An identity for the joint distribution of order statistics and its applications

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Received 12 May 1993; revised 21 April 1994

Abstract

It is shown that the joint distribution function of several order statistics can be expressed in terms of the distribution functions of the maximum (or the minimum) order statistics of suitable subsamples. The result is used to derive explicit expressions for the expectations of functions of order statistics from certain families of distributions which include the exponential distribution and the power function distribution. The results generalize earlier work by the authors for a single order statistic.

AMS classification: 62G30

Keywords: Order statistics; Product moments; Exponential distribution; Power function distribution

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be independent random variables with distribution functions $F_1, F_2, \ldots, F_n$ respectively and let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics.

We set $N = \{1, 2, \ldots, n\}$. If $S \subset N$ then S' will denote the complement of S in N and $|S|$ will denote the cardinality of S. The rth order statistic for the set $\{X_i | i \in S\}$ is denoted by $X_{r:S}$. The distribution function of $X_{r:S}$ is denoted by $H_{r:S}(x)$. When there is no possibility of confusion we will write $X_{r:|S|}$ instead of $X_{r:S}$ and $H_{r:|S|}$ instead of $H_{r:S}$.

The distribution of the rth order statistic $X_{r:n}$ can be expressed as a linear combination of the distribution functions of the maximum (or the minimum) order statistic for various subsets of $\{X_1, X_2, \ldots, X_n\}$. In the jargon of reliability theory this simply means that the reliability of an 'r out-of n' system can be expressed as a linear combination of the reliabilities of all possible series (or parallel) systems formed out of the n components. We state the result next; for a proof see Balasubramanian et al. (1991).

Theorem 1. *For* $1 \leqslant r \leqslant n$ *and* $n \geqslant 2$:

(i) $H_{r:n}(x) = \sum_{j=0}^{n-r} \sum_{S \subset N, |S|=n-j} (-1)^{r-|S|} \binom{|S|-1}{|S|-r} H_{|S|:S}(x).$

(ii) $H_{r:n}(x) = \sum_{j=0}^{r-1} \sum_{S \subset N, |S|=n-j} (-1)^{n-r+1-|S|} \binom{|S|-1}{n-r} H_{1:S}(x).$

Theorem 1 also follows from the well-known work of Feller (1968). One can obtain the distribution function of $X_{r:n}$ using (5.2) of Ch. 4 there to obtain (i) of Theorem 1. The motivation behind obtaining the result in Balasubramanian et al. (1991) was to use it to get explicit expressions for the moments of functions of order statistics for distributions such as the exponential, for which the minimum order statistic is again exponential; or the power function distribution, for which the maximum order statistic again has a power function distribution. The purpose of this paper is to obtain a generalization of Theorem 1 to the case of the joint distribution function of several order statistics. We will show that the joint distribution function of $X_{r_1:n} < X_{r_2:n} < \cdots < X_{r_k:n}$ where $r_1 < r_2 < \cdots < r_k$ at $x_1 < x_2 < \cdots < x_k$ can be expressed as a linear combination of terms of the type

$$H_{|S_1|:S_1}(x_1) H_{|S_2|:S_2}(x_2) \cdots H_{|S_k|:S_k}(x_k) \tag{1}$$

or of the type

$$H_{1:S_1}(x_k) H_{1:S_2}(x_{k-1}) \cdots H_{1:S_k}(x_1) \tag{2}$$

where $S_1, S_2, \ldots, S_k$ are pairwise disjoint nonempty subsets of N. We indicate how the result may be used to get explicit expressions for the product moments of functions of order statistics from distributions such as the exponential or the power function distribution.

2. The main result

Let $1 \leqslant r_1 < r_2 < \cdots < r_k \leqslant n$ and let $H_{r_1,r_2,\ldots,r_k:n}(x_1,x_2,\ldots,x_k)$ denote the joint distribution function of $X_{r_1}, X_{r_2}, \ldots, X_{r_k}$.

Theorem 2. *For* $1 \leqslant r_1 < r_2 < \cdots < r_k \leqslant n$; $k \leqslant n$ *we have*

$$H_{r_1,r_2,\ldots,r_k:n}(x_1,x_2,\ldots,x_k) = \sum (-1)^{\sum_{i=1}^{k}(r_i - i|S_i|)} \prod_{i=1}^{k} \binom{|S_i|-1}{\sum_{j=1}^{i}|S_j| - r_i} H_{|S_i|:S_i}(x_i),$$

for any $x_1 < x_2 < \cdots < x_k$; *where the summation is over all collections* $\{S_1, S_2, \ldots, S_k\}$ *of pairwise disjoint nonempty subsets of* N.

Note that in Theorem 2 as well as in the rest of the paper we take the binomial coefficient $\binom{p}{q}$ to be zero whenever $q > p$ or $q < 0$.

We can give a result similar to Theorem 2 using the minimum order statistics in various subsets. We give the result next.

Theorem 3. *For* $1 \leqslant r_1 < r_2 < \cdots < r_k \leqslant n$; $k \leqslant n$ *we have*

$$H_{r_1,r_2,\ldots,r_k:n}(x_1,x_2,\ldots,x_k) = \sum (-1)^{\sum_{i=1}^{k}\{n-r_i+1-i|S_i|\}} \prod_{i=1}^{k} \binom{|S_i|-1}{n-\sum_{j=1}^{i-1}|S_j|-r_{k+1-i}} H_{1:S_{k+1-i}}(x_i),$$

for any $x_1 < x_2 < \cdots < x_k$; *where the summation is over all possible collections* $\{S_1,\ldots,S_k\}$ *of pairwise disjoint nonempty subsets of* N.

Allowing $x_1 \to \infty, x_2 \to \infty, \ldots, x_k \to \infty$, *Theorems 2 and 3 give*

$$\sum (-1)^{\sum_{i=1}^{k}\{r_i-i|S_i|\}} \prod_{i=1}^{k} \binom{|S_i|-1}{\sum_{j=1}^{i}|S_j|-r_i} = 1$$

and

$$\sum (-1)^{\sum_{i=1}^{k}\{n-r_i+1-i|S_i|\}} \prod_{i=1}^{k} \binom{|S_i|-1}{n-\sum_{j=1}^{i-1}|S_j|-r_{k+1-i}} = 1$$

respectively. These combinatorial identities may be of independent interest.

The proof of Theorem 2 is deferred to Section 4. The proof of Theorem 3 is similar.

3. Expectation of functions of order statistics

In this section we make use of the identities of Section 2 to obtain expressions for expectation of functions of order statistics.

Suppose the random variable X has an arbitrary distribution function $F(x)$. Define the following two families of distribution functions with a positive parameter λ.

Family I: $F^{\lambda}(x) = [F(x)]^{\lambda}, \quad \lambda > 0.$

Family II: $F_{\lambda}(x) = 1 - [1 - F(x)]^{\lambda}.$

In the survival analysis literature this family is known as the proportional hazard rate model.

Let $X^{(\lambda)}$ have distribution function $F^{\lambda}(x)$. Let $X_1, X_2, \ldots, X_n$ be independently distributed as $X^{(\lambda_1)}, X^{(\lambda_2)}, \ldots, X^{(\lambda_n)}$ respectively. Then

$$\begin{aligned} H_{|S|:S}(x) &= \prod_{i\in S} F^{\lambda_i}(x) \\ &= \prod_{i\in S} [F(x)]^{\lambda_i} \\ &= [F(x)]^{\lambda_S} = F^{\lambda_S}(x), \end{aligned}$$

where

$$\lambda_S = \sum_{i \in S} \lambda_i.$$

Similarly, let $X_{(\lambda)}$ have distribution function $F_\lambda(x)$. If $X_1, X_2, \ldots, X_n$ are independently distributed as $X_{(\lambda_1)}, X_{(\lambda_2)}, \ldots, X_{(\lambda_n)}$ respectively, then

$$\begin{aligned} H_{1:S}(x) &= 1 - \prod_{i \in S}[1 - F_{\lambda_i}(x)] \\ &= 1 - \prod_{i \in S}[1 - F(x)]^{\lambda_i} \\ &= 1 - [1 - F(x)]^{\lambda_S} = F_{\lambda_S}(x), \end{aligned}$$

where

$$\lambda_S = \sum_{i \in S} \lambda_i.$$

It follows from Theorems 2 and 3 that

$$\begin{aligned} &E\{g(X_{r_1:n}, \ldots, X_{r_k:n})\} \\ &= \sum (-1)^{\sum_{i=1}^{k}(r_i - i|S_i|)} \prod_{i=1}^{k} \binom{|S_i| - 1}{\sum_{j=1}^{i}|S_j| - r_i} g^*(\lambda_{S_1}, \ldots, \lambda_{S_k}) \end{aligned} \tag{3}$$

and

$$\begin{aligned} &E\{g(X_{r_1:n}, \ldots, X_{r_k:n})\} \\ &= \sum (-1)^{\sum_{i=1}^{k}(n - r_i + 1 - i|S_i|)} \prod_{i=1}^{k} \binom{|S_i| - 1}{n - \sum_{j=1}^{i-1}|S_j| - r_{k+1-i}} g_*(\lambda_{S_1}, \ldots, \lambda_{S_k}), \end{aligned} \tag{4}$$

where the summation is over pairwise disjoint subsets $S_1, S_2, \ldots, S_k$ of N,

$$\begin{aligned} g^*(\lambda_1, \ldots, \lambda_k) = {} & E\{g(X^{(\lambda_1)}, \ldots, X^{(\lambda_k)}) | X^{(\lambda_1)} < X^{(\lambda_2)} < \cdots < X^{(\lambda_k)}\} \\ & \times \Pr(X^{(\lambda_1)} < X^{(\lambda_2)} < \cdots < X^{(\lambda_k)}) \end{aligned}$$

and

$$\begin{aligned} g_*(\lambda_1, \ldots, \lambda_k) = {} & E\{g(X_{(\lambda_1)}, \ldots, X_{(\lambda_k)}) | X_{(\lambda_1)} < X_{(\lambda_2)} < \cdots < X_{(\lambda_k)}\} \\ & \times \Pr(X_{(\lambda_1)} < X_{(\lambda_2)} < \cdots < X_{(\lambda_k)}). \end{aligned}$$

Here g^* and g_* are assumed to exist and to be finite.

An example of Family I

1. Consider

$$F(x) = \left(\frac{x}{\theta}\right), \quad 0 \leqslant x \leqslant \theta; \;\; \theta > 0,$$

then

$$F^{\lambda}(x) = \left(\frac{x}{\theta}\right)^{\lambda}, \quad 0 \leqslant x \leqslant \theta; \; \theta, \lambda > 0,$$

which is a power function distribution.

If

$$g(x_1, \ldots, x_k) = \prod_{i=1}^{k} x_i^{\alpha_i}, \quad \alpha_i \geqslant 0, \; i = 1, 2, \ldots, k,$$

then

$$g^*(\lambda_1, \ldots, \lambda_k) = \underbrace{\int \cdots \int}_{0 \leqslant x_1 < \cdots < x_k \leqslant \theta} \prod_{i=1}^{k} \left\{ \left(\frac{\lambda_i}{\theta}\right) x_i^{\alpha_i} \left(\frac{x_i}{\theta}\right)^{\lambda_i - 1} \right\} dx_i$$

$$= \prod_{i=1}^{k} \left\{ \frac{\lambda_i \theta^{\alpha_i}}{\sum_{j=1}^{i} (\alpha_j + \lambda_j)} \right\}.$$

Hence from (3), we get

$$E\left\{\prod_{i=1}^{k} X_{r_i:n}^{\alpha_i}\right\}$$

$$= \sum (-1)^{\sum_{i=1}^{k} \{r_i - i|S_i|\}} \prod_{i=1}^{k} \binom{|S_i| - 1}{\sum_{j=1}^{i} |S_j| - r_i} \prod_{i=1}^{k} \left\{ \frac{\lambda_{S_i} \theta^{\alpha_i}}{\sum_{j=1}^{i} (\alpha_j + \lambda_{S_j})} \right\}, \tag{5}$$

where the summation is over pairwise disjoint subsets $S_1, S_2, \ldots, S_k$ of N.

An example of Family II

1. Consider

$$F(x) = 1 - \exp\{-x\}, \quad x \geqslant 0,$$

then

$$F_{\lambda}(x) = 1 - \exp\{-\lambda x\}, \quad x \geqslant 0, \; \lambda > 0,$$

which is an exponential distribution.

If

$$g(x_1, \ldots, x_k) = \exp\left\{\sum_{i=1}^{k} t_i x_i\right\},$$

then

$$g_*(\lambda_1, \ldots, \lambda_k) = \underbrace{\int \cdots \int}_{0 \leqslant x_1 < \cdots < x_k \leqslant \infty} \exp\left\{\sum_{i=1}^{k} t_i x_i\right\} \prod_{i=1}^{k} \{\lambda_i \exp\{-\lambda_i x_i\}\} \mathrm{d}x_i.$$

Making the transformation $x_1 = u_1, x_2 = u_1 + u_2, \ldots, x_k = u_1 + u_2 + \cdots + u_k$, we have

$$g_*(\lambda_1, \ldots, \lambda_k)$$
$$= \left\{\prod_{i=1}^{k} \lambda_i\right\} \int_0^\infty \cdots \int_0^\infty \exp - \left\{\left(\sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} t_i\right) u_1 + \left(\sum_{i=2}^{k} \lambda_i - \sum_{i=2}^{k} t_i\right) u_2 + \cdots + (\lambda_k - t_k) u_k\right\} \mathrm{d}u_1\, \mathrm{d}u_2 \cdots \mathrm{d}u_k$$
$$= \frac{\prod_{i=1}^{k} \lambda_i}{\sum_{i=1}^{k}(\lambda_i - t_i)\sum_{i=2}^{k}(\lambda_i - t_i)\cdots(\lambda_k - t_k)}.$$

Hence from (4) the joint moment generating function of $X_{r_1:n}, \ldots, X_{r_k:n}$ is given by

$$\Psi(t_1, \ldots, t_k)$$
$$= \sum (-1)^{\sum_{i=1}^{k}\{n - r_i + 1 - i|S_i|\}} \prod_{i=1}^{k} \binom{|S_i| - 1}{n - \sum_{j=1}^{i-1}|S_j| - r_{k+1-i}}$$
$$\times \frac{\prod_{i=1}^{k} \lambda_{S_i}}{\sum_{i=1}^{k}(\lambda_{S_i} - t_i)\sum_{i=2}^{k}(\lambda_{S_i} - t_i)\cdots(\lambda_{S_k} - t_k)}, \tag{6}$$

where the summation is over pairwise disjoint subsets $S_1, S_2, \ldots, S_k$ of N.

Expression (6) is similar to the joint moment generating function of all order statistics derived in Bapat and Beg (1989).

In particular for $k = 2$, (6) reduces to

$$\Psi(t_1, t_2)$$
$$= \sum (-1)^{\sum_{i=1}^{2}\{n - r_i + 1 - i|S_i|\}} \prod_{i=1}^{2} \binom{|S_i| - 1}{n - \sum_{j=1}^{i-1}|S_j| - r_{3-i}}$$
$$\times \frac{\lambda_{S_1}}{\lambda_{S_1} + \lambda_{S_2}} \left[1 - \left(\frac{t_1 + t_2}{\lambda_{S_1} + \lambda_{S_2}}\right)\right]^{-1} \left[1 - \frac{t_2}{\lambda_{S_2}}\right]^{-1}.$$

It is easy to see that

$$\Psi(t_1, t_2)$$
$$= \sum (-1)^{\sum_{i=1}^{2}\{n - r_i + 1 - i|S_i|\}} \prod_{i=1}^{2} \binom{|S_i| - 1}{n - \sum_{j=1}^{i-1}|S_j| - r_{3-i}}$$
$$\times \frac{\lambda_{S_1}}{\lambda_{S_1} + \lambda_{S_2}} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i} \binom{i}{k} \frac{t_2^{j+i-k} t_1^k}{\lambda_{S_2}^j (\lambda_{S_1} + \lambda_{S_2})^i}$$

and hence

$$
\begin{aligned}
&E\{X_{r_1:n}^{\alpha_1}X_{r_2:n}^{\alpha_2}\} \\
&= \sum(-1)^{\sum_{i=1}^{2}\{n-r_i+1-i|S_i|\}}\prod_{i=1}^{2}\binom{|S_i|-1}{n-\sum_{j=1}^{i-1}|S_j|-r_{3-i}} \\
&\quad\times\frac{\alpha_1!\alpha_2!\lambda_{S_1}}{\lambda_{S_1}+\lambda_{S_2}}\sum_{i=0}^{\alpha_2}\binom{\alpha_1+i}{\alpha_1}\frac{1}{\lambda_{S_2}^{\alpha_2-i}(\lambda_{S_1}+\lambda_{S_2})^{\alpha_1+i}}.
\end{aligned}
$$

For $\alpha_1 = \alpha_2 = 1$, the above reduces to

$$
\begin{aligned}
&E\{X_{r_1:n}X_{r_2:n}\} \\
&= \sum(-1)^{\sum_{i=1}^{2}\{n-r_i-1-i|S_i|\}}\prod_{i=1}^{2}\binom{|S_i|-1}{n-\sum_{j=1}^{i-1}|S_j|-r_{3-i}} \\
&\quad\times\frac{\lambda_{S_1}(\lambda_{S_1}+3\lambda_{S_2})}{\lambda_{S_2}(\lambda_{S_1}+\lambda_{S_2})^3},
\end{aligned}
$$

where the summation is over pairwise disjoint subsets S_1, S_2 of N.

4. Proof of the main result

We will need the following result.

Lemma 1. *Let* $f(t) = a_0 + a_1t + \cdots + a_nt^n$ *and let* $b_r = \sum_{i=r}^{n} a_i$, $r = 0, 1, \ldots, n$. *Then the coefficient of* t^r *in* $tf(t)/t-1$, $|t| > 1$, *is* b_r.

Proof. It is easily verified that

$$tf(t) - b_0 = (t-1)(b_0 + b_1t + \cdots + b_nt^n)$$

and therefore, when $|t| > 1$,

$$
\begin{aligned}
\frac{tf(t)}{t-1} &= b_0 + b_1t + \cdots + b_nt^n + \frac{b_0}{t-1} \\
&= b_0 + b_1t + \cdots + b_nt^n + b_0\left\{\frac{1}{t} + \frac{1}{t^2} + \cdots\right\}
\end{aligned}
$$

and the result follows. □

We now introduce some notation.

Let A_{ij}, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k$ be events such that $A_{i1} \subset A_{i2} \subset \cdots \subset A_{ik}$ for every i. For $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k \leqslant n$, let

$$P_{[\alpha_1,\ldots,\alpha_k]} = Pr\{\text{exactly}\,\alpha_j \quad A_{ij}'s \text{ occur}, \; j = 1, 2, \ldots, k\}.$$

Note that since $A_{ir} \subset A_{\overline{ir+1}}$, $r = 1, 2, \ldots, k-1$, the condition $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k$ is necessary for $P_{[\alpha_1,\ldots,\alpha_k]}$ to be positive.

Let

$$P^*_{[\alpha_1,\ldots,\alpha_k]} = \Pr\{\text{at least } \alpha_j\ A'_{ij}s \text{ occur},\ j = 1,2,\cdots,k\}$$
$$= \sum_{i_1 \geqslant \alpha_1} \cdots \sum_{i_k \geqslant \alpha_k} P_{[i_1,i_2,\ldots,i_k]}. \tag{7}$$

Define

$$S_{\beta_1,\ldots,\beta_k} = \sum \Pr\{\cap_{j=1}^k \cap_{i \in I_j} A_{ij}\},$$

where the summation is over $I_1, I_2, \ldots, I_k$, disjoint subsets of $\{1,2,\ldots,n\}$, with cardinalities $\beta_1, \beta_2, \ldots, \beta_k$ respectively.

We wish to express $P^*_{[\alpha_1,\ldots,\alpha_k]}$ in terms of $S_{\beta_1,\ldots,\beta_k}$.

Consider

$$G(t_1,\ldots,t_k) = E\prod_{i=1}^n \{t_1\chi_{A_{i1}} + t_2(\chi_{A_{i2}} - \chi_{A_{i1}}) + \cdots + t_k(\chi_{A_{ik}} - \chi_{A_{\overline{ik-1}}}) + \chi_{A_{ik}}\}$$
$$= \sum_{j_1}\sum_{j_2}\cdots\sum_{j_k} t_1^{j_1} t_2^{j_2} \cdots t_k^{j_k} P_{[j_1, j_1+j_2, \ldots, j_1+j_2+\cdots+j_k]},$$

where χ_A is an indicator function. Changing $t_1 = u_1u_2\cdots u_k$, $t_2 = u_2u_3\cdots u_k, \ldots, t_k = u_k$, we get

$$G(t_1,\ldots,t_k) = H(u_1,\ldots,u_k)$$
$$= \sum_{j_1}\sum_{j_2}\cdots\sum_{j_k} u_1^{j_1} u_2^{j_1+j_2} \cdots u_k^{j_1+\cdots+j_k} P_{[j_1, j_1+j_2, \ldots, j_1+j_2+\cdots+j_k]}$$
$$= \sum_{\alpha_1}\sum_{\alpha_2}\cdots\sum_{\alpha_k} u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_k^{\alpha_k} P_{[\alpha_1,\ldots,\alpha_k]},$$

where the last summation is over $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k$. We have by (7) and by Lemma 1,

$$P^*_{[\alpha_1,\ldots,\alpha_k]} = \text{coefficient of } u_1^{\alpha_1}\cdots u_k^{\alpha_k} \text{ in}$$
$$\frac{u_1\cdots u_k}{(u_1-1)\cdots(u_k-1)} H(u_1,\ldots,u_k). \tag{8}$$

Now

$$G(t_1,\ldots,t_k) = E\prod_{i=1}^n \{(t_1-t_2)\chi_{A_{i1}} + (t_2-t_3)\chi_{A_{i2}} + \cdots + (t_{k-1}-t_k)\chi_{A_{\overline{ik-1}}}$$
$$+ (t_k-1)\chi_{A_{ik}} + 1\}$$
$$= \sum_{j_1}\cdots\sum_{j_k} (t_1-t_2)^{j_1}\cdots(t_k-1)^{j_k} S_{j_1,j_2,\ldots,j_k}$$

$$= \sum_{j_1} \cdots \sum_{j_k} (u_1 - 1)^{j_1} (u_2 - 1)^{j_2} \cdots (u_k - 1)^{j_k}$$
$$\times u_2^{j_1} u_3^{j_1+j_2} \cdots u_k^{j_1+\dots+j_{k-1}} S_{j_1, j_2, \cdots, j_k}. \tag{9}$$

Since $G(t_1, \dots, t_k) = H(u_1, \dots, u_k)$ we get from (7), (8) and (9) that

$$P^*_{[\alpha_1,\dots,\alpha_k]} = \text{coefficient of } u_1^{\alpha_1} \cdots u_k^{\alpha_k} \text{ in (9)}$$
$$= \sum_{j_1} \cdots \sum_{j_k} \binom{j_1 - 1}{\alpha_1 - 1} \binom{j_2 - 1}{\alpha_2 - j_1 - 1} \binom{j_3 - 1}{\alpha_3 - j_1 - j_2 - 1}$$
$$\times \cdots \times \binom{j_k - 1}{\alpha_k - j_1 - j_2 - \cdots - j_{k-1} - 1} (-1)^{\sum_{i=1}^k j_i - \sum_{i=1}^k \alpha_i} S_{j_1, j_2, \dots, j_k} \tag{10}$$

Identity (10) is a generalization of Eq. (5.2) of Feller (1968, p. 109).

In order to obtain Theorem 2 from (10) we set $A_{ij} = \{X_i < x_j\}$, where $x_1 < x_2 < \cdots < x_k$. We make a rearrangement of the terms and write $j_1, j_2, \dots, j_k$ in terms of $|S_1|, |S_2|, \dots, |S_k|$.

Acknowledgements

The authors would like to thank the referees for some useful comments which led to a more general form of the main result.

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