SOME INEQUALITIES FOR NORMS OF COMMUTATORS*

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Abstract. Let A, B be positive operators and let f be any operator monotone function. We obtain inequalities for $|||f(A)X - Xf(B)|||$ in terms of $|||f(|AX - XB|)|||$ for every unitarily invariant norm. The case $X = I$ was considered by T. Ando [*Math. Z.*, 197 (1988), pp. 403–409], and some of our results reduce to his results in this special case. Some related inequalities are obtained.

Key words. operator monotone functions, unitarily invariant norms, singular values, commutators

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1. Introduction. The aim of this paper is to present commutator versions of some perturbation inequalities proved by Ando [1] and by Jocic and Kittaneh [4]. For simplicity, we will state our results first for $n \times n$ matrices and then point out the small modifications needed to extend their validity to operators on a Hilbert space.

Let A, B be positive (semidefinite) matrices, f any nonnegative operator monotone function on $[0, \infty)$, and $|||\cdot|||$ any unitarily invariant norm. Then we have the following inequality due to Ando [1]:

$$|||f(A) - f(B)||| \leq |||f(|A - B|)|||. \tag{1}$$

Here, $|X|$ denotes $(X^*X)^{1/2}$.

Our first theorem is the following extension of this result.

THEOREM 1. *Let A, B be positive matrices. Let X be any matrix and let $s_j(X), 1 \leq j \leq n$ be the decreasingly ordered singular values of X. Then for every nonnegative operator monotone function f and for every unitarily invariant norm we have*

$$|||f(A)X - Xf(B)||| \leq \frac{1 + s_1^2(X)}{2} \left|\left|\left| f\left(\frac{2}{1 + s_n^2(X)} |AX - XB|\right)\right|\right|\right|. \tag{2}$$

After this we prove another inequality, which implies the following.

THEOREM 2. *Let A, B be positive matrices and let X be any contraction* (*i.e.*, $||X|| := s_1(X) \leq 1$). *Then for every nonnegative operator monotone function f and for every unitarily invariant norm we have*

$$|||f(A)X - Xf(B)||| \leq \frac{5}{4} |||f(|AX - XB|)|||. \tag{3}$$

For the special case of the operator norm $||\cdot||$ and the power functions $f(t) = t^r$, $0 < r \leq 1$ the inequality (3) has been proven by Pedersen [8].

Note that while the choice $X = I$ reduces the inequality (2) to (1) the same is not the case with (3). It is an interesting open question to decide whether the constant

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5/4 occurring here could be replaced by 1. We show that for 2×2 matrices this can indeed be done.

Section 2 of this paper contains the proofs of these results, several related inequalities, and some remarks. We then obtain extensions, in the same spirit, of the following result from [4]: If A, B are Hermitian, then for every positive integer m

$$\left|\left|\left|(A-B)^{2m+1}\right|\right|\right| \leq 2^{2m} \left|\left|\left|A^{2m+1} - B^{2m+1}\right|\right|\right|. \tag{4}$$

The extension we obtain is the following.

THEOREM 3. *Let A, B be Hermitian and let X be any matrix. Then for every positive integer m and for every unitarily invariant norm*

$$||| \, |AX - XB|^{2m+1} ||| \leq \frac{\left(1 + s_1^2(X)\right)^{2m+1}}{1 + s_n^2(X)} |||A^{2m+1}X - XB^{2m+1}|||. \tag{5}$$

If X is a contraction we have

$$||| \, |AX - XB|^{2m+1} \, ||| \leq 2^{2m} \left(\frac{5}{4}\right)^{2m+1} |||A^{2m+1}X - XB^{2m+1}|||. \tag{6}$$

2. Proofs and remarks. We will use standard facts about unitarily invariant norms and singular values (see, e.g., [3]) and about operator monotone functions [9]. Recall that if f is a nonnegative operator monotone function on $[0, \infty)$ then it has an integral representation

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{\lambda t}{\lambda + t} d\mu(\lambda), \tag{7}$$

where $\alpha, \beta \geq 0$ and μ is a positive measure. We will repeatedly use the identity

$$f(UAU^*) = Uf(A)U^*, \tag{8}$$

valid for all unitary operators U, Hermitian operators A, and functions f whose domain contains the spectrum of A. (In the infinite-dimensional case $f(A)$ is defined via the spectral theorem for all *measurable* functions f. The representation (7) shows that operator monotone functions are infinitely differentiable.)

LEMMA 4. *For every positive A, unitary U, and nonnegative operator monotone function f on $[0, \infty)$ we have*

$$|||f(A)U - Uf(A)||| \leq |||f(|AU - UA|)|||. \tag{9}$$

Proof. Using the unitary invariance of $|||\cdot|||$, the relation (8), and the inequality (1) we have

$$\begin{aligned} |||f(A)U - Uf(A)||| &= |||f(A) - Uf(A)U^*||| \\ &= |||f(A) - f(UAU^*)||| \\ &\leq |||f(|A - UAU^*|)||| \\ &= |||f(|AU - UA|)|||. \qquad \square \end{aligned}$$

LEMMA 5. *Let X, Y, Z be any three matrices. Then*

$$|||f(|XYZ|)||| \leq |||f(\|X\| \, \|Z\| \, |Y|)||| \tag{10}$$

for any monotone increasing function f on $[0,\infty)$.

Proof. It is an easy consequence of the min–max principle that

$$s_j(XYZ) \le \|X\|\ \|Z\|\ s_j(Y) \quad \text{for all } j.$$

Hence,

$$\begin{aligned} s_j\left(f(|XYZ|)\right) &= f\left(s_j(XYZ)\right) \\ &\le f\left(\|X\|\ \|Z\|\ s_j(Y)\right) \\ &= s_j\left(f(\|X\|\ \|Z\|\ |Y|)\right). \end{aligned}$$

This is more than adequate to ensure (10). □

The special case $A = B$, $X = X^*$. We will first prove the inequality (2) in this special case. Let

$$U = (X - i)\,(X + i)^{-1} \tag{11}$$

be the Cayley transform of X; U is unitary and its spectrum does not contain the point 1. We have

$$X = i(1 + U)(1 - U)^{-1} = 2i(1 - U)^{-1} - i. \tag{12}$$

So, we can write

$$\begin{aligned} &|||f(A)X - Xf(A)||| \\ &= \left|\left|\left|f(A)\left(2i(1-U)^{-1} - i\right) - \left(2i(1-U)^{-1} - i\right)f(A)\right|\right|\right| \\ &= 2\left|\left|\left|f(A)(1-U)^{-1} - (1-U)^{-1}f(A)\right|\right|\right| \\ &= 2\left|\left|\left|(1-U)^{-1}\left(f(A)U - Uf(A)\right)(1-U)^{-1}\right|\right|\right| \\ &\le 2\left\|(1-U)^{-1}\right\|^2 |||f(A)U - Uf(A)||| \\ &\le 2\left\|(1-U)^{-1}\right\|^2 |||f\left(|AU - UA|\right)|||, \end{aligned} \tag{13}$$

using Lemma 4. Now use (12) to obtain

$$\|(1 - U)^{-1}\|^2 = \left\|\frac{X + i}{2}\right\|^2 = \frac{1 + s_1^2(X)}{4}. \tag{14}$$

Also note that

$$\begin{aligned} |||f\left(|AU - UA|\right)||| &= \left|\left|\left|f\left(\left|A\left(1 - 2i(X+i)^{-1}\right) - \left(1 - 2i(X+i)^{-1}\right)A\right|\right)\right|\right|\right| \\ &= \left|\left|\left|f\left(2\left|(X+i)^{-1}A - A(X+i)^{-1}\right|\right)\right|\right|\right| \\ &= \left|\left|\left|f\left(2\left|(X+i)^{-1}(AX - XA)(X+i)^{-1}\right|\right)\right|\right|\right| \\ &\le \left|\left|\left|f\left(2\|(X+i)^{-1}\|^2\,|AX - XA|\right)\right|\right|\right| \end{aligned} \tag{15}$$

using Lemma 5. Finally, note that

$$\|(X + i)^{-1}\|^2 = \frac{1}{1 + s_n^2(X)}. \tag{16}$$

The proof of (2) in the special case is completed by combining (13), (14), (15), and (16).

Proof of Theorem 1. The general case follows from the special one by a much-used trick. Let

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$

Then C is positive and Y is Hermitian. The singular values of Y are the same as those of X (but each counted twice now). The special case of the theorem applied to C in place of A and Y in place of X leads to the inequality (2). ∎

Proof of Theorem 2. Let t be any nonzero real number. Then the inequality (2) with tX in place of X gives

$$|||f(A)X - Xf(B)||| \;\leq\; \frac{1+t^2 s_1^2(X)}{2|t|} \left|\left|\left| f\left(\frac{2|t|}{1+t^2 s_n^2(X)}\,|AX - XB|\right)\right|\right|\right|. \tag{17}$$

Let $||X|| \leq 1$. Put $t = 1/2$ in (17) to get

$$|||f(A)X - Xf(B)||| \;\leq\; \frac{5}{4}\left|\left|\left| f\left(\frac{4}{4+s_n^2(X)}\,|AX - XB|\right)\right|\right|\right|. \tag{18}$$

Since f is operator monotone, the inequality (3) follows from (18). ∎

Remark 1. With slight modifications, the results above carry over to operators in an infinite-dimensional Hilbert space. We need to replace $s_1(X)$ by $\|X\|$ in (14) and in the subsequent discussion. In (16) we need to replace $s_n(X)$ by $\inf_{||\psi||=1} ||X\psi||$, and in the subsequent discussion we need to replace it by $\inf_{||\psi||=1} ||Y\psi||$, where

$$Y = \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}.$$

Note that $\inf_{||\psi||=1} ||X\psi||$ is equal to zero if X is compact and is equal to $||X^{-1}||^{-1}$ if X is invertible.

Remark 2. In [6], Mathias showed that Ando's inequality (1) is true if f is a nonnegative *matrix monotone function of order* n on $[0, \infty)$. (This means that f is assumed to be order preserving on positive semidefinite matrices of order n only, while an *operator monotone function* is one which is matrix monotone of order n for all n.) Our proof shows that the inequalities (2) and (3) in the special case $A = B$ and $X = X^*$ are true for all functions f that are matrix monotone of order n. The proof for the general case works if f is matrix monotone of order $2n$.

Remark 3. The special case in which $f(t) = t^r$, $0 < r \leq 1$, and the norm is the operator norm has been studied before. In [7] it was shown that for every positive A and for every X

$$\|A^r X - XA^r\| \;\leq\; (1-r)^{r-1}\, ||X||^{1-r}\, \|AX - XA\|^r, \quad 0 < r \leq 1. \tag{19}$$

It was mentioned in that paper that Haagerup showed that the factor $(1-r)^{r-1}$ occurring in (19) could be replaced by $(\sin r\pi)/\pi r(1-r)$. This, and some extensions, were also proven in [2]. Pedersen [8], using arguments like the ones we have used, showed that the factor $(1-r)^{r-1}$ can be replaced by $5/4$. He remarks that for the special case $r = 1/2$ this can be reduced further to $2/\sqrt{\pi}$. In some special situations our inequality (2) can give better results. For example, this is so when $||X\| = 1$ and $s_n(X) > .76$.

Remark 4. For 2×2 matrices, the factor $5/4$ occurring in the inequality (3) can be replaced by 1. To see this, let

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \qquad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}.$$

Then

$$|f(A)X - Xf(A)| = \begin{bmatrix} |(f(a_1) - f(a_2))\, x_{21}| & 0 \\ 0 & |(f(a_1) - f(a_2))\, x_{12}| \end{bmatrix}$$

and

$$f\left(|AX - XA|\right) = \begin{bmatrix} f\left(|(a_1 - a_2)\, x_{21}|\right) & 0 \\ 0 & f\left(|(a_1 - a_2)\, x_{12}|\right) \end{bmatrix}.$$

So, it is enough to show that if $|x| \leq 1$, then

$$|(f(a_1) - f(a_2))\, x| \;\leq\; f\left(|(a_1 - a_2)\, x|\right). \tag{20}$$

It follows from the representation (7) that $cf(t) \leq f(ct)$ for $0 \leq c \leq 1$. So, if $x = ce^{i\theta}$, we have

$$\begin{aligned} |(f(a_1) - f(a_2))\, x| &= c\left|(f(a_1) - f(a_2))\, e^{i\theta}\right| \\ &\leq cf\left(\left|(a_1 - a_2)\, e^{i\theta}\right|\right) \\ &\leq f\left(|(a_1 - a_2)\, x|\right). \end{aligned}$$

Our next proposition shows that if we replace the operator norm with the Hilbert–Schmidt norm, then the first factor on the right-hand side of the inequality (19) can be replaced by 1.

PROPOSITION 6. *Let A, B be positive and let X be any matrix. Then for $0 < r < 1$,*

$$||A^r X - XB^r||_2 \;\leq\; ||X||_2^{1-r}\, \|AX - XB\|_2^r. \tag{21}$$

Proof. As in Theorem 1, the general case follows from the special case $A = B$. Assume, without loss of generality, that A is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then

$$\begin{aligned} ||A^r X - XA^r||_2^2 &= \sum_{i,j} \left|\left(\lambda_i^r - \lambda_j^r\right) x_{ij}\right|^2 \\ &\leq \sum_{i,j} |\lambda_i - \lambda_j|^{2r} |x_{ij}|^2 \\ &= \sum_{i,j} |\lambda_i - \lambda_j|^{2r} |x_{ij}|^{2r} |x_{ij}|^{2(1-r)} \\ &\leq \left(\sum_{i,j} |\lambda_i - \lambda_j|^2 |x_{ij}|^2\right)^r \left(\sum_{i,j} |x_{ij}|^2\right)^{1-r} \\ &= ||AX - XA||_2^{2r} \;\; ||X||_2^{2(1-r)}. \end{aligned}$$

We have used Hölder's inequality to arrive at our last inequality. ∎

The inequality (21) is valid for operators on Hilbert space. Let X be any Hilbert–Schmidt operator and A any positive operator. By a theorem of Weyl and von Neumann [5, p. 525] A can be expressed as a diagonal operator plus a Hilbert–Schmidt operator with arbitrarily small Hilbert–Schmidt norm. So, the same proof gives the inequality (21) in this case as well.

Following the same arguments as Ando [1] we can derive the following generalization of Theorem 2 in that paper.

THEOREM 7. *Let g be an increasing function on $[0,\infty)$ such that $g(0) = 0$, $\lim_{t\to\infty} g(t) = \infty$, and the inverse function of g is operator monotone. Then for all $A, B \geq 0$ and for all X,*

$$\frac{1+s_n^2(X)}{2}\left|\left|\left|g\left(\frac{2}{1+s_1^2(X)}\,|AX-XB|\right)\right|\right|\right| \;\leq\; |||g(A)X-Xg(B)|||. \tag{22}$$

Once again, first replacing X by tX and then making the special choice $t = 1/2$, we get from this

$$\frac{4+s_n^2(X)}{4}\left|\left|\left|g\left(\frac{4}{4+s_1^2(X)}\,|AX-XB|\right)\right|\right|\right| \leq |||g(A)X-Xg(B)|||\,. \tag{23}$$

Since g is monotonically increasing, we obtain from this the following theorem.

THEOREM 8. *Let $A, B \geq 0$ and let X be any operator with $||X|| \leq 1$. Then for every function g satisfying the conditions of Theorem 7 we have*

$$\left|\left|\left|g\left(\frac{4}{5}\,|AX-XB|\right)\right|\right|\right| \;\leq\; |||g(A)X-Xg(B)|||\,. \tag{24}$$

In particular, for every $r \geq 1$ we have

$$|||\,|AX-XB|^r\,||| \leq \left(\frac{5}{4}\right)^r |||A^rX-XB^r\,|||. \tag{25}$$

We remark that should it be possible to replace the factor $5/4$ by 1 in inequality (3), then the same could be done in (24) and (25).

The proof of Theorem 3 is analogous to that of Theorem 1. We leave the details to the reader.

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REFERENCES

[1] T. ANDO, *Comparison of norms $|||f(A)-f(B)|||$ and $|||f(|A-B|)|||$*, Math. Z., 197 (1988), pp. 403–409.

[2] K. BOYADZHIEV, *Some inequalities for generalized commutators*, Publ. Res. Inst. Math. Sci., 26 (1990), pp. 521–527.

[3] I. C. GOHBERG AND M. G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, AMS, Providence, RI, 1969.

[4] D. JOCIC AND F. KITTANEH, *Some perturbation inequalities for self-adjoint operators*, J. Operator Theory, 31 (1994), pp. 3–10.

[5] T. KATO, *Perturbation Theory for Linear Operators*, 2nd ed., Springer-Verlag, Berlin, 1976.

[6] R. MATHIAS, *Concavity of monotone matrix functions of finite order*, Linear Multilinear Algebra, 27 (1990), pp. 129–138.

[7] C. L. OLSEN AND G. K. PEDERSEN, *Corona C^* - algebras and their applications to lifting problems*, Math. Scand., 64 (1989), pp. 63–86.

[8] G. K. PEDERSEN, *A Commutator Inequality*, preprint, Copenhagen University.

[9] M. ROSENBLUM AND J. ROVNYAK, *Hardy Classes and Operator Theory*, Oxford University Press, New York, 1985.