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A STUDY OF ADMISSIBILITY THROUGH EXTERIOR BOUNDARY VALUE PROBLEM

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By

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A thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the award of Doctor of Philosophy

Calcutta

1976

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My greatest personal debt is to Professors J.K.Ghosh, C.R.Rao and D. Basu for teaching me the basic elements of statistics. I have benefitted a great deal from their lectures and the discussions I have had with them. I would consider this work rewarding if they found anything interesting in this dissertation.

This work was done mainly under the supervision of Professor J.K.Ghosh. I gratefully acknowledge his help and advice at various stages of this work. I take this opportunity to thank Dr. B.V.Rao for his suggestions at the final stages.

Finally, I would like to thank the Research and Training School of Indian Statistical Institute for providing me with all the facilities during the course of the work.

Calcutta,
July 1976

C. Srinivasan

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I N T R O D U C T I O N

The study of admissible decision procedures began in late forties when Wald introduced the concept to characterize the minimal complete class of decision procedures. Starting with the pioneering work of Abraham Wald, **there** has been considerable contribution to this area over the last three decades. However, most of the articles in this field dealt with specific decision procedures and studied their admissibility. It was Stein [1] who first characterized, admissible decision procedures. Farrell ([2], [3]) generalized the result of Stein. In spite of the works of Stein and Farrell the problem of deciding whether a given decision procedure is admissible or not remained difficult even in smooth set ups. The reason for this is the necessary and sufficient conditions given by Stein-Farrell's theorem are not easy to check.

A major contribution towards this problem was made my Brown [1] in 1971. In this brilliant article, he showed that the study of the admissibility of generalized Bayes estimators, under quadratic loss, of the mean of a multivariate normal distribution could be linked up with a calculus of variation problem. By establishing a relation between the calculus of variation problem and the recurrence of a diffusion process, he characterized

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the admissible estimators under some conditions. This characterization yields easily verifiable conditions. Apart from this, the link between admissibility and the recurrence of the associated diffusion process is interesting and novel. In this dissertation we extend the results of Brown [1]. We give a chapterwise summary below.

In Chapter I we deal with exterior boundary value problems and relate them to a calculus of variation problem. These exterior boundary value problems play crucial in the rest of this dissertation.

The main results of Chapter II are generalizations of Brown's results (Brown [1]). Further, our proofs differ from his. While he goes through diffusion processes, we resort to exterior boundary value problems. Our method gives a shorter proof of the main theorem of Brown [1].

Chapter III is devoted to the admissible estimators of the mean of multivariate normal distribution. First, we consider the problem of improving inadmissible estimators. Using a result of Stein [5] we show how to construct generalized Bayes minimax estimators for dimensions $m \geq 3$. We also completely establish the nonexistence of proper Bayes minimax estimators for $m = 3$ and 4. Our result generalizes Strawderman's [2].

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The latter part of the chapter contains results on the admissibility of estimators and also deals with coordinatewise estimation.

In Chapter IV we show that the method of Brown could be used to obtain necessary and sufficient conditions for the admissibility of generalized Bayes estimators of the natural parameter of exponential formly. Theorems similar to those in Chapter II are obtained under same conditions.

In the last chapter we relate the exterior boundary value problem to almost admissibility of estimators. Through this we are able to obtain generalizations of results of Karlin [1] and Zidek [1] to higher dimensions.

In conclusion, we would like to remark that there are still quite a few problems regarding the admissibility of estimators of the mean of normal distribution yet to be solved. Especially, the problem of improving an inadmissible estimator seems challenging. Our efforts in this direction have not been completely rewarding.

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CHAPTER 1

EXTERIOR BOUNDARY VALUE PROBLEM

§ 0. Introduction and Summary

A brief account of elliptic partial differential equations and boundary value problems, which play a vital role in our study of admissibility, is presented in this chapter. This chapter, which is essentially introductory in nature, contains four sections. In section 1, we state, without proof, certain properties of solutions of elliptic partial differential equations. The material is drawn from Miranda [1]. Section 2 deals with boundary value problems on unbounded domains. We consider two boundary value problems and present results regarding their solvability. The material of this section is essentially based on a paper of Meyers and Serrin [1]. A calculus of variation problem on an unbounded domain is considered in section 3. Its relation to an exterior boundary value problem is studied. The main result of this section is the fundamental tool for the study of admissibility. In section 4, we outline briefly some generalizations of results in the previous sections.

§ 1. Second Order Elliptic Partial Differential Equations

We introduce in this section elliptic partial differential equations and list, without proof, a few properties of their solutions.

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Let E^m be the m-dimensional Euclidean space. We assume throughout this section and the following ones of this chapter that $m \geq 2$. Let $\Omega \subseteq E^m$ be a region (i.e. Ω is open and connected) consider m^2+m+1 real functions $a_{ik}(x)$, $b_i(x)$, $c(x)$ $(i,k = 1,2,\ldots,m)$ defined in Ω. We shall denote by L the linear differential operator of the second order :

$$L = \sum_{i,k=1}^{m} a_{ik} \frac{d^2}{dx_i \, dx_k} + \sum_{i=1}^{m} b_i \frac{d}{dx_i} + c$$

Supposing that (a_{ik}) matrix is symmetric we have the following definition.

Definition : L is said to be uniformly Elliptic if the a_{ik} are measurable in Ω and if there exists a constant $a_o > 0$ such that for $x \in \Omega$ and $\xi \in E^m$ the following holds

$$a_o \sum_{i=1}^{m} \xi_i^2 \leq \sum_{i,k=1}^{m} a_{ik}(x) \, \xi_i \, \xi_k \leq a_o^{-1} \sum_{i=1}^{m} \xi_i^2$$

We will be dealing with, in our statistical problems, linear differential operators for which the (a_{ik}) matrix is identity $I_{m \times m}$ and $c(x) \equiv 0$. So we shall take, unless the contrary is mentioned, L to be of the form $L = \sum_{i=1}^{m} \frac{d^2}{dx_i \, dx_i} + \sum b_i(x)$.

The results we present below, though more generally true, are for this operator.

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Let u be a twice continuously differentiable function defined on Ω. Assume that $b_i(x)$ $(i = 1 \ldots m)$ are bounded in Ω. We state below a maximum modulus principle.

<u>Theorem 1.1</u> (Strong Maximum Principle). If $Lu \geq 0$ $(Lu \leq 0)$ then u can not have a relative positive maximum (negative relative minimum) unless u is a constant.

If we assume Ω to be bounded we have the following maximum principle.

<u>Theorem 1.2</u> If $Lu \geq 0$ $(Lu \leq 0)$ in Ω and u is continuous in $\overline{\Omega}$ (the closure of Ω) then

$$u(x) \leq \max_{x \in \partial\Omega} u(x) \qquad \left(u(x) \geq \min_{x \in \partial\Omega} u(x) \right)$$

For proofs of the above theorems see Miranda [1].

<u>Remark 1.1</u> By $\partial\Omega$ we mean the boundary of Ω and $\overline{\Omega} = \Omega \cup \partial\Omega$. Moreover if u is a non-constant function, one can show that the inequalities in Theorem 1.2 are strict.

The next two results, known as Harnack inequalities, are on the local properties of positive solutions of $Lu = 0$ in Ω.

<u>Theorem 1.3</u> Let S be a compact subset of Ω. Then there exists a constant K depending only on S and Ω such that

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the inequality

$$u(x) \leq K\, u(y)$$

holds for any two points x, y in S.

<u>Theorem 1.4</u> Under the conditions of Theorem 1.3 there exists a constant $b > 0$ such that

$$\sup_{x \in S} \left\| \frac{\nabla u(x)}{u(x)} \right\| < b$$

For proofs of the above theorems one may refer to Serrin, [ ]. Two comments are in order in regard to the Harnack inequalities. The versions we have given are specialized to our set up and they could be stated more generally. The second one is the constant in the inequalities is in some sense absolute. More precisely if $\Omega' \supset \Omega \supset S$ and u is a solution (positive) in Ω' then the constant K may be chosen independent of Ω'.

We end this section with an existence result for a boundary value problem on spheres.

Let $\Omega = S_r = \{x : \|x\| < r\}$ and L be defined on S_r. Let ϕ be a given continuous function on $S_r = \{x : \|x\| = r\}$. Then the following theorem can be proved and for proof one may refer to Miranda [1].

<u>Theorem 1.5</u> There exists a unique continuous function u defined on $\bar{S}_r$ such that $Lu = 0$ in S_r and $u = \phi$ on S_r.

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The above theorem holds good for general regions Ω with smooth boundary. We do not need the general result. However, we will have occasion to use the following result. If $\Omega = S_{r_1} - S_{r_2}$, $r_1 > r_2$ (i.e. Ω is an annulus) and if ϕ_1 and ϕ_2 are two given continuous functions on the two boundaries of Ω, then there exists a unique continuous function u defined on $\overline{\Omega}$ such that $Lu = 0$ on Ω and $u = \phi_1$ and ϕ_2 on the two boundaries.

Finally, the unique solution given by Theorem 1.5 has another interesting property. Considering the calculus of variation problem of minimizing the integral

$$\int_{S_r} \|\nabla v(x)\|^2 g(x)\, dx$$

with respect to $v \varepsilon V = \{v(x) ; v: E^m \to E^1,$ v is defined and continuous on $\overline{S}_r$ and $v(x) = \phi(x)$ on $S_r\}$, where $g(x)$ is a smooth real valued positive function and $\phi(x)$ is continuous on S_r. This calculas of variation problem has a unique minimizing function in V and it is given by the unique solution of Theorem 1.5 with the operator $Lu = g(x) \Sigma \frac{d^2u}{d x_i\, d x_i} + \nabla g(x) \nabla u$. Thus, if u_o is the unique solution of $Lu = 0$ with $u(x) = \phi(x)$ on

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S_r, we have

$$\int_{S_r} \|\nabla u_0(x)\|^2 g(x)\, dx = \inf_{v \varepsilon V} \int_{S_r} \|\nabla v(x)\|^2 g(x)\, dx.$$

We will be using this property of solutions of boundary value problems in the next two sections of this chapter. It plays a crucial role in the proofs of the results presented there.

§ 2. Exterior Boundary Value Problem

We present in this section an account of boundary value problems for elliptic equations on an unbounded region. One may refer to Meyers and Serrin [1] for proofs of the results stated.

Let E be the region $\{x : \|x\| > 1\}$. Let L be defined on E. Let ϕ be a given continuous function on the finite boundary $\partial E = \{x : \|x\| = 1\}$ of E. We consider the following two problems.

Problem 1 (BPI). To find a unique solution to $Lu = 0$ on E satisfying the boundary conditions $u(x) = \phi(x)$ on E and $u(x) \to 0$ as $\|x\| \to \infty$.

Problem 2 (BP II). To find a unique bounded non-negative solution to $Lu = 0$ on E satisfying $u(x) = \phi(x)$ on E and $\sup_{x \varepsilon E} u(x) \leq \max_{x \varepsilon \partial E} |\phi(x)|$.

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Below we give a few necessary and sufficient conditions for the solvability of these two problems. Before doing so we give a general result and some definitions.

<u>Theorem 2.1</u> There always exists a solution to $Lu = 0$ on satisfying the boundary condition $u(x) = \phi(x)$ on ∂E .

<u>Proof</u> : Let S_n denote the sphere of radius n. Let u_n be the solution of $Lu_n = 0$ on $S_n \cap E$ satisfying the boundary conditions $u_n = \phi$ on ∂E and $u_n = 0$ on ∂S_n. Such a solution exists and is unique follows from section 1. Now using Schauder's interior and boundary estimates for u_n's (See Miranda [1]) one can show that there exists a subsequence of $\{u_n\}$ which converges to a function u defined on $E \cup \partial E$. This function has the necessary properties.

q.e.d.

<u>Definition</u> : A function $v(x)$ is a barrier at infinity for L on E if (i) v is defined and positive in some neighbourhood of infinity (ii) $v(x) \to 0$ as $\|x\| \to \infty$ and (iii) $Lv(x) \leq 0$.

<u>Definition</u> : A non-negative function $\delta(t)$, $0 < t_0 \leq t < \infty$ is called a Dini function if the integral $\int^{\infty} \delta(t) \frac{dt}{t}$ exists and is finite.

<u>Theorem 2.2</u> Problem I (BP I) is solvable if and only if L has a barrier at infinity.

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The next result is a sufficient condition for Problem I to be solvable which we will have occasion to use.

Therorem 2.3 If $\sum_{i=1}^{m} x_i b_i(x) \geq (2-m) + \varepsilon(\|x\|)$ in some neighbourhood of infinity, where $\varepsilon(r)$ is such that

$$\Delta(t) = \exp\left\{-\int^{t} \varepsilon(s)\,\frac{ds}{s}\right\}$$

is a Dini function, then Problem I is solvable.

We now go on to Problem II and present some results regarding its solvability. First observe that there always exists at least one bounded solution for Problem II. This follows from Theorem 2.1 and maximum modulus principle.

Theorem 2.4 Problem II (BP II) is solvable if and only if for every solution u of $Lu = 0$ on E and $u(x) = \phi(x)$ on ∂E

$$\max_{\|x\|=R} |u(x)| \geq \limsup_{\|x\| \to \infty} |u(x)|$$

for all sufficiently large R.

The next result is a sufficient condition for Problem II to be solvable.

Theorem 2.5 If $\sum_{i=1}^{m} x_i b_i(x) \leq (2-m) + \varepsilon(\|x\|)$ in some neighbourhood of ∞, where $\varepsilon(r)$ is such that

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$$\Delta(t) = \exp\left\{- \int^{t} \frac{\varepsilon(s)}{s} ds\right\}$$

is not a Dini function, then Problem II is solvable.

Remark 2.1 : The boundary value problems BPI and BPII are in some sense dual to each other. In fact, (See Meyers and Serrin [1]), if $m = 2$, then either BPI or BPII is solvable for

$$Lu = \Delta u + \sum_{i=1}^{2} b_i(x) \frac{du}{dx_i} = 0.$$

In our statistical problems, we will be dealing with the above boundary value problems for which the boundary data $\phi \equiv 1$. So henceforth, unless otherwise specified, we take $\phi \equiv 1$. We end this section with a necessary and sufficient condition for the solvability of BPII with $\phi \equiv 1$.

Let u_n be the solution of $Lu = 0$ on $S_n \cap E$ as defined in the proof of Theorem 2.1 satisfying $u_n = 1$ on ∂E and $u_n = 0$ on ∂S_n.

Theorem 2.6 Problem II (BPII) is solvable if and only if every convergent subsequence of $\{u_n\}$ converges uniformly on compacta to 1.

Proof : ''Only if''. Suppose there exists a subsequence of $\{u_n\}$ which converges uniformly on compacta to a function $u_o \not\equiv 1$. It follows from maximum modulus principle that u_o

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is a bounded solution of $Lu = 0$ on E with $u(x) = 1$ on ∂E . This contradicts the fact that BPII is solvable.

''if''. Suppose every convergent subsequence of $\{u_n\}$ converges uniformly on compacta to 1. Let, if possible, $u_o \not\equiv 1$ be a function such that $Lu_o = 0$ on E , $u_o(x) = 1$ on ∂E and $u_o(x) \leq 1$. An application of the maximum modulus principle shows that $u_n \leq u_o$ $\forall$ n. This contradicts the hypothesis.

q.e.d.

§ 3. A Calculus of Variation Problem

This section deals with a calculus of variation problem in an unbounded domain. We shall see in the next chapter that the study of admissibility is related to a calculus of variation problem. We shall study here the relation between the calculus of variation problem and exterior boundary value problem II.

Let $f(x)$ be a piecewise differentiable positive function defined on whole of E^m. Let K be an unbounded closed convex subset of E^m. Let J be the following class of functions :

$$J = \{ j : j : E^m \to R, \ j(x) \geq 0, \ j(x) = 1 \text{ for } \|x\| \leq 1,$$

$$j(x) \text{ is differentiable and } \sup_{\{x: \|x\| = r, \ x\in K\}} j(x) \rightrightarrows 0 \text{ as } r \to \infty \}$$

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Consider the following calculus of variation problem.

$$\inf_{j \varepsilon J} \int_{\|x\|>1} \|\nabla j\|^2 f(x)\, dx$$

We want to find a necessary and sufficient condition for the minimizing function j_o of the above calculus of variation problem to be 1. i.e. To find a necessary and sufficient condition for $\inf_{j \varepsilon J} \int \|\nabla j\|^2 f(x)\, dx = 0$.

We set about obtaining this characterization below. We first prove a result for $K = E^m$. In the next result we consider the case K is an arbitrary unbounded closed convex set. This is done mainly for the purpose of clarity.

<u>Theorem 3.1</u> Let $K = E^m$. A necessary and sufficient condition for $\inf_{j \varepsilon J} \int \|\nabla j(x)\|^2 f(x)\, dx = 0$ is BPII is solvable for the elliptic differential equation $Lu = \Delta u + \nabla u \cdot \frac{\nabla f}{f} = 0$.

<u>Proof</u> : We shall use the characterization Theorem 2.6 of BPII.

<u>Sufficiency</u>. Let $\{u_n\}$ be a sequence of functions such that $Lu_n = 0$ for $1 < \|x\| < n$ $u_n(x) = 1$ for $\|x\| \le 1$, $u_n(x) = 0$ for $\|x\| \ge n$. Assume without loss of generality $u_n \to 1$ uniformly on compacta (If $\{u_n\}$ is not a convergent sequence take a subsequence of it). Plainly, $u_n \varepsilon J$ for all n. By applying Gauss' divergence theorem and using the fact that u_n's

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are solutions of $Lu = 0$ in their respective domains one can show that, for fixed $R > 1$,

$$\lim_{n\to\infty} \int_{1\leq\|x\|\leq R} \|\nabla u_n(x)\|^2 f(x)dx = \lim_{n\to\infty} \int_{1\|x\|\leq R} \|\nabla u_n(x)-\nabla 1\|^2 f(x)dx= \tag{3.1}$$

Also, it follows from section 1 $\int \|\nabla u_n(x)\|^2 f(x)dx$ decreases as $n \to \infty$.

Suppose, now, $\inf_{j\varepsilon J} \int_{\|x\|>1} \|\nabla j(x)\|^2 f(x)dx > 0$. Then $\lim_{n\to\infty} \int_{\|x\|>1} \|\nabla u_n(x)\|^2 f(x)dx = \varepsilon > 0$, because $\{u_n\} \subset J$. Let n_1 be such that $\int_{\|x\|\geq 1} \|\nabla u_{n_1}(x)\|^2 f(x)dx < \frac{3}{2}\varepsilon$. Let $n_2 > n_1$ be so large such that $\int_{1<\|x\|\leq n_1} \|\nabla u_{n_2}(x)\|^2 f(x)dx < \varepsilon/8$. This is possible in view of (3.1). Also, $\int_{\|x\|>1} \|\nabla u_{n_2}(x)\|^2 f(x)dx \leq \frac{3}{2}\varepsilon$.

Let $u = \dfrac{u_{n_1} + u_{n_2}}{2}$. The minimizing property of u_{n_2} implies that $\int_{\|x\|>1} \|\nabla u\|^2 f(x)dx \geq \int_{\|x\|>1} \|\nabla u_{n_2}(x)\|^2 f(x)dx > \varepsilon$. On the otherhand, using Schwartz inequality and the choice of u_{n_2} it is easy to show $\int_{\|x\|\geq 1} \|\nabla u(x)\|^2 f(x)dx < \varepsilon$. This is a contradiction.

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<u>Necessary part</u>. We shall show that if BP II is not solvable then $\inf_{j \varepsilon J} \int \|\nabla j(x)\|^2 f(x)dx > 0$.

So assume BP II is not solvable. Then there exists a solution to $Lu = 0$ on E satisfying $u(x) = 1$ on ∂E, $u(x) \leq 1$ such that $u \not\equiv 1$. Let $\{u_n\}$ be a sequence of solutions, as defined in the sufficiency part, such that $u_n \to u$ uniformly on compacta. We shall now prove that

$$0 < K_o = \int_{\|x\|>1} \|\nabla u(x)\|^2 f(x)dx \leq \inf_{j \varepsilon J} \int_{\|x\|>1} \|\nabla j(x)\|^2 f(x)dx \tag{3.2}$$

Suppose not. Then there exists a $j_o \varepsilon J$ such that

$$\int_{\|x\|>1} \|\nabla j_o(x)\|^2 f(x)dx < K_o - \delta \quad \text{for some } \delta > 0. \tag{3.3}$$

Let k_n be a sequence of functions satisfying

$$L\, k_n(x) = 0 \quad \text{for } 1 < |x| < n$$

$$k_n(x) = 1 \quad \text{for } \|x\| = 1$$

$$= j_o(x) \text{ for } \|x\| \geq n$$

By the minimizing property of $k_n(x)$ in the region $1 < \|x\| < n$ it follows

$$\int_{\|x\|>1} \|\nabla k_n(x)\|^2 f(x)dx \leq \int_{\|x\|>1} \|\nabla j_o(x)\|^2 f(x)dx < K_o - \delta \tag{3.4}$$

for all n.

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Let $R > 0$ be such that

$$\int_{1<\|x\|\leq R} \|\nabla u(x)\|^2 f(x)dx \geq K_o - \delta/2 \tag{3.5}$$

It is easily seen, by maximum modulus principle, that

$0 \leq K_n(x) - u_n(x) \leq \sup_{\|x\|=n} j_o(x)$ for x such that $1 \leq \|x\| \leq n$.

This, combined with the fact that $\sup_{\|x\|=r} j_o(x) \to 0$ as $r \to \infty$, enables us to show that $(k_n(x) - u_n(x)) \to 0$ as $n \to \infty$ uniformly on compacta. Hence $k_n(x) \to u(n)$ as $n \to \infty$ uniformly on compacta. Now, if we show

$$\int_{1\leq\|x\|\leq R} (\|\nabla u(x)\|^2 - \|\nabla k_n(x)\|^2) f(x)dx \to 0 \text{ as } n \to \infty \tag{3.6}$$

it would lead to a contradiction because of (3.4) and (3.5). The proof of (3.6) follows easily by an application of Gauss' divergence theorem and the Harnack inequality (Theorem 1.4.). This complets the proof of necessary part. q.e.d.

<u>Remark 3.1</u> The proof of (3.6) can also be obtained by appealing to Schauder's estimates for $\{K_n\}$ and u instead of using the Gauss' divergence theorem.

<u>Remark 3.2</u> The proof of the above theorem is very similar to that of a theorem proved by L. Brown [1], However his set up is different from ours. His arguments go through the transience

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and recurrence of a related diffusion process.

The general case is considered in the next theorem.

Theorem 3.2 A necessary and sufficient condition for

$\inf_{j \varepsilon J} \int \|\nabla j(x)\|^2 f(x)dx = 0$ if and only if BP II is solvable.

Proof : ''Sufficiency''. Exactly same as in Theorem 3.1.

''Necessity''. Assume $\inf_{j \varepsilon J} \int \|\nabla j(x)\|^2 f(x)\, dx = 0$. We shall give the proof in three stages.

(a) Let $\{k_n(x)\}$ be a sequence of functions satisfying

$$L\, k_n(x) = 0 \quad \text{for } 1 < \|x\| < n$$

$$k_n(x) = 1 \quad \text{for } \|x\| = 1$$

$$= \phi_n(x) \text{ for } \|x\| = n$$

$$= 0 \quad \text{for } \|x\| > n$$

where $\phi_n(x)$ is a smooth function such that $\phi_n(x) = 1$ for $x \not\varepsilon K$ and $1 > \phi_n(x) \geq \varepsilon > 0$ for $x \varepsilon K$ and $\|x\| = n$ with equality holding at some point and $\varepsilon < 1$.

By hypothesis there exists a sequence $\{j_n\} \subset J$ such that $\int \|\nabla j_n(x)\|^2 f(x)\, dx \downarrow 0$. Let $\{u_{m_n}\}$ be a sequence of functions satisfying $Lu_{m_n} = 0$ for $1 < \|x\| < m_n$ $u_{m_n}(x) = 1$ for $\|x\| = 1$ and $u_{m_n}(x) = j_n(x)$ for $\|x\| \geq m_n$, where m_n is so chosen

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to satisfy $\sup\limits_{\{x\,:\,\|x\|=m_n,\, x\in K\}} j_n(x) \leq \varepsilon$.

The minimizing property of u_{m_n} implies

$$\int \|\nabla u_{m_n}(x)\|^2 f(x)dx \leq \int \|\nabla j_n(x)\|^2 f(x)dx.$$

Assume without loss of generality that $\{m_n\}$ is an increasing sequence. It follows from Schauder's estimates that u_{m_n} converges to a solution u of L in E^m satisfying $u(x) = 1$ for $\|x\| = 1$.

From a result of Serrin's on lower semicontinuity (See Morrey [1]) it follows that for every compact set

$$\int_C \|\nabla u(x)\|^2 f(x)dx \leq \liminf_{m_n \to \infty} \int \|\nabla u_{m_n}\|^2 f(x)dx$$

$$\leq \liminf_{n \to \infty} \int \|\nabla j_n(x)\|^2 f(x)dx = 0$$

This implies $u(x) \equiv 1$. On the other hand

$K_{m_n}(x) \geq u_{m_n}(x)$ for $1 \leq \|x\| \leq m_n$ by maximum modulus principle.

Therefore $K_{m_n}(x)$ converges uniformly on compacta to 1.

(b) Let $\{\varphi_n(x)\}$ be a sequence of functions satisfying $L\,\varphi_n(x) = 0$ for $1 < \|x\| < n$, $\varphi_n(x) = 1$ for $\|x\| = 1$ and $\varphi_n(x) = 0$ for $\|x\| \geq n$. Let $\{V_n(x)\}$ be another sequence

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such that $Lv_n = 0$ for $1 < \|x\| < n$, $v_n(x) = 1$ for $\|x\| = 1$ and $v_n(x) = W_n(x)$ for $\|x\| = n$ where $W_n(x)$ is a smooth function satisfying $\varepsilon < W_n(x) \leq 1 - \varepsilon$, $0 < \varepsilon < \frac{1}{2}$ and

$W_n(x) = 1-\varepsilon$ for $x \notin K$, $\|x\| = n$ and $W_n(x) > \varepsilon > 0$ for $x \in K$, $\|x\| = n$.
Suppose $v_n(x)$ converges to 1 uniformly on compacta. Then $V_n(x) = \frac{1}{\varepsilon}\,[v_n(x) - (1 - \varepsilon)]$ also converges to 1 uniformly on compacta. Moreover $L\, V_n(x) = 0$ for $1 < \|x\| < n$. By maximum modulus principle it is easy to see that

$\varphi_n(x) \geq V_n(x)$ for $1 \leq \|x\| \leq n$ for all n. Hence
$\varphi_n(x) \to 1$ uniformly on compacta.

(c) Observe that the result in (a) goes through if the boundary functions ϕ_n's of K_n's satisfy $\phi_n = 1 - \varepsilon_n$ for $x \notin K_n$ $\|x\| = n$ and $1 - \varepsilon_n > \phi_n(x) \geq \varepsilon_n$ for $x \in K$, $\|x\| = n$, where ε_n monotonically decreases to zero. Now, using (b) we can get a subsequence of $\{\varphi_n\}$ which converges to 1 uniformly on compacta. But $\{\varphi\}$ is a monotonically increasing sequence. Therefore $\varphi_n \to 1$ uniformly on compacta.

q.e.d

As we have already mentioned [Remark 3.2] L. Brown [1] proves a similar result. But, it appears, we have to work harder for our general case than he does in his characterization

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theorem (See Theorem 4.3.1 of L. Brown [1]). His set up is such that he is able to get hold of a function directly which gives a lower bound for

$$\inf_{j \in J} \int \|\nabla j(x)\|^2 f(x)\, dx.$$

We do not have any such function. Probably, this is the reason for the lengthy proof of the above theorem.

Our next result, which will be used in the proof of the main theorem of chapter 2, gives a necessary condition for the solvability of BP II. We need certain facts for the proof of the next result which we develop below.

Consider the one dimensional calculus of variation problem

$\inf_{j \in J} \int_{x \geq 1} \left|\frac{d}{dx} j(x)\right|^2 f(x)dx.$ If this infimum is positive, then the unique minimizing function exists and is given by

$$j_o(x) = c \int_x^\infty \frac{1}{f(x)}\, dx \qquad (3.7)$$

where c is a normalizing constant so as to make $j_o(x) = 1$ for $\|x\| = 1$. Conversely, if $j_o(x)$ as defined (3.7) is finite and goes to zero as $x \to \infty$, then the infimum is positive. One can in fact show

$$L\, j_o = f(x) \frac{dj_o(x)}{d^2} + \frac{dj_o(x)}{dx} \cdot \frac{df(x)}{dx} = 0 \quad \text{for} \quad x > 1$$

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whenever the infimum is positive. If $m \geq 2$ and $f(x)$ is spherically symmetric (i.e. $f(x) = f(\|x\|)$), then $j_o(x)$ defined by

$$j_o(x) = c \int_{\|x\|}^{\infty} \frac{1}{\|x\|^{m-1}} \frac{1}{f(\|x\|)} d\|x\| \qquad (3.8)$$

has similar properties as in the one dimensional case.

Now we are in a position to state and prove our next theorem. Let, $f_R(r, \phi)$ denote $f(x)$ in plar co-ordinates. Assume that ϕ is normalized.

<u>Theorem 3.3</u> : If there is a measurable set $Q \subseteq \{\phi\}$ with $\int_Q d\phi > 0$ such that

$$\sup_{\phi \varepsilon Q} \int_1^{\infty} \frac{1}{r^{m-1}} \frac{1}{f_R(r,\phi)} dr < \infty$$

then BP II is not solvable.

<u>Proof</u> : It suffices to show, by Theorem 3.2, that $\inf_{j \varepsilon J} \int_{\|x\| \geq 1} \|\nabla j\|^2 f(x)dx > 0$. Let, if possible, $\{j_n\} \subseteq J$ be such that $\int_{\|x\| \geq 1} \|\nabla j_n\|^2 f(x)dx \to 0$ as $n \to \infty$ and $j_n(x) = 1$, for $\|x\| = 1$ and $j_n(x) = 0$ for $\|x\| > n$. Now, observe that

$$\int_{\|x\|>1} \|\nabla j_n(x)\|^2 f(x)\, dx$$

$$\geq \int_Q \left(\int_r \left\| \frac{d}{dr} j_n(r, \phi) \right\|^2 f_R(r, \phi)\, dr \right) d\phi.$$

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But for $\phi \in Q$,

$$\int_r \left\| \frac{d}{dr} j_n(r,\phi) \right\|^2 r^{m-1} f_R(r, \phi)\, dr$$

$$\geq \int \left\| \frac{d}{dr} c(\phi) \int_r^\infty (r^{m-1} f_R(r,\phi))^{-1} dr \right\|^2 r^{m-1} f_R(r,\phi) dr$$

where $c(\phi) = \int_1^\infty (r^{m-1} f_R(r, \phi))^{-1} dr$. Hence

$$\int_{\|x\|>1} \| \nabla j_n(x) \|^2 f(x) dx \geq \int_Q \left(\int_1^\infty (r^{m-1} f_R(r,\phi))^{-1} dr \right) dQ > 0$$

for all n.

Hence the result. q.e.d.

§ 4. Some Generalisations

The results of sections 3 and 4 could be stated for uniformly elliptic equations of the form

$$Lu = \sum_{i,k} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \nabla f(x). \nabla u = 0 \qquad (4.1)$$

We will have occasions to deal with the general equation. We briefly spellout in this section how the results would get transformed in the general set up.

Let us begin with the exterior boundary value problems BP I and BPII. Both these problems could be posed for this general equation. Theorem 2.2 and 2.4 remain unaltered in the general case. However, theorems 2.3 and 2.5 undergo a change.

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Let $M = \frac{1}{r^2} \sum_{i,k}^{m} a_{ik} x_i x_k$, where $r = \|x\|$. Define a function $\underline{A}^*(x)$ by $A^* = \frac{\sum_{i=1}^{m} a_{ii} + x.\nabla f(x)}{M}$

The analogues of Theorems(2.3)and (2.4) are respectively.

Theorem 4.1 : If $A^*(x) \geq 2 + \varepsilon(\|x\|)$ in some neighbourhood of infinity, where $\varepsilon(r)$ is such that $\Delta(t) = \exp\{-\int^t \varepsilon(s) \frac{ds}{s}\}$ is a Dini function, then BP I is solvable.

Theorem 4.2 : If $A^*(x) \leq 2 + \varepsilon(\|x\|)$ in some neighbourhood of infinity, where $\varepsilon(r)$ is such that $\Delta(t) = \exp\{-\int^t \varepsilon(s) \frac{ds}{s}\}$ is not a Dini function, the BP II is sovable.

Let us now consider a calculus of variation problem of the form

$$\inf_{j \varepsilon J} \int \left(\sum_{\substack{i=1 \\ k=1}}^{m} \frac{\partial j}{\partial x_i} a_{ik} \frac{\partial j}{\partial x_k} \right) f(x) dx \qquad (4.2)$$

where (a_{ik}) is a positive definite matrix.

The results of section (3) will go through for (4.2) if the BP II is considered for the operator

$$Lu = \sum a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \nabla u . \frac{\nabla f(x)}{f(x)} .$$

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CHAPTER II

ADMISSIBLE ESTIMATORS AND EXTERIOR BOUNDARY VALUE PROBLEM

§ 0. Introduction and Summary

In this chapter we shall be concerned with the problem of characterizing admissible estimators of the mean of a multivariate normal distribution, under quadratic loss function. This problem, which is not new, has drawn the attention of many, and, over years, there have been many articles which studied the admissibility of estimators. Stein, in 1955-56, gave a fillip to this study with his fundamental papers (Stein [1], Stein [2], See also, James and Stein [1]). Later, Brown [1] formulated the problem of characterizing admissibility in terms of a problem in calculus of variations and obtained, under some conditions, necessary and sufficient conditions for the admissibility of a generalized Bayes estimator. Actually, the idea of using calculus of variations to study the problem of admissibility was due to Stein [3]. However, it was Brown [1] who proved the first result in this line. We generalize this result in this chapter and obtain a fairly complete characterization of admissibility of generalized Bayes estimators.

This chapter is divided into nine sections. The first section gives the basic notations and concepts. In section 2,

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we formulate the problem and state the main characterization theorem along with the assumptions. We discuss the assumptions in section 3 and show that Brown's assumption easily implies ours. Section 4 contains purely technical results which are needed to the prove the main theorem. The proof of the main theorem is given in section 5. In section 6, we relate the calculus of variations problem with diffusion processes. A generalization of the main theorem for spherically symmetric estimators is proved in section 7. This result is very general and covers almost all estimators in the spherically symmetric case. Some statistical results are given in section 8. Finally, in the last section we give examples and make some general comments.

§ 1. Basic Notations

Let X be an m-dimensional normal random variable with unknown mean θ and the identity matrix as the dispersion matrix. The density of X is denoted by

$$p_\theta(x) = (2\pi)^{-m/2} \exp(-\frac{1}{2} \sum_{i=1}^{m} (x_i - \theta_i)^2)$$

with respect to m-dimensional Lebesgue measure on E^m. Let $\delta = (\delta_1, \ldots, \delta_m)$ denote the estimate of $\theta = (\theta_1, \ldots, \theta_m)$. We take $L(\theta, \delta) = ||\theta - \delta||^2$ as the loss function. (The symbol

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$||\cdot||$ stands for the Euclidean norm on E^m). As usual, for an estimator $\delta(\cdot)$, the risk function $R(\cdot,\cdot)$ is defined by

$$R(\theta,\delta) = E_\theta L(\theta, \delta(x)) = \int L(\theta,\delta(x))\, p_\theta(x)dx.$$

Let G be any nonnegative Borel measure on E^m. Suppose $\int p_\theta(x)\, G(d\theta) < \infty$ almost everywhere with respect to Lebesgue measure. The generalized Bayes estimator δ_G with respect to the measure G is defined by

$$\delta_G(x) = \frac{\int \theta\, p_\theta(x)\, G(d\theta)}{\int p_\theta(x)\, G(d\theta)} \qquad (1.1)$$

If G is finite, δ_G is nothing but the usual Bayes estimator with respect to the prior G. i.e.

$$\int R(\theta, \delta_G(x))\, G(d\theta) = \inf_\delta \int R(\theta,\delta)\, G(d\theta)$$

We will be interested only in measures G for which $g^*(x) = \int p_\theta(x)\, G(d\theta) < \infty$ a.e. with respect to Lebesgue measure. For such a measure G, the generalized Bayes estimator δ_G is well defined and, differentiation under the integral sign yields

$$\frac{\nabla g^*(x)}{g^*(x)} = \delta_G(x) - x \qquad (1.2)$$

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A nonfinite measure G such that $g^*(x) < \infty$ a.e. is known as improper prior.

Since $L(\theta,t)$ is strictly convex in t the class of non-randomized estimators form a complete class. This enables us to confine to the class of non-randomized estimators in the above formulation.

An estimator $\delta(\cdot)$ is called admissible if, for any other estimator δ', $R(\theta,\delta') \leq R(\theta,\delta)$ for all θ implies $R(\theta,\delta) \equiv R(\theta,\delta')$. In fact it can be shown (Farrell [1]), using the convexity of the loss function L, that $\delta' = \delta$ almost everywhere with respect to Lebesgue measure.

For a given generalized prior F, let K_F denote the closed convex hull of the support of F. For any point x in E^m define

$$d(x) = \inf \{\|x-y\| : y \in K_F\}$$

$$K_F^{\alpha} = \{x : d(x) \leq \alpha\}$$

for $\alpha \geq 0$ $(K_F^0 = K_F)$. Plainly, if $\pi(x)$ denotes the projection of x into K_F, then $d(x) = \| x - \pi(x) \|$.

Finally, if $u : E^m \rightarrow E^1$, we shall say u is piecewise differentiable if there exists a collection (countable) of disjoint open sets $\{O_i\}$ such that $E^m = \bigcup_{i=1}^{\infty} \bar{O}_i$ and u is

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continuous on E^m and continuously differentiable in O_i, $i = 1,2, \ldots$.

§ 2. The Problem

The basic problem of this chapter is to find verifiable necessary and sufficient conditions for an estimator $\delta(\cdot)$ of Θ to be admissible. Sacks [1] has shown, for dimension $m = 1$, that generalized Bayes estimators form a complete class (See also Farrell [2]). Using a continuity theorem for Laplace transforms Brown [1] has given a short proof of this fact for m-dimensional normal problem. We record this as a theorem below.

Theorem 2.1 : If an estimator $\delta(\cdot)$ is admissible for Θ then there exists a Borel measure F on E^m such that $f^*(x) < \infty$ a.e. and $\delta(x) = \delta_F(x)$ a.e. with respect to Lebesgue measure.

Thus our study on admissibility can be confined to generalized Bayes estimators. The central aim of this chapter is to find necessary and sufficient conditions on $f^*(x)$ (hence on F) for δ_F to be admissible. Thoughout the remainder of this chapter F is a fixed non-negative Borel measure with unbounded support. (If the support is bounded then δ_F is a Bayes procedure and hence admissible).

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The basic tool for our study is the necessary and sufficient condition for admissibility due to Stein [1]. See also Farrell [3] and [4]. The version given below is due to Farrell.

<u>Theorem 2.2</u> (Stein-Farrell)

An estimator δ is admissible if and only if there exist a sequence of finite Borel measures $\{G_n\}$ satisfying

(i) G_n has compact support (2.1)

(ii) The supports of G_n increase to E^m as $n \to \infty$

(iii) There exists a compact set C and a constant $\beta > 0$ such that $G_n(C) \geq \beta > 0$ for all n (2.2)

(iv) $\int (R(\theta,\delta) - R(\theta,\delta_{G_n}))\, G_n(d\theta) \to 0$ as $n \to \infty$ (2.3)

Using the definition of (1.1) and interchanging the order of integration in (2.3) we have, as in Brown [1]

$$\int (R(\theta,\delta_F) - R(\theta,\delta_{G_n}))G_n(d\theta) = \int \| \delta_F(x) - \delta_{G_n}(x) \|^2 g_n^*(x)dx \quad (2.4)$$

where $g_n^*(x) = \int p_\theta(x)\, G_n(d\theta)$. Defining $h_n^{\frac{1}{2}}(x) = \left(\frac{g_n^*(x)}{f^*(x)}\right)^{\frac{1}{2}}$

and using (1.2) one gets, after some algebra,

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$$\int \| \delta_F(x) - \delta_{G_n}(x) \|^2 g_n^*(x)dx = \int \| \nabla h_n^{\frac{1}{2}}(x) \|^2 f^*(x)dx \tag{2.5}$$

This identity, which plays a crucial role, may be viewed as fundamental for our study. The connection between the exterior boundary value problem and admissibility arises through this. This was first observed by Brown [1]. We outline the relation below.

We can take, without loss of generality, C to be $S_1 = \{\theta : \|\theta\| \leq 1\}$ and $\beta = 1$ in the regularity condition (iii) of Theorem 2.2. As pointed out by Brown [1], this implies $h_n^{\frac{1}{2}}(x) \geq 1$ for $\|x\| \leq 1$ for all n (if necessary normalize the measure F on the unit ball). Hence, without loss of generality, we can assume

$$h_n(x) = \frac{g_n^*(x)}{f^*(x)} \geq 1 \quad \text{for} \quad \|x\| \leq 1 \tag{2.6}$$

The condition (i) of Theorem 2.2 has an interesting implication. Let G be a Borel measure with compact support. Brown has proved the following result.

<u>Theorem 2.3</u> Let $\alpha \geq 0$ be any fixed number. Then

$$\lim_{r \to \infty} \sup_{\{x \,:\, x \in K^{\alpha},\, \|x\| \geq r\}} \frac{g^*(x)}{f^*(x)} = 0$$

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It immediately follows from Theorem 2.3 that Condition (i) of Theorem (2.2) implies the following boundary condition at infinity for h_n's

$$\lim_{r\to\infty} \sup_{\{x : x\varepsilon K^{\alpha}, \|x\| \geq r\}} h_n^{\frac{1}{2}}(x) = 0 \quad \forall\, n \tag{2.7}$$

Let us now consider the problem of minimizing

$$\int \| \nabla j(x) \|^2 f^*(x)dx$$

with respect to j, where j is a non-negative real valued **piecewise** differentiable function defined on E^m **satisfying** the constraints

$$\text{(i)} \quad j(x) = 1 \quad \text{for} \quad \|x\| \leq 1 \tag{2.8}$$

$$\text{(ii)} \quad \lim_{r\to\infty} \sup_{\{x : x\varepsilon K^{\alpha}, \|x\| \geq r\}} j(x) = 0 \tag{2.9}$$

We shall denote by J the class of all **piecewise** differentiable functions $j \geq 0$ satisfying (2.8) and (2.9). Plainly,

$$\int (R(\theta,\delta_F)-R(\theta,\delta_{G_n}))\, G_n(d\theta) \geq \inf_{j\varepsilon J} \int \| \nabla j(x) \|^2 f^*(x)dx \tag{2.10}$$

So, if δ_F is admissible, by virtue of (2.3) and the fact that $\frac{g_n^*(x)}{f^*(x)}$ is **piecewise** differentiable for every n, we have

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$$\inf_{j \varepsilon J} \int \| \nabla j(x) \|^2 f^*(x)\, dx = 0 \qquad (2.11)$$

Therefore (2.11) is a necessary condition for the admissibility of δ_F. At this stage the question arises whether (2.11) is also a sufficient condition. The main result of this chapter is that it is so under some conditions.

Brown proved the sufficiency of (2.11) under the assumption that $\| \frac{\nabla f^*(x)}{f^*(x)} \|$ is bounded in K_F. This assumption amounts to saying that the risk of δ_F is bounded in K_F. This also implies all moments of the posteriori distribution are bounded in K_F. Our result, the statement of which is given below, is a generalization of Brown's. Our techniques are different from Brown's. For details see the discussion in section 5. However, the idea behind the proof is similar to that of his. It could be stated succinctly as follows. If (2.11) holds, then we could get hold of, by appealing to Theorem 3.2 of chapter 1, a smooth sequence of functions $\{j_n\}$ such that

$$\lim_{n \to \infty} \int \| \nabla j_n(x) \|^2 f^*(x)\, dx = 0$$

Now, use this sequence $\{j_n\}$ of functions to manufacture a sequence of finite measures $\{G_n\}$ with the properties listed

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in Theorem 2.2 and then verify (2.3) for this sequence.

We now state the basic assumptions and the main theorem

Assumptions

(i) $\Delta \log f^*(x) < B \quad \forall x \in E^m$ where

$$\text{(ii)} \quad \|\delta_F(x)\| \leq \|x\| + K \tag{2.12}$$

Theorem 2.3 (Main Theorem)

A necessary condition for δ_F to be admissible is that BP II be solvable. Conversely, if BP II is solvable for $L_f u = 0$ and (2.12) holds then δ_F is admissible.

A part of the theorem has already been establised. That is, if δ_F is admissible then (2.11) holds, which in turn, by Theorem 3.2 of Chapter I, implies BP II is solvable.

Before we go on to the proof, we shall discuss the assumptions and compare it with Brown's. We postpone the proof to section 4.

§ 3. Discussion on the assumptions

The assumption (i) is equivalent to saying that the posteriori risk is bounded. Indeed,

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$$\Delta \log f^*(x) = \frac{\int \|x-\theta\|^2 e^{-\frac{1}{2}\|x-\theta\|^2} f(d\theta)}{f^*(x)} - \left\|\frac{\nabla f^*(x)}{f^*(x)}\right\|^2 - m$$

$$= \frac{\int \|\theta-\delta_F(x)\|^2 e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta)}{f^*(x)} - m$$

Brown assumes $\left\|\frac{\nabla f^*(x)}{f^*(x)}\right\| < C$ for $x \in K_F$. We shall show that $\left\|\frac{\nabla f^*(x)}{f^*(x)}\right\| < C$ in K_F implies $\Delta \log f^*(x) < B$.

Suppose $x \in K_F$. Plainly,

$$\Delta \log f^*(x) \leq \frac{\int \|x-\theta\|^2 e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta)}{f^*(x)}$$

$$\leq \frac{\int e^{\|x-\theta\|} e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta)}{f^*(x)}$$

$$\leq \frac{K_1 \int \int_{\|\xi\|<K+1} e^{-\frac{1}{2}\|x+\xi-\theta\|^2} F(d\theta)d\xi}{f^*(x)} \qquad (3.1)$$

for some constants $K_1 > 0$ and $K > 0$. The last step follows from a lemma of Brown [1]. Now,

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$$(3.1) \leq K_1 \int_{\|\xi\|<K+1} \frac{f^*(x+\xi)d\xi}{f^*(x)} \leq K_1 e^{K_2 \left\| \frac{\nabla f^*(x)}{f^*(x)} \right\|} \int_{\|\xi\|<K+1} d\xi \qquad (3.2)$$

$$\leq C_1$$

where $C_1 > 0$ is a constant such that $K_1 e^{K_2} \int^{C}_{\|\xi\|<K+1} d\xi < C_1$.

The last inequality in (3.2) is obtained by Taylor **expansion** of $\log f^*(x+\xi)$ upto the first derivative and Lemma 3.2.3 of Brown [1]. Thus we have shown $\Delta \log f^*(x)$ is bounded in K_F.

For $x \notin K_F$ we proceed as follows. Let $\pi(x)$ be the projection of x onto K_F. It suffices to prove $\left|\Delta \log \frac{f^*(x)}{f^*(\pi(x))}\right|$ is bounded, in view of the fact that $\Delta \log f^*(\pi(x)) \leq C_1$. Assume without loss of generality that $x = (-d(x), 0, \ldots, 0)$, $\pi(x) = 0$ and $K_F \subseteq \{\theta : \theta_1 \geq 0\}$. (This can always be done. Consider the hyperplane tangent to the boundary of K_F at $\pi(x)$. Clearly, x lies on the normal to the tangent plane. Now rotate and translate the space so that the normal to the tangent plane coincides with the axis $(-1,0,\ldots,0)$ and $\pi(x) = 0$). Then

$$\frac{f^*(x)}{f^*(0)} = \int e^{-\frac{1}{2}\|-x_1-\theta_1\|^2} e^{-\frac{1}{2}\sum_2^m \theta_i^2} F(d\theta) \quad \text{where } -x_1 = d(x) \qquad (3.3)$$

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and

$$\Delta \log \frac{f^*(x)}{f^*(0)} \leq \frac{\int \theta_1^2 e^{-x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} e^{-\frac{1}{2}\sum_2^m \theta_i^2} F(d\theta)}{\int e^{-x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} e^{-\frac{1}{2}\sum_2^m \theta_i^2} F(d\theta)}$$

$$- \left[\frac{\int \theta_1 e^{-x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} e^{-\frac{1}{2}\sum_2^m \theta_i^2} F(d\theta)}{\int e^{-x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} e^{-\frac{1}{2}\sum_2^m \theta_i^2} F(d\theta)} \right]^2 \qquad (3.4)$$

Note that $f^*(0) = \int e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)$ under the new co-ordinate system. Now, conditioning with respect to θ_1 and integrating with respect to the other variables we have

$$\Delta \log \frac{f^*(x)}{f^*(0)} \leq \frac{\int_0^\infty \theta_1^2 e^{-x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} F_1(d\theta_1)}{\int_0^\infty e^{-x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} F_1(d\theta_1)}$$

$$- \left[\frac{\int_0^\infty \theta_1 e^{-x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} F_1(d\theta_1)}{\int_0^\infty e^{-x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} F_1(d\theta_1)} \right]^2 \qquad (3.5)$$

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where $F_1(d\theta_1) = \int e^{-\frac{1}{2}\sum_2^m \theta_i^2} F(d\theta_2,\ldots,d\theta_m/d\theta_1)$. (Note that fixing θ_1 amounts to fixing a hyperplane). Also, observe that $f^*(0) = \int e^{-\frac{1}{2}\theta^2} F(d\theta)$ is observed inside thus normalizing the measures in (2.17).

It is easily seen, by integration by parts,

$$\Delta \log \frac{f^*(x)}{f^*(0)} \leq \frac{\int_0^\infty \theta_i^2 e^{-\theta_1 x_1} \tilde{F}(\theta_1)d\theta_1}{\int_0^\infty e^{-\theta_1 x_1} \tilde{F}(\theta_1)d\theta_1} - \frac{\int_0^\infty \theta_1 e^{-\theta_1 x_1} \tilde{F}(\theta_1)d\theta_1}{\int_0^\infty e^{-\theta_1 x_1} \tilde{F}(\theta_1)d\theta_1} \qquad (3.6)$$

where $\tilde{F}(\theta_1) = \int_0^{\theta_1} e^{-\frac{1}{2}t^2} F_1(dt)$. A word of caution is in order at this stage. In the above expression (3.5), $\tilde{F}(0)$ may be positive. We have assumed that $\tilde{F}(0) = 0$ for convenience (to avoid cumbersome expressions). The argument that follows does not depend on the value of $\tilde{F}(0)$. Let us now take a close look at $\tilde{F}(\theta_1)$ and study its properties. First of all note that $0 \leq \tilde{F}(\theta_1) \leq 1$ and non-decreasing s.t. $\tilde{F}(\infty) = 1$. This follows from the fact that $f^*(0) = \int e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)$ has been

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so absorbed in (2.17) as to normalize $\tilde{F}(\theta_1)$. The second property, which is more important to us, gives the precise implication of the assumption $\left\| \frac{\nabla f^*(y)}{f^*(y)} \right\| < C$ for $y \in K_F$ on $\tilde{F}(\theta_1)$. We have already observed that there exists a constant K_1 (depending only on C) such that $\Delta \log f^*(y) < K_1$ for $y \in K_F$. Thus the posteriori variance at y is bounded by $K_1 + m$.

Therefore, by Chebyshev's inequality, there exists a constant K_2 (depending only on C, K_1 and m) such that

$$\int_{\|\delta_F(y)-\theta\| \leq K_2} \frac{e^{-\frac{1}{2}\|y-\theta\|^2} F(d\theta)}{f^*(y)} \geq \frac{1}{4} \tag{3.7}$$

Since, $\|\delta_F(y)-y\| = \left\| \frac{\nabla f^*(y)}{f^*(y)} \right\| < C$, there exists another constant K_3 (depending on K_2 and C) such that

$$\int_{\|y-\theta\| \leq K_3} \frac{e^{-\frac{1}{2}\|y-\theta\|^2} F(d\theta)}{f^*(y)} \geq \frac{1}{4} \tag{3.8}$$

Observe that (3.8) is true for all $y \in K_F$ and the constant K_3 does not depend on y. Now, taking y to be $\pi(x)$ in (3.8) and linearly transforming the space so that $\pi(x) = 0$ we have

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$$\int_{\|\theta\| \le K_3} \frac{e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)}{f^*(0)} \geq \frac{1}{4} \tag{3.9}$$

in the transformed space. Going back to $\tilde{F}(\theta_1)$ now it follows from (3.9) that there exists a constant $K > 0$ (depending on K_3) such that $\tilde{F}(\theta_1) \geq \frac{1}{4}$ for $\theta_1 \geq K$. This is the bearing of the assumption $\left\| \frac{\nabla f^*(y)}{f^*(y)} \right\| < C$ for $y \in K_F$ on $\tilde{F}(\theta_1)$. Now, to show $\Delta \log f^*(x)$ is bounded above for $x \notin K_F$, we shall give an estimate for the first term in (3.6) using (3.9) as follows.

$\Delta \log f^*(x) \leq$

$$\leq \frac{\int_0^\infty \theta_1^2 e^{-\theta_1 x_1} \tilde{F}(\theta_1)\, d\theta_1}{\int_0^\infty e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1} = \frac{\int_0^{2K} \theta_1^2 e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1 + \int_{2K}^\infty \theta_1^2 e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1}{\int_0^{2K} e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1 + \int_{2K}^\infty e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1}$$

$$= \frac{\dfrac{\int_0^{2K} \theta_1^2 e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1}{\int_0^{2K} e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1} + \dfrac{\int_{2K}^\infty \theta_1^2 e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1}{\int_0^{2K} e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1}}{1 + \int_{2K}^\infty e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1 \Big/ \int_0^{2K} e^{-\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1}$$

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$$\leq \frac{(2K)^2 \int_0^{2K} \frac{\theta_1^2}{(2K)^2} e^{-\theta_1 x_1} \tilde{F}(\theta_1)d\theta_1}{\int_0^{2K} e^{-\theta_1 x_1} \tilde{F}(\theta_1)d\theta_1} + \frac{\int_{2K}^{\infty} \theta_1^2 e^{-\theta_1 x_1} d\theta_1}{\int_K^{2K} e^{-\theta_1 x_1} \tilde{F}(\theta_1)d\theta_1}$$

$$\leq (2K)^2 + \frac{(2K)^2 \frac{e^{-2Kx_1}}{x_1} + 2(2K)\frac{e^{-2Kx_1}}{x_1^2} + \frac{2e^{-2Kx_1}}{x_1^3}}{\frac{1}{4}\left[\frac{e^{-Kx_1}}{x_1} - \frac{e^{-2Kx_1}}{x_1}\right]} \tag{3.10}$$

In the last step, the numerator of second term follows from integration by parts and in the denominator of the same term we have used the fact $\tilde{F}(\theta_1) \geq \frac{1}{4}$ for $\theta_1 \geq K$. Therefore,

$$\Delta \log f^*(\mathbf{x}) \leq (2K)^2 (1+4) \quad \text{for, say,} \quad x_1 > 1 \tag{3.11}$$

For $x_1 \leq 1$ (that is $d(x) = \| x-\pi(x) \| \leq 1$), it follows from a lemma of Brown [1] (Lemma 3.3.3) that

$$\left\| \frac{\nabla f^*(x_1)}{f^*(x_1)} \right\| \leq \xi(1 + d(x)) \leq 2\xi \tag{3.12}$$

where the constant ξ depends only on C and m (the dimension). Thus, it follows from (3.2), (3.11) and (3.12) that $\Delta \log f^*(x)$ is bounded. We shall record this as a theorem below.

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<u>Theorem 3.1</u> Let $\left\|\frac{\nabla f^*(x)}{f^*(x)}\right\| < C$ for $x \varepsilon K_F$. Then there exists a constant K (depending only on C and m) such that

$$\Delta \log f^*(x) \leq K \quad \text{for all} \quad x \varepsilon E^m.$$

<u>Remark 3.2</u> It is interesting note that for any given $1 > \delta > 0$,

$$\theta_1 > \delta \Rightarrow F(\theta_1) > 0.$$

To prove this let H_{θ_1} denote the hyperplane given by $H_{\theta_1} = \{(t_1 \ldots t_m) : t_1 = \theta_1\}$. Thus H_o coincides with the tangent plane at $\pi(x)$. Suppose there exists a $\delta_o > 0$ such that $\tilde{F}(\theta_1) = 0$ for $\theta_1 \leq \delta_o$. Consider, now, the set $H_{\delta_o} = \bigcup_{\theta < \delta_o} H_\theta$. Plainly $F(H_{\delta_o} \cap K_F) = 0$. Therefore the closure of $K_F - H_{\delta_o}$, which is a convex set, is strictly contained in K_F and contains the support of the measure F. This contradicts the fact that K_F is the smallest closed convex set containing the support of F. Thus $\tilde{F}(\theta_1) > 0$ for $\theta_1 > 0$.

The second assumption $\|\delta_F(x)\| \leq \|x\| + K$, also follows easily from Brown's assumption $\left\|\frac{\nabla f^*(x)}{f^*(x)}\right\| < C$ for $x \varepsilon K_F$. For $x \varepsilon K_F$, Brown's assumption trivially implies $\|\delta_F(x)\| \leq \|x\| + C$. On the other hand for $x \not\varepsilon K_F$, we go about proving as follows.

$$\|\delta_F(x)\| \leq \|\delta_F(x) - \delta_F(\pi(x))\| + \|\delta_F(\pi(x))\| \tag{3.13}$$

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where $\pi(x)$ is the projection of x. Now, if $\hat{f}(x)$ denotes $\int e^{\theta x} e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)$, then

$$\delta_F(x) - \delta_F(\pi(x)) = \nabla \log \frac{\hat{f}(x)}{\hat{f}(\pi(x))} \tag{3.14}$$

Again, we can assume $x = (-d(x), 0, \ldots, 0)$ and $\pi(x) = 0$ as in the proof of the previous theorem (Theorem 3.1). An estimate for $\| \nabla \log \frac{f(x)}{f(\pi(x))} \|$ is contained in the argument that led to Theorem 3.1. To see this, using the same notations as before, it follows from (3.12) that for $d(x) \leq 1$

$$\| \nabla \log \frac{f(x)}{f(\pi(x))} \| = \| \nabla \log \frac{f(-d(x))}{f(0)} \| \leq 2\xi \tag{3.15}$$

where ξ depends only on C and m. For x such that $d(x) > 1$, the bound given in (3.11) is also an upper bound for $\| \nabla \log \frac{f(x)}{f(\pi(x))} \|$ i.e.

$$\| \nabla \log \frac{f(x)}{f(\pi(x))} \| \leq (2K)^2 \cdot 5 \tag{3.16}$$

To prove (3.16) one has to note that the bound which led to (3.11) is actually an upper bound for

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$$\frac{\int \theta_1^2 e^{-\theta_1 d(x)} e^{-\frac{1}{2}\theta_1^2} e^{-\frac{1}{2}\sum_2^m \theta_i^2} F(d\theta)}{\int e^{-\theta_1 d(x)} e^{-\frac{1}{2}\theta_1^2} e^{-\frac{1}{2}\sum_2^m \theta_i^2} F(d\theta)} \qquad (3.17)$$

But (3.17) is greater than or equal to $\| \nabla \log \frac{f(x)}{f(\pi(x))} \|^2$. Hence, there exists a constant K_1 (depending only on C and m) such that

$$\| \delta_F(x) \| \leq \| \delta_F(\pi(x)) \| + K_1.$$

But $\| \delta_F(\pi(x)) \| \leq \| \pi(x) \| + C \leq \| x \| + C$ because $\pi(x)$ is the projection of x onto K_F and the origin is in K_F. Thus there exists a constant K (depending only on C and m) such that

$$\| \delta_F(x) \| \leq \| x \| + K. \qquad (3.18)$$

Thus we have the following result.

<u>Theorem 3.3</u>: Let $\| \frac{\nabla f^*(x)}{f^*(x)} \| < C$. Then there exists a constant K (depending only on C and m) such that

$$\| \delta_F(x) \| \leq \| x \| + K \quad \text{for all} \quad x \in E^m.$$

<u>Remark 3.4</u> The assumption (ii) is a condition on the growth of $\frac{\nabla f^*(x)}{f^*(x)}$ in the direction of $\frac{\delta_F(x)}{\|\delta_F(x)\|}$. So it is a smoothness condition on the measure. Statistically, one could interpret

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assumption (ii) as a condition on the expansion of the estimator $\delta_F(x)$ in relation to x. Of course it is more meaningful to talk of shrinking or expansion in one dimensional case rather than in higher dimension. It is not at all clear how one should define shrinking or expansion in higher dimensions. However, we shall see later in this chapter, that assumption (ii) could be dispensed with in the spherically symmetric case.

<u>Remark 3.5</u> The fact that assumption (ii) is essentially a condition on the second derivatives of log $f^*(x)$ is brought out by the following. Let $\hat{f}(x) = \int e^{\theta x} e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)$ and $A(x)$ denote the second derivative matrix of log $\hat{f}(x)$. Suppose $A(x)$ satisfies the condition

$$Z'A(x)Z \leq 1 + \frac{\gamma(\|x\|)}{\|x\|} \quad \text{for all } x \text{ and for all } Z \tag{3.19}$$

where Z is a unit vector and $\gamma(\|x\|)$ is a non-negative dini function (See Chapter 1 for the definition of dini function).

It is easy to see, by integrating along the line segments, that (3.19) implies assumption (ii). A measure satisfying (3.19) can have at the most exponential growth rate in any direction at a given point. There are other conditions on $A(x)$ which would imply the assumption (ii). We shall not pause here to catalog them.

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It is not surprising at all that Brown's assumption implies our assumptions. The condition that $\left\|\frac{\nabla f^*(x)}{f^*(x)}\right\| < C$ for $x \in K_F$ is so stringent that it implies that all absolute moments of the form

$$\frac{\int \|x-\theta\|^K e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta)}{f^*(x)}, \qquad K \geq 1$$

are bounded for $x \in K_F$. Indeed $\frac{\int e^{K\|x-\theta\|} e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta)}{f^*(x)}$ is bounded for $x \in K_F$. This is very stringent compared to our assumption $\Delta \log f^*(x) < B$ which is equivalent to

$$\frac{\int \|\theta-\delta_F(x)\|^2 e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta)}{f^*(x)} < B + m.$$

We conclude this section with a comment on the proof of Theorem 3.1. A close look at the **argument** preceding the theorem reveals that we have used only the behaviour of $f^*(y)$ on the boundary of K_F to prove $\Delta \log f^*(x) < B$ and $\|\delta_F(x)\| \leq \|x\| + K$ for $x \notin K_F$. i.e. For $x \notin K_F$, we have used the fact $\left\|\frac{\nabla f^*(\pi(x))}{f^*(\pi(x))}\right\| < C$. Thus we have the following result.

Theorem 3.6: Let $\Delta \log f^*(x) < B$ and $\|\delta_F(x)\| \leq \|x\| + K$ for $x \in K_F$. Suppose $\left\|\frac{\nabla f^*(x)}{f^*(x)}\right\| < C$ for $x \in \Gamma_{K_F}$ where

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Γ_{K_F} is the boundary of K_F. Then there exist constants B_1 and K_1 (depending only on B, K, C and m) such that

(i) $\Delta \log f^*(x) < B_1$ for all $x \varepsilon E^m$

(ii) $\|\delta_F(x)\| \leq \|x\| + K_1$ for all $x \varepsilon E^m$

§ 4. Some Technical Results

In this section we develop some technical lemmas which would be needed in the proof of the main theorem.

The first result gives lower bound for functions of the form

$$\frac{\int_{\|\theta\| \leq n} e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta)}{f^*(x)}$$

for x such that $\|x\| = n$.

Lemma 4.1 Assume (2.12). Let r be sufficiently large (say $r > 10(K+B)$). Then there exists a constant $K_0 > 0$ (depending only on K and B) such that

$$\inf_{\{x : \|x\|=r\}} \frac{\int_{\|\theta\|\leq r} e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta)}{f^*(x)} \geq K_0 > 0 \tag{4.1}$$

Proof : Let S_r denote the sphere of radius r. Let x be a point on S_r i.e. $\|x\| = r$. Let x_0 be the point on the

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line segment $[0, x]$ (the line segment joining 0 and x) such that $\|x_o\| = r - (K+B+1)$. Note that $\|x - x_o\| = (K+B+1)$. It follows from the assumption (ii) of (2.12) that $\|\delta_F(x_o)\| < r - K+B$. Now, observe that by Chebyshev's inequality we have

$$\int_{\|\theta - \delta_F(x_o)\| \leq B+1} \frac{e^{-\frac{1}{2}\|x_o - \theta\|^2} F(d\theta)}{f^*(x_o)} \geq 1 - \frac{B}{(B+1)^2} \geq \frac{1}{B+1} > 0.$$

We shall use $f^*(x_o)$ to get a lower bound for

$$\int_{\|\theta\| \leq r} \frac{e^{-\frac{1}{2}\|x - \theta\|^2} F(d\theta)}{f^*(x)}$$ as follows.

Expanding $\log f^*(x)$ by Taylor expansion about x_o we get

$$\log f^*(x) = \log f^*(x_o) + (x - x_o) \frac{\nabla f^*(x_o)}{f^*(x_o)} + (x - x_o)' \, Q(\xi)(x - x_o) \tag{4.2}$$

where ξ is a point on the line segment $[x, x_o]$ and $Q(\xi)$ is the second derivative matrix. Moreover, $Q(\xi) = A(\xi) - I$ where $A(\xi)$ is a positive semidefinite matrix and I is the $m \times m$ identity matrix. The matrix $A(\xi)$ is a variance covariance matrix whose (i,j)-th element is given by

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$$\left\{\frac{\int \theta_i\,\theta_j\, e^{\theta\cdot\xi}\, e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)}{\int e^{\theta\xi}\, e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)}\right\} - \left\{\frac{\int \theta_i\, e^{\theta\xi}\, e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)}{\int e^{\theta\xi}\, e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)}\right. \times$$

$$\times \left.\frac{\int \theta_j\, e^{\theta\xi}\, e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)}{\int e^{\theta\xi}\, e^{-\frac{1}{2}\|\theta\|^2} F(d\theta)}\right\} \qquad (4.3)$$

Therefore,

$$(x-x_0)'\; Q(\xi)(x-x_0) \leq (K+B+1)^2\; m^2(B+m) \qquad (4.4)$$

In (4.4) we have used the fact $\|x-x_0\| = (K+B+1)$ and the diagonal elements of $A(\xi)$ are bounded by $(B+m)$. The latter fact follows from $\Delta \log f^*(\xi) < B$ and $\Delta \log f^*(\xi)$ is the sum of the diagonal elements $Q(\xi)$.

Hence it follows from (4.2) and (4.4)

$$\frac{f^*(x_0)}{f^*(x)} \geq e^{-(x-x_0)\frac{\nabla f^*(x_0)}{f^*(x_0)}}\; e^{-(K+B+1)^2\, m^2(B+m)} \qquad (4.5)$$

Now consider

$$\frac{\int_{\|\theta\|\leq r} p_\theta(x)\; F(d\theta)}{f^*(x_0)} = \frac{\int_{\|\theta\|\leq r} e^{-\frac{1}{2}\|x-x_0+x_0-\theta\|^2} F(d\theta)}{f^*(x_0)} \qquad (4.6)$$

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where $p_\theta(x)$ is the normal density.

$$(4.6) \geq e^{-\frac{1}{2}(K+B+1)^2} \int_{\|\delta_F(x_o)-\theta\|<B+1} \frac{e^{-(x-x_o)(x_o-\delta_F(x_o)+\delta_F(x_o)-\theta)} p_\theta(x_o)F(d\theta)}{f^*(x_o)} \quad (4.7)$$

$$\geq e^{-\frac{1}{2}(K+B+1)^2} e^{(x-x_o)\left(\frac{\nabla f^*(x_o)}{f^*(x_o)}\right)} \times$$

$$\times \frac{\int_{\|\delta_F(x_o)-\theta\|<B+1} e^{-\|x-x_o\|\|\delta_F(x_o)-\theta\|} p_\theta(x_o)F(d\theta)}{f^*(x_o)} \quad (4.8)$$

$$\geq e^{-\frac{1}{2}(K+B+1)^2} e^{(x-x_o)\left(\frac{\nabla f^*(x_o)}{f^*(x_o)}\right)} e^{-(K+B+1)(B+1)} \frac{1}{(B+1)} \quad (4.9)$$

In obtaining (4.7) we have used the assumption to conclude $\|\delta_F(x_o)\| \leq r-\overline{B+1}$ and hence $\{\theta: \|\delta_F(x_o)-\theta\| < B+1\} \subset \{\theta: \|\theta\| < r\}$.

In (4.8), we have written $(\delta_F(x_o)-x_o)$ as $\dfrac{\nabla f^*(x_o)}{f^*(x_o)}$. In the last step we have used Chebyshev's inequality.

Therefore, using (4.5) and (4.9), we have for x such that $\|x\| = r$,

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$$\frac{\int_{\|\theta\|\leq r} p_\theta(x)\, F(d\theta)}{f^*(x)} = \frac{\int_{\|\theta\|\leq r} p_\theta(x)\, F(d\theta)}{f^*(x_o)} \cdot \frac{f^*(x_o)}{f^*(x)}$$

$$\geq \frac{e^{-\frac{1}{2}(K+B+1)^2 - (K+B+1)^2}}{B+1}\, e^{(x-x_o)\frac{\nabla f^*(x_o)}{f^*(x_o)}}$$

$$\times\ e^{-(K+B+1)^2 m^2 (B+m)}\, e^{(x-x_o)\frac{\nabla f^*(x_o)}{f^*(x_o)}}$$

$$\geq \frac{1}{B+1}\, e^{-(K+B+1)^2(\frac{3}{2}+m^2(B+m))} \tag{4.10}$$

Now set: $K_o = \frac{1}{B+1}\, e^{-(K+B+1)^2(2+m^2(B+m))}$.

This completes the proof of the lemma. q.e.d.

As a corollary to Lemma 4.1 we have the following result.

Corollary 4.2 Assume (2.12). Let $\{u_n\}$ be a sequence of functions satisfying

$$\begin{aligned} L_f\, u_n &= 0 \quad \text{for } 1 < \|x\| < n \\ u_n &= 1 \quad \text{for } \|x\| \leq 1 \\ &= \varepsilon \quad \text{for } \|x\| = n \\ &= 0 \quad \text{for } \|x\| > n \end{aligned}$$

where $1 > \varepsilon > 0$ is fixed.

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$$L_f\, u_n = 0 \quad \text{for } 1 < \|x\| < n$$
$$u_n = 1 \quad \text{for } \|x\| \leq 1$$
$$= \epsilon \quad \text{for } \|x\| = n$$
$$= 0 \quad \text{for } \|x\| > n$$

where $1 > \epsilon > 0$ is fixed. Then there exists a constant $K_o' > 0$ such that

$$\inf_{\{x:\ \|x\| \leq n\}} \frac{\int_{\|\theta\| \leq n} u_n(\theta) p_\theta(x) F(d\theta)}{f^*(x)} \geq K_o' \text{ for all sufficiently large } n.$$

<u>Proof</u> : By maximum modulus principle $u_n(\theta) \geq \epsilon$ for $\|\theta\| \leq n$ and we have

$$\int_{\|\theta\| \leq n} \frac{u_n(\theta) p_\theta(x) F(d\theta)}{f^*(x)} \geq \epsilon \int_{\|\theta\| \leq n} \frac{p_\theta(x) F(d\theta)}{f^*(x)}$$

Therefore,

$$\inf_{\{x:\ \|x\| \leq n\}} \int_{\|\theta\| \leq n} \frac{u_n(\theta)\ p_\theta(x)\ F(d\theta)}{f^*(x)}$$

$$\geq \epsilon \inf_{\{x:\ \|x\| \leq n\}} \int_{\|\theta\| \leq n} \frac{p_\theta(x)\ F(d\theta)}{f^*(x)} \geq \epsilon K_o \tag{4.11}$$

Taking $K_o' = \epsilon K_o$ we have the lemma. This completes the prove.

q.e.d.

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The next lemma is a rather standard technical result.

<u>Lemma 4.3</u> : Let u be a piecewise differentiable function on E^m. Then there exists a constant $C_2>0$ such that

$$\int (u(\theta)-u(x))^2 p_\theta(x)\, dx \leq C_2 \int \| \nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x)\, dx$$

$$+ C_2 \int \| \nabla u(x)\|^2 p_\theta(x)\, dx.$$

<u>Proof</u> : Write x in polar co-ordinates around θ i.e. $x = (r(x), \phi)$ where $r(x) = \|x-\theta\|$. Assume that ϕ has been normalized. By Schwartz inequality we have

$$(u(\theta) - u(x))^2 \leq r(x) \int_0^{r(x)} \| \nabla u(s,\phi)\|^2 ds$$

Therefore, denoting $r(x)$ by r, we have

$$\int(u(\theta)-u(x))^2 e^{-\frac{1}{2}\| x-\theta\|^2} dx \leq \int r(\int_0^r \| \nabla u (s,\phi) \|^2 ds) e^{-\frac{1}{2}r^2} r^{m-1} dr\, d\phi$$

$$= \int_0^\infty \| \nabla u(s)\phi \|^2 \int_s^\infty r^m e^{-\frac{1}{2}r^2} dr\, ds d\phi \qquad (4.12)$$

Now integrating by parts we have

$$\int_s^\infty r^m e^{-\frac{1}{2}r^2} dr \leq C_2 (s^{m-1} e^{-\frac{1}{2}s^2} + e^{-\frac{1}{2}s^2})$$

for some constant $C_2 > 0$.

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Therefore,

$$(4.12) \leq C_2\left[\int_0^\infty \|\nabla u(s,\phi)\|^2 s^{m-1} e^{-\frac{1}{2}s^2} ds d\phi + \int_0^\infty \|\nabla u(s,\phi)\|^2 e^{-\frac{1}{2}s^2} ds d\theta\right]$$

$$= C_2\left[\int \|\nabla u(x)\|^2 p_\theta(x) \, dx + \int \|\nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x) dx\right]$$

Hence the lemma.

q. e. d.

Lemma 4.4 : Let ρ be a constant such that $0 < \rho < \frac{1}{2}$. Then there exist constants K_1 and K_2 such that

$$\int_{\|x-\theta\| \leq \rho} \|\nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x) \, dx \leq K_1 \int_{\|x-\theta\| < K_2} \|\nabla u(x)\|^2 p_\theta(x) dx.$$

Proof : Fix θ. Define a density function $r(\theta,x)$ by

$$r(\theta,x) = C \, I(\theta,x) \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x)$$

where $I(\theta,x) = 1$ if $\|x-\theta\| \leq \rho$ and $= 0$ otherwise and C is the normalizing constant so that $\int r_\theta(x) dx = 1$. Note that C depends only on ρ. Define a new density function $s(\theta,x)$ by setting

$$s(\theta,x) = \int \cdots \int r(\theta,t_1) \, r(t_1,t_2) \cdots r(t_{2\ell}, x) \, dt_1 \cdots dt_\ell$$

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where $\lambda \geq 1$ is a fixed integer. Plainly $\int s(\theta,x)dx = 1$. Moreover, $s(\theta,x) = 0$ for $\|x-\theta\| > \frac{2\lambda}{\rho}$ and $s(\theta,x)$ is bounded.

The bound of $s(\theta,x)$, say K_3, depends only on ρ, m and λ. It is also easy to see

$$\int \|\nabla u(x)\|^2 \gamma(\theta,x)dx = \int \|\nabla u(x)\|^2 s(\theta,x)dx \tag{4.13}$$

Now,

$$\begin{aligned}\int_{\|x-\theta\|\leq} \|\nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x)\, dx &= \int \|\nabla u(x)\|^2 \gamma_\theta(x)dx \\ &= \int \|\nabla u(x)\|^2 s(\theta,x)dx \\ &= \int_{\|x-\theta\|\leq \frac{2\lambda}{\rho}} \|\nabla u(x)\| s(\theta,x)dx \\ &\leq K_3 \int_{\|x-\theta\|\leq \frac{2\lambda}{\rho}} \|\nabla u(x)\|^2 dx\end{aligned} \tag{4.14}$$

Since,

$$e^{-\frac{1}{2}\|x-\theta\|^2} \geq e^{-\frac{1}{2}(\frac{2\lambda}{\rho})^2} \quad \text{for } x \text{ in } \{x : \|x-\theta\| \leq \frac{2\lambda}{\rho}\},$$

we have

$$(4.13) \leq K_3\, e^{\frac{1}{2}(\frac{2\lambda}{\rho})^2} \int_{\|x-\theta\|\leq \frac{2\lambda}{\rho}} \|\nabla u(x)\|^2 e^{-\frac{1}{2}\|x-\theta\|^2} dx$$

Letting $K_1 = K_3\, e^{\frac{1}{2}(\frac{2\lambda}{\rho})^2}$ and $K_2 = \frac{2\lambda}{\rho}$, the lemma follows.

q.e.d.

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Corollary 4.5 : Let u be a peicewise differentiable function in E^m. Then there exists a constant $K > 0$ such that

$$\int (u(\theta)-u(x))^2 \, p_\theta(x) \, dx \leq K \int \|\nabla u(x)\|^2 \, e^{-\frac{1}{2}\|x-\theta\|^2} dx$$

Proof : By lemma (4.3) we have

$$\int (u(\theta)-u(x))^2 \, p_\theta(x) \, dx \leq C_2 \int \|\nabla \; u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x) +$$

$$+ C_2 \int \|\nabla u(x)\|^2 \, p_\theta(x) \, dx$$

Now, write,

$$\int \| \quad u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x) \, dx = \int_{\|x-\theta\| \leq \rho} \|\nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x) dx$$

$$+ \int_{\|x-\theta\| > \rho} \|\nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x) dx \tag{4.15}$$

where $\rho = p$ is a fixed positive constant, say $\frac{1}{4}$.

The first term in the right side of (4.15) can be bounded using Lemma 4 .4 as follows

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$$\int_{\|x-\theta\| \le \frac{1}{4}} \|\nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_\theta(x)dx \le K_1 \int_{\|x-\theta\| \le K_2} \|\nabla u(x)\|^2 p_\theta(x)dx$$

$$\le K_1 \int \|\nabla u(x)\|^2 p_\theta(x)dx \qquad (4.16)$$

The second term in right side of (4.15) is bounded by

$$4^m \int p_\theta(x) \; \|\nabla \; u(x)\|^2 \, dx \qquad (4.17)$$

Combining (4.16) and (4.17) we have

$$\int (u(\theta)-u(x))^2 \, p_\theta(x) \; dx \le (C_2+K_1+4^m) \int \|\nabla u(x)\|^2 \, p_\theta(x) \; dx.$$

q.e.d.

Hence the corollery.

§ 5. Proof of the Main Theorem and Other Results.

After developing all the necessary auxiliary results we come to the proof of the main theorem now.

Proof of Theorem 2.3 (Main Theorem)

We give the proof of the sufficiency. Recall the basic assumptions (2.12) i.e. $\Delta \log f^*(x) < B$ and (ii) $\|\sigma_F(x)\| \le \|x\| + K$.

Assume that BP II is solvable for the Elliptic equation $L_f u = 0$. Then there exists a sequence $\{u_n\}$ of functions satisfying $L_f \, u_n = 0$ for $1 < \|x\| < n$, $u_n(x) = 1$ for $\|x\| \le 1$

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and $u_n(x) = \varepsilon$ for $\|x\| = n$, $u_n = 0$ for $\|x\| > n$, which converge to 1 uniformly on compacta. Define a sequence of finite measures $\{G_n\}$ by setting $G_n(d\theta) = u_n(\theta)\, F(d\theta)$. Plainly, $G_n(S_1) = F(S_1) \geq 1$ for all n (S_1 denotes the unit sphere). Moreover, the supports of G_n increase to E^m.

Let δ_{G_n} be the Bayes procedure with respect to G_n. Thus $\delta_{G_n}(x) = \dfrac{\nabla g_n^*(x)}{g_n^*(x)} + x$ for all x. We shall show,

$$\int (R(\theta,\delta_F) - R(\theta,\delta_{G_n}))\, G_n(d\theta) \longrightarrow 0 \quad \text{as} \quad n \to \infty \qquad (5.1)$$

Computation similar to that in section 2 yields

$$\int (R(\theta,\delta_F) - R(\theta,\delta_{G_n}))\, G_n(d\theta) = \int \left\|\nabla\left(\frac{g_n^*(x)}{f^*(x)}\right)^{\frac{1}{2}}\right\|^2 f^*(x)\, dx$$

$$= \int \left[\int (u_n(\theta) - u_n(x))\left(\theta - x - \frac{\nabla f^*(x)}{f^*(x)}\right) \frac{p_\theta(x) F(d\theta)}{f^*(x)}\right]^2$$

$$\times \frac{1}{\left(\frac{g_n^*(x)}{f^*(x)}\right)} f^*(x)\, dx \qquad (5.2)$$

$$(5.2) = \int_{\|x\| \leq n} \left[\int (u_n(\theta) - u_n(x))\left(\theta - x - \frac{\nabla f^*(x)}{f^*(x)}\right) \frac{p_\theta(x)\, F(d\theta)}{f^*(x)}\right]^2$$

$$\times \frac{1}{(g_n^*(x)/f^*(x))} \cdot f^*(x)\, dx$$

$$+ \int_{\|x\| > n} \left\|\nabla\left(\frac{g_n^*(x)}{f^*(x)}\right)^{\frac{1}{2}}\right\|^2 f^*(x)\, dx \qquad (5.3)$$

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We shall show that the solvability of BP II implies that the first term in the right side of (5.3) goes to zero as $n \to \infty$. The second term, the tail, goes to zero independently as $n \to \infty$.

Consider the first term. It follows from corollary 4.2 that there exists a constant K_o (depending only B, K, m and ε) such that $\frac{f^*(x)}{g_n^*(x)} \leq K_o$ for all $x \in \{y : \|y\| \leq n\}$. Therefore

$$\int_{\|x\|\leq n} [\int (u_n(\theta)-u_n(x))(\theta - x - \frac{\nabla f^*(x)}{f^*(x)}, \frac{p_\theta(x)}{f^*(x)} F(d\theta)]^2 \frac{1}{\frac{g_n^*(x)}{f^*(x)}} f^*(x)dx$$

$$\leq K_o \int_{\|x\|\leq n} [\int (u_n(\theta)-u_n(x))(\theta - x - \frac{\nabla f^*(x)}{f^*(x)}, \frac{p_\theta(x)\ F(d\theta)}{f^*(x)}]^2 f^*(x) \tag{5.4}$$

Now, by an application of Schwertz inequality, it follows

$$[\int (u_r(\theta)-u_n(x))(\theta - x - \frac{\nabla f^*(x)}{f^*(x)}) \frac{p_\theta(x)\ F(d\theta)}{f^*(x)}]^2$$

$$\leq \int (u_n(\theta)-u_n(x))^2 \frac{p_\theta(x)\ F(d\theta)}{f^*(x)}$$

$$\times \int \|\theta - x - \frac{\nabla f^*(x)}{f^*(x)}\|^2 \frac{p_\theta(x)\ F(d\theta)}{f^*(x)} .$$

Since $\Delta \log f^*(x) < B$,

$$\int \|\theta - x - \frac{\nabla f^*(x)}{f^*(x)}\|^2 \frac{p_\theta(x)\ F(d\theta)}{f^*(x)} = \Delta \log f^*(x) + m \leq (B+m).$$

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Hence,

$$(5.4) \leq (B+m)\cdot K_o \int\int_{\|x\| \leq n} (u_n(\theta)-u_n(x))^2 \frac{p_\theta(x)\ F(d\theta)}{f^*(x)} f^*(x)dx$$

$$= (B+m)\ K_o \int\int_{\|x\| \leq n} (u_n(\theta)-u_n(x))\ p_\theta(x)\ dx\ F(d\theta).$$

From Corollary 4.5 it follows that for some constant K_1 (depending only on m)

$$(5.4) \leq (B+m)K_o\cdot K_1 \int_{\|x\| \leq n} \|\nabla u_n(x)\|^2\ p_\theta(x)\ F(d\theta)dx$$

$$\leq (B+m)\ K_o\cdot K_1 \int_{\|x\| \leq n} \|\nabla u_n(x)\|^2 f^*(x)dx. \qquad (5.5)$$

The right side of (5.5) goes to zero as $n \to \infty$. This follows from the solvability BP II for $L_f u = 0$.

Let us now deal with the second term (the tail) of (5.3). The second term can be bounded as follows.

$$\int_{\|x\| \geq n} [\int u_n(\theta)(\theta - x - \frac{\nabla f^*(x)}{f^*(x)}) \frac{p_\theta(x)\ F(d\theta)}{g_n^*(x)}]^2\ g_n^*(x)\ dx$$

$$\leq \int_{\|x\| \geq n} \int u_n(\theta)\ \|\theta - x - \frac{\nabla f^*(x)}{f^*(x)}\|^2\ p_\theta(x)\ F(d\theta)dx \qquad (5.6)$$

(We have used Schwartz inequality in (5.6) with respect to the measure $\frac{p_\theta(x)\ F(d\theta)}{g_n^*(x)}$,

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Since $u_n(\theta) = 0$ for $||\theta|| > n$, we have

$$(5.6) \leq c \int_{||x||>n} ||x||^2 \int_{||\theta||\leq n} e^{-\frac{1}{2}(||x||-||\theta||)^2} F(d\theta)\, dx \qquad (5.7)$$

Observe that the assumption $||\delta_F(x)|| \leq ||x|| + K$ implies $\frac{x}{||x||} \nabla \log f^*(x) \leq K$. Therefore integrating along the line segment x we get $f^*(x) \leq e^{||x|| K}$. We shall now show that $\int e^{-||\theta|| K_1} F(d\theta) < \infty$ for some constant $K_1 > 0$.

$$\int e^{-||\theta|| K_1} F(d\theta) = \frac{1}{(2\pi)^{m/2}} \int e^{-||\theta|| K_1} \left(\int e^{-\frac{1}{2}||x-\theta||^2} dx\right) F(d\theta)$$

$$\leq \frac{1}{(2\pi)^{m/2}} \int\int e^{-\frac{1}{2}||x-\theta||^2} e^{-||x|| K_1} e^{K_1 ||x-\theta||} dx\, F(d\theta)$$

(5.8)

A lemma of Brown's (See Brown [1]) implies

$$e^{-\frac{1}{2}||x-\theta||^2} e^{K_1 ||x-\theta||} \leq K_2 \int_{||\xi|| \leq K_2+1} e^{-\frac{1}{2}||x-\theta+\xi||^2} d\xi \,. \qquad (5.9)$$

Substituting the right side of (5.9) in (5.8) we get

$$\int\int e^{-\frac{1}{2}||x-\theta||^2} e^{-||x|| K_1} e^{K_1 ||x-\theta||} dx\, F(d\theta)$$

$$\leq K_2 \int\int\int_{||\xi|| \leq K_2+1} e^{-\frac{1}{2}||x-\xi-\theta||^2} d\xi\, e^{-||x|| K_1} dx\, F(d\theta)$$

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$$= K_2 \int\int_{\|\xi\|\leq K_2+1} f^*(x-\xi)\, e^{-\|x\| K_1}\, dx\, d\xi \tag{5.10}$$

But $f^*(x) \leq e^{\|x\| K}$. Therefore

$$(5.10) \leq K_2 \int_{\|\xi\|\leq K_2+1} \int e^{(\|x\| + \|\xi\|) K}\, e^{-\|x\| K_1}\, dx\, d\xi$$

$$\leq K_3 \int e^{\|x\| K} \cdot e^{-\|x\| K_1}\, dx \tag{5.11}$$

If we choose $K_1 = K+1$ (5.11) is finite. The constant K_3 is an upper bound for $K_2 \int_{\|\xi\|\leq K+1} d\xi\, e^{K_2 \cdot K}$.

Therefore,

$$\int_{\|x\|\geq n} \|x\|^2 \int_{\|\theta\|\leq n} e^{-\frac{1}{2}(\|x\|-\|\theta\|)^2} F(d\theta)$$

$$\leq \int_{\|x\|\geq n} \|x\|^2 e^{nK_1} \int_{\|\theta\|\leq n} e^{-\|\theta\| K_1} e^{-\frac{1}{2}(\|x\|-\|\theta\|)^2} F(d\theta) dx \tag{5.12}$$

Now observe that $e^{-\frac{1}{2}(\|x\|-\|\theta\|)^2}$ is a concare function in $\|\theta\|$ for $\|\theta\| < \|x\|$. Let F_n be the probability measure defined by

$$F_n(d\theta) = \frac{e^{-\|\theta\|(K+1)} F(\theta)}{\int_{\|\theta\|\leq n} e^{-\|\theta\|(K_1+1)} F(d\theta)}, \quad \text{for } \|\theta\| \leq n.$$

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Let $C_n = \int_{\|\theta\| \leq n} e^{-\|\theta\|(K_1+1)} F(d\theta)$ and

$$b_n = \frac{1}{C_n} \int_{\|\theta\| \leq n} \|\theta\| \, e^{-\|\theta\|(K_1+1)} F(d\theta)$$

It is an easy fact to check that $\limsup_{n \to \infty} \frac{1}{n} b_n < a < 1$.

Hence, by Jensen's inequality we have,

$$\int_{\|\theta\| \leq n} e^{-\|\theta\|(K_1+1)} e^{-\frac{1}{2}(\|x\| - \|\theta\|)^2} F(d\theta) \leq C_n e^{-\frac{1}{2}(\|x\| - na)^2} \tag{5.13}$$

for all sufficiently large n. Therefore,

$$(5.12) \leq C_n \int_{\|x\| \geq n} \|x\|^2 e^{nK_1} e^{-\frac{1}{2}(\|x\| - na)^2} dx$$

$$\leq C_n e^{nK_1} e^{-\frac{1}{4}n^2(1-a)^2} \int_{\|x\| \geq n} \|x\|^2 e^{-\frac{1}{4}(\|x\| - na)^2} dx$$

$$\leq C_n \cdot (na+2)^{m+1} e^{nK_1} \cdot e^{-\frac{1}{4}n^2(1-a)} \longrightarrow 0 \tag{5.14}$$

It follows from (5.14) that (5.6) goes to zero as $n \to \infty$. Thus we have shown that right side of (5.3) goes to zero as $n \to \infty$. This completes the proof of the main theorem.

q.e.d.

Remark 5.1 : Note that we do not actually need the assumption $\|\delta_F a\| \leq \|x\| + K$ to show that (5.7) goes to zero. It is possible to prove it from the condition $\Delta \log f^*(x) < B$. Therefore, the

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condition $\| \delta_F(x) \| \leq \|x\| + K$, is needed only to prove lemma 4.1 and is not used anywhere else in the proof of the main theorem.

As a corollary to our theorem we have Brown's result. We record this below.

Corollary 5.2 : Let $\| \frac{\nabla f^*(x)}{f^*(x)} \| < C$ for $x \varepsilon K_F$. Then δ_F is admissible if BP II is solvable for $L_f u = 0$.

We have already observed in Theorem 3.5 that our assumptions can be stated in a different form when $K_F \neq E^m$. Thus we have the following theorem.

Theorem 5.3 : An estimator δ_F, generalized Bayes with respect to F, is admissible if the following conditions are satisfied.

(i) $\Delta \log f^*(x) < B$ for $x \varepsilon K_F$

(ii) $\|\delta_F(x)\| \leq \|x\| + K$ for $x \varepsilon K_F$

(iii) $\| \frac{\nabla f^*(x)}{f^*(x)} \| < B_1$ for $x \varepsilon \Gamma$ where Γ is the boundary of K_F

(iv) BP II is solvable for $L_f\, u = 0$.

We end this section with a few comments. Our proof avoids the construction of smooth minimizing solutions on which Brown's heavily depends. To prove a result similar to that of Lemma 4.1, Brown needs smooth solutions of the boundary value problem. His

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method can not be extended to the general case (the case when $\left\|\frac{\nabla f^*(x)}{f^*(x)}\right\|$ is not bounded) because he uses a Harnack inequality which is valid only for solutions of Elliptic equations with bounded coefficients. The author believes the condition that $\|\delta_F(x)\| < \|x\| + K$ could be relaxed. To do that, we believe one should get hold of a sequence of regions expanding to E^m which may not be spheres as in our case. The choice of such regions will very much depend on the behaviour of $\delta_F(x)$ or $f^*(x)$.

§ 6. Diffusion Processes and Admissibility:

The elliptic partial differential operator L_f which plays a crucial role in our admissibility problem is intimately related to Diffusion processes. Indeed, associated with every smooth elliptic operator $Lu = \sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u}{\partial x_i}$ there exists a local diffusion process Z_t with local mean $\{b_i(x)\}$ and variance $(a_{ij}(x))$ (See Ito and Mckean [1], Brown [2], Varadhan and Stroock [1]). By smoothness of L we mean that the coefficients of L should satisfy some mild regularity conditions. One can show that BP II for L is solvable if and only if Z_t is recurrent i.e. Prob $\{\inf_t \|Z_t^*\| \leq 1\} = 1$ where Z_t^* is the diffusion process starting at x. See Brown [1], for a proof of this fact.

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The above facts enable us to state our main theorem in terms of the recurrence of a diffusion process.

Theorem 6.1 : Let δ_F be a generalized Bayes procedure. Z_t be the diffusion process associated with L_f. A necessary condition for δ_F to be admissible is Z_t is recurrent. Furthermore, if Z_t is recurrent and (i) $\Delta \log f^*(x) < B$, (ii) $\|\delta_F(x)\| \leq \|x\| + K$ then δ_F is admissible.

Note that the usual best invariant estimator, $\delta_1(x) = x$, is the generalized Bayes estimator with respect to the m-dimensional Lebesgue measure. The differential operator which corresponds to this prior is $Lu = u$ and the diffusion associated with it is a version of Brownian motion run with $\frac{1}{2}$ speed clock). Thus it follows from Theorem 6.1 that $\delta_1(x)$ is admissible if and only if the Brownian motion is recurrent. It is well known that Brownian motion is recurrent for dimension $m \leq 2$ and transient when $m \geq 3$ (Z_t is transient when Z_t is not recurrent). The result that δ_1 is admissible for $m \leq 2$ and inadmissible for $m \geq 3$ is already known (Stein [2], Stein and James [1]). We find this relation with Brownian motion interesting. We shall show later in this chapter that BP II for $Lu = \Delta u$ is solvable when $m \leq 2$ and not solvable when $m \geq 3$.

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§ 7. A generalization in the Spherically Symmetric Case

A generalization of our main theorem is possible in the spherically symmetric case. A close look at the proof of the main theorem reaveals that we have used the assumption $||\delta_F(x)|| \leq ||x|| + k$ at two points, in proving lemma 4.1 and tackling the tail in the proof of the main theorem. We shall show, below, that in the spherically symmetric case this assumption could be dispensed with and the main theorem could be derived from the assumption $\Delta \log f^*(x) < B$. Let F be a spherically symmetric measure on E^m such that $f^*(x) = f^*(||x||) < \infty$ for all x

Theorem 7.1 : Let $\Delta \log f^*(x) < B$ Furthermore, if BP II is solvable for $L_f u = 0$ then ∂_F is admissible.

First we shall prove Lemma 4.1 in this case and then go to the proof of the theorem.

Lemma 7.2 : Suppose BPII is solvable for $L_f u = 0$. Then

$$\liminf_{||x|| \to \infty} \frac{x}{||x||} \cdot \frac{\nabla f^*(||x||)}{f^*(||x||)} - \frac{(2-m)}{||x||} - \frac{\partial}{||x||} \leq 0$$

where $1 > \partial > 0$.

Proof:- Since BPII is solvable for $L_f u = 0$ BPI is not solvable for $L_f u = 0$ i.e. there does not exist a solution u_0 for $L_f u = 0$ such that $u(||x||) = 1$ for $||x|| = 1$ and

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and $u(||x||) \to 0$ as $||x|| \to \infty$. This follows from Chapter I Now, a sufficient condition for BPI to be solvable for $L_f u = 0$ is

$$\bar{x} \cdot \frac{\nabla f^*(x)}{f^*(x)} \geq \frac{(2-m)}{||x||} + \frac{\partial}{||x||}$$ for all sufficiently large $||x||$. Hence the lemma. q.e.d.

Lemma 7.3 : Suppose BPII is solvable for $L_f u = 0$. Then there exists a sequence r_n and a constant $k_o > 0$ such that

$$\inf_{x:||x|| \leq r_n} \int_{||\theta|| \leq r_n} \frac{e^{-\frac{1}{2}||x-\theta||^2} F(d\theta)}{f^*(x)} \geq k_o \quad \text{for all } n \qquad (7.1)$$

Proof: By Lemma .2 there exists a sequence r_k , $r_k \uparrow \infty$ as $k \to \infty$, such that

$$\frac{f^{*\prime}(r_k)}{f^*(r_k)} \leq \frac{(2-m)}{r_k} + \frac{\partial}{r_k} \qquad \text{for all } k.$$

Let $r_n = r_k + B + 1$. Since the infimum of the left side of (7.1) is attained on the boundary it suffices to prove

$$\int_0^{r_n} \frac{p_\theta(\bar{x}) \quad F(d||\theta||)}{f^*(r_n)} \geq k_o \quad \text{for all } n. \qquad (7.2)$$

where $\bar{x}$ is such that $||\bar{x}|| = r_n$.

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We adopt the following notations in this proof. Let $p_\theta(r)$ denote $p_\theta(x)$ such that $||x|| = r$. By $\delta_F(r)$ we mean $\delta_F(x)$ such that $||x|| = r$. Moreover, an expression of the form $(r_1 - r_2)$ means line segment.

As in lemma 4.1, we shall obtain a lower bound for

$$\frac{\int_0^{r_n} p_\theta(r_n)F(d\theta)}{f^*(r_n - B-1)} \cdot \frac{f^*(r_n - B - 1)}{f^*(r_n)} .$$

Observe that $r_k = r_n - B - 1$ and $\delta_F(r_k) \leq r_k + \frac{2-m}{r_k} + \frac{\partial}{r_k}$ and hence

$$\int_0^{r_n} p_\theta(r_n)F(d\theta) = \int_0^{r_n} \exp\left(-\tfrac{1}{2}(r_n-r_k)^2\right).$$

$$\exp\left[(r_n-r_k)(r_k-\delta_F(r_k)\ + \delta_F(r_k)-\theta)\right]p_\theta(r_k)F(d\theta)$$

$$\geq \exp\left[-\tfrac{1}{2}(B+1)^2\right]\exp\left[(r_k-\delta_F(r_k))(r_n-r_k)\right]$$

$$\int_0^{r_n} \exp\left((r_n-r_k)(\delta_F(r_k)-\theta\right)\ p_\theta(r_k)F(d\theta) \qquad (7.3)$$

It follows, therefore, from (7.3)

$$\frac{\int_0^{r_n} p_\theta(r_n)F(d\theta)}{f^*(r_k)} \geq K_0' \exp\left(-\tfrac{1}{2}(B+1)^2\right)\exp\left[(r_n-r_k)(r_k-\delta_F(r_k))\right] \qquad (7.4)$$

where K_0' (depending only on B and m) is a constant such that

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$$\frac{\int_0^{r_n} p_\theta(r_k).\exp[(r_n-r_k)(\delta_F(r_k)-\theta)]F(d\theta)}{f^*(r_k)} \geq K_o' > 0 . \qquad (7.5)$$

Note that in the above step (7.5) we have used Chebyshev inequality as in Lemma 4.1 and the fact $|r_n-r_k| \leq B+1$.

Using an argument similar to that in Lemma 4.1, we have

$$\frac{f^*(r_n - B - 1)}{f^*(r_n)} \geq K_1 \exp[-(r_n-r_k)(r_k - \delta_F(r_k))] \qquad (7.6)$$

where K_1 depends only on B and m. The lemma how follows from (7.5) and (7.6) easily q.e.d.

Having proved a result similar to Lemma 4.1, we now come to the proof of Theorem 7.1 .

Proof of Theorem 7.1

Let u_n be a sequence of solutions of $L_f u = 0$ such that $L_f u_n = 0$ for $1 < ||x|| < n$, $u_n = 1$ for $||x|| \leq 1$ and $u_n = \epsilon$ for $||x|| = n$. u_n is set to be zero outside S_n. Define a sequence of finite measures by setting $G_n(d\theta) = u_n(\theta)F(d\theta)$. Let, as usual, δ_{G_n} denote the Bayes procedure with respect to G_n. To prove the theorem it suffices to show

$$\lim_{n \to \infty} \int (R(\theta, \delta_F) - R(\theta, \delta_{G_n}))\, G_n(d\theta) = 0 \qquad (7.7)$$

for some sequence n. Take the sequence n to be the

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sequence $\{r_n\}$ as defined in Lemma 7.3. The rest of the proof is exactly as in the proof of the main theorem. So, it follows from Lemma 7.3 and the hypothesis that BPII is solvable for $L_f u = 0$ that

$$\lim_{n \to \infty} \int_{||x|| \leq n} || \int (u_n(\theta) - u_n(x))(\theta - x - \frac{\nabla f^*(x)}{f^*(x)}) \frac{p_\theta(x)}{f^*(x)} F(d\theta) ||^2 f^*(x) dx = 0 \tag{7.8}$$

Therefore, to show (7.7), it remains to prove

$$\lim_{n \to \infty} \int_{||x|| > n} || \int (u_n(\theta) - u_n(x))(\theta - x - \frac{\nabla f^*(x)}{f^*(x)}) \frac{p_\theta(x)}{f^*(x)} F(d\theta) ||^2 f^*(x) dx = 0 \tag{7.9}$$

Now, observe that $\Delta \log f^*(x) < B$ implies $\frac{x}{||x||} \frac{\nabla f^*(x)}{f^*(x)} < B_1 ||x||$ where B_1 is a constant depending only on B and m. Moreover, there exists a constant ϵ $(0 < \epsilon < 1)$ such that $\int e^{-\frac{1}{2}\epsilon ||\theta||^2} F(d\theta) < \infty$. To prove this, we proceed as follows. Clearly, $\log f^*(x) \leq B_1 ||x||^2/2$. Therefore, we have

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$$\int f^*(x) e^{\frac{-(B_1+1)||x||^2}{2}} dx = \int e^{\frac{-(B_1+1)||x||^2}{2}} \int e^{-\frac{1}{2}||x-\theta||^2} F(d\theta)dx \tag{7.10}$$

$$= \int\left(\int e^{-\frac{1}{2}||x-\theta||^2} e^{\frac{-(B_1+1)||x||^2}{2}} dx\right)F(d\theta) \tag{7.11}$$

$$= \int\int e^{-\frac{1}{2}||x\sqrt{2+B_1} - \frac{\theta}{\sqrt{2+B_1}}||^2} dx\, e^{-\frac{1}{2}\epsilon||\theta||^2} F(d\theta)$$

$$= C.\int e^{-\frac{1}{2}\epsilon||\theta||^2} F(d\theta) \tag{7.12}$$

where C is a constant depending on B_1 and $\epsilon = \frac{(1+B_1)}{(2+B_1)} < 1$.

It is easily seen that the left side of (7.10) is finite and hence (7.12) is finite. Note that in obtaining (7.11) we have used Tonelli's theorem.

The proof of (7.9) is now easy. By an argument similar to that in the proof of the main theorem, it follows

$$\int_{||x|| > n} ||\int (u_n(\theta)-u_n(x))(\theta - x - \frac{p_\theta(x)}{f^*(x)}) F(d\theta)||^2 f^*(x)dx$$

$$\leq C \int_{||x|| > n} ||x||^2 \int_{||\theta|| \leq n} e^{-\frac{1}{2}(||x|| - ||\theta||)^2} F(d\theta) . \tag{7.13}$$

Now from the concavity of $e^{-\frac{1}{2}(||x|| - ||\theta||)^2}$ for $||\theta|| < ||x||$

and by (7.12) we have

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$$\lim_{n \to \infty} \int_{||x|| > n} ||x||^2 \int_{||\theta|| \leq n} e^{-\frac{1}{2}(||x||-||\theta||)^2} F(d\theta) = 0 \tag{7.14}$$

This completes the proof of the theorem. q.e.d.

§ 8. Some Admissibility Results

In this section we present various criteria for the admissibility (or inadmissibility) of a given generalized Bayes estimator δ_F. The conditions are interms of solvability of BPII (or BPI) for $L_f u = 0$. We do not consider the case $m = 1$. It is similar to the case of spherical symmetry. So we assume, in what follows that $m \geq 2$.

8.1 Spherically symmetric case

We assume, throughout this subsection, that F is spherically symmetric. Hence $f^*(x)$ ($f^*(||x||) = f^*(x)$) is also spherically symmetric and consequently any solution of $L_f u = 0$ is also spherically symmetric.

Theorem 8.1.1. Let $f^*(r) = f^*(||x||)$. If

$$\int_1^\infty \frac{1}{f^*(r)} \frac{1}{r^{m-1}} dr < \infty \tag{8.1}$$

then δ_F is inadmissible. If the above integral is infinite and $\Delta \log f^*(x)$ is bounded then δ_F is admissible.

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Proof:- If (8.1) holds, then the function $u(r) = \int_r^\infty \frac{1}{f^*(t)} \frac{1}{t^{m-1}} dt$ is a barrier at ∞. Hence, BPI is solvable and δ_F is inadmissible. Conversely, if $\int_1^\infty \frac{1}{f^*(t)} \frac{1}{t^{m-1}} dt = \infty$ then the function $u(r) = \int_1^r \frac{1}{f^*(t)} \frac{1}{t^{m-1}} dt$ is an antibarrier at ∞. Therefore, BPII is solvable and it follows from Theorem 7.1 that δ_F is admissible. This completes the proof of the theorem. q.e.d.

As a corollary to the above theorem we have the following result.

Corollary 8.1.1: If there exists a $L > 0$ such that

$$\frac{f^{*\prime}(r)}{f^*(r)} \geq \frac{(2-m)}{r} + \frac{\partial_1(r)}{r} \qquad \text{for all } r > L$$

where $\partial_1(t)$ is a non-dinifunction (see Chapter I for the definition of a dinifunction) then δ_F is inadmissible. Conversely, if

$$\frac{f^{*\prime}(r)}{f^*(r)} \leq \frac{(2-m)}{r} + \frac{\partial_2(r)}{r} \qquad \text{for all } r > L$$

where $\partial_2(t)$ is a dinifunction, and $\Delta \log f^*(x)$ is bounded then δ_F is admissible.

If F is the Lebesgue measure λ on E^m (note that the Lebesgue measure is spherically is symmetric) then it follows from the above results that δ_λ is admissible for $m \leq 2$ and inadmissible for $m \geq 3$. We have already noted that $\delta_\lambda(x) = x$ i.e. δ_λ is the best invariant estimator.

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8.2. General results in m dimensions

This section contains several criteria for admissibility in the general m dimensional case. We assume in what follows that $||\delta_F(x)|| \leq ||x|| + k$. We again remind the reader that this condition is needed only to prove admissibility and is not needed to prove inadmissibility.

Theorem 8.2.1: If there exists a $L > 0$ such that

$$\frac{x}{||x||} \frac{\nabla f^*(x)}{f^*(x)} \geq \frac{(2-m)}{||x||} + \frac{\partial_1(||x||)}{||x||} \quad \text{for all} \quad ||x|| > L \tag{8.2.1}$$

where $\partial_1(t)$ is a non dinifunction, then δ_F is inadmissible. If

$$\frac{x}{||x||} \frac{\nabla f^*(x)}{f^*(x)} \leq \frac{(2-m)}{||x||} + \frac{\partial_2(||x||)}{||x||} \quad \text{for} \quad ||x|| > L \tag{8.2.2}$$

where $\partial_2(t)$ is a dinifunction and $\Delta \log f^*(x)$ is bounded then δ_F is admissible.

Proof: Suppose (8.2.1) holds. We shall produce a barrier at ∞ for $L_f u = 0$. It would then follow BPI is solvable and hence δ_F is inadmissible. Let $v(r)$ be a function defined as follows

$$v(r) = \int_r^\infty \left\{ e^{-\int_1^t \frac{\partial_1(s)}{s} ds} \right\} \frac{dt}{t} \tag{8.2.3}$$

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Plainly $V(r)$ is a positive function and $V(r) \to 0$ as $r \to \infty$. We shall show that $V(||x||)$ is a barrier at ∞.

$$L_f V(||x||) \doteq \Delta V(||x||) + \frac{\nabla f^*(x)}{f^*(x)} V(||x||)$$

$$= V''(||x||) + \frac{m}{||x||} V'(||x||) - \frac{V'(||x||)}{||x||} + \frac{x\, \nabla f^*(x)}{||x||\, f^*(x)} V'(||x||)$$

$$= V''(||x||) + \left(\frac{x}{||x||} \frac{\nabla f^*(x)}{f^*(x)} + \frac{(m-1)}{||x||}\right) V'(||x||)$$

$$= V''(||x||) + (1 + \partial_1(||x||)) \frac{V'(||x||)}{||x||} \qquad (8.2.4)$$

In obtaining (8.2.4) we have used that fact $V'(||x||) < 0$ and (8.2.1). It is easy to see, by substituting (8.2.3) in (8.2.4), that

$$V''(||x||) + (1 + \partial_1(||x||)) \frac{V'(||x||)}{||x||} = 0 .$$

Therefore V is a barrier at ∞.

To prove the other part assume (8.2.2). Let $\tilde{V}(r)$ be defined as

$$\tilde{V}(r) = \int_1^r e^{-\int_1^t \frac{\partial_2(s)}{s} ds} \frac{dt}{t}$$

Clearly, $\tilde{V}(r)$ is a positive function and $\tilde{V}(r) \to \infty$ as $r \to \infty$. Moreover $\tilde{V}'(r) > 0$. Using this and (8.2.3) it is easy to check

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$$L_f \tilde{V}(||x||) \leq 0$$

Therefore, $\tilde{V}(||x||)$ is an antibarrier for $L_f u = 0$. This implies BPII is solvable for $L_f u = 0$ and hence δ_F is admissible. q.e.d.

We could catalog some more results of this type using the results in Chapter I - for example, if there is $B \subset \{\phi\}$ with $\int_B d\phi > 0$ (here (r, ϕ) denote spherical co-ordinates) such that $\int \frac{1}{f^*(r,\phi)} \frac{1}{r^{m-1}} dr < \infty$ for all $\phi \in B$ then δ_F is inadmissible - but we willlnot pause here to do that.

§9. Examples and General Comments

We have already seen, in section 3, that our main theorem contains completely Brown's result. In this section we give a few examples to indicate that there are quite a lot of measures, especially in higher dimensions, with Weirstrass transforms which violate Brown's condition but satisfy our conditions.

Example: Let $F(d\theta) = e^{-\frac{1}{4}||\theta||^2} d\theta$. Clearly F is a finite measure and its transform

$$f^*(x) = \int e^{-\frac{1}{2}||x-\theta||^2} e^{-\frac{1}{4}||\theta||^2} d\theta = e^{-\frac{1}{6}||r||^2}.$$

Plainly, $f^*(x)$ violates Brown's condition. However,

$\Delta \log f^*(x) = (-\frac{1}{6}||x||^2) = -\frac{1}{3}.m.$ Therefore $\Delta \log f^*(x)$ is bounded. Now $\delta_F(x) = -\frac{1}{3}x + x = \frac{2}{3}x$. Hence $||\delta_F(x)|| \leq ||x||$. Since F is finite, δ_F is admissible.

One can give a plethora of examples involving finite measures where Brown's condition is not satisfied but our conditions are satisfied. Indeed, we have not been able to find any finite measure which violates our conditions.

The next few examples are non-finite measures which violate Brown's condition but whose admissibility can be checked by our main theorem.

Example 2. Let F be an absolutely continuous measure on E^2 whose density is given by

$$\phi(\theta_1, \theta_2) = \phi_1(\theta_1).\phi_2(\theta_2) = c.e^{-\frac{1}{2}\theta_2^2} .$$

It is easy to check that

$$\int \phi(\theta_1, \theta_2) d\theta_1 d\theta_2 = \infty .$$

The transform $f^*(x)$ is given by $c_1 e^{-\frac{1}{4} x_2^2}$. Plainly,

$||\frac{\nabla f^*(x)}{f^*(x)}|| = |\frac{x_2}{2}|$ which is not bounded in E^2.

On the other hand $\Delta \log f^*(x) = -\frac{1}{2}$. Also,

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$||\delta_F(x)||^2 = ||(x_1, \frac{-x_2}{2})||^2 = |x_1|^2 + \frac{1}{4}|x_2|^2 \leq ||x||^2$. So $f^*(x)$ satisfies our conditions. Since

$$\frac{x}{||x||} \frac{\nabla f^*(x)}{f^*(x)} = \frac{-\frac{x_2^2}{2}}{||x||} < 0,$$ it follows from Theorem 3.2 that δ_F is admissible.

The next example deals with a measure whose support is not the whole plane.

Example 3. Let $F = F_1 + F_2$ be a measure on E^2 defined as follows

$$F_1(d\theta) = \phi(\theta_1, \theta_2) d\theta_1 \, d\theta_2$$

where $\phi(\theta_1, \theta_2) = 1$ if $-1 < \theta_1 < 1$, $\theta_2 < 0$

$= 0$, otherwise .

That is, F_1 is the restriction of the Lebesgue measure to the strip $\{(\theta_1, \theta_2): -1 \leq \theta_1 \leq 1, \theta_2 < 0\}$. The measure F_2 is defined by

$$F_2(d\theta) = \phi(\theta_2) d\theta_1 \, d\theta_2 \qquad -1 \leq \theta_1 \leq 1, \quad -\infty < \theta_2 < \infty$$

$= 0$, elsewhere.

where $\phi(\theta_2) = e^{-\frac{1}{2}\theta_2^2}$.

It is easy to check that the support of F is the convex set $\{(\theta_1, \theta_2): -1 \leq \theta_1 \leq 1\}$ and F is not a finite measure.

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Clearly,

$$f^*(x) = f_1^*(x) + f_2^*(x)$$

$$= f_1^*(x) + e^{-\frac{1}{4}x_2^2} \int_{-1}^{1} e^{-\frac{1}{2}(x_1 - \theta_1^2)} d\theta_1$$

$$= \int_{-1}^{1} e^{-\frac{1}{2}(x_1-\theta_1)^2} \int_{-\infty}^{0} e^{-\frac{1}{2}(x_2-\theta_2)^2} d\theta_2\, d\theta_1 +$$

$$+ e^{-\frac{1}{4}x_2^2} \int_{-1}^{1} e^{-\frac{1}{2}(x_1-\theta_1)^2} d\theta_1$$

$$= \int_{-1}^{1} e^{-\frac{1}{2}(x_1-\theta_1)^2} d\theta_1 \left[e^{-\frac{1}{4}x_2^2} + \int_{-\infty}^{0} e^{-\frac{1}{2}(x_2-\theta_2)^2} d\theta_2\right]$$

It is easily seen that, $\left|\left|\frac{\nabla f^*(x)}{f^*(x)}\right|\right|$ is not bounded on the strip $\left\{(x_1\ x_2): -1 < x_1 < 1,\ \ x_2 > 0\right\}$ and hence it is not **bounded in** K_F. It can be checked, after going through some computation, that $\Delta \log f^*(x)$ is bounded and $||\delta_F(x)|| \leq ||x|| + K$. To verify the admissibility of δ_F we appeal to Theorem 8.2.1. It is easily checked that $x \cdot \frac{\nabla f^*(x)}{f^*(x)} \leq 0$ for all sufficiently large x_1 and x_2.

One could go about listing a lot of similar examples where the admissibility of the estimator can be checked by our theorem and not by Brown's.

We have some more examples, where the underlying measures are discrete, violating Brown's condition and yet yielding

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admissible estimators by our theorem. We refrain from cataloging them for the simple reason that the computations involved, though not difficult, are very tedious.

We end this chapter with some general comments. In spite of the broad scope of our theorem, we do believe there exist estimators which are admissible but do not satisfy our assumptions. However, we have not been successful in obtaining such estimators. We do have examples in which the estimators do not satisfy our assumptions. But we find it hard to verify their admissibility. In some of the examples the estimators turn out to be inadmissible.

Finally, we have stated and proved our theorem for a normal distribution with variance covariance matrix as identity matrix. We could have assumed the dispersion matrix to be an arbitrary, but fixed, positive definite matrix. Then we can reduce the problem, by orthogonal transformation, to one in which the dispersion matrix is diagonal. Our proof would go through in this case with little modification. We could have also assumed that the loss function is a quadratic form of a fixed positive definite matrix A. This would entail a change in the differential operator. In this case the differential operator L_f would be of the form

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$$\sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \frac{\nabla f^*(x)}{f^*(x)} A \nabla u(x)$$

where $A = (a_{ij})$.

The exterior boundary value problem can be stated for this operator and conditions for their solvability can be given. Our proof is such that it would go through in this set up with some modifications.

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CHAPTER III

ON THE ADMISSIBILITY OF ESTIMATORS OF THE MEAN OF A MULTIVARIATE NORMAL DISTRIBUTION

§ 0. Introduction and Summary

The material in this chapter could be considered as applications based on the results of the previous chapter. This chapter consists of two parts. In the first part, we consider the problem of improving inadmissible estimators. Using a result of Stein [5] on the representation of the risk function, we show that superharmonic functions give rise to generalized Bayes minimax procedures. We also prove that there do not exist proper Bayes minimax estimators for dimensions 3 and 4. This generalizes the result of Strawdermen [2]. Moreover, our method is different from his. However, we have not been successful in proving similar results in the general case. We consider, in the second part, the admissibility of a given estimator. This part consists of results which are generalizations of results of Strawderman and Cohen [1] and Brown [1].

This chapter contains four sections. The first section gives the notations and concepts. The second

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section is on minimax estimators and related results. In the third section we consider the admissibility of a given estimator. We also study shrinkers and expanders. The last section contains results on coordinate by coordinate estimation.

§ 1. Basic concepts and notations.

Let X denote the m dimensional normal random variable with mean $\theta \in E^m$ and the dispersion matrix identity $I_{m \times m}$. An estimator of θ with respect to the quadratic loss function $L(\theta, t) = \|\theta - t\|^2$ is denoted by $\delta(x)$. The generalized Bayes estimator with respect to a σ-finite measure F such that $f^*(x) = \int e^{-\frac{1}{2}\|x-\theta\|^2} F(d\theta) < \infty$ for all x, is denoted by δ_F. We have already seen, in Chapter II, $\delta_F(x) = \frac{\nabla f^*(x)}{f^*(x)} + x$.

Let L_f denote, as usual, the elliptic differential operator $L_f u = \Delta u + \frac{\nabla f^*(x)}{f^*(x)} u$, where $u: E^m \to R$ is a twice continuously differentiable function. A real valued twice continuously differentiable function $v: D \to R$, where D is a domain in E^m (a domain is an open connected set), is called a super solution of L_f in D if $L_f v \leq 0$ in D. We will also use the following standard terminology if L

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if L is such that $Lu = \Delta u$. A twice continuously differentiable real valued function u, defined on a domain $D \subset E^m$, is called super harmonic, harmonic, or subharmonic according as $\Delta u \leq 0$, $\Delta u = 0$ or $\Delta u \geq 0$ in D.

We will have occasion to use the following result in this chapter. Let $R \subset E^m$ be a domain with smooth boundary. Let u and v be real valued twice continuously differentiable functions defined on $\bar{R}$. The following identity, known as Green's identity, can be proved using Divergence Theorem.

Green's Identity

$$\int_R (u \Delta v - v \Delta u)\, dx = \int_{\partial R} (u D_n v - v D_n u)\, d\sigma \; .$$

Where ∂R is the boundary of R and $d\sigma$ is the surface measure. Here $D_n f$ denotes the directional derivative of f in the direction of the outer normal n to the surface of R.

§ 2. Minimax Estimators

It is a well known fact, for arbitrary dimension m that $\delta_o(x) = x$ is minimax estimator for the underlying problem. Since $\delta_o(x)$ is admissible for $m \leq 2$, it is the only minimax estimator for that case. However there do exist

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other minimax estimator (other than $\delta_o(x)$) for $m \geq 3$, by virtue of inadmissibility of $\delta_o(x)$ when $m \geq 3$. So a natural question arises at this stage. The question is how to construct, when $m \geq 3$, minimax estimators different from $\delta_o(x)$. We solve this problem in this section. Using a result of Stein [5] we give a method of constructing minimax procedures.

We assume throughout this section that $m \geq 3$. The result of Stein's (Stein [5]), which plays a crucial role in what follows, is given below.

Let $g: E^m \to R_+ = [0, \infty)$ be a twice continuously differentiable function. Let $\delta(x)$ be any estimator given by $\delta(x) = \nabla \log g(x) + x$.

<u>Theorem 2.1</u> (Stein)

A sufficient condition for $\delta(x)$ to be minimax is $g^{\frac{1}{2}}(x)$ is super-harmonic i.e. $\Delta g^{\frac{1}{2}}(x) \leq 0$ in E^m.

Proof: Observe that in order to show that $\delta(x)$ is minimax it suffices to prove $R(\theta,\delta) \leq R(\theta,\delta_o) = m$. It is easy to show integrating by parts

$$R(\theta,\delta) = m - E_\theta\left[\left|\left| \frac{\nabla g(x)}{g(x)} \right|\right| - \frac{2\Delta g(x)}{g(x)} \right] \qquad (2.1)$$

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where the expectation is taken with respect to m-dimensional normal distribution with mean θ. Now, if $g^{\frac{1}{2}}(x)$ is super-harmonic it is easily seen that

$$E_\theta\left[\left\|\frac{\nabla g(x)}{g(x)}\right\|^2 - \frac{2\Delta g(x)}{g(x)}\right] \leq 0 .$$

Therefore, $R(\theta,\delta) \leq m = R(\theta, \delta_o)$. Hence the theorem.

q.e.d.

Remark 2.2. Stein observed (2.1) and showed that if $g(x)$ is super-harmonic then $\delta(x)$ is minimax. It is easy to see $\Delta g(x) \leq 0$ implies $g^{1/2}(x)$ is super-harmonic. Secondly, the above representation (2.1) can be given a shorter and elegant proof using Gauss-Divergence theorem

As noted by Stein one can use Theorem 2.1 to show that James-Stein estimator $\delta(X) = X(1 - \frac{m-2}{||x||^{m-2}})$ dominates $\delta_o(x)$ in the sense $R(\theta,\delta) \leq R(\theta,\delta_o)$. Indeed, $\delta(x)$ can be written as $x + \nabla \log g(x)$, where $g(x) = \frac{1}{||x||^{m-2}}$. It is well known $\frac{1}{||x||^{m-2}}$ is a harmonic function, except at origin, for $m \geq 3$.

Let us now go in to the problem of constructing generalized Bayes minimax estimators. Inadmissibility of δ_o, we observed in Chapter II, is equivalent to the solvability of BPI for $Lu = u = 0$. Therefore, there exists a unique

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function $j_o(x) \geq 0$ such that

$$\Delta j_o(x) = 0 \quad \text{for} \quad \|x\| > 1$$

satisfying the boundary conditions $j_o(x) = 1$ for $\|x\| = 1$ and $\lim_{\|x\| \to} j_o(x) = 0$. Set $j_o(x) = 1$ for $\|x\| < 1$. We can use j_o to construct a generalized Bayes estimator as follows. Define a measure on E^m by setting $G(d\theta) = j_o(\theta)d\theta$. Clearly,

$$g^*(x) = \int e^{-\frac{1}{2}\|x-\theta\|} G(d\theta) < \infty \quad \text{for all} \quad x.$$

Let $\delta_G(x) = x + \nabla \log g^*(x)$. $\delta_G(x)$ by the generalized Bayes estimator with respect to G. Our next theorem says that δ_G dominates δ_o.

<u>Theorem 2.3:</u> δ_G is minimax i.e. $R(\theta, \delta_G) \leq R(\theta, \delta_o)$.

Proof:- It suffices to prove, by (2.1), $g^*(x)$ is superharmonic.

$$\Delta g^*(x) = \int \Delta_x e^{-\frac{1}{2}\|x-\theta\|} j_o(\theta)d\theta$$

$$= \int_{\|\theta\|<1} \Delta_x e^{\frac{1}{2}\|x\|} p_\theta(x) j_o(\theta)d\theta +$$

$$\int_{\|\theta\|<1} e_x^{-\frac{1}{2}\|x-\theta\|} p_\theta(x) j_o(\theta)d\theta \qquad (2.2)$$

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By Green's identity we have

$$\int_{\|\theta\|<1} \Delta_x \, p_\theta(x) \, j_o(\theta) d\theta = \int_{\|\theta\|<1} p_\theta(x) \, \Delta j_o(\theta) d\theta$$

$$+ \int_{\|\theta\|=1} j_o(\theta) \, D_{n_1} p_\theta(x) \, d\sigma - \int_{\|\theta\|=1} p_\theta(x) \, D_{n_1} j_o(\theta) d\sigma \qquad (2.3)$$

where $D_{n_1} f$ is the derivative of f in the direction of the outer normal n_1 to the surface of $\|\theta\| \leq 1$ and $d\sigma$ is the surface measure.

In obtaining (2.3) we have used $\Delta_x \, p_\theta(x) = \Delta_\theta \, p_\theta(x)$.

Similarly, applying the Green's identity to the second term in (2.2) we get

$$\int_{\|\theta\|>1} \Delta_x \, p_\theta(x) \, j_o(\theta) d\theta = \int_{\|\theta\|>1} p_\theta(x) \, \Delta j_o(\theta) d\theta$$

$$+ \int_{\|\theta\|=1} D_{n_2} \, p_\theta(x) \, j_o(\theta) d\sigma - \int_{\|\theta\|=1} p_\theta(x) \, D_{n_2} \, j_o(\theta) d\sigma \qquad (2.4)$$

where $D_{n_2} f$ is the directional derivative of f in the direction of the outer normal to the surface of $\|\theta\| \geq 1$ i.e. $n_2 = -n_1$. Observe that $D_{n_2} j_o(\theta) > 0$ for $\|\theta\| = 1$, $D_{n_1} j_o(\theta) = 0$ for $\|\theta\| = 1$ and $D_{n_1} \, p_\theta(x) = - D_{n_2} \, p_\theta(x)$

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for $||\theta|| = 1$. Therefore, combining (2.3) abd (2.4), we have

$$\Delta g(x) = \int \Delta j_o(\theta)\, p_\theta(x)d\theta - \int_{||\theta||=1} p_\theta(x)\, D_{n_2} j_o(\theta) d\sigma < 0$$

since $\Delta j_o = 0$ for $||\theta|| > 1$ and $||\theta|| < 1$.

This completes the proof of the theorem. q.e.d.

Remark 2.4: We have used, in (2.3) and (2.4) of the above proof, that $j_o(\theta)$ has directional derivatives at $||\theta|| = 1$. It can be shown that if the coefficients of the differential operator of an elliptic boundary value problem are smooth on the closure of its domain (of definition) and if the boundary function is smooth, then any solution of the boundary value problem has directional derivative, in the direction of the outer normal, on the boundary. See, for a proof, Ladyshenskaya and Uraltseva [1].

Remark 2.5: In the above construction, we could use any smooth superharmonic function to obtain procedures better than δ_o so long as it gives rise to a generalized Bayes procedure.

It is interesting to note that the estimator $\delta_G(x)$, defined above, is also admissible To see this we appeal to Theorem 7.1 (or Theorem 2.3, the main theorem) of Chapter II

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Clearly, $\Delta \log g^*(x) = \frac{\Delta g^*(x)}{g^*(x)} - \| \frac{\nabla g^*(x)}{g^*(x)} \|^2 \leq 0$ by Theorem 2.3 above. Moreover, the measure G is easily seen to satisfy the growth condition. So, it suffices to check the solvability of BP II for $L_g u = \Delta u + \frac{\nabla g^*(x)}{g^*(x)} \nabla u = 0$. First observe that $j_o(\theta) = \frac{1}{\|\theta\|^{m-2}}$ for $\|\theta\| \geq 1$. Therefore G is spherically symmetric and it is easily checked that $g_R^*(r) = 0(\frac{1}{r^{m-2}})$ where $g_R^*(r)$ is $g^*(\|x\|)$(Here the angle $\phi = \frac{x}{\|x\|}$ is suppressed). Hence, it follows from section 8 of Chapter II that BP II is solvable for $L_g u = 0$. Therefore δ_G is admissible.

Thus we have given a method of constructing generalized Bayes minimax estimators. The next two results are on the existence of proper Bayes minimax estimators. Strawderman [2] showed that there do not exist spherically symmetric proper Bayes minimax estimators for $m = 3$ or 4. We prove below there do not exist proper Bayes minimax estimators of any kind for $m = 3$ or 4. Moreover our proof is short and elegant.

Theorem 2.6 There do not exist proper Bayes minimax estimators for dimension $m = 3$ or 4.

Proof:- Suppose there exists a finite prior measure G such that δ_G is minimax. Then we have $R(\theta, \delta_G) \leq m \ \forall \ \theta$

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Moreover,

$$R(\theta, \partial_G) = m + C\int \left\| \frac{\nabla g^*(x)}{g^*(x)} \right\|^2 p_\theta(x)dx + 2C\int \frac{\nabla g^*(x)}{g^*(x)} (x-\theta)p_\theta(x)dx \tag{2.5}$$

where $C = \frac{1}{(2\pi)^{m/2}}$ Now integrating by parts we have

$$R(\theta, \partial_G) - m = \| C \nabla_\theta \int \log g^*(x)p_\theta(x)dx \|^2 +$$

$$2C \Delta_\theta \int \log g^*(x)p_\theta(x)\, dx \leq 0 \tag{2.6}$$

In the last step we have used Schwartz inequality.
Setting $\psi(\theta) = C \int \log g^*(x)\, p_\theta(x)\, dx$, (2.6) becomes

$$2\Delta\psi(\theta) + \|\nabla\psi(\theta)\|^2 \leq 0 \quad \text{for all } \theta \tag{2.7}$$

It follows from (2.7) that $e^{\frac{1}{2}\psi(\theta)}$ is superharmonic
But $u(\theta) = \frac{1}{\|\theta\|^{m-2}}$ is a harmonic function for $\|\theta\| > 0$
Therefore, by maximum modulus principle, there exists a constant $C_1 > 0$ such that

$$e^{\frac{1}{2}\psi(\theta)} \geq C_1 u(\theta) \quad \text{for } \|\theta\| \geq 1 \tag{2.8}$$

Hence, by Jensen's inequality, we have

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$$C \int g^*(x)\, p_\theta(x)\, dx \geq C_1^2\, u^2(\theta) = C_1^2 \frac{1}{||\theta||^{2(m-2)}} \tag{2.9}$$

and

$$C \int\int_{||\theta||>1} g^*(x)\, p_\theta(x)\, dx d\theta \geq C_1^2 \int_{||\theta||>1} \frac{1}{||\theta||^{2(m-2)}}\, d\theta \tag{2.10}$$

The right side of (2.10) is infinity for $m = 3$ and 4, which inturn implies that $\int g^*(x) dx = \infty$. This contradicts that G is a proper prior measure. Hence the theorem. q.e.d.

Strawderman [3] showed the existence of proper Bayes minimax estimators for dimension $m \geq 5$ He considered estimators Bayes with respect to the prior probability measures given by $\theta \sim N_m(0, I(\frac{\lambda}{1-\lambda})^{-1})$, $0 < \lambda \leq 1$ and λ has density $\frac{\lambda^{-a}}{1-a}$ for $\frac{1}{2} \leq a < 1$. These Bayes estimators are of the form

$$X\left(1 - \left(\frac{m+2-2a}{||x||^2} - \frac{2\, e^{-\frac{1}{2}||x||^2}}{||x||^2 \int_0^1 \lambda^{(m-a)/2}\, e^{-\lambda||x||^2/2}\, d\lambda}\right)\right)$$

They belong to the class of minimax estimators, constructed by Baranchik [1], which are of the type

$$X\left(1 - \gamma\left(\tfrac{1}{2}||x||^2\right) \frac{m-2}{||x||^2}\right),$$

where $0 \leq \gamma \leq 2$ and γ is a nondecreasing function of $||x||^2$. We can give a very simple proof of the minimaxity of Baranchik

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type estimators, using Theorem 2.1, without going into the evaluation of their risk.

Assume, to avoid technical details, that γ is a differentiable function. (Note that a nondecreasing function is differentiable almost every-where). Let

$$g(x) = \exp\left(\psi\left(\frac{m-2}{2}\log\frac{1}{||x||^2}\right)\right)$$

Where is such that $\nabla\psi = -\gamma\left(\frac{1}{2}||x||^2\right)(m-2)\frac{X}{||x||^2}$.

Clearly, the estimator

$$X + \frac{\nabla g(x)}{g(x)} = x\left[1 - \gamma\left(\frac{1}{2}||x||^2\right)\frac{(m-2)}{||x||^2}\right]$$

We shall now prove that $g^{\frac{1}{2}}(x)$ is superharmonic

$$\Delta g^{\frac{1}{2}}(x) = \Delta\exp\left(\frac{1}{2}\psi\left(\frac{(m-2)}{2}\log\frac{1}{||x||^2}\right)\right)$$

$$=\left(\exp\left(\frac{1}{2}\psi\left(\frac{(m-2)}{2}\log\frac{1}{||x||^2}\right)\right)\right)\left(\frac{1}{2}\Delta\psi\left(\frac{m-2}{2}\log\frac{1}{||x||^2}\right)\right.$$

$$\left.+\frac{1}{4}||\nabla\psi\left(\frac{m-2}{2}\log\frac{1}{||x||^2}\right)||^2\right) \tag{2.5}$$

The quantity in the bracket of the right side of (2.5) can be written as

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$$\frac{1}{2}\left[\psi''\left((m-2)\log\frac{1}{||x||}\right)+\psi'\left((m-2)\log\frac{m-11}{||x||}\right)\frac{m-1}{||x||}\right]+$$

$$+\frac{1}{4}\left|\left|\psi'\left((m-2)\log\frac{1}{||x||}\right)\right|\right|^2 \qquad (2.6)$$

Where $\psi' = \frac{\partial}{\partial||x||}\psi$ and $\psi'' = \frac{\partial^2}{\partial||x||^2}\psi$

But $\psi'(||x||) = -\gamma(||x||)\frac{(m-2)}{||x||}$ and

$$\psi''(||x||) = -\gamma'(||x||)\frac{(m-2)}{||x||}+\gamma(||x||)\frac{m-2}{||x||^2}.$$

Therefore (2.6) is equal to

$$-\frac{1}{2}\gamma'(||x||)\frac{(m-2)}{||x||}+\frac{1}{2}\frac{\gamma(||x||)(m-2)}{||x||^2}-\frac{1}{2}\frac{\gamma(||x||)(m-2)(m-1)}{||x||^2}$$

$$+\frac{1}{4}\frac{(\gamma(||x||)(m-2))^2}{||x||^2} \qquad (2.7)$$

Since $0 \leq \gamma \leq 2$, it follows that (2.7) is less than or equal to zero. This proves that $g^{\frac{1}{2}}(x)$ is superharmonic and hence Baranchik type estimator is minimax. Finally, if $\gamma(||x||)$ is not given to be differentiable, its monotone nature ensures that $\gamma(||x||)$ has derivatives almost everywhere and hence the above argument goes through.

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We have not been successful in constructing proper Bayes minimax estimators generally. It is possible to construct, using subharmonic functions, Bayes procedures which are minimax outside a compact set. A method of constructing such procedures is as follows. Let $j(||\theta||)$ be a non-negative function such that $\int j(||\theta||)d\theta < \infty$ and $j^{1/2}(||\theta||)$ is superharmonic almost everywhere. Moreover, assume $\frac{j'(||\theta||)}{j(||\theta||)} = O(\frac{1}{||\theta||})$. Note that such superharmonic functions exist. Consider the Bayes estimator δ_G given by the finite measure $G(d\theta) = j(\theta)d\theta$. It is not difficult to show that δ_G is minimax outside a compact set. This is achieved by showing $g^{1/2}(x)$, where $g(x) = \int p_\theta(x)\, G(d\theta)$ is superharmonic outside a compact set. We shall not go into the proof of this result.

Next, we consider the problem of constructing procedures which dominate a given inadmissible generalized Bayes estimator. Let F be a σ-finite measure on E^m such that $f^*(x) < \infty$. Suppose the generalized estimator δ_F given by F is inadmissible. Then, by Theorem 2.3 of Chapter II, the operator

$$L_f j = \Delta j + \frac{\nabla f^*(x)}{f^*(x)} \nabla j\,(x) = 0$$

has a nontrivial solution in the exterior domain $x: ||x|| > 1$ satisfying the boundary condition $j(x) = 1$ for $||x|| = 1$ and $j(x) < 1$ for $||x|| > 1$. Let us denote this solution by $j_o(x)$

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We can use this solution to construct a procedure dominanting δ_F as follows.

By Theorem 2.1, the risk of δ_F can be written as

$R(\theta, \delta_F) = m + E_\theta[\frac{2\Delta f^*}{f^*} - ||\frac{\nabla f^*}{f^*}||^2]$. Using this it can easily be shown that an estimator $\delta_g(x) = x + \frac{\nabla g(x)}{g(x)}$ is better than δ_F if $L_f \frac{g^{1/2}(x)}{f^{*1/2}(x)} \leq 0$. Appealing to this fact, we can immediately prove the following theorem. Let j_o be as above and set $j_o(x) = 1$ for $||x|| \leq 1$.

Theorem 2.3. Let δ_F be inadmissible. Then, the estimator $\delta(x)$ given by $\delta(x) = \frac{\nabla j_o(x)f^*(x)}{j_o(x)f^*(x)} + x$ improves upon $\delta_F(x)$ i.e. $R(\theta, \delta) \leq R(\theta, \delta_F)$ for all θ

Proof:- Observe that $L_f j_o \leq 0$ almost everywhere with respect to Lebesgue measure. Therefore, $L_f(\frac{j_o f^{*\frac{1}{2}}}{f^*}) \leq 0$ almost everywhere This completes the proof. q.e.d

Remark 2.4: It is easily seen, from the above theorem that one can use any smooth function j such that $L_f j \leq 0$ almost everywhere to manufacture an estimator better than δ_F.

The dominating procedures, given by the above method, need not in general be generalized Baye's estimators. Following a suggestion of Brown [1], we could consider the generalized Bayes estimator δ_G given by the measure $G(d\theta) = j_o(\theta)F(d\theta)$ as a competitor for δ_F. Brown [1]

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conjectured that δ_G is admissible and better than δ_F. We have not been able to prove this generally. However, in the spherically symmetric case, it is possible to prove that δ_G, constructed in the above manner, is admissible if $\Delta \log f^*(x) < B$. The proof of this fact is not difficult and follows from the results of section 8 of Chapter II. We refrain from giving proofs of such results since our results are not complete in this direction. We end this section with the following simple result.

<u>Theorem 2.5</u>: Let δ_F be an inadmissible generalized Bayes estimator such that $f^{*1/2}(x)$ is subharmonic. Then any generalized Bayes estimator δ_G, with $\Delta g^{*1/2}(x) \leq 0$, is better than δ_F.

Proof:- Since $f^{*1/2}(x)$ is subharmonic, $R(\theta, \delta_f) \geq m$. Therefore, if δ_G is such that $g^{*1/2}(x)$ is superhermonic, $R(\theta, \delta_G) \leq m \leq R(\theta, \delta_F)$

q.e.d.

3. On Admissibility of Estimators

In this section we give some applications based on the characterization theorem of Chapter II. We observed in the last chapter that the class of generalized Bayes estimators form a complete class. Therefore, in order to verify whether a given estimator is admissible we should know whether it is generalized Bayes or not. Strawderman and Cohen [1] have given necessary and sufficient conditions for an estimator to be generalized Bayes in the univariate case and the spherically symmetric case.

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We extend their results here and also study some properties of shrinkers and expanders

3.1 Univariate and Spherically Symmetric Estimators

We are considering spherically symmetric estimators and one dimensional estimators in the some section because of the similarity in their behaviour and treatment.

The following two results have been proved by Strawderman and Cohen [1].

Theorem 3.1.1: A one dimensional estimator $\delta(x)$ is generalized Bayes if and only if

$$e^{\int_0^x (\delta(y)-y)} = \int e^{-\frac{1}{2}||x-\theta||^2} F(d\theta) \quad \text{for all} \quad x$$

where F is a σ-finite measure on E'.

As a corollary to the above theorem, an alternate condition for one dimensional estimator to be generalized Bayes can be given as follows.

Theorem 3.1.2: A one dimensional estimator $\delta(x)$ is generalized Bayes if and only if $\exp [\int_0^x \delta(y)dy]$ is the moment generating function of a probability measure.

Using these two theorems and Theorem 7.1 of Chapter II we can give necessary and sufficient conditions for an estimator $\delta(x)$ to be admissible, purely interms of $\delta(x)$. Recall that Theorem 7.1 of the previous chapter has one regularity condition.

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That is $\Delta \log f^*(x) < B \forall x$. In particular, if a one dimensional estimator $\delta(x)$ is generalized Bayes with respect to a measure F, then the above condition is equivalent to $\delta'(x) < B+1$ where $\delta'(x)$ is the derivative of $\delta(x)$. Thus we have the following theorem.

<u>Theorem 3.1.3:</u> A one dimensional estimator $\delta(x)$ is admissible if the following conditions are satisfied

i) $\delta(x)$ is generalized Bayes

ii) $\delta'(x) < B \ \forall x$

iii) $\int_1^\infty \frac{1}{g(x)} dx = \int_{-\infty}^{-1} \frac{1}{g(x)} dx = \infty$

where $g(x) = e^{\int_0^x (\delta(y) - y) dy}$. Conversely, $\delta(x)$ is admissible only if (i) and (iii) are satisfied.

<u>Proof:</u> Follows immediately from Theorems 3.1.1 and 7.1 of Chapter II.

A similar result holds in spherically symmetric case. The following theorem has been proved by Strawderman and Cohen [1].

<u>Theorem 3.1.4:</u> A spherically symmetric estimator $\delta(x) = h(||x||^2)x$ is generalized Bayes if and only if

$$e^{\frac{1}{2}\int_0^{||x||^2} (h(y)-1)dy} = \int e^{-\frac{1}{2}||x-\theta||^2} F(d\theta) < \infty \ \forall x$$

for some spherically symmetric σ-finite measure F.

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We are now in a position to characterize admissible spherically symmetric estimators.

Theorem 3.1.5: A spherically symmetric estimator $\delta(x) = h(||x||)x$ is admissible if

i) $\delta(x)$ is generalized Bayes

ii) $\nabla \cdot \delta(x) = \text{Div}\ \delta(x) < B\ \forall x$

iii) $\int_1^\infty \frac{1}{r^{m-1}} \frac{1}{g(r)} dr = \infty$

where $g(r) = \exp [\frac{1}{2} \int_0^{r^2} (h(y)-1)dy]$. Conversely, (i) and (iii) are necessary for δ to be admissible.

Proof:- Follows from, Theorem 3 1.4 and Theorem .1 of Chapter II.

Remark:- The condition (ii) $\nabla \cdot \delta(x) = \sum_{i=1}^{m} \frac{d}{dx_i} \delta_i(x)$, where $\delta_i(x)$ is the i^{th} component of the vector $\delta = h(||x||)x$, is equivalent to the assumption $\Delta \log f^*(x) < B$. Here $f^*(x) = g(||x||)$. So; we could have stated the condition (ii) as $\Delta \log g(||x||) < B$ in a seemingly different form.

In their paper Strawderman and Cohen have proved results similar to Theorems 3.1.3 and 3.1.5 using Brown's theorem. Our theorems are more general then theirs.

They have studied one dimensional estimators of the form $\delta(x) = x + \alpha(x)$ such that $\exp [\int_0^x \alpha(y)dy]$ is the Laplace

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transform of a probability distribution function $F(\theta)$. Such estimators are generalized Bayes. We have the following, easily provable result on such estimators.

<u>Theorem 3.1.6:</u> $\delta(x)$ is inadmissible if $\liminf_{x \to \infty} \alpha'(x) > 0$ and admissible if $\limsup_{x \to \infty} \alpha'(x) < 0$.

Proof:- First, note that $\alpha'(x) = \frac{d}{dx}\alpha(x)$ exists, because $\delta(x)$ is generalized Bayes and therefore we have

$\alpha(x) = \frac{f^{*\prime}(x)}{f^{*}(x)}$, where $f^{*}(x)$ is the Weirrstrass transform of a measure F. The condition $\liminf_{x \to \infty} \alpha'(x) > 0$ implies $\exists$ $c > 0$ such that $\alpha'(x) > c$ for all large x It is now easy to check the inadmissibility by showing $\int_{1}^{\infty} \frac{1}{f^{*}(x)} dx < \infty$. Similarly, the other part is proved. q.e.d.

One can show the inadmissibility of an estimator by proving that it is not generalized Bayes. Strawderman and Cohen have shown that the class of estimators $\delta(x) = x - ax/(b + x^2)$, $a > 0$, $b \geq 0$ are not admissible because they are not generalized Bayes. We have nothing to add in this connection.

3.2 <u>Non-Spherically Symmetric Case</u>

In the non-spherically symmetric case it is not easy to give characterization of generalized Bayes estimators which involve easily verifiable conditions as in the spherically symmetric case.

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We consider in this section estimators of the form $\delta(x) = h(x)\,x$ where $h(x) : E^m \to E^1$.

Theorem 3.2.1: An estimator $\delta(x) = h(x)x$ is generalized Bayes if and only if there exists a σ-finite measure F such that

$$\exp\left[\int_0^{||x||} (h(t,\varphi) - 1)t\,dt\right] = \int e^{-\frac{1}{2}||x-\theta||^2}\, dF(\theta)\ \forall\, x \qquad (3.2.1)$$

where $\varphi = \frac{x}{||x||}$.

Proof:- If (3.2.1) holds, differentiating with respect to x, it is easy to see that $\delta(x)$ is generalized Bayes. Conversely, if $\delta(x)$ is generalized Bayes, then integration along the line segments yields (3.2.1). q.e.d.

We can now state an easily provable result regarding the admissibility of estimators of the form $\delta(x) = h(x)x$. Assume that $h(x) \leq 1 + \frac{K}{||x||}$ for some K. This assumption implies

$||\delta(x)|| \leq ||x|| + k$

Theorem 3.2.2: An estimator $\delta(x) = h(x)x$ is admissible if

i) $\delta(x)$ is generalized Bayes

ii) $\Delta \log g(x) < \infty$ where $g(x) = \exp \int_0^{||x||} (h(t, \frac{x}{||x||}) - 1)t\,dt$

iii) BPII is solvable for $L_g u = \Delta u + \frac{\nabla g}{g} \nabla u = 0$

Conversely, $\delta(x)$ is admissible only if (i) and (iii) hold.

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Proof:- Follows immediately from Theorem 3.2.1 and Theorem 2.3 of Chapter II.

It is possible to construct estimators which are not generalized Bayes in spherically symmetric case. For example estimators of the form $\delta(x) = X(1 - \frac{a}{b+\|x\|^2}) = xh(\|x\|)$ where $a > 0$ and $b \geq 0$ have non-removable singularities and hence cannot be extended to the complex plane as an analytic function. If $\delta(x)$ were generalized Bayes, $\exp[\frac{1}{2}\int_0^{t^2} h(y)dy]$ is a Laplace transform of a measure. This is a contradiction because any Laplace transform is analytic in its domain of definition. Therefore $\delta(x)$ is not generalized Bayes. Similarly one can show that estimators of the form $\delta(x) = X(1 - \frac{a}{b+x'Ax})$, where $a > 0$, $b \geq 0$ and A is a $m \times m$ positive definite matrix, are not generalized Bayes because of non-removable singularities.

3.3 Expanders and Shrinkers

We consider expanders and shrinkers of specific form in this section.

Definition 3.3.1: We define a one dimensional shrinker to be an estimator of the form $\delta(x) = x - \epsilon(x)$, where $\epsilon(x) \geq 0$ for $x \geq 0$ and $\epsilon(x) \leq 0$ for $x \leq 0$.

We have the following result which is a generalization of a result due to Strawderman and Cohen.

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<u>Theorem 3.3.2</u>: Any shrinker $\delta(x)$ such that $\delta'(x) < B$ is admissible i.e. Any shrinker $\delta(x)$ with bounded posteriori risk is admissible.

Proof:- Let $\delta(x)$ be a generalized Bayes shrinker such that δ' is bounded. To show that $\delta(x)$ is admissible, it suffices to prove by virtue of Theorem 3.1.3

$$\int^{\infty} \frac{1}{\exp[\int_0^x (\delta(y)-y)dy]} dx = \infty = \int^{-\infty} \frac{1}{\exp[\int_0^x (\delta(y)-y)dy]} dx \qquad (3.3.1)$$

It is easy to see, from the definition of a shrinker, $\int_0^x (\delta(y)-y)dy \leq 0$ for any $-\infty \leq x \leq \infty$ and hence $\exp[\int_0^x (\delta(y)-y)dy] \leq 1$. Now, (3.3.1) easily follows. Hence the theorem. q.e.d.

We could define a shrinker to be an estimator $\delta(x)$ of the form $\delta(x) = h(x) \cdot x$ where $0 \leq h(x) \leq 1$. This definition is more restrictive than the Definition 3.3.1. However, this enables one to define shrinkers towards a given point as given below.

<u>Definition 3.3.3</u>. An estimator $\delta(x)$ is called a 'shrinker towards x_0' if $\delta(x) = h(x)x + (1-h(x))x_0$ where $0 \leq h(x) \leq 1$.

We have the following result about shrinkers towards x_0.

<u>Theorem 3.3.4</u>: Any generalized Bayes estimator $\delta(x)$ which is a shrinker towards x_0 is admissible if $\delta'(x) < B$.

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Proof:- It follows from a lemma of Strawderman and Cohen that if an estimator $\delta_1(y)$ is generalized Bayes or (and) admissible, then so is $\delta_1(y + a) - a$ for any $-\infty < a < \infty$. Let, now, $\delta(x)$ be a generalized Bayes shrinker towards x_0 such that $\delta'(x) < B$. Then, it is easy to see, the estimator $\delta_2(x)$, given by $\delta_2(x) = \delta(x + x_0) - x_0$, is a shrinker in the sense of definition 3.3.1. Moreover $\delta_2'(x) < B$. Now, appeal to Theorem 3.3.2 to complete the proof.

q.e.d.

Next, we define an expander.

<u>Definition 3.3.5</u>. An estimator $\delta(x)$ is called an expander if $\delta(x) = h(x)x$ where $h(x) \geq 1$.

We could have, following Strawderman and Cohen, defined an expander to be an estimator of the form $\delta(x) = x + \epsilon(x)$ where $\epsilon(x) \geq 0$ for $x \geq 0$ and $\epsilon(x) \leq 0$ for $x \leq 0$. This definition coincides with Definition 3.3.5. The following result is easily proved using Theorem 3.1.3.

<u>Theorem 3.3.6:</u> A generalized Bayes expander $\delta(x) = x(1+\psi(x))$, $\psi(x) \geq 0$ such that $\delta'(x) < B$ is admissible if and only if

$$\int^{\infty} \frac{1}{\exp[\int_0^x \psi(y) y \, dy]} dx = \int_{-\infty} \frac{1}{\exp[\int_0^x \psi(y) y \, dy]} dx = \infty$$

As a corollary to the above theorem we have the following result

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the proof of which follows easily.

Corollary 3.3.7: A generalized Bayes expander of the form $\delta(x) = h(x)x$ is inadmissible if $\liminf_{x \to \infty} h(x) > 1$.

Remark 3.3.8: Note that the proof of the necessary part of Theorem 3.3.6 does not require the assumption $\delta'(x) < B$. We have stated the theorem in that form for convenience.

There do exist many expanders which are admissible. Strawderman and Cohen have given an example of an expander which is admissible. Indeed, one could construct a plthora of admissible expanders using Theorem 3.3.6. For example estimators given by the improper prior measure with density $\frac{dF(\theta)}{d\theta} = 1 + \frac{1}{|\theta|^k}$ for $|\theta| \geq 1$ and $= 1$ for $|\theta| < 1$, where $k \geq 1$ are easily verified to be admissible expanders.

Finally, we have the following two results on shrinkers and expanders in the general case. We define an estimator $\delta(x) = h(x) \cdot x$ to be a shrinker if a $h(x) \leq 1$ and an expander if $h(x) > 1$.

Theorem 3.3.9: A generalized Bayes shrinker $\delta(x) = h(x) \cdot x$ such that $x \nabla h(x) < B$ is admissible if

$$||x||^2 (h(x) - 1) \leq (2 - m) + \gamma(||x||) \quad \text{for } ||x|| > M > 0 \qquad (3.3.2)$$

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where $\gamma(||x||)$ is a dinifunction and M is a constant.

Proof:- It suffices to check the conditions of the main characterization theorem of Chapter II, because (3.3.2) implies that BPII is solvable. Clearly, $\delta(x)$, being shrinker, satisfies the growth condition $||\delta(x)|| \leq ||x|| + k$. Indeed, $||\delta(x)|| \leq ||x||$. It is now easy to check that the condition $\Delta \log f^*(x)$ is bounded is equivalent to $x \nabla h(x)$ is bounded. Hence, the theorem follows from the main theorem of Chapter II. q.e.d.

Remark 3.3.10: This result is comparable to a theorem of Cohen [1] which gives a sufficient condition for the admissibility of an estimator of the form $\delta(x) = Ax$ where A is a $m \times m$ matrix, A part of Cohen's result follows from the above theorem. However, we would prove Cohen's result.

Theorem 3.3.10: Completely in the next section where we consider co-ordinatewise co-ordinate estimation.

On generalized Bayes expanders we have the following inadmissibility result.

Theorem 3.3.11: A generalized Bayes expander $\delta(x) = h(x) \cdot x$ is inadmissible if

$$||x||^2 (h(x) - 1) \geq (2 - n) + \gamma(||x||) \quad \text{for large } ||x||. \tag{3.3.3}$$

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where $\gamma(\|x\|)$ is not a Dini function.

Proof:- Follows immediately from Theorem 8.2.1 of Chapter II.

It is well known that a generalized Bayes estimator $\delta_F(x)$ is Bayes if and only if $\int f^*(x)dx < \infty$. Strawderman and Cohen have given sufficient conditions for a generalized Bayes spherically symmetric estimator $\delta(x) = h(\|x\|)x$ to be Bayes interms of $h(\|x\|)$. We can easily extend their results to the estimators of the form $\delta(x) = h(x)\cdot x$. However, we do not pause here to do so.

We end this section with a comment on the admissibility of estimators of the form

$$\delta(x) = x_0 + h(x - x_0)(x - x_0) \tag{3.3.4}$$

where x_0 is fixed. It is easy to show that an estimator $\delta(x)$ is admissible or generalized Bayes if and only if the estimator $\delta_a^*(x) = \Theta a + \delta(x - a)$ is admissible or generalized Bayes. Indeed, if $\delta(x)$ is generalized Bayes with respect to $F(\theta)$ then $\delta_a^*(x)$ is generalized Bayes with respect to $F(\theta + a)$, the translate of $F(\theta)$. Using this fact and the previous results of this section, one can easily give sufficient and necessary conditions, in terms of $h(y)$, for the estimator of the type (3.3.4) to be admissible.

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§4. Coordinate by Coordinate Estimation

The problem of coordinate by coordinate estimation can be explained, succinctly as follows. Suppose we have two normal populations with dimensions m_1 and m_2, respectively. Let $x = (x_1, \ldots, x_{m_1})$ and $y = (y_1, \ldots, y_{m_2})$ be the observation from the two populations. Further, suppose $\delta_1(x)$ and $\delta_2(y)$ are admissible estimators of the mean vectors of the two populations respectively. We want to study the admissibility of $\delta = (\delta_1, \delta_2)$ as an estimator of the mean of $(m_1 + m_2)$ dimensional problem. (Note that the loss function is quadratic.)

Brown [1] considered this problem and showed that there exists a one dimensional generalized Bayes estimator δ on E^1 such that the estimator $\delta^{(m)}$ on E^m, defined by $\delta^{(m)} = (\delta(x_1), \ldots, \delta(x_m))$, is admissible. He also proved a version of the following result.

<u>Theorem 4.1</u>: Let δ_1 be any admissible estimator on E^{m_1} and δ_2 be any proper Bayes estimator on E^{m_2}. Then, the estimator δ on $E^{m_1+m_2}$, given by $\delta(x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}) = (\delta_1(x_1, \ldots, x_{m_1}), \delta_2(y_1, \ldots, y_{m_2}))$, is admissible.

Proof:- The proof is similar to the one given by Brown and we omit it.

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As a corollary to Theorem 4.1, as mentioned in section 3 we have the following result of Cohen [1]. Let X be a dimensional random variable with mean θ. Let A be $m \times m$ matrix.

Corollary 4.2. The estimator $\delta(x) = Ax$ is admissible if A is symmetric and the characteristic roots of A, say a_i, satisfy, $\theta \leq a_i \leq 1$, with equality at one for at most two of the roots.

Proof: First observe that Ax is admissible if and only if $(P'AP)x$ is admissible for any orthogonal $m \times m$ matrix P. So, without loss of generality, we assume A is a diagonal matrix with $(a_1, \ldots, a_m)$ as the diagonal elements. If all the diagonal elements $a_1, \ldots, a_n$, are strictly less than one, it is easily checked that the estimator Ax is Bayes with respect to the prior (proper) given by

$$\prod_{i=1}^{m} \left(\frac{1-a_i}{a_i}\right)^{1/2} e^{-\left(\frac{1-a_i}{a_i}\right)\theta_i^2/2} d\theta, \ldots, d\theta_m \quad (4.1)$$

and hence admissible. Now, assume $a_1 = 1$, $a_2 = 1$ and $a_i < 1$ for $i > 2$. The case, when only one characteristic root is equal to 1 is similar. Clearly, the estimator $\delta(x)$ is given by $(x_1, x_2, a_3x_3, \ldots, a_mx_m)$. We, now write $\delta(x)$ as $(\delta_1(x_1, x_2),$ $\delta_2(x_3, \ldots, x_m))$, where $\delta_1(x_1, x_2) = (x_1, x_2)$ and $\delta_2(x_3, \ldots x_m) =$ $(a_3x_3, \ldots, a_mx_m)$. Now, $\delta_1(x_1, x_2)$ is admissible on E^2 (i.e. $m_1 = 2$) and δ_2 is Bayes with respect to a finite measure

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similar to (4.1) on E^{m_2} (ie $m_2 = m-2$). Therefore, by Theorem 4.1, $\delta(x)$ is admissible on E^m. This completes the proof of the corollary.

Remark 4.3: The fact that if all a_i's are less than 1, then Ax is Bayes with respect to the prior (4.1) is due to Cohen. Secondly, note that if any a_i is zero, then the marginal prior puts all the mass at $\theta_i = 0$.

Remark 4.4: The converse of Corollary 4.2 is also true. That is, Ax is admissible only if A is symmetric and its eigen values lie in the interval [0,1]. This has been proved by Cohen. He has shown that if A is not symmetric then Ax is not generalized Bayes and hence, not admissible. If A is symmetric but its eigen values do not lie in [0,1], then it follows from section 8 of Chapter II that Ax is inadmissible.

Finally, we consider the following problem posed by Brown [1]. Is there an estimator δ_3 on E^1 which is not genuine Bayes such that for any admissible estimator δ_3 on E^{m-1}, the estimator $\delta(x,y) = (\delta_2(x), \delta_3(y))$, where $x \in E^{m-1}$ $y \in E^1$, is admissible?

We answer this question, below, under some conditions.

Let δ_2 be any admissible estimator on E^{m-1} satisfying the conditions (2.12) of Chapter II. Let F be the measure on E^{m-1} with respect to which δ_2 is generalized Bayes. Then

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$\Delta \log f^*(x) < B$ and $\|\delta_2(x)\| \leq \|x\| + K$ for some constants B and K. Assume $x \frac{\nabla f^*(x)}{f^*(x)} \leq (2 - \overline{m-1})$ outside a compact set in E^{m-1}. We shall exhibit an estimator $\delta_3(y)$ on E^1 such that $(\delta_2(x), \delta_3(y))$ is admissible. Let G be a measure on E^1 with continuously differentiable density $g(\theta)$ given by

$$
\begin{aligned}
g(\theta) &= 1 && \text{for } |\theta| < (1+c) \\
&= \frac{1}{\theta \log \theta} && \text{for } \theta > (1+c) \\
&= \frac{1}{\theta^2} && \text{for } \theta < -(1+c)
\end{aligned}
$$

where c is a positive constant. Clearly, the measure G is not finite since $\int_{\theta > (1+c)} \frac{1}{\theta \log \theta} d\theta = \infty$. Let $g^*(y) = \int p_\theta(y) G(d\theta)$.

It follows from a result of Brown [1] (Lemma 3.4.1) $|\nabla \log g^*(y)|$ is bounded since $|\nabla \log g(\theta)|$ is bounded. Therefore, $\Delta \log g^*(y)$ is bounded. Moreover, integrating by parts one can show that

$$y \frac{\nabla g^*(y)}{g^*(y)} \leq -1 \qquad \text{for all large } y \quad y \in E^1. \tag{4.2.1}$$

Let $\delta_3(y)$ be the generalized Bayes estimator with respect to G. Then the estimator $\delta(x, y) = (\delta_2(x), \delta_3(y))$ $x \in E^{m-1}$ $y \in E^1$, is generalized Bayes with respect to the measure $H = F \times G$ on E^m. Also, $h^*(x, y) = f^*(x) \cdot g^*(y)$. We shall now show that $\delta(x,y)$ satisfies the conditions (2.12) of Chapter II. It is easy to

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see that $\Delta \log h^*(x,y) = \Delta \log f^*(x) \cdot g^*(y)$. is bounded since $\Delta \log f^*(x) < B$ and $\Delta \log g^*(y)$ is bounded. It remains to verify $||\delta(x,y)|| \leq ||(x,y)|| + K_1$ for same $K_1 > 0$. But this follows easily from the assumption that $||\delta_2(x)|| \leq ||x|| + K$ and (4.2). Therefore, to prove the admissibility of $\delta(x,y)$ it suffices, by Theorem 2.3 of Chapter II, to show that BP II is solvable for the equation $L_h u = \Delta u + \frac{\nabla h^*(x,y)}{h^*(x,y)} \nabla u = 0$. To see this, observe that

$$(x,y) \frac{\nabla h^*(x,y)}{h^*(x,y)} = x \frac{\nabla f^*(x)}{f^*(x)} + y \frac{\nabla g^*(y)}{g^*(y)} \leq (2 - \overline{m-1}) - 1 \quad (4.3)$$

outside a compact set. The above step follows from our assumption $x \cdot \frac{\nabla f^*(x)}{f^*(x)} = (2 - \overline{m-1})$ and (4.2). Therefore, by Theorem 8.2.1 of Chapter II, BPII is solvable for $L_h u = 0$ and hence $\delta(x, y)$ is admissible.

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CHAPTER IV

ADMISSIBLE ESTIMATORS OF THE NATURAL PARAMETER OF EXPONENTIAL DISTRIBUTION

§ 0. Introduction and Summary

The main theme of this chapter is a generalization of the results of Chapter II to exponential family under some conditions. We show that the method adopted by Brown [1], to characterize the admissible estimators in the normal set up, can be extended to the problem of estimating the natural parameter of an exponential family under quadratic loss function. The members of the exponential family we consider are Weirstrass transforms of a σ-finite measure absolutely continuous with respect to m-dimensional Lebesgue measure. We prove a result which characterizes admissibility of generalized Bayes estimators in terms of the solvability of an exterior Boundary Value Problem. The proofs of most of the results are similar to that of Chapter II. So our exposition, in this chapter, is not in detail. However, if a proof involved some modifications we have spelt it out clearly.

This chapter contains four sections. The first gives the notations. We have adopted here the notations of Chapter II mostly. In section 2 we pose the problem. Section 3 contains the main characterization theorem along with other technical results which are needed to prove it. In section 4 we give some applications and general comments.

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§ 1. Notation

Let μ be a σ-finite measure on E^m with density $q(x)$ with respect to the m dimensional Lebesgue measure. Let $\Theta = \{\theta : \int e^{-\frac{1}{2}\|x-\theta\|^2} q(x)dx < \infty\}$. X denotes a random variable, taking values in E^m. Assume X is distributed according to the probability measure P_θ whose density with respect to the Lebesgue measure is given by

$$\beta(\theta)\ e^{-\frac{1}{2}\|x-\theta\|^2}\ q(x) \quad \text{a.e } [dx] \tag{1.1}$$

where $\theta \in \Theta$ and $\beta(\theta)$ is the normalizing function given by

$\frac{1}{\beta(\theta)} = \int e^{-\frac{1}{2}\|x-\theta\|^2} f(x)d\,x$. We shall denote $e^{-\frac{1}{2}\|x-\theta\|^2}$ by $p_\theta(x)$.

Observe that the family P_θ is an exponential family dominated by Lebesgue measure. It is also well known that Θ is a convex subset of E^m. We assume $\Theta = E^m$ throughout this chapter. Let $\delta = (\delta_1, \ldots, \delta_m)$ denote an estimate of $\theta = (\theta_1, \ldots, \theta_m)$. We take the loss function $L(\theta, \delta)$ to be quadratic i.e.

$L(\theta, \delta) = \|\theta - \delta\|^2$. The risk function of δ is denoted by $R(\theta, \delta)$. As usual, the generalized Bayes estimator of θ with respect to a σ-finite measure F on E^m, if it exists, is given by

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$$\delta_F(x) = \frac{\int \theta\, p_\theta(x)\ \beta(\theta) F(d\theta)}{\int p_\theta(x)\ \beta(\theta) F(d\theta)} \quad \text{a.e. } [q(x)\ dx] \tag{1.2}$$

Let $f^*(x) = \int p_\theta(x)\ \beta(\theta)\ F(d\theta)$. Then $\delta_F(x)$ is given by

$$\delta_F(x) = \frac{\nabla f^*(x)}{f^*(x)} + x \tag{1.3}$$

Let K_F denote the closed convex hull of the support of F and let $d(x)$, $\pi(x)$ be as defined in Chapter II.

§ 2. The Basic Problem

We are interested in studying the admissibility of estimators of the natural parameter θ under the loss function $L(\theta, \delta)$. Since the loss function is convex we can confine our study to non-randomized estimators. Moreover, it follows from Theorem 3.1 of Bown [1], that every admissible estimator is generalized Bayes in the set up of this chapter. Thus

Theorem 2.1 If δ is an admissible estimator for θ then there exists a σ-finite measure F on E^m such that $\delta(x) = \delta_F(x)$ a.e. $[q(x)dx]$.

The proof of Theorem 2.1 is similar to that of Brown's. Indeed, the above theorem (Theorem 2.1) can be generalized to the case where μ is an arbitrary σ-finite measure. The proof

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of this general result involves a minor modification of the argument given by Brown [1].

In view of Theorem 2.1, it suffices to consider generalized Bayes estimators for our study. Using Stein-Farrell Theorem (Theorem 2.2, Chapter II), we have that a generalized Bayes estimator δ_F is admissible if and only if there exists a sequence of finite measures G_n with compact supports such that

$$\text{(i)} \quad G_n(S_o) \geq 1 \quad \text{for} \quad n \geq 1 \tag{2.1}$$

$$\text{(ii)} \quad \lim_{n\to\infty} \int (R(\theta,\delta_F)-R(\theta,\delta_{G_n}))G_n(d\theta)=\int \|\delta_F(x)-\delta_{G_n}(x)\|^2 g_n^*(x)\,q(x)\,dx=0 \tag{2.2}$$

By a computation similar to that in Section 2 of Chapter II we can show

$$\int (R(\theta,\delta_F)-R(\theta,\delta_{G_n}))G_n(d\theta) = \int \|\nabla h_n(x)\|^2 f^*(x)\, q(x)dx \tag{2.3}$$

where $h_n^2(x) = \dfrac{g_n^*(x)}{f^*(x)}$. Moreover, it follows by an argument similar to that in Chapter II

$$h_n^2(x) = \frac{g_n^*(x)}{f^*(x)} \geq 1 \qquad \text{for} \quad \|x\| \leq 1 \tag{2.4}$$

and by Theorem 2.3 of Chapter II (Lemma 3.5.1 of Brown [1])

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$$\lim_{r\to\infty} \sup_{\{x\,:\,x\varepsilon K^{\alpha},\ \|x\|\geq r\}} h_n(x) = 0$$

Thus, we have as in normal set up

$$\int (R(\theta,\delta_F)-R(\theta,\delta_{G_n}))G_n(d\theta) \geq \inf_{j\varepsilon J} \int \|\nabla j(x)\|^2 f^*(x)\, q(x)dx \qquad (2.6)$$

where J is the class of functions as defined in section 2 of Chapter II i.e. J consists of piecewise differentiable functions j satisfying

$$\text{(i)} \quad j(x) \geq 1 \quad \text{for} \quad \|x\| \leq 1$$

$$\text{(ii)} \quad \lim_{r\to\infty} \sup_{\{x\,:\,x\varepsilon K^{\alpha},\ \|x\|\geq r\}} j(x) = 0 \qquad (2.7)$$

Therefore, δ_F is inadmissible if the right side of (2.6) is positive. We shall prove, in the next section, that the converse holds under some conditions.

The purpose of this chapter, as already mentioned in the introduction, is to show that the method suggested by Brown [1] to study the admissibility of generalized Bayes estimators in the normal set up can be extended to exponential family and we can obtain a sufficient condition for admissibility through the exterior boundary value problem suggested by the calculus of variation problem

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$$\inf_{j \varepsilon J} \int \| \nabla j(x) \|^2 f^*(x) \; q(x) \; dx. \qquad (2.8)$$

We have already seen in Chapter I that the above calculus of variation problem is related to an exterior boundary value problem. We proved that (2.8) is zero if and only if BP II is solvable for $L_f u = \Delta u + (\frac{\nabla f^*(x)}{f^*(x)} + \frac{\nabla q(x)}{q(x)}) \nabla u = 0$ provided q(x) is continuously differentiable. In the next section we link the solvability of BP II for L_f with admissibility of δ_F under some conditions.

§ 3. Main Theorem

In this section we prove that δ_F is admissible of BP II is solvable. Before we state the theorem we list our assumptions.

Assumptions :

(I) q(x) is positive, continuously differentiable,

and $\| \frac{\nabla q(x)}{q(x)} \| < B_1$

(II) $\| \frac{\nabla f^*(x)}{f^*(x)} \| < B_2$ for $x \varepsilon K_F$.

Theorem 3.1 (Main Theorem)

A necessary condition for an estimator δ_F to be admissible is that there exist a nonnegative measure F on E^m such that

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$f^*(x) = \int p_\theta(x)\ \beta(\theta)\ F(d\theta) < \infty$ and $\delta = \delta_F(x)$ almost everywhere (with respect to Lebesgue measure). Furthermore

(A) If BP II is not solvable for $L_f = \Delta u + \left(\frac{\nabla f^*(x)}{f^*(x)} + \frac{\nabla q(x)}{q(x)}\right) u=0$ then δ_F is inadmissible.

(B) If BP II is solvable for $L_f u = 0$ and assumptions I and II hold then δ_F is admissible.

A part of the theorem has already been proved. In section 2 we observed that if δ is admissible then δ is generalized Bayes with respect to a σ-finite measure F. Also, if BP II is not solvable for $L_f\ u = 0$ then (2.8) is positive and hence δ_F is inadmissible. Therefore it remains to prove (B). We shall give a proof of it after proving some technical results. The lemmas that follow are analogous to the technical results given in Chapter II (Section 4) and therefore we shall avoid details wherever possible. We assume, in what follows, that (I) and (II) hold.

The first lemma is due to Brown [1].

Lemma 3.2

Given a constant K_1 there exists a $0 < K_2 < \infty$ such that for all $x, \theta \in E^m$

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$$e^{K_1 \|x-\theta\|} p_\theta(x) \leq K_2 \int_{\|\xi\| < K_1+1} p_{\theta+\xi}(x)\, d\xi \qquad (3.1)$$

Proof : See Brown [1].

The next lemma has been proved Section 4 Chapter II.

<u>Lemma 3.3</u> Under assumption (II) there exists a constant K_0^1 (depending only on B_2 and m) such that

$$\inf_{\{x: \|x\| \leq r\}} \frac{\int_{\|\theta\| \leq r} p_\theta(x)\, \beta(\theta)\, F(d\theta)}{\int p_\theta(x)\, \beta(\theta)\, F(d\theta)} \geq K_0^1 \qquad (3.2)$$

for all sufficiently large r.

Proof : It follows from Theorems 3.1 and 3.3 of Chapter II that $\Delta \log f^*(x) < B_3$ and $\|\delta_F(x)\| \leq \|x\| + K$ for all $x \in E^m$ where B_3 and K are constants which depend only on B_2 and m. Now, appeal to Lemma 4.1 of Chapter II to complete the proof.

q.e.d.

<u>Lemma 3.4</u> There exist constants K_3 and K_4 (depending only on B_1 and m) such that

$$\int (u(\theta)-u(x))^2 p_\theta(x) q(x) dx \leq K_3 \int \int_{\|\xi\| \leq K_4+1} \|\nabla u(x)\|^2 p_{\theta+\xi}(x) q(x)\, d\xi\, dx$$

(3.3)

where u is a piecewise differentiable function.

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Proof : It follows from assumption I, $\frac{q(x)}{q(\theta)} \leq K_5 \, e^{B_1 \|x-\theta\|}$.

Therefore,

$$\int (u(\theta)-u(x))^2 p_\theta(x) q(x)\,dx \leq K_5 q(x) \int (u(\theta)-u(x)) p_\theta(x) e^{B_1\|x-\theta\|}\,dx \tag{3.4}$$

Hence, by lemma 3.2 we have

$$\int (u(\theta)-u(x))^2 p_\theta(x) q(x)\,dx \leq K_5 \cdot K_6 q(\theta) \int \int_{\|\xi\|\leq B_1+1} (u(\theta)-u(x)^2 \times \times p_{\theta+\xi}(x)\, d\xi\, dx \tag{3.5}$$

where K_6 is a constant depending only on B_1. Now, it follows from a lemma of Brown [Lemma 5.5.3, Brown [1]),

$$\int (u(\theta)-u(x)^2 p_{\theta+\xi}(x)\,dx \leq K_7 \int \int_{\|\eta\|<2B_1+3} \|\nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_{\theta+\xi+\eta}(x)\,dx \tag{3.6}$$

where K_7 is a constant depending only on B_1. Therefore, combining (3.4),(3.5) and (3.6), we have

$$\int (u(\theta)-u(x)^2 p_\theta(x) q(x)\,dx \leq$$

$$\leq K_8 q(\theta) \int_{\|\xi\|<B_1+1} \int_{\|\eta\|\leq 2B_1+3} \|\nabla u(x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} \times \times p_{\theta+\xi+\eta}(x)\, d\eta\, d\xi\, dx \tag{3.7}$$

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Now fixing ξ and η and using an argument similar to that in Lemma 4.4 of Chapter II we have

$$\int \|\nabla u\ (x)\|^2 \frac{1}{\|x-\theta\|^{m-1}} p_{\theta+\xi+\eta}(x)dx \le$$

$$\le K_9 \int \|\nabla u\ (x)\|^2 p_{\theta+\xi+\eta}(x)\ dx \tag{3.8}$$

In obtaining (3.8) we also use the fact $\|\xi\| \le B_1 + 1$, $\|\eta\| \le 2B_1 + 3$.

Therefore,

$$\int (u\ (\theta)-u\ (x))^2\ p_\theta(x)q(x)dx \le K_8 \cdot K_9\ q(\theta) \int_{\|\xi\|\le B_1+1} \int_{\|\eta\|\le 2B_1+3} \int \nabla u(x) \times$$

$$\times p_{\theta+\xi+\eta}(x)\ d\eta\ d\xi\ dx \tag{3.9}$$

and using the assumption (I) again we have

$$\int (u\ (\theta)-u_n(x))^2\ p_\theta(x)q(x)dx \le K_8 \cdot K_9 \int_{\|\xi\|\le B_1+1} \int_{\|\eta\|\le 2B_1+3} \int \|\nabla u(x)\|^2 \times$$

$$e^{B_1\|x-\theta\|} p_{\theta+\xi+\eta}(x)q(x)d\eta d\xi dx \tag{3.10}$$

$$\le K_{10} \int_{\|\xi\|\le B_1+1} \int_{\|\eta\|\le 2B_1+3} \int_{\|\psi\|\le B_1+1} \int \|\nabla u\ (x)\|^2 p_{\theta+\xi+\eta+}(x) \times$$

$$\times q(x)d\eta d\xi d\psi\ dx. \tag{3.11}$$

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Now, using the inequality

$$p_{\theta+\xi+\eta+\psi}(x) = e^{-\frac{1}{2}\|x-\theta+\xi+\eta+\psi\|^2} \le e^{-\frac{1}{2}\|x-\theta\|^2} e^{\|x-\theta\|(\|\xi\|+\|\eta\|+\|\psi\|)}$$

we have

$$(3.11) \le K_{10} \cdot K_{11} \int \|\nabla u_n(x)\|^2 p_\theta(x)\, e^{\|x-\theta\|(4B_1+5)}\, dx \qquad (3.12)$$

where $K_{11} = (\int_{\|\xi\|<2B_1+3} d\xi)^3$. (3.3) now follows from (3.12) and Lemma 3.2. This completes the proof.

q.e.d

Then next result is essentially contained in Lemma 5.3.1 of Brown [1]. The proof that is given is a minor modification of the proof given by Brown.

Lemma 3.5 : Suppose BP II is solvable for $L_f u = 0$. Then there exists a measure F_1 such that $\sup_{x\in E^m} \left\|\frac{\nabla f_1^*(x)}{f_1^*(x)}\right\| < B_3$ and BP II is solvable for $L_{f_1} u = \Delta u + \left(\frac{\nabla f_1^*(x)}{f_1^*(x)} + \frac{\nabla q(x)}{q(x)}\right) \nabla u = 0.$

Proof : Define $F_1 = F + f^*(\pi(\theta)\, e^{-(B_1+1)d(\theta)}\, d\theta$. Following the same argument as in Brown [1], it is easy to check that

$\left\|\frac{\nabla f_1^*(x)}{f_1^*(x)}\right\| < B_3$ where B_3 is a constant depending only on B_1, B_2 and m.

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It remains to prove that BP II is solvable for $L_{f_1} u = 0$. We shall show this, in view of Theorem 3.1 of Chapter I, by proving that for each $\varepsilon_1 > 0$ there exists a piecewise differentiable function, j_1, satisfying

$$j_1(x) \geq 1 \qquad \text{for} \quad \|x\| \leq 1 \tag{3.13}$$

$$\lim_{r\to\infty} \sup_{\|x\|=r} j_1(x) = 0 \tag{3.14}$$

$$\int \|\nabla j_1(x)\|^2 f_1^*(x)\, q(x)\, dx < \varepsilon_1 \tag{3.15}$$

Since BP II is solvable, given $\varepsilon > 0$ there is an $R > 0$ and a piecewise differentiable function j satisfying

$$j(x) \geq 1 \qquad \text{for} \quad \|x\| \leq 1$$

$$j(x) = 0 \qquad \text{for} \quad \|x\| \geq R$$

$$\int \|\nabla j(x)\|^2 f^*(x)\, q(x)\, dx < \varepsilon . \tag{3.16}$$

Let $K^1 = \{x : d(x) \leq 1\}$ and $\pi_1(x)$ be the projection of x onto K^1. Let $d_1(x) = \|x - \pi_1(x)\|$. Since the mapping $x \to (\pi_1(x), d_1(x))$ one to one we can transform the variables in the integral (3.16) to get

$$\varepsilon \geq \int_{1<d(x)<2} \|\nabla j(x)\|^2 f^*(x) q(x) dx = \int_{0<d_1<1} \|\nabla j(x^{-1}(\pi_1, d_1))\|^2 \times$$

$$\times f^*(x^{-1}(\pi_1, d_1))\, q(x^{-1}(\pi_1, d_1)) \times$$

$$\times J(\pi_1, d_1)\, d\pi_1\, dd_1 \tag{3.17}$$

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where J_1 is the Jacobian and $d\pi_1$ is the appropriate suface measure on the boundary of K^1. Moreover, $1 \leq J \leq d^{m-1} = (1+d_1)^{m-1}$. Therefore, it follows from (3.17) that there is β, $1 \leq \beta \leq 2$, such that

$$\int \| \nabla j(x^{-1}(\pi_1, \beta)) \|^2 f^*(x^{-1}(\pi_1, \beta))\, q(x^{-1}(\pi_1, \beta))\, d\pi_1 \leq \varepsilon \tag{3.18}$$

With β as above, consider K^β and let $\pi_\beta(x)$ be the projection on K^β and $d_\beta(x) = \|\pi_\beta(x) - x\|$. Define

$$j_1(x) = j(\pi_\beta(x)) \ \exp(-\alpha d_\beta(x)) \tag{3.19}$$

where $\alpha > 0$ is to be chosen later. Plainly, j_1 satisfies the conditions (3.13) and (3.14). Write

$$\int \| \nabla j_1(x) \|^2 f_1^*(x) q(x) dx = \int_{K^\beta} \|\nabla j_1(x)\|^2 \ f_1^*(x) q(x) dx$$

$$+ \int_{E^m - K^\beta} \| \nabla j_1(x) \|^2 \ f_1^*(x) q(x) dx \tag{3.20}$$

It follows, by an argument similar to the one given by Brown [1], that

$$\int_{K^\beta} \| \nabla j_1(x) \|^2 \ f_1^*(x) \ g(x) \ dx \leq K_2 \ \varepsilon \tag{3.21}$$

where K_2 is a constant (depending only on B_1, B_2 and m).

For the second integral on the right of (3.20) we have by changing variables

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$$\int_{E^m - K^\beta} \|\nabla j_1(x)\|^2 f_1^*(x) q(x)\,dx$$

$$= \int_{d_\beta > 0} \|\nabla j_1(x^{-1}(\pi_\beta, d_\beta))\|^2 f_1^*(x^{-1}(\pi_\beta, d_\beta)) q(x^{-1}(\pi_\beta, d_\beta)) \times J_2(\pi_\beta, d_\beta)\, d\pi_\beta\, dd_\beta \qquad (3.22)$$

where J_2 is the Jacobian and $J_2 \leq (d_\beta + 1)^{m-1}$. Moreover, it can be shown (see Brown [1]) that

$$\|\nabla j_1(x)\| \leq \|\nabla j(\pi_\beta(x))\|\, e^{-\alpha d_\beta(x)} + \alpha\, j(\pi_\beta(x))\, e^{-\alpha d_\beta(x)} \qquad (3.23)$$

Now, note that $f_1^*(x) = f^*(x) + \phi^*(x)$ where $\phi^*(x) = \int p_\theta(x)\, f^*(\pi(\theta))\, e^{-(B_1+1)d(\theta)}\, d\theta$. It follows by a result of Brown (Lemma 3.1.4, [1]) that

$$f^*(x) \leq \exp\left(-\frac{d_\beta^2(x)}{2}\right) f^*(\pi_\beta(x)).$$

Furthermore, a result of Brown (Lemma 3.4.1, [1]) implies

$$\phi^*(x) \leq (\exp B_3^2 + 2^{m/2})\, f^*(\pi(x))\, e^{-(B_1+1)d(x)} \leq C\, e^{-(B_1+1)d_\beta(x)} \times f^*(\pi_\beta(x))$$

where C is a constant depending on B_3 and m.

Hence there is a constant C_1 such that

$$f_1^*(x) \leq C_1 \cdot C^{-(B_1+1)d_\beta(x)} f^*(\pi_\beta(x)). \qquad (3.24)$$

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Moreover, by assumption (I) we have

$$q(x) \leq e^{Bd_\beta(x)} \quad q(\pi_\beta(x)) \tag{3.25}$$

Therefore, it follows from (3.23), (3.24) and (3.25)

$$\int_{E^m - K^\beta} \| \nabla j_1(x) \|^2 f_1^*(x) \; q(x) dx$$

$$\leq C_1 \int \int_{d_\beta > 0} \| \nabla j(\pi_\beta) \|^2 e^{-(1+\alpha) d_\beta} f^*(\pi_\beta) q(\pi_\beta) (d_\beta + 1)^{m-1} d\pi_\beta dd_\beta$$

$$+ C_1 \alpha \int \int_{d_\beta > 0} j(\pi_\beta) \; e^{-(1+\alpha) d_\beta} f^*(\pi_\beta) q(\pi_\beta) (d_\beta + 1)^{m-1} d\pi_\beta dd_\beta \tag{3.26}$$

By (3.17) the first integral in the first term on the right of (3.26) is less than ε. On the other hand, since $j(x) = 0$ for $\|x\| \geq R$, the second integral is finite and bounded by an absolute constant.

Therefore, for α sufficiently small

$$\int_{E^m - K_\beta} \| \nabla j_1(x) \|^2 f_1^*(x) \; q(x) dx \leq C_3 \; \varepsilon.$$

This comples the proof of the lemma.

q.e.d.

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<u>Lemma 3.6</u> Let $\{u_n\}$ be a sequence of functions satisfying

$L_{f_1} u_n = 0$ for $1 < \|x\| < n$, $u_n = 1$ for $\|x\| \leq 1$, $u_n = \varepsilon$,

$\|x\| = n$, where $1 > \varepsilon > 0$, and $u_n = 0$ for $\|x\| > n$.

Then there exists a constant K_o (depending on ε, B_2 and m) such that

$$\inf_{\{x : \|x\| \leq n\}} \frac{\int u_n(\theta)\, p_\theta(x)\, \beta(\theta)\, F(d\theta)}{\int p_\theta(x)\, \beta(\theta)\, F(d\theta)} \geq K_o$$

for all sufficiently large n.

Proof : Similar to Corollary 4.2 of Chapter II.

We now come to the proof of the main theorem. The argument is similar to that of main theorem of Chapter II and we shall not go into details.

<u>Proof of Theorem 3.1</u>

Let BP II be solvable for $L_f u = \Delta u + \left(\frac{\nabla f^*}{f^*} + \frac{\nabla q}{q}\right)\nabla u = 0$. Then, by Lemma 3.5, BP II is solvable for $L_{f_1} u = 0$. Therefore, we have a sequence of functions $\{u_n\}$ satisfying $L_{f_1} u_n = 0$ for $1 < \|x\| < n$, $u_n = 1$ for $\|x\| \leq 1$, $u_n = \varepsilon_o$ for $\|x\| = n$ $(0 < \varepsilon_o < 1)$ and $u_n = 0$ for $\|x\| > n$. Let $\{G_n\}$ be a sequence of finite measures defined by $G_n(d\theta) = u_n(\theta)\, F(d\theta)$. We shall show that

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$$\lim_{n\to\infty} \int (R(\theta,\delta_F) - R(\theta,\delta_{G_n}))\, G_n(d\theta) = 0 \tag{3.27}$$

It follows from section 2

$$\int (R(\theta,\delta_F)-R(\theta,\delta_{G_n}))G_n(d\theta)=\int \left\| \nabla\left(\frac{g_n^*(x)}{f^*(x)}\right)\right\|^2 \frac{1}{g_n^*(x)/f^*(x)}\, f^*(x)\cdot q(x)dx \tag{3.28}$$

Therefore, to show (3.27) it suffices to prove

$$\lim_{n\to\infty} \int_{\|x\|\le n} \left\|\nabla\left(\frac{g_n^*(x)}{f^*(x)}\right)\right\|^2 \frac{1}{g_n^*/f^*(x)}\, f^*(x)\; q(x)dx = 0 \tag{2.29}$$

$$\lim_{n\to\infty} \int_{\|x\|>n} \left\| \nabla\left(\frac{g_n^*(x)}{f^*(x)}\right)^{\frac{1}{2}} \right\|^2 f^*(x)q(x)dx = 0 \tag{3.30}$$

Consider (3.29). It follows from Lemma 3.6 that there exists a constant C_1 such that

$$\int_{\|x\|\le n} \left\| \nabla\left(\frac{g_n^*(x)}{f^*(x)}\right)\right\|^2 \frac{1}{g_n^*(x)/f^*(x)}\cdot f^*(x)\; q(x)dx \le$$

$$\le C_1 \int_{\|x\|\le n} \left\| \nabla\left(\frac{g_n^*(x)}{f^*(x)}\right)\right\|^2 f^*(x)q(x)dx \tag{3.31}$$

Now, by lemma 3.4 we have

$$(3.31) \le C_2 \int_{\|x\|\le n} \| \nabla u_n(x)\|^2 \int_{\|\xi\|<C_3+1} f^*(x+\xi)\, d\xi \quad q(x)dx \tag{3.32}$$

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Since $f^*(x) \leq f_1^*(x)$, where $f_1^*(x)$ is given by the measure F_1 defined in Lemma 3.5,

$$(3.31) \leq C_2 \int_{\|x\|\leq n} \|\nabla u_n(x)\|^2 \int_{\|\xi\|<C_3+1} f_1^*(x+\xi)d\xi \quad q(x)dx$$

$$\leq C_3 \int_{\|x\|\leq n} \|\nabla u_n(x)\|^2 f_1^*(x) \; q(x) \; dx \qquad (3.33)$$

The step (3.33) follows from the fact that $\left\|\frac{\nabla f_1^*(x)}{f_1^*(x)}\right\| < B_3$.

(3.29) now follows from (3.33) and the solvability of BP II for $L_f u = 0$. Therefore it remains to prove (3.30). This follows along the same lines as given in the proof of the main theorem of Chapter II if we observe $q(x) = (e^{B_1\|x\|})$. The latter fact is easily seen to follow from assumption (I).

This completes the proof of the theorem.

q.e.d.

§ 4. Applications and General Comments

In this section we consider, briefly, some statistical applications of the main theorem of this chapter.

The problem of characterizing generalized Bayes estimators of the natural parameter under quadratic loss function, in exponential families has been studied by Strawderman [5]. He

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has obtained necessary and sufficient conditions for a given estimator to be generalized Bayes. The results are quite analogous to that in the normal set up (See Strawderman and Cohen [1]). We have nothing to add to this.

Using our main theorem of this chapter we can list many criteria (as we have shown in section 8, Chapter II), for the admissibility of a given generalized Bayes estimator δ_F, through $f^*(x)\ q(x)$. Indeed, all the results of section 8, Chapter II, can be stated and proved in the set up of this chapter (Of course, under the assumptions (I)and(II)). The proofs are similar. We give below a couple of results of that sort.

The first result is in spherically symmetric case. Assume the density $q(x)$ and the measure F are spherically symmetric.

<u>Theorem 4.1</u> : Under the assumptions (I) and (II), a necessary and sufficient condition for δ_F to be admissible is

$$\int_1^{\infty} \frac{1}{f^*(r)\ q(r)\ r^{m-1}}\, dr = \infty \qquad (4.1)$$

Proof : Note that (4.1) implies that BP II is solvable for $L_T u = 0$ and conversely. Now appeal to Theorem 3.1.

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Remark 4.2 : Observe that assumptions (I) and (II) are not needed for the necessary part.

The next theorem is for the general case.

Theorem 4.3 : Assume (I) and (II). Then a sufficient conditi[on] for δ_F to be admissible is

$$x \frac{\nabla(f^*(x)\ q(x))}{f^*(x)\ q(x)} \leq (2-m) + \mathcal{J}(\|x\|) \tag{4.2}$$

for all sufficiently large $\|x\|$, where $\mathcal{J}$ is a dini function.

Proof : (4.2) implies that BP II is solvable for $L_f u = 0$ and hence the theorem.

We can also consider the problems treated in sections 3 and 4 of Chapter III in this set up. Since the results are analogous and the proofs are similar to that in the normal set up, we do not pause here to do so.

Finally, one can give a lot of examples of exponential distributions which satisfy the conditions of this chapter. For example, $q(x) = \|x\|^k$ or $e^{\|x\|}$. The class of exponential distributions, we consider in this chapter, though fairly rich, does not cover all. There are many exponential distributions which do not satisfy our conditions. We do not know how to prove a result similar to Theorem 3.1 in such cases.

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CHAPTER V

A SUFFICIENT CONDITION FOR ALMOST ADMISSIBILITY OF ESTIMATORS

§0. Introduction and Summary

We present in this chapter some sufficient conditions for almost admissibility of generalized Bayes estimators, under quadratic loss function in terms of solvability of an exterior boundary value problem. Using a theorem of Stein [6], on almost admissibility of an estimator, Zidek [1] obtained a sufficient condition for almost admissibility for dimension $m = 1$ by relating it to a one dimensional calculus of variation problem. We generalize his result to arbitrary dimension under some conditions. We have also generalized, under some conditions, a result of Karlin for exponential family to m dimensions.

This chapter contains three sections. Section 1 is on basic notations and preliminaries. In section 2 we study the admissibility of the estimators of a function the natural parameter of exponential family under some conditions. In section 3 we present a generalization of a result of Zidek to m dimensions.

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§1. Definitions and Preliminaries

Let $(\mathcal{X}, \mathcal{B})$ be a measurable space and X a random variable taking its values in $\mathcal{X}$. Assume X is distributed according to an unknown but unique member of a family probability distribution P_θ indexed by a parameter set $\Theta \subset E^m$. The problem is to estimate a vector valued function $g(\theta)$, $g: \Theta \to E^m$, with quadratic loss function $L(\theta, t) = \|\theta - t\|^2$, $t \in E^m$. We assume that the family $\{P_\theta : \theta \in \Theta\}$ is dominated by a σ-finite measure μ on $(\mathcal{X}, \mathcal{B})$. Let $p_\theta(x)$ denote the density of P_θ with respect to μ. Assume $p_\theta(x)$ is jointly measurable in its arguments θ and x. Let Π be a σ-finite measure on the Borel subsets of Θ satisfying

$$\int (1 + \|g(\theta)\|^2) p_\theta(x) \pi(d\theta) < \infty \quad \text{a.e.} \quad [\mu] \tag{1.1}$$

Then the generalized Bayes estimator δ_π of $g(\theta)$ with respect to Π exists and is given by

$$\delta_\pi(x) = \frac{\int g(\theta) p_\theta(x) \Pi(d\theta)}{\int p_\theta(x) \Pi(d\theta)} \qquad \text{a-e } [\mu] \tag{1.2}$$

Let δ be an estimator of $g(\theta)$. We denote its risk function by $R(\theta, \delta)$. δ is called almost admissible with respect to Π, if δ_1 is an estimator of $g(\theta)$ satisfying $R(\theta, \delta_1) \leq R(\theta, \delta)$

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for all $\theta \in \Theta$ then $R(\theta,\delta_1) = R(\theta,\delta)$ a.e. $[\Pi]$, clearly, any Bayes estimator is almost admissible with respect to its prior. For a result giving conditions under which almost admissibility implies admissibility see Zidek [1].

We will assume, throughout this chapter, that Π is absolutely continuous with respect to the m - dimensional Lebesgue measure λ and $\Theta = E^m$. Let $\pi(\theta)$ denote the density of Π. Then the formal posteriori distribution of θ, given $X = x$, has a density with respect to λ and it will be denoted by $p(\theta/x)$. Let $E^{\theta}(\cdot)$ and E_{π}^{X} denote the expectations with respect to P_{θ} and the formal posteriori distribution. We assume $\pi(\theta) > 0 \ \forall \theta \in E^m$.

The results of this chapter will be consequences of the following theorem due to Stein [6]. A proof of this for dimension $m = 1$ is given in Zidek [1].

Let S_r *be the sphere of radius* r with origin as centre. Let J_r denote the class of all non-negative real valued functions j on E^m satisfying $j(\theta) \geq 1$ on S_r and $\int j(\theta)R(\theta, \delta_{\pi})\, \pi(\theta)d\theta < \infty$.

<u>Theorem 1.1</u>: The estimator δ_{π} is almost admissible with respect to Π if for every S_r and $\epsilon > 0$, there exists $j \in J_r$ such that

$$E \ \| \int E_{\pi}^{X}(j(\theta)(g(\theta) - \delta_{\pi}))\|^2 / E_{\pi}^{X}(j) < \epsilon \qquad (1.3)$$

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where E is the expectation with respect to the σ finite measure $p_\theta(x)\pi(\theta)d\theta\, d\mu(x)$.

Proof:- Suppose δ_π is not almost admissible with respect to Π. Let δ_o be an estimator such that $R(\theta,\delta_o) \leq R(\theta,\delta_\pi)$ with strict inequality holding on a set of positive Π measure. Then there exist constant $d > 0$, a set B_d and sphere $S_r \subset E^m$ such that $\pi(S_r \cap B_d) > 0$ and $R(\delta_\pi,\theta) - R(\delta_o,\theta) \geq d$ for $\theta \in B_d$. Now, choose $\epsilon < d\,\pi(S_r \cap B_d)$ and a $j \in Jr$ satisfying (1.3). Then

$$\epsilon < d\,\pi(S_r \cap B_d) \leq \int_{S_r\, B_d} (R(\theta,\delta_\pi) - R(\theta,\delta_o))j(\theta)\,\pi(\theta)d\theta$$

$$\leq \int R(\theta,\delta_\pi)j(\theta)\,\pi(\theta)d\theta$$

$$- \inf_\delta \int R(\theta,\delta)j(\theta)\,\pi(\theta)d\theta \quad (1.4)$$

But the right side of (1.4) is the left side of (1.3). Therefore (1.4) is less than ϵ, which is a contradiction. This completes the proof of the theorem. q.e.d.

§2. Almost Admissible Estimators in Exponential Family

In this section we consider the exponential family and obtain a sufficient condition for almost admissibility of estimators interms of solvability of an exterior boundary value problem.

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Let μ be a σ - finite measure on $\mathcal{X} = E^m$. Let $p_\theta(x) = \beta(\theta)e^{x\theta}$ and $\Theta = \left\{ \theta : \frac{1}{\beta(\theta)} = \int e^{\theta x} \mu(dx) < \infty \right\}$. Θ is, in general, a convex subset of E^m. We assume $\Theta = E^m$. Suppose we are interested in estimating a continuous vector valued function $g(\theta) = \theta \; h(\|\theta\|)$ where h is a real valued function. Let Π be the measure with density

$$\pi(\theta) = \exp\left[-\int_0^{\|\theta\|} t\,h(t)\,dt\right]/\beta(\theta) . \tag{2.1}$$

Suppose

$$\int \beta(\theta)e^{x\theta}\pi(\theta)d\theta < \infty \quad \text{a.e } [\mu] \tag{2.2}$$

Then Π is a prior measure. Moreover,

$$\nabla(\beta(\theta)e^{x\theta}\pi(\theta)) = (x - g(\theta))\; \beta(\theta)e^{x\theta}\; \pi(\theta) \tag{2.3}$$

Therefore, if

$$\beta(\theta)e^{x\theta}\; \pi(\theta) \to 0 \quad \text{as} \quad \theta \to \infty \tag{2.4}$$

then the generalized Bayes estimator of $g(\theta)$ with respect to Π is given by $\delta_\pi(x) = x$. This follows from (2.1) and integration by parts.

<u>Theorem 2.1:</u> Under (2.2) and (2.4), X is an admissible estimator of $g(\theta)$ if BP II is solvable for the elliptic equation

$$L_\pi u = \nabla u + \frac{\nabla \pi}{\pi} \nabla u = 0 .$$

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Proof:- We appeal to Theorem 1.1. Let S_r be, following the notation of Theorem 1.1, the sphere of radius r. Let $j \in J_r$ be a continuously differentiable function vanishing outside a compact set. Then

$$\mathrm{Cov}_{\pi}^{X}(j^2, g) = \frac{\int j^2(\theta)\, \nabla(e^{\theta x}\beta(\theta)\pi(\theta))d\theta}{\int e^{\theta x}\beta(\theta)\pi(\theta)\, d\theta} \tag{2.5}$$

Integrating, the numerator of (2.5), by parts we have

$$\mathrm{Cov}_{\pi}^{X}(j^2, g) = \frac{\int \nabla(j^2(\theta))e^{\theta x}\beta(\theta)\pi(\theta)d\theta}{\int e^{\theta x}\beta(\theta)\pi(\theta)d\theta} \tag{2.6}$$

Hence

$$E\left\{ \|\mathrm{Cov}_{\pi}^{X}(j^2, g)\|^2 / E_{\pi}^{X}(j^2)\right\} = E \left\| \frac{\int \nabla j^2(\theta)e^{\theta x}\beta(\theta)\pi(\theta)d\theta}{\int e^{\theta x}\beta(\theta)\pi(\theta)d\theta}\right\|^2 \Big/ E_{\pi}^{X}(j^2) \tag{2.7}$$

Applying Schwartz inequality to the right side of (2.7) we have

$$E\left\{ \left\| \frac{\int \nabla j^2(\theta)e^{\theta x}\beta(\theta)\pi(\theta)d\theta}{\int e^{\theta x}\beta(\theta)\pi(\theta)d\theta}\right\| \Big/ E_{\pi}^{X}(j^2)\right\}$$

$$\leq 4E\left\{ \frac{\int \|\nabla j(\theta)\|^2 e^{\theta x}\pi(\theta)\beta(\theta)d\theta}{\int e^{\theta x}\pi(\theta)\beta(\theta)d\theta}\right\}$$

$$4\int \|\nabla j(\theta)\|^2 \pi(\theta)d\theta \tag{2.8}$$

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Thus, we have related in (2.8) the almost admissibility of X with a calculus of variation problem i.e. X is almost admissible for $g(\theta)$ if for every r.

$$\inf_{j \in \tilde{J}_r} \int \| \nabla j(\theta) \|^2 \pi(\theta) d\theta = 0 \qquad (2.9)$$

where $\tilde{J}_r = \{ j : j \in J_r,\ j(\theta) = 1$ on S_r, j is continuously differentiable$\}$. The calculus of variation problem in (2.9) can be related to BPII for L_π as follows. First observe that, by Theorem 3.1 of Chapter I, (2.9) holds if the sequence of functions j_n, satisfying $L_\pi j_n(x) = 0$ for $r < \|x\| < n+r$ $j_n(x) = 1$ for $\|x\| \leq r$ and $j_n(x) \geq 0$ for $\|x\| \geq r+n$, converge uniformly on compacta to 1. Now, since BPII is solvable for L_π there exists a sequence of functions $\{u_n\}$, satisfying $L_\pi u_n = 0$ for $1 < \|x\| < n+r$, $u_n(x) = 1$ for $\|x\| \leq 1$, $u_n(x) = 0$ for $\|x\| \geq n+r$, converge uniformly on compacta to 1. It follows from maximum modulus principle, that $u_n(x) \leq j_n(x)\ \forall\ x$. Therefore $j_n(x)$ converges to 1 uniformly on compacta. Hence, if BPII is solvable for L_π then (2.9) holds for every r and X is almost admissible. This completes the proof of the theorem. q.e.d.

<u>Remark 2.2.</u> Note that, since $g(\theta)$ is continuous and the underlying family of distributions is an exponential family, $R(\theta,\delta)$ is continuous in θ. Therefore it follows from Theorem 2.1 that X

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is also admissible because the prior measure π is equivalent to Lebesgue measure.

A one dimensional version of the above result has been obtained by Zidek [1]. We shall now apply the above theorem to study the admissibility of estimators of the mean vector of the exponential family.

Assume μ is spherically symmetric. Then the normalizing function $\beta(\theta)$ is also spherically symmetric. The mean vector of the exponential family is given by $\frac{-\nabla\beta(\theta)}{\beta(\theta)} = -\frac{\theta}{\|\theta\|} \frac{\beta'(\|\theta\|)}{\beta(\|\theta\|)}$

We want to obtain a sufficient condition for the admissibility of the estimators of the form $\frac{x}{1+\lambda}$, $\lambda > 0$.

Note that if (2.2) and (2.4) hold with $\pi(\theta) = \beta^{\lambda+1}(\theta)$ then $\frac{x}{\lambda+1}$ is the generalized Bayes estimator with respect to the measure $\pi(\theta)d\theta$. Thus we have the following result.

<u>Theorem 2.3:</u> Suppose (2.2) and (2.4) hold for $\pi(\theta) = \beta^{\lambda+1}(\theta)$. Then a sufficient condition for $\frac{x}{\lambda+1}$ to be admissible is BPII be solvable for $L_\pi u \cong 0$.

Proof:- Follows immediately from Theorem 2.1 and Remark 2.2. As a corollary to the above theorem we have

<u>Corollary 2.4:</u> Suppose (2.2) and (2.4) hold for $\pi(\theta) = \beta^{\lambda+1}(\theta)$.

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Then a sufficient condition for $\frac{x}{\lambda+1}$ to be admissible is

$$\int_1^{\infty} \frac{1}{\beta^{\lambda}(||\theta||)} \frac{1}{||\theta||^{m-1}} d||\theta|| = \infty . \qquad (2.3)$$

The above corollary is a generalization of a result of Karlin [1] to m dimensions. He proved the result for one dimensional case. Moreover, Karlin does not assume that $\frac{x}{\lambda+1}$ is generalized Bayes whereas our method needs this assumption. Cheng Ping [1] has considered estimators of the form $a+bx$, $b>0$ and has given a sufficient condition for their admissibility in the one dimensional case. His sufficient condition is similar to that of Karlin's. One can generalize his result to m dimensions similarly.

We would like to note that our set up is quite restrictive on two counts. Firstly, we have assumed that $\Theta = E^m$. We have already observed that Θ is a convex set which may not be whole of E^m. The reason for this assumption is that we do not know how to solve an exterior boundary value problem on an arbitrary convex set with an unbounded boundary. It is not even clear how to formulate a boundary value problem on such a domain. However, we can treat the case of estimation of the mean vector $\frac{-\nabla\beta(\theta)}{\beta(\theta)}$ more generally and this will be done in the next section. Secondly, we have confined ourselves to estimators which are

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generalized Bayes. We do not know whether an admissible estimator ought to be generalized Bayes in our set up. However, for dimension $m = 1$ one can show, using the results of Farrell [2] that an admissible estimator of $\frac{-\beta'(\omega)}{\beta(\omega)}$ is generalized Bayes. We prove this result below. The proof is essentially contained in Farrell's paper (Farrell [2]). The notation we use below is for this proof only.

Let $(R, \mathcal{B}, \mu)$ be a σ-finite measure space on the real line R equipped with Borel σ-algebra. Consider the family of all exponential distributions with density $f(x,\omega) = \beta(\omega)\, e^{x\omega}$ with respect to μ where $\Omega = \{\omega : \int e^{x\omega} d\mu(x) < \infty\}$ and $\beta^{-1}(\omega) = \int e^{x\omega} \mu(dx)$. Plainly $\Omega \subset R$ is an interval. The upper and lower end points, say $\bar{\omega}$ and $\underline{\omega}$, of Ω may or may not belong to Ω and $\bar{\omega}$ ($\underline{\omega}$) may be finite or $+\infty$ (finite or $-\infty$). The loss function is $W(\omega, t) = \left(\frac{-\beta'(\omega)}{\beta(\omega)} - t\right)^2$. We assume that the end points $\underline{\omega}$ and $\bar{\omega}$ do not belong to Ω. i.e. Ω is an open interval.

<u>Theorem 2.5</u>: An estimator δ is admissible only if δ is generalized Bayes i.e. there exists a measure F on $(R, \mathcal{B})$, not necessarily finite, s.t.

$$\delta(x) = \frac{\int \frac{-\beta(\omega)}{\beta(\omega)} f(x,\omega)\, dF(\omega)}{\int f(x,\omega)\, dF(\omega)} \quad \text{a.e. } x[\mu].$$

We give the proof of this theorem after a series of lemmas.

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Let $V_1(\omega, t) = \frac{d}{dt} W(\omega, t)$. Observe that if δ' is a decision procedure Bayes with respect to a probability measure λ on Ω, then $t \in$ support of $\delta'(x, \cdot)$ satisfies the equation

$$0 = \int V_1(\omega, t)\ f(x, \omega)\lambda(d\,\omega)$$

for almost all $x\ [\mu]$.

Define the normalizing function $V(\omega) = \left|\frac{\beta'(\omega)}{\beta(\omega)}\right| + 1$. It is easily checked that if $E \subset (-\infty, \infty)$ is a compact set, then $\frac{V_1(\omega, t)}{V(\omega)}$ is uniformly continuous function of $(\omega, t) \in \Omega \times E$.

Throughout we assume that

$$\int V(\omega)\, f\,(x, \omega)\, \lambda\,(d\,\omega) < \infty \quad \text{a.e.} \quad x[\mu]$$

This assumption is equivalent to the supposition that a Bayes procedure has finite risk. Also, note that $\frac{-\beta'(\omega)}{\beta(\omega)}$ is a monotone function and $\lim_{\omega \to \bar{\omega}} \frac{\beta'(\omega)}{\beta(\omega)} = \infty$ and $\lim_{\omega \to \underline{\omega}} \frac{-\beta'(\omega)}{\beta(\omega)} = -\infty$. Therefore

$$\lim_{\omega \to \bar{\omega}} \frac{V_1(\omega, t)}{V(\omega)} = 1 \quad \text{and} \quad \lim_{\omega \to \underline{\omega}} \frac{V_1(\omega, t)}{V(\omega)} = -1.$$

We need the following result of Farrel [3] in our proof.

<u>Theorem 2.6</u> [Farrel]

A decision procedure δ is admissible iff

i) δ is non-randomized

ii) there exists a sequence of decision procedures δ_n Bayes

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with respect to finite measures λ_n having compact supports D_n $(n \geq 1)$ such that $D_n \uparrow \Omega$ satisfying

(ii)(a) there exists a compact set $E_0 \subset \Omega$, $\inf_{n \geq 1} \lambda_n(E_0) \geq 1$

(ii)(b) for every compact set $E \subset \Omega$, $\sup_{n \geq 1} \lambda_n(E) < \infty$

(c) $\int (R(\omega,\delta) - R(\omega,\delta_n))\, \lambda_n(d\omega) \to 0$

(d) $R(\omega,\delta_n) \to R(\omega,\delta)\ \forall\ \omega$.

Define a sequence of probability measures on Ω by setting

$$\nu_n(x,E) = \frac{1}{k_n(x)} \int_E V(\omega)\, f(x,\omega)\lambda_n(d\,\omega)$$

where $k_n(x) = \int V(\omega) f(x,\omega)\lambda_n(d\omega)$ and λ_n's are the finite measures given by Theorem 2.6. Obtain a compactification Ω^* of Ω such that Ω^* is metrizable and Ω is a Borel subset of Ω^* as in Farrel [2] and extend the measures $\nu_n(x,\cdot)$ to Ω^*. Denote the extension of $\nu_n(x,\cdot)$ by itself. In future we use only these extended measures.

Let $F_n(E) = \int_E V(\omega)\lambda_n(d\omega)$. Since $\sup_{n \to 1} \lambda_n(E) < \infty$ for compact E, there exists a subsequence n_i such that F_{n_i} converge weakly to a σ-finite measure F', the weak convergence being with respect to the class of continuous functions on Ω vanishing outside compact sets. Assume without loss of generality that $F_n \to F'$ weakly (in the above sense).

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Lemma 2.7: There exists a subsequence $\{n_i\}$ such that

$$\nu_{n_i}(x,\cdot) \to \nu(x,\cdot) \text{ weakly for almost all } x\ [\mu].$$

Proof: Observe that $\nu_n(x,\cdot)$ $(n \geq 1)$ are continuous bilinear forms on the Banach space $L_1(R, \mathcal{B}, \mu) \times C(\Omega^*)$, where $C(\Omega^*)$ is the Banach space of continuous real valued functions on Ω^* equipped with supremum norm. Since $L_1(R, \mathcal{B}, \mu)$ is separable. the unit ball of continuous bilinear forms is sequentially compact, Now a standard diagonalization argument along with the fact that μ is σ - **finite** gives the result.

Lemma 2.8: For almost all $x, y[\mu]$

$$\liminf_{n \geq 1} \frac{k_n(x)}{k_n(y)} > 0 .$$

Proof: Plainly

$$\frac{k_n(x)}{k_n(y)} = \int_\Omega \frac{f(x,\omega)}{f(y,\omega)} \nu_n(y, d\omega) .$$

If Ω is a finite interval, then the result is trivial. If Ω is an infinite interval the argument is similar to that in Lemma 4.2 of Farrel [2].

Lemma 2.9: $\nu(x, \Omega) > 0$ for almost all $x[\mu]$.

Proof:

Case (i) Ω is a finite interval with end points $\underline{\omega}$ and $\overline{\omega}$.

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Suppose $\nu(x,\Omega) = 0$ on an x-set of positive μ-measure. Then

$$0 = \int_{\Omega^*} \frac{V_1(\omega,t)}{V(\omega)} \nu(x,d\omega) = \int_{\Omega} \frac{V_1(\omega,t)}{V(\omega)} \nu(x,d\omega) + \int_{\{\underline{\omega}\}} \frac{V_1(\omega,t)}{V(\omega)} \nu(x,d\omega)$$

$$+ \int_{\{\bar{\omega}\}} \frac{V_1(\omega,t)}{V(\omega)} \nu(x,d\omega) .$$

Since $\frac{V_1(\omega,t)}{V(\omega)} = -1$ at $\bar{\omega}$ and $+1$ at $\underline{\omega}$, we have

$$\nu(x, \{\bar{\omega}\}) = \nu(x, \{\underline{\omega}\})$$ on a x set of positive μ-measure.

It follows from a result of Farrell (Section 4, [2]) that this is not possible in our set up.

Case (ii) Ω is an infinite interval.

If $\underline{\omega} \neq -\infty$ ($\bar{\omega} = +\infty$) then by Lemma 4.3 of Farrel [2]

$\nu(x,\{-\infty\}) = 0$ ($\nu(x, \{\infty\}) = 0$). Therefore if $\nu(x,\Omega) = 0$, on a non-null x-set $\nu(x,\{\infty\}) = 1$ ($\nu(x,-\{-\infty\}) = 1$). Then

$$0 = \int_{\{\infty\}} \nu(x,d\omega) + \int_{\Omega} \frac{V_1(\omega,t)}{V(\omega)} \nu(x,d\omega) = C.$$ A contradiction.

Lemma 2.10: For almost all $x(\mu)$ $\limsup_{n \geq 1} K_n(x) < \infty$.

Proof: It is well known that the necessary condition (vi) of Theorem 2.6, proved by Farrell [3] and the convexity of the loss function imply that there exists a subsequence $\{\delta_{n_i}\}$ of $\{\delta_n\}$ given in Theorem 2.6, such that $\delta = \text{weak limit}_{i \to \infty} \delta_{n_i}$. There is no lack of generality in assuming $\{\delta_{n_i}\}$ to be $\{\delta_n\}$. It follows from

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Lemmas 2.7, 2.8, and 2.9 and Lemma 3.2 of Farrell [2] that there exists an open set $U \subset \Omega$ having compact closure in Ω such that

$$0 < \liminf_{n \to \infty} \frac{\lambda_n(U)}{K_n(x)} \leq \limsup_{n \to \infty} \frac{\lambda_n(U)}{K_n(x)} < \infty$$

for almost all $x[\mu]$. Since $\sup_{n \geq 1} \lambda_n(\bar{U}) < \infty$ by Theorem 2.6 we have $\limsup K_n(x) < \infty$ for almost all $x[\mu]$.

<u>Lemma 2.11</u>: For almost all $x[\mu]$ and all $t \in (-\infty, \infty)$

$$\text{I} \quad \lim_{n \to \infty} \int_\Omega f(x, \omega) = \int_\Omega f(x, \omega) F'(d\omega)$$

$$\text{II} \quad \lim_{n \to \infty} \int_\Omega \frac{V_1(\omega, t)}{V(\omega)} f(x, \omega) dF_n(d\omega) = \int_\Omega \frac{V_1(\omega, t)}{V(\eta)} f(x, \omega) F\,(d\omega).$$

The limits are finite.

Proof: We shall give the proof of II. Proof of I is similar. If $\Omega = (\underline{\omega}, \bar{\omega})$ is a finite interval then $f(x, \omega)$ is a bounded continuous function of ω. Since $F_n \to F'$ weakly the result follows (Note $\frac{V_1(\omega, t)}{V(\omega)}$ is bounded continuous). Suppose now, Ω is an infinite interval $(-\infty, \infty)$. Then for any $0 < A < \infty$,

$$\int_{-A}^{A} \frac{V_1(\omega, t)}{V(\omega)} f(x, \omega) F_n(d\omega) \to \int_{-A}^{A} \frac{V_1(\omega, t)}{V(\omega)} f(x, \omega) F'(d\omega)$$

$$= \int_{-A}^{A} V_1(\omega, t) f(x, \omega) F(d\omega)$$

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We complete the proof by establishing uniform integrability i.e.

$$(*)\quad \lim_{A\to\infty}\ \limsup_{n\to\infty} \int_A^\infty \left|\frac{V_1(\omega,t)}{V(\omega)}\right| f(x,\omega)\, F_n(d\omega) = 0$$

Let X be the set of x s.t. (*) fails. We shall show that $\mu(X) = 0$. Suppose not. Then there is an x' s.t. $\mu(x', \infty) > 0$, and

$$\lim_{A\to\infty}\ \limsup_{n\to\infty} \int_A^\infty \left|\frac{V_1(\omega,t)}{V(\omega)}\right| f(x',\omega)\, F_n(d\,\omega) > 0.$$

But for $y > x'$,

$$\int_A^\infty \left|\frac{V_1(\omega,t)}{V(\omega)}\right| f(x\ ,\omega) F_n(d\omega) \le \sup_{\omega\ge A} \left[\frac{|V_1(\omega,t)|}{V(\omega)}\, e^{(x'-y)\omega}\right] \int_A^\infty f(y,\omega) F_n(d\omega)\ .$$

Now taking limsup and letting $A \to \infty$ we find, since

$$\sup_{\omega\to A} \frac{|V_1(\omega,t)|}{V(\omega)}\, e^{(x'-y)\omega} \to 0 \quad \text{as } A \to \infty, \text{ that}$$

$$\limsup_{n\to\infty} \int_{-\infty}^\infty f(y,\omega) F_n(d\,\omega) = \infty.$$

This contradicts Lemma 2.10. Therefore $\mu(X) = 0$. The other cases $\Omega = (-\infty, \bar{\omega})$ and $\Omega = (\underline{\omega}\ , \infty)$ are similar.

We now complete the proof of the assertion that 'if δ is admissible then it is generalized Bayes'.

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Proff of Theorem 2.5.

Let $F_1 \subset R$ be the measurable set with $\mu(F_1) = 0$ such that for $x \notin F_1$ the convergence in Lemma 2.5 takes place. Let $F_2 \subset R$ be the null set such that if $x \notin F_2$ and $t \in$ support of $\delta(x,\cdot)$ there exist sequences $\{n_i\}$ and $\{t_{n_i}\}$, $t_{n_i} \in$ support $\delta_{n_i}(x,\cdot)$, $t_{n_i} \to t$. (See Lemma 2.1 of Farrell [2]. If $x \notin F_1 \cup F_2$ then there are sequence $\{n_i\}$ and $\{t_{n_i}\}$, $t_{n_i} \in$ support $\delta_{n_i}(x,\cdot)$ and $t_{n_i} \to t$ and also convergence in Lemma 2.5 holds. Therefore

$$0 = \int \frac{V_1(\omega, t_{n_i})}{V(\omega)} f(x,\omega) F_{n_i}(d\omega) \quad \text{and}$$

since $\dfrac{V_1(\omega,t_{n_i})}{V(\omega)} \to \dfrac{V_1(\omega,t)}{V(\omega)}$ uniformly in ω, by Lemma 2.11.

We have

$$0 = \lim_{i\to\infty} \int \frac{V_1(\omega,t_{n_i})}{V(\omega)} f(x,\omega)F_{n_i}(d\omega) = \int \frac{V_1(\omega,t)}{V(\omega)} f(x,\omega)F'(d\omega)$$

$$= \int V_1(\omega,t) f(x,\omega) F(d\omega).$$

This co~~mplete~~s the proof.

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§3. A Sufficient condition for almost admissibility

In this section we study the almost admissiblity in the general set up and give a sufficient condition, under some assumptions, using an exterior boundary value problem. Our result is a generalization of a result of Zidek [1] to m-dimensions.

We assume throughout this section that $\Theta = E^m$ and the prior measure Π is absolutely continuous with respect to the m-dimensional Lebesgue measure. Let the density $\pi(\theta)$ of Π be positive everywhere and once continuously differentiable.

Let $(s,\emptyset)$ denote θ in polar coordinates. Define,

$$M(x,\theta) = \int_s^\infty \|\eta\|^{m-1} \|\eta-\delta_\pi\| \frac{p_{(\|\eta\|,\emptyset)}(x)\ \pi(\|\eta\|,\emptyset)\, d\|\eta\|}{s^{m-1}\ p_{(s,\emptyset)}(x)\ \pi(s,\emptyset)} .$$

and $h(t) = E(M^2(x,\theta)/\theta = t)$.

Furthermore, we assume

I ____ lmost all $x\ [\mu]$, the set $\{\theta\ ;\ p_\theta(x) > 0\}$ has non-empty interior

II $h(t)$ is once continuously differentiable.

Then, we have the following theorem.

Theorem 3.1: Under assumptions (I) and (II), δ_π is almost admissible for θ if BPII is solvable for $L_\pi u = \Delta u + (\frac{\nabla\pi}{\pi} + \frac{\nabla h}{h})\nabla u = 0$.

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Proof: We use Theorem 1.1. Let S_r, in the notation of Theorem 1.1, be the sphere of radius r with centre as the origin. Let j be any nonnegative, continuously differentiable function vanishing outside a compact set such that $j(\theta) = 1$ on S_r. Now, consider

$$\mathrm{Cov}_{\pi}^{X}(j^2,\eta) = \int j^2(\eta)(\eta - \delta_{\pi}) \frac{p_{\eta}(x)\pi(\eta)d\eta}{\int p_{\eta}(x)\pi(\eta)d\eta} \tag{3.1}$$

Expressing η in polar coordinates as $(r,\emptyset)$ we have

$$\|\mathrm{Cov}_{\pi}^{X}|(j^2,\eta)\| \leq \int \left(\int_0^r \|\nabla j^2(r,\emptyset)\| ds\right) \times$$

$$\|r\emptyset - \delta_{\pi}\| \frac{p_{(r,\emptyset)}(x)\pi(r,\emptyset)dr\, d\emptyset}{\int p_{\eta}(x)\ \pi(\eta)d\eta} \tag{3.2}$$

The right side of (3.2) can be written as

$$\int \|\nabla j^2(s,\emptyset)\| \left(\int_s^{\infty} r^{m-1}\|r\emptyset - \delta_{\pi}\| p_{(r,\emptyset)}(x)\pi(r\emptyset)dr\right)ds\,d\emptyset. \tag{3.3}$$

Therefore,

$$\|\mathrm{Cov}_{\pi}^{X}(j^2,\eta)\| \leq 2\int \|\nabla j(s,\emptyset)\|\, j(s,\emptyset)$$

$$\frac{\left(\int_s^{\infty} \|r\emptyset - \delta_{\pi}\|\, r^{m-1}\, p_{r\emptyset}(x)\pi(r\emptyset)dr\right)p(s\emptyset/x)}{p(s\emptyset/x)\int p_{\theta}(x)\pi(\theta)d\theta}\, ds\,d\emptyset \tag{3.4}$$

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where $p(s\phi/x)$ is the posteriori density and according to a convention that will be adopted here $\frac{p^{1/2}(\theta/x)}{p(\theta/x)} = 0$ when $p(\theta/x) = 0$. Moreover, note that the bracketed quantity in (3.4) is s^{m-1} $M(x, s\phi)$ and it is well defined in view of the assumption (I). Multiplying and dividing the integrand in (3.4) by s^{m-1} and applying Schwartz inequality we have

$$||\mathrm{Cov}_\pi^X (j^2,\eta)||^2 \leq 4\int ||\nabla j(s,\phi)||^2 s^{m-1}[M(x,s\phi)]^2 p(s\phi/x)ds d\phi$$

$$\int j^2(s,\phi)s^{m-1} p(s\phi/x)dsd\phi.$$

$$\frac{||\mathrm{Cov}_\pi^X (j^2,\eta)||^2}{E_\pi^X (j^2)} \leq \frac{4}{E_\pi^X (j^2)} \int ||\nabla j(t)||^2 [M(x,t)]^2 p(t/x)dt.$$

$$\cdot \int j^2(t)p(t/x)dt.$$

$$= 4 \int ||\nabla j(t)||^2 [M(x,t)]^2 p(t/x)dt \qquad (3.5)$$

since $E_\pi^X(j^2) = \int j^2(\theta)p(\theta/x)d\theta$.

Hence,

$$E\left(\frac{||\mathrm{Cov}_\pi^X(j^2,\eta)||^2}{E_\pi^X (j^2)}\right) \leq 4 \cdot \int ||\nabla j(t)||^2 h(t)\pi(t)dt \qquad (3.6)$$

since $E[M^2(x,t)\, P(t/x)] = \int M^2(x,t) \cdot \pi(t)p_t(x)d\mu(x)$.

Therefore, if for every $\epsilon > 0$ and $r > 0$, there exists i such that

$$\int ||\nabla j(\theta)||^2 h(\theta)\pi(\theta) d\theta < \epsilon$$

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and $j \geq 1$ on S_r then δ_π is admissible by Theorem 1.1. This implies that δ_π is almost admissible if

$$\inf_{j \in \bar{J}_r} \int \|\nabla j(\theta)\|^2 \pi(\theta) h(\theta) d\theta = 0 \qquad (3.7)$$

where $\bar{J}_r = \{ j : j$ is continuously differentiable, $j = 1$ on S_r and j vanishes outside a compact set $\}$.

We have already seen in section 2 (Theorem 2.1) the above calculus of variation problem is related to BPII for

$L_\pi u = \Delta u + (\frac{\nabla \pi}{\pi} + \frac{\nabla h}{h}) \nabla u = 0$ and (3.7) holds if BPII is solvable for $L_\pi u = 0$. This comoletes the proof of the theorem.

q.e.d.

In addition to assumptions (I) and (II) if $h(\theta)$ is bounded then we have the following result.

<u>Theorem 3.2:</u> Assume (I) and (II). Furthermore if $h(\theta)$ is bounded then a sufficient condition for almost admissibility of δ_π is that BPII be solvable for $Lu = \Delta u + \frac{\nabla \pi}{\pi} \cdot \nabla u = 0$.

Proof: Observe that if $h(\theta) < M$ for some $M > 0$, then it follows from the proof of Theorem 3.1

$$E \frac{\|Cov_\pi^X(j^2, \eta)\|^2}{E_\pi^X(j^2)} \leq 4 \int \|\nabla j(\theta)\|^2 h(\theta) \pi(\theta) d\theta$$

$$< 4M \int \|\nabla j(\theta)\|^2 \pi(\theta) d\theta \quad . \qquad (3.8)$$

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Now, it follows from Theorem 3.1 of Chapter I, that

$$\inf_{j \in \bar{J}_r} \int \| \nabla j(\theta) \|^2 \pi(\theta) d\theta = 0$$

if BPII is solvable for $Lu = \triangle u + \frac{\nabla \pi}{\pi} \nabla u = 0$.

Combining this fact with (3.8) we have the Theorem. q.e.d.

We can use the above theorem to get a sufficient condition for the admissibility of generalized Bayes estimators of the mean of an exponential family even if the natural parameter space is not the whole of E^m. This is shown below.

Let μ be such that its support is not contained in any lower dimensional set. Consider the exponential family of distributions $dP_\eta(x) = \beta(\eta) e^{\eta x} d\mu(x)$ where $\eta \in \mathfrak{N} = \{\eta: \beta^{-1}(\eta) = \int e^{x\eta} d\mu(x) < \infty\}$. Let $\theta(\eta) = \frac{-\nabla\beta(\eta)}{\beta(\eta)}$ be the mean vector. Since the support of μ is not contained in a lower dimensional set, the second derivative matrix of $\log \beta(\eta)$ is positive definite. Therefore, the mapping $\eta \to \theta(\eta)$ is 1-1 and smooth (i.e. continuously differentiable.) Let ψ be the function such that $\psi(\theta(\eta)) = \eta$. Using ψ, we can transform the underlying the probability distributions in to another exponential family with parameter θ as follows:

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$$\beta(\eta)e^{\eta x}d\mu(x) = \beta(\psi(\theta))e^{\psi(\theta)\cdot x}d\mu(x) = dQ_{\theta}(x) .$$

Therefore estimating the mean of $\{P_\eta\}$ is same as estimating the parameter θ of $\{Q_\theta\}$. Moreover, any prior measure Π on $\mathcal{N}$ can be carried over to Θ. Therefore, any generalized Bayes estimator of $\frac{-\nabla\beta(\eta)}{\beta(\eta)}$ is generalized Bayes estimator for θ. Now, suppose $\mathcal{N}$ is an open set. Then it is easy to show that $\Theta = E^m$ Therefore, we can use Theorem 3.1 to get a sufficient condition for the admissibility of the generalized Bayes estimators of $\frac{-\nabla\beta(\eta)}{\beta(\eta)}$ if $\mathcal{N}$ is an open convex set.

We end this chapter with a few comments on Theorem 3.6. Zidek [1] gave a sufficient condition for almost admissibility when $m = 1$ by relating the problem to a calculus of variation problem. Our theorem is a generalization of Zidek's Theorem to m-dimensions. However, our assumption II is more stringent than Zidek's for $m = 1$. This is because we relates the admissibility problem with the calculus of variation problem (step (3.5)) and obtains a condition for (3.6) whereas we go a step-further and relate it to the exterior boundary value problem consequently, it is easier to verify his conditions that ours.

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