_n^{-1}\ n^{1/2}\ \bar{\sigma}_n|^{-1}$$

$$\leq b.w.\ \exp\left(-\frac{t^2}{2}(1-3r)\right)$$

Finally

(2.2.34) $$I_3 = |\exp(-\tfrac{1}{2}\ t^2+\tfrac{1}{2}t^2 s_n^{-2}\ \bar{\sigma}_n^2)\ \Phi(-B_n(t)-t\ s_n^{-1}\ n^{1/2}\ \bar{\sigma}_n)-\Phi(-t)|$$

$$\leq |\exp(-\tfrac{1}{2}t^2\ +\tfrac{1}{2}t^2\ s_n^{-2}\ n\ \bar{\sigma}_n^2)\ \{\Phi(-B_n(t)-t\ s_n^{-1}\ n^{1/2}\ \bar{\sigma}_n)-\Phi(-t)\}|$$

$$+\ |\exp(-\tfrac{1}{2}t^2 + \tfrac{1}{2}t^2\ s_n^{-2}\ n\ \bar{\sigma}_n^2)\ -\ 1|\ \Phi(-t).$$

$$\leq b_1\ \exp\left\{-\tfrac{1}{2}t^2(1-\ s_n^{-2}n\ \bar{\sigma}_n^2)\right\}\left\{|B_n(t)|\ +t\ |1-\ s_n^{-1}\ n^{1/2}\ \bar{\sigma}_n|\right\}$$

$$\times \exp(-t^2/2)\ +\ b_1|\exp\{(-t^2/2)(1-s_n^{-2}\ n\ \bar{\sigma}_n^2)\}\ -1|\ t^{-1}.e^{-t^2/2}$$

$$\leq b.w.\ \exp\left[-\tfrac{1}{2}t^2\ (1-3r)\right]$$ from (2.2.25) and (2.2.27)

The theorem now follows from (2.2.10) and (2.2.32) - (2.2.34).

<u>Proof of theorem</u> 2.2.2. The result is trivially true for $K = 1$ by using the same trucation as of theorem 2.2.1. For $K > 1$, first note that for $t > 0$

$$(2.2.35)\quad \bar{\Phi}(-t) \leq t^{-1}(2\pi)^{-\frac{1}{2}}\exp(-t^2/2) = t^{-1}(2\pi)^{-\frac{1}{2}}\exp\left[-\frac{t^2(K-1)}{2K} - \frac{t^2}{2K}\right]$$

$$\leq t^{-1}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{t^2}{2K}\right)(n^{c/2}\, u(\sqrt{n}))^{-\frac{1}{2}(K-1)}$$

$$\leq b\,(n^{c/2}\, u(\sqrt{n}))^{-\frac{1}{2}(K-1)}\, t^{-2(K+1)}$$

The rest of the proof of the theorem for $t > 0$ follows the lines of Michel (1976) by taking $r = (2K(K+1))^{-1}$ and

$$(2.2.36)\quad h = t^{-1}\, n^{-1/2}\left[\left(\frac{c}{2}\log n + \log u(\sqrt{n})\right)(K-1) + 2K(K+1)\log t\right].$$

For $t < 0$, the proof is similar.

<u>Remark</u> 2.2.1. In the special case of i.i.d rv's when $u(x) = 1$ for all x, $c > 0$, Michels theorem 1 follows as a special case of our theorem 2.2.1 by taking $K = 2(1+c^{-1})$. However, with the same choice of K theorem 2 of Michel does not follow from (2.2.2). However, again in the case of i.i.d. rv's, under the conditions of theorem 2.2.2 one can for $c > 0$ obtain instead of (2.2.2) the bound

$$(2.2.37)\quad |F_n(t)-\bar{\Phi}(t)| \leq b.\, n^{-(Kc+2)/4}\,(u(\sqrt{n}))^{-(Kc+2)/2c}\, t^{-2Kc} + nP(|X_1| > r\, n^{1/2}\,|t|)$$

taking $0 < r < \min [c(Kc-2)^{-1}, (2+c)(Kc)^{-2}]$ and $h = t^{-1}n^{-1/2}$ $\cdot$ $[(Kc-2)(\frac{1}{2}\log n + c^{-1}\log u(\sqrt{n})) + (Kc)^2 \log t]$ for $K >$ Then taking $K = 2(1+c^{-1})$ one obtains a stronger form of Michel' (1976) theorem 2.

From theorems 2.2.1 and 2.2.2 (by a proper choice of $K > 0$) it is easy to derive the following non uniform Berry-Essee theorem which generalises theorem 3 of Michel (1976) and theorem 3 of Nagaev (1965) Included also is a corresponding uniform Berry-Esseen theorem of Katz (1963).

Theorem 2.2.3 There exists a constant b (> 0) dependin only on c and u, such that for all t

(2.2.38) $$|F_n(t) - \Phi(t)| \leq b(1 + |t|^{2+c})^{-1} [n^{c/2} u(\sqrt{n}) \wedge n^{1/}$$

where $x \wedge y = \min(x,y)$.

Remark 2.2.1a. The order of t in (2.2.38) can be impro in general. From theorems 2.2.1 and 2.2.2 it is easy to obtain

$$|F_n(t) - \Phi(t)| \leq b [n^{c/2} u(\sqrt{n}) \wedge n^{1/2}]^{-1} |t|^{-K} + \sum_{i=1}^{n} P(|X_{ni}| > r$$

$$\leq b [n^{c/2} u(\sqrt{n}) \wedge n^{1/2}]^{-1} |t|^{-K}$$

$$+ b_1 n^{-c/2} |t|^{-(2+c)} (u(r\sqrt{n}\, t))^{-1}$$

for any $K > 0$ and $b\ (> 0)$ being a constant depending on K.
Now for $c \geq 1$, we have

$$|F_n(t) - \Phi(t)| \leq b_2\, n^{-1/2}\, (1 + |t|^{(2+c)})^{-1}\, (1 + u(t))^{-1}$$

and for $0 \leq c < 1$

$$|F_n(t) - \Phi(t)| \leq b_2 \left[n^{c/2}\, u(\sqrt{n}) \right]^{-1} (1+|t|^{(2+c)})^{-1} \left(1+ \frac{u(r\sqrt{n}\,t)}{u(\sqrt{n})}\right)^{-1}$$

Which is further improvement of (2.2.38).

Next theorems 2.2.1 and 2.2.2 are used for proving moment type convergences of $Y_n = |s_n^{-1} S_n|$ to those of $T = |N(0,1)|$

Related results of Von Bahr (1965) and Michel (1976) are special cases of the following theorem, where $\lambda_1, \lambda_2, \lambda_3$ are arbitrary positive constants.

<u>Theorem</u> 2.2.4. Suppose that the assumptions of the theorem 2.2.1 are satisfied with $u'(x) < \lambda_1 + \lambda_2 x^{\lambda_3}$ $\forall$ $x \in N^c$, $x > 0$, where N is a countable set and points of N (with $\pm$ sign.) are continuity points of F_n, then

$$(2.2.39) \quad |E(Y_n^{2+c}\, u(Y_n)) - E(T^{2+c}\, u(T))| = O(n^{-c^*})$$

where $c^* = \frac{1}{2} \min(c, 1)$.

Further for $c < 1$ with $\lim_{x \to \infty} u(x) = \infty$ and $\lim_{x \to 0} u(x) = 0$, if $|X_{ni}|^{2+c} u(X_{ni})$ are uniformly integrable (u, i) then the above order can be made $o(n^{-c^*})$.

<u>Proof</u>. Let $h(x) = x^{2+c} u(x)$, $x \geq 0$ then $h'(x) = \lambda_1 + \lambda_2 x^{\lambda_3}$, x

Now since the points of N are continuity points of F_n, contribution from those points to $h(Y_n)$ is zero. Also $EH(Y_n) = \int_{(0,\infty)-N} h'(t) P(|Y_n| > t)\, d\lambda(t)$. Similarly for $Eh(T)$. Therefore

$$(2.2.40) \qquad |Eh(Y_n) - Eh(T)| \leq \int_{(0,\infty)-N} h'(t) |P(|Y_n| \leq t) - P(|T| < t)| d\lambda(t)$$

$$= O(n^{c/2} u(\sqrt{n}) \wedge n^{1/2})^{-1} + \sum_{i=1}^{n} \int_0^{\infty} h'(t) P(|X_{ni}| > rs_n t) dt$$

by using theorems 2.2.1 and 2.2.2 with sufficiently large value of K. Now

$$(2.2.41) \qquad \sum_{i=1}^{n} \int h'(t)\, P(|X_{ni}|/rs_n > t) dt = \sum_{i=1}^{n} Eh(X_{ni}/rs_n)$$

$$= r^{-(2+c)}\, s_n^{-(2+c)} \sum_{i=1}^{n} E|X_{ni}|^{2+c}\, u(X_{ni}/rs_n)$$

(2.2.39) therefore follows from (2.0.2), (2.1.1), (2.2.40) and (2.2.41).

When $\{|X_{ni}|^{2+c} u(X_{ni})\}$ are u.i and $\lim_{x \to 0} u(x) = 0$, then $E|X_{ni}|^{2+c} u(X_{ni}/rs_n) = o(1)$ uniformly in i, $1 \leq i \leq n$, as $n \to \infty$. Also for $c < 1$ 1st term of r.h.s. of (2.2.40) is $o(n^{-c^*})$ since $\lim_{x \to \infty} u(x) = \infty$. Hence the 2nd part of the theorem.

The bound (2.2.39) might not appear very useful when $c = 0$. But even in that case under u.i. assumption and $\lim_{x \to \infty} u(x) = \infty$, $\lim_{x \to 0} u(x) = 0$, the l.h.s of (2.2.40) converges to zero.

Erickson (1973) derived L_p-versions of the Berry-Esseen theorem. Our next theorem also provides a non uniform L_p version, which is stronger than the corresponding uniform version, although the assumptions and final results are different from Erickson's. We write $\|.\|_p$ for the usual L_p-norm with respect to lebesgue measure.

Theorem 2.2.5. Suppose the assumptions of theorem 2.2.1 and 2.2.2 are satisfied. Then for $p \geq 1$

$$(2.2.42) \qquad \|(1+|t|)^{2+c-q/p}(F_n(t)-\Phi(t))\|_p = O(n^{c/2}u(\sqrt{n}) \wedge n^{1/2})^{-1}$$

for any $q > 1$.

Proof. Note that

$$(2.2.43) \qquad \|(1+|t|)^{2+c-q/p}(F_n(t)-\Phi(t))\|_p = \int_{-\infty}^{\infty} (1+|t|)^{p(2+c)-q} \times |F_n(t)-\Phi(t)|^p dt$$

Using theorem 2.2.3 with the observation $\lim_{t \to \infty} (1+t^m)/(1+t)^m = 1$ $m > 0$ and

$$(2.2.44) \qquad \int_{-\infty}^{\infty} (1+|t|)^{-q} dt < \infty$$ for $q > 1$, the desired conclusion follows.

The next two theorems investigate whether the tail probabilities

(2.2.45) $\quad 1 - F_n(t_n) \sim \Phi(-t_n) \quad$ as $\quad t_n \to \infty$.

(By $a(n) \sim b(n)$ we mean $a(n)/b(n) \to 1$ as $n \to \infty$)

We shall see that as a consequence of theorem 2.2.6, one can easily establish probabilities of moderate deviations (see Rubin and Sethuraman (1965), Michel (1974) and Michel (1976)) in the special case $t_n = (c \log n)^{1/2}$.

<u>Theorem</u> 2.2.6. Suppose that the condition of theorem 2.2.' are satisfied. Then for a sequence $\{t_n\}$ $t_n \to \infty$. with

(2.2.46) $\quad t_n^2 - c \log n - 2(c+1) \log |t_n| - 2 \log u(r\, s_n\, t_n) \to$

(2.2.45) holds.

Further, if the sequence $\{ |X_{ni}|^{2+c}\, u(X_{ni}) \}$ is u.i then (2.2.45) holds even if l.h.s of (2.2.46) is bounded above by a positive constant

Proof of the above theorem and the following follows the lines o theorems 4 and 5 of Michel (1976) with an application of theorems 2.2.1 and 2.2.2, observing that, for $t > 0$

$$(2.2.47)\qquad \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n t) = O(n^{c/2}\, u(r\, s_n t)\, t^{2+c})^{-1}$$

$$= o(n^{c/2}\, u(r\, s_n t)\, t^{2+c})^{-1}$$

$$\text{if } \left\{ |X_{ni}|^{2+c} u(X_{ni}) \right\} \text{ is u.i.}$$

The following theorem states that such a strong conclusion may not be possible if

$$(2.2.48)\qquad t_n^2 - c \log n - 2(c+1)\log|t_n| - 2\log u(r\, s_n t_n) \to \infty$$

<u>Theorem</u> 2.2.7 Suppose that the conditions of the theorems 2.2.1 and 2.2.2 are satisfied. Then for a sequence t_n ($\to \infty$) satisfying (2.2.48)

$$(2.2.49)\qquad 1-F_n(t_n) = o(t_n^{-(2+c)}\, n^{-c/2}\, u^{-1}(r\, s_n t_n)) \text{ if}$$

$$\left\{ |X_{ni}|^{2+c}\ u(X_{ni}) \right\} \text{ is u.i.}$$

$$= O(t_n^{-(2+c)}\, n^{-c/2}\, u^{-1}(r\, s_n t_n)), \text{ otherwise}$$

<u>Remark</u> 2.2.2. Suppose

$$(2.2.50)\qquad 0 < \underline{\lim}_{x\to\infty} \frac{u(\lambda x)}{u(x)} \le \overline{\lim}_{x\to\infty} \frac{u(\lambda x)}{u(x)} < \infty \quad \forall\ \lambda > 0$$

then (2.2.46) amd (2.2.48) reduces to

$$t_n^2 - c\log n - 2(c+1)\log|t_n| - 2\log u(\sqrt{n}\, t_n) \to -\infty$$

and

$$t_n^2 - c\log n - 2(c+1)\log|t_n| - 2\log u(\sqrt{n}\, t_n) \to \infty.$$

in view of (2.0.2) . Also the order of approximation in the first line of (2.2.49) reduces to $o(t^{-(2+c)}\, n^{-c/2}\, u^{-1}(\sqrt{n}\, t_n))$ if the u.i assumption holds.

The condition (2.2.50) is satisfied for $u(x) = \log^m(1+|x|)$ $m \geq 0$, $\log\log(e+|x|)$ and in general for slowly varying functions.

As an example consider the case when $u(x) = \log^m(1+|x|)$, $m \geq 0$, c and m not both zeroes. Then (2.2.46) reduces to

(2.2.51) $$t_n^2 - c \log n - (c+2m+1)\log\log n \to -\infty \text{ if } c > 0$$

and

(2.2.52) $$t_n^2 - 2m \log\log n - \log\log\log n \to -\infty \text{ if } c = 0$$

From (2.2.52) with an application of Tomkin's theorem (see Tomkins 1971 or Stout (1974) P-261) it therefore follows that laws of the iterated logarithm for s_n, row sum in triangular array, holds if $m = (1+\varepsilon)$ for some $\varepsilon > 0$.

$$\text{i.e., if } \sup_{n\geq 1} \max_{1\leq i\leq n} E X_{ni}^2 \{\log(1+|X_{ni}|)\}^{1+\varepsilon} < \infty$$

which is incidentally the best known result even for independent case.

Remark 2.2.3. The condition (2.0.2) can be relaxed for the proof of theorem 2.2.6. As for example consider the case $c > 0$,

$u(x) = 1$. It is clear from the proof of Michel (1976) that it suffices to have $w = n^{-\varepsilon}$ for some $\varepsilon > 0$ in the theorem 2.2.1. Thus, from (2.2.16) we need

$$\frac{t^3}{6s_n^3} E|Y_i|^3 e^{t|Y_i|/s_n} \leq b.n^{-1}.n^{-\varepsilon} \exp(\tfrac{5}{4} r t^2)$$

i.e., $s_n^{-3} E|Y_i|^3 \leq b.n^{-(1+\varepsilon)} e^{rt^2/4}$

Using (2.2.15), we need

$$s_n^{-2-c'} \leq b.\ n^{-(1+\varepsilon)}, \quad c' = \min(c,1) \text{ and } \varepsilon > 0 \text{ is arbitrary}$$

i.e., $s_n \geq b.\ n^{(1+\varepsilon)/(2+c')}$

Hence to prove theorem 2.2.6, (2.0.2) may be replaced by

$$s_n \geq b.\ n^{(1+\varepsilon)/(2+c')}.$$

2.3 NON UNIFORM RATES WHEN ALL THE MOMENTS OF X_{ni}'S EXIST

In this section we consider the triangular array X_{ni}, $1 \leq i \leq n$, $n \geq 1$ under the same set up of section 2.1 except that instead of (2.1.1) we assume (2.0.1) to be satisfied with g having sharper growth than any power of $|x|$ but with growth less than or equal to $\exp(s|x|)$, $s > 0$. More specifically we assume (2.0.1) to hold with g satisfying

$$(2.3.1) \qquad K(c)|x|^c + L(c) < g(x) \leq \exp(s|x|) \quad \forall \; c > 0$$

and some $s > 0$ where $K(c)$ and $L(c)$ are constants depending only on c and $x^{-1} \log g(x)$ is nonincreasing for $x > x_0$ (≥ 0).

In other words the cases when all the finite moments exist but the moment generating function of X_{ni}'s do not necessarily exist is the subject of study. Examples of such functions are $g(x) = \exp(\log^m(1+|x|))$, $m > 1$, $g(x) = \exp(|x|^\alpha)$, $0 < \alpha \leq 1$, etc.

As applications of these non-uniform bounds the range of the values of t_n where $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$ is found. This gives a clear picture about the variation of the normal approximation zone depending on the functional form of g. Consequently the earlier results of Cramér (1938) are obtained under milder conditions (see theorem 2.4.5). These non-uniform bounds are further utilised to obtain stronger form of the L_p version of the Berry-Esseen theorem as compared with the earlier section and to prove certain types of moment convergences. Apart from the so called large deviations, it is also shown that 'too large' deviation type results (see Bahadur 1960) can be obtained in limiting sense (theorem 2.4.6).

2.4 THE RESULTS ON ROW SUMS OF RANDOM VARIABLES IN TRIANGULAR ARRAY

The following theorem states the rate of convergence of $F_n(t)$ to $\Phi(t)$ depending on n and t when t is in a neighbourhood of the origin.

Theorem 2.4.1 Let (2.0.1), (2.0.2) and (2.3.1) hold. Then for

(2.4.1) $$1 \leq t^2 \leq 2(\log |t| + \log g(r\, s_n\, t))$$

with $|t| \leq \epsilon_2 \sqrt{n}$, ϵ_2 (> 0) small, there exists constant $b > 0$ depending on r, $0 < r < 1/2$ such that

(2.4.2) $$|F_n(t)-\Phi(t)| \leq b.\exp(-\tfrac{1}{2}t^2)\ |t|^{-1}|\exp(O(n^{-1/2}|t|^3))-1|$$
$$+ b\exp(-\tfrac{1}{2}t^2 + O(n^{-1/2}\,|t|^3)).n^{-1/2}$$
$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n\, |t|).$$

Remark 2.4.1 The second term in the r.h.s. of (2.4.2) can very well be dropped, but it is written in conformity with theorem 2.4.3. For $t^2 \leq 1$ one may use uniform bound $O(n^{-1/2})$ since all the moments exist. This comment holds for theorem 2.4.3 also.

Proof of the theorem. w.o.l.g assume $t > 0$. As in section 2.2 we have the following representation

(2.4.3) $$P(s_n^{-1} S_n' > t) = A_n(t) \int_{B_n(t)}^{\infty} \exp(-t\, s_n^{-1} n^{1/2} \bar{\sigma}_n z) dH_n(z)$$

where the symbols have the same meaning as in section 2.3. Under (2.0.1), (2.0.2) and (2.3.1) one has the following estimates.

(2.4.4) $$|E\, Y_1| = o\,(r\, s_n t\, g(r\, s_n t))^{-1}$$

(2.4.5) $$0 \leq E\, X_{ni}^2 - E\, Y_i^2 = o\,(g(r\, s_n t))^{-1}.$$

Then for any p (≥ 2) fixed, denoting $F_{n,i}$ the distribution function of X_{ni}

(2.4.6) $$\int_{|x|<rs_n t} |x|^p \exp(t|x|/s_n)\, dF_{n,i}(x)$$

$$\leq K(\varepsilon) \int_{|x|<rs_n t} \exp\ (1+\varepsilon)t|x|/s_n \ dF_{n,i}(x)$$

by Holders inequality in view of (2.0.1) and (2.3.1), where $K(\varepsilon)$ is a constant depending on ε (> 0) and ε can be made arbitrarily small.

Now,

$$\exp\ (1+\varepsilon)t\ |x|/s_n \leq x^2 g(x)$$ pointwise within the range

$$e \vee x_0 < |x| < r\, s_n t$$

(to avoid the origin where the inequality is false, we take $x > e \vee x_0$)

if $(1+\varepsilon)t \leq \dfrac{s_n}{|x|} \log (x^2 g(x))$ for $e \vee x_o < |x| < r\, s_n t$

which is satisfied for large n if

$$(2.4.7) \qquad t^2 \leq \frac{1}{r(1+\varepsilon)} \left[\log g(r\, s_n t) + 2\log(r\, s_n t)\right]$$

since $x^{-1} \log g(x)$ is non-increasing in $x > x_o$.

Hence letting $r = (2(1+\varepsilon))^{-1}$ $(< 1/2)$ we have

$$(2.4.8) \qquad \sup_{n \geq 1} \max_{1 \leq i \leq n} \int_{|x|<rs_n t} |x|^p \exp(t\, |x|/s_n) dF_{n,i}(x)$$

$$\leq K_1 + K(\epsilon) \sup_{n \geq 1} \max_{1 \leq i \leq n} \int x^2 g(x) dF_{n,i}(x) \text{ for some } K_1 < \infty$$

$< \infty$

if t satisfies (2.4.1).

In what follows t satisfies (2.4.1) and b is a constant whose values may be different at different equations. Using (2.4.4) - (2.4.8) alongwith (2.0.2)

$$(2.4.9) \qquad \left|f_i(t) - 1 - \frac{t^2}{2s_n^2} EX_{ni}^2 \right| \leq b\, n^{-1} n^{-1/2} t^3$$

Similarly

$$(2.4.10) \qquad \left|EY_i \exp (t\, s_n^{-1} Y_i) - t\, s_n^{-1} EX_{ni}^2 \right| \leq b\, n^{-1} t^2.$$

and

$$(2.4.11) \qquad \left|EY_i^2 \exp (t\, s_n^{-1} Y_i) - EX_{ni}^2 \right| \leq b\, n^{-1/2} t.$$

Hence from (2.2.6), (2.2.7) and (2.4.9) - (2.4.11) one gets

$$(2.4.12) \qquad m_i(t) = f_i^{-1}(t)\, E\,[Y_i \exp(tY_i/s_n)] = ts_n^{-1}\, EX_{ni}^2 + O(n^{-1}t^2$$

$$(2.4.13) \qquad m_i^2(t)+\sigma_i^2(t) = f_i^{-1}(t)\, E\,[Y_i^2 \exp(tY_i/s_n)] = EX_{ni}^2 + O(n^{-1/2}$$

And hence

$$(2.4.14) \qquad \bar{m}_n(t) = \frac{1}{n}\sum_{i=1}^{n} m_i(t) = t\, n^{-1}s_n + O(n^{-1}t^2)$$

$$(2.4.15) \qquad \bar{\sigma}_n^2(t) = n^{-1}\sum_{1}^{n} \sigma_i^2(t) = n^{-1}s_n^2 + O(n^{-1/2}\, t)$$

Also from (2.4.9)

$$(2.4.16) \qquad \sum_{i=1}^{n} \log f_i(t) = t^2/2 + O(n^{-1/2}\, t^3)$$

Hence from (2.2.11), (2.2.12), (2.4.14) and (2.4.16),

$$(2.4.17) \qquad A_n(t) = \prod_{i=1}^{n} f_i(t) \exp(-ts_n^{-1}\, n\, \bar{m}_n(t)) = \exp(-t^2/2 + O(n^{-1/2}\, t^3))$$

$$(2.4.18) \qquad |B_n(t)|\, \bar{\sigma}_n(t) \leq b\, n^{-1/2}\, t^2$$

we are now ready to estimate the following difference

$$(2.4.19) \qquad |P(s_n^{-1} S_n' \leq t) - \Phi(t)| = |P(s_n^{-1} S_n' > t) - \Phi(-t)|$$

$$= |A_n(t) \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1}\, n^{1/2}\, \bar{\sigma}_n(t) z)\, dH_n(z) - \Phi(-t$$

$$\leq I_1 + I_2 + I_3$$

where

$$(2.4.20) \qquad I_1 = |A_n(t) \int_{B_n(t)}^{\infty} \exp(-t\, s_n^{-1} n^{1/2} \bar{\sigma}_n z) d(H_n(z)-\Phi(z))|$$

$$(2.4.21) \qquad I_2 = |A_n(t)-\exp(-t^2/2)| \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1} n^{1/2} \bar{\sigma}_n z) d\Phi(z)$$

$$(2.4.22) \qquad I_3 = |\exp(-t^2/2) \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1} n^{1/2} \bar{\sigma}_n z) d\Phi(z) - \Phi(-t)|$$

Using (2.4.15), (2.4.17), (2.4.18), the Berry-Esseen theorem and C_δ-inequality with $\delta = 3$ (see also (2.2.32))

$$(2.4.23) \qquad I_1 \leq A_n(t) \exp(-ts_n^{-1} n^{1/2} B_n(t)\bar{\sigma}_n) \sup_z |H_n(z)-\Phi(z)|$$

$$\leq b\, n^{-1/2} \exp(-t^2/2 + O(n^{-1/2} t^3))$$

Next

$$(2.4.24) \qquad I_2 = |A_n(t)-\exp(-t^2/2)| \exp(\tfrac{1}{2}t^2 s_n^{-2} n\bar{\sigma}_n^2) \Phi(-B_n(t)-ts_n^{-1} n^{\frac{1}{2}} \bar{\sigma}_n)$$

$$\leq b|A_n(t)-\exp(-t^2/2)| \exp(\tfrac{1}{2}t^2 s_n^{-2} n\bar{\sigma}_n^2)\, e^{-\frac{1}{2}[B_n(t)+ts_n^{-1}n^{\frac{1}{2}}\bar{\sigma}_n]^2}$$

$$\times |B_n(t)+ t\, s_n^{-1} n^{1/2} \bar{\sigma}_n|^{-1}$$

$$\leq b\, e^{-t^2/2} |e^{O(t^3/\sqrt{n})} - 1| \exp[\tfrac{1}{2} B_n^2(t)$$

$$+2|B_n(t)|ts_n^{-1} n^{1/2} \bar{\sigma}_n] \cdot |B_n(t) + ts_n^{-1} n^{1/2} \bar{\sigma}_n|^{-1}$$

From (2.4.18) note that

(2.4.24a) $|B_n(t)| \leq b\, n^{-1}\, t^2 \leq \varepsilon_1\, t$ since $t < \varepsilon_2\, n^{1/2}$

and $\varepsilon_1 = b \cdot \varepsilon_2$ can be made arbitrarily small with small choice of ε_2. Hence

(2.4.25) $$|B_n(t) + t\, s_n^{-1}\, n^{1/2}\, \bar{\sigma}_n|^{-1} = O(t^{-1})$$

Also from (2.4.18)

(2.4.26) $$\tfrac{1}{2} B_n^2(t) + 2|B_n(t)|\, t\, s_n^{-1}\, n^{1/2}\, \bar{\sigma}_n$$

$$\leq \tfrac{1}{2} b^2 n^{-1}\, t^4 + b\, n^{-1/2}\, t^3\, s_n^{-1}\, n^{1/2}\, \bar{\sigma}_n$$

Hence from (2.4.24) - (2.4.26)

(2.4.27) $$I_2 \leq b.t^{-1} \exp(-t^2/2)\, |\exp(O(n^{-1/2}\, t^3)) - 1|$$

Finally following the lines of (2.2.34)

(2.4.28) $$I_3 = |\exp(-\tfrac{1}{2}t^2 + \tfrac{1}{2}t^2\, s_n^{-2}\, n\, \bar{\sigma}_n^2)\Phi(-B_n(t) - t\, s_n^{-1}\, n^{1/2}\, \bar{\sigma}_n) - \Phi(-$$

$$\leq b \exp\left\{-\tfrac{1}{2}t^2(1 - s_n^{-2}\, n\, \bar{\sigma}_n^2)\right\}\left(|B_n(t)| + t|1 - s_n^{-1}\, n^{1/2}\, \bar{\sigma}_n|\right.$$

$$.\exp(-t^2/2) + b|\exp(-\tfrac{1}{2}t^2(1 - s_n^{-2} n\, \bar{\sigma}_n^2)) - 1|t^{-1}.\, e^{-t^2/2}$$

From (2.4.15) note that, in view of (2.0.2)

(2.4.29) $$s_n^{-2}\, n\, \bar{\sigma}_n^2 = 1 + b\, n^{-1/2}\, t$$

Hence from (2.4.24a), (2.4.28) and (2.4.29) we have

$$(2.4.30) \quad I_3 \leq b.\exp(O(n^{-1/2} t^3)) \exp(-t^2/2) \; n^{-1}t^2$$

$$+ b|\exp(O(n^{-1/2} t^3)) - 1| \; t^{-1} \exp(-t^2/2)$$

$$\leq b. \; t^{-1} \exp(-t^2/2) \; |\exp(O(n^{-1/2} t^3)) - 1|$$

The theorem now follows form (2.4.19), (2.4.23), (2.4.27) and (2.4.30).

<u>Remark</u> 2.4.2. In some special cases of g, ε in (2.4.7) can be taken to be zero i.e., $r = 1/2$ in those cases. e.g. if $g(x) = \exp(|x|)$ then for $t > 0$

$$|x|^p \exp(t|x|/s_n) < x^2 g(x), \quad 0 \leq x_o < x \leq r\, s_n t$$

if $t|x|/s_n < |x| - (p-2) \log |x|$

i.e., if

$$t < s_n \left(1 - \frac{p-2}{|x|} \log |x|\right)$$

which is always true when

$$t \leq s_n \left(1 - \frac{p-2}{|x_o|} \log |x_o|\right)$$

It may further be noted that for general g the region (2.4.1) may be extended to $t^2 \leq M(\log|t| + \log g(r\, s_n t))$, $M > 0$ arbitrary by sufficiently small choice of r in (2.4.7)

Remark 2.4.3. The value ε_2 in theorem 2.4.1 is immeterial when $g(x) = o(\exp(s|x|)$ $\forall$ $s > 0$, for in that case (2.4.1) asserts $t = o(n^{1/2})$. But when $g(x) = \exp(s|x|)$ for some $s > 0$, (2.4.1) tells us $t \leq s\, s_n$; hence the value of ε_2 matters in that case. Proof of the above theorem leads us to conclude that ε_2 is basically determined through the counstant b in the relation $\varepsilon_1 = b\, \varepsilon_2$ of (2.4.24a) and b can be taken to be (see (2.4.8)) $(c/6) \sup_n \max_i E\, |X_{ni}|^3 \exp(s|x_{ni}|)$ * if $t < s' s_n$, $s' < s$ where $c = \sup_{n \geq 1} (\sqrt{n}/s_n)^3$

*(which is finite with an application of Holders inequality as all the moments of X_{ni} exist and from (2.0.1))

The ultimate value of ε_2 , constraining the value of ε_1 in (2.4.24a) so that (2.4.25) holds turns out to be

$$[c \sup_n \max_i E\, |X_{ni}|^3 \exp(s' |X_{ni}|)]^{-1} .$$

When $g(x) = \exp(s|x|)$. Thus theorem 2.4.1 is valid for

$$t < [\, c \sup_n \max_i E(|X_{ni}|^3 \exp(s'|X_{ni}|)^{-1} \wedge s' c^{-1/3}]\, n^{1/2}$$

$s' < s$ when $g(x) = \exp(s|x|)$.

Similarly the order of 1st and 2nd terms of the r.h.s. of (2.4.2) i.e., $\exp(O(n^{-1/2} |t|^3)) = \exp(K\, n^{-1/2} |t|^3)$ with

$$K = c \sup_n \max_i E|X_{ni}|^3 \exp(s'|X_{ni}|)$$

For theorem 2.4.3 similarly we have, $K = c \sup_n \max_i E\, X_{ni}^4 \exp(s' . |X_{ni}|)$, $c = \sup_{n \geq 1} (\sqrt{n}/s_n)^4$.

Noting that moment generating function (m.g.f) of a r.v X exists around a neighbourhood of the origin implies that $E(\exp(s|X|)) < \infty$ for some $s > 0$, a few observations immediate from theorem 2.4.1 are listed below

Theorem 2.4.2 If the m.g.f of X_{ni}, $n \geq 1$, $1 \leq i \leq n$ exist and uniformly bounded around a fixed nbhd of the origin then

(2.4.31) $$1 - F_n(t_n) \sim \Phi(-t_n) \quad \text{if} \quad t_n = o(n^{1/6}).$$

Remark 2.4.4. When X_{ni}'s have identical distribution the above reduces to a theorem of Cramér (1938). Subsequently we will show that even in the case of triangular array the condition of theorem 2.4.2 can be substancially relaxed to obtain the same conclusion (see theorem 2.4.5)

Proof of theorem 2.4.2. In view of the well known result $\Phi(-x) \sim (2\pi)^{-1/2} x^{-1} \exp(-x^2/2)$, $x \to \infty$, it suffices to show that

$$(2.4.32) \qquad t_n \exp(t_n^2/2)\,(1 - F_n(t_n) - \Phi(-t_n)) = o(1).$$

which follows from theorem 2.4.1 as $n^{-1/2}\, t_n^3 = o(1)$ and

$$(2.4.33) \qquad \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n t) \leq bt^{-2}(g(rs_n t))^{-1} \cdot \sup_n \max_i EX_{ni}^2\, g(X_{ni}) I(|X_{ni}| > rs_n t)$$

$$= O(t^{-2} \exp(-r\, s_n t)$$

$$= o(t^{-1} \exp(-t^2/2)) \quad \text{as } t = o(n^{1/6})$$

Remark 2.4.5 From the proof of the theorem 2.4.1 it follows that the truncation of the random variables is not necessary when the m.g.f exist. Hence the calculations (2.4.33) may be omitted in this case.

Remark 2.4.6 The normal approximation zone can be extended to $o(n^{1/4})$ when

$$(2.4.34) \qquad EX_{ni}^3 = 0 \quad \forall\ n \geq 1 \quad 1 \leq i \leq n.$$

Then we have the following

Theorem 2.4.3 Under the assumptions of theorem 2.4.1 and (2.4.34), for $|\leq t^2 \leq 2(\log|t| + \log g(r\, s_n t))$ with $|t| \leq \varepsilon_2 \sqrt{n}$ ε_2 (> 0) small, there exists constant b depending or r, $0 < r < 1/2$ such that

$$(2.4.35)\quad |F_n(t) - \Phi(t)| \leq b \exp(-t^2/2)\ |t|^{-1}\ |\exp(O(n^{-1}t^4)) - 1|$$
$$+ b \exp(-t^2/2 + O(n^{-1}t^4))\ n^{-1/2}$$
$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n |t|).$$

Remark 2.4.7 The 2nd term in the r.h.s of (2.4.35) ensures that the overall order of n, $(-\infty < t < \infty)$, cannot be less than $n^{-1/2}$.

Proof of the theorem 2.4.3 The proof essentially follows the same lines as that of theorem 2.4.1. We indicate the necessary modifications.

$$(2.4.26)\quad |f_i(t) - 1 - \frac{t^2}{2s_n^2} EX_{ni}^2| \leq \frac{t}{s_n} |EY_i| + \frac{t^2}{2s_n^2} E(X_{ni}^2 - Y_i^2)$$
$$+ \frac{t^3}{3!\ s_n^3} |EY_i^3| + \frac{t^4}{4!\ s_n^4} EY_i^4 \exp(t|Y_i|/s_n)$$

Now from (2.4.34)

$$E\, Y_i^3 = -E\, X_{ni}^3\, I\,(|X_{ni}| > r\, s_n t)$$

So.

$$(2.4.37)\quad |E\, Y_i^3| \leq \int_{|X_{ni}|>rs_nt} |X_{ni}|^{3+a} (r\, s_n t)^{-a}\, dF(X_{ni})$$
$$= O\,(t^{-a}\, n^{-a/2})$$ since all the moments of X_{ni} exist.

Hence taking $a = 1$

(2.4.38) $$|f_i(t) - 1 - \frac{t^2}{2s_n^2} E X_{ni}^2 | \leq b\, n^{-1} n^{-1} t^4$$

Similarly

(2.4.39) $$|E\, Y_i \exp(t\, s_n^{-1} Y_i) - t\, s_n^{-1} E\, X_{ni}^2| \leq b\, n^{-3/2} t^3$$

(2.4.40) $$|E\, Y_i^2 \exp(t\, s_n^{-1} Y_i) - E\, X_{ni}^2 | \leq b\, n^{-1} t^2$$

Using the above estimates

(2.4.41) $$m_i(t) = t\, s_n^{-1} E\, X_{ni}^2 + 0(n^{-3/2} t^3)$$

(2.4.42) $$m_i^2(t) + \sigma_i^2(t) = E\, X_{ni}^2 + 0(n^{-1} t^2)$$

And therefore

(2.4.43) $$\bar{m}_n(t) = t\, n^{-1} s_n + 0(n^{-3/2} t^3)$$

(2.4.44) $$\bar{\sigma}_n^2 = n^{-1} s_n^2 + 0(n^{-1} t^2)$$

(2.4.45) $$\sum_{i=1}^{n} \log f_i(t) = t^2/2 + 0(t^4/n).$$

Next

(2.4.46) $$A_n(t) = \exp\left(-t^2/2 + 0(t^4/n)\right)$$

(2.4.47) $$|B_n(t)|\, \bar{\sigma}_n = 0(n^{-1} t^3)$$

And finally

(2.4.48) $$I_1 \leq b \exp(-t^2/2 + 0(n^{-1}t^4))\, n^{-1/2}$$

$$(2.4.49) \qquad I_2 \leq b \exp(-t^2/2)\,|\exp(O(n^{-1}t^4)) - 1|\, t^{-1}$$

$$(2.4.50) \qquad I_3 \leq b \exp(-t^2/2)\,|\exp(O(n^{-1}t^4)) - 1|\, t^{-1}$$

This proves the theorem.

Using theorem 2.4.3 and following the lines of proof of theorem 2.4.2 when the m.g.f of X_{ni}'s are uniformly bounded around a fixed nbhd of the origin one proves remark 2.4.6.

As a consequence of theorems 2.4.1 and 2.4.3 we may obtain normal approximation zones for general g which will be helpful to obtain normal approximation results known so far, under weaker assumptions.

<u>Theorem</u> 2.4.4 Under the assumptions (2.0.1), (2.0.2) and (2.3.1) for a sequence $\{t_n\}$ satisfying

(i) $t_n = o(n^{1/6})$

and (ii) $t_n^2 - 2(\log t_n + \log g(r\, s_n t_n)) \to -\infty, \quad 0 < r < 1/2$

$$1 - F_n(t_n) \sim \Phi(-t_n) \quad \text{as} \quad t_n \to \infty.$$

Further if (2.4.34) is satisfied, (i) may be replaced by

$$t_n = o(n^{1/4}).$$

(r may be taken to be 1/2 in some cases according to remark 2.4.2)

<u>Proof</u>. The proof is immediate from theorems 2.4.1 and 2.4.3 along the lines of theorem 2.4.2 with the following observation

$$(2.4.51)\quad \sum_{i=1}^{n} P(|X_{ni}|> rs_n t) \leq b\, t^{-2}(g(rs_n t))^{-1}$$

$$\times \sup_{n\geq 1} \max_{1\leq i\leq n} E\, X_{ni}^2\, g(X_{ni}) I(|X_{ni}|> rs_n t)$$

$$= 0\,(t^{-2}\,(g(r\,s_n t))^{-1}\,)$$

$$= o(t^{-1}\,e^{-t^2/2})\quad \text{as } t = t_n \text{ satisfies (ii).}$$

<u>Remark</u> 2.4.8 If the sequence $\{X_{ni}^2\, g(X_{ni})\}$ is unifromly integrable then the conclusion of theorem 2.4.4 holds even if l.h.s of (ii) is bounded above, since

$$(2.4.52)\quad \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n t) = o(t^{-2}(g(r\,s_n t))^{-1})$$

in that case.

Let us calculate the normal approximation zone when $g(x) = \exp\,(s|x|^{\alpha}\,)$, $s > 0$, $0 < \alpha \leq 1$. Letting $t = t_n \to \infty$ from (ii) we have

$$t^2 \leq 2s(r\,s_n t)^{\alpha}\quad ,\quad t > 0.$$

i.e., $$t^{2-\alpha} \leq 2s\, r^{\alpha}\, n^{\alpha/2}\, \lambda^{\alpha}\quad \text{where } \lambda^2 = \inf\,(s_n^2/n) > 0$$

i.e., $$t \leq (2s\, r^{\alpha}\, \lambda^{\alpha})^{\frac{1}{2-\alpha}}\, n^{\alpha\,/\,[2(2-\alpha)]}$$

Note that $\frac{\alpha}{2(2-\alpha)} = \frac{1}{6}, \frac{1}{4}$ if $\alpha = \frac{1}{2}, \frac{2}{3}$ respectively.

Therefore, in view of theorem 2.4.4. We have the following.

Theorem 2.4.5 The conclusion of theorem 2.4.2 remains valid under the relaxed condition

(2.4.53) $$\sup_{n\geq 1} \max_{1\leq i\leq n} E\left\{X_{ni}^2 \exp(s|X_{ni}|^{1/2})\right\} < \infty \text{ for some } s>0$$

Similarly under (2.4.34) the conclusion of remark 2.4.6 holds even if

(2.4.54) $$\sup_{n\geq 1} \max_{1\leq i\leq n} E\left\{X_{ni}^2 \exp(s|X_{ni}|^{2/3})\right\} < \infty \text{ for some } s>0$$

Remark 2.4.9 Since g has growth more than any power bound it is immaterial whether we consider $x^2 g(x)$ or $g(x)$. We prefered to consider $x^2 g(x)$ rather than $g(x)$ because of following two reasons. Firstly, it is known that the conclusion on the rates of convergence cannot be achieved unless we assume a bit more than the existance of the 2nd moment (see e.g. Katz (1963)). Therefore we wanted to base our conclusion on rates solely on the excess of x^2 viz. $g(x)$. Also the estimate (2.4.4), (2.4.5) etc. take nice form if we consider $x^2 g(x)$.

Next note that excessive deviations of the type $a\sqrt{n}$, $a > 0$ can also be handled as per the following theorem.

<u>Theorem</u> 2.4.6 Under the assumptions of theorem 2.4.2 for $t_n = \varepsilon\sqrt{n}$, $\varepsilon > 0$ small

$$1 - F_n(t_n) \leq b\, t_n^{-1} \exp\left\{-\frac{t_n^2}{2}(1 + K\varepsilon)\right\} \tag{2.4.55}$$

under additional assumption (2.4.34)

$$1 - F_n(t_n) \leq b\, t_n^{-1} \exp\left\{-\frac{t_n^2}{2}(1 + K\varepsilon^2)\right\} \tag{2.4.56}$$

Further in special case of iid random variables, one has

$$\exp\left\{-\frac{t_n^2}{2}(1 + O(\varepsilon))\right\} \leq 1-F_n(t_n) \leq bt_n^{-1}\exp\left\{-\frac{t_n^2}{2}(1+K\varepsilon\right. \tag{2.4.57}$$

and under (2.4.34)

$$\exp\left\{-\frac{t_n^2}{2}(1+O(\varepsilon^2))\right\} \leq 1-F_n(t_n) \leq bt_n^{-1}\exp\left\{-\frac{t_n^2}{2}(1+K\varepsilon^2)\right\} \tag{2.4.58}$$

where K is defined in remark 2.4.3.

<u>Proof</u>. Since the m.g.f of X_{ni} exist, the third term of r.h.s. of (2.4.2) does not appear (see ramark 2.4.5). Hence from theorem 2.4.1

$$1 - F_n(t_n) \leq \Phi(-t_n) + bt_n^{-1}\exp\left\{-\frac{t_n^2}{2} + K\, n^{-1/2}\, t_n^3\right\} \tag{2.4.59}$$

$$= b\, t_n^{-1} \exp\left\{-\frac{t_n^2}{2}(1 + K\varepsilon)\right\}$$

Similarly (2.4.56) follows from theorem 2.4.3.

For lower class inequality in the special case of iid random variables, w.o.l.g assuming $EX_1^2 = 1$, we have from Chernoff's theorem (see Bahadur (1971)).

(2.4.60) $$\lim_{n\to\infty} \frac{2}{\varepsilon^2 n} \log P(\overline{X}_n > \varepsilon) = \frac{2}{\varepsilon^2} \log \inf_{t\geq 0} E\, e^{t(X_1-\varepsilon)}$$

Now

(2.4.61) $$E\, e^{t(X_1-\varepsilon)} \geq E\left\{1 + t(X_1-\varepsilon) + \frac{t^2}{2}(X_1-\varepsilon)^2 + \frac{t^3}{6}(X_1-\varepsilon)^3\right\}$$

$$= 1 - \varepsilon t + \frac{t^2}{2}(1+\varepsilon^2) + \frac{t^3}{6}(\mu_3 - 3\varepsilon - \varepsilon^3),$$

where $\mu_3 = EX_1^3$. Differenciating the r.h.s of (2.4.61) w.r.t t and equating it to zero we have

(2.4.62) $$-\varepsilon + t(1+\varepsilon^2) + \frac{t^2}{2}(\mu_3 - 3\varepsilon - \varepsilon^3) = 0$$

The above quadratic equation admits two solutions which tend to 0 and $-\frac{2}{\mu_3}$ as $\varepsilon \to 0$ (seen by putting $\varepsilon = 0$ in (2.4.62)) of which we may ignore the second one as the 2nd derivative of r.h.s of (2.4.61).

(2.4.63) $$1 + \varepsilon^2 + t(\mu_3 - 3\varepsilon - \varepsilon^3) \to -1 \text{ at } t = -\frac{2}{\mu_3} \text{ as } \varepsilon \to 0$$

indicating that supremum (and not infimum) is attained at the second solution . Therefore we will assume $t \to 0$ as $\varepsilon \to 0$

Now let $v = \frac{t-\varepsilon}{\varepsilon}$ i.e., $t = \varepsilon + v\varepsilon$.

Now from (2.4.62)

(2.4.64) $$(t - \varepsilon) + t\varepsilon^2 + \frac{t^2}{2}(\mu_3 - 3\varepsilon - \varepsilon^3) = 0$$

Dividing both sides by t and noting $(t-\varepsilon)/t = v(1+v)$

(2.4.65) $$\frac{v}{1+v} + \varepsilon^2 + \frac{t}{2}(\mu_3 - 3\varepsilon - \varepsilon^3) = 0$$

using the fact that $t \to 0$ as $\varepsilon \to 0$ we obtain $v \to 0$ as $\varepsilon \to 0$. That is

(2.4.66) $$t = \varepsilon + o(\varepsilon)$$

Finally the following rearrangement of (2.4.62)

(2.4.67) $$t = \left\{ \varepsilon - \frac{t^2}{2}(\mu_3 - 3\varepsilon - \varepsilon^3) \right\} / (1+\varepsilon^2)$$

alongwith (2.4.66) implies that

(2.4.68) $$t = \varepsilon + \bar{O}(\varepsilon^2) \text{ or } \varepsilon + O(\varepsilon^3) \text{ according as}$$

$$\mu_3 \neq 0 \text{ and } \mu_3 = 0.$$

one may check that the 2nd derivative of r.h.s of (2.4.61) is positive at these values of t for sufficiently small values of ε, ensuring that infimum is attained at these values.

Putting these approximate solutions to the r.h.s of (2.4.61) we obtain from (2.4.60)

(2.4.69) $$\lim_{n \to \infty} \frac{2}{\varepsilon^2 n} \log P(\bar{X}_n > \varepsilon) \geq \frac{2}{\varepsilon^2} \log \left\{ 1 - \frac{\varepsilon^2}{2} - O(\varepsilon^3) \right\} \quad \text{if } \mu_3 \neq 0$$

$$\geq \frac{2}{\varepsilon^2} \log \left\{ 1 - \frac{\varepsilon^2}{2} - O(\varepsilon^4) \right\} \quad \text{if } \mu_3 = 0$$

Hence

$$\lim_{n\to\infty} \frac{2}{\varepsilon^2 n} \log P(\bar{X}_n > \varepsilon) = -1 + O(\varepsilon) \quad \text{or} \quad -1 + O(\varepsilon^2)$$

according as $\mu_3 \neq 0$
or $\mu_3 = 0$.

Hence the results.

<u>Remark</u> 2.4.10. From Bernstein inequality one may obtain

$$1 - F_n(t_n) = P(s_n^{-1} S_n > \varepsilon\sqrt{n}) \leq \exp\left\{-\frac{t_n^2}{2}(1 + f(\varepsilon))\right\}$$

where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. But the rate at which $f(\varepsilon) \to 0$ remains unspecified. Similarly Bahadur (1960) proves for iid case

$$1 - F_n(t_n) = \exp\left\{-\frac{t_n^2}{2}(1 + f(n,\varepsilon))\right\}$$

where $\lim_{\varepsilon\to 0} \lim_{n\to\infty} f(n, \varepsilon) = 0$

Theorem 2.4.6 is therefore an improvement of their results providing us an order of $f(n, \varepsilon)$. Also since the constant K can be estimated, the contribution from t_n^{-1} in (2.4.55) and (2.4.56) is not insignificant.

The following theorem states the order of $1 - F_n(t_n)$ for $t_n = o(n^{1/2})$

Theorem 2.4.7 Under the assumption of theorem 2.4.2 for $t_n = o(n^{1/2}) = \varepsilon_n \sqrt{n}$, where $\varepsilon_n \to 0$ as $n \to \infty$

(2.4.70) $$1 - F_n(t_n) \leq b\, t_n^{-1} \exp\left\{-\frac{t_n^2}{2}(1 + K\varepsilon_n)\right\},$$

under the additional assumption (2.4.34)

(2.4.71) $$1 - F_n(t_n) \leq b\, t_n^{-1} \exp\left\{-\frac{t_n^2}{2}(1 + K\varepsilon_n^2)\right\},$$

when the X_{ni} 's are iid rv's one may further have

(2.4.72) $$1 - F_n(t_n) \geq b \exp\left\{-\frac{t_n^2}{2}(1 + o(1))\right\}.$$

Proof. As proofs of (2.4.70) and (2.4.71) are similar to those of (2.4.55) and (2.4.56), we only prove (2.4.72).

Note that $P(\overline{X}_n > \varepsilon_n) \geq P(\overline{X}_n > \varepsilon)$ where for large n $\varepsilon_n = n^{-1/2} t_n = o(1) < \varepsilon$ (> 0) fixed.

i.e., $$\frac{2}{\varepsilon_n^2 n} \log P(\overline{X}_n > \varepsilon_n) \geq \frac{2}{\varepsilon^2 n} \log P(\overline{X}_n > \varepsilon)$$

$$\liminf_{n \to \infty} \frac{2}{\varepsilon_n^2 n} \log P(\overline{X}_n > \varepsilon_n) \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{2}{\varepsilon^2 n} \log P(\overline{X}_n > \varepsilon) =$$

Hence $$1 - F_n(t_n) \geq b \exp\left\{-\frac{t_n^2}{2}(1 + o(1))\right\}.$$

The following theorem provides the non-uniform rates of convergence in the complementary zone of theorem 2.4.1.

Theorem 2.4.8 For $t^2 \geq 2(\log|t| + \log g(r\, s_n t))$ with $x^{-1} \log g(x) \to 0$ as $x \to \infty$ we have

(2.4.73) $$|F_n(t)-\Phi(t)| = O(|t| g(rs_n t))^{-1+\varepsilon_{n,t}} + \sum_{i=1}^{n} P(|X_{ni}|>rs_n|t|)$$

where

(2.4.74) $$\varepsilon_{n,t} = O(|t|^{-1} s_n^{-1} \log(|t| g(r\, t\, s_n)) \to 0 \text{ as } n \to \infty$$

Proof. w.o.l.g assume $t > 0$. Since $\Phi(-t) \leq bt^{-1}\exp(-t^2/2)$ and $1 - F_n(t) \leq P(s_n^{-1} S_n' > t) + \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n t)$

it is enough to show that

(2.4.75) $$P(s_n^{-1} S_n' > t) = O(|t|\, g(r\, s_n t))^{-1+\varepsilon_{n,t}}$$

Recall that $S_n' = \sum_{i=1}^{n} Y_i$ where $Y_i = X_{ni}\, I(|X_{ni}| \leq r\, s_n t)$.

Now by the Bernstein inequality

(2.4.76) $$P(s_n^{-1} S_n' > t) \leq \prod_{i=1}^{n} \beta_i \exp(-h\, s_n t) \text{ where } \beta_i = E(\exp(h\, Y_i))$$

Let $h = 2\, t^{-1} s_n^{-1} \log(t\, g(r\, s_n t))$, then $h \to 0$ as $n \to \infty$

Also

(2.4.77) $\beta_i = E(\exp(h\, Y_i))$

$$\leq 1 + h|EY_i| + \frac{h^2}{2} EX_{ni}^2 + \frac{h^3}{6} E(|Y_i|^3 \exp(h|Y_i|))$$

$$(\because EY_i^2 \leq E X_{ni}^2)$$

$$= 1 + \frac{h^2}{2}(EX_{ni}^2 + Kh) + o(n^{-1}) \text{ using } (2.4.4),$$

and $0 < K < \infty$ as shown below,

(2.4.78) $E|Y_i|^3 \exp(h|Y_i|) \leq E(\exp((1+\varepsilon_1)h|Y_i|))\ K(\varepsilon_1)$ where ε_1 can be made arbitrarily small (see 2.4.6) .

Now,

(2.4.79) $\exp((1+\varepsilon)\, hx) \leq x^2 g(x)$ pointwise in the range

$$e \vee x_o < |x| < r\, s_n t$$

which is satisfied if

$$(1+\varepsilon)h \leq x^{-1}\log(x^2 g(x)) \left(\geq (r\, s_n t)^{-1}\log\left[(r\, s_n t)^2 g(r\, s_n t)\right]\right.$$

since $x^{-1}\log g(x) \searrow$ in x, we need

$$2(1+\varepsilon)r \log(t\, g(rs_n t)) \leq \log\left\{(rs_n t)^2 g(rs_n t)\right\}$$

which is true for $n \geq n_o$ and $r < 1/2$.

Thus, from (2.0.1), (2.4.78) and (2.4.79), it follows that $0 < K < \infty$. (see also (2.4.8)). Hence,

$$(2.4.80) \quad \sum_{i=1}^{n} \log \beta_i \le \sum_{i=1}^{n} \frac{h^2}{2}(E\, X_{ni}^2 + K\, h) + b$$

$$= \frac{h^2}{2}(s_n^2 + n\, K\, h) + b$$

$$= 2t^{-2}\{\log(t\, g(rs_n t))\}^2 (1 + O(h)) + b$$

$$\le \log(t\, g(rs_n t))\,(1 + O(h)) + b$$

$$\text{since} \quad t^2 \ge 2\{\log|t| + \log g(r\, s_n t)\}$$

Hence

$$(2.4.81) \quad \prod_{i=1}^{n} \beta_i \le b\,\{t\, g(r\, s_n\, t)\}^{1+O(h)}$$

so

$$(2.4.82) \quad P(s_n^{-1} S_n > t) = O(t\, g(rs_n t))^{-1+\varepsilon_{n,t}} \quad \text{from (2.4.76)}$$

and hence the theorem.

<u>Remark</u> 2.4.11 In the case $x^{-1}\log g(x) \to s$ (> 0) as $x \to \infty$ e.g. when $g(x) = \exp(s|x|)$, for $t = o(n^{1/2})$ we may use theorem 2.4.1 or theorem 2.4.3 and for

$$t^2 \ge 2\{\log|t| + \log g(r\, s_n t\, K_n)\} \quad (\text{i.e. } |t| \ge s s_n K_n = o(n^{1/2}))$$

we have

$$(2.4.83) \quad |F_n(t) - \Phi(t)| = O(|t|\, g(rs_n t\, K_n))^{-1+O(K_n)} + \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|)$$

where K_n is any sequence such that $K_n \to 0$ as $n \to \infty$

Proof of (2.4.83) follows the same lines as that of theorem 2.4.8 with the following choice of h

(2.4.84) $h = 2\, t^{-1} s_n^{-1} \log(t\, g(r\, s_n t\, K_n))$, ($\to 0$ as $n \to \infty$).

As a consequence of theorems 2.2.1, 2.4.1, 2.4.8 (remark 2.4.10) we may obtain following nonuniform bound over the entire range of t , $-\infty < t < \infty$.

Theorem 2.4.9 Let (2.0.1), (2.0.2) and (2.3.1) hold. Also let for some λ_1, λ_2, λ_3, ε positive constants

(2.4.85) $[g(rs_n t)]^{-1+\varepsilon} \leq \lambda_1 n^{-1/2} [g(\lambda_2 t)]^{-1}$

for all sufficiently large n, when $x^{-1}\log g(x) \to 0$ as $x \to \infty$ with t satisfying $t^2 \geq 2(\log|t| + \log g(rs_n t))$

(2.4.86) $[g(rs_n tK_n)]^{-1+\varepsilon} \leq \lambda_3\, n^{-1/2} [g(\lambda_2 t)]^{-1}$

for all sufficiently large n, when $x^{-1}\log g(x) \to s$ (> 0) as $x \to \infty$ with t satisfying $t^2 \geq 2(\log|t| + \log g(rs_n tK_n))$ where K_n is some sequence converging to zero.

Then

(2.4.87) $$|F_n(t) - \Phi(t)| \leq b\, n^{-1/2} [g(\lambda_2 t)]^{-1} + \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|).$$

Further if

(2.4.88) $\quad [t^2 g(rs_n t)]^{-1} \leq b\, n^{-1/2} [g(\lambda_2 t)]^{-1}$ for $t > t_o (\geq 0)$

then

(2.4.89) $\quad |F_n(t) - \Phi(t)| \leq b\, n^{-1/2} [\, g(\lambda_2 t)]^{-1}$

<u>Proof of the theorem</u> To tackle the first and the second terms in the r.h.s. of (2.4.2), theorem 2.4.1, note that with $0 < p < 1$ ($p < 1$ letting $t = o(n^{1/2})$ therein)

$$\exp(-\tfrac{p}{2}t^2) = n^{-1/2} \exp(-\tfrac{p}{2}t^2 + \tfrac{1}{2}\log n)$$

$$\leq n^{-1/2} \exp(-at^2) \quad , \quad 0 < a < p/2$$

if $-at^2 \geq -pt^2/2 + \frac{1}{2}\log n$ i.e., if $t^2 \geq (p-2a)^{-1} \log n$.
Since $\exp(-at^2) \leq b\,[g(\lambda_2 t)]^{-1}$ from (2.3.1) , 1st part of the above theorem i.e., (2.4.87) in case $t^2 \geq (p-2a)^{-1}\log n$ follows from theorems 2.4.1 and 2.4.8 (remark 2.4.10 with $K_n = o(\log n)^{-1}$ say) alongwith the assumption (2.4.85) ((2.4.86)).

For $t^2 \leq (p-2a)^{-1}\log n$, (2.4.87) follows from theorem 2.2.1 choosing c therein sufficiently large ($\because$ all the moments of X_{ni} exist). (See also Michel (1976) theorem 1)
Alternatively note that $|\exp(O(|t|^3/\sqrt{n})) - 1| = O(n^{-1/2}|t|^3)$

for $t^2 \leq (p-2a)^{-1}\log n$.

Hence $|F_n(t)-\Phi(t)| \leq b\, n^{-1/2} e^{-rt^2}$ for some $r > 0$, $t_o^2 \leq t^2 \leq K \log n$, $K > 0$ arbitrary. For $t^2 < t_o^2$ the assertion follows from the uniform bound $n^{-1/2}$ for $|F_n(t)-\Phi(t)|$. Finally (2.4.89) follows from (2.4.88) as

$$\sum_{i=1}^{n} P(|X_{ni}| > rs_n t) \leq n(rs_n t)^{-2} [g(rs_n t)]^{-1} \times \sup_n \max_i E[X_{ni}^2 g(X_{ni})]$$

And hence the theorem.

<u>Remark</u> 2.4.12 Assumptions (2.4.85), (2.4.86) and (2.4.88) intutively follows from the fact that g has growth more than any power bound. All these conditions are satisfied for $g(x) = |x| \exp(\log^m(1 + |x|))$, $m > 1$; $g(x) = |x| \exp(|x|^\gamma)$, $0 < \gamma \leq 1$ etc. where $\lambda_2 > 0$ can be made arbitrarily large and $\varepsilon > 0$ arbitrarily small

From the theorem 2.4.9 it is possible to obtain the following non uniform L_p version of the Berry-Esseen theorem.

<u>Theorem</u> 2.4.10 Under the assumptions of the theorem 2.4.9

(2.4.90) $$\|g(\lambda_2 t)(1 + |t|)^{-q/p} (F_n(t)-\Phi(t)\|_p = O(n^{-1/2})$$

for any $p \geq 1$ and $q > 1$.

Proof: Proof of the above follows from theorem 2.4.9 along with the fact that

$$\int_{-\infty}^{\infty} (1 + |t|)^{-q} \, dt < \infty .$$

Theorem 2.4.9 may further be utilised to find the rate of convergence of expectations of some function based on $Y_n = |s_n^{-1} S_n|$ to that of $T = |N(0, 1)|$.

Theorem 2.4.11: Under the assumptions of theorem 2.4.9 and

(2.4.91) $$\frac{d}{dx}[x^2 g(x)] \leq \lambda_1 g(\lambda_2 x)(1+x)^{-q} + \lambda_3 \;\forall\, x \in N^c, \; x \geq 0$$

for some $\lambda_1, \lambda_3, 0, q > 1, \lambda_2$ same as that of theorem 2.4.9 and where N is a countable set and points of N (with $\pm$ sign) are continuity points of F_n, one has

(2.4.92) $$\left| E(Y_n^2 \, g(Y_n)) - E(T^2 \, g(T)) \right| = O(n^{-1/2})$$

Proof: Proof of the above theorem follows from (2.4.89) along the lines of theorem 2.2.4.

2.5 NON UNIFORM RATES WHEN m.g.f NECESSARILY EXIST BUT THE RANDOM VARIABLES ARE NOT NECESSARILY BOUNDED

In earlier sections we covered the spectrum of g extending upto moment generating function. Since it is known that $s_n^{-1} S_n \xrightarrow{\mathcal{D}} N(0, 1) = T$, it is natural to ask the question, given that $E\, g(T) < \infty$ when does $|E\, g(s_n^{-1} S_n) - Eg(T)| \to 0$ as $n \to \infty$. For example

$$Eg(T) < \infty \text{ if } g(x) = O((1+|x|)^{-1-\delta} \exp(x^2/2)) \text{ for some } \delta > 0$$

In the following section we partially cover that spectrum of g which has higher growth than $\exp(s|x|)$ for all fixed s but lies below $(1+|x|)^{-1-\delta} \exp(x^2/2)$.

Unlike preceeding section, the conditions on the triangular array will be a bit stringent. Here we shall assume

$$(2.5.1) \qquad E\, X_{ni} = 0 \quad \forall\ n \geq 1, \quad 1 \leq i \leq n$$

and

$$(2.5.2) \qquad E\, X_{ni}^2 = s_n^2/n \ (>0), \quad 1 \leq i \leq n, \quad n \geq 1.$$

In other words (2.5.2) means that variance of the random variables X_{ni} is uniform for all i, in a row. This is satisfied for example in the iid case

With the assumption that all the odd order moments are vanishing i.e.

(2.5.3) $EX_{ni}^{2m+1} = 0 \quad \forall \; n \geq 1 \quad 1 \leq i \leq n, \quad m = 1,2,3 \ldots$

We shall show that a sharper result is possible. As one may note, our results cover the situation of symmetric iid r v's.

Since it is known that the normal approximation zone cannot, in general, be extended further compared to those obtained in section 2.4 , even if random variables are bounded (see e.g. Feller 1966) we shall be content with theorems lime 2.4.1 and 2.4.3 around a nbhd of the origin. Also we obtain a theorem analogous to 2.4.8. However, because of the stringency of our assumptions sharper estimates for the differences of tail probabilities are available.

As applications, these non uniform bounds will be used to obtain moment type convergences as indicated above and to obtain stronger non-uniform L_p version of the Berry-Esseen theorem.

2.6 THE RESULTS ON THE ROW SUMS OF RANDOM VARIABLES IN A TRIANGULAR ARRAY

We start with the following theorem.

Theorem 2.6.1 Let $\{X_{ni} : 1 \leq i \leq n, \ n \geq 1\}$ be a triangular array of random variables where variables within an array are independent and satisfy (2.5.1) - (2.5.3) and

(2.6.1) $$\sup_{n \geq 1} \max_{1 \leq i \leq n} \left(\frac{n}{s_n^2}\right)^m E X_{ni}^{2m} \leq \ell^{-m} \frac{(2m)!}{m!}, \quad 1 < \ell \leq$$

$m = 1, 2, 3 \ldots$

then there exists a constant $b(>0)$ such that

(2.6.2) $$|F_n(t) - \Phi(t)| \leq b \exp(-t^2(1-\ell^{-1})), \quad -\infty < t < \infty$$

Proof. Since $1 - \Phi(t) \leq b\,|t|^{-1} \exp(-t^2/2)$ sufficient to show that

(2.6.3) $$P(s_n^{-1} S_n > t) \leq \exp\{-t^2(1-\ell^{-1})\}$$

Now

(2.6.4) $$P(s_n^{-1} S_n > t) \leq \prod_{i=1}^{n} \beta_i \exp(-h\, s_n t) \quad \text{where}$$

(2.6.5) $$\beta_i = E\,[\exp(h X_{ni})], \quad i = 1, 2, \ldots, n.$$

Let $h = t/s_n$ then

(2.6.6) $P(s_n^{-1} S_n > t) \leq \left(\prod_{i=1}^{n} \beta_i \right) \exp(-t^2)$

Now in view of (2.5.3)

(2.6.7) $\beta_i = E[\exp(hX_{ni})] \leq \exp(h^2 s_n^2 / \ell n)$

since

(2.6.8) $$1 + \frac{h^2}{2!} E X_{ni}^2 + \frac{h^4}{4!} E X_{ni}^4 + \ldots + \frac{h^{2m}}{(2m)!} E X_{ni}^{2m} + \ldots$$

$$\leq 1 + \frac{h^2 s_n^2}{\ell n} + \frac{1}{2!} \left[\frac{h^2 s_n^2}{\ell n} \right]^2 + \ldots + \frac{1}{m!} \left[\frac{h^2 s_n^2}{\ell n} \right]^m + \ldots$$

and from (2.6.1)

(2.6.9) $E X_{ni}^{2m} \leq \frac{(2m)!}{m!} \ell^{-m} (s_n^2/n)^m$.

Note that for $m = 1$, $(s_n^2/n =) EX_{ni}^2 \leq 2\ell^{-1} s_n^2/n$ restricting $\ell \leq 2$ in (2.6.1).

Hence from (2.6.6) and (2.6.7)

(2.6.10) $P(s_n^{-1} S_n > t) \leq \exp(-h^2 s_n^2 (1 - \ell^{-1}))$

$$= \exp(-t^2 (1 - \ell^{-1}))$$

Hence the theorem.

<u>Remark</u> 2.6.1 If odd order moments are non zero still we may have

(2.6.11) $\frac{1}{2}[E e^{hX_{ni}} + E e^{-hX_{ni}}] =$ l.h.s of (2.6.8) $\leq \exp(h^2 s_n^2/\ell n)$

from (2.6.7) under assumption (2.6.1)

Since $E\, e^{-hX_{ni}} \geq 0$ we have from (2.6.11)

(2.6.12) $$E\, e^{hX_{ni}} \leq 2 \exp (h^2 s_n^2/\ell n)$$

Therefore from (2.6.6)

(2.6.13) $$P(s_n^{-1} S_n > t) \leq 2^n \exp\{-t^2(1 - \ell^{-1})\}$$

Hence we have the following

<u>Theorem</u> 2.6.2. If the assumption (2.5.3) is omitted in theorem 2.6.1 then

(2.6.14) $$|F_n(t) - \Phi(t)| \leq b.\, 2^n \exp\{-t^2(1 - \ell^{-1})\}$$

We continue to assume that odd order moments are zero, our next theorem states moment type convergences of $Y_n = |s_n^{-1} S_n|$ to that of $T = |N(0, 1)|$

<u>Theorem</u> 2.6.3. Let the assumptions of the theorem 2.6.1 alongwith (2.0.2) be satisfied. Let $g : (-\infty, \infty) \to (0, \infty)$ $g(x)$ even, $g(0) = 0$ be such that $E\, g(T) < \infty$ and

(2.6.15) $$g'(x) = 0 \left\{ (1+x)^{-1-\delta} \exp (x^2/(1-\ell^{-1})) \right\} , x>0$$

Then

(2.6.16) $$|E\, g(Y_n) - E\, g(T)| = 0(n^{-p^*}) \quad \text{where}$$

$$p^* = \left\{ \frac{\delta}{3+(\delta \vee 1)} \wedge \frac{1}{2} \right\}$$

Proof. Under (2.6.1) since m.g.f of X_{ni} exist (as $\beta_i < \infty$ for fixed h from 2.6.7) we have, in view of (2.0.2) and (2.5.3) with m = 1, by an application of theorem 2.4.3 with the last term in the r.h.s of (2.4.35) delated (see remark 2.4.5)

(2.6.17) $$|F_n(t)-\Phi(t)| \leq b \exp(-t^2/2)\, r_n , \quad 0 < |t| < M_n$$

where $r_n = (n^{-1} M_n^3) \vee (n^{-1/2})$, $M_n = 0(n^{1/4})$, as, for $t = 0(n^{1/4})$, $|t|^{-1} \, |\exp(0(n^{-1} t^4)) - 1| = 0(n^{-1}|t|^3)$

Again from theorem 2.6.1

(2.6.18) $$|F_n(t)-\Phi(t)| \leq b \exp(-t^2(1-\ell^{-1})), \quad M_n \leq |t| < \infty$$

Hence with the representation

(2.6.19) $$|Eg(Y_n) - Eg(T)| \leq \int_0^\infty g'(t)\,|P(|s_n^{-1}S_n| \leq t) - P(|N(0,1)| \leq t)|\,dt$$

and that

$$\int_0^\infty (1+x)^{-1-\delta}\, dx < \infty , \qquad \int_{M_n}^\infty (1+x)^{-1-\delta}\, dx = 0(M_n^{-\delta})$$

we have

(2.6.20) $$|Eg(Y_n) - Eg(T)| = 0(r_n) + 0(M_n^{-\delta})$$

Equating the order of $M_n^{-\delta}$ and r_n

$$\therefore\ n^{-1} M_n^3 = M_n^{-\delta} \quad \text{i.e.,} \quad M_n = n^{1/(3+\delta)}$$

$$\Rightarrow \quad M_n = n^{(\frac{1}{3+\delta} \wedge \frac{1}{2\delta})}$$

$$n^{-1/2} = M_n^{-\delta} \quad \text{i.e.,} \quad M_n = n^{1/2\delta}$$

in the case $\delta \geq 1$ (see 2.6.17), giving an order

$$(2.6.21) \qquad M_n^{-\delta} = n^{-[\frac{\delta}{3+\delta} \wedge \frac{1}{2}]} \qquad \text{for} \quad (2.6.20)$$

For $0 < \delta \leq 1$ letting $M_n = n^{1/4}$ we have the order

$$(2.6.22) \qquad M_n^{-\delta} = n^{-\delta/4} \quad \text{for} \quad (2.6.20), \quad 0 < \delta \leq 1$$

(2.6.21) , (2.6.22) completes the proof.

The following theorem provides a non uniform L_p version of the Berry-Esseen theorem

<u>Theorem</u> 2.6.4 Under the assumption (2.0.2) and those of theorem 2.6.1 for any $p \geq 1$

$$(2.6.23) \qquad \|\exp(t^2(1-\ell^{-1}))(1+|t|)^{-(\delta+1/p)} (F_n(t)-\Phi(t))\|_p = O(n^{-p^*})$$

where $\delta > 0$ and p^* is defined in theorem 2.6.3.

<u>Proof</u>. Note that

$$\text{L.H.S of (2.6.23)} = \left[\int_{-\infty}^{\infty} (1+|t|)^{-(1+\delta p)} \left\{ \exp(t^2(1-\ell^{-1})) \times (F_n(t)-\Phi(t)) \right\}^p dt \right]^{1/p}$$

The rest follows along the lines of theorem 2.6.3.

Next we consider moment convergence and L_p version of the Berry-Esseen theorem when the assumption (2.5.3) is not satisfied i.e., odd order moments are non-zero.

Theorem 2.6.5 Let the assumptions (2.0.2), (2.5.1), (2.5.2) and (2.6.1) be satisfied with some $\ell > 1$. Then for any $g : (-\infty, \infty) \to (0, \infty)$, $g(x)$ even, $g(0) = 0$, such that $E\, g(T) < \infty$ and

(2.6.24) $\quad g'(x) = O(\exp(x^2/f(x))), \quad 0 < x < \infty$ where

$f : (0, \infty) \to (0, \infty)$, nondecreasing, with $\lim_{x \to \infty} f(x) = \infty$

the following holds.

(2.6.25) $\quad |E\, g(Y_n) - E\, g(T)| = O(n^{-1/2}).$

Proof. Since m.g.f of X_{ni} exist under 2.6.1, delating the last term of r.h.s of (2.4.2) theorem 2.4.1 (for an explanation see remark 2.4.5) and following the 1st part of the proof of theorem 2.4.9 we have

$$(2.6.26) \quad |F_n(t)-\Phi(t)| \leq b\, n^{-1/2} \exp(-at^2) \quad \text{for}$$

$$t^2 \geq (p-2a)^{-1} \log n$$

$$0 < a < p/2, \quad p < 1, \text{and} \quad t = o(n^{1/2})$$

Similarly for $t^2 \leq (p-2a)^{-1} \log n$ we have from theorem 2.2.1 choosing c therein sufficiently large

$$(2.6.27) \quad |F_n(t)-\Phi(t)| \leq b\, n^{-1/2} \exp(-a_1 t^2) \quad \text{for some} \quad a_1 > 0$$

Finally from theorem 2.6.2

$$(2.6.28) \quad |F_n(t) - \Phi(t)| \leq b\, 2^n \exp\{-t^2(1-\ell^{-1})\}$$

$$\leq b\, n^{-1/2} \exp(-a_2 t^2), \quad a_2 > 0$$

if $\quad n^{1/2}\, 2^n \exp\{-t^2(1-\ell^{-1})\} \leq \exp(-a_2 t^2)$

if, $\quad t^2\{1 - \ell^{-1} - a_2\} \geq n \log 2 + \frac{1}{2} \log n\,; \quad 1-\ell^{-1}-a_2 > 0$

if $\quad t^2 \geq \lambda^2 n \quad$ for some λ depending on ℓ and a_2.

Finally for the zone $o(n^{1/2}) < t < \lambda n^{1/2}$ we may use remark 2.4.11 with $g(x) = \exp(|x|)$. Incidentally remark 2.4.5 applies there also, and therefore from (2.4.83) and (2.0.2)

(2.6.29) $\quad |F_n(t)-\Phi(t)| \leq b \exp(-K_n n^{1/2} t)$ where K_n is some sequence converging to zero.

Now we need show that

(2.6.30) $$\exp(-K_n n^{1/2} t) \leq n^{-1/2} t^{-2} \exp(-t^2/f(t)),$$

$$K_n n^{1/2} < t < \lambda n^{1/2}.$$

i.e., $\quad -K_n n^{1/2} t + \frac{1}{2} \log n + 2 \log t \leq -t^2/f(t).$

Which is true if

$$f(t) \geq [K_n n^{1/2} t^{-1} - \tfrac{1}{2} t^{-2} \log n - 2t^2 \log t]^{-1},$$

$$K_n n^{1/2} < t < \lambda n^{1/2}$$

since f is nondecreasing the above is true if

$$f(K_n n^{1/2}) \geq [K_n/\lambda - \frac{1}{2\lambda^2} n^{-1} \log n - 2\lambda^{-2} n^{-1} \log(\lambda n^{1/2})]^{-1}$$

Since the above choice of f depends on K_n and n (which can be interpreted as inverse fn of K_n assming the map $n \to K_n$ is one to one) and since the choice of $K_n \to 0$ is arbitrary it follows that we may take $f(t)$, nondecreasing with $\lim_{t\to\infty} f(t) = \infty$ arbitrarily.

Hence for $K_n n^{1/2} < t < \lambda n^{1/2}$ we have from (2.6.29) and (2.6.30)

(2.6.32) $$|F_n(t) - \Phi(t)| \leq b\, n^{-1/2}\, t^{-2} \exp(-t^2/f(t))$$

Finally combining (2.6.26) - (2.6.28) and (2.6.32) we have for $-\infty < t < \infty$, alongwith uniform bound $O(n^{-1/2})$ for $|F_n(t)-\Phi(t)|$

(2.6.33) $$|F_n(t) - \Phi(t)| \leq b\, n^{-1/2} (1+t)^{-2} \exp(-t^2/f(t))$$

The required assertion now follows from the representation (2.6.19) and (2.6.24).

From (2.6.33) it is further possible to obtain a non-uniform L_p-version of the Berry-Esseen theorem.

Theorem 2.6.6 Under assumptions (2.0.2), (2.5.1), (2.5.2) and (2.6.1) with some $\ell > 1$, the following holds

(2.6.34) $$\| \exp(t^2/f(t)) \; (F_n(t) - \Phi(t)) \|_p = O(n^{-1/2})$$

where $\lim_{t \to \infty} f(t) = \infty$.

Proof Proof of the above follows from (2.6.33) and that

$$\int_{-\infty}^{\infty} (1 + t^2)^{-1} dt < \infty .$$

We remark that it suffices to consider the values of t_n for which $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$; as the values of t_n for which $F_n(t_n) \sim \Phi(t_n)$, $t_n \to -\infty$ can be dealt similarly.

CHAPTER 3.

RATES OF CONVERGENCE FOR GENERAL NON-LINEAR STATISTICS (T_n)

3.1 Introduction In this chapter we consider non-linear statistics of the form

(3.1.1) $$T_n = s_n^{-1} S_n + R_n \quad \text{where} \quad S_n = \sum_{i=1}^{n} X_{ni}$$

$X_{n1}, X_{n2}, \ldots X_{nn}$ being independent random variables satisfying (2.0.1) and (2.0.2) . Representation of the above type is fairly general and are obtainable, for example via. Hájek's projection lemma (see Hájek (1968)). Under suitable assumptions on the moments of R_n , we shall show that the results of Chapter 2 may be extended to include T_n.

Results on T_n analogous to those obtained in sections (2.2) - (2.6) are proved in sections (3.2) - (3.4). In sections (3.5), (3.6) we cite examples of T_n and verify the assumptions on R_n. Finally section (3.7) exploits the fact that non-uniform rates in finite population may be obtainable via a representation like (3.1.1).

3.2 NON-UNIFORM RATES FOR T_n WHEN SOME FINITE MOMENTS OF X_{ni} EXIST

In this section results for T_n analogous to those obtained in section 2.2 are proved. Assume (2.0.2) and (2.1.1). Further suppose that R_n satisfies

(3.2.1) $\quad E(R_n^{2m}) = 0(n^{-m}(\log n)^h)$ for some $h \geq 0$ (h may be a function of m but not of n) m being a positive integer, $c < 2m \leq c+2$.

Let

(3.2.1a) $\quad a_n(t) = |t|^d (n^{c/2} u(\sqrt{n}))^{-\eta}$ where $d > 1$ and $\eta > 0$

to be chosen later. Then for $t^2 \leq K[\frac{1}{2} c\log n + \log u(\sqrt{n})]$, theorem 2.2.1 yields

(3.2.2)
$$|P(s_n^{-1} S_n \leq t \pm a_n(t)) - \Phi(t \pm a_n(t))|$$
$$\leq b \exp[-\frac{1}{2}(t \pm a_n(t))^2(1-3r)][n^{c/2}u(\sqrt{n}) \wedge \sqrt{n}]^{-1}$$
$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n |t \pm a_n(t)|)$$

Note $a_n(t) \to 0$, as $n \to \infty$ for $t^2 \leq K[\frac{1}{2}c \log n + \log u(\sqrt{n})]$, so $|t \pm a_n(t)|^2 = t^2(1+o(1))$.

without any loss of generality assume $t > 0$, as $t < 0$ can be handled similarly.

Then

(3.2.3) $$|\Phi(t \pm a_n(t)) - \Phi(t)| \leq a_n(t)\,(2\pi)^{-1/2} \times \exp\left[-\tfrac{1}{2}(t - a_n(t))^2\right] \leq b\, t^d\,(n^{c/2}\, u(\sqrt{n}\,))^{-\eta}\, \exp(-t^2/2\,).$$

Again using Markov's inequality and (3.2.1) one has

(3.2.4) $$P(|R_n| > a_n(t)) \leq a_n^{-2m}(t)\, E\, R_n^{2m} = O(t^{-2md}(n^{c/2}\, u(\sqrt{n}\,))^{2m\eta}\, n^{-m}\, (\log n)^h\,)$$

If $c > 0$, choose η such that

(3.2.4a) $$n^{-c\eta/2} = n^{-mc\eta}\, n^{-m}$$

i.e., $\eta = \frac{2m}{(2m+1)c}$. For $c = 0$, $\eta\ (> 0)$ can be chosen arbitrarily.

From (3.2.2) -(3.2.4), if $c > 0$, one gets for $t^2 \leq K[\,\frac{c}{2} \log n + \log u(\sqrt{n}\,)]$,

(3.2.5) $$|P(T_n \leq t) - \Phi(t)| \leq b[\,\exp(-\tfrac{1}{2}t^2(1-3r))(n^{c/2}u(\sqrt{n}\,)) \wedge n^{1/2})^{-1} + |t|^d\, n^{\frac{-m}{2m+1}}\,(u(\sqrt{n}\,))^{-\frac{2m}{(2m+1)c}}\, \exp(-t^2/2)$$

$$+ \quad |t|^{-2md} \; n^{\frac{-m}{2m+1}} \; (u(\sqrt{n}\,))^{\frac{4m^2}{(2m+1)c}} \; (\log n)^h \;]$$

$$+ \sum_{i=1}^{n} P(\,|X_{ni}| > r\, s_n |t - a_n(t)|\,).$$

Now for $t > 0$,

$t - a_n(t) = t(1 - t^{d-1}(n^{c/2}\, u(\sqrt{n}\,))^{-\eta}\,)$. This equals zero if either $t = 0$ or $t = (n^{c/2}\, u(\sqrt{n}\,))^{\eta/(d-1)}$. Since the last value of t lies outside the region $t^2 \leq K[\frac{c}{2} \log n + \log u(\sqrt{n})]$ and since

$$\inf_n \quad \inf_{0<t^2<K[\frac{c}{2}\log n + \log u(\sqrt{n}\,)]} |1-t^{-1}a_n(t)| = \lambda = \lambda(K,c,u,d)>0$$

we have

(3.2.6)
$$|P(T_n \leq t) - \Phi(t)| \leq [\exp(-\tfrac{1}{2}t^2(1-3r))(n^{c/2}u(\sqrt{n}) \wedge n^{\frac{1}{2}})^{-1}$$

$$+ \; |t|^d \; n^{\frac{-m}{2m+1}} \; (u(\sqrt{n}\,))^{\frac{-2m}{(2m+1)c}} \; \exp(-t^2/2)$$

$$+ \; |t|^{-2md} \; n^{\frac{-m}{2m+1}} \; (u(\sqrt{n}\,))^{\frac{4m^2}{(m+1)c}} \; (\log n)^h]\; b$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\,\lambda\, s_n |t|\,)$$

For $c = 0$ similarly we have

(3.2.7)
$$|P(T_n \leq t) - \Phi(t)| \leq b\,[\exp(-\tfrac{t^2}{2}(1-3r))(u(\sqrt{n}\,))^{-1}$$

$$+ |t|^{d} (u(\sqrt{n}))^{-\eta} \exp(-t^2/2) + |t|^{2md}(u(\sqrt{n}))^{2m\eta}$$

$$.n^{-m}(\log n)^{h}] + \sum_{i=1}^{n} P(|X_{ni}| > r \lambda s_n |t|)\}$$

for some $\lambda > 0$, where $\eta > 0$ is arbitrary.

We are now in a position to state

<u>Theorem</u> 3.2.1 Let $\{X_{ni}\}$ be a sequence of r.v's satisfying (2.0.2) and (2.1.1). Let $T_n = s_n^{-1} S_n + R_n$ where $S_n = \sum_{i=1}^{n} X_{ni}$; $s_n^2 = \sum_{i=1}^{n} E X_{ni}^2$, $(n \geq 1)$ and R_n satisfies (3.2.1). Then for $t_n^2 \leq K[\frac{c}{2} \log n + \log u(\sqrt{n})]$, (3.2.6) holds if $c > 0$ and (3.2.7) holds if $c = 0$.

Yet another form of the inequality is obtainable by a different choice of $\eta > 0$. For $c > 0$ let $\eta > 0$ be such that $mc\eta - m < -c/2$ i.e., $\eta < (m - \frac{1}{2}c)/(mc)$; $c < 2m \leq c+2$. Then from (3.2.2) - (3.2.4) it follows that for some $\gamma > 0$,

(3.2.8) $$|P(T_n \leq t) - \Phi(t)| \leq b\, n^{-\gamma} [\exp(-t^2/2) + n^{-c/2}] + \sum_{i=1}^{n} P(|X_{ni}| > r\lambda s_n |t|)$$

Note that putting $|t| = (c \log n)^{1/2}$ one obtains probabilities of moderate deviations; (3.2.7) and (3.2.8) will further be utilised to have a theorem analogous to theorem 2.2.6.

Next we derive an error bound for $t^2 > K[\frac{c}{2}\log n + \log u(\sqrt{n})]$. First obtain from theorem 2.2.2 with $t > 0$

(3.2.9) $$|F_n(t \pm a_n(t)) - \Phi(t \pm a_n(t))|$$

$$\leq b[n^{c/2}u(\sqrt{n})]^{-(K-1)/2}|t - a_n(t)|^{-2(K+1)}$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r s_n|t - a_n(t)|)$$

Note that

(3.2.10) $$|\Phi(t \pm a_n(t)) - \Phi(t)| \leq a_n(t)\,\phi(t - a_n(t))\,, \quad \phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Hence for $t \neq a_n(t)$,

(3.2.11) $$|P(T_n \leq t) - \Phi(t)| \leq b[n^{c/2}u(\sqrt{n})]^{-(K-1)/2}$$

$$\times |t - a_n(t)|^{-2(K+1)}$$

$$+ [n^{c/2}u(\sqrt{n})]^{-\eta}\, t^d \exp[-\tfrac{1}{2}(t-a_n(t))^2]$$

$$+ \sum_{i=1}^{n} P(|X_{ni}|>r s_n|t-a_n(t)|) + P(|R_n(t)| > a_n(t))$$

Note that the solutions of $t - a_n(t) = 0$ are $t = 0$ and $t = [n^{c/2}u(\sqrt{n})]^{-\eta/(d-1)}$. So $|t - a_n(t)| \geq \lambda t$ if $|1 - t^{-1}a_n(t)| > \lambda$ where λ $(0 < \lambda < 1)$ may depend on K, and is at our choice.

Hence, for $t \notin [(1 \pm \lambda)n^{c/2} u(\sqrt{n})]^{\eta/(d-1)}$, one has

(3.2.12) $$|P(T_n \le t) - \Phi(t)| \le b\,[n^{c/2} u(\sqrt{n})]^{-(K-1)/2} |t|^{-2(K+1)}$$

$$+ (n^{c/2} u(\sqrt{n}))^{-\eta} |t|^d \exp[-\tfrac{1}{2}\lambda^2 t^2]$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r \lambda s_n |t|)$$

$$+ b_2(n^{-c/2-\varepsilon'} |t|^{-2(K+1)}) \text{ for some } \varepsilon' > 0.$$

To justify (3.2.12) note that

$$P(|R_n| > a_n(t)) \le (a_n(t))^{-2m} E R_n^{2m}$$

$$= O(n^{-m}(\log n)^h (n^{c/2} u(\sqrt{n}))^{2m\eta} |t|^{-2md})$$

$$= O(n^{-c/2-\varepsilon'} |t|^{-2(K+1)})$$

for some $\varepsilon' > 0$, if $m(1-c\eta) > \frac{c}{2} + \varepsilon'$ i.e., $1-c\eta > (c+2\varepsilon')/2m$

i.e., $\eta < (2m-c-2\varepsilon')/2m$

Since $c < 2m \le c+2$ this choice of η (> 0) is possible for some $\varepsilon' \varepsilon (0, (2m-c)/2)$; d is chosen in such a way that

$$2md > 2(K+1) \quad \text{i.e.,} \quad d > (K+1)/m$$

Again for $t \in [(1 \pm \lambda) n^{c/2} u(\sqrt{n})]^{\eta/(d-1)}$, choose

$a_n^*(t) = |t|^{d^*} (n^{c/2}u(\sqrt{n}))^{-\eta}$, $(1 < d^* < d)$, and get a similar inequality as (3.2.12) with d^* replacing d for $t \notin [(1\pm\lambda)\, n^{c/2} u(\sqrt{n}))]^{\eta/(d^*-1)}$.

Since the two intervals

$$((1 \pm \lambda)n^{c/2}u(\sqrt{n}))^{\eta/(d-1)} \quad \text{and} \quad ((1 \pm \lambda)n^{c/2}\, u(\sqrt{n}))^{\eta/(d^*-1)}$$

can be made disjoint, one gets (3.2.12) for all $t^2 > K[\frac{c}{2}\log n + \log u(\sqrt{n})]$. Now

(3.2.13) $$\exp[-\tfrac{1}{2}\lambda^2t^2] = \exp[-\tfrac{1}{2}\lambda^2(1-a)t^2] \,.\exp[-\tfrac{1}{2}\lambda^2at^2]$$
$$\leq [n^{c/2}u(\sqrt{n})]^{-K\lambda^2a/2} \exp[-\tfrac{1}{2}\lambda^2(1-a)t^2]$$

choose 'a' such that $(\eta + K\lambda^2 a/2) = (K-1)/2$ i.e., $a = \frac{K-1}{K\lambda^2} - 2\eta$

For adequate choice of λ and η one has $0 < a < 1$.

Hence the 2nd term of the r.h.s of (3.2.12) is less than or equal to

$$[n^{c/2}\, u(\sqrt{n})]^{-(K-1)/2}\, |t|^d \exp[-\tfrac{1}{2}\lambda^2(1-a)t^2]$$

we now state the following

<u>Theorem</u> 3.2.2 Let the conditions of theorem 3.2.1 hold, then for $t^2 > K[\frac{c}{2}\log n + \log u(\sqrt{n})]$, we have, for some $\varepsilon' > 0$,

$$(3.2.14) \qquad |P(T_n \leq t) - \Phi(t)| \leq b\,[n^{c/2}u(\sqrt{n})]^{-(K-1)/2}|t|^{-2K+1}$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\lambda s_n |t|)$$

$$+ b_2(n^{-c/2} \quad {}^{-\varepsilon'} \quad |t|^{-2(K+1)})$$

In view of theorems 3.2.1 and 3.2.2 with K sufficiently large we have for $c > 0$, $t \in (-\infty, \infty)$,

$$(3.2.15) \qquad |P(T_n \leq t) - \Phi(t)| \leq b\,[(n^{c/2}u(\sqrt{n}))^{-1}$$

$$+ n^{\frac{-m}{2m+1}}(u(\sqrt{n}))^{c\frac{4m^2}{(2m+1)}}(\log n)^h]\,(1+|t|^{2+c})^{-1}$$

where $2m$ is the largest even integer $\leq (c+2)$, and for $c = 0$

$$(3.2.16) \qquad |P(T_n \leq t) - \Phi(t)| \leq b\,[u(\sqrt{n})]^{-1}(1+t^2)^{-1}, \quad t \text{ real}$$

<u>Remark</u> 3.2.1 The non uniform bound (3.2.15) can be improved when from uniform Berry-Esseen bound one knows that

$$(3.2.17) \qquad \sup_t |P(T_n \leq t) - \Phi(t)| = O((n^{c/2}u(\sqrt{n}) \wedge n^{1/2})^{-1})$$

Then without using (3.2.6) one may directly use (3.2.14) alongwith (3.2.17) to obtain

$$(3.2.18) \qquad |P(T_n \leq t) - \Phi(t)| \leq b(n^{c/2}u(\sqrt{n}) \wedge n^{1/2})^{-1}(\log n)^{g(c)}$$

$$\times (1+|t|^{2+c})^{-1}$$

In the light of (3.2.6), (3.2.7), (3.2.8) and (3.2.14) it is also possible to obtain theorems analogous to 2.2.4 to 2.2.7 Without going into the detailed proof we now state the results as follows.

<u>Theorem</u> 3.2.3 Suppose that the assumptions concerning $u(x)$ of theorem 2.2.4 hold. Let $Y_n = |T_n|$ and $T = |N(0, 1)|$. Then

(3.2.19)
$$|E(Y_n^{2+c}u(Y_n)) - E(T^{2+c}u(T))| = \begin{cases} O\left(n^{-c/2} \vee n^{\frac{-m}{(2m+1)}} (u(\sqrt{n}))^{\frac{4m^2}{(2m+1)c}} (\log n)^h\right) & \text{if } c > 0 \\ o\left(n^{-c/2} \vee n^{\frac{-m}{(2m+1)}} (u(\sqrt{n}))^{\frac{4m^2}{(2m+1)c}} (\log n)^h\right) & \text{if } c>0 \text{ and if } |X_{ni}|^{2+c} u(X_{ni}) \text{ is u.i} \\ o(1) & \text{if } c = 0 \text{ and if } |X_{ni}|^{2+c}u(X_{ni}) \text{ is u.i}, \lim_{x\to 0} u(x)=0 \text{ and } \lim_{x\to\infty} u(x) = \infty \end{cases}$$

Note that if a uniform Berry-Esseen bound (3.2.17) is known by using (3.2.18) instead of (3.2.6) sometimes it is possible to obtain sharper orders in (3.2.19) for $c > 0$. This comment holds for the next theorem also.

Denote $G_n(t) = P(T_n \le t)$.

Theorem 3.2.4 Under the assumptions of theorem 3.2.1

(3.2.20) $$\|(1+|t|)^{2+c-q/p}(G_n(t)-\Phi(t))\|_p$$

$$= O(n^{-c/2} \vee n^{\frac{-m}{(2m+1)}} (\log n)^h) \quad \text{if } c > 0$$

$$= O(u(\sqrt{n}))^{-1} \quad \text{if } c = 0$$

for $p \geq 1$ and any $q > 1$.

Remark 3.2.2 If $c > 0$, we take $u(x) = 1$ in (2.1.1) to obtain (3.2.10).

Using (3.2.8) in the case $c > 0$ and (3.2.7) with $\eta = 1$ for $c = 0$ we have the following theorems.

Theorem 3.2.5. Under the assumptions of theorem 3.2.1, for a sequence $\{t_n\}$, $t_n \to \infty$ with

(3.2.21) $$t_n^2 - c \log n - 2(c+1)\log|t_n| - 2\log u(r\, s_n t_n) \to -\infty$$

We have

(3.2.22) $$1 - G_n(t_n) \sim \Phi(-t_n)$$

Further if the sequence $\{|X_{ni}|^{2+c} u(X_{ni})\}$ is u.i. then (3.3.22) holds even if l.h.s of (3.3.21) is bounded above.

Theorem 3.2.6 Suppose that the conditions of the theorems 3.2.1 and 3.2.2 are satisfied. Then for a sequence $\{t_n\}$, $t_n \to \infty$ with

$$(3.2.23) \qquad t_n^2 - c \log n - 2(c+1)\log |t_n| - 2 \log u(r\, s_n t_n) \to \infty$$

we have

$$(3.2.24) \qquad 1 - G_n(t_n) = o(t_n^{-(2+c)} n^{-c/2}\, u^{-1}(rs_n t_n)) \quad \text{if } |X_{ni}|^{2+c} u(X_{ni}) \text{ is u.i.}$$

$$= O(t_n^{-(2+c)} n^{-c/2} u^{-1}(rs_n t_n)) \quad \text{otherwise.}$$

As a concluding remark of this section we may note that remark 2.2.2 and 2.2.3 apply as well to the theorems 3.2.5 and 3.2.6.

3.3 NON-UNIFORM RATES FOR T_n WHEN ALL THE MOMENTS OF X_{ni} EXIST

In this section results for T_n analogous to those obtained in sections 2.3 - 2.4 are obtained. Assume (2.0.1), (2.0.2) and (2.3.1). Further suppose that R_n satisfies

$$(3.3.1) \qquad E(R_n^{2m}) \leq c(2m)\, n^{-m} (\log n)^{hm} \quad \text{for some } h \geq 0, \quad m = 1,2,3, \ldots$$

where $c(.)$ is constant depending on m, under suitable restriction on c and h we shall show that the results of section 2.4 may be extended to include T_n. w.o.l.g let $t > 0$. Note that due to the representation (3.1.1)

$$(3.3.2) \qquad |P(T_n \leq t) - \Phi(t)| \leq P(s_n^{-1} S_n \leq t \pm a_n(t)) - \Phi(t \pm a_n(t))|$$

$$+ |\Phi(t \pm a_n(t)) - \Phi(t)| + P(|R_n| > a_n(t))$$

where $a_n(t) > 0$ will be choosen accordingly. Now

$$(3.3.3) \qquad P(|R_n| > a_n(t)) \leq \exp[-A_n \, a_n(t)] \, E \, [\exp(A_n |R_n|)]$$

It will be shown that if

$$(3.3.4) \qquad c(2m) \leq (2m)! \; L^{2m} \quad \text{for some} \quad L > 0$$

for $A_n = \varepsilon \, n^{1/2} (\log n)^{-h/2}$ and for some $\varepsilon > 0$

$$(3.3.5) \qquad \sup_n \; E \, (\exp \, (A_n \, |R_n| \,)) \quad < \infty$$

so that

$$(3.3.5a) \qquad P(|R_n| > a_n(t)) \; = \; O[\exp(-A_n \, a_n(t))]$$

First note that

$$(3.3.6) \quad \exp(A_n|R_n|) + \exp(-A_n|R_n|) = 2 \left[1 + \sum_{m=1}^{\infty} \frac{(A_n|R_n|)^{2m}}{(2m)!} \right]$$

Taking expectation both sides and noting that $E \exp(-A_n|R_n|) > 0$

we have, in view of (3.3.1) and (3.3.4)

(3.3.7) $$E \exp(A_n |R_n|) \leq 2 \left[1 + \sum_{m=1}^{\infty} (\varepsilon L)^{2m} \right]$$

Hence for $\varepsilon < L^{-1}$ r.h.s of (3.3.7) is a convergent geometric series free from n, therefore we have (3.3.5) under (3.3.4).

We use this result in obtaining a normal approximation zone for T_n.

Let $a_n(t) = n^{-\gamma}$ where $\gamma > 0$ will be chosen later. Then for the 1st term of the r.h.s of (3.3.2) we have from theorem 2.4.1 with t satisfying (2.4.1)

(3.3.8) 1st term in the r.h.s of (3.3.2) $\leq b \exp(-(t \pm n^{-\gamma})^2/2)$

$$\times |t \pm n^{-\gamma}|^{-1} |\exp(O(|t \pm n^{-\gamma}|^3 n^{-1/2}))-1|$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r s_n |t \pm n^{-\gamma}|)$$

$$\leq b \exp(-t^2/2) | \exp(O(|t|^3 n^{-1/2})) -1| \, |t|^{-1}$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r \lambda s_n |t|) \quad \text{for } t = O(n^{\gamma})$$

with some $0 < \lambda < 1$.

(3.3.9) 2nd term $\leq b . n^{-\gamma} \exp(-t^2/2)$

$$= o(|t|^{-1} \exp(-t^2/2)) \quad \text{for } t = o(n^{\gamma})$$

(3.3.10) $\quad$ 3rd term $\leq b \exp(-\varepsilon\, n^{\frac{1}{2}-\gamma} (\log n)^{-h/2})$

(3.3.8) - (3.3.10) with $\gamma = 1/6$ implies the following theorem along the lines of theorem 2.4.4 and remark 2.4.8.

<u>Theorem</u> 3.3.1 Suppose (2.0.1), (2.0.2), (2.3.1), (3.3.1) and (3.3.4) hold. Then for a sequence $\{t_n\}$ satisfying

i) $t_n = o(n^{1/6})$ if $h = 0$

$\leq \varepsilon' n^{1/6}(\log n)^{-h/4}$ if $h > 0$ for some $0 < \varepsilon' < \varepsilon$

and

ii) $t_n^2 - 2(\log t_n + \log g(r\lambda s_n t_n)) \to -\infty$, $0 < r\lambda < 1/2$

the following holds

(3.3.11) $\quad 1 - P(T_n \leq t_n) \sim \Phi(-t_n)$ as $t_n \to \infty$.

Further if the sequence $\{X_{ni}^2\, g(X_{ni})\}$ is uniformly integrable then (3.3.11) holds even if l.h.s of ii) is bdd above.

Now let us have a different choice of $a_n(t)$ viz. $a_n(t) = \alpha t$ $0 < \alpha < 1$. In that case with $t\ (> 0)$ satisfying (2.4.1) we have, for the 1st term of the r.h.s of (3.3.2), from theorem 2.4.1 ,

(3.3.12) $\quad$ 1st term of r.h.s of (3.3.2) $\leq b \exp\left[-\frac{t^2}{2}(1-\alpha)^2 + kn^{-1/2}t^3\right]$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r(1-\alpha)t\, s_n) \text{ for some } k > 0.$$

(3.3.13) 2nd term $\leq$ b exp $[-(1-\alpha)^2 t^2/2] \alpha t$

(3.3.14) 3rd term $\leq$ b exp $[-\varepsilon \alpha t n^{1/2} (\log n)^{-h/2}]$

and hence we have the following theorem.

Theorem 3.3.2 Suppose (2.0.1) with $g(x) = \exp(s|x|)$ for some $s > 0$, (2.0.2), (3.3.1) and (3.3.4) hold. Then

(3.3.15) $P(T_n > t_n) \leq b \exp\left\{-\frac{t_n^2}{2}(1 + o(1))\right\}$

for $t_n = o(n^{1/2}(\log n)^{-h/2})$, $t_n \to \infty$.

proof of the above theorem is similar to that of theorem 2.4.6 letting $\alpha \to 0$.

For $h = 0$, noting that for $t = \varepsilon'\sqrt{n}$, $\alpha t n^{1/2} = \alpha t^2/\varepsilon'$, letting $\varepsilon' \to 0$ along the lines of theorem 2.4.6 with the observation that l.h.s of (3.3.16) is independent of α (and hence finally letting $\alpha \to 0$) we have the following theorem.

Theorem 3.3.3 Let the conditions of theorem 3.3.2 hold with $h = 0$, then for $t_n = \varepsilon'\sqrt{n}$, $\varepsilon' > 0$

(3.3.16) $\limsup_{\varepsilon' \to 0} \limsup_{n \to \infty} (t_n^2/2)^{-1} \log P(T_n > t_n) \leq -1$

Next with the same choice of $a_n(t)$ i.e., $a_n(t) = \alpha t$, the following theorem and remark 3.3.1 follow from theorem 2.4.8 remark 2.4.11 and (3.3.2).

<u>Theorem</u> 3.3.4 Suppose (2.0.1), (2.0.2), (2.3.1), (3.3.1) and (3.3.4) hold, then for $t^2 \geq 2(\log|t| + \log g(r\, s_n t))$ with $x^{-1} \log g(x) \to 0$, we have for some $b(>0)$ and any α, $0 < \alpha < 1$

$$(3.3.17) \quad |P(T_n \leq t) - \Phi(t)| \leq O(|t| g(r\alpha s_n t))^{-1+\varepsilon_{n,t}}$$

$$+ \exp(-(1-\alpha)^2\, t^2/2)\, \alpha\, |t|\, b$$

$$+ b \exp\left[-\alpha |t| n^{1/2} (\log n)^{-h/2}\right] +$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r(1-\alpha)\, s_n\, |t|)$$

where $\varepsilon_{n,t}$ is defined by (2.4.74).

<u>Remark</u> 3.3.1 Suppose (2.0.1), (2.0.2), (2.3.1), (3.3.1) and (3.3.4) hold, then for $t^2 \geq 2(\log|t| + \log g(rs_n tk_n))$ with any sequence $k_n \to 0$ we have, for $x^{-1}\log g(x) \to s\ (>0)$

$$(3.3.18) \quad |P(T_n \leq t) - \Phi(t)| \leq O(|t| g(r\alpha s_n t k_n))^{-1+O(k_n)}$$

$$+ b \exp(-(1-\alpha)^2\, t^2/2)$$

$$+ b\exp\left[-\alpha |t| n^{1/2} (\log n)^{-h/2}\right]$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r(1-\alpha) s_n |t|)$$

To obtain a non-uniform bound over the entire range of t, i.e., $-\infty < t < \infty$, we proceed as in theorem 2.4.9. Let t_0 be some positive constant. Note that for

$t^2 > k \log n$ where k is sufficiently large, (3.3.21) follows the same lines as that of theorem 2.4.9 where one uses (3.3.12) - (3.3.14), theorem 3.3.4 and remark 3.3.1. For $t_o^2 < t^2 \leq k \log n$, one uses (3.3.2) with $a_n(t) = \lambda n^{-1/2}(\log n)^{h/2+1}|t|$; in this case (3.3.21) becomes almost immediate when α is replaced by $\alpha_n = \lambda n^{-1/2}(\log n)^{h/2+1}$ in (3.3.12) - (3.3.14) and noting that $|\exp(O(|t|^3 n^{-1/2})) - 1| = O(|t|^3 n^{-1/2})$ for $t^2 \leq k \log n$. For $|t| < t_o$ (> 0) one uses an uniform bound $n^{-1/2}(\log n)^{h/2+1}$ as in theorem 3.5.2. Hence the following

Theorem 3.3.4 Suppose (2.0.1), (2.0.2), (2.3.1), and (3.3.4) hold alongwith

(3.3.19) $$[g(r\,\alpha\,s_n t)]^{-1+\varepsilon} \leq \lambda_1 n^{-1/2}(\log n)^{\frac{h}{2}+1}\,[g(\lambda t)]^{-1}$$

for all sufficiently large n when $x^{-1}\log g(x) \to 0$ as $x \to \infty$, with t satisfying $t^2 \geq 2(\log|t| + \log g(r\,s_n t))$

(3.3.20) $$[g(r\,\alpha\,s_n t\,k_n)]^{-1+\varepsilon} \leq \lambda_2\, n^{-1/2}(\log n)^{\frac{h}{2}+1}\,[g(\lambda t)]^{-1}$$

for all sufficiently large n, when $x^{-1}\log g(x) \to s$ (> 0) as $x \to \infty$ with t satisfying $t^2 \geq 2(\log|t| + \log g(r\,s_n t\,k_n)$, where k_n is some sequence $k_n \to 0$, $\varepsilon, \alpha, \lambda, \lambda_1, \lambda_2$ some +ve constants.

Then

(3.3.21) $$|P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/2} (\log n)^{\frac{h}{2}+1} [g(\lambda t)]^{-1} + \sum_{i=1}^{n} P(|X_{ni}| > r(1-\alpha) s_n |t|)$$

Further if, for some $t_1 > 0$

(3.3.22) $$[t^2 g(r(1-\alpha) s_n t))]^{-1} \leq b\, n^{-1/2} (\log n)^{\frac{h}{2}+1} [g(\lambda t)]^{-1}$$

for all $t > t_1$

then

(3.3.23) $$|P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/2} (\log n)^{\frac{h}{2}+1} [g(\lambda t)]^{-1}, \quad -\infty < t < \infty$$

As a consequence of theorem 3.3.4 we may have the following two theorems the proof of which are similar to those of theorem 2.4.10 and theorem 2.4.11.

<u>Theorem</u> 3.3.5 Under the assumptions of theorem 3.3.4

(3.3.24) $$\|g(\lambda t)(1+|t|)^{-q/p} (G_n(t) - \Phi(t))\|_p = O(n^{-1/2} (\log n)^{\frac{h}{2}+1})$$

for $p \geq 1$ and any $q > 1$.

<u>Theorem</u> 3.3.6 Under the assumptions of theorem 3.3.4 and

(3.3.25) $$\frac{d}{dx} [x^2 g(x)] \leq \lambda_1\, g(\lambda x)(1+|x|)^{-q} + \lambda_2 \quad \forall\; x \geq 0$$

and for some $\lambda_1, \lambda_2 > 0$, $q > 1$, λ being same as that of theorem 3.3.4, one has

$$(3.3.26) \quad |E(T_n^2\, g(T_n)) - E(T^2\, g(T))| = O(n^{-1/2}(\log n)^{\frac{h}{2}+1})$$

where $T = |N(0,1)|$

See remark 2.4.12 for some examples of g satisfying the conditions of the above theorems.

Remark 3.3.2 (3.3.5) essentially needs the existance of moment generating function of normalised R_n and this is ensured by the assumption (3.3.4). This assumption has only been utilised to have an estimate of $P(|R_n| > a_n(t))$. In the same fashion it can be shown that if $c(2m) \leq (2\gamma_1 m)!\, L^m$ for some $L > 0$ and $\gamma_1 > 1$ then $P(|R_n| > a_n(t)) = O(\exp(-(A_n . a_n(t))^{1/\gamma_1}))$, $A_n = \epsilon n^{1/2}(\log n)^{-h/2}$ for some $\epsilon > 0$. The technique may be briefly described as follows. By Markov's inequality,

$$P(|R_n| > a_n(t)) \leq [a_n(t)]^{-k}\, E|R_n|^k .$$

Now to have the optimal order differenciate r.h.s w.r.t k after setting the estimate of $E|R_n|^k$. For details of the technique Chapter 5 may be consulted where dependent processes is considered. (See also (3.7.31) - (3.7.34))

As a result all the theorems of this chapter may be modified under weaker conditions on R_n, e.g. $c(2m) \leq (2\gamma_1 m)!\, L^{2m}$ $\gamma_1 > 1$ or $c(2m) \leq \exp[(2m)^{\gamma}]$, $\gamma > 1$ etc. Since the basic technique remains the same, details are omitted.

3.4 NON-UNIFORM RATES FOR T_n WHEN m.g.f OF X_{ni} NECESSARILY EXIST

This section generalises the results of section 2.5 - 2.6 for non linear statistics T_n. Assume (3.3.1) with

(3.4.1) $c(2m) \leq m!\, L^{2m}$ for some $L > 0$.

Then note that

(3.4.2) $$\sup_n E\left[\exp(\lambda\, n^{1/2}(\log)^{-h/2}|R_n|)^2\right]$$

$$= \sup_n \left[1 + \sum_{m=1}^{\infty} (\lambda n^{1/2}(\log n)^{-h/2})^{2m}\, E\, R_n^{2m}/m!\right]$$

$$\leq 1 + \sum_{m=1}^{\infty} (\lambda L)^{2m} < \infty \quad \text{if} \quad 0 < \lambda < L^{-1}$$

Consequently

(3.4.3) $$P(|R_n| > a_n(t)) = O(\exp(-(\lambda n^{1/2}(\log n)^{-h/2}\, a_n(t))^2)),$$

$$0 < \lambda < L^{-1}$$

Hence from (3.3.2) with $t_0^2 \leq t^2 \leq k \log n$ (k may be arbitrarily large) letting $a_n(t) = n^{-1/2}(\log n)^{(h+1)/2}\, t\, \lambda^{-1}$ and w.o.l.g letting $t > 0$

(3.4.4) $$|P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/2}\, t^2 \exp(-t^2/2)$$

$$+ bn^{-1/2}(\log n)^{(h+1)/2} t \exp(-t^2/2) + bn^{-1/2}\exp(-t^2/2)$$

for $t^2 \leq t_o^2$ (> 0), one , however, uses uniform bound $b\ n^{-1/2}$. $.(\log n)^{(h+1)/2}$. Hence (3.4.4) for $t^2 \leq k \log n$. (for similar calculations see (3.3.8) - (3.3.10))

For $t \geq k \log n$ under the assumptions of theorem 2.6.1 one has, using (2.6.2) and (3.3.2) (with the same choice of $a_n(t)$ as above)

$$(3.4.5) \quad |P(T_n \leq t) - \Phi(t)| \leq b\ n^{-1/2} \exp\{-t^2(1-\ell^{-1}).r\}$$

$$+ b\ n^{-1/2}(\log n)^{(h+1)/2} \exp(-t^2/2) + b\ n^{-1/2}\exp(-t^2/2)$$

where $0 < r < 1$,

since, $\exp(-t^2(1-\ell^{-1})) \leq n^{-1/2}\exp(-t^2(1-\ell^{-1})r)$ if

$$t^2 \geq [2a(1-r)]^{-1} \log n, \quad 0 < r < 1.$$

Which can be ensured choosing k sufficiently large.

Note that trucation of the variables X_{ni} are not needed as m.g.f exists, so the term $\Sigma P(|X_{ni}| > r\ s_n t)$ is omitted (see remark 2.4.5)

As a consequence of (3.4.4) and (3.4.5) we have the following non uniform bound over the entire range of t

<u>Theorem</u> 3.4.1 Under the assumptions of theorem 2.6.1 and (3.3.1) with (3.4.1) $\exists b > 0$ depending on r such that

$$(3.4.6) \qquad |P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/2}(\log n)^{(h+1)/2} \times \exp\,(-t^2(1-\ell^{-1})r)\ .$$

for all r, $0 < r < 1$, $-\infty < t < \infty$.

Subsequently the following two theorems are immediate from (3.4.6) noting that $0 < r < 1$ is arbitrary.

<u>Theorem</u> 3.4.2 Under the assumptions of theorem 3.4.1 for any $g : (-\infty,\infty) \to (0, \infty)$, $g(x)$ even, such that $E\, g(T) < \infty, g(0)=0,$ $T = |N(0, 1)|$, and

$$(3.4.7) \qquad g'(x) = O(\exp(-x^2(1-\ell^{-1})r)), \qquad 0 < x < \infty$$

and for some r, $0 < r < 1$

the following holds

$$(3.4.8) \qquad |E\, g(T_n) - E\, g(T)| = O(n^{-1/2}(\log n)^{(h+1)/2})$$

prof of the above follows from (2.4.93).

<u>Theorem</u> 3.4.3 Under the assumtion of theorem 3.4.1

$$(3.4.9) \qquad \|\exp(t^2(1-\ell^{-1})r)(G_n(t)-\Phi(t))\|_p = O(n^{-1/2}(\log n)^{(h+1)/2}) \quad \text{for any } p \geq 1 \text{ and } 0 < r < 1.$$

Next we consider the case when the assumption (2.5.3) in theorem 2.6.1 is not satisfied i.e., we consider the case when odd order moments of X_{ni} are non vanishing. As before for $t^2 \leq k \log n$, it is possible to obtain (3.4.4) since theorem 2.6.1 is not used there. However for $t^2 \geq k \log n$ we may use (2.6.33) in (3.3.2) with the same choice of $a_n(t)$ viz. $a_n(t) = n^{-1/2} (\log n)^{(h+1)/2} |t| \lambda^{-1}$ to obtain

$$(3.4.10) \quad |P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/2}\, t^{-2} \exp(-t^2/f(t))$$

$$+ b\, n^{-1/2} (\log n)^{(h+1)/2} \exp(-t^2/2) + b n^{-1/2} \exp(-t^2/2)$$

for $t^2 \geq k \log n$, where $\lim_{t \to \infty} f(t) = \infty$.

Hence combining (3.4.4) and (3.4.10) it is possible to obtain the following non uniform bound

<u>Theorem</u> 3.4.4 Under assumptions (2.0.2), (2.5.1), (2.5.2) (2.6.1) and (3.3.1) with (3.4.1) $\exists$ a constant b (> 0) depending on $f(t)$, $f:(0,\infty) \to (0,\infty)$, nondecreasing and $\lim_{t \to \infty} f(t) = \infty$, such that

$$(3.4.11) \quad |P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/2} (\log n)^{(h+1)/2} \exp(-t^2/f(t))$$

$$-\infty < t < \infty.$$

Consequently the following two theorems on moment type convergence and on non-uniform L_p version of the Berry-Esseen theorem follows from (3.4.11) noting that $f(t)$ therein is arbitrary.

Theorem 3.4.5 Under the assumptions of theorem 3.4.4 for any $g : (-\infty, \infty) \to (0, \infty)$, $g(x)$ even, $g(0) = 0$, such that $E\, g(T) < \infty$ and

$$(3.4.12) \qquad g'(x) = O(\exp(x^2/f(x)) , \quad 0 < x < \infty , \ f: (0,\infty) \to (0,\infty)$$

nondecreasing and $\lim_{t\to\infty} f(t) = \infty$, the following holds

$$(3.4.13) \qquad |E\, g(T_n) - Eg(T)| = O(n^{-1/2}(\log n)^{(h+1)/2})$$

Theorem 3.4.6 Under the assumption of theorem 3.4.4, for any $p \geq 1$,

$$(3.4.14) \qquad \|\exp(t^2/f(t)) \,.\, (G_n(t)-\Phi(t))\|_p = O(n^{-1/2}(\log n)^{(h+1)/2})$$

3.5 RATES OF CONVERGENCE FOR L-STATISTICS

Let $X_{1n} \leq X_{2n} \leq \ldots \leq X_{nn}$ denote the order statistics corresponding to n iid v.v's $X_1, X_2, \ldots, X_n$ having a common distribution function F. Consider linear combination of function of order statistics of the form

$$(3.5.1) \qquad T_n = \sum_{i=1}^{n} c_{in}\, h(X_{in})$$

where the c_{in}'s are constant and h is some measureble function. Let $H = h \circ F^{-1} \circ G$ where $G(x) = 1 - \exp(-x)$. Also, let $z_{1n} \leq \ldots \leq z_{nn}$ denote the order statistics corresponding to n iid r.v's $z_1, z_2, \ldots, z_n$ each having the distribution function $G(x)$. Then T_n is identically distributed as $\sum_{i=1}^{n} c_{in} H(z_{in})$. This representation is due to Chernoff, Gastwirth and Johns (1967). Note that z_{in} has the same distribution as $\sum_{j=1}^{i} z_j/(n-j+1)$ and hence

$$(3.5.2) \qquad \upsilon_{in} = E(z_{in}) = \sum_{j=1}^{i} (n-j+1)^{-1}, \qquad 1 \leq i \leq n$$

Assuming that H is differenciable, by the first mean value theorem, one has

$$(3.5.3) \qquad (T_n - \mu_n)/s_n = s_n^{-1} L_n + R_n \quad \text{where}$$

$$(3.5.4) \qquad \mu_n = \sum_{i=1}^{n} c_{in}\, H(\upsilon_{in})$$

(3.5.5) $$L_n = \sum_{i=1}^{n} c_{in} H'(\upsilon_{in})(z_{in}-\upsilon_{in})$$

(3.5.6) $$R_n = s_n^{-1} \sum_{i=1}^{n} c_{in}(z_{in}-\upsilon_{in}) [H'(\theta_n z_{in}+(1-\theta_n)\upsilon_{in})- H'(\upsilon_{in})]$$

for some $0 < \theta_n < 1$; $s_n^2 = \sum_{i=1}^{n} \alpha_{in}^2$, $\alpha_{in} = (n-i+1)^{-1} \sum_{j=i}^{n} c_{jn} H'(\upsilon_{jn})$

$1 \leq i \leq n$. Note that $\sum_{i=1}^{n} c_{in} H'(\upsilon_{in})(z_{in}-\upsilon_{in})$ has the same distribution as $U_n = \sum_{i=1}^{n} \alpha_{in}(z_{i\ell}-1)$ and $s_n^2 = \text{var}(U_n)$.

The above expansion was used by Bjerve (1977) in obtaining a uniform Berry-Esseen theorem of $O(n^{-1/2})$ for trimmed L-statistics. Helmers (1977) obtained the same rate of convergence for general L-statistics under different conditions. The asymptotic normality of T_n was proved by Chernoff et al (1967)

Our aim in this section is to develop non-uniform Berry-Esseen bounds for T_n and to obtain theorems analogous to theorems (3.3.1), (3.3.5) and (3.3.6).

The following assumptions are made.

I. $\sup_{n\geq 1} \max_{1\leq i\leq n} |c_{in}| < \infty$;

II. H is differenciable and $\sup_{0<x<\infty} |H'(x)| < \infty$;

III. H' is Lipschitz of order one over $(0, \infty)$;

IV. $\underline{\lim}_{n\to\infty} n^{-1} s_n^2 > 0$.

It is now easy to see that (2.0.2) and (2.0.1) are satisfied with $g(x) = \exp(|x|)$ since m.g.f of the exponential distribution exists. To have an estimate of $c(2m)$ first observe that in view of I and III

$$(3.5.7) \quad E|R_n|^{2m} \leq L^m n^{-m} E\left[\sum_{i=1}^{n} (z_{in} - \upsilon_{in})^2\right]^{2m}$$

$$= L^m n^{-m} E\left[\sum_{i=1}^{n} \left\{\sum_{j=1}^{i} (z_j - 1)/(n-j+1)\right\}^2\right]^{2m}$$

$$= L^m n^{-m} E\left[\sum_{j=1}^{n} (z_j - 1)^2/(n-j+1) + \right.$$

$$\left. +2 \sum_{j=1}^{n-1} \sum_{j'=j+1}^{n} (z_j - 1)(z_{j'} - 1)/(n-j+1)\right]^{2m}$$

Where in the above and in what follows L is a generic constant which is not dependent on m and n. Also

$$(3.5.8) \quad E\left(\sum_{j=1}^{n} (z_i - 1)^2/(n-j+1)\right)^{2m} = \sum_{j_1} \cdots \sum_{j_{2m}} \frac{E\left[(z_{j_1} - 1)^2 \cdots (z_{j_{2m}} - 1)^2\right]}{(n-j_1+1) \;\cdots\; (n-j_{2m}+1)}$$

Using Holder's inequality 2m times,

$$(3.5.9) \quad E\left[(z_{j_1} - 1)^2 \cdots (z_{j_{2m}} - 1)^2\right] \leq \prod_{i=1}^{2m} E^{1/2m} (z_{j_i} - 1)^{4m} .$$

But $E(z_1 - 1)^{4m} \leq (4m)!$. Hence from (3.5.8) and (3.5.9)

(3.5.10) $$E\Big(\sum_{j=1}^{n}(z_j-1)^2/(n-j+1)\Big)^{2m} \le (4m)!\Big(\sum_{j=1}^{n}\frac{1}{n-j+1}\Big)^{2m}$$

$$\le (4m)!\,(\log n)^{2m}$$

Further

(3.5.11) $$E\Big[\sum_{j=1}^{n-1}\sum_{j'=j+1}^{n}(z_j-1)(z_{j'}-1)/(n-j+1)\Big]^{2m}$$

$$= \sum_{j_1=1}^{n-1}\sum_{j_1'=j_1+1}^{n}\cdots\sum_{j_{2m}=1}^{n-1}\sum_{j_{2m}'=j_{2m}+1}^{n}\frac{E(z_{j_1}-1)(z_{j_1'}-1)\ldots(z_{j_{2m}}-1)(z_{j_{2m}'}-1)}{(n-j_1+1)\;\ldots\;(n-j_{2m}+1)}$$

Note that if any one of the pairs (j_i, j_i') occurs only once, then the expectation vanishes, and hence every suffix should occur at least twice to make a non-zero contribution.

Subject to the condition that each pair of suffixes occurs at least twice the maximum number of pairs that can occur is m. Also applying Holders inequality $4m$ times,

(3.5.12) $$|E(z_{j_1}-1)(z_{j_1'}-1)\;\ldots\;(z_{j_{2m}}-1((z_{j_{2m}'}-1)|$$

$$\le \prod_{j=1}^{4m} E^{1/4m}\,(z_j-1)^{4m} = E(z_1-1)^{4m} \le (4m)!$$

In view of the fact that $\sum_{i_1}\ldots\sum_{i_K}\frac{1}{(n-i_1+1)^{p_1}\ldots(n-i_K+1)^{p_K}}\downarrow$ as $p_j(>0)\uparrow$, $j=1\ldots K$ and that maximum number of pairs is m each occuring at least twice, we have

(3.5.13) l.h. s of (3.5.11) $\le$

$$\leq (4m)! \sum_{j_1=1}^{n-1} \sum_{j_1'=j+1}^{n} \cdots \sum_{j_m=1}^{n-1} \sum_{j_m'=j_m+1}^{n} \frac{1}{(n-j_1+1)^2 \cdots (n-j_m+1)^2}$$

$$= (4m)! \left(\sum_{j=1}^{n-1} \frac{1}{(n-j+1)} \right)^m \leq (4m)! \ (\log n)^m$$

And hence, finally

(3.5.14) $$E\, R_n^{2m} \leq (4m)! \ n^{-m} (\log n)^{2m} L^{2m} \quad \text{for some } L > 0.$$

We now proceed to have theorems analogous to 3.3.1, 3.3.5 and 3.3.6. Note that according to remark 3.3.2

(3.5.15) $$P(|R_n| > a_n(t)) = O(\exp(-(A_n a_n(t))^{1/\upsilon_1}))$$

In this case $\upsilon_1 = 2$, $A_n = \varepsilon\, n^{1/2} (\log n)^{-1}$ as $h = 2$ here, (see 3.3.1). Therefore letting $a_n(t) = n^{-\upsilon} (\log n)^{\lambda}$, $\lambda, \upsilon > 0$ to be chosen later, we have

(3.5.16) $$P(|R_n| > a_n(t)) = O(\exp(-(\varepsilon\, n^{\frac{1}{2}-\upsilon} \log n^{-1+\lambda})^{1/2}))$$

$$= o(|t|^{-1} \exp(-t^2/2))$$

$$\text{for } t = o(n^{\frac{1}{2}-\upsilon} (\log n)^{-1+\lambda})^{1/4}$$

Also

(3.5.17) $$|\Phi(t \pm a_n(t)) - \Phi(t)| \leq b\, n^{-\upsilon} (\log n)^{\lambda} \exp(-t^2/2)$$

$$= o(|t|^{-1} \exp(-t^2/2)), \quad \text{for } t = o(n^{\upsilon} (\log n)^{-\lambda}).$$

Now equating $n^{\upsilon}(\log n)^{-\lambda} = (n^{\frac{1}{2}-\upsilon}(\log n)^{-1+\lambda})^{1/4}$

which gives $\upsilon = 1/10$ and $\lambda = 1/5$ the following theorem follows along calculations (3.3.8) - (3.3.10) of theorem 3.3.1.

Theorem 3.5.1 Under assumptions I-IV for T_n defined in (3.5.1)

(3.5.18) $$1 - P(T_n \leq t_n) \sim \Phi(-t_n) \quad \text{for} \quad t_n = o(n^{1/10} (\log n)^{-1/5}), \quad t_n \to \infty.$$

Now letting $a_n(t) = \varepsilon\, n^{-1/2}(\log n)^3 |t|$ in (3.5.15) the following theorems follows in view of the following observation

(3.5.19) $$P(|R_n| > a_n(t)) = O(\exp(-\varepsilon^{1/4}\sqrt{|t|}\ \log n)) \quad \text{for} \quad a_n(t) = \varepsilon n^{-1/2}(\log n)^3 |t|$$

$$\leq bn^{-1/2}\exp(-\lambda\sqrt{|t|}) \quad \text{if} \quad |t| > t_o, \text{ for some } t_o \text{ depending on } \lambda.$$

Hence $|P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/2} (\log n)^3 \exp(-\lambda\sqrt{|t|})$

for $|t| > t_o$

But a uniform bound $n^{-1/2}(\log n)^3$ is available from the relation

(3.5.19a) $$||F(X+Y)-\Phi|| \leq ||F(X) - \Phi|| + (2\pi)^{-1/2} a_n + P(|Y| > a_n).$$

for any two r.v. X and Y and $a_n > 0$. Now choose $X + Y = T_n$, $Y = R_n$ and $a_n = \varepsilon\, n^{-1/2}(\log n)^3$ to obtain uniform bound $n^{-1/2}(\log n)^3$ proceeding as in (3.5.19), Hence the following

<u>Theorem</u> 3.5.2 Under assumptions I - IV for T_n defined in (3.5.1) for any $\lambda > 0$ $\exists$ b > 0 (depending on λ) s.t.

(3.5.20) $$|P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/2}(\log n)^3 \exp(-\lambda\sqrt{|t|}\,)$$

$$-\infty < t < \infty.$$

In view of theorem 3.5.2 we have the following two theorems

<u>Theorem</u> 3.5.3 Under the assumptions of theorem 3.5.2, for any $\lambda > 0$

(3.5.21) $$|E\, T_n^2 \exp(\lambda\sqrt{|T_n|}\,) - E\, T^2 \exp(\lambda\sqrt{|T|}\,)\,|$$

$$= O(n^{-1/2} (\log n)^3\,)$$

<u>Theorem</u> 3.5.4 Under the assumptions of theorem 3.5.2 for any $\lambda > 0$ and $p \geq 1$

(3.5.22) $$\|\exp(\lambda\sqrt{|t|}\,)(G_n(t) - \Phi(t))\|_p = O(n^{-1/2} (\log n)^3)\ ,$$

The conditions assumed on H are satisfied when for example $F(x) = G(x) = 1 - \exp(-x)$ and h is the identity map. In the case of trimmed L statistics $T_n = \sum_{[n\alpha]+1}^{[n\beta]} c_{in}\, H(z_{in})$

$0 < \alpha < \beta < 1$, and weaker condition on H suffices. All we need assume there is $\sup_{a<x<b} |H'(x)| < \infty$ and H' is Lipschitz of order 1 on $[a, b]$ where $a < -\log(1-\alpha)$, $b > -\log(1-\beta)$ alongwith I and IV.

3.6 RATES OF CONVERGENCES FOR U-STATISTICS

Let $\{X_n , n \geq 1\}$ be a sequence of independent but not necessarily identically distributed random variables. A U-statistic with kernel ϕ and degree r, based on $X_1, X_2, \ldots, X_n$ $(n \geq r)$, is defined by

$$(3.6.1) \qquad U_n = \binom{n}{r}^{-1} \sum_{1 \leq i_1 < \ldots < i_r \leq n} \phi(X_{i_1}, \ldots, X_{i_r})$$

where the kernel ϕ is symmetric in its arguments.

In this section we provide a Hoeffding (1961) decomposition for U-statistics in the non iid case and use this in deriving certain uniform Berry-Esseen bound for U-statistics with some non uniform Berry-Esseen bound analogous to those obtained in sections 3.2 - 3.4. Finally for Kernels with degree 2, a weak invariance principle is proved for U-statistics in the non iid case.

First a few notations are introduced. Assume w.o.l.g that $E\,\phi(X_{i_1}, \ldots, X_{i_r}) = 0$ for all $1 \leq i_1 < \ldots < i_r \leq n$. To keep the notations simple we show the decomposition only for r = 3, although it can very well be generalised. Let

$$(3.6.2) \qquad \phi^{i_1}_{i_2,i_3}(x_{i_1}) = \phi^{i_1}_{i_2,i_3}(x_{i_1}) = E\left[\phi(X_{i_1},X_{i_2},X_{i_3}) \mid X_{i_1}=x_{i_1}\right]$$

$$1 \leq i_1 \neq i_2 \neq i_3 \leq n.$$

$$(3.6.3)\quad \phi_{i_3}^{i_1,i_2}(x_{i_1},x_{i_2}) = E\left[\phi(X_{i_1},X_{i_2},X_{i_3}) \mid X_{i_1}=x_{i_1},X_{i_2}=x_{i_2}\right]$$

$$1 \le i_1 \ne i_2 \ne i_3 \le n;$$

$$(3.6.4)\quad \psi_{i_3}^{i_1,i_2}(x_{i_1},x_{i_2}) = \phi_{i_3}^{i_1,i_2}(x_{i_1},x_{i_2}) - \psi_{i_2,i_3}^{i_1}(x_{i_1}) - \psi_{i_1,i_3}^{i_2}(x_{i_2}$$

$$1 \le i_1 \ne i_2 \ne i_3 \le n$$

$$(3.6.5)\quad \psi^{i_1,i_2,i_3}(x_{i_1},x_{i_2},x_{i_3}) = \phi(x_{i_1},x_{i_2},x_{i_3}) - \psi_{i_2,i_3}^{i_1}(x_{i_1})$$

$$-\psi_{i_1,i_3}^{i_2}(x_{i_2}) - \psi_{i_1,i_2}^{i_3}(x_{i_3}) - \psi_{i_3}^{i_1,i_2}(x_{i_1},x_{i_2})$$

$$-\psi_{i_2}^{i_1,i_3}(x_{i_1},x_{i_3}) - \psi_{i_1}^{i_2,i_3}(x_{i_2},x_{i_3})$$

$$1 \le i_1 \ne i_2 \ne i_3 \le n$$

Writing

$$(3.6.6)\quad \bar{\psi}_n^{(1)}(X_i) = \binom{n-1}{2}^{-1} \sum_{\substack{1\le j<\ell\le n \\ j\ne i \\ \ell\ne i}} \psi_{j,\ell}^{i}(X_i) \quad ,$$

$$\bar{\psi}_n^{(2)}(X_i,X_j) = (n-2)^{-1} \sum_{k=1(\ne i\ne j)}^{n} \psi_k^{i,j}(X_i,X_j)$$

$$(3.6.7)\quad V_n^{(1)} = n^{-1}\sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i) \ , \quad V_n^{(2)} = \binom{n}{2}^{-1} \sum\sum_{1\le i<j\le n} \bar{\psi}_n^{(2)}(X_i,X_j)$$

and

$$(3.6.8)\quad V_n^{(3)} = \binom{n}{3}^{-1} \sum\sum\sum_{1\le i_1<i_2<i_3\le n} \psi^{i_1,i_2,i_3}(X_{i_1},X_{i_2},X_{i_3})$$

One has the representation

$$(3.6.9) \qquad U_n = 3\,V_n^{(1)} + 3\,V_n^{(2)} + V_n^{(3)}$$

The following facts can be easily verified

$$(3.6.10) \qquad E\,\psi^{i_1}_{i_2,i_3}(X_{i_1}) = 0, \quad E\,[\psi^{i_1,i_2}_{i_3}(X_{i_1},X_{i_2})\,|\,X_{i_1} = x_{i_1}] = 0$$

a.e.,

$$(3.6.11) \qquad E\,[\psi^{i_1,i_2,i_3}(X_{i_1},X_{i_2},X_{i_3})\,|\,X_{i_1} = x_{i_1},\; X_{i_2} = x_{i_2}] = 0 \quad \text{a.e.,}$$

$$(3.6.13) \qquad E\,[\psi^{i_1,i_2,i_3}(X_{i_1},X_{i_2},X_{i_3})\,|\,X_{i_1} = x_{i_1}] = 0 \quad \text{a.e.}$$

It follows from (3.6.10) - (3.6.12) that

$$(3.6.13) \qquad E\,[\,\psi^{i_1}_{i_k,i_\ell}(X_{i_1})\;\psi^{i_1,i_2}_{i_m}\;(X_{i_1},\,X_{i_2})] = 0\,;$$

for any $1 \le i_k \ne i_\ell \ne i_1 \le n$, $1 \le i_m \ne i_1 \ne i_2 \le n$;

$$(3.6.14) \qquad E\,[\,\psi^{i_1,i_2}_{i_r}\;(X_{i_1},\,X_{i_2})\cdot\;\psi^{i_1,i_3}_{i_s}\;(X_{i_1},\,X_{i_3})] = 0\,,$$

for any $1 \le i_r \ne i_1 \ne i_2 \le n$, $1 \le i_s \ne i_1 \ne i_3 \le n$;

$$(3.6.15) \qquad E\,[\,\psi^{i_1}_{i_k,i_\ell}(X_{i_1})\;\;\psi^{i_1,i_2,i_3}\;(X_{i_1},\,X_{i_2},\,X_{i_3})] = 0,$$

for any $1 \le i_k \ne i_\ell \ne i_1 \le n$, $1 \le i_1 \ne i_2 \ne i_3 \le n$;

(3.6.16) $$E\left[\psi_{i_r}^{i_1,i_2}(X_{i_1}, X_{i_2})\ \psi^{i_1',i_2',i_3'}(X_{i_1'}, X_{i_2'}, X_{i_3'})\right] = 0$$

for any $1 \leq i_r \neq i_1 \neq i_2 \leq n$, $1 \leq i_1' \neq i_2' \neq i_3' \leq n$,

whenever $\{i_1, i_2\}$ and $\{i_1', i_2', i_3'\}$ are disjoint ;

(3.6.17) $$E\left[\psi^{i_1,i_2,i_3}(X_{i_1},X_{i_2},X_{i_3})\ \psi^{i_1',i_2',i_3'}(X_{i_1'},X_{i_2'},X_{i_3'})\right] = 0$$

whenever $(i_1, i_2, i_3) \neq (i_1', i_2', i_3')$.

Henceforth, unless otherwise mentioned, we work with U-statistics with kernel ϕ and degree 3. The generalisation to an arbitrary r (>3) is immediate. It is assumed without loss of generality that

(3.6.18) $$E\phi(X_{i_1}, X_{i_2}, \ldots, X_{i_r}) = 0, \quad 1 \leq i_1 \neq \ldots \neq i_r \leq n.$$

We now prove a lemma which gives moment bounds for $V_n^{(2)}$ and $V_n^{(3)}$ when ϕ has uniformly bounded (2m) th moment.

<u>Lemma</u> 3.6.1 If (3.6.18) holds and

(3.6.19) $$\max_{1\leq i_1<i_2<i_3\leq n} E|\phi(X_{i_1}, X_{i_2}, X_{i_3})|^{2m} < \infty$$

then

(3.6.20) $$E(V_n^{(2)})^{2m} \leq n^{-2m} L^m \max_{1\leq i_1<i_2<i_3\leq n} E|\phi(X_{i_1}, X_{i_2}, X_{i_3})|^{2m}$$

(3.6.21) $$E(V_n^{(3)})^{2m} \leq n^{-3m} L^m \max_{1\leq i_1<i_2<i_3\leq n} E|\phi(X_{i_1}, X_{i_2}, X_{i_3})|^{2m}$$

Where in the above and in what follows L (> 0) is a generic constant independent of m and n.

Proof.

(3.6.22) $$E(V_n^{(2)})^{2m} = \binom{n}{2}^{-2m} \sum_{1\leq i_1<j_1\leq n} \cdots \sum_{1\leq i_{2m}<j_{2m}\leq n} E[\bar{\psi}_n^{(2)}(X_{i_1}, X_{j_1}) \cdots \bar{\psi}_n^{(2)}(X_{i_{2m}}, X_{j_{2m}})]$$

Note that if a pair of suffixes (i_k, j_k) occurs exactly once in $(\{i_1, j_1\}, \ldots, \{i_{2m}, j_{2m}\})$, then in view of (3.6.14)

(3.6.23) $$E[\bar{\psi}_n^{(2)}(X_{i_1}, X_{j_1}) \cdots \bar{\psi}_n^{(2)}(X_{i_{2m}}, X_{j_{2m}})] = 0$$

Subject to the condition that each pair of suffixes (i_k, j_k) occurs at least twice, the maximum number of suffixes that can occur is $2m$. Also by repeated application of Holders inequality

(3.6.24) $$|E\, \bar{\psi}_n^{(2)}(X_{i_1}, X_{j_1}) \cdots \bar{\psi}_n^{(2)}(X_{i_{2m}}, X_{j_{2m}})| \leq \prod_{k=1}^{2m} [E^{2m}\, \bar{\psi}_n^{(2)}(X_{i_k}, X_{j_k})]^{1/2m}$$

Now applying Jensens and C_{2m} inequalities one gets

$$(3.6.25) \quad E^{2m}\, \bar{\psi}_n^{(2)}(X_{i_k}, X_{j_k}) \leq L^m \max_{1 \leq i_1 < i_2 < i_3 \leq n} E|\phi(X_{i_1}, X_{i_2}, X_{i_3})|^{2m}$$

Therefore, since the maximum number of suffixes that can occur is $2m$, each ranging from 1 to n, we have from (3.6.22), (3.4.24) and (3.6.25)

$$(3.6.26) \quad E(V_n^{(2)})^{2m} \leq \binom{n}{2}^{-2m} n^{2m} L^m \max_{1 \leq i_1 < i_2 < i_3 \leq n} E|\phi(X_{i_1}, X_{i_2}, X_{i_3})|^{2m}$$

Hence (3.6.20). Similarly (3.6.21) can be shown. The lemma follows

Remark 3.6.1 The method of proof of this lemma is essentially adapted from Funk (1970) and Grams and Serfling (1973). They considered respectively the one sample situation in the iid case and the c sample situation.

NON UNIFORM RATES FOR U-STATISTICS

From the above lemma, writing

$$(3.6.27) \quad n^{1/2}\, U_n/(3\sigma_n) = s_n^{-1} \sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i) + R_n \,, \quad \text{where}$$

$$(3.6.28) \quad s_n^2 = \sum_{i=1}^{n} E\, \bar{\psi}^2(X_i) \,, \quad \sigma_n^2 = n^{-1} s_n^2$$

$$\text{and} \quad R_n = n^{1/2} \sigma_n^{-1} \left(V_n^{(2)} + \tfrac{1}{3} V_n^{(3)}\right)$$

We may have, under $\inf_n \sigma_n^2 > 0$

$$(3.6.29) \quad ER_n^{2m} \leq n^{-m} \sup_{n \geq 3} \max_{1 \leq i_1 < i_2 < i_3 \leq n} E\, \phi^{2m}(X_{i_1}, X_{i_2}, X_{i_3})\, L^m$$

Also note, that from (3.6.2), (3.6.6) using c_δ-inequality and Jensen's inequality for conditional expectation

$$(3.6.30) \quad \sup_{n \geq 3} \max_{1 \leq i_1 < i_2 < i_3 \leq n} E|\phi(X_{i_1}, X_{i_2}, X_{i_3})|^\delta \leq k(\delta)$$

$$\Rightarrow E|\bar{\psi}_n^{(1)}(X_i)|^\delta \leq k(\delta)\, L^\delta, \quad \delta > 0,$$

$$\forall\ i = 1, \ldots, n.$$

Comparing (3.6.29) with (3.2.1), (3.3.1), (3.3.4), (3.3.27) and (3.4.1) we may have analogous theorems for U-statistics depending on the stringency on the assumption on the moments of the kernel ϕ. For example if $(c+2)$ th moments are uniformly bounded for ϕ for some fixed $c > 0$, we may obtain the results of the section 3.2 (letting $u(x) \equiv 1$).

Similarly if

$$(3.6.31) \quad \sup_{n \geq 3} \max_{1 \leq i_1 < i_2 < i_3 \leq n} \phi^{2m}(X_{i_1}, X_{i_2}, X_{i_3}) \leq (2m)!\, L^m, \quad m \geq 1$$

then we may obtain results of the section 3.3. Finally if

$$(3.6.32) \qquad \sup_{n \geq 3} \max_{1 \leq i_1 < i_2 < i_3 \leq n} \phi^{2m}(X_{i_1}, X_{i_2}, X_{i_3}) \leq m!\, L^m , \qquad m \geq 1,$$

then results of the section 3.4 may be obtained.

A slightly weaker uniform bound is associated with the above non uniform bounds ; see e.g. (3.2.15), (3.2.16), (3.3.23), (3.4.6) and (3.4.11). In the following we find out sharpest uniform Berry-Esseen bound for U-statistics in the non iid case. In what follows K denotes a generic constant and

$$\bar{\gamma}_n(\delta) = n^{-1} \sum_{i=1}^{n} E|\bar{\psi}_n^{(1)}(X_i)|^{2+\delta} .$$

<u>Theorem</u> 3.6.1 If

$$(3.6.33) \qquad \sup_{n \geq 3} \max_{1 \leq i_1 < i_2 < i_3 \leq n} E|\phi(X_{i_1}, X_{i_2}, X_{i_3})|^{2+\delta} \leq K < \infty$$

for some $\delta > 0$, and

$$(3.6.34) \qquad \inf_{n \geq 1} \min_{1 \leq i \leq n} E[\bar{\psi}_n^{(1)}(X_i)]^2 > 0,$$

then

$$(3.6.35) \qquad |P(n^{1/2} U_n/(3\sigma_n) \leq x) - \Phi(x)| \leq K n^{-\delta'/2} \bar{\gamma}_n(\delta') \sigma_n^{-(2+\delta')}$$

where $\delta' = \min(\delta, 1)$ and $\Phi(x)$ is the distribution function of a $N(0, 1)$ variable.

Remark 3.6.2 When the X_i's are iid, the conditions (3.6.33) and (3.6.34) simplify. Under these conditions with $\delta = 1$, Callert and Janssen (1978) proved (3.6.35). Our method of proof uses their argument as well as Esseen's classical lemma.

Proof of theorem 3.6.1 First write

(3.6.36) $$n^{1/2} U_n/(3\sigma_n) = s_n^{-1} \sum_{i=1}^{n} \bar{\psi}_n(X_i) + R_n$$

where $R_n = n^{1/2} \sigma_n^{-1} (V_n^{(2)} + \frac{1}{3} V_n^{(3)})$. Write $T_n^{-1} = n^{-\delta'/2} \bar{\gamma}_n(\delta)$. $\sigma_n^{-(2+\delta')}$, using lemma 3.6.1 and Marcov's inequality

(3.6.37) $$P\left(\tfrac{1}{3} n^{1/2} \sigma_n^{-1} |V_n^{(3)}| > T_n^{-1}\right) \leq k\, T_n^2\, n\, \sigma_n^{-2} E(V_n^{(3)})^2$$

$$= O(n^{\delta'-2} \bar{\gamma}_n^{-2}(\delta') \sigma_n^{2(2+\delta')})$$

But since (3.6.33) and (3.6.34) imply that $\sigma_n = O_e(1)$, $\bar{\gamma}_n(\delta') = O_e(1)$ where O_e denotes the exact order, the r.h.s of (3.6.37) can be written as $o(n^{-\delta'/2} \bar{\gamma}_n(\delta') \sigma_n^{-(2+\delta')})$. Further, following Chan and Wierman (1977), decompose $Y_n = n^{1/2} \sigma_n^{-1} V_n^{(2)}$ as

(3.6.38) $$Y_n = \Delta_{n1} + \Delta_{n2} \quad \text{where} \quad \Delta_{n1} = n^{\frac{1}{2}} \sigma_n^{-1} \binom{n}{2}^{-1} \sum\sum_{1 \leq i < j \leq c_n} \bar{\psi}^{(2)}(X_i, X_j)$$

and $$\Delta_{n2} = n^{1/2} \sigma_n^{-1} \binom{n}{2}^{-1} \sum_{j=c_n+1}^{n} \sum_{i=1}^{j-1} \bar{\psi}_n^{(2)}(X_i, X_j), \quad c_n = [n - 3n^{\frac{1}{2}} \log n]$$

the integer part of $(n-3n^{1/2} \log n)$. Now for fixed n, writing $\xi_j = \sum_{i=1}^{j-1} \bar{\psi}_n^{(2)}(X_i, X_j)$ and $\mathbb{F}_j$ the σ-algebra generated by $\xi_1, \xi_2, \ldots \xi_j$ $(j \geq 2)$, if follows from (3.6.10) that $\{\xi_j, \mathbb{F}_j ; j \geq 2\}$ forms a martingale sequence. Also, writing $W_k = \sum_{j=2}^{k} \xi_j$ $(k \geq 2)$ if follows that $E(W_{k+1} \mid W_2, \ldots, W_k) = W_k$ a.e. for all $k \geq 2$. Hence repeated application of martingale inequality of Dharmadhikari, Fabian and Jogdeo (1968) gives

$$(3.6.39) \quad E|n^{-1/2}\binom{n}{2} \sigma_n \Delta_{n2}|^{2+\delta} \leq K(n-c_n)^{1+\delta/2} (n-c_n)^{-1}$$

$$\sum_{j=c_n+1}^{n} (j-1)^{1+\delta/2} (j-1)^{-1} \sum_{i=1}^{j-1} E| \bar{\psi}_n^{(2)} (X_i X_j)|^{2+\delta}$$

using c_δ and Jensens inequalities for conditional expectations one gets

$$(3.6.40) \quad E | \bar{\psi}_n^{(2)}(X_i, X_j) |^{2+\delta} \leq K .$$

Hence from (3.6.39), (3.6.40) and Markov's inequality,

$$(3.6.41) \quad P(|\Delta_{r2}| > T_n^{-1}) \leq T_n^{2+\delta} E |\Delta_{r2}|^{2+\delta}$$

$$\leq K T_n^{2+\delta} n^{-2(2+\delta)} n^{1+\delta/2} (n-c_n)^{1+\delta/2} n^{1+\delta/2}$$

$$\leq K n^{\delta'(2+\delta)/2} n^{-2-\delta} (n^{1/2} \log n)^{1+\delta/2}$$

$$\leq K n^{-1/2-\delta/4} (\log n)^{1+\delta/2} = o(T_n^{-1}).$$

In view of (3.6.36), (3.6.37) and (3.6.41) it suffices to show that

(3.6.42) $$\left| P\left(s_n^{-1} \sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i) + \Delta_{n1} \leq x \right) - \Phi(x) \right|$$

$$\leq K \bar{\gamma}_n(\delta') \, \sigma_n^{-(2+\delta')} \, n^{-\delta'/2}$$

But using Esseen's lemma and the notation $\eta_z(t)$ for the characteristic function of a random variable z, one gets

(3.6.43) l.h.s of (3.6. 42)

$$\leq \int_{-T_n}^{T_n} \left| \eta_{s_n^{-1} \sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i) + \Delta_{n1}}(t) - \exp(-t^2/2) \right| |t|^{-1} dt + K T_n^{-1}$$

Using a result of Esseen (1945),

(3.6.44) $$\int_{-T_n}^{T_n} \left| \eta_{s_n^{-1} \sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i)}(t) - \exp(-t^2/2) \right| |t|^{-1} dt \leq K T_n^{-1}$$

Thus it remains to show that

(3.6.45) $$\int_{0}^{T_n} \left| \eta_{s_n^{-1} \sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i)}(t) - \eta_{s^{-1} \sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i) + \Delta_{n1}}(t) \right| t^{-1} dt \leq K T_n^{-1}$$

as the other integral, namely the one over the range $(-T_n, 0)$

can be handled similarly. Now choosing ε (> 0) such that $T_n^{1/2} > \varepsilon$ for $n \geq n_o$ (say), proceed as in Chan and Wierman (1977) to obtain the inequality

$$(3.6.46) \quad \int_0^{\varepsilon T_n^{1/2}} \left| \eta_{s_n^{-1} \sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i)}(t) - \eta_{s_n^{-1} \sum_{i=1}^{n} \bar{\psi}_n^{(1)}(X_i)+\Delta_{n1}}(t) \right| t^{-1} dt$$

$$\leq K \int_0^{\varepsilon T_n^{1/2}} t^{-1} \left[\left| E\left(\exp(its_n^{-1} \sum_{j=1}^{n} \bar{\psi}_n^{(1)}(X_j))\Delta_{n1}\right) \right| + \frac{1}{2}t^2 E\Delta_{n1}^2 \right] dt$$

Now

$$(3.6.47) \quad E\left[\exp\left(its_n^{-1} \sum_{j=1}^{n} \bar{\psi}_n^{(1)}(X_j)\right) \Delta_{n1}\right]$$

$$\leq n^{1/2} \sigma_n^{-1} \binom{n}{2}^{-1} \sum_{i \leq j < j' \leq c_n} \sum \left\{ \prod_{k \neq j \neq j'} \eta_{\bar{\psi}_n^{(1)}(X_k)}(t\, s_n^{-1}) \right\}$$

$$\times E\left[\exp(its_n^{-1}(\bar{\psi}_n^{(1)}(X_j) + \bar{\psi}_n^{(1)}(X_{j'}))\ \bar{\psi}_n^{(2)}(X_j, X_{j'})\right]$$

But for $t \leq T_n$, $ts_n^{-1} \leq \varepsilon\, n^{\delta'/2} \sigma_n^{2+\delta'} \bar{\gamma}_n^{-1}(\delta') \leq K \varepsilon n^{(\delta'-1)/2} \leq K\varepsilon$

so,

$$(3.6.48) \quad \prod_{k \neq j \neq j'} \eta_{\bar{\psi}_n^{(1)}(X_k)}(t\, s_n^{-1}) \leq \prod_{k \neq j \neq j'} \exp\left(-\frac{1}{2}t^2 s_n^{-2} E(\bar{\psi}_n^{(1)}(X_k))^2\right.$$

$$= \exp\left(-\frac{1}{2}t^2\right) \exp\left[\frac{1}{2}t^2 s_n^{-2}\left\{E\left[\psi_n^{(1)}(X_j)\right]^2 + E\left[\bar{\psi}_n^{(1)}(X_{j'})\right]^2\right\}\right]$$

$$\leq K \exp\left(-\tfrac{1}{2} t^2\right)$$

Also from (3.6.14), (3.6.33) and (3.6.34),

(3.6.49) $$E\Delta_{n1}^2 \leq n\,\sigma_n^{-2} \binom{n}{2}^{-2} \sum_{1\leq i<j\leq c_n}\sum E\left[\bar{\psi}_n^{(2)}(X_i, X_j)\right]^2$$

$$= O(n^{-1})$$

Now, proceeding as Callert and Janssen (1978), it follows from (3.6.46) - (3.6.49) that

(3.6.50) l.h.s. of (3.6.46)

$$\leq Kn^{1/2}\sigma_n^{-1}\, s_n^{-2} \binom{n}{2}^{-1} \sum_{1\leq i<j\leq c_n}\sum E\left|\left[\bar{\psi}_n^{(1)}(X_i)\,\bar{\psi}_n^{(1)}(X_j)\bar{\psi}_n^{(2)}(X_i, X_j)\right|\right.$$

$$+ O(T_n\, n^{-1})$$

$$\leq Kn^{-1/2}\sigma_n^{-3}\,\bar{\gamma}_n(\delta) + O(n^{\delta'/2\,-1}\;\bar{\gamma}_n^{-1}(\delta')\;\sigma_n^{2+\delta'})$$

$$= O\,(T_n^{-1})$$

Finally rêpeating the steps of Callert and Janssen (1978) it can be shown that

(3.6.51) $$\int_{\varepsilon T_n^{1/2}}^{T_n} \left|\eta_{s_n^{-1}\sum_{i=1}^{n}\bar{\psi}_n^{(1)}(X_i)}(t) - \eta_{s_n^{-1}\sum_{i=1}^{n}\bar{\psi}_n^{(1)}(X_i)+\Delta_{n1}}(t)\right| |t|^{-1}dt$$

$$\leq K\,T_n^{-1}$$

The proof of theorem 3.6.1 is complete.

In the speical case when ϕ has degree 2 write

$$\psi_j^i(x_i) = \phi_j^i(x_i) = E[\phi(X_i, X_j) | X_i = x_i],$$

$$\psi_2(x_i, x_j) = \phi(x_i, x_j) - \psi_j^i(x_i) - \psi_i^j(x_j), \quad 1 \leq i \neq j \leq n.$$ Also

let $\bar{\psi}_n(x_i) = (n-1)^{-1} \sum_{\substack{j=1 \\ (\neq i)}}^{n} \psi_j^i(x_i)$, $i = 1, \ldots, n$. Then U_n has the representation

$$U_n = (2/n) \sum_{i=1}^{n} \bar{\psi}_n(x_i) + R_n^* .$$

where $R_n^* = \binom{n}{2}^{-1} \sum\sum_{1 \leq i < j \leq n} \psi_2(x_i, x_j)$. If $\mathbb{F}_n$ denotes the σ-algebra generated by $X_1, \ldots, X_n$, then it can be easily verified that $\{R_n^*, \mathbb{F}_n ; n \geq 2\}$ is a martingale sequence. Using a result of Mcleish (1974) and following the lines of Miller and Sen (1972), one can establish a weak invariance principle for U-statistics.

3.7 RATES OF CONVERGENCE IN FINITE POPULATION

Consider a finite population of N units $A = \{1,2,\ldots, N\}$ and let 'a' be a subset of A. 'a' may be considered as outcomes of a simple random sampling from the finite population A.

Denoting an a consisting of K elements by a_K and probability of a_K by $P(a_K)$ we have simple random sampling of size n defined by the following probabilities

$$(3.7.1) \qquad P(a_K) = \begin{cases} \binom{N}{n}^{-1} & \text{if } K = n \\ 0 & \text{otherwise,} \end{cases}$$

on the other hand poisson sampling of mean size n is defined as follows

$$(3.7.2) \qquad P(a_K) = \left(\frac{n}{N}\right)^K \left(1 - \frac{n}{N}\right)^{N-K}$$

Let $y_1 \ldots y_N$ be a sequence of reals and let

$$(3.7.3) \qquad \xi = \sum_{i \varepsilon a_n} y_i$$

where a_n is the outcome of a simple random sampling of size n. Then the random variable ξ has the finite mean

$$(3.7.4) \qquad E\,\xi = \frac{n}{N} \sum_{i=1}^{N} y_i$$

and variance

$$(3.7.5) \qquad V(\xi) = \frac{n}{N} \cdot \frac{N-n}{N-1} \sum_{i=1}^{N} (y_i - \overline{Y})^2 \quad , \quad \overline{Y} = \frac{1}{N} \sum_{i=1}^{N} y_i$$

In view of the relation

$$(3.7.6) \qquad \left(\frac{n}{N}\right)^K \left(1 - \frac{n}{N}\right)^{N-K} = \left[\binom{N}{K} \left(\frac{n}{N}\right)^K \left(1 - \frac{n}{N}\right)^{N-K} \right] \binom{N}{K}^{-1}$$

observing that poisson sampling may be interpreted as simple random sampling of size K, where K is a binomial random variable attaining the value K with probability $\binom{N}{K} \left(\frac{n}{N}\right)^K \left(1-\frac{n}{N}\right)^{N-K}$ Hájek (1960) obtained necessary and sufficient condition for asymptotic normality of ξ and the conditions obtained agree with those [redacted] derived by Erdös and Rényi (1959). Hájek, however, considers convergence to other limiting distributions also.

Following Hájek (1960) we shall split $\eta = \xi - E\,\xi = \sum_{i \varepsilon a_n} (y_i - \overline{Y})$ into two parts, viz.,

$$(3.7.7) \qquad \eta = \eta^* + (\eta - \eta^*) \quad \text{where} \quad \eta^* = \sum_{i \varepsilon a_K} (y_i - \overline{Y}) = \sum_{i=1}^{N} \zeta_i$$

where

$$(3.7.8) \qquad \zeta_i = (y_i - \overline{Y})\, I\,(a_K \ni i)$$

Note that conditional on $K = k$, ζ_i's are iid r.v 's

Now,

$$(3.7.9) \qquad \eta - \eta^* = \begin{cases} 0 & \text{if } K = n \\ \sum_{i \varepsilon a_n - a_K} (y_i - \overline{Y}) & \text{if } K < n \\ \sum_{i \varepsilon a_K - a_n} (y_i - \overline{Y}) & \text{if } K > n \end{cases}$$

and can be treated as remainder. Note that we have adopted the same notation as that of Hájek except we use a_n, a_K instead of s_n, s_K of Hájek to avoid canfusion of these notation of earlier chapters.

Now note that

$$(3.7.10) \qquad E\left\{(\eta-\eta^*)^{2m} \mid K=k\right\} = E\left[\left\{\sum_{i=1}^{\ell} (y_i-\overline{Y})\right\}^{2m} \mid K=k\right]$$

where $\ell = |k-n|$.

Expanding, we have from (3.7.10)

$$(2.7.11) \qquad E\left\{(\eta-\eta^*)^{2m} \mid K=k\right\} = E\left\{\sum_{i_1} \dots \sum_{i_{2m}} z_{i_1} \dots z_{i_{2m}}\right\},$$

$$z = y - \overline{Y} .$$

Note that if any one of the suffixes $i_1 \dots i_{2m}$ occurs only once then the expectation vanishes. Hence each suffix should occur at least twice to make a non-zero contribution.

Subject to the condition that each suffix occurs at least twice the maximum number of suffixes that can occur is m. Also applying Holders inequality 2m times

$$E\,|z_{i_1} \cdots z_{i_{2m}}| \leq \prod_{j=1}^{2m} (E\, z_{i_j}^{2m})^{1/2m}$$

Therefore

$$(3.7.12) \qquad E\,[(\eta-\eta^*)^{2m} / K=k] \leq \ell^m E\, z_{i_1}^{2m} = \ell^m E(y-\overline{Y})^{2m}$$

where y is a random variable taking values $y_1,\ldots,y_N$ each with probability $1/N$. From (3.7.12) we finally have

$$(3.7.13) \qquad E(\eta-\eta^*)^{2m} \leq E(y-\overline{Y})^{2m}\, E|K-n|^m$$

Since K is the sum of N iid Barnoulli r.v's using theorems 2.6.3 and 2.6.5 we have

$$(3.7.14) \qquad E\exp(r\,X^2) < \infty \text{ for some } r > 0, \text{ where } X = \frac{K-n}{\sqrt{var(K)}}$$

if $\frac{n}{N} = \frac{1}{2}$ (so that odd order moments of X are zeroes).

and

$$(3.7.15) \qquad E\exp(|X|^{2-\varepsilon}) < \infty \text{ if } \frac{n}{N} \neq \frac{1}{2}, \text{ for any } \varepsilon > 0.$$

Now for any r.v' X, we shall show the following

<u>Lemma</u> 3.7.1

(3.7.16) $$E \exp(s|X|^{1/\upsilon}) < \infty \quad \text{for some} \quad s > 0$$

$$\Rightarrow \quad E|X|^m \leq L^m m^{m\upsilon} \quad \text{for some} \quad L > 0$$

where $\upsilon > 0$, $m = 1,2,\ldots$,

and

(3.7.17) $$E \exp \{\log(1+|X|)\}^{\upsilon/(\upsilon-1)} < \infty \Rightarrow E|X|^m \leq L^m e^{m^\upsilon}$$

for some $L > 0$, where $\upsilon > 1$, $m=1,2,\ldots$

<u>Proof</u>: For (3.7.16), note that

$$\exp(s|X|^{1/\upsilon}) \geq \frac{|X|^{p/\upsilon}}{p!\ s^p} \qquad p = 1,2,\ldots$$

Now $$\frac{|X|^{p/\upsilon}}{p!\ s^p} \geq \frac{|X|^m}{p!\,s^p}$$ if $p/\upsilon \geq m$, e.g., if

$$p = [\upsilon m] + 1$$

Therefore, with this choice of p

(3.7.18) $$E|X|^m = E|X|^m I(|X| \leq 1) + E|X|^m I(|X| > 1)$$

$$\leq 1 + ([\upsilon m]+1)!\ s^{[\upsilon m]+1}\ E \exp(s|X|^{1/\upsilon})$$

Now $E \exp(s|X|^{1/\upsilon}) < \infty$ and using Stirlings approximation for factorials, $([\upsilon m]+1)! = ([\upsilon m]+1)([\upsilon m])! \leq L^m m^{m\upsilon}$

for some $L > 0$.

And hence from (3.7.18)

$$E|X|^m \leq L^m m^{m\upsilon} \quad \text{for some } L > 0.$$

For (3.7.17), also note

(3.7.19) $$\exp\,(\log(1+|X|))^{\upsilon/(\upsilon-1)} = (1+|X|)^{\{\log(1+|X|)\}^{1/(\upsilon-1)}}$$

Now

(3.7.20) $$E|X|^m < E(1+|X|)^m = E(1+|X|)^m\, I(\log^{1/(\upsilon-1)}(1+|X|) \geq m)$$

$$+\; E(1+|X|)^m\, I(\log^{1/(\upsilon-1)}(1+|X|) < m)$$

The first term of the r.h.s of (3.7.20) is finite from l.h.s of (3.7.17) and from (3.7.19). Again note that

$$\log^{1/(\upsilon-1)}(1+|X|) < m \Rightarrow \log(1+|X|) < m^{\upsilon-1} \Rightarrow m\log(1+|X|) < m^{\upsilon}$$

$$\Rightarrow (1+|X|)^m < e^{m^{\upsilon}}.$$

Hence the second term of the r.h.s of (3.7.20) is bounded above by $\exp(m^{\upsilon})$ and therefore (3.7.17) follows.

Throughout this chapter we shall assume $n/N \to \lambda$, $(0 < \lambda < 1)$ of which $n/N = 1/2$ is a particular case.

Since $var(K) = 0_e(n)$ where 0_e denotes the exact order. We have

(3.7.21) $$E|K-n|^m \leq n^{m/2} m^{(1+\varepsilon)m/2} L^m \quad \text{under (3.7.15)}$$

$$\leq n^{m/2} m^{m/2} L^m \quad \text{under (3.7.14)}$$

Hence a normalised version of $\sum_{i\varepsilon a_n} y_i$ (call it T_n) can be written in the following form, see e.g. (3.7.7).

(3.7.22) $$T_n = \eta^*/\sqrt{var(\eta^*)} + (\eta-\eta^*)/\sqrt{var(\eta^*)}$$

$$= [var(\eta^*)]^{-1/2} \sum_{i=1}^{N} \xi_i + R_n ,$$

$$R_n = (\eta-\eta^*)/\sqrt{var(\eta^*)}$$

where the 1st component is standardised sum of N independent random variables, see (3.7.8) , and 2nd component is remainder; under the assumption

(3.7.23) $$\inf_n \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{Y})^2 > 0$$

We have

(3.7.24) $$var(\eta^*) = n(1-\frac{n}{N}) \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{Y})^2$$

$$= 0_e(n)$$

Hence, from definition of R_n in (3.7.22), (3.7.24) and (3.7.13), we have,

$$(3.7.25) \qquad E\, R_n^{2m} \le n^{-m/2}\, m^{(1+\varepsilon)m/2}\, L^m\, E(y-\bar{Y})^{2m} \qquad \text{under } (3.7.15)$$

and

$$(3.7.26) \qquad E\, R_n^{2m} \le n^{-m/2}\, m^{m/2}\, L^m\, E(y-\bar{Y})^{2m} \qquad \text{under } (3.7.14)$$

where

$$(3.7.27) \qquad E(y-\bar{Y})^{2m} = \frac{1}{N} \sum_{i=1}^{N} (y_i-\bar{Y})^{2m}$$

under the assumption that

$$(3.7.28) \qquad d^* \le \max_{1 \le i \le N} |y_i-\bar{Y}| \le d^*$$

i.e., the values of $y-\bar{Y}$ are bounded above and below .

We have from (3.7.25) - (3.7.28)

$$(3.7.29) \qquad E\, R_n^{2m} \le n^{-m/2}\, m^{(1+\varepsilon)m/2}\, L^m \qquad \text{under } (3.7.15), \text{ for some } L > 0,$$

and

$$(3.7.30) \qquad E\, R_n^{2m} \le n^{-m/2}\, m^{m/2}\, L^m \qquad \text{under } (3.7.14), \text{ for some } L > 0.$$

To have an estimate of $P(|R_n| > \lambda_n(t))$, note that if (3.7.30) is true then

$$(3.7.31) \qquad P(|R_n| > \lambda_n(t)) \le [\lambda_n(t)]^{-2m}\, ER_n^{2m}$$

$$\leq \quad [\lambda_n(t)]^{-2m}\, n^{-m/2}\, m^{m/2}\, L^m$$

$$\leq \exp\left[-2m \log \lambda_n(t) - \frac{m}{2}\log n + \frac{m}{2}\log m + m \log L\right]$$

Differenciating the quantity appearing in exponent w.r.t. m and equating it to zero, we have

(3.7.32) $$-2 \log \lambda_n(t) - \frac{1}{2} \log n + \frac{1}{2} \log m + \frac{1}{2} + \log L = 0$$

or

(3.7.33) $$m = \exp\left[4 \log \lambda_n(t) + \log n - 1 - 2 \log L\right]$$

$$= \lambda_n^4 (t)\, n\, e^{-1}\, L^{-2}$$

Hence from (3.7.31) and (3.7.32)

(3.7.34) $$P(|R_n| > \lambda_n(t)) \leq \exp(-m/2) = \exp\left[-\lambda_n^4(t) n L\right], \text{ from } (3.7.33)$$

for some $L > 0$.

In the above we conveniently ignore that m may not be an integer.

Similarly if (3.7.29) holds, then

(3.7.35) $$P(|R_n| > \lambda_n(t)) \leq \exp\left[-(\lambda_n^2(t)\, n^{1/2})^{2-\varepsilon}\, L\right]$$

To obtain normal approximation zone we recall (3.3.2). Note that, in view of (3.7.28), f_i's defined in (3.7.8) are bounded. Therefore, taking $g(x) = \exp(|x|)$ and $\lambda_n(t) = n^{-\upsilon}$

(the notation is $a_n(t)$ in section 3.2), we may obtain in place of (3.3.8)-(3.3.10) the followings

(3.7.36) 1st term in r.h.s. of (3.3.2)

$$\leq b\ |t|^{-1} \exp(-t^2/2)|\exp(O(|t|^3\ n^{-1/2}))-1|$$

for $|t| = O(n^{\upsilon})$

(The term $\sum_{i=1}^{N} P(|\xi_i| > r\ n^{1/2}|t \pm n^{-\upsilon}|)$ is absent since ξ_i's are bounded, see remark 2.4.5)

(3.7.37) 2nd term in r.h.s. of (3.3.2) $\leq b\ n^{-\upsilon} \exp(-t^2/2)$

and

(3.7.38) 3rd term in r.h.s. of (3.3.2)

$$\leq \exp(-L\ n^{1-4\upsilon}) \quad \text{if} \quad n/N = 1/2$$

$$\leq \exp(-L\ n^{(1-4\upsilon)(1-\varepsilon)}) \quad \text{if} \quad n/N \to \lambda, \quad 0 < \lambda < 1$$

for some $L > 0$,

from (3.7.34)-(3.7.35) taking $\lambda_n(t) = n^{-\upsilon}$, $\upsilon > 0$.

Letting $\upsilon = 1/6$ we may obtain the following theorem parallel to theorem 3.3.1

<u>Theorem</u> 3.7.1 Let T_n defined in (3.7.22) be a standardised sample sum from a finite population of size N. Let (3.7.23) and (3.7.28) hold. Then for a sequence $\{t_n\}$ satisfying

$$t_n = o(n^{1/6}) \quad \text{if} \quad n/N = 1/2$$
$$= O(n^{\frac{1}{6}-\varepsilon}) \quad \text{if} \quad n/N \to \lambda,\ 0 < \lambda < 1, \quad \varepsilon > 0$$

arbitrarily small

the following holds

$$1 - P(T_n \leq t_n) \sim \Phi(-t_n) \quad , \quad t_n \to \infty,$$

With a different choice of $\lambda_n(t)$ viz. $\lambda_n(t) = \alpha t$, $0 < \alpha < 1$, $t > 0$, for T_n defined in (3.7.22) we may have (3.3.12) and (3.3.13) in the following form

(3.7.39) 1st term in r.h.s. of (3.3.2)

$$\leq b \exp\left[- \frac{t^2}{2} (1-\alpha)^2 + K n^{-1/2} t^3 \right]$$

(The term $\sum_{i=1}^{N} P(|\xi_i| > r(1\pm\alpha) t n^{1/2})$ is absent since ξ_i's are bounded, see remark 2.4.5)

(3.7.40) 2nd term in r.h.s of (3.3.2)

$$\leq b \exp(-(1-\alpha)^2 t^2/2) \alpha t$$

However (3.3.14) changes to , in view of (3.7.34) and (3.7.35),

(3.7.41) 3rd term in r.h.s. of (3.3.2)

$$\leq b \exp(-\alpha^4 L t^4 n) \quad \text{if} \quad n/N = 1/2$$
$$\leq b \exp(-\alpha^{4-\varepsilon} L t^{4-\varepsilon} n^{1-\varepsilon}) \quad \text{if } n/N \to \lambda,\ 0 < \lambda < 1.$$

Subsequently, following two theorems follow along the lines of theorems 3.3.2 and 3.3.3.

Theorem 3.7.2 Under (3.7.23) and (3.7.28), for T_n defined in (3.7.22);

(3.7.42) $$P(T_n > t_n) \leq b \exp\left[-\frac{t_n^2}{2}(1+o(1))\right]$$

$$\text{for } t_n = o(n^{1/2}), \quad t_n \to \infty$$

Remark 3.7.1 Unlike theorem 3.3.2 where $t_n = o(n^{1/2}(\log n)^{-h}$ with $R_n = 0_p(n^{-1/2}(\log n)^{h/2})$; in (3.7.42) $t_n = o(n^{1/2})$ although $R_n = 0_p(n^{-1/4})$ because of sharper order of t in (3.7.41).

The following theorem deals excessive deviations of the type $an^{1/2}$, $a > 0$ and the proof is similar to that of theorem 3.3.3.

Theorem 3.7.3 Under the assumptions of theorem 3.7.2, for $t_n = \varepsilon' n^{1/2}$, $\varepsilon' > 0$

(3.7.43) $$\limsup_{\varepsilon' \to 0} \limsup_{n \to \infty} (t_n^2/2)^{-1} \log P(T_n > t_n) \leq -1$$

Next we obtain moment type convergences and non uniform L_p version of the Berry-Esseen theorem. W.o.l.g. let $t > 0$. For $t^2 \leq K \log n$ proceeding like (3.4.4) (where notation $a_n(t)$ is used instead of $\lambda_n(t)$) with $\lambda_n(t) = \lambda'(n^{-1} t^2 \log n)^{1/4}$, for some $\lambda' > 0$, if $n/N = 1/2$ and using (3.7.34) instead of (3.4.3) we obtain

$$(3.7.44)\quad |P(T_n \le t)-\Phi(t)| \le b\, n^{-1/2}\, t^2 \exp(-t^2/2)$$

$$+\, bn^{-1/4}(\log n)^{1/4}\, t^{1/2} \exp(-t^2/2) + b\, n^{-1/2} \exp(-ct^2)$$

for some $c > 0$

For $t^2 \ge K \log n$, with the same choice of $\lambda_n(t)$ and proceeding like (3.4.10), we obtain

$$(3.7.45)\quad |P(T_n \le t)-\Phi(t)| \le b\, n^{-1/2}\, t^{-2} \exp(t^2/f(t))$$

$$+\, b\, n^{-1/4}(\log n)^{1/4} \exp(-t^2/2) + b\, n^{-1/2} \exp(-t^2/2)$$

where $f : (0, \infty) \to (0, \infty)$ nondecreasing with $\lim_{t \to \infty} f(t) = \infty$.

Also note that proceeding like (3.5.19a) with $\lambda_n = \lambda'(n^{-1} \log n)^{1/4}$ we may obtain an uniform bound $O(n^{-1/4}(\log n)^{1/4})$

For $n/N \to \lambda$, $0 < \lambda < 1$, (3.7.44) and (3.7.45) hold with 2nd term of r.h.s. of the above equations changed to $b\, n^{-1/4+\varepsilon}\, .\, t^{1/2+\varepsilon} \exp(-t^2/2)$, letting $\lambda_n(t) = \lambda'(n^{-1}\, t^{2+\varepsilon})^{1/4}$. Also the uniform bound changes to $O(n^{-1/4+\varepsilon})$, letting $\lambda_n = \lambda'\, n^{-1/4}$. Hence we have the following .

<u>Theorem</u> 3.7.4 Let (3.7.23) and (3.7.28) hold, Then $\exists$ a constant b (> 0) depending on $f(t)$ $f : (0,\infty) \to (0,\infty)$, f nondecreasing with $\lim_{t \to \infty} f(t) = \infty$, such that for all real t

$$(3.7.46) \quad |P(T_n \leq t) - \Phi(t)| \leq b\, n^{-1/4}(\log n)^{1/4} \exp(-t^2/f(t)) \quad \text{if } n/N = 1/2$$

$$\leq b(\varepsilon)\, n^{-1/4+\varepsilon} \exp(-t^2/f(t)) \quad \text{if } n/N \to \lambda,\ 0 < \lambda < 1$$

where $\varepsilon > 0$ can be made arbitrarily small.

Subsequently, from the above theorem we may obtain analogous theorems of theorems 3.4.5 and 3.4.6 with the order of (3.4.13) and (3.4.14) changed to $O(n^{-1/4}(\log n)^{1/4})$ or $O(n^{-1/4+\varepsilon})$ $\varepsilon > 0$ arbitrarily small, depending on whether $n/N = 1/2$ or $n/N \to \lambda$, $0 < \lambda < 1$.

As a concluding remark of this section we may note that the condition (3.7.28) may be relaxed. Still we may use (3.7.25) and (3.7.26) to have an estimate of $P(|R_n| > \lambda_n(t))$. As for example under the assumption $\sup_N E|y-\bar{Y}|^{2+c} < \infty$ for some $c > 0$, from (3.7.25) and (3.7.26) we may have $E\, R_n^{2m} = O(n^{-m/2})$ where $c < 2m < c+2$ and results analogous to those of section 3.2 may be obtained by the same technique used in that section. Similarly if $\sup_N E|y-\bar{Y}|^{2+c} < \infty$ for all $c > 0$, then setting the bound for $\sup_N E(y-\bar{Y})^{2m}$ in (3.7.25) and (3.7.26) and following the technique used in (3.7.31)-(3.7.34) we may obtain estimate of $P(|R_n| > \lambda_n(t))$ and from there analogous results of section 3.3 may be obtained. Similarly we may obtain analogous results of section 3.4 even with out assuming (3.7.28). Since the basic technique remains the same, we omit details.

CHAPTER 4.

RATES OF CONVERGENCE FOR m-DEPENDENT PROCESS

4.1 Inteoduction Results of Chapter 2 are generalised for strictly stationary m-dependent process under similar assumptions. Let $\{X_n, n \geq 1\}$ be a stationary m-dependent process with

$$(4.1.1) \quad E X_1 = 0 , \quad EX_1^2 + 2 \sum_{i=1}^{m-1} EX_1 X_{1+i} = 1 \quad \text{and} \quad Eg(X_1) < \infty$$

where in the 2nd section of this chapter we shall consider

$$(4.1.2) \qquad g(x) = |x|^{2+c} u(x) , \quad c \geq 0$$

and u satisfying conditions mentioned in (2.1.1).

In the 3rd section we shall consider a higher spectrum of g viz. (2.3.1)

The technique used can be briefly described as follows. The partial sum $(X_1 + \ldots + X_n)$ will be divided into two types of blocks. The long blocks can be treated with the same procedure as for iid random variables whereas the contribution from the short blocks can be shown to be sufficiently negligible.

To be specific we consider the following blocking procedure which will be followed throughout this chapter.

For $p \geq m$ with $k = [n/(p+m)]$ where $[x]$ denotes the integer part of x and $\ell' = n-k(p+m)$ if $n > k(p+m)$ define

$$\xi_i = \sum_{j=1}^{p} X_{(i-1)(p+m)+j}$$

(4.1.3)
$$\eta_i = \sum_{j=1}^{m} X_{ip+(i-1)m+j} \qquad i = 1, 2, \ldots, k$$

$$\xi_{k+1} = \sum_{j=1}^{\ell'} X_{k(p+m)+j} \quad \text{or } 0 \text{ according as } \ell' \geq 1 \text{ or not.}$$

Now write ξ_1 in the following form

(4.1.4)
$$\begin{aligned}\xi_1 = X_1 &+ X_{m+1} + X_{2m+1} + \ldots \\ + X_2 &+ X_{m+2} + X_{2m+2} + \ldots \\ + \ldots &+ \ldots + \ldots \\ + X_m &+ X_{2m} + \ldots \\ = z_1 &+ z_2 + \ldots + z_m\end{aligned}$$

Where each z_i is a sum of $[p/m]$ on $[p/m] + 1$ independent components. Let $g_1(x) = g(x/m)$. Then

(4.1.5)
$$Eg_1(p^{-1/2}\xi_1) = Eg\Big(\frac{1}{m}p^{-1/2}\sum_{i=1}^{m} z_i\Big)$$

$$\leq Eg\big(p^{-1/2}\sup_{1\leq i\leq m}|z_i|\big) \quad \because g(x) \text{ is non-decreasing in } |x|$$

$$\leq E\sum_{i=1}^{m} g(p^{-1/2}|z_i|) = \sum_{i=1}^{m} Eg(p^{-1/2}|z_i|)$$

$$= \sum_{i=1}^{m} Eg(p^{-1/2}z_i), \quad \because g \text{ is an even function.}$$

$< \infty$ uniformly in p, from theorems 2.2.4, 2.4.11 under appropriate assumptions.

Also note that for a triangular array Y_{ni}, $1 \leq i \leq n$, $n \geq 1$.

with independent components in an array with zero mean, following the same proof as those of theorems 2.2.4, 2.4.11 and 2.6.3 under appropriate assumptions on g, for a sequence a_n with $a_n \to \infty$ as $n \to \infty$

$$(4.1.6) \quad Eg(s_n^{-1} S_n)\, I(s_n^{-1} S_n > a_n) = Eg(T)\, I(T > a_n) + o(1)$$
$$= o(1) \quad , T = N(0,1).$$

(Note that the function $h(x) = g(x)\, I(x>a_n)$ is not differentiable at $x = a_n$. In case a_n is a point of continuity of F_n we can directly use theorems 2.2.4 and 2.4.11. Otherwise noting that since a distribution function may have at most countably many discontinuities the set of all points of discontinuity of F_n $n \geq 1$ is countable. Now select a point a'_n, where F_n, $n \geq 1$ are continuous, from $(a_n - \epsilon , a_n)$, such that $a_n - \epsilon = 0_e(a_n)$ and then use theorems 2.2.4 and 2.2.11 with $h(x) = g(x)\, I(x > a'_n)$.)

Similarly we can show

$$(4.1.7) \quad Eg_1(p^{-1/2} \xi_1)\, I(p^{-1/2} \xi_1 > a_n)$$
$$= Eg(\tfrac{1}{m}\, p^{-1/2} \sum_{i=1}^{m} z_i)\, I\, (p^{-1/2} \sum_{i=1}^{m} z_i > a_n)$$
$$\leq Eg(p^{-1/2} \sup_i |z_i|\, I(p^{-1/2} \sup_i |z_i| > a_n/m)$$

$$\leq E \sum_{i=1}^{m} g(p^{-1/2}|z_i|)\ I(p^{-1/2}|z_i| > a_n/m)$$

$$= \sum_{i=1}^{m} Eg(p^{-1/2}\ z_i)\ I(p^{-1/2}\ |z_i| > a_n/m\ )$$

$$= o(1) \qquad \text{as} \quad n \to \infty.$$

These facts are to be used repeatedly later.

4.2 NON UNIFORM RATES FOR $g(x) = |x|^{2+c}u(x)$; $c \geq 0$

We shall consider $c > 0$ and $c = 0$ separately. Let us first consider the case $c = 0$. The following theorem states non-uniform rates of convergence of $|F_n(t)-\Phi(t)|$ to zero around a nbhd of origin.

<u>Theorem</u> 4.2.1 Let $\{X_n\}$ be a stationary m-dependent process satisfying (4.1.1) and (4.1.2) with $c = 0$. Then for $1 \leq t^2 \leq \lambda \log u(\sqrt{n})$ there exist constants b, b_1, $r > 0$ depending on u and λ such that

(4.2.1) $$|F_n(t)-\Phi(t)| \leq b\ w \exp\left[-\tfrac{1}{2}t^2(1-3r)\right] + o(g_1(t\sqrt{n/k}\ f(n)))^{-1} + o((k+1)(g_1(r\sqrt{k}\ t))^{-1}) + b_1\ f(n)\exp(-t^2/2)$$

where $w = (u(r\, k^{1/2}\, t))^{-1}$ and $f(n)$ is any +ve sequence converging to zero such that $t\sqrt{n/k}\, f(n) \to \infty$.

(For $t^2 \leq 1$ we may use uniform bound $O(f(n) \vee (g_1(\sqrt{n/k}\, f(n))^{-1} \vee ((k+1)(g_1(\sqrt{k}\,)))^{-1}$ as in theorem 3.5.2, see (3.5.19a))

Proof. For $t_n = t \pm f(n)$ we shall complete the proof by showing the following

$$(4.2.2) \qquad |\Phi(t_n) - \Phi(t)| \leq b_1 f(n) \exp(-t^2/2)$$

$$(4.2.3) \qquad P(\,|\sum_{i=1}^{k} \eta_i| > t\, n^{1/2} f(n)) = o(g_1(t\sqrt{n/k}\, f(n)))^{-1}$$

$$(4.2.4) \qquad |P(\sum_{1}^{k+1} \xi_i > t_n\, n^{1/2}) - \Phi(-t_n)\,|$$

$$\leq b\, w \exp(-\tfrac{1}{2} t^2 (1-3r)) + o((k+1)(g(r\sqrt{k}\, t))^{-1})$$

proof of (4.2.2) is trivial,

For (4.2.3) note that since

$$(4.2.5) \qquad Eg_1(\eta_1) = Eg(\tfrac{1}{m}(X_1 + \ldots + X_m)) \leq Eg(\sup_{1 \leq i \leq m} |X_i|) \leq E \sum_{i=1}^{m} g(|X_i|)$$

$$= \sum_{i=1}^{m} Eg(X_i) = m\, Eg(X_1) < \infty.$$

We have from theorem 2.2.4 combined with an argument like (4.1.7)

$$(4.2.6)\quad Eg_1\left(k^{-1/2}\sum_1^k \eta_i\right) I\left(k^{-1/2}\sum_1^k \eta_i > t\sqrt{n/k}\, f(n)\right) = o(1)$$

whenever $t\sqrt{n/k}\, f(n) \to \infty$

and therefore by Markov's inequality (4.2.3) follows.

Finally to prove (4.2.4) note that we have from (4.1.6)

$$(4.2.7)\quad Eg_1(p^{-1/2}\xi_1) \leq \sum_{i=1}^{m} Eg(p^{-1/2} z_i) < \infty \quad \text{uniformly in } p.$$

Hence using same technique as for the proof of the theorem 2.2.1

$$(4.2.8)\quad \left|P\left(\sum_1^{k+1} \xi_i > t_n n^{1/2}\right) - \Phi(-t_n)\right|$$

$$\leq b\, w \exp\left[-\frac{1}{2} t^2(1-3r)\right] + \sum_{i=1}^{k+1} P(p^{-1/2}|\xi_i| > r\, k^{1/2} t)$$

where the last term may be written as $b(k+1)\, P(p^{-1/2}|\xi_1| > r\, k^{1/2}t)$.

(4.2.4) now follows along (4.1.7) and Markov's inequality.

As a consequence of theorem 4.2.1 we obtain a zone where $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$. Since $1 - \Phi(t_n) \sim \frac{1}{\sqrt{2\pi}} t_n^{-1} \exp(-t_n^2/2)$ as $t_n \to \infty$ from the terms on the r.h.s of (4.2.1) we have following restriction on t.

From the 1st term

(4.2.9) $t \exp(t^2/2)\, w \exp\left(-\frac{1}{2} t^2(1-3r)\right) = o(1)$ gives

$$t^2 \leq \frac{2}{3r}(-\log t - \log w - M_n) \text{ for a sequence } M_n \to \infty.$$

and choosing r sufficiently small this gives

(4.2.10) $t^2 \leq -\lambda \log w = \lambda \log u(r\, k^{1/2} t)$ where λ may be arbitrarily large.

From 2nd term

(4.2.11) $t \exp(t^2/2)\,(g_1(t\sqrt{n/k}\, f(n)))^{-1} = O(1)$ states

(as $g_1(x) = g(\frac{1}{m}x) = (\frac{x}{m})^2\, u(\frac{x}{m})$).

(4.2.12) $t^2 \leq 2(\log t + 2\log(\sqrt{n/k}\, f(n)) + \log u(t\sqrt{n/k}\, f(n)/m)) + M$ for some $M > 0$.

Let

(4.2.12a) $f(n) = n^{-\varepsilon'/2}$, $k = n^{1-\varepsilon}$, $\varepsilon > \varepsilon' > 0$.

Note that for $t > 0$, $t\sqrt{n/k}\, f(n)$ as in (4.2.6), $\to \infty$ for this choice of $f(n)$ and k, (4.2.12) then reduces to

(4.2.13) $t^2 \leq \varepsilon'' \log n$ for some $\varepsilon'' > 0$

Next from the 3rd term

(4.2.14) $(k+1)(g_1(r\sqrt{k}\, t))^{-1}\, t \exp(t^2/2) = O(1)$ gives

(4.2.15) $\quad t^{-1} \exp(t^2/2)\,(u(r\sqrt{k}\,t/m))^{-1} = O(1)$ i.e.,

(4.2.16) $\quad t^2 \leq 2(\log t + \log u(r\sqrt{k}\,t/m)) + M$ for some $M > 0$

$$= 2(\log t + \log u(r\, n^{1/2 - \varepsilon/2}\, t/m)) + M\,,$$

taking $k = n^{1-\varepsilon}$.

Finally the 4th term states

(4.2.17) $\quad t^2 = o(f(n))^{-2}$ i.e., $t^2 = o(n^{\varepsilon'})$

Hence combining (4.2.10), (4.2.13), (4.2.16) and (4.2.17) we have the determining equation as (4.2.16) (as the region determined by the other equations are larger than this).

Observing that $\varepsilon > 0$ in (4.2.16) is arbitrary and r , m are some +ve constants, we have the following theorem .

<u>Theorem</u> 4.2.2 Under the assumptions of theorem 4.2.1

$$1 - F_n(t_n) \sim \Phi(-t_n)\ , \quad t_n \to \infty \quad \text{if}$$

(4.2.18) $\quad t^2 \leq 2(\log t + \log u(n^{1/2-\varepsilon}\, t)) + M$,for some $M > 0$

where $\varepsilon > 0$ can be made arbitrarily small.

<u>Remark</u> 4.2.1. The corresponding zone for independent random variables in a triangular arrary was

$$t^2 \leq 2(\log t + \log u(r\, n^{1/2}\, t)) + M$$

To prove moment type convergence theorems, we will, however, prefer the following form of (4.2.1)

$$(4.2.19)\quad |F_n(t)-\Phi(t)| \leq b\, w \exp\left[-\tfrac{1}{2}t^2(1-3r)\right] + P\left(\left|\sum_{i=1}^{k}\eta_i\right| > |t|\, n^{1/2} f(n)\right) + b(k+1)\, P\left(p^{-1/2}|\xi_1| > r\, k^{1/2}|t|\right) + b\, f(n)\exp(-t^2/2).$$

The following theorem states non uniform rates of convergence to the complementary zone of theorem 4.2.1.

<u>Theorem</u> 4.2.3 Under the assumptions of theorem 4.2.1, for $t^2 \geq \lambda \log u(\sqrt{n})$, there exist constants $b, r > 0$ depending on u, λ such that

$$(4.2.20)\quad |F_n(t)-\Phi(t)| \leq b\left[u(\sqrt{k})\right]^{-(\lambda-1)/2} |t|^{-2(\lambda+1)} + P\left(\left|\sum_{i=1}^{k}\eta_i\right| > |t|\, n^{1/2} f(n)\right) + b(k+1)P(p^{-1/2}|\xi_1|>rk^{1/2}|t|) + bf(n)\exp(-t^2/2).$$

The proof of the above theorem follows from theorem 2.2.2 and along the lines of theorem 4.2.1.

Combining theorems 4.2.1 and 4.2.3 with (4.2.12a) we have the following theorem which generalises Katz's result.

Theorem 4.2.4 Under the assumptions of theorem 4.2.1 for every $\varepsilon > 0$ there exists a constant b depending on ε and u such that

$$(4.2.21) \quad |F_n(t)-\Phi(t)| \leq b(1+t^2)^{-1}(u(n^{1/2-\varepsilon}))^{-1}, \quad -\infty < t < \infty$$

As a consequence of the above we have the following non-uniform L_p version of the Berry-Esseen theorem extending a result of Erickson (1973).

Theorem 4.2.5 Under the assumptions of theorem 4.2.1 for $p \geq 1$ and any $q > 1$

$$(4.2.22) \quad \|(1+t^2)^{1-q/p}(F_n(t)-\Phi(t))\|_p = O(u(n^{1/2-\varepsilon}))^{-1}$$

where $\varepsilon > 0$ is arbitrary.

Next we prove a moment type convergence theorem using (4.2.19) and (4.2.20).

Theorem 4.2.6 Under the assumption of theorem 4.2.1 and $\lim_{x \to 0} u(x) = 0$ with $u'(x) < \lambda_1+\lambda_2 x^{\lambda_3}$, $0 < x < \infty$, for some $\lambda_1, \lambda_2, \lambda_3 > 0$.

(4.2.23) $|EY_n^2 u_1(Y_n) - ET^2 u_1(T)| = o(1)$ where $Y_n = n^{-1/2} S_n$, $T = N(0,1)$, and $u_1(x) = u(x/m)$.

<u>Proof</u> Let $h(x) = x^2 u_1(x)$, $x \geq 0$, then $h(0) = 0$ and $h'(x) < \lambda_1 + \lambda_2 x^{\lambda_3}$ for some $\lambda_1, \lambda_2, \lambda_3 > 0$ and therefore using the following representation

$$(4.2.24) \quad |Eh(Y_n) - Eh(T)| \leq \int_0^\infty h'(t) |P(|n^{-1/2} S_n| \leq t) - P(|N(0,1)| \leq t)| \, dt$$

Dividing the zone $(0, \infty)$ into $(0, \lambda \log u(\sqrt{n}))$ and $[\lambda \log u(\sqrt{n}), \infty)$ with sufficiently large λ and using (4.2.19) and (4.2.20) with (4.2.12a) we have

$$(4.2.25) \quad |Eh(Y_n) - Eh(T)| \leq O(u(n^{1/2-\varepsilon}))^{-1} + b \int_0^\infty h'(t) \, P(|\sum_1^k \eta_i| > t\, n^{1/2} f(n)) + b(k+1) \int_0^\infty h'(t) \, P(p^{-1/2} |\xi_1| > r\, k^{1/2} t) \, dt$$

Now the 2nd term of the r.h.s of (4.2.25),

$$= b\, Eh\,[|\sum_{i=1}^k \eta_i| > n^{1/2} f(n)] = b\, Eh\,[k^{-1/2} |\sum_1^k \eta_i| > n^\varepsilon]$$

for some $\varepsilon > 0$ and hence from (4.1.6) it is $o(1)$.

Similarly, writing $Y = p^{-1/2}|\xi_1|$, the third term of the r.h.s of (4.2.25)

$$= b(k+1)\ Eh(Y/rk^{1/2}) = b(k+1)r^{-2}k^{-1}\ Eu_1(Y/rk^{1/2})\ Y^2.$$

Note that $EY^2u_1(Y) < \infty \Rightarrow EY^2u_1(Y)\ I(|Y| > a) = o(1)$ as $a \to \infty$ from (4.1.7).

Also $EY^2u_1(Y/r\ k^{1/2})\ I(|Y| < a) \to 0$ for every fixed a if $k \to \infty$ ($\because$ $u_1(x) = u(x/m) \to 0$ as $|x| \to 0$) Hence the third term is also $o(1)$ and the theorem follows.

CASE $c > 0$

Here to start with we shall truncate the random variables X_i at $t\ n^{1/2}$ i.e., $X_i' = X_i\ I(|X_i| > t\ n^{1/2})$ and define the usual blocking procedure (4.1.3) with X_i replaced by X_i' . Note that the properties (4.1.5) - (4.1.7) are still preserved as the corresponding results are known for triangular array. The truncation of the random variables is done so as to use a moment inequality (lemma 1) of Babu Ghosh and Singh (1978) for obtaining a sharper bound than (4.2.3) and $(k+1)P(p^{-1/2}|\xi_1| > r\ k^{1/2}t)$.

Using the said inequality for (4.2.3), with η_i as sum of m truncated X_i's

(4.2.26) $P(|\sum_{i=1}^{k} \eta_i| > t\, n^{1/2} f(n))$

$$\leq D(v)(tn^{1/2}f(n))^{-v}\,[(km)^{v/2} + km(n^{1/2}t)^{v-(2+c)}]$$

$$= D(v)\,[m^{v/2}(t\sqrt{n/k}\, f(n))^{-v} + kmf(n)^{-v}(n^{1/2}t)^{-(2+c)}]$$

Similarly

(4.2.27) $(k+1)P(p^{-1/2}|\xi_1| > rk^{1/2}t)$

$$\leq (k+1)\, D(v)\, (r(kp)^{1/2}t)^{-v}\,[p^{v/2} + p(n^{1/2}t)^{v-(2+c)}]$$

$$= (k+1)\, D(v)\, [(n^{1/2}\, t)^{-(2+c)} + (kt)^{-v}]$$

It therefore follows that choosing v sufficiently large say $v = v_0$ depending on c and $f(n) = n^{-\varepsilon/v_0}$ $0 < \varepsilon < 1$ it is possible to make both (4.2.26) and (4.2.27) $O(k(n^{1/2}t)^{-(2+c)}\, n^{\varepsilon})$.

Hence letting $k = n^{1-\varepsilon-\varepsilon'}$, $\varepsilon' > 0$, $1-\varepsilon-\varepsilon' > 0$, we have along the same lines of theorem 4.2.1, the following.

<u>Theorem</u> 4.2.7 Let $\{X_n\}$ be a stationary m-dependent process satisfying (4.1.1) and (4.1.2) with $c > 0$. Then for $t^2 \leq \lambda(\frac{c}{2} \log n + \log u(\sqrt{n}))$ there exist constants b, γ and $r > 0$ (depending on λ, c and u) such that

(4.2.28) $|F_n(t)-\Phi(t)| \leq b\, n^{-\gamma} \exp(-(t^2/2)(1-3r))$

$+ n P(|X_1|>n^{1/2}|t|) + b|t|^{-(2+c)}\, n^{-c/2-\varepsilon'}$

As a consequence of this theorem let us obtain a zone where $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$. Since $\Phi(-t) \sim (2\pi)^{-1/2} t^{-1}\exp(-t^2/2)$, $t \to \infty$, restriction on t_n from the 1st term of the r.h.s of (4.2.28) turns out

$t \exp(t^2/2)\, n^{-\gamma} \exp(-t^2(1-3r)/2) = o(1)$, or

(4.2.29) $t^2 \leq \frac{2}{3r}(\gamma \log n - \log t - M_n)$ for a sequence $M_n \to \infty$.

Hence for sufficiently small choice of r (4.2.29) states

(4.2.30) $t^2 \leq (c+1) \log n$

Similarly from the 2nd term in the r.h.s of (4.2.28) we get

$$t \exp(t^2/2)\, n\, P(|X_1| > n^{1/2}t) = o(1)$$

i.e., $t \exp(t^2/2)\, n(n^{1/2}t)^{-(c+2)} (u(n^{1/2}t))^{-1} = O(1)$

$(\because E|X_1|^{2+c} u(X_1) I(|X_1| > n^{1/2}t) = o(1))$

i.e., $t^{-(c+1)} n^{-c/2} \exp(t^2/2) (u(n^{1/2}t))^{-1} = O(1)$ or

(4.2.31) $$t^2 \leq 2\left\{\frac{c}{2}\log n + (c+1)\log t + \log u(n^{1/2}t)\right\} + M$$

for some $M > 0$

Restriction from the 3rd term in the r.h.s of (4.2.28)

$$t^{-(c+1)} \exp(t^2/2)\, n^{-c/2-\varepsilon'} = o(1) \quad \text{i.e.,}$$

(4.2.32) $$t^2 \leq (c+2\varepsilon'')\log n + 2(c+1)\log t + M \ , \ 0 < \varepsilon'' < \varepsilon$$

Because of the structural restriction $u(x) < |x|^{\varepsilon} + L$ for all $\varepsilon > 0$ with some $L > 0$ (see 2.1.1.) it follows that the determining equation so that $1 - F_n(t_n) \sim \Phi(-t_n)$ holds is (4.2.31). Hence the following

<u>Theorem</u> 4.2.8 Under the assumptions of theorem 4.2.7, a zone of t_n such that $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$, holds is

(4.2.33) $$t^2 \leq c \log n + 2(c+1)\log t + 2\log u(n^{1/2}t) + M \ ,$$

for some $M > 0$.

The above theorem states that contribution of the function u to the normal approximation zone may be incorporated even for m-dependent process. We shall not proceed to prove moment convergences and L_p versions of the Berry-Esseen theorem for the case $c > 0$ as they follow from more general results of Babu, Ghosh and Singh (1978).

4.3 NON UNIFORM RATES OF CONVERGENCE TO NORMALITY FOR m-DEPENDENT PROCESS WHEN ALL THE MOMENTS EXIST

This section extends the results of sections 2.3 - 2.4 for m-dependent process. Here we shall assume (4.1.1) with g satisfying (2.3.1). As before the blocking technique will be used to find non-uniform rates, using these non-uniform rates. We shall show that normal approximation zone can be extended upto $o(n^{1/10})$ under the existance of moment generating function, under additional assumption of $EX_1^3 = 0$, we shall further extend the zone upto $o(n^{1/8})$. These non-uniform rates will be further utilised to deal 'too large' deviation in limiting case and to prove moment type convergences and non uniform L_p-versions of the Berry-Esseen theorem.

The following theorem provides the non-uniform rates of convergence around a nbhd of the origin.

Theorem 4.3.1: Let (4.1.1) with $g(x)$ replaced by $x^2 g(x)$ and g satisfying (2.3.1) holds for a stationary m-dependent process X_n. Also let g satisfy the conditions of theorem 2.4.11. Then for

$1 \leq t^2 \leq 2(\log|t| + \log g_1(rk^{1/2}t)$ there exist constants b, b_1 depending on g such that

$$(4.3.1)\quad |F_n(t)-\Phi(t)| \leq b|t_n|^{-1}\exp(-t_n^2/2)|\exp(O(|t_n|^3k^{-1/2}))-1| + o(g_1(t\sqrt{n/k}\,f(n))^{-1} + o((g_1(r\,k^{1/2}t))^{-1}) + b_1f(n)\exp(-t^2/2),\ 0<r<1/2,$$

where $t_n = (t \pm f(n))$, $f(n)$ is a sequence $f(n) \to 0$ as $n \to \infty$ and $g_1(x) = g(x/m)$. (For $t^2 \leq 1$ one may use uniform bound obtained by putting $t = 1$ in (4.3.1) and this is obtainable proceeding like (4.3.2) to (4.3.4) with $t = 1$, see also (3.5.19a).)

<u>Proof</u>: Recall the blocking procedure (4.1.3). We shall complete the proof by showing

$$(4.3.2)\quad |\Phi(t_n) - \Phi(t)| \leq b_1\, f(n)\exp(-t^2/2)$$

$$(4.3.3)\quad P(|\sum_{i=1}^{k}\eta_i| > t\,n^{1/2}f(n)) = o(g_1(t\sqrt{n/k}\,f(n)))^{-1} \text{ and}$$

$$(4.3.4)\quad |P(\sum_{i=1}^{k+1}\xi_i > t_n\,n^{1/2}) - \Phi(-t_n)| \leq b|t_n|^{-1}\exp(-t_n^2/2)|\exp(O(|t_n|^3\,k^{-1/2})) - 1| + o((g_1(r\,k^{1/2}t))^{-1})$$

proof of (4.3.2) is trivial and that of (4.3.3) is similar to that of (4.2.3). Finally (4.3.4) follows from theorem 2.4.1 along with (4.2.4) and noting that

$$b(k+1)\,P(p^{-1/2}|\xi_1| > rk^{1/2}t) = o[(k+1)(rk^{1/2}t)^{-2}g_1^{-1}(rk^{1/2}t)]$$

from (4.1.5), replacing g by $x^2g(x)$

$$= o(t^{-2}\,g_1^{-1}(rk^{1/2}t)) = o(g_1^{-1}(rk^{1/2}t)) \text{ as } t^2 \geq 1.$$

We shall obtain normal approximation zone as a consequence of this result. In subsequent steps $\varepsilon > 0$ is a constant whose value may differ at different equations. w.o.l.g let $t > 0$

First note that for $t = o(k^{1/4})$

(4.3.5) $|\exp(O(|t_n|^3 k^{-1/2}))-1| = |\exp(O(|t|^3 k^{-1/2}))-1|$

as $t_n = t \pm f(n)$, $f(n) \to 0$.

Since $\Phi(-t) \sim \frac{1}{\sqrt{2\pi}} t^{-1}\exp(-t^2/2)$, $t \to \infty$, for normal approximation zone, the following are the restrictions on t

From the 1st term in the r.h.s of (4.3.1), in view of (4.3.5) ,

(4.3.6) $t^2 \leq \varepsilon \log g_1(r\, k^{1/2} t)$ with $t = o(k^{1/6})$

From 2nd term of r.h.s of (4.3.1)

(4.3.7) $t^2 \leq \varepsilon \log g_1(t\sqrt{n/k}\, f(n)) - \log t$

which is true if $t^2 \leq \varepsilon \log g_1(t\sqrt{n/k}\, f(n))$ where the last inequality follows from the fact that g and hence g_1 has growth more than any power bound, and ε in the second inequality need not be the same as the first.

Similarly from the third term of r.h.s of (4.3.1)

(4.3.8) $t^2 \leq \varepsilon \log g_1(rk^{1/2}t) - \log t$

which is true if $t^2 \leq \varepsilon \log g_1(r\, k^{1/2}t)$.

And from the 4th term

(4.3.9) $t = o(f(n))^{-1}$ i.e., $tf(n) \leq r_n$ where $r_n \to 0$ as $n \to \infty$

Combining (4.3.9) with (4.3.8) we have

(4.3.10) $t^2 \leq \varepsilon \log g_1(r_n\, (n/k)^{1/2})$

Since (4.3.6) is more stringent than (4.3.8), we have, from (4.3.5) - (4.3.10) the following theorem

<u>Theorem</u> 4.3.2 Under the assumptions of theorem 4.3.1 we have $1 - F_n(t_n) \sim \Phi(-t_n)$ $t_n \to \infty$, if for some $\varepsilon > 0$ and two positive sequence r_n, c_n converging to zero with $kp \sim n$.

(4.3.11) $t_n^2 \leq \sup_k \min \left\{ \varepsilon \log g_1(r_n\sqrt{n/k}),\ \varepsilon \log g_1(rk^{1/2}t_n),\ c_n\, k^{1/6} \right\}$

Let us calculate the zone (4.3.11) for some special functional form of g. For $g(x) = \exp(s|x|^{\gamma})$, $0 < \gamma \leq 1$, $s > 0$

$t^2 \leq \varepsilon \log g_1(r_n\sqrt{n/k})$ states $t^2 \leq \varepsilon(r_n(\sqrt{n/k}))^{\gamma}$, $\varepsilon > 0$ is arbitrary.

i.e.,

(4.3.12) $t = o(n/k)^{\gamma/4}$

Also $t^2 \leq \varepsilon \log g_1(r\, k^{1/2}\, t) = \varepsilon\, (k^{1/2}\, t)^{\gamma}$

i.e.,

(4.3.13) $t \leq \varepsilon\, k^{\gamma/2(2-\gamma)}$

And hence we have from (4.3.11)

(4.3.14) $t = o\left((n/k)^{\gamma/4},\ k^{\frac{1}{6}} \wedge \frac{\gamma}{2(2-\gamma)} \right)$

Equating the brackated terms with the observation $kp \sim n$

$(n/k)^{\gamma/4} = k^{\gamma/2(2-\gamma)}$ i.e., $n = k^{\frac{2}{(2-\gamma)}+1}$ i.e, $k = n^{\frac{2-\gamma}{4-\gamma}}$

i.e., $k^{\gamma/\,2(2-\gamma)} = n^{\gamma/\,2(4-\gamma)}$

Similarly $(n/k)^{\gamma/4} = k^{1/6}$ states $k^{1/6} = n^{\gamma/\,2(3\gamma+2)}$

And hence we have the normal approximation zone as

(4.3.15) $t_n = o\left(n^{\frac{\gamma}{2(4-\gamma)}} \wedge \frac{\gamma}{2(3\gamma+2)} \right),\ t_n \to \infty.$

For $\gamma = 1$ we have $t_n = o(n^{1/10})$.

Now consider a different spectrum of g. Let $g(x)$ $= \exp\,(s \log\,(1+|x|))^{\gamma}$, $\gamma > 1$, $s > 0$.

Then $t^2 \leq \varepsilon \log g_1(r_n \sqrt{n/k})$ states $t^2 \leq \varepsilon [\log(r_n \sqrt{n/k})]^{\gamma}$

i.e.,

(4.3.16) $t^{2/\gamma} \leq \varepsilon \log p + \log r_n \leq \varepsilon \log p$ (choosing $r_n \to 0$ appropiately)

Also $t^2 \leq \varepsilon \log g_1(r\, k^{1/2} t)$ states $t^2 \leq \varepsilon [\log(rk^{1/2}t)]^{\gamma}$

or, $t^{2/\gamma} - \varepsilon \log t \leq \varepsilon \log k$, which is true if

(4.3.17) $t^{2/\gamma} \leq \log k$. From (4.3.16) and (4.3.17) equating $p \sim k \sim n^{1/2}$, we have $t^{2/\gamma} \leq \varepsilon \log n$ and hence

(4.3.18) $t^2 \leq \varepsilon (\log n)^{\gamma}$ for some $\varepsilon > 0$ turns out to be normal approximation zone.

(which satisfies $t = o(k^{1/6}) = o(n^{1/12})$ of (4.3.11)).

Next we prove a few excessive deviation type results. To be precise we find out a zone of t for which $1 - F_n(t)$ $= b \exp\{-\frac{t^2}{2}(1+o(1))\}$. Singh (1978) proves that $1-F_n(t)$ $= b \exp\{-\frac{t^2}{2}(1+\delta_{n,\varepsilon})\}$ for $t = \varepsilon n^{1/2}$, $\varepsilon > 0$ where $\lim_{\varepsilon \to 0} \overline{\lim_{n\to\infty}} |\delta_{n,\varepsilon}| = 0$. The following theorem generalises the result providing a rate for $\delta_{n,\varepsilon}$. Also the proof is considerably simplified.

<u>Theorem</u> 4.3.3 For a stationary m-dependent process $\{X_n\}$ with $EX_1 = 0$, $EX_1^2 + 2\sum_{i=1}^{m-1} EX_1X_{1+i} = 1$ and $E\exp(s|X_1|) < \infty$ for some $s > 0$ we have

$$(4.3.19) \qquad P(n^{-1/2} S_n > t) = b\, t^{-1} \exp\left\{-\frac{t^2}{2}(1+\lambda_{n,t})\right\} \qquad t > 0$$

where $\lambda_{n,t} = o(1)$ uniformly in t for $t = o(n^{1/2})$

$= O(\varepsilon)$ for $t = \varepsilon\, n^{1/2} c(\varepsilon)$ where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(e.g. one may take $c(\varepsilon) = \varepsilon^{1/K}$ where $K > 0$ may be arbitrary large)

<u>Proof of the theorem.</u> In view of (4.2.7) with $g(x) = \exp(s|x|)$ for some $s > 0$, it follows from the work of Plachky and Stienback (1975) etc. that a Chernoff type theorem holds for the triangular array of the random variables $\{\xi_i/p^{1/2}\}$ $i = 1,\ldots, k+1$. Consequently by theorem 2.4.6 and 2.4.7, for $t \le \varepsilon k^{1/2}$ with $\varepsilon_{1,p} \to 0$ as $p \to \infty$, we have

$$(4.3.20) \qquad P\left(k^{-1/2} \sum_{i=1}^{k+1} \xi_i/\sqrt{p} > t(n/kp)^{1/2}(1 \pm \varepsilon_{1,p})\right)$$

$$= b\, t^{-1}\exp\left\{-\frac{t^2}{2}(1+\lambda'_{k,t})\right\}$$

Foot note: To be precise, in theorem 4.3.3 we need $E X_1^2 e^{s'|X_1|} < \infty$ for some $s'>0$. This is implied by $E\exp(s|X_1|) < \infty$, $s > 0$ by an application of Hölders inequality where $s' < s$.

where $\lambda'_{k,t} = o(1)$ uniformly in t for $t = o(k^{1/2})$

$= O(\varepsilon)$ for $t = \varepsilon\, k^{1/2}$, $\varepsilon > 0$.

Also since η_i's are iid with finite m.g.f around a nbhd of the origin (as m.g.f of X_1 is finite around a nbhd of origin and η_1 is sum of m X_i's) we have

(4.3.21) $$P(k^{-1/2} \sum_{i=1}^{k} \eta_i > t\, \varepsilon_{1,p} (n/k)^{1/2})$$

$$= b(\varepsilon_{1,p}\, t\sqrt{n/k})^{-1} \exp\left\{-\tfrac{1}{2}(\varepsilon_{1,p}\, t\sqrt{n/k})^2 (1+\lambda'_{k,t_1})\right\}$$

where $t_1 = \varepsilon_{1,p}\, t\sqrt{n/k}$.

Now r.h.s of (4.3.21) is negligible compared to r.h.s of (4.3.20) if

$$(\varepsilon_{1,p}\, t\sqrt{n/k}\,)^2 / t^2 \to \infty \quad \text{as} \quad p \to \infty.$$

i.e., if $\varepsilon^2_{1,p}\, p \to \infty$ as $p \to \infty$

Hence with this choice combining (4.3.20) and (4.3.21) we have

(4.3.22) $$P(n^{-1/2} S_n > t) = b\, t^{-1} \exp\left\{-\frac{t^2}{2}(1 + \lambda'_{k,t})\right\}$$

Let us now choose $p = c^{-2}(\varepsilon)$ with $c(\varepsilon) \to 0$ as $\varepsilon \to 0$.
Then $k = n/(p+m) = O_e\,(c^2(\varepsilon)n)$

and $\lambda'_{k,t} = o(1)$ for $t = o(k^{1/2}) = o(n^{1/2})$

$= O(\varepsilon)$ for $t = \varepsilon k^{1/2} = O_e(\varepsilon\, n^{1/2}\, c(\varepsilon))$

Hence the theorem.

We now attempt to find a zone where $1 - F_n(t) \leq b\, t^{-1} \cdot \exp\left\{-\frac{t^2}{2}(1+o(1))\right\}$, $t \to \infty$, for general g.

Theorem 4.3.4 Under the assumptions of theorem 4.3.1 for $t^2 \leq \lambda \log g_1(r\, n^{1/2}\, \varepsilon_n t)$ for some $\lambda > 0$ and any positive sequence $\varepsilon_n \to 0$ the following holds

(4.3.23) $$1 - F_n(t) \leq b\, t^{-1} \exp\left\{-\frac{t^2}{2}(1+o(1))\right\} \text{ for some } b > 0,\ t \to \infty.$$

Proof. Since $\Phi(-t) \leq b\, t^{-1} \exp(-t^2/2)$, $t \to \infty$, in view of (4.3.4)

(4.3.24) $$P\left(k^{-1/2} \sum_{i=1}^{k+1} \xi_i/\sqrt{p} > t(n/kp)^{1/2}(1 \pm \varepsilon_n)\right)$$

$$\leq b\, t^{-1} \exp\left\{-\frac{t^2}{2}(1+o(1))\right\} + o\left\{(g_1(rk^{1/2}t))^{-1}\right\}$$

$$\text{for } t^2 \leq 2\log g_1(rk^{1/2}t)$$

Now

(4.3.25) $$(g_1(rk^{1/2}t))^{-1} \leq t^{-1}\exp(-t^2/2) \quad \text{if}$$

$$t^2 \leq 2(\log g_1(r\, k^{1/2}t) - \log t)$$

$$\leq \lambda \log g_1(r\, k^{1/2}t) \text{ for some } \lambda > 0.$$

Similarly

(4.3.26) $$P(|\sum_{i=1}^{k}\eta_i| > n^{1/2}t\,\varepsilon_n) = P(k^{-1/2}|\sum_{1}^{k}\eta_i| > t(n/k)^{1/2}\varepsilon_n)$$

$$\leq b\,t_1^{-1}\exp\{-\frac{t_1^2}{2}(1+o(1))\} + o(\,g_1^{-1}(rk^{1/2}t_1))$$

where $t_1 = t(n/k)^{1/2}\varepsilon_n$

for $t_1^2 \leq 2\log g_1(r\,k^{1/2}\,t_1)$

Also

(4.3.27) $$g_1^{-1}(rk^{1/2}t_1) \leq b\,t_1^{-1}\exp(-t_1^2/2) \quad \text{if } t_1^2 \leq \lambda\log g_1(rk^{1/2}$$

for some $\lambda > 0$.

From (4.3.24) - (4.3.27) the result follows letting $k = 0_e(n)$

The following theorem provides non-uniform rates in the complementary zone of theorem 4.3.1

<u>Theorem</u> 4.3.5 For a stationary m-dependent process satisfying the assumptions of theorem 4.3.1 with $x^{-1}\log g(x) \to 0$ as $x \to \infty$, for $t^2 \geq 2\{\log g_1(r\,k^{1/2}t)+\log|t|\}$, the following ho

(4.3.28) $$|F_n(t)-\Phi(t)| \leq O(|t|g_1(r\,k^{1/2}t))^{-1+o(1)}$$

$$+\,b(k+1)\,P(p^{-1/2}|\xi_1| > r\,k^{1/2}|t|)+\,b\,k\,P(|\eta_1|>rk^{1/2}|t|$$

<u>Proof</u>. w.o.l.g assume $t > 0$. Since $\Phi(-t) \leq bt^{-1}\exp(-t^2/2)$ $= 0$ (r.h.s of (4.3.28)) it is enough to show that

(4.3.29) $P(n^{-1/2} S_n > t) \leq O(tg_1(r\, k^{1/2}\, t))^{-1+o(1)}$

$+ b(k+1)\, P(p^{-1/2}|\xi_1|>rk^{1/2}t) + b\, kP(|\eta_1|>r\, k^{1/2}\, t)$

Now in view of (4.2.7) we have from theorem 2.4.8 for $t^2 \geq 2\{\log g_1(rk^{1/2}t) + \log|t|\}$ and $\varepsilon_n \to 0$

(4.3.30) $P(k^{-1/2} \sum_1^{k+1} \xi_i/\sqrt{p} > t(n/kp)^{1/2}(1 \pm \varepsilon_n))$

$\leq O(tg_1(rk^{1/2}t))^{-1+o(1)} + b(k+1)\, P(p^{-1/2}|\xi_1|> rk^{1/2}t)$

And for $\varepsilon_n(n/k)^{1/2} > 1$, similarly we have, for the same region of t,

(4.3.31) $P(k^{-1/2}|\sum_1^k \eta_i| > t\, \varepsilon_n(n/k)^{1/2}) \leq P(k^{-1/2}|\sum_1^k \eta_i| > t)$

$\leq O(tg_1(r\, k^{1/2}t))^{-1+o(1)} + b\, k\, P(|\eta_1| > r\, k^{1/2}t))$

Hence by a proper choice of ε_n, say $\varepsilon_n = 2(k/n)^{1/2}$ we have (4.3.29) from (4.3.30) and (4.3.31).

For the case $x^{-1}\log g(x) \to s$ (> 0) we have the following remark from remark 2.4.11 following the lines of theorem 4.3.5.

Remark 4.3.1 For $x^{-1}\log g(x) \to s$ (> 0), we have, for $t^2 \geq 2\{\log|t| + \log g_1(r\, k^{1/2}\, \varepsilon_n t)\}$ with any positive sequence $\varepsilon_n \to 0$

$$(4.3.32) \quad |F_n(t)-\Phi(t)| \le O(tg_1(r\, k^{1/2}\, \varepsilon_n t))^{-1+o(1)}$$

$$+ b(k+1)P(|\xi_1/\sqrt{p}\,| > r\, k^{1/2}|t|) + b\, k\, P(|\eta_1| > r\, k^{1/2}|t|)$$

Combining theorems 4.3.1 and 4.3.5 / remark 4.3.1 we may have a non uniform bound over the entire range of t, $-\infty < t < \infty$. Let $k \sim n^{1/2}$ and $(f(n))^{-1} = n^{1/4-\varepsilon}\lambda$ for some λ and $\varepsilon > 0$, Then assuming

$$(4.3.33) \quad (g_1(\lambda\, n^{\varepsilon}\, t))^{-1} \le b\, n^{-1/4+\varepsilon}\, (g_1(\lambda^* t))^{-1} \quad \text{for some}$$

$$b,\ \lambda^* > 0, \text{ and } -\infty < t < \infty,$$

and

$$(4.3.34) \quad (g_1(rn^{1/4}t))^{-1} \le b\, n^{-1/4+\varepsilon}\, (g_1(\lambda^* t))^{-1}$$

we have from (4.3.1), for $t^2 \le 2(\log g_1(r\, n^{1/4}t) + \log |t|)$. the following

$$(4.3.35) \quad |F_n(t)-\Phi(t)| \le b|t_n|^{-1} \exp(-t_n^2/2)|\exp(O(|t_n|^3\, n^{-1/4}))-1|$$

$$+ b\, n^{-1/4+\varepsilon}\, g_1^{-1}(\lambda^* t) + bn^{-1/4+\varepsilon} \exp(-t^2/2)$$

Also note that for (4.3.28) and (4.3.32)

$$(4.3.36) \quad (k+1)\, P(|\xi_1/\sqrt{p}\,| > r\, k^{1/2}|t|) = O(g_1^{-1}(r\, k^{1/2}\, t))$$

$$= O(g_1^{-1}(r\, n^{1/4}\, t))$$

and

(4.3.37) $$k\,P(|\eta_i| > r\,k^{1/2}|t|) = O(g_1^{-1}(r\,k^{1/2}\,t))$$

$$= O(g_1^{-1}(r\,n^{1/4}\,t)).$$

Therefore, if for some $\varepsilon^* > 0$ and $|t|>t_o$, $t_o>0$ arbitrary; we have

(4.3.38) $$[|t|\,g_1(r\,n^{1/4}t)]^{-1+\varepsilon^*} \leq b\,n^{-\frac{1}{4}+\varepsilon}\,[g_1(\lambda^* t)]^{-1}$$

when $\lim_{x\to\infty} x^{-1}\log g(x) = 0$

and

(4.3.39) $$[\,|t|\,g_1(r\,\varepsilon_n\,n^{1/4}t)]^{-1+\varepsilon^*} \leq b\,n^{-\frac{1}{4}+\varepsilon}\,[g_1(\lambda^* t)]^{-1}$$

when $\lim_{x\to\infty} x^{-1}\log g(x) = s\ (>0)$

then in view of (4.3.34), (4.3.36) and (4.3.37), (4.3.28) and (4.3.32) take the form

(4.3.40) $$|F_n(t) - \Phi(t)| \leq b\,n^{-\frac{1}{4}+\varepsilon}\,[g_1(\lambda^* t)]^{-1}$$

Hence we have the following theorem in the same lines of theorem 2.4.9

<u>Theorem</u> 4.3.6 For a stationary m-dependent process satisfying the assumptions of theorem 4.3.1 and (4.3.33), (4.3.34), (4.3.38) and (4.3.39) we have

(4.3.41) $$|F_n(t)-\Phi(t)| \leq b\,n^{-\frac{1}{4}+\varepsilon}\,[g_1(\lambda^* t)]^{-1}, \quad -\infty < t < \infty$$

Consequently we may have the following two theorems on moment type convergences and non uniform L_p version of the Berry-Esseen theorem

Theorem 4.3.7   Under the assumptions of theorem 4.3.6 and

$$(4.3.42) \quad \frac{d}{dx}\left[x^2 g_1(x)\right] = O\left(g_1(\lambda^* x)(1+|x|)^{-q}\right), \quad -\infty < x < \infty, \ q>1$$

one has

$$(4.3.43) \quad |EY_n^2 g_1(Y_n) - ET^2 g_1(T)| = O(n^{-\frac{1}{4}+\varepsilon}), \quad T = |N(0,1)|, \quad Y_n = |n^{-1/2} S_n|$$

Theorem 4.3.8   Under the assumptions of theorem 4.3.6, for any $p \geq 1$ and $q > 1$

$$(4.3.44) \quad \| g_1(\lambda^* t)(1+|t|)^{-q/p}(F_n(t)-\Phi(t)) \|_p = O(n^{-\frac{1}{4}+\varepsilon})$$

Next we consider the case when $EX_1^3 = 0$. Note that using the same blocking technique it is possible to obtain, when all the moments of X_1 exist,

$$(4.3.45) \quad |F_n(t)-\Phi(t)| \leq b\, n^{-\frac{1}{4}+\varepsilon}(1+|t|)^{-c} \qquad -\infty < t < \infty$$

where $c > 0$ may be made arbitrarily large and $\varepsilon > 0$ arbitrarily small, b depends on ε and c.

Now note that

(4.3.46) $$E(n^{-1/2}S_n)^3 = \int_0^\infty x^3\, dF_n(x) + \int_{-\infty}^0 x^3\, dF_n(x)$$

Also

(4.3.47) $$\int_0^\infty x^3\, dF_n(x) = x^3(1-F_n(x))\Big|_0^\infty + \int_0^\infty 3x^2(1-F_n(x))\, dx$$

$$= \int_0^\infty 3x^2(1-F_n(x))\, dx$$

Similarly

(4.3.48) $$\int_{-\infty}^0 x^3\, dF_n(x) = -\int_{-\infty}^0 3x^2 F_n(x)\, dx$$

Letting $c = 4$ in (4.3.45) and noting that $\int_{-\infty}^\infty x^3 d\Phi(x) = 0$ we therefore have from (4.3.46)

(4.3.49) $$|E\,(n^{-1/2}\, S_n)^3| \leq b\, n^{-\frac{1}{4}+\varepsilon}$$

And hence

(4.3.50) $$|E(p^{-1/2}\, \xi_1)^3| \leq b\, p^{-\frac{1}{4}+\varepsilon}$$

where $\varepsilon > 0$ is arbitrary.

Note that in Chapter 2, if $E\, X_{ni}^3 = O(p^{-\alpha})$, $p = p(n)$ and $\alpha > 0$ then in place of (2.4.9) we have, expanding f_i upto the 4th term (see also (2.4.38))

(4.3.51) $$|f_i - 1 - \frac{t^2}{2s_n^2} E\, X_{ni}^2| \leq b\, n^{-1}\, n^{-1/2}\, p^{-\alpha}\, t^3 + b\, n^{-1} n^{-1} t^4$$

Then proceeding as in theorem 2.4.3 it is possible to obtain instead of (2.4.35) the following

$$(4.3.52) \quad |F_n(t) - \Phi(t)| \leq b \exp(-t^2/2)\ |t|^{-1}$$
$$\times\ |\exp(O(n^{-1/2}\ p^{-\alpha}\ |t|^3 + n^{-1}\ t^4)) - 1|$$
$$+ b \exp(-t^2/2 + O(n^{-1/2}\ p^{-\alpha}\ |t|^3 + n^{-1}\ t^4))\ n^{-1/2}$$
$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\ s_n |t|).$$

Therefore for the m-dependent process the equation (4.3.1) changes to

$$(4.3.53) \quad |F_n(t) - \Phi(t)| \leq b \exp(-t_n^2/2)\ |t_n|^{-1}$$
$$\times \exp(O(|t_n|^3\ k^{-1/2}\ p^{-\frac{1}{4}+\varepsilon} + t^4\ k^{-1})) - 1|$$
$$+ b\ k^{-1/2} \exp(-t_n^2/2 + O(|t_n|^3\ k^{-1/2}\ p^{-\frac{1}{4}+\varepsilon} + k^{-1} t^4))$$
$$+ o(g_1(t\sqrt{n/k}\ f(n))^{-1} + o(g_1(r\ k^{1/2} t))^{-1})$$
$$+ b_1\ f(n) \exp(-t^2/2)$$

for $1 \leq t^2 \leq 2(\log g_1(r\ k^{1/2} t) + \log |t|)$

Recalling the steps used to prove theorem 4.3.2 we have, in view of (4.3.53) the following theorem

Theorem 4.3.9 Under the assumptions of theorem 4.3.1 and $EX_1^3 = 0$ we have $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$, if for some $\varepsilon^* > 0$ and two positive sequences r_n, c_n converging to zero with $k.p \sim n$ and $\varepsilon > 0$ arbitrary small the following holds

$$(4.3.54)\quad t_n^2 \le \sup_k \min \{ \varepsilon^* \log g_1(r_n \sqrt{n/k}), \varepsilon^* \log g_1(r\, k^{1/2} t_n), c_n k^{1/2}, c_n (k^{1/2}\, p^{1/4-\varepsilon})^{1/3} \}.$$

If we calculate the zone (4.3.54) for $g(x) = \exp(s|x|^\gamma)$, $0 < \gamma < 1$, $s > 0$ then (4.3.14) reduces to

$$(4.3.55)\quad t = o\{ (n/k)^{\gamma/4} \wedge (k^{1/2}\, p^{\frac{1}{4}-\varepsilon})^{1/3} \wedge k^{\gamma/2(2-\gamma)} \wedge k^{\frac{1}{4}} \}$$

Since $kp \sim n$ we can write (4.3.55) as

$$(4.3.56)\quad t = o\{ (n/k)^{\gamma/4} \wedge (n^{\frac{1}{4}-\varepsilon} \cdot k^{\frac{1}{4}+\varepsilon})^{1/3} \wedge (k^{(\gamma/(2(2-\gamma)) \wedge 1/4)}) \}$$

Note that $\frac{\gamma}{2(2-\gamma)} \gtrless \frac{1}{4} \iff \gamma \gtrless \frac{2}{3}$

Therefore for $\gamma \le \frac{2}{3}$, $\gamma/2(2-\gamma) \wedge 1/4 = \gamma/2(2-\gamma)$ and $(n/k)^{\gamma/4} = k^{\gamma/2(2-\gamma)}$ gives $k = n^{(2-\gamma)/(4-\gamma)}$ with

$(n/k)^{\gamma/4} = n^{\gamma/2(4-\gamma)}$

For $\gamma > \frac{2}{3}$ $\left(\frac{n}{k}\right)^{\gamma/4} = k^{1/4}$ gives $k = n^{\gamma/(\gamma+1)}$

with $(n/k)^{\gamma/4} = n^{\gamma/4(\gamma+1)}$.

Hence $(n/k)^{\gamma/4} = k^{(\gamma/2(2-\gamma) \wedge 1/4)}$ gives $k = n^{(2-\gamma)/(4-\gamma)} \vee n^{\frac{\gamma}{\gamma+1}}$

with $(n/k)^{\gamma/4} = n^{\gamma/2(4-\gamma)} \wedge n^{\gamma/4(\gamma+1)}$

Similarly $(n/k)^{\gamma/4} = \left(n^{\frac{1}{4}-\varepsilon} k^{\frac{1}{4}+\varepsilon}\right)^{1/3}$ gives $k = n^{\frac{3\gamma-1}{3\gamma+1}+\varepsilon}$ ($\varepsilon>0$ arb.)

with $(n/k)^{\gamma/4} = n^{\gamma/2(3\gamma+1)-\varepsilon}$

Hince letting $k = n^{(2-\gamma)/(4-\gamma)} \vee n^{\gamma/(\gamma+1)} \vee n^{\gamma/2(3\gamma+1)-\varepsilon}$

in (4.3.56) we have $t = o(n/k)^{\gamma/4}$ i.e.,

(4.3.57) $t = o(n^{\gamma/2(4-\gamma)} \wedge n^{\gamma/4(\gamma+1)} \wedge n^{\gamma/2(3\gamma+1)-\varepsilon})$

where $\varepsilon > 0$ is arbitrary.

For $\gamma = 1$ i.e., when m.g.f of X_1 exists we have $t = 0(n^{1/8-\varepsilon})$.

As a concluding remark of this chapter we may note that possible extension of the results obtained can be made to the non-stationary m-dependent process also as the corresponding results are known for triangular array of independent random variables from chapter 2 which has been utilised as the basic tool in the present chapter.

CHAPTER 5.

RATES OF CONVERGENCE TO NORMALITY FOR NON-STATIONARY ϕ-MIXING PROCESS

5.1 Introduction:

Let $\{X_n, n \geq 1\}$ be a non stationary ϕ-mixing process defined on a probability space (Ω, A, P).

Define $S_n = \sum_{i=1}^{n} X_i$, $\sigma_n^2 = V(S_n)$, $F_n(t) = P(S_n \leq t\,\sigma_n)$

Assume that

(5.1.1) $E(X_n) = 0$ for all $n \geq 1$

(5.1.2) $\phi_n \ll \exp(-\lambda n)$ for some $\lambda > 0$

(5.1.3) $\sup_{i \geq 1} E|X_i|^m \leq f(m)$, $m = 1, 2, \ldots$

where $f : (0, \infty) \to (0, \infty)$ is nondecreasing, satisfying

$$\sup_{1 \leq j \leq m} f(j)\, f(m+1-j) \leq f(m+1) \quad \text{and}$$

(5.1.4) $\inf_{n \geq 1} n^{-1/2} \sigma_n^2 > 0$

Under various types of moment bounds we shall study the nonuniform rates of convergence to normality of $F_n(t)$ to $\Phi(t)$ where $\Phi(t)$ is the $N(0, 1)$ distribution.

In section 2 we prove a lemma which states the order of m^{th} absolute moment S_n in terms of $f(n)$. This lemma which is of specific importance of its own, is the basic tool for proving the results in section 3 where we deal non uniform rates.

5.2 THE LEMMA

Lemma 5.2.1 Let $\{X_n, n \geq 1\}$ be a non stationary ϕ-mixing process satisfying (5.1.1) - (5.1.3). Then there exist a constant $L\ (> 1)$, depending only on ϕ, such that for all positive integer u and $h \geq 0$.

$$(5.2.1) \qquad E\left|\sum_{i=1}^{u} X_{i+h}\right|^m \leq u^{m/2}\, m!\, L^m f(m), \qquad m \geq 1$$

Proof of the lemma: The lemma is proved by induction. Define

$$(5.2.2) \qquad C(u,m,h) = E\left|\sum_{i=1}^{u} X_{i+h}\right|^m \text{ and } C(u,m) = \sup_{h\geq 0} C(u,m,h)$$

Then it follows from Babu, Ghosh and Singh (1978, lemma 1) that

$$(5.2.3) \qquad C(u, m) \leq u^{m/2} k(m)$$

Hence the lemma is true for $m \leq m_o$, where m_o may be taken sufficiently large by adjusting L with $k(m_o)$. Specifically we take $L = \max \left\{ 1, \max_{m \leq m_o} (m! \, , f(m))^{-1/m} k^{1/m}(m) \right\}$. Therefore assuming the lemma to be true for $m \leq m_o$ (m_o sufficiently large) we shall show it to hold for (m_o+1). For simplicity of notations we write $m = m_o$.

Fix an integer $h \geq 0$. Define $S_u = \sum_{i=1}^{u} X_{i+h}$,

$S_{u,t} = (S_{2u+t} - S_{u+t})$ and $S'_{u,t} = \sum_{i=1}^{t} X_{i+u+h}$

Now

$$(5.2.4) \quad E(|S_u+S_{u,t}|)^{m+1} \leq E(|S_u|+|S_{u,t}|)^{m+1}$$

$$= \sum_{j=0}^{m+1} \binom{m+1}{j} E(|S_u|^{m+1-j} \, |S_{u,t}|^j \,)$$

$$\leq 2C(u, \, m+1) + \sum_{j=1}^{m} \binom{m+1}{j} E(|S_u|^{m+1-j} \, |S_{u,t}|^j \,)$$

using a lemma of Ibragimov (1962) e.g., see lemma 1 P-170 of Billingsley

$$(5.2.5) \quad |E(|S_u|^{m+1-j} |S_{u,t}|^j) - E(|S_u|)^{m+1-j} \, E(|S_{u,t}|)^j|$$

$$\leq 2 \, \phi^{(m+1-j)/(m+1)} (t) \quad C(u,m+1)$$

Therefore

$$(5.2.6) \quad E(|S_u + S_{u,t}|)^{m+1} \leq 2C(u, m+1) + 2C(u,m+1) \sum_{j=1}^{m} \phi^{1-j/(m+1)}(t)$$

$$+ \sum_{j=1}^{m} \binom{m+1}{j} E\,|S_u|^{m+1-j}\, E|S_{u,t}|^j .$$

Now

$$(5.2.7) \quad \sum_{j=1}^{m} \binom{m+1}{j} \phi^{1-j/(m+1)}(t) = \phi(t) \left[\{1+\phi^{-1/(m+1)}(t)\}^{m+1} - \phi^{-1}(t) - 1 \right]$$

$$\leq \left[1 + \phi^{1/(m+1)}(t)\right]^{m+1}$$

$$\leq \left[1 + e^{-\lambda_1 t/(m+1)}\right]^{m+1} \quad \because \ \phi(t) \leq e^{-\lambda_1 t}$$

for some $\lambda_1 > 0$.

$$\leq (1 + c)^{m+1} \quad \text{letting } t = t_o = \lambda_2 m \text{ where } c > 0$$

can be made arbitrarily small with sufficiently large choice of λ_2. Therefore denoting $(1+c)^{m+1} = \lambda(m) = \lambda$ and assuming (5.2.1) for $j \leq m$ i.e.,

$$(5.2.8) \quad C(u,j) \leq u^{j/2}\, j!\, L^j f(j) \quad , \text{ we have}$$

$$(5.2.9) \quad E(|S_u + S_{u,t}|)^{m+1} \leq 2(1+\lambda)\, C(u,m+1) + u^{(m+1)/2} (m+1)!\, .L^{m+1} \sum_{j=1}^{m} f(j)\, f(m+1-j)$$

$$\leq 2(1+\lambda)\, C(u, m+1) + u^{(m+1)/2} (m+1)!\, L^{m+1} m\, f(m+1), \text{ from}$$

2nd part of (5.1.3)

Therefore

(5.2.10) $$C(2u,\ m+1) = E|S_u + S_{u,t} + S'_{u,t} - S'_{2u,t}|^{m+1}$$

$$\leq \left[\left\{ 2(1+\lambda)\ C(u,m+1) + u^{(m+1)/2}\ (m+1)!\ L^{m+1}\ m\ f(m+1) \right\}^{\frac{1}{m+1}} + 2t \sup_{i \geq 1} E^{1/(m+1)} |X_i|^{m+1} \right]^{m+1}$$

$$\leq \left\{ 2(1+\lambda)\ C(u,m+1) + u^{(m+1)/2} (m+1)!\ L^{m+1} m\ f(m+1) \right\} (1+\varepsilon_1)^{m+1}$$

where

(5.2.11) $$\varepsilon_1 = \frac{2t \sup_{i \geq 1} E^{1/(m+1)} |X_i|^{m+1}}{u^{1/2}\ L \left\{ f(m+1)\ (m+1)! \right\}^{1/(m+1)}}$$

$$\leq a\ u^{-1/2}\ L^{-1} \text{ for some } a > 0, \text{ since } t = \lambda_2 m \text{ and } (m!)^{1/m} \geq e^{-1} m.$$

So ε_1 can be made arbitrarily near to zero (uniformly in m and u) by a sufficiently large choice of L.

Writing $b = (1+\varepsilon_1)^m$ we have repeating (5.2.11) r times for $u = 2^r$.

(5.2.12) $$C(2^r,\ m+1) \leq 2(1+\lambda)\ b\ C(2^{r-1},\ m+1) + (2^{r-1})^{(m+1)/2} .b\ f_1(m+1)$$

where $f_1(m+1) = (m+1)!\ L^{m+1}\ m\ f(m+1)$

....

$$\leq [2(1+\lambda)b]^r\, C(1,\ m+1) + (2^{r-1})^{(m+1)/2}\, f_1(m+1)b$$

$$+ (2^{r-2})^{(m+1)/2}\, f_1(m+1)\, b^2\, 2(1+\lambda) + \ldots\ ..$$

$$\leq [2(1+\lambda)\, b]^r \sup_{i\geq 1} E|X_i|^{m+1} + f_1(m+1)\, b\, 2^{(r-1)(m+1)/2}$$

$$\cdot\ (1 + b\, 2^{-(m+1)/2}\, 2(1+\lambda))^{-1}$$

In order that the above is less than or equal to $u^{(m+1)/2}\, L^{m+1}\cdot$ $(m+1)!\ f(m+1)$, first we may need

$$[2(1+\lambda)b]^r \sup_{i\geq 1} E|X_i|^{m+1} \leq \frac{1}{2}\, 2^{r(m+1)/2}\, L^{m+1}\ (m+1)!\ f(m+1)$$

which is true if

$$[2(1+\lambda)b]^r \leq 2^{r(m+1)/2} \quad \text{as} \quad m \geq 1 \quad \text{and} \quad L > 1$$

i.e., if

(5.2.13) $$2(1+\lambda)b \leq 2^{(m+1)/2}$$

i.e., if $2[1 + (1+c)^{m+1}]\ (1+\varepsilon_1)^{m+1} \leq 2^{(m+1)/2}$

i.e., if $2[1 + (1+c)^{m+1}] \leq [\sqrt{2}/(1+\varepsilon_1)]^{m+1}$

i.e., if $2 \leq [\sqrt{2}/(1+\varepsilon_1)]^{m+1} - 2(1+c)^{m+1}$

Which is true for all large m since r.h.s. $\to \infty$ as $m \to \infty$ if ε_1 and c are near to zero.

For the 2nd term of the r.h.s of (5.2.12) note that

$b\, 2^{-(m+1)/2}\, 2(1+\lambda) < 1$ if $b\, 2(1+\lambda) < 2^{(m+1)/2}$ which, in view of (5.2.13) is true for small choice of ε_1 and c.

And hence $\left[1 - b\, 2^{-(m+1)/2}\, 2(1+\lambda) \right]^{-1} = \lambda_4 < \infty$.

So we have

$$\lambda_4\, f_1(m+1) b\, 2^{(r-1)(m+1)/2} \leq \frac{1}{2}\, 2^{r(m+1)/2}\, L^{m+1}\, (m+1)!\, f(m+1)$$

if $\lambda_4\, (1+\varepsilon_1)^{m+1} \leq (2m)^{-1}\, 2^{(m+1)/2}$

i.e., if $\lambda_4 < (2m)^{-1} \left[\sqrt{2} / (1+\varepsilon_1) \right]^{m+1}$

which is again true for all large m as r.h.s. $\to \infty$ as $m \to \infty$, if $\varepsilon_1 < \sqrt{2} - 1$.

It is now easy to show that the sum of two terms of the r.h.s of (5.2.12) is less than or equal to

$$2^{r(m+1)/2}\, L^{m+1}\, (m+1)!\, f(m+1)$$

for all sufficiently large m.

Hence the lemma is proved for all u of the form $u = 2^r$.

For general u we use binary decomposition of u as in Ghosh and Babu (1977) to show

$$C(u, m) \leq u^{m/2} L^m 2^{m/2} m! f(m)$$

$$= u^{m/2} L_1^m m! f(m), \quad \text{where } L_1 = L\sqrt{2}$$

Hence the lemma is proved for general u.

Remark 5.2.1. There are a choice of constants involved in the proof of the lemma. First of all choose λ_2 in (5.2.7) large so that c therein small, once λ_2 is fixed 'a' in (5.2.11) is fixed and therefore choose L large so that ε_1 in (5.2.11) is small uniformly in m and u.

5.3 RATES OF CONVERGENCE

In this section we shall study the non-uniform rates for different choices of f. As a matter of fact we shall consider the following two types of bounds viz.

(5.3.1) $$\sup_{i \geq 1} E|X_i|^m \leq f(m) = L^m \exp(\upsilon\, m \log m)$$

for some $L > 0$ and $\upsilon \geq 0$.

and

(5.3.2) $$\sup_{i\geq 1} E|X_i|^m \leq f(m) = L^m \exp(m^\upsilon)$$

for some $L > 0$ and $\upsilon > 1$

It is apperent from (3.7.16) - (3.7.17) that (5.3.1) and (5.3.2) are implied respectively by

(5.3.3) $$\sup_{i\geq 1} E \exp(s|X_i|^{1/\upsilon}) < \infty \quad , \text{ for some } s > 0$$

and

(5.3.4) $$\sup_{i\geq 1} E \exp\left[\log(1+|X_i|)\right]^{\upsilon/(\upsilon-1)} < \infty.$$

possible extention of the technique used can be made to bounds other than (5.3.1) and (5.3.2) also.

That, f satisfies the requirement mentioned in (5.1.3) can be verified as follows,

For f defined in (5.3.2) note that

$$\sup_{1\leq j\leq m} f(j)\, f(m+1-j) = L^{m+1} \sup_{1\leq j\leq m} \exp(j^\upsilon + (m+1-j)^\upsilon)$$

$$\leq L^{m+1} \exp\left[(m+1)^\upsilon\right] \quad \text{if}$$

$$\left(\frac{j}{m+1}\right)^\upsilon + \left(1 - \frac{j}{m+1}\right)^\upsilon \leq 1, \quad 1 \leq j \leq m. \quad \text{i.e., if}$$

(5.3.5) $$x^\upsilon + (1-x)^\upsilon \leq 1, \qquad 0 < x < 1$$

Since the function $g(x) = x^{\upsilon} + (1-x)^{\upsilon}$ is symmetric around the point $1/2$, it suffices to consider the range $1/2 < x < 1$.

Now

$$g'(x) = \upsilon\, x^{\upsilon-1} - \upsilon(1-x)^{\upsilon-1} \uparrow x \quad \text{for} \quad x > 1/2, \quad \upsilon > 1$$

Hence supremum of g is attained at the end point $x = 1$ and there $g(x) = 1$. Hence (5.3.5) is satisfied. Similarly for (5.3.2) assertion on f may be proved.

We now describe the blocking technique that will be used throughout the chapter.

Let $p = p(\alpha, n) = [n^{\alpha}]$, $q = q(\beta, n) = [n^{\beta}]$, $k = k(\alpha, \beta, n) = [n/(p+q)]$ and $\ell = n-k(p+q)$ where $0 < \beta \leq \alpha < 1$ will be chosen accordingly.

Put,

(5.3.6)
$$\begin{cases} \xi_i = \xi_{ni} = \sum_{j=1}^{p} X_{(i-1)(p+q)+j} \\ \eta_i = \eta_{ni} = \sum_{j=1}^{q} X_{ip + (i-1)q + j} \\ \xi_{k+1} = \xi_{n,k+1} = \sum_{j=1}^{\ell} X_{k(p+q)+\ell} \text{ or } 0 \text{ according as } \ell \geq 1 \text{ or not.} \end{cases}$$

Also let $U_n = \sum_{i=1}^{k+1} \xi_i$, $U_n' = \sum_{i=1}^{k} \eta_i$ and $t_n = (t \pm n^{-\lambda})$

where $\lambda > 0$ to be chosen later.

First we consider the moment bound (5.3.1). The following theorem states nonuniform rates of convergence in an interval containing the origin.

<u>Theorem</u> 5.3.1 Let $\{ X_n , n \geq 1 \}$ be a non-stationary ϕ-mixing process satisfying (5.1.1) - (5.1.4) and (5.3.1). Then for

(5.3.7) $$t^2 \leq M k^{1/(2\upsilon+1)} , \quad M > 0$$

with $|t| < \varepsilon k^{1/2}$, $\varepsilon > 0$ small, there exists constant $b > 0$ (depending on M) such that for any $\lambda > 0$ and some $a > 0$ (depending on υ and ϕ)

(5.3.8) $$\begin{aligned} |F_n(t) - \Phi(t)| \leq\ & b|t|^{-1}\exp(-t^2/2)|\exp(O(|t|^3 k^{-1/2}))-1| \\ & + b\, n^{-\lambda} \exp(-t^2/2) \\ & + b \exp\left[-a \{|t|\, n^{(\alpha-\beta-2\lambda)/2}\}^{1/(\upsilon+1)}\right] \\ & + b \exp\left[-a \{|t|\, n^{(1-\alpha)/2}\}^{1/(\upsilon+1)}\right] \end{aligned}$$

Proof of the theorem : w.o.l.g. assume $t > 0$.

Since the proof is long we shall complete it in a few parts. It will be shown

(5.3.9) $$|\Phi(t_n) - \Phi(t)| \le b\, n^{-\lambda} \exp(-t^2/2)$$

(5.3.10) $$P(|U_n'| > t\, n^{-\lambda}\sigma_n) \le b \exp\left[-a \{t\, n^{(\alpha-\beta-2\lambda)/2}\}^{1/\upsilon+1)}\right]$$

and

(5.3.11) $$|P(U_n > t_n\, \sigma_n) - \Phi(-t_n)| \le b\, t^{-1} \exp(-t^2/2)\, |\exp(O(t^3\, k^{-1/2})) - 1| + b \exp\left[-a \{t\, n^{(1-\alpha)/2}\}^{1/(\upsilon+1)}\right]$$

for $t^2 \le M\, k^{1/(2\upsilon+1)}$ with $t < \varepsilon\, k^{1/2}$ and $t_n = (t \pm n^{-\lambda})$.

These will complete the proof.

Proof of (5.3.9) is trivial.

For (5.3.10) note the following. From lemma 5.2.1 and (5.3.1) denoting L to be an arbitrary positive constant

(5.3.12) $$\begin{aligned} P(|U_n'| > t\, n^{-\lambda}\sigma_n) &\le (t\, n^{-\lambda}\, \sigma_n)^{-m}\, E|U_n'|^m \\ &\le t^{-m}\, n^{-m(1/2-\lambda)}\, L^m\, n^{(1-\alpha+\beta)m/2}\, m!\, e^{m\upsilon \log m} \\ &\le t^{-m}\, n^{-(\alpha-\beta-2\lambda)m/2}\, L^m\, e^{(\upsilon+1)m \log m} \end{aligned}$$

Hence

$$(5.3.13)\qquad \log P(|U_n'| > t\, n^{-\lambda}\sigma_n) \leq -m \log t - \frac{m}{2}(\alpha-\beta-2\lambda)\log n$$

$$+ (\upsilon+1)\, m \log m + m \log L .$$

We try to find an optimal value of m so that r.h.s of (5.3.12) is a minimum.

Differenciating the r.h.s of (5.3.13) w.r.t m and equating it to zero, we have

$$(5.3.14)\qquad -\log t - \frac{1}{2}(\alpha-\beta-2\lambda)\log n + (\upsilon+1)\log m + (\upsilon+1) +$$

$$+ \log L = 0$$

with a value of m

$$(5.3.15)\qquad m = \{n^{(\alpha-\beta-2\lambda)/2}\; t\, L^{-1}\}^{1/(\upsilon+1)}\; e^{-1}$$

In the above we conveniently ignore that m may not be an integer.

Therefore from (5.3.13) and (5.3.14) we have

$$(5.3.16)\qquad \log P(U_n'| > t\, n^{-\lambda}\sigma_n) \leq -m(\upsilon+1)$$

$$= -(\upsilon+1)\, e^{-1} \{n^{(\alpha-\beta-2\lambda)/2}\; t\, L^{-1}\}^{1/(\upsilon+1)}$$

and hence we have (5.3.10).

Finally to prove (5.3.11), let $\xi_i' = p^{-1/2}\,\xi_i$ and $\xi_{io} = \xi_i'\, I\,(|\xi_i'| \leq s\, t_n (k+1)^{1/2})$ where $s\ (>0)$ will be chosen accordingly. Applying lemma 5.2.1 once again and following the steps used to prove (5.3.10), we have

$$(5.3.17)\quad \left|P\left(\sum_{i=1}^{k+1} \xi_i' > t_n\, p^{-1/2}\sigma_n\right) - P\left(\sum_{i=1}^{k+1} \xi_{io} > t_n\, p^{-1/2}\,\sigma_n\right)\right|$$

$$\leq \sum_{i=1}^{k+1} P(|\xi_i'| > s(k+1)^{1/2}\, t_n)$$

$$\leq (k+1)\,\exp\left\{-a(t\, n^{(1-\alpha)/2})^{1/(\upsilon+1)}\right\}$$

$$\leq \exp\left\{-a(t\, n^{(1-\alpha)/2}\ \ {}^{1/(\upsilon+1)}\right\} \quad \text{for some } a > 0.$$

Next write $C_n = (\sigma_n^2/(k+1)p\,)^{1/2}$, $b_k = n\,\sigma_n^{-1}\, t_n\, k^{-1}\, p^{-1/2}$

$$f_i = E(\exp\,(b_k\,\xi_{io}))\,, \qquad g_k = E(\exp\,(b_k \sum_{i=1}^{k} \xi_{io})$$

$$m_i = f_i^{-1}\, E(\xi_{io}\exp\,(b_k\,\xi_{io}))\,, \qquad \overline{m} = (k+1)^{-1}\sum_{i=1}^{k+1} m_i \quad \text{and}$$

$$\overline{\sigma}^2 = (k+1)^{-1}\sum_{i=1}^{k+1}\left[\, f_i^{-1} E\,(\xi_{io}^2\,\exp(b_k\,\xi_{io})) - m_i^2\,\right]$$

Then after some routine steps we have the following representation

(5.3.18) $$P\left(\sum_{i=1}^{k+1} \xi_{io} > (k+1)^{1/2} t_n c_n\right) = A_k \int_{B_k}^{\infty} \exp(-D_k x)\, dH_k(x)$$

where $A_k = g_k \exp\left(-b_k \sum_{i=1}^{k+1} m_i\right)$

$$B_k = (b_k n^{-1} \sigma_n^2 - \bar{m})(k+1)^{1/2} / \bar{\sigma}$$

$$D_k = b_k (k+1)^{1/2} \bar{\sigma} \quad \text{and} \quad H_k(x) = G_k(\bar{\sigma} (k+1)^{1/2} x + (k+1)\bar{m})$$

Note that from (5.1.1), $E\, \xi_i' = 0$ and $E|\xi_i'|^m < \infty$ for every fixed m, uniformly in i and p, from lemma 5.2.1. So

(5.3.19) $$|E\, \xi_{io}| = |E\, \xi_i' - E\, \xi_i' I(|\xi_i'| > s\, t_n (k+1)^{1/2})|$$

$$= |E\, \xi_i' I(|\xi_i'| > s\, t_n (k+1)^{1/2})| = 0(k^{-2}).$$

Next we shall show

(5.3.20) $$E|\xi_{io}|^3 \exp(b_k |\xi_{io}|) < \infty \quad \text{for } t \text{ satisfying (5.3.7).}$$

Note that

(5.3.21) $$b_k = 0_e(k^{-1/2}\, t) \quad \text{and}$$

(5.3.22) $$E|\xi_{io}|^3 \exp(b_k |\xi_{io}|) \leq K(\varepsilon)\, E \exp\{(1+\varepsilon)\, b_k |\xi_{io}|\}$$

by Holders inequality, since all the moments of ξ_i' and hence of ξ_{io} exist; where $K(\varepsilon)$ is a constant depending on ε.

ε (> 0) can be made arbitrarily small.

Now

(5.3.23) $$E \exp\{(1+\varepsilon)\, b_k\, |\xi_{io}|\} \le E \exp(C\, t\, (k+1)^{-1/2}\, |\xi_{io}|)$$

for some $C > 0$ from (5.3.21)

$$= 1 + \int_0^{s(k+1)^{1/2}t} C\, t(k+1)^{-1/2} \exp\{C\, t(k+1)^{-1/2}\, L_1\} \cdot P(|\xi_{io}| > L_1)\, dL_1$$

(where one uses for $h \ge 0$, $h(0) = 0$; $Eh(x) = \int_0^\infty h'(t) P(|X|>t)\,dt$ and the fact that upper bound of $|\xi_{io}|$ is $s(k+1)^{1/2}\, t$)

$$\le 1 + \int_0^{s(k+1)^{1/2}t} C\, t(k+1)^{-1/2} \exp\{Ct(k+1)^{-1/2}\, L_1\} \cdot \exp(-a\, L_1^{1/(\upsilon+1)})\, dL_1$$

by lemma 5.2.1.

Since the integrand is a monotone function reaching its maximum at the upper end point of the integral, we have, multiplying the maximum of the integrand with the length of the interval of the integration

(5.3.24) $$E \exp\{(1+\varepsilon)\, b_k |\xi_{io}|\} \le 1 + C\, st^2 \exp\{C\, st^2 - a(k^{1/2}t)^{\frac{1}{\upsilon+1}}\}$$

for some $a > 0$.

Now the r.h.s of (5.3.24) is finite if for some C

$$C\,s\,t^2 - a(k^{1/2}\,t)^{1/(\upsilon+1)} < \infty$$

i.e., $C\,s\,t^{2-1/(\upsilon+1)} \leq a\,k^{1/2(\upsilon+1)}$

i.e., $t \leq (\frac{a}{Cs})^{(\upsilon+1)/(2\upsilon+1)}\,k^{1/2(2\upsilon+1)}$

Hence chosing s (> 0) sufficiently small (5.3.24) follows from (5.3.7).

We are now ready to estimate

$$(5.3.25)\quad f_i = E\exp(b_k\,\xi_{io})$$

$$= 1 + b_k E\,\xi_{io} + \frac{b_k^2}{2} E\,\xi_{io}^2 + O(\frac{b_k^3}{6} E|\xi_{io}^3|\exp(b_k|\xi_{io}|)$$

$$= 1 + \frac{b_k^2}{2} E\,\xi_{io}^2 + O(b_k^3)$$

$$= 1 + \frac{b_k^2}{2} E\,\xi_{io}^2 + O(k^{-3/2}\,t^3) \quad \text{from } (5.3.19)\text{-}(5.3.21)$$

Therefore

$$(5.3.26)\quad \sum_{i=1}^{k+1} \log f_i = \frac{b_k^2}{2} \sum_{i=1}^{k+1} \xi_{io}^2 + O(k^{-1/2}\,t^3)$$

Further following the lines of proof of lemma 6 of Ghosh and Babu (1977), one gets

(5.3.27) $$\left| (k+1)^{-1} \sum_{i=1}^{k+1} E\, \xi_{io}^2 - n^{-1} \sigma_n^2 \right| = 0(n^{-1})$$

From (5.3.26), (5.3.27) one gets

(5.3.28) $$\sum_{i=1}^{k+1} \log f_i = t_n^2/2 + 0(k^{-1/2} t^3)$$

Also

(5.3.29) $$m_i = f_i^{-1} E\, \xi_{io} \exp(b_k\, \xi_{io}) = b_k\, E\, \xi_{io}^2 + 0\left(\frac{b_k^2}{2} E|\xi_{io}^3| \cdot \exp(b_k |\xi_{io}|)\right)$$

$$= b_k\, E\, \xi_{io}^2 + 0(k^{-1} t^2)$$

Hence

(5.3.30) $$\bar{m} = (k+1)^{-1} \sum_{i=1}^{k+1} m_i = b_k (k+1)^{-1} \sum_{i=1}^{k+1} E\, \xi_{io}^2 + 0(k^{-1} t^2)$$

$$= b_k\, n^{-1} \sigma_n^2 + 0(k^{-1} t^2)$$

$$= (k+1)^{-1} \sigma_n\, t_n\, n^{-1/2} + 0(k^{-1} t^2)$$

And

(5.3.31) $$E\, \xi_{io}^2 \exp(b_k\, \xi_{io}) = E\, \xi_{io}^2 + 0(b_k), \quad \text{therefore}$$

(5.3.32) $$f_i^{-1} E\, \xi_{io}^2 \exp(b_k\, \xi_{io}) - m_i^2 = E\, \xi_{io}^2 - m_i^2 + 0(b_k), \quad \text{so}$$

(5.3.33) $$\bar{\sigma}^2 = n^{-1} \sigma_n^2 + 0(k^{-1/2} t)$$

Using the above estimates finally we have

(5.3.34) $$B_k = 0(k^{-1/2} t^2)$$

(5.3.35) $$D_k = t_n(1 + O(k^{-1/2}\ t))$$

Now arguing as in Babu, Ghosh and Singh (1978) one can show

(5.3.36) $$|g_k - \prod_{i=1}^{k+1} f_i| \leq k(k+1)\ \phi_q \prod_{i=1}^{k+1} (f_i + 2\phi_q \exp(s\ n\ \sigma_n^{-1} t_n^2 \cdot (p(k+1))^{-1/2}))_.$$

$$= O(n^{-2}) \exp\ (t_n^2/2)$$

where one uses the fact that $\phi(n) \leq \exp(-\lambda n)$ for some $\lambda > 0$, and

(5.3.37) $$\|H_k - \Phi\| = O(k^{-1/2})_.$$ Also

(5.3.38) $$A_k = g_k \exp(-b_k \sum_{i=1}^{k+1} n_i) = \exp\left\{\frac{t_n^2}{2} + O(k^{-1/2}\ t^3) - t_n^2\right\}$$

$$= \exp\left\{(-t_n^2/2) + O(k^{-1/2}\ t^3)\right\}$$

Finally we write

(5.3.39) $$|A_k \int_{B_k}^{\infty} \exp(-D_k x) dH_k(x) - \Phi(-t_n)| \leq I_1 + I_2 + I_3$$

where

(5.3.40) $$I_1 = A_k | \int_{B_k}^{\infty} \exp(-D_k x)\ d(H_k(x) - \Phi(x))|$$

$$\leq A_k \exp\ (-B_k\ D_k)\ \sup_x\ |H_k(x) - \Phi(x)|$$

$$\leq b\ k^{-1/2} \exp(-t^2/2 + O(k^{-1/2}\ t^3))$$

(5.3.41) $$I_2 = |A_k - \exp(-t_n^2/2)| \int_{B_k}^{\infty} \exp(-D_k\, x)\, d\Phi(x)$$

$$\leq b|A_k - \exp(-t_n^2/2)| \int_{B_k}^{\infty} \exp(-D_k\, x - x^2/2)\, dx$$

$$= b|A_k - \exp(-t_n^2/2)| \int_{B_k}^{\infty} \exp\left\{-\tfrac{1}{2}(x+D_k)^2 + \tfrac{1}{2} D_k^2\right\} dx$$

$$= b|A_k - \exp(-t_n^2/2)| \exp(D_k^2/2) \int_{B_k+D_k}^{\infty} \exp(-t^2/2)\, dt$$

$$\leq b|A_k - \exp(-t_n^2/2)| \exp(D_k^2/2) \exp\left(-\tfrac{1}{2}(B_k+D_k)^2\right)|B_k+D_k|^{-1}$$

$$\leq b|A_k - \exp(-t_n^2/2)| \exp(B_k^2/2 + 2|B_k D_k|)\, |B_k + D_k|^{-1}$$

Now $|B_k| \leq b\, t^2\, k^{-1/2} \leq \varepsilon_1 t$ since $t < \varepsilon_2 k^{1/2}$ for some $\varepsilon_2 > 0$

Hence $B_k + D_k \geq -\varepsilon_1 t + t(1 + O(k^{-1/2}t)) \geq (1-\varepsilon)t$ where ε (> 0) can be made arbitrarily small choosing ε_2 (> 0) sufficiently small. Hence

(5.3.42) $$I_2 \leq b \exp(-t^2/2)\, |\exp(O(t^3\, k^{-1/2})) - 1|\, t^{-1}$$

as $B_k^2 = O(t^4/k)$

$B_k D_k = O(t^3\, k^{-1/2})$

Finally

$$(5.3.43)\quad I_3 = \left|\exp(-t_n^2/2)\int_{B_k'}^{\infty}\exp(-D_k x)d\Phi(x) - \Phi(-t_n)\right|$$

$$\leq b\left|\exp(-t_n^2/2 + D_k^2/2)\,\Phi(-B_k - D_k) - \Phi(-t_n)\right|$$

$$\leq b\, t^{-1}\exp(-t^2/2)\left|\exp(O(t^3 k^{-1/2})) - 1\right|$$

Hence (5.3.11) follows from (5.3.17) and (5.3.39)-(5.3.43) completing the proof of the theorem.

As a consequence of theorem 5.3.1 let us find a zone where $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$. Since $\Phi(-t) \sim (2\pi)^{-1/2} \cdot t^{-1}\exp(-t^2/2)$ it follows that the restriction on t from the 1 st term of r.h.s. of (5.3.8) is

$$(5.3.44)\quad t = o(k^{1/6}) \text{ and } t \leq M k^{1/2(2\upsilon+1)} \text{ with } t < \varepsilon_2 k^{1/2}$$

$\varepsilon_2 > 0$ small.

Restriction on t from 2nd term of r.h.s of (5.3.8) is

$$(5.3.45)\quad t = o(n^{\lambda})$$

Denoting ε to be an arbitrary positive constant, we have, restriction on t from the 3rd term of r.h.s of (5.3.8) is

$$(5.3.46)\quad t^2 \leq \varepsilon\left\{t\, n^{(\alpha-\beta-2\lambda)/2}\right\}^{1/(\upsilon+1)} - \log t$$

$$\leq \varepsilon \{t\, n^{(\alpha-\beta-2\lambda)/2}\}^{1/(\upsilon+1)}$$

i.e., $t^{2-1/(\upsilon+1)} \leq \varepsilon\, n^{\frac{1}{2(\upsilon+1)}(\alpha-\beta-2\lambda)}$ i.e.,

(5.3.47) $t \leq \varepsilon\, n^{(\alpha-\beta-2\lambda)/2(2\upsilon+1)}$

Similarly restriction from the 4th term

(5.3.48) $t \leq \varepsilon\, n^{(1-\alpha)/2(2\upsilon+1)}$

Equating the powers of n in the r.h.s of (5.3.47) and (5.3.48) and letting $\beta \to 0$ we obtain $1-\alpha = \alpha - 2\lambda$ with a value of α,

(5.3.49) $\alpha = \lambda + 1/2$

For (5.3.44), note that $1/6 \gtrless \frac{1}{2(2\upsilon+1)} \Leftrightarrow \upsilon \gtrless 1$
Hence for $\upsilon > 1$, we have, from (5.3.44) in view of $k = O_e(n^{1-\alpha})$

$t = o(n^{(1-\alpha)/2(2\upsilon+1)})$, which has already been considered in (5.3.48). Therefore for $\upsilon > 1$ putting the value of α from (5.3.49) to (5.3.48) and equating the power of n with that of (5.3.45) we obtain

$$\lambda = (1/2 - \lambda)/2(2\upsilon+1), \quad \text{i.e.,} \quad \lambda = 1/2(4\upsilon+3)$$

So, for $\upsilon > 1$ we have $t = O(n^{(1/2(4\upsilon+3))-\varepsilon})$, where $\varepsilon > 0$

can be made arbitrarily small by sufficiently small choice of β in (5.3.47). For $\upsilon \leq 1$ since $(2(2\upsilon+1))^{-1} \geq 1/6$ we shall consider (5.3.44) which is more stringent than (5.3.48).

From (5.3.44) we have $t = o(n^{(1-\alpha)/6})$. Equating the power of n with that of (5.3.47) letting $\beta \to 0$, and that of (5.3.45) we have

(5.3.50) $$\frac{1-\alpha}{6} = \frac{\alpha-2\lambda}{2(2\upsilon+1)} = \lambda$$

1st and 3rd elements of (5.3.50) when equated gives $\alpha = 1-6\lambda$ and with this value 2nd and 3rd terms of (5.3.50) states

(5.3.51) $$\lambda = (2(2\upsilon+1) + 8)^{-1}$$

We now summerise the results in the following theorem.

<u>Theorem</u> 5.3.2 Under the assumptions of theorem 5.3.1, we have $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$ for $t_n = O(n^{C^*-\varepsilon})$, $\varepsilon > 0$ arbitrary and $C^* = \min\left\{(2(4\upsilon+3))^{-1}, (2(2\upsilon+1)+8)^{-1}\right\}$.

For a zone of t where $1 - F_n(t) \leq b \exp\left\{-\frac{t^2}{2}(1+o(1))\right\}$ following the proof of theorem 5.3.1 with $t_n = t(1\pm\varepsilon)$, $0 < \varepsilon < 1$, (5.3.8) takes the following form

$$(5.3.52) \quad |F_n(t)-\Phi(t)| \leq b|t|^{-1} \exp(-t^2/2)|\exp(O(|t|^3k^{-1/2}))-1|$$

$$+ b\exp(-t^2/2) + b\exp\left[-a\left\{|t|\,n^{(\alpha-\beta)/2}\right\}^{1/(\upsilon+1)}\right]$$

$$+ b\exp\left[-a\left\{|t|\,n^{(1-\alpha)/2}\right\}^{1/(\upsilon+1)}\right]$$

for $t^2 \leq M\,k^{1/(2\upsilon+1)}$ with $|t| < \varepsilon_2 k^{1/2}$, $\varepsilon_2 > 0$ small.

In this case, noting that $\Phi(-t) \sim (2\pi)^{-1/2}\, t^{-1}$. $\exp(-t^2/2)$ as $t \to \infty$, we find the region of t for which $1 - F_n(t) \leq b\exp\left[-\frac{1}{2}t^2(1+o(1))\right]$ holds.

Restriction from the 1st term of r.h.s of (5.3.52)

$$(5.3.53) \quad t^2 \leq M\,k^{1/(2\upsilon+1)} \text{ with } t = o(k^{1/2}) \text{ so that } |t|^3k^{-1/2}=o(t^2)$$

From 2nd term of r.h.s of (5.3.52) there is no restriction on t. Restriction from 3rd term

$$(5.3.54) \quad t = O(n^{\frac{\alpha}{2(2\upsilon+1)}-\varepsilon})$$

where $\varepsilon > 0$ can be made arbitrarily small choosing β small.

From the 4th term of r.h.s. of (5.3.52) restriction on t

$$(5.3.55) \quad t = O(n^{(1-\alpha)/2(2\upsilon+1)})$$

Let $\alpha = 1/2$, then from (5.3.54) and (5.3.55) we get

(5.3.56) $$t = O(n^{\frac{1}{4(2\upsilon+1)}-\varepsilon})$$

Since $k = O_e(n^{1-\alpha})$ we have from (5.3.53)

(5.3.57) $$t \leq M\, n^{1/4(2\upsilon+1)} \quad \text{with} \quad t = o(n^{1/4})$$

comparing (5.3.56) and (5.3.57) we have $t = O(n^{\frac{1}{4(2\upsilon+1)}-\varepsilon})$

Hence we have the following theorem.

<u>Theorem</u> 5.3.3 Under the assumptions of theorem 5.3.1

(5.3.58) $$1 - F_n(t_n) \leq b \exp\left\{-\frac{t_n^2}{2}(1+o(1))\right\}, \quad t_n \to \infty \text{ for}$$

$t_n = O(n^{\frac{1}{4(2\upsilon+1)}-\varepsilon})$ where $\varepsilon > 0$ is arbitrary.

To prove moment type convergences and nonuniform L_p verion of the Berry-Esseen theorem we need rates of convergence in the complementary zone of t of theorem 5.3.1. The following theorem states rates of convergence for $t^2 > C_1\, k^{1/(2\upsilon+1)}$ for any $C_1 > 0$.

<u>Theorem</u> 5.3.4 Under the assumptions of theorem 5.3.1, for $t^2 > C_1\, k^{1/(2\upsilon+1)}$, $C_1 > 0$, we have,

(5.3.59) $\quad |F_n(t)-\Phi(t)| \leq b \exp\left\{-|t|^{1/(\upsilon+2)} \; n^{\varepsilon} a\right\}$

for some $\varepsilon, a > 0$.

<u>Proof:</u> Recall (5.3.6). Define $U_n = \sum_{i=1}^{k} \xi_i$, $U_n' = \sum_{i=1}^{k} \eta_i$ and $T_n = \xi_{k+1}$. w.o.l.g let $t > 0$.

By lemma 5.2.1 and the procedure adopted to prove (5.3.10) we have

(5.3.60) $\quad P(|T_n| > t\, n^{1/2}) \leq b \exp\left\{-a(t\, n^{(1-\alpha)/2})^{1/(\upsilon+1)}\right\}$

Let $y = t^{-\alpha'} n^{-1/2} k^{\alpha'/2(2\upsilon+1)}$, $0 < \alpha' < 1$ to be chosen later

$$\xi_j^* = \xi_j I(|\xi_j| < y^{-1}) \quad \text{and} \quad U_n^* = \sum_{j=1}^{k} \xi_j^*$$

Then

(5.3.61) $\quad |P(U_n > t\, n^{1/2}) \quad P(U_n^* > t\, n^{1/2})| \leq \sum_{i=1}^{k} P(|\xi_i| > y^{-1})$

$$\leq b \exp\left\{-a(t^{\alpha'} n^{1/2 - \alpha/2 - \alpha'(1-\alpha)/2(2\upsilon+1)})^{1/(\upsilon+1)}\right\}$$

following the same lines of (5.3.60).

Now by Marcov's inequality and the lemma 2 of Ghosh, Babu and Singh (1978), one has,

(5.3.62) $\quad P(U_n^* > t\, n^{1/2}) \leq e^{-yt\, n^{1/2}} E(e^{yU_n^*})$

$$\leq \exp\left\{-t^{1-\alpha'} k^{\alpha'/2(2\upsilon+1)}\right\} \prod_{i=1}^{k} s_j \; .$$

where

(5.3.63) $$s_j = 2\,e\,\phi_p + E(e^{y\xi_j^*})$$

$$\leq 2\,e\,\phi_p + 1 + |E\,\xi_j^*|y + 2e\,y^2\,E(\xi_j^*)^2$$

To estimate s_j note that

(5.3.64) $$|E\,\xi_j^*| \leq |E\,\xi_j\,I(|\xi_j| > y^{-1})|$$

$$\leq y^m\,E|\xi_j|^{m+1} \quad \text{for every fixed} \quad m \geq 1$$

$$\leq b\exp\left\{-a(t^{\alpha'}\,n^{1/2}\;{}^{-\alpha/2}\;{}^{-\alpha'(1-\alpha)/2(2\upsilon+1)})^{1/(\upsilon+1)}\right\}$$

adopting the same procedure as that of (5.3.10).

Also note that $ky\exp\left\{-a(t^{\alpha'}\,n^{1/2}\;{}^{-\alpha/2}\;{}^{-\alpha'(1-\alpha)/2(2\upsilon+1)})^{1/(\upsilon+1)}\right\} \ll 1$.

And hence

(5.3.65) $$\log\prod_{i=1}^{k} s_i = 0(1 + y^2\sum_{i=1}^{k} E\,\xi_i^2) = 0(1 + k\,p\,y^2)$$

$$= 0(1)$$

since $k\,p\,y^2 = 0_e(n\,y^2) = 0_e(t^{-2}\,k^{1/(2\upsilon+1)})^{\alpha'} = 0(1)$

as $t^2 \geq C_1\,k^{1/(2\upsilon+1)}$ for some $C_1 > 0$.

Therefore from (5.3.62)

(5.3.66) $$P(U_n^* > t\,n^{1/2}) \leq b\exp\left\{-t^{1-\alpha'}\,k^{\alpha'/2(2\upsilon+1)}\right\}$$

Equating the exponent power of t from (5.3.61) and (5.3.66)

We obtain

$$1 - \alpha' = \alpha'/(\upsilon+1) \quad \text{or} \quad \alpha' = \frac{\upsilon+1}{\upsilon+2} .$$

With this choice of α', noting that

$$\frac{1}{2} - \frac{\alpha}{2} - \frac{\alpha'(1-\alpha)}{2(2\upsilon+1)} = \frac{1}{2}(1-\alpha)\left\{1 - \frac{\alpha'}{2(2\upsilon+1)}\right\} > 0 ,$$

$$\text{for } \alpha' = \frac{\upsilon+1}{\upsilon+2} \quad \text{as} \quad \alpha < 1 ,$$

we finally obtain from (5.3.60), (5.3.61) and (5.3.66).

(5.3.67) $$P(S_n > 3\, t\, n^{1/2}) \leq b \exp\left\{-t^{1/(\upsilon+2)}\, n^{\varepsilon}\, a\right\} \quad \text{for some } \varepsilon > 0,\ a > 0,$$

which proves the theorem in view of $\inf_{n \geq 1} n^{-1} \sigma_n^2 > 0$ and $\bar{\Phi}(-t) \leq b\, t^{-1} \exp(-t^2/2)$, $t > 0$.

As a consequence of theorem 5.3.1 and 5.3.4, we may obtain a non uniform bound over the entire range of t $-\infty < t < \infty$. For that choosing $\lambda = \frac{1}{6} - \varepsilon$, $\alpha = 2/3$ and letting $\beta \to 0$ in (5.3.8) alongwith (5.3.59) we obtain the following non-uniform bound proceeding as in theorem 2.4.9.

Theorem 5.3.5 Under the assumptions of theorem 5.3.1, one has for any $\lambda^* > 0$,

(5.3.68) $$|F_n(t)-\Phi(t)| \leq b\, n^{-\frac{1}{6}+\varepsilon} \exp(-\lambda^*|t|^{1/(\upsilon+2)})$$ where $\varepsilon > 0$ may be made arbitrarily small $-\infty < t < \infty$.

Subsequently we may obtain the following two theorems on moment convergence and on L_p version of the Berry Esseen theorem proof of which are similar to those of theorems 2.4.10 and 2.4.11.

Theorem 5.3.6 Under the assumptions of theorem 5.3.1, for any $p \geq 1$, $\lambda^* > 0$ and $q > 1$, one has

(5.3.69) $$\|\exp(\lambda^*|t|^{1/(\upsilon+2)})(1+|t|)^{-q/p}(F_n(t)-\Phi(t))\|_p = o(n^{-\frac{1}{6}+\varepsilon})$$

Theorem 5.3.7 Let $g : (-\infty, \infty) \to (0, \infty)$ be even with

(5.3.70) $$g'(x) = O(\exp(\lambda^* x^{1/(\upsilon+2)})(1+x)^{-q})$$ for some $q > 1$, $\lambda^* > 0$ and $0 < x < \infty$, $g(0)=0$.

Then under the assumptions of theorem 5.3.1 one has

(5.3.71) $$|Eg(\sigma_n^{-1} S_n) - Eg(T)| = O(n^{-\frac{1}{6}+\varepsilon})$$, $\varepsilon > 0$ is arbitrary , $T = N(0,1)$.

Next we shall find out the normal approximation zone when the third moment of the random variables are vanishing. i.e., $EX_i^3 = 0$, $i = 1, \ldots, n$. In this case proceeding as in the case of m-dependent process we have the following in place of (5.3.8).

(5.3.72) $$|F_n(t)-\Phi(t)| \leq b|t|^{-1} \exp(-t^2/2)|\exp(O(|t|^3 k^{-1/2} \cdot p^{-(1/6)+\varepsilon} + t^4 k^{-1})) - 1|$$

$$+ b\, k^{-1/2} \exp(-t^2/2 \; + O(|t|^3 k^{-1/2} p^{-(1/6)+\varepsilon} + t^4 k^{-1}))$$

$$+ b\, n^{-\lambda} \exp(-t^2/2) + b \exp\left[-a \left\{|t|\, n^{(\alpha-\beta-2\lambda)/2}\right\}^{1/(\upsilon+1)}\right]$$

$$+ b \exp\left[-a \left\{|t|\, n^{(1-\alpha)/2}\right\}^{1/(\upsilon+1)}\right]$$

Consequently (5.3.44) changes to (which takes care of 2nd term of the r.h.s of (5.3.72) as well)

(5.3.73) $$t = O(k^{1/6} p^{(1/18)-\varepsilon}), \quad \varepsilon > 0 \text{ is arbitrary},$$

$$t = o(k^{1/4}), \quad t \leq M k^{1/2(2\upsilon+1)} \text{ with } t \leq \varepsilon_2 k^{1/2}$$

$$\varepsilon_2 > 0 \quad \text{small.}$$

And (5.3.45), (5.3.47) and (5.3.48) remain the same, we write these equations again

(5.3.74) $$t = o(n^{\lambda})$$

(5.3.75) $t \le \varepsilon\, n^{(\alpha-\beta-2\lambda)/2(2\upsilon+1)}$

(5.3.76) $t \le \epsilon\, n^{(1-\alpha)/2(2\upsilon+1)}$

A simplified version of (5.3.73) is

(5.4.77) $t = O(n^{(1-\alpha)/6 \;+\alpha/18 \;-\varepsilon}) = O(n^{(3-2\alpha)/18 \;-\varepsilon})$,

$t = o(n^{(1-\alpha)/4})$, $t \le M\, n^{(1-\alpha)/2(2\upsilon+1)}$ with

$t \le \varepsilon_2\, n^{(1-\alpha)/2}$.

Since the third and 4th condition on t in equation (5.3.77) is redundant in view of more stringent condition (5.3.76), as $\upsilon \ge 0$, the final set of restrictions on t is

(5.3.78) $t = O(n^{(3-2\alpha)/18 \;-\varepsilon})$ with $t = o(n^{(1-\alpha)/4})$

(5.3.79) $t = o(n^{\lambda})$

(5.3.80) $t \le \varepsilon\, n^{(\alpha-\beta-2\lambda)/2(2\upsilon+1)}$

and

(5.3.81) $t \le \varepsilon\, n^{(1-\alpha)/2(2\upsilon+1)}$

where in the above sequal and in what follows $\varepsilon > 0$ is arbitrary (small) positive constant

Note that $\upsilon \gtrless 1/2 \iff 2(2\upsilon+1) \gtrless 4$. So, for $\upsilon > 1/2$ we shall consider (5.3.81) instead of $t = o(n^{(1-\alpha)/4})$ of (5.3.78).

In this case equating the powers of n from (5.3.79) - (5.3.81) and letting $\beta \to 0$ i.e.,

(5.3.82) $\lambda = (\alpha - 2\lambda)/2(2\upsilon+1) = (1-\alpha)/2(2\upsilon+1)$

(5.3.83) $\lambda = (2(4\upsilon+3))^{-1}$ with $\alpha = \lambda + 1/2$ (< 1)

and this value of λ satisfies: $\lambda \leq (3-2\alpha)/18 - \varepsilon$ (of (5.3.78)) as $\upsilon > 1/2$.

Hence for $\upsilon > 1/2$ with the choice (5.3.83) we have from (5.3.78) - (5.3.81) $t = O(n^{(2(4\upsilon+3))^{-1} - \varepsilon})$ where $\varepsilon > 0$ is arbitrary small letting $\beta \to 0$ in (5.3.80).

Next consider $\upsilon \leq 1/2$. In this case since $4 \geq 2(2\upsilon+1)$ we can ignore (5.3.81) in view of the 2nd equation of (5.3.78). Hence, as before, letting $\beta \to 0$ in (5.3.80) we have to select α, λ in such a way that minimum of $(3-2\alpha)/18$, $(1-\alpha)/4$, λ, $(\alpha-2\lambda)/2(2\upsilon+1)$ is maximised.

Equating the first and the last two elements we have

(5.3.84) $\lambda = 3/2(4\upsilon+13)$ with $\alpha = 6(\upsilon+1)/(4\upsilon+13)$

and this choice of λ and α satisfies; $\lambda \leq (1-\alpha)/4$ in view of $\upsilon \leq 1/2$. Hence for $\upsilon \leq 1/2$ we have $t = O(n^{3/2(4\upsilon+13) - \varepsilon})$ where $\varepsilon > 0$ is arbitrarily small. We now summerise the results as follows

Theorem 5.3.8 Under additional assumption $E X_i^3 = 0$, $i = 1,\ldots, n$ alongwith the assumptions of theorem 5.3.1 we have $1 - F_n(t_n) \sim \bar{\Phi}(-t_n)$ for $t_n = O(n^{C^*-\varepsilon})$, $t_n \to \infty$

where $C^* = \min\{(2(4\upsilon+3))^{-1}, 3/2(4\upsilon+13)\}$

and $\varepsilon > 0$ may be made arbitrarily small.

In (5.3.72) letting $\lambda = (1/4)-\varepsilon$, $\alpha = 3/4$ and $\beta \to 0$ alongwith theorem 5.3.4 it is possible to obtain the following theorem following the lines of theorem 5.3.5.

Theorem 5.3.9 Under the assumptions of theorem 5.3.8 for any $\lambda^* > 0$, one has

$$(5.3.85) \qquad |F_n(t) - \bar{\Phi}(t)| \leq b\, n^{-(1/4)+\varepsilon} \exp(-\lambda^* |t|^{1/(\upsilon+2)})$$

where $\varepsilon > 0$ may be made arbitrarily small

$-\infty < t < \infty$.

Consequently, under the assumptions of theorem 5.3.8 the order in theorems 5.3.6 and 5.3.7 can be sharpened to $n^{-(1/4)+\varepsilon}$ $0 < \varepsilon < 1/4$.

Next we consider the moment bound (5.3.3). The following theorem states non-uniform rates of convergence in an interval containing the origin

Theorem 5.3.10 Let the assumptions (5.1.1) - (5.1.4) and (5.3.2) hold for a non-stationary $\emptyset$-mixing process. Then for

(5.3.86) $$1 \leq t^2 \leq M (\log n)^{\upsilon/(\upsilon-1)} ,$$

there exists constant b (> 0) depending on M, such that for any $\lambda > 0$, $\varepsilon > 0$

(5.3.87) $$|F_n(t)-\Phi(t)| \leq b|t|^{-1} \exp(-t^2/2)| \exp(O(|t|^3 k^{-1/2}))-1|$$

$$+ b\, n^{-\lambda} \exp(-t^2/2)$$

$$+ b \exp\left[-(\upsilon-1) \left\{ \frac{(1-\varepsilon)}{\upsilon}\left(\frac{1}{2}(\alpha-\beta-2\lambda)\log n + \log |t|\right) \right\}^{\frac{\upsilon}{(\upsilon-1)}}\right]$$

$$+ b \exp\left[-(\upsilon-1) \left\{ \frac{(1-\varepsilon)}{\upsilon}\left(\frac{1}{2}(1-\alpha)\log n + \log |t|\right) \right\}^{\upsilon/(\upsilon-1)}\right]$$

Proof of the theorem: The proof of the above theorem is similar to that of theorem 5.3.1. We mention only the necessary modifications. From lemma 5.2.1 and (5.3.2), denoting L to be an arbitrary positive constant, note that for $t > 0$

(5.3.88) $$P(|U_n'| > t\, n^{-\lambda}\sigma_n) \leq (t\, n^{-\lambda}\, \sigma_n)^{-m}\, E|U_n'|^m$$

$$\leq t^{-m}\, n^{-m(\frac{1}{2}-\lambda)}\, L^m\, n^{(1-\alpha+\beta)m/2}\, m!\, \exp(m^\upsilon)$$

$$\leq t^{-m}\, n^{-(\alpha-\beta-2\lambda)m/2}\, L^m\, \exp(m^\upsilon + m\log m)$$

Foot note: For $t^2 \leq 1$ one may use the uniform bound obtained by putting $t = 1$ in (5.3.87) and that is obtainable following the proof of the theorem 5.3.10 with $t = 1$, see also (3.5.19a).

Therefore

(5.3.89) $\log P(|U_n'| > t\, n^{-\lambda} \sigma_n)$

$$\leq -m \log t - \frac{m}{2}(\alpha-\beta-2\lambda)\log n + m \log L + (m^{\upsilon} + m \log m)$$

Differenciating the r.h.s of (5.3.89) w.r.t m and equating it to zero, we have

(5.3.90) $- \log t - \frac{1}{2}(\alpha-\beta-2\lambda)\log n + \log L + (\upsilon m^{\upsilon-1} + 1 + \log m)$

$$= 0.$$

In view of the fact that $\log m = o(m^{\upsilon-1})$, $\upsilon-1 > 0$ we have solution of (5.3.90) as

(5.3.91) $m = \left[\frac{1}{(\upsilon \pm \varepsilon)} \left\{ \frac{1}{2}(\alpha-\beta-2\lambda)\log n + \log t - \log L - 1\right\}\right]^{\frac{1}{\upsilon-1}}$

where $\varepsilon > 0$ may be made arbitrarily small

Note that the 2nd derivative from (5.3.90) is $\upsilon(\upsilon-1)\, m^{\upsilon-2} + \frac{1}{m} > 0$ as $\upsilon > 1$.

Hence from (5.3.89) and (5.3.90)

(5.3.92) $\log P(|U_n'| > t\, n^{-\lambda} \sigma_n) \leq -(\upsilon-1)m^{\upsilon} - m \leq -(\upsilon-1)m^{\upsilon}$

$$\leq -(\upsilon-1) \left[\left\{ \frac{1}{(\upsilon+\varepsilon)} \left\{ \frac{1}{2}(\alpha-\beta-2\lambda)\log n + \log t - \log L - 1\right\}\right\}\right]^{\frac{\upsilon}{\upsilon-1}}$$

$$\leq -(\upsilon-1)\left[\frac{1}{(\upsilon+\varepsilon)}\left\{\tfrac{1}{2}(1-\varepsilon')(\alpha-\beta-2\lambda)\log n+\log t\right\}\right]^{\upsilon/(\upsilon-1)}$$

where $\varepsilon' > 0$ can be made arbitrarily small as $n \to \infty$

$$\leq -(\upsilon-1)\left[\frac{(1-\varepsilon)}{\upsilon}\left(\tfrac{1}{2}(\alpha-\beta-2\lambda)\log n+\log t\right)\right]^{\upsilon/(\upsilon-1)}$$

where $\varepsilon > 0$ can be made arbitrarily small

Similarly, for (5.3.17), note that

(5.3.93) $$\left|P\left(\sum_{i=1}^{k+1}\xi_i' > t_n\, p^{-1/2}\sigma_n\right) - P\left(\sum_{i=1}^{k+1}\xi_{io} > t_n\, p^{-1/2}\sigma_n\right)\right|$$

$$\leq \sum_{i=1}^{k+1} P(|\xi_i'| > s(k+1)^{1/2}t_n) \leq (k+1)(s(k+1)^{1/2}t_n)^{-m}E|\xi_i'|^m$$

$$\leq \exp\left[-(\upsilon-1)\left\{\frac{(1-\varepsilon)}{\upsilon}\left(\tfrac{1}{2}(1-\alpha)\log n+\log t\right)\right\}^{\upsilon/(\upsilon-1)}\right]$$

following (5.3.92)

Finally for (5.3.23) note that

(5.3.94) $$E\exp\{(1+\varepsilon)b_k|\xi_{io}|\} \leq 1+\int_0^{s(k+1)^{1/2}t} Ct\,(k+1)^{-1/2}\exp\{Ct(k+1)^{-\frac{1}{2}}L_1\}$$

$$\cdot\, P\{|\xi_{io}| > L_1\}\, dL_1 .$$

Dividing the range of integration $(0, s(k+1)^{1/2}t)$ into $(0, \lambda)$ and $[\lambda, s(k+1)^{1/2}t)$ where λ is a constant to be chosen later and noting that the integration over the 1st interval is a finite quantity for all k , we have

$$(5.3.95)\quad E \exp\{(1+\varepsilon)b_k\,|\xi_{io}|\} \leq \lambda_1 + \int_{\lambda}^{s(k+1)^{1/2}t} Ct\,(k+1)^{-1/2}\,.$$

$$.\exp\{Ct(k+1)^{-1/2}L_1\}\; P\{|\xi_{io}| > L_1\}\, dL_1$$

$$\leq \lambda_1 + \int_{\lambda}^{s(k+1)^{1/2}t} Ct\,(k+1)^{-1/2} \exp\{Ct(k+1)^{-1/2}L_1\}.$$

$$.\exp\{-\delta(\log L_1 - \log L)^{\upsilon/(\upsilon-1)}\}\, dL_1$$

for some $\delta > 0$, following (5.3.92).

Choose λ so large such that $(\log \lambda - \log L)^{\upsilon/(\upsilon-1)}$ is defined. R.H.S. of (5.3.95) is finite if (following (5.3.24))

$$C\,st^2 \exp\{C\,st^2 - \delta \log(s(k+1)^{1/2}t\,/L)^{\upsilon/(\upsilon-1)}\} < \infty$$

i.e., if $C\,st^2 \leq \delta(\log n)^{\upsilon/(\upsilon-1)}$ for some δ and C

$\because \log k = O(\log n)$

i.e., if $t^2 \leq (\frac{\delta}{Cs})(\log n)^{\upsilon/(\upsilon-1)}$

which covers the region (5.3.87), choosing $s > 0$ sufficiently small. Hence for t^2 satisfying (5.3.87), (5.3.95) is finite (uniformly in k).

The rest of the proof is similar to that of theorem 3.5.1.

As a consequence of the above theorem we may proceed to find normal-approximation zone. In view of the fact that $\Phi(-t) \sim (2\pi)^{-1/2} t^{-1} \exp(-t^2/2)$, $t \to \infty$ restriction from the 1st term on the r.h.s of (5.3.87) is

$$(5.3.96) \qquad t = o(k^{1/6}) = o(n^{(1-\alpha)/6})$$

That from 2nd term is

$$(5.3.97) \qquad t = o(n^{\lambda})$$

Restriction from the 3rd term of the r.h.s of (5.3.87) is

$$t^2/2 + \log t \leq (\upsilon-1) \left\{ \frac{(1-\varepsilon)}{\upsilon} (\tfrac{1}{2}(\alpha-\beta-2\lambda)\log n + \log t) \right\}^{\frac{\upsilon}{\upsilon-1}}$$

which, in view of $\log t = o(\log n)$ from (5.3.96) and (5.3.97) reduces to

$$(5.3.98) \qquad t^2 \leq 2(\upsilon-1) \left\{ \frac{(1-\varepsilon)}{\upsilon} \frac{1}{2}(\alpha-\beta-2\lambda)\log n \right\}^{\upsilon/(\upsilon-1)}$$

where $\varepsilon > 0$ can be made arbitrarily small.

Similarly restriction from the 4th term of the r.h.s of (5.3.87) is

$$(5.3.99) \qquad t^2 \leq 2(\upsilon-1) \left\{ \frac{(1-\varepsilon)}{\upsilon} \frac{1}{2} (1-\alpha)\log n \right\}^{\upsilon/(\upsilon-1)}$$

Letting $\alpha = 1/2$ and $\beta, \lambda \to 0$ we have from (5.3.98) and (5.3.99)

(5.3.100) $$t^2 \leq 2(\upsilon-1)\left(\frac{1-\varepsilon}{4\upsilon}\log n\right)^{\upsilon/(\upsilon-1)}, \quad \varepsilon > 0$$

can be made arbitrary small. Note that (5.3.100) satisfies (5.3.96) and (5.3.97).

Hence we have the following theorem.

Theorem 5.3.11 Under the assumptions of theorem 5.3.10 we have $1 - F_n(t_n) \sim \Phi(-t_n)$, $t_n \to \infty$ if

(5.3.101) $$t_n^2 \leq 2(\upsilon-1)(1-\varepsilon)\left\{\frac{1}{4\upsilon}\log n\right\}^{\upsilon/(\upsilon-1)}, \quad \upsilon > 1$$

where $\varepsilon > 0$ can be made arbitrarily small.

Next we prove a non uniform bound on the complementary zone of t of theorem 5.3.1. As before, the proof of the following theorem is similar to that of theorem 5.3.4.

Theorem 5.3.12 Under the assumptions of theorem 5.3.10, for $t^2 \geq C_1(\log n)^{\upsilon/(\upsilon-1)}$, $C_1 > 0$, and for any $\varepsilon > 0$ there exists b (> 0) depending on C_1 and ε such that

(5.3.102) $$|F_n(t)-\Phi(t)| \leq b\exp\left\{-(\upsilon-1)\left(\frac{1-\varepsilon}{\upsilon}\left(\frac{1}{2}\log n+\log|t|\right)\right)^{\frac{\upsilon}{\upsilon-1}}\right\}$$

Proof: Define U_n, U_n' and T_n as in theorem 5.3.4 and w.o.l.g let $t > 0$.

By lemma 5.2.1 and the procedure adopted to prove (5.3.93) we have

(5.3.103) $$P(|T_n| > tn^{1/2}) \leq b\exp\{-(\upsilon-1)(\frac{1-\varepsilon}{\upsilon}(\frac{1}{2}(1-\alpha)\log n + \log t))^{\upsilon/(\upsilon-1)}\}$$

Let $y = t^{-\alpha'} n^{-1/2} (\log n)^{\alpha'\upsilon/2(\upsilon-1)}$, $0 < \alpha' < 1$

$$\xi_i^* = \xi_j I(|\xi_j| < y^{-1}) \text{ and } U_n^* = \sum_{j=1}^{k} \xi_j^*$$

Then

(5.3.104) $$|P(U_n > t\, n^{1/2}) - P(U_n^* > t\, n^{1/2})| \leq \sum_{i=1}^{k} P(|\xi_i| > y^{-1})$$

$$\leq \sum_{i=1}^{k} P(p^{-1/2}|\xi_i| > n^{(1-\alpha)/2}\, t^{\alpha'} (\log n)^{\alpha'\upsilon/(\upsilon-1)})$$

$$\leq \exp\left[-(\upsilon-1)\left\{\frac{(1-\varepsilon)}{\upsilon}(\frac{1}{2}(1-\alpha)\log n + \alpha' \log t)\right\}^{\upsilon/(\upsilon-1)}\right]$$

following the same procedure as used in (5.3.93),

Proceeding as (5.3.62) one has

(5.3.105) $$P(U_n^* > t\, n^{1/2}) \leq \exp\{-t^{1-\alpha'} (\log n)^{\alpha'\upsilon/(\upsilon-1)}\} \prod_{i=1}^{k} s_j$$

where s_j is given by (5.3.63) i.e.,

(5.3.106) $$s_j = 2\, e\, \phi_p + E(e^{y\xi_j^*})$$

$$\leq 2\, e\, \phi_p + 1 + |E\xi_j^*|y + 2\, e\, y^2 E\, \xi_j^{*2}$$

From (5.3.64)

(5.3.107) $|E\,\xi_j^*| \leq y^m E\,|\xi_j|^{m+1}$ for every fixed $m \geq 1$

$$\leq \exp\left[-(\upsilon-1)\left\{\frac{1-\varepsilon}{\upsilon}\left(\frac{1}{2}(1-\alpha)\log n + \alpha'\log t\right)\right\}^{\upsilon/(\upsilon-1)}\right]$$

following the same technique as for (5.3.104).

Also note that $ky \exp\left[-(\upsilon-1)\left\{\frac{1-\varepsilon}{\upsilon}\left(\frac{1}{2}(1-\alpha)\log n + \alpha'\log t\right)\right\}^{\upsilon/(\upsilon-1)}\right]$

$$\ll 1.$$

And hence

(5.3.108) $\log \prod_{i=1}^{k} s_i = O(1 + y^2 \sum_{i=1}^{k} E\xi_i^2) = O(1 + kpy^2) = O(1)$

since $kpy^2 = O_e(ny^2) = O_e(t^{-2}(\log n)^{\upsilon/(\upsilon-1)})^{\alpha'} = O(1)$

as $t^2 \geq C_1(\log n)^{\upsilon/(\upsilon-1)}$

Therefore from (5.3.105)

(5.3.109) $P(U_n^* > t\,n^{1/2}) \leq b \exp\left\{-t^{1-\alpha'}(\log n)^{\alpha'\upsilon/(\upsilon-1)}\right\}$

we finally obtain from (5.3.103), (5.3.104) and (5.3.109)

(5.3.110) $P(S_n > 3t\,n^{1/2}) \leq b \exp\left[-(\upsilon-1)\left\{\frac{1-\varepsilon}{\upsilon}\left(\frac{1}{2}(1-\alpha)\log n\right.\right.\right.$

$$\left.\left.\left.+ \alpha'\log t\right)\right\}^{\upsilon/(\upsilon-1)}\right]$$

This proves, the theorem, letting $\alpha, \alpha' \to 0$ noting that $\varepsilon > 0$ is arbitrary, $\inf_{n\geq 1} n^{-1}\sigma_n^2 > 0$ and $\Phi(-t) \leq b\, t^{-1} . \exp(-t^2/2)$, $t \to \infty$.

Using (5.3.87) or a version of (5.3.72) when (5.3.2) holds instead of (5.3.1), in the case $EX_i^3 = 0$; alongwith (5.3.102) we may obtain non uniform bound over the entire region of t. Note that since $\upsilon > 1$, $\log n = o(\log n)^{\upsilon/(\upsilon-1)}$. And hence putting $\lambda = \frac{1}{6} - \varepsilon$, $\alpha = 2/3$ and letting $\beta \to 0$ in (5.3.87) ; $\lambda = \frac{1}{4} - \varepsilon$, $\alpha = 3/4$ and $\beta \to 0$ for the version of (5.3.72) we obtain the following non uniform bound in view of (5.3.102)

<u>Theorem</u> 5.3.13 Under the assumptions of theorem 5.3.10 for any $\varepsilon > 0$ and $\lambda > 0$ there exists a constant $b > 0$ depending on ε and λ such that

(5.3.11) $$|F_n(t) - \Phi(t)| \leq b\, n^{-\frac{1}{6}+\varepsilon} \exp\left[-(\upsilon-1)\left\{\lambda + \frac{1-\varepsilon}{\upsilon}\log(1+|t|)\right\}^{\frac{\upsilon}{\upsilon-1}}\right]$$

further if $E\,X_i^3 = 0 \ \forall\ i$ then

(5.3.112) $$|F_n(t) - \Phi(t)| \leq b\, n^{-\frac{1}{4}+\varepsilon} \exp\left[-(\upsilon-1)\left\{\lambda + \frac{1-\varepsilon}{\upsilon}\log(1+|t|)\right\}^{\frac{\upsilon}{\upsilon-1}}\right]$$

Subsequently, in view of theorem 5.3.13, we may prove moment type convergences and non-uniform L_p versions of the Berry-Esseen theorem.

Theorem 5.3.14 Under the assumptions of theorem 5.3.10, for any $\varepsilon > 0$, $\lambda > 0$ and $p \geq 1$

(5.3.113) $$\| \exp \left[(\upsilon-1) \left\{ \lambda + \frac{1-\varepsilon}{\upsilon} \log |t| \right\}^{\upsilon/(\upsilon-1)} \right] (F_n(t) - \Phi(t)) \|_p$$

$$= O(n^{-\frac{1}{6}+\varepsilon})$$

$$= O(n^{-\frac{1}{4}+\varepsilon}) \quad \text{if} \quad E X_i^3 = 0, \quad i=1,\ldots,n$$

Theorem 5.3.15 Under the assumptions of theorem 5.3.10, for an even function $g : (-\infty, \infty) \to (0, \infty)$ with $g(0) = 0$ satisfying

(5.3.114) $$g'(x) = O(\exp \left[(\upsilon-1) \left\{ \lambda + \frac{1-\varepsilon}{\upsilon} \log (1+x) \right\}^{\upsilon/(\upsilon-1)} \right]), \quad x>0$$

one has, for $T = N(0, 1)$,

(5.3.115) $$|E\, g(\sigma_n^{-1} S_n) - E\, g(T)| = O(n^{-\frac{1}{6}+\varepsilon})$$

$$= O(n^{-\frac{1}{4}+\varepsilon}), \quad \varepsilon>0 \text{ arbitrary}$$

$$\text{if } EX_i^3 = 0, \quad i = 1,\ldots, n.$$

Proofs of above two theorems are similar to theorems 2.4.10 and 2.4.11 and are omitted.

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