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AN INFINITE SERIES OF ADJUSTED ORTHOGONAL DESIGNS WITH REPLICATION TWO

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Abstract: We construct a series of adjusted orthogonal designs with $n+1$ rows, $2n$ columns and n^2+n treatments, where n is an integer other than 6. It is shown that members of this series are E-optimal when $n \geq 5$. This is the first series of adjusted orthogonal designs with constant replication number, and that too, at the minimum possible level. We also provide efficient designs in the cases $n = 2, 3$ and 4. The column designs of the above series are not regular graph designs. Since they are shown to be E-optimal, they provide an infinite series of counter examples to John and Mitchell's (1977) conjecture.

Key words and phrases: Adjusted orthogonal design, row-column design, rectangular design, E-optimality, regular graph design.

1. Introduction

Shah and Eccleston (1986) introduced the class of adjusted orthogonal row-column designs. Since then the equireplicate designs in this class have gained much importance because of the simplicity of their analysis and because many of them are optimal. (See Shah and Eccleston (1986) for their properties, Shah and Sinha (1989, chapter 4) and Bagchi and Shah (1989) for optimality results).

Let us recall that an equireplicate row column design with replication r is said to be adjusted orthogonal, if

$$M'N = rJ,$$

where M and N are the treatment-row and treatment-column incident matrices of the design and J is the all-one matrix of appropriate order.

Now, not many adjusted orthogonal designs are available in the literature. So far, the only known series are those in Agarwal (1966), Shah and Raghavarao (1980), John and Eccleston (1986) Eccleston and Street (1990), Bagchi and Berkum (1991) and Bagchi (1996). Among these, the members of the first , fifth and sixth are known to be optimal and those of the second are highly efficient.

In the present paper, we construct a new series and prove its E-optimality. Specifically, we prove :

Theorem 1.1. *Let n be an integer $\geq 2, n \neq 6$. Then there exists a row-column design with $n+1$ rows, $2n$ columns and n^2+n treatments such that*
(i) *it is adjusted orthogonal* (AO),
(ii) *its row design is a linked block design* (LBD) *and*
(iii) *its column design is the dual of a balanced rectangular design of type two.*

By a balanced rectangular design of type 2 (RD2) we mean a PBIBD with rectangular association scheme with two rows and $\lambda_1 = \lambda_2 - 2 = \lambda_3 - 1$. (See Bagchi (1994).)

We also prove :

Theorem 1.2. *The dual of an* RD2 *is* E-*optimal whenever it has at least* 10 *blocks.*

This together with Theorem 4.4.2 of Shah and Sinha (1989) proves the following.

Corollary. *The members of the series described in Theorem* 1.1 *are* E-*optimal when $n \geq 5$.*

We now note the following remarkable features of this series of designs.

1. This is the first series of AO row-column designs with constant replication number. This constant, again, is the minimum possible value for a connected equireplicate design. Thus, these designs are extremely cost-effective as they allow the maximum number of treatments to be tested on a given set up.
2. The members of the series are quite plentiful, as for every positive integer other than 6, there is a design.
3. The column designs are not regular graph designs (RGD). (For the definition see John and Mitchell (1977).) So, here is another infinite series (the earlier one was in Bagchi (1994)) of designs which are counter examples to John and Mitchell's conjecture.

An adjusted orthogonal (AO) design need not always be good (it can even be disconnected!). To have an optimality property, it must have good marginals. Now the following question arises. Suppose there are two row-column designs d_1 and d_2, of which only d_1 is AO. Both have the same row design, but the column design of d_2 is better (with respect to some optimality criterion) than that of d_1. Which one is better (with respect to that criterion)? This seems to be a difficult question. (Note that this is relevant in the search for optimal designs as AO designs with specified marginals need not exist, even when both marginals exist. See Section 3.2 for examples.) In Section 3.2 we have studied a few parametric set ups where this type of situation arises. In this connection we find examples of nonisomorphic designs with the same marginals. Interestingly, their performances differ considerably.

2. The Proof of Theorem 1.1

Case 1. $n = 2$. The following array represents the proposed two-way design when $n = 2$.

1	4	5	2
2	5	3	6
3	6	1	4

Case 2. $n = 3$. In this case the following is a two-way design as proposed.

A	4	7	5	8	2
2	5	C	9	3	6
3	B	9	1	4	7
1	6	8	A	C	B

Case 3. $n \geq 4, n \neq 6$. In all these cases there are two mutually orthogonal latin squares of order n, say L^1 and L^2 with a common transversal. Indeed, in all these cases – except possibily when $n = 10$ – there are three mutually orthogonal latin squares of order n (see Brouwer(1978) for instance) any two of which may be taken to be L^1 and L^2; then the positions in which any given symbol occurs in the third square is a common transversal of L^1, L^2. When $n = 10$, the following example (taken from Rouse Ball and Coxeter (1986)) shows the existence of L^1, L^2 (here the entry ij in a position indicates that i occurs in L^1 and j in L^2 in that position):

00	82	95	48	76	23	51	39	17	64
28	11	03	96	50	87	34	62	49	75
59	30	22	14	97	61	08	45	73	86
84	69	41	33	25	98	72	10	56	07
67	05	79	52	44	36	90	83	21	18
32	78	16	89	63	55	47	91	04	20
15	43	80	27	09	74	66	58	92	31
93	26	54	01	38	19	85	77	60	42
71	94	37	65	12	40	29	06	88	53
46	57	68	70	81	02	13	24	35	99

For simplicity we assume that the symbols of L^1, L^2 are integers from 0 to $n - 1$, and that (without loss of generality) the common transversal is the main diagonal occupied (in both L^1 and L^2) by the symbols in natural order. That is,

$$L^1_{ii} = i = L^2_{ii}, \quad \text{for } 0 \leq i \leq n - 1.$$

This may be achieved by suitable permutations of rows, columns and symbols.

Now we present the two-way design as follows. Let V denote the treatment set, R the set of rows and C the set of columns. Let $f(i,j), i \in R, j \in C$ denote the treatment assigned to the position (i,j). Take $I = \{0, 1, \cdots n-1\}$, $V = I \cup (I \times I)$, $R = I \cup \{\infty\}$, $C = I^+ \cup I^-$, where $I^\pm = \{i^\pm : i \in I\}$ are two disjoint copies of I, and ∞ is a new symbol. Then for i and j in I we assign

$$\begin{aligned} f(\infty, i^-) &= i, \\ f(\infty, i^+) &= (i,i), \\ f(j, i^-) &= (x,y), \\ f(j, i^+) &= \begin{cases} (i,j), & \text{if } i \neq j, \\ i, & \text{if } i = j, \end{cases} \end{aligned}$$

where $x, y \in I$ are determined by $L^1_{x,y} = i$ and $L^2_{x,y} = j$.

An easy but tedious verification shows that this is an adjusted orthogonal two-way design, its row design is an LBD and its column design is the dual of a rectangular design with $v = 2n,\ b = n^2 + n,\ r = n + 1,\ k = 2,\ \lambda_1 = 0,\ \lambda_2 = 2$ and $\lambda_3 = 1$, based on a $2 \times n$ rectangular association scheme.

That the designs constructed for the cases $n = 2$ and 3 satisfy the above properties is easy to verify.

3. Optimality Study

3.1. Proof of Theorem 1.2

We first introduce a few notations.

Notation 3.1.

(i) $\mathcal{D}(b,k,v)$ denotes the class of all connected block designs with b blocks; each of size k and v treatments, where $k < v$.

(ii) $r = bk/v$.

(iii) If $d \in \mathcal{D}(b,k,v)$ is equireplicate (and hence has replication number r) then the parameters of the dual design of d are denoted by b^*, k^* and v^*. Thus $b^* = v, k^* = r$ and $v^* = b$ and the replication number is $r^* = k$.

(iv) $\lambda^* = [k(r-1)/(b-1)]$. Here $[x]$ denotes the integral part of x.

(v) For a design d in $\mathcal{D}(b,k,v), N_d$ will denote its treatment-block incidence matrix, $r_{di}, 1 \le i \le v$, the replication numbers arranged in increasing order, λ_{dij} the (i,j) entry of $N_d N_d^T$ and μ_d the minimum positive eigenvalue of the C-matrix of d. We drop d from the suffix when there is no sense of confusion.

(vi) d^* denotes the dual of RD2. (For definition of RD2, see the paragraph following Theorem 1.1.) It is easy to see that

$$k\mu_{d^*} = \{k(r-1) - 3/2\}b/(b-1). \tag{3.1}$$

In view of Theorem 3.1 of Bagchi (1994) and the well-known relation between the C-matrices of a design and its dual, it suffices to show :

Proposition 3.2. *Let d be an unequally replicated design in $\mathcal{D}(b,k,v)$. Then*

$$\mu_d \leq \mu_{d^*}, \quad \textit{when} \quad b \geq 10.$$

Remark. With a little modification in the proof, this proposition can be strenthened to include the case $b = 10$. But since RD2 is not E-optimal when $v = 8$, Theorem 1.2 can not be strengthened.

In order that RD2 can belong to $\mathcal{D}(b^*,k^*,v^*)$, we assume $k(r-1) = b/2 + 1 \bmod (b-1)$, so that, using Notation 3.1 (iv), we get

$$k(r-1) = \lambda^*(b-1) + b/2 + 1. \tag{3.2}$$

Before going to the proof of Proposition 3.2 we recall a few well-known inequalities, for the proof of which we refer to Cheng (1980), Jacroux (1980) or Constantine (1981). For d in $\mathcal{D}(b,k,v)$, we have

$$k\mu_d \leq \begin{cases} vr_{di}(k-1)(v-1)^{-1}, & \text{(i)} \\ (r_{di}+r_{dj})(k-1)/2 + \lambda_{dij}, & \text{(ii)} \\ v\{(r_{di}+r_{dj})(k-1)/2 - \lambda_{dij}\}(v-2)^{-1}. & \text{(iii)} \end{cases} \tag{3.3}$$

Let us fix an unequally replicated design d in $\mathcal{D}(b,k,v)$. Clearly $r_{d1} \leq r-1$.

Proof of Proposition 3.2.
Case 1. $r \geq 3$.

In view of 3.3 (i) and (3.1), it suffices to show $2(r-1)v(b+k-r-1) - 3b(v-1) \geq 0$, which would follow from the following:

$$2(r-1)(b+k-r-1) \geq 3b. \tag{3.4}$$

Since $k < v$, $r < b$ then r can take the values $3, 4, \ldots, b-1$. Now the coefficient of r^2 in the L.H.S. of (3.4) is negative, so that it is minimum at one of the boundaries. It is easy to check that (3.4) holds for both $r = 3$ and $r = b-1$ whenever $k \geq 2$ and $b \geq 8$. Hence the result.
Case 2. $r = 2$.

Here

$$v = (b/2)k, \quad r_1 = 1. \tag{3.5}$$

Case 2.1. $r_2 = 1$. Here, $\lambda_{12} = 0$ or 1.
Case 2.1(a). $\lambda_{12} = 0$.

By 3.3 (ii) we have $k\mu_d \leq k-1$. So, we have to show that $k \geq b/2+1$, which follows from (3.2).

Case 2.1(b). $\lambda_{12} = 1$.

Using (3.3) (iii), we get $k\mu_d \leq v(k-2)(v-2)^{-1}$. So, we need to show

$$2(k-v)(b-1) \leq (v-2)(k-3b/2), \tag{3.6}$$

which trivially holds when $\lambda^* \geq 1$, by (3.2). Again, in the case $\lambda^* = 0$, $k = b/2+1$ (see (3.2)), so that (3.6) is reduced to $\{(v-2)+2(k-v)\}(b-1) \leq 0$, which holds because of (3.5).

Case 2.2. (The remaining case) $r_2 \geq 2$.

In this case, we have $r_i = 2$, $2 \leq i \leq v-1$ and $r_v = 3$. Let B denote the unique block containing treatment 1. Let

$$\begin{aligned} &U_j = \{2 \leq i \leq v-1 : i \text{ appears } j \text{ times in } B\}, \\ &u_j = |U_j|. \end{aligned} \tag{3.7}$$

Then

$$u_0 + u_1 + u_2 = v - 2 \tag{3.8}$$

and

$$u_1 + 2u_2 = k - 1 - \ell, \tag{3.9}$$

where ℓ is the number of times v appears in B, $\ell = 0, 1, 2$ or 3.

From (3.5), (3.8) and (3.9) it follows that

$$u_0 \geq v - k - 1 \geq 4k - 1. \tag{3.10}$$

It is easy to see that in order to prove the required result, it suffices to find a $v \times 1$ vector $\boldsymbol{x}$ with $x'\mathbf{1}_v = 0$, such that

$$\frac{\boldsymbol{x}'(kC_d)\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} \leq k\mu_{d^*}. \tag{3.11}$$

For this purpose we choose the following vector: $\boldsymbol{x}_1 = u_0$, $\boldsymbol{x}_i = -\mathbf{1}$, $i \in U_0$, $\boldsymbol{x}_i = \mathbf{0}$, $i \neq 1$, $i \notin U_0$. Then

$$\frac{\boldsymbol{x}'(kC_d)\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} \leq (k-1)\{1 + (u_0+1)^{-1}\},$$

so that it suffices to show that

$$(k-1)/(u_0+1) - 1 \leq (k - 3b/2)/(b-1).$$

By (3.10), the L.H.S. ≤ 0. So, in the case when $\lambda^* \geq 1$, the R.H.S. ≥ 0 and the proof is complete.

In the remaining case, however, this inequality does not hold. We therefore go to a more involved computation. Since from now on we have $k = b/2 + 1$ and $v = k(k-1)$, the following holds:

$$k\mu_{d^*} = k - 1. \tag{3.12}$$

Let $\overline{C}_d$ denote the "averaged version" of C_d, "averaged separately over U_0, U_1 and U_2". (For the meaning of these terms, see Constantine (1981).) Then the form of $k\overline{C}_d$ is as given below:

a_1	$0 \cdot J_{1\times u_0}$	$-1 \cdot J_{1\times u_1}$	$-2 \cdot J_{1\times u_2}$	$-\ell$
$0 \cdot J_{u_0\times 1}$	$a_2 I_{u_0} + b_2 J_{u_0}$	$b_3 J_{u_0\times u_1}$	$0 \cdot J_{u_0\times u_2}$	$b_4 J_{u_0\times 1}$
$-1 \cdot J_{u_1\times 1}$	$b_3 J_{u_1\times u_0}$	$a_3 I_{u_1} + c_2 J_{u_1}$	$-2 \cdot J_{u_1\times u_2}$	$-(\ell+\varepsilon) J_{u_1\times 1}$
$-2 \cdot J_{u_2\times 1}$	$0 \cdot J_{u_2\times u_0}$	$-2 \cdot J_{u_2\times u_1}$	$a_4 I_{u_2} - 4 \cdot J_{u_2}$	$-2\ell \cdot J_{u_2\times 1}$
$-\ell$	$b_4 J(1\times u_0)$	$-(\ell+\varepsilon) J_{1\times u_1}$	$-2\ell \cdot J_{1\times u_2}$	a_5

where

$$\begin{aligned} &a_1 = k-1, \quad a_2 + b_2 = d + \delta,\ \delta = 0 \text{ or } 2, \\ &a_3 + c_2 = d, \quad a_4 = d+2, \quad \text{with } d = 2(k-1), \\ &a_5 \le 3(k-1) - \ell(\ell-1), \\ &0 \le -b_2 \le 4,\ 0 \le -b_3 \le 2,\ 1 \le -c_2 \le 2, \text{ and } \varepsilon \ge 0. \end{aligned} \tag{3.13}$$

Since $k\overline{C}_d . \mathbf{1}_v = 0$, we also have

$$a_2 + u_0 b_2 - \delta + u_1 b_3 + b_4 = 0, \tag{3.14}$$

$$u_0 b_3 = -2(k-1) + 1 - (u_1 - 1)c_2 + 2u_2 + \ell + \varepsilon \tag{3.15}$$

and

$$u_0 b_4 = -a_5 + \ell + u_1(\ell + \varepsilon) + u_2 2\ell.$$

Using the upper bound for a_5 from (3.13) in the last equation, we get

$$u_0 b_4 \ge -3(k-1) + \ell(\ell-1) + \ell(1 + u_1 + 2u_2) + u_1\varepsilon.$$

This, with the help of (3.9) simplifies to the following:

$$u_0 b_4 \ge (k-1)(\ell-3) + u_1\varepsilon. \tag{3.16}$$

Substituting in (3.14) the value of b_3 from (3.15) and the lower bound for b_4 from (3.16) and using the fact that $c_2 < 0$ (see (3.13)), we get

$$u_0(u_0-1)b_2 \le (k-1)(2u_1 - 2u_0 + 3 - \ell) - u_1(1 + 2u_2 + \ell). \tag{3.17}$$

In view of (3.11) and (3.12), it suffices to find a vector $\boldsymbol{x}$ with $\boldsymbol{x}'\mathbf{1}_v = 0$ such that

$$\boldsymbol{x}'(k\overline{C}_d)\boldsymbol{x} - (k-1)\boldsymbol{x}'\boldsymbol{x} \leq 0. \tag{3.18}$$

We choose the following vector $\boldsymbol{x} = (x_1, \ldots, x_v)'$, $x_1 = u_0 - 2u_2 = u$ say, $x_i = -1$, $i \in U_0$, $x_i = 2$, $i \in U_2$, $x_i = 0$ for remaining i's. So, we have

$$\begin{aligned}&\boldsymbol{x}'(k\overline{C}_d)\boldsymbol{x} - (k-1)\boldsymbol{x}'\boldsymbol{x}\\ &= (k-1)(u_0+4u_2) - 8u_0 - u_0\delta + 2[u_0(u_0-1)b_2 - 4u_2u - 16u_2(u_2-1)]. \end{aligned}\tag{3.19}$$

Using (3.17) and the fact that $u_1, u_2 \geq 0$, we have

$$\text{R.H.S.} \leq (k-1)(4u_1 + 4u_2 - 3u_0 + 6 - 2\ell) - 8u_2u.$$

From (3.9), it follows that $u \geq 0$. Hence all we have to show is that the coefficient of $k-1$ is negative.

Again using (3.9), we find that $u_0 - u_2 \geq k^2 - 2k - 1$, so that the coefficient of $k-1$ is

$$\begin{aligned}&\leq \ 4(u_1 + 2u_2) - 3(u_0 - u_2) + 6\\ &\leq \ -3k^2 + 10k + 5, \text{ by } (3.10).\end{aligned}$$

This is ≤ 0 as $k = b/2 + 1 \geq 6$.

This completes the proof.

3.2. The smaller values of n

In this section we consider the cases $n = 2, 3, 4$. In the case $n = 3$, we have obtained an E-optimal design. In the other cases no design could be proved to be optimal. We have therefore searched for efficient designs (with regard to A- and D-criteria as well) . In the process we have made some interesting observations regarding the relative performances of non AO designs with good marginals and AO designs with not-so-good marginals.

In all the optimaity statements that follow, the competing class is the equireplicate class unless otherwise stated.

Notation 3.2.1. The row-treatment and column-treatment incidence matrices of a row-column design with p rows, q columns and v treatments are denoted by $M(v \times p)$ and $N(v \times q)$ respectively. The C-matrix of an **equireplicate** design d is given by

$$C_d = rI_v - q^{-1}MM' - p^{-1}NN' + v^{-1}rJ(v \times v),$$

where $r = pq/v$.

Definition 3.2.2. A row-column design is said to be commutative (respectively have i common eigenvectors) if MM' and NN' commute (respectively have i common eigenvectors).

Notation 3.2.3. For a row-column design d, A_d denotes the sum of reciprocals and D_d denotes the product of the positive eigenvalues of C_d.

Before we study the cases $n = 2, 3$ and 4, let us calculate the eigen values of the C-matrix of the design (d_0, say) constructed in Section 2.

For a fixed n, let C_1 and C_2 denote the C-matrices of the row design and the column design respectively of d_0 . Then it is easy to see that the spectrum of C_1 is given by $2^{n^2-1}((n+1)/n)^n$ and the spectrum of C_2 can be obtained from Bagchi (1994) as

$$2^{n^2-n+1}\Big(\frac{n+2}{n+1}\Big)^{n-1}\Big(\frac{n}{n+1}\Big)^{n-1}.$$

Since the design is adjusted orthogonal, it follows that (see Shah and Eccleston (1986)) the spectrum of C_{d_0} is

$$2^{(n-1)^2}\Big(\frac{n+1}{n}\Big)^{n}\Big(\frac{n+2}{n+1}\Big)^{n-1}\Big(\frac{n}{n+1}\Big)^{n-1}.$$

Case $n = 2 : p = 3, q = 4$ and $v = 6$.

In this case, if there existed an AO design with both marginals LBD, then that would be optimal with regard to all convex decreasing optimality criteria. (See Bagchi and Shah (1989).) However, it can be easily seen that such a design does not exist.

In the class of equireplicate and binary designs, up to isomorphism there are exactly three designs. Among these, there is only one AO design, namely d_0, the member with $n = 2$ in the series of Theorem 1.1. The other two designs are shown below.

$$d_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 6 \\ 4 & 3 & 6 & 5 \end{bmatrix} \quad \text{and} \quad d_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 6 \\ 4 & 6 & 5 & 3 \end{bmatrix}.$$

Note that d_1 and d_2 have the same marginals (both LBD's) but they are non isomorphic : d_1 is commutative while d_2 is not.

The performances of these designs are presented in the following table.

Table 3.1.

Design(d)	μ_d	A_d	D_d	# of Common eigenvectors	Comment
d_0	0.67	4.084	4	6	d_2 is best
d_1	0.83	3.867	4.44	6	w.r.t each of
d_2	0.91	3.847	4.47	2	A-,D-,E-criteria

Case $n = 3 : p = 4, q = 6, v = 12$.

In this case, the best (w.r.t. any convex decreasing criterion) row design would be an LBD. A column design is E-optimal if it is the dual of a GDD with 3 groups and $\lambda_1 = 0, \lambda_2 = 1$, say GDD(3,0). (See Theorem 2.1 (a) of Bagchi and Cheng (1993).) Thus, by Theorem 4.4.2 of Shah and Sinha (1989) we have:

An AO design with row design as an LBD and the column design as the dual of GDD(3,0) is E-optimal in the general class, if it exists.

We have been able to construct a design with the above properties, which is displayed below, denoted by d^*. Shah and Puri have contructed a design with the row design and the column design same as those of d^*, but it is not AO. Interestingly enough, we have found two other non-AO designs with the same properties and neither of them is isomorphic to the design of Shah and Puri.

We now make a comparative study of the performances of these designs.

$$d^* = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 1 & 2 & 9 & 10 \\ 5 & 12 & 10 & 11 & 8 & 4 \\ 11 & 6 & 12 & 9 & 3 & 7 \end{bmatrix}.$$

Let d_1 denote the design constructed by Shah and Puri which is presented in page 82 of Shah and Sinha (1989). Let

$$d_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 1 & 2 & 9 & 10 \\ 5 & 11 & 9 & 12 & 4 & 7 \\ 12 & 6 & 11 & 10 & 8 & 3 \end{bmatrix} \quad \text{and} \quad d_3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 1 & 2 & 9 & 10 \\ 5 & 6 & 12 & 11 & 8 & 7 \\ 11 & 12 & 10 & 9 & 3 & 4 \end{bmatrix}.$$

Thus, in this case we have four (at least!) non isomorphic designs having the same marginals, and one of them AO. We compare them below.

Table 3.2.

Design	μ_{d_1}	A_d	D_d	# of common eigenvectors	Comment
d_0	0.75	9.42	33.3	12	d^* is best w.r.t. A-
d^*	1.00	8.08	42.6	12	and D-criteria also.
d_1	0.91	8.47	32.7	8	Next best is d_1 w.r.t.
d_2	0.57	8.67	32.0	10	E-criterion and d_3 w.r.t.
d_3	0.83	8.37	35.6	12	A- and D-criteria

Case $n = 4 : p = 5, q = 8, v = 20$.

Here again, the best row-design is an LBD. Let us look at the column set up. The dual set up has $v^* = 8, r^* = 5, k^* = 2$.

Now there are three regular graphs of degree 5 with 8 vertices. These are the complements of the following graphs:
G_1 = Union of two disjoint four cycles.
G_2 = Union of one 3-cycle and another 5-cycle.
G_3 = The 8-cycle.

Let R_i denote the RGD with the edges of the complement of G_i as blocks. Then we know that:
(i) R_2 is E-optimal. (See Bagchi and Cheng (1993) : Theorem 2.1 (c)(iv).)
(ii) R_1 as well as R_3 has the next largest value of μ. (That μ of R_1 (R_3) is at least as large as any non- RGD follows by simple calculations using (3.3) (iii).)

Let $\overline{R}_i$ denote the dual of R_i. In the set up with $b = 8,\ k = 5$, and $v = 20$ an unequally replicated design has $\mu(C_d) \le 0.8 = \mu(\overline{R}_1)$. (Follows from 3.3 (i).)

Thus we have,
(i) $\overline{R}_2$ is E-optimal in $\mathcal{D}(8, 5, 20)$,
(ii) $\overline{R}_1$ (also $\overline{R}_3$) is next best.

Now, let us go back to our row-column set up. From the above discussion, we find:
(i) An AO design with row design an LBD and column design $\overline{R}_2$ is E-optimal.
(ii) The next largest value of μ among AO designs is 0.8.

We do not know whether the design described in (i) exists. The best design that we could find with marginals same as that of (i) is shown below. We call it d_1:

$$d_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 6 & 5 & 7 & 8 \\ 13 & 14 & 1 & 2 & 9 & 10 & 12 & 11 \\ 16 & 15 & 20 & 19 & 13 & 4 & 14 & 3 \\ 18 & 8 & 17 & 12 & 10 & 20 & 6 & 16 \\ 7 & 17 & 9 & 18 & 15 & 11 & 5 & 19 \end{bmatrix}.$$

Further, we have obtained an AO design with row design an LBD and column design $\overline{R}_1$, which we present below:

$$d_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 13 & 14 & 2 & 1 & 10 & 9 & 12 & 11 \\ 15 & 16 & 19 & 20 & 3 & 4 & 14 & 13 \\ 17 & 8 & 9 & 18 & 15 & 12 & 5 & 20 \\ 7 & 18 & 17 & 10 & 11 & 16 & 19 & 6 \end{bmatrix}.$$

We have another AO design the column design of which is the dual of a non

RGD. It's row design is again the same LBD as d_1 and d_2:

$$d_3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 1 & 2 & 11 & 12 & 13 & 14 \\ 15 & 16 & 13 & 14 & 3 & 4 & 18 & 17 \\ 17 & 18 & 19 & 20 & 9 & 10 & 5 & 6 \\ 7 & 8 & 20 & 19 & 16 & 15 & 12 & 11 \end{bmatrix}.$$

We now compare these designs.

Table 3.3.

Design(d)	μ	A_d	D_d	# of common eigenvectors	Comment
d_0	0.8	13.95	1105.9	20	d_2 is best w.r.t.
d_1	0.7	13.84	1192.4	15	all criteria while
d_2	0.8	13.66	1327.1	20	d_0 and d_3 are
d_3	0.8	13.754	1254.4	20	equally good w.r.t.
					E-criterion

We have investigated a few other designs, the performances of which are worse than those given above, regarding all three criteria. It appears that 0.8 is the maximum value of μ, which would mean that d_0, the member of our series, is actually E-optimal, although we cannot prove it.

Let us now summarise our findings for the three cases, $n = 2, 3, 4$.

1. In each case, we find non-isomorphic designs having the same marginals. Their performances differ quite a bit. (For $n = 4$, we had constructed two other designs having the same marginals as d_1. We have not presented these as their performances are poorer than those of the ones presented.)

2. Since an AO design is commutative, one might expect that the more the number of common eigen vectors (of MM' and NN'), the better the performance. But it need not be so. Consider the case $n = 2$. The design d_1 is not only commutative but its C-matrix has three (out of five) positive eigenvalues the same as that of the hypothetical best design, (say d^*) whereas for d_2 only one eigenvalue of C_{d_2} equals that of C_{d^*}. Still d_2 is better than d_1 regarding all the three criteria. In the case $n = 3$, however, we have the opposite picture. Here, among the three designs $d_i, 1 \leq i \leq 3$, having the same marginals, d_3 is the best regarding A- and D-criteria. We observe that d_3 is commutative and is the "closest" to adjusted orthogonality in the sense that there is only one eigenvector of C_{d_3} which correspond to positive eigenvalues for both MM' and NN'. As a result, the spectrum of C_{d_3} differs from the spectrum of C_{d^*} in only two eigenvalues. However d_3 is not the next best w.r.t. the E-criterion. So the

question arises whether there exists a non-AO design which is "closest" to d^* w.r.t. all of A-,D- and E-criteria ? In the case $n = 4$ we could not construct any design which is "close" to the hypothetical best design.

Thus it is more or less clear that either an AO design with one best and another second best marginal or a non AO design with both marginals best is likely to be optimal (in the absence of an AO design with both marginals best, of course). But to be able to say which one wins the race in what situation, further study is needed.

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