A RAPID METHOD OF ESTIMATING BASAL AREA IN TIMBER SURVEY—AN APPLICATION OF INTEGRAL GEOMETRY TO AREAL SAMPLING PROBLEMS*

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1. The Bitterlich's method (Bitterlich, 1948) or the variable plot radius method (Grosenbaugh, 1952)—its literal translation is the angular count sampling method—of estimating tree basal area per acre is a very ingenious one. It is true that his method does not require the measurement of plot radius or tree diameter at breast height, but it is assumed that the cross-section of each is a circle and the tree itself is transparent. Otherwise, his method might give a biased estimate.

In our new rapid method—the distance count sampling method—these two assumptions are removed and only the convexity of the cross section is assumed instead. In his method, only the estimate of the total of areas of cross sections is given, whereas in our method, the total number of trees, the total of perimeters and the total of areas are given simultaneously. A method of removing the bias due to the boundary effect is also given in this article.

2. In our method, to estimate basal area ϕ for a stand, the cruiser simply visits a set of points sampled at random, and counts at each point every tree whose distance appears to be less than assigned value r_i when viewed through the range-finder. ($i=1, 2, \dots, N \geqq 3$). If the ground at a visited point slopes, the r_i is multiplied by the secant of the angle of the slope in the direction of viewed tree, i.e., its horizontal distance should be always less than r_i. For every i, assume that n_i points are visited and k_i trees are counted, crediting overlaps, then the average $k_i/n_i = \hat{p}_i$ is an almost unbiased estimate of

$$p_i = (\phi + r_i\lambda + \nu\pi r_i^2)/T \qquad \dots \quad (1)$$

where λ and ν are the sum of the length of boundary of each cross section at breast height and the total number of trees on the land in question, the area of which is T. The unknown parameters ϕ, λ and ν are estimable as the solutions of the linear simultaneous equations (1). If the parameter ν, viz. the total number of trees in the land, is known, the fundamental equation is put in the following form:

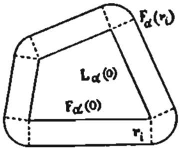
$$p_i T - \nu\pi r_i^2 = \phi + \lambda r_i, \quad (i = 1, 2, \dots, N \geqq 2). \qquad \dots \quad (2)$$

* Read at the Autumnal Meeting, Mathematical Society of Japan, November, 1952.

3. The proof is simple, if we consider the area $F_\alpha(r_1)$ of a parallel oval at the distance r_1 for the α-th oval, whose area is $F_\alpha(0)$ with the boundary of length $L_\alpha(0)$. Then there exists the Steiner's relation (Blaschke, 1916, 1935, 1937) viz.

$$F_\alpha(r_1) = F_\alpha(0)+r_1 L_\alpha(0)+\pi r_1^2, \qquad (\alpha = 1, 2, \ldots, \nu), \qquad \ldots \quad (3)$$

from which we deduce the fundamental relation



$$\sum_{\alpha=1}^{\nu} F_\alpha(r_1) = p_1 T$$

$$= \sum_{\alpha=1}^{\nu} F_\alpha(0)+r_1 \sum_{\alpha=1}^{\nu} L_\alpha(0)+\nu\pi r_1^2 \qquad \ldots \quad (4)$$

$$= \phi+r_1\lambda+\nu\pi r_1^2.$$

To count a tree is equivalent to plot an observational point in its parallel area. This is our basic idea.*

In actual survey, it would be

$$\phi/\nu<\pi r_1^2, \qquad \ldots \quad (5)$$

but the precision of estimates $\hat{p}_1$ should be so high that the small ϕ might not be "covered up" by the too large $\nu\pi r_1^2$.

4. If we take into consideration the boundary effect on the estimates, and use the same notation for point set and its measure, the correction is given by

$$R_\alpha(r_1) = F_\alpha(r_1)-F_\alpha(r_1)\cap T \qquad \ldots \quad (6)$$

Then the estimate $\hat{p}_1 T = A_1$ is an unbiased estimate of the total of $F_\alpha(r_1)\cap T$. To remove this bias, we should only select at random n_1 points within an enlarged area which contains a parallel curve of the land boundary at the distance r_1. In this case the boundary curve of the original land need not be an oval. However, this enlargement may increase the variance of estimate $\hat{p}_1$. Fortunately this bias would be negligiblely small, if the actual land is sufficiently large and not too slender. The sampling method is needed just for such a land.

5. If observations are performed at 3 levels of r_1, the estimates are given be

$$\Delta\hat{\phi} = A_1 r_2 r_3(r_3-r_2)+A_2 r_3 r_1(r_1-r_3)+A_3 r_1 r_2(r_2-r_1), \qquad \ldots \quad (7)$$

$$\Delta\hat{\lambda} = A_1(r_2^2-r_3^2)+A_2(r_3^2-r_1^2)+A_3(r_1^2-r_2^2), \qquad \ldots \quad (8)$$

$$\pi\Delta\hat{\nu} = A_1(r_3-r_2)+A_2(r_1-r_3)+A_3(r_2-r_1), \qquad \ldots \quad (9)$$

* The fixed ovals, i.e., the cross sections, and the movable oval, i.e., the circle of radius r_1 are the open and bounded point sets.

where we put

$$\Delta = (r_1-r_2)(r_2-r_3)(r_3-r_1). \qquad \dots \quad (10)$$

If we put further

$$r_1 = d, \quad r_2 = 2d \quad \text{and} \quad r_3 = 3d, \qquad \dots \quad (11)$$

then the following simple formula are obtained:

$$\hat{\phi} = A_3 - 3(A_2 - A_1), \qquad \dots \quad (12)$$

$$\hat{\lambda} = (8A_2 - 5A_1 - 3A_3)/(2d), \qquad \dots \quad (13)$$

and

$$\hat{\nu} = (A_1 + A_3 - 2A_2)/(2\pi d^2). \qquad \dots \quad (14)$$

Note that $\hat{\phi}$ is independent of d. Although this result is simple, but this design may not be "optimal" to estimate ϕ. To define the optimal design, we must take into consideration various circumstances which we might meet in actual survey. However, we shall confine ourselves to discuss only the variance of estimate.

6. The sampling error is estimated as a random fluctuation of counts per visited point (see Appendix B). We shall denote the standard deviation in population by σ with variate in bracket. Then we can put

$$r_1 = 1, \; r_2 = 1+X \; (X>0), \text{ and } \; r_3 = 1+X+Y \; (Y>0) \qquad \dots \quad (15)$$

and

$$\sigma(A_2) = (1+\xi)\sigma(A_1), \; \sigma(A_3) = (1+\eta)\sigma(A_1), \qquad \dots \quad (16)$$

without loss of generality, because ϕ is a function of the ratio $r_1 : r_2 : r_3$. Then, if the estimates $\hat{p}_i$'s are mutually uncorrelated—generally speaking, it need not be so—the ratio $\sigma^2(\hat{\phi})/\sigma^2(A_1)$ is given by

$$f(X, Y) = \left(\frac{1+X}{X}\right)^2 \left(\frac{1+X+Y}{X+Y}\right)^2 + \frac{(1+\xi)^2(1+X+Y)^2}{X^2Y^2} + \frac{(1+\eta)^2(1+X)^2}{Y^2(X+Y)^2} \qquad \dots \quad (17)$$

As X and Y are positive the equations

$$XY-(1+\xi)(1+X+Y) = (X-1)(Y-1)-2-\xi(1+X+Y), \qquad \dots \quad (18)$$

and

$$Y(X+Y)-(1+\eta)(1+X) = (1+X+Y)(Y-1)-\eta(1+X) \qquad \dots \quad (19)$$

suggest that if both ξ and η are non-negative and finite*, X and Y should be greater than 1 to minimize $f(X, Y)$. $f(X, Y)$ is always greater than 1 and is equal to 1, when both X and Y tend to infinity. Thus the greater both X and Y are, the better. In

* We assume here that ξ and η are finite. But in general ξ and η are an increasing function of X and of $(X+Y)$ respectively. Hence $f(X, Y)$ may or may not have an extremum. We owe this remark to Dr. D. J. Finney.

actual survey the magnitude of r_1 may be limited by the accuracy of range-finder and the number of countable trees.

If r_1 and r_3 are fixed simultaneously, there exists in general an optimal solution for r_2. Let r_1, r_2 and r_3 be 1, x and b respectively and assume the variances of A_i are all the same, then the variance ratio $\sigma^2(\hat{\phi})/\sigma^2(A_1)$ is equal to

$$f(x) = \frac{b^2x^2}{(1-b)^2(1-x)^2}+\frac{b^2}{(1-x)^2(x-b)^2}+\frac{x^2}{(1-b)^2(x-b)^2}, \quad \dots \quad (20)$$

where $1<x<b$.

The solution of $df/dx = 0$ will be that of

$$g(x) = bx(x-b)^3+b(1-b)^2(2x-b-1)+x(x-1)^3 = 0. \quad \dots \quad (21)$$

As

$$g(1) = -g(b) = 2b(1-b)^3<0, \quad \dots \quad (22)$$

there should be at least one y between 1 and b for which

$$g(y) = 0. \quad \dots \quad (23)$$

For example, if $r_1 = 1$ and $r_3 = 6$, then $y = 3.90$. We may put $r_2 = 4$, because there is no remarkable change of $f(x)$ near $x=y$.

7. As an illustration, we shall show an experimental result obtained by Miss Y. Kuroiwa. The parameters used are given in the Tables 1 and 2.

Table 1.

	ϕ	λ	ν	ϕ/ν
True	7,300	1,344	18	406
Estimated	5,600	1,568	12.5	448
Corrected	5,896	1,536	15.5	380

She selected at random $N=100$ points with coordinates (x_j, y_j). $(j=1,2,\dots,100)$. These 200 figures were selected at random from a rectangular population. The figures in the 3rd row of the Table 1 are the estimates corrected by (6). The total area T is 56,000. She used the same point for three levels of r.

Table 2.

			estimated	
r_1	p_1	$\hat{p}_1$	S.D.	C.V.
10	0.471	0.45	0.0500	11.1
20	1.014	0.94	0.0547	5.82
30	1.866	1.57	0.0756	4.82

8. If the cross section of tree be circular, we have from (4) the following formula:

$$p_1T = \pi \sum_{s=1}^{\nu} (a_s+r_1)^2, \quad \dots \quad (24)$$

where a_α is the radius of the α-th cross section. In the Bitterlich's method, r_i is so selected that the ratio $(a_\alpha+r_i)/a_\alpha$ is constant for all trees.

In this case the formula (24) can be put in another useful form. Let m_a, σ_a^2 and $G(a)$ be the population mean, the population variance and the distribution function of radius a_α for the land (the population). Then by definition we have

$$\phi = \pi \sum_{\alpha=1}^{\nu} a_\alpha^2 = \nu\pi \int_{-\infty}^{\infty} a^2 dG(a) = \nu\pi(\sigma_a^2+m_a^2) \qquad \dots \quad (25)$$

and

$$\lambda = 2\pi \sum_{\alpha=1}^{\nu} a_\alpha = \nu 2\pi \int_{-\infty}^{\infty} a dG(a) = 2\nu\pi m_a, \qquad \dots \quad (26)$$

and accordingly

$$\begin{aligned} p_i T &= \nu\pi(\sigma_a^2+m_a^2+2m_a r_i+r_i^2) \\ &= \nu\pi[\sigma_a^2+(m_a+r_i)^2], \qquad (i=1,2). \end{aligned} \qquad \dots \quad (27)$$

After some reduction we have for $r_1 \neq r_2$

$$m_a = \tfrac{1}{2}\left\{\left(\frac{p_2-p_1}{r_2-r_1}\right)\frac{T}{\nu\pi} - (r_1+r_2)\right\}, \qquad \dots \quad (28)$$

$$\sigma_a^2 = \tfrac{1}{2}\left\{(p_1+p_2)\frac{T}{\nu\pi} - \{(r_1+m_a)^2+(r_2+m_a)^2\}\right\} \qquad \dots \quad (29)$$

These two equations show that m_a and σ_a are estimable, if $p_i T$ and ν, or p_i and ν/T are estimable.

9. We shall show another experimental result obtained by Miss Y. Kuroiwa, where the total number of trees is known. In the area of size $T=56{,}000$ in which there are 23 trees of various size and shape, she selected at random $N=100$ points and counted the number of trees in prescribed manner. The true values and their estimates are given in Table 3.

Table 3

	ϕ	λ
True	9,813	1,846
Estimated	13,331	1,472

Table 4

r_i	p_i	$\hat{p}_i$	estimated S.D.	estimated C.V.
10	0.6338	0.63	0.0569	9.03
20	1.3500	1.28	0.0645	5.04

In this case the parameters were estimated by the following formulas:

$$\phi = \left(\frac{r_2\hat{p}_1-r_1\hat{p}_2}{r_2-r_1}\right)T+\nu\pi r_1 r_2 \qquad \dots \quad (30)$$

and

$$\hat{\lambda} = \left(\frac{\hat{p}_2-\hat{p}_1}{r_2-r_1}\right) T - \nu\pi(r_1+r_2). \quad \ldots \quad (31)$$

Let the true error of $\hat{p}_i$ be ϵ_i, viz. $\hat{p}_i = p_i+\epsilon_i$, then we have

$$\frac{\hat{\phi}-\phi}{\phi} = \frac{r_2\epsilon_1-r_1\epsilon_2}{r_2-r_1}\left(\frac{T}{\phi}\right) \quad \ldots \quad (32)$$

and

$$\frac{\hat{\lambda}-\lambda}{\lambda} = \frac{\epsilon_2-\epsilon_1}{r_2-r_1}\left(\frac{T}{\lambda}\right) \quad \ldots \quad (33)$$

These formulas may be useful to determine the size of sample, if we can get the crude estimates of ϵ_i's. They show us also that the positive correlation of two estimates $\hat{p}_1$ and $\hat{p}_2$ is favourable to reduce the relative errors of $\hat{\phi}$ and $\hat{\lambda}$.

10. If we define a parameter C called "the shape factor" by

$$F_a(0) = CL_a^2(0)+e_a, \quad \sum_{a=1}^{\nu} e_a = 0 \quad \ldots \quad (34)$$

then the variance of perimeters $L_a(0)$ is given by

$$\sigma_L^2 = \frac{1}{C}\left(\frac{\phi}{\nu}\right) - \left(\frac{\lambda}{\nu}\right)^2 \quad \ldots \quad (35)$$

C is less than or equal to $1/4\pi$ by the isoperimetric inequality. The variance of areas $F_a(0)$ will be given by the following approximate formula

$$\sigma_F^2 = 4C\left(\frac{\phi}{\nu}\right)\sigma_L^2 = 4\left[\left(\frac{\phi}{\nu}\right)^2 - C\left(\frac{\phi}{\nu}\right)\left(\frac{\lambda}{\nu}\right)^2\right] < 4\left(\frac{\phi}{\nu}\right)^2 \quad \ldots \quad (36)$$

if $\nu\sigma_L/\lambda$ is not so large.

11. In our rapid method, if there is a tree, which should be counted, behind another tree, the usual method of topographical survey is applicable. It is a very easy operation to determine whether the tree in question lies in or contacts with the circle of radius r_i, the visited point being the center. We assume here the simple random sampling method. However, the p_i is estimable by other sampling methods, say, by the systematic sampling method with random start.

The same idea may be applied to other field surveys. Fo example, to estimate the acreages of different crops simultaneously without measuring the actual size of each area. The enumerator simply counts the number of areas on which such and such crop grows, within the circle of radius r_i, the point selected at random being the centre. In the haematological techniques, to estimate the characteristics of the

Price-John's curve, the technician should only count the number of erythrocytes in the view of microscope. He need not measure the diameter of each cell. There may be further the crystallographic, or the histological applications.

We use the movable circle as a movable oval. But there may be other convex closed curves, say square, triangle etc. (Hadwiger, 1948; Nöbeling, 1949) provided that the fundamental formula is not so simple as ours.

Dr. D. J. Finney has kindly read the first manuscript and pointed out our inaccuracy of statement about the behavior of $f(X, Y)$ at infinity. The author wishes to express his gratitude to him.

APPENDICES

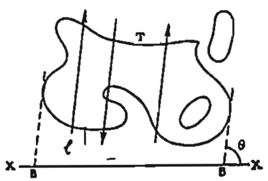
A. Simplified or distance count line sampling method

1. In the distance count sampling method the cruiser visits several points which are selected at random. However, in actual timber survey, to visit the selected points is not so easy. If we could assume that the cross section of tree be circular, we might simplify the method of survey as follows.

In the simplified method the cruiser only walks along every line which is selected at random and is parallel to a fixed direction l. He counts every tree whose distance appears to be less than r_i $(i=1, 2, \ldots, N > 2)$.

2. Let the area in question be T, and settle a base line XX at breast height which is used as a reference line. The cruiser should walk along a line which makes a constant angle θ with XX. To make sure, the same area might be scanned in two ways, viz. $\theta=0°$ and $\theta=90°$, say.

The area T may be neither simply connected nor convex. But, to reduce sampling error, it would be better to divide T into several sub-areas, which are simple in shape and homogeneous in respect of number of trees per acre. We shall consider in what follows one of these sub-areas and name it a new T.



Consider on XX a segment BB which should contain all projections of the cross sections at breast height parallel to the line l.

To estimate the total number of trees in T, say ν, and the mean diameter δ/ν of these trees, we use the fundamental equations

$$p_i b = \delta + 2\nu r_i, \qquad \dots \quad (A.1)$$

where b is the length of segment BB and an almost unbiased estimate of p_i is the total number of counted trees, crediting overlaps, divided by the number of parallel lines along which the cruiser walked.

The band of width $2r_i$, the line l being its central line, may be called the strip of width $2r_i$. It is essential to count the total number of trees within or being in contact with this strip, so that we may use any other means instead of range-finder. As an actual line has a non-zero width, to reduce the bias due to this non-zero width we should count as one only the tree which lies outside the strip but contacts with the right or left boundary of a strip. The similar note is valid for the movable circle.

The length of each strip does not contribute anything to p_i directly, as is seen in (A.1). However, it may contribute to the cost function of the survey and may or may not affect the variance of the estimate $\hat{p}_i$.

The intersection points of central lines and the segment BB should be selected by using random sampling numbers from a rectangular population. The method of removing bias is just like the one introduced in the text.

If the cruiser counts the number of trees on one side of the line l, the numerical coefficient of the second term of (A.1) should be replaced by 1.

As is easily seen, the cross section of the tree need not be circular. If not, δ is the sum of the projection of the cross section at breast height parallel to l on BB.

3. When $N=2$, the estimates are given as follows:

$$\hat{\delta} = b(r_2\hat{p}_1 - r_1\hat{p}_2)/(r_2 - r_1), \qquad \dots \quad (A.2)$$

and

$$\hat{\nu} = (b/2)(\hat{p}_2 - \hat{p}_1)/(r_2 - r_1). \qquad \dots \quad (A.3)$$

The relative errors of these estimates are given by (32) and (33), in which ϕ, λ and T should be replaced by δ, 2ν and b respectively.

To make the relative error of $\hat{\delta}$ small, we have to make (b/δ) small and the correlation between $\hat{p}_1$ and $\hat{p}_2$ positively large. The similar minimizing conditions of the relative error ν is obvious from (33).

4. As an illustration, we shall show a statistical model experiment done by Miss Y. Kuroiwa.

Table A—1

	δ	ν	δ/ν
True	3,350	280	12
Estimated	3,643	261	14.0

Table A—2

r_i	p_i	$\hat{p}_i$	estimated S.D.	estimated C.V.
0	13.96	15.18	0.486	3.2
5	25.63	26.06	0.614	2.4

In her experiment $b=240$ and the number of sampled lines was 50.

The similar simplified method is possible for the Bitterlich's method.

B. On the variance function

1. Suppose the observed frequency of counted trees per point is k_t $(t=0,1,2,\ldots,\nu)$, viz. we count no tree k_0 times, one tree k_1 times, two trees k_2 times etc. and let the expectation of k_t be NP_t. Then we have the multinomial variates $(k_0, k_1, \ldots, k_\nu)$ and

$$\sum_{t=0}^{\nu} P_t = 1, \qquad \sum_{t=0}^{\nu} k_t = N, \qquad \ldots \quad (B.1)$$

and

$$\sum_{t=0}^{\nu} t\, k_t = N\hat{p} \quad (= K, \text{ say}). \qquad \ldots \quad (B.2)$$

For the sake of simplicity, we drop here the suffix i.

As is easily proved, the moment generating function of a linear from

$$w = \sum_{t=0}^{\nu} h_t\, k_t, \qquad (h_t = \text{constant}), \qquad \ldots \quad (B.3)$$

is

$$\psi(\theta) = \left(\sum_{t=0}^{\nu} P_t e^{h_t \theta} \right)^N \qquad \ldots \quad (B.4)$$

Hence the mean and the variance of the estimate $\hat{p}$ in population are

$$m\{\hat{p}\} = \sum_{t=0}^{\nu} P_t\, t \qquad (= \bar{t}, \text{ say}) \qquad \ldots \quad (B.5)$$

and

$$\sigma^2\{\hat{p}\} = \sum_{t=0}^{\nu} P_t (t-\bar{t})^2/N \qquad \ldots \quad (B.6)$$

respectively. From (B.6) we have an inequality

$$\sigma^2\{\hat{p}\} < \nu^2/(4N). \qquad \ldots \quad (B.7)$$

2. As to use the same point for different levels of r_i is economical from the viewpoint of the journey cost, we shall consider two different values of the suffix i, say 1 and 2 at the same point and assume that $r_1 < r_2$. Then $P_t(r_i)$ wil be represented as follows:

$$P_t(r_1) = P_{tt} + P_{tt+1} + \cdots + P_{t\nu}, \qquad \ldots \quad (B.8)$$

and

$$P_t(r_2) = P_{0t} + P_{1t} + \cdots + P_{tt}, \qquad \ldots \quad (B.9)$$

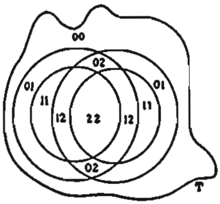
where P_{ts} means the expectation of k_{ts}/N and k_{ts} means the number of points, at which we count t trees for $r = r_1$ and s trees for $r = r_2$.

If we define

$$P_{ts} = 0 \quad \text{for all} \quad t > s \qquad \dots \text{(B.10)}$$

and

$$k_t = 0 \quad \text{for all} \quad t > s. \qquad \dots \text{(B.11)}$$



then (B.8) and (B.9) will be put in the following compact forms:

$$P_t(r_1) = \sum_{s=0}^{\nu} P_{ts} \qquad \dots \text{(B.12)}$$

and

$$P_t(r_2) = \sum_{t=0}^{\nu} P_{ts}. \qquad \dots \text{(B.13)}$$

Then we have by definition

$$K(r_1) = Np_1 = \sum_{t=0}^{\nu} t \sum_{s=0}^{\nu} k_{ts} \qquad \dots \text{(B.14)}$$

and

$$K(r_2) = Np_2 = \sum_{s=0}^{\nu} s \sum_{t=0}^{\nu} k_{ts}. \qquad \dots \text{(B.15)}$$

As the variates (k_{ts}) is distributed multinomially, the formula (B.4) gives us the mean and the variance of $K(r_1)$ in the population which need not be written down.

The covariance function between $K(r_1)$ and $K(r_2)$ is proved to be

$$\mathrm{cov}\{K(r_1), K(r_2)\} = N \sum_{t=0}^{\nu} \sum_{s=0}^{\nu} ts\,[P_{ts} - \sum_{j=0}^{\nu} \sum_{i=0}^{\nu} P_{tj}P_{is}], \qquad \dots \text{(B.16)}$$

which shows that the covariance is positive when $\nu = 1$ but it is not necessarily positive when $\nu > 1$.

C. On Mahalanobis' empirical formula

If we use a square of size a_i^2 ($i=1,2,\ldots,N>3$) instead of a circle of a radius r_i as a movable oval, we shall have the following fundamental formula:

$$p_i T = \phi + \frac{2\lambda}{\pi} a_i + \nu a_i^2. \qquad \ldots \quad (C.1)$$

According to Prof. P. C. Mahalanobis' private communication, he found a semi-logical, semi-empirical formula during his survey of acreage in Bengal:

$$p = 2A + 4\sqrt{A} \qquad \ldots \quad (C.2)$$

where $A(=a_i^2$ in our notation) is a geographical area of the sample unit of square shape in acres. According to his study this formula holds well for $A = 2 \sim 100$ acres. According to his rough estimation in his case

$$T = 75{,}000 \text{ sq. mi.} = 48 \times 10^6 \text{ acres}$$

and

$$\nu = 12 \times 10^7 \text{ plots.}$$

Thus

$$\nu/T = 2.5 \text{ acre}^{-1},$$

which is approximately equal to 2, the first coefficient of his formula.

$\phi/T = 1$ is practically negligibly small in his case, if A is large, say $A > 4$.

We have no means of deducing numerical coefficient 4 in his empirical formula, but if we assume his figure, we can obtain the mean length of the boundary of a plot, viz., λ/ν.

From

$$\frac{2\lambda}{\pi \times 75{,}000}\sqrt{\frac{1}{640}} = 4,$$

we have

$$\lambda = 12 \times 10^6 \text{ miles}$$

and

$$\frac{\lambda}{\nu} = \frac{12 \times 10^6}{12 \times 10^7} \text{ mi.} = \frac{1}{10} \text{ mi.} = 16 \times 10 \text{ m.}$$

The last figure is a plausible one, because if we assume roughly that the shape of each plot is a square, then the area of a square, the perimeter of which is 160 m, is $16 \times 10^2 \text{m}^2$. On the other hand, the mean area is estimated as T/ν which is 0.4 acres or $16 \times 10^2 \text{m}^2$ which should be greater than or equal to the former estimate $16 \times 10^2 \text{m}^2$. The more precise experimental comparison will be given elsewhere.*

Our assumption of "convexity" can be removed, if necessary. In such a case we should only use the kinematic principal formula (W. Blaschke and L. A. Santaló).

* In the Seminar in the Training Course, Indian Statistical Institute on 31 January 1953.

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