DETERMINATION OF PROBABILITY MEASURES THROUGH GROUP ACTIONS

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CONTENTS

INTRODUCTION

One of the fundamental problems in Measure Theory is the following: given a measurable space $(X, \mathcal{B}_X)$, to find subclasses $\mathcal{D}$ of $\mathcal{B}_X$ such that whenever for two probability measures μ and ν on $(X, \mathcal{B}_X)$, $\mu(B) = \nu(B)$ for every $B \in \mathcal{D}$, then $\mu(B) = \nu(B)$ for every $B \in \mathcal{B}_X$. The first basic theorem of Measure Theory, viz., the Caratheodory Extension Theorem says that any sub-algebra $\mathcal{D}$ of $\mathcal{B}_X$ which generates $\mathcal{B}_X$ has the above mentioned property.

Let $(X, \mathcal{B}_X)$ be a given measurable space. A subclass $\mathcal{D}$ of $\mathcal{B}_X$ is called a determining class for a class $\mathcal{P}$ of probability measures on $(X, \mathcal{B}_X)$ if for $\mu,\nu \in \mathcal{P}$, whenever $\mu(B) = \nu(B)$ for every $B \in \mathcal{D}$, then $\mu(B) = \nu(B)$ for every $B \in \mathcal{B}_X$. The problem of finding determining classes, other than the ones assured by the Caratheodory Extension Theorem, has been of interest. One of the earlier results in this direction is due to Cramer and Wold [4]. They considered the case $X = \mathcal{R}^n$ and showed that the class $\mathcal{D} = \{S_{\underline{t},s} \mid \underline{t} \in \mathcal{R}^n, s \in \mathcal{R}\}$ is a determining class for the class of all probability measures on $\mathcal{R}^n$, where for $\underline{t} = (t_1,\ldots,t_n) \in \mathcal{R}^n$ and $s \in \mathcal{R}$, $S_{\underline{t},s} = \{\underline{x}=(x_1,x_2,\ldots x_n) \in \mathcal{R}^n \mid \sum_{i=1}^{n} t_i x_i < s\}$. In the case, when X is a metric space and $\mathcal{B}_X$ is the σ-algebra of Borel subsets of X, various determining classes (which utilize the topology of X) are known for $\mathcal{M}(X)$, the class of all probability measures on $(X, \mathcal{B}_X)$. For example one knows that the class $\mathcal{O}$ of all open subsets of X is a determining class for $\mathcal{M}(X)$. If X is a complete separable metric space, then the class $\mathcal{K}$ of all compact subsets of X is a determining class for $\mathcal{M}(X)$. Another class of subsets in a metric space X which is of interest is the class S of all closed balls in X. It turns out that, in general S is not a determining class for $M(X)$. Davis [5] has shown that one can construct a compact metric space and two distinct Borel probability measures on it which agree on all closed balls. The problem of finding metric spaces X in which the class S is a determining class for $M(X)$ has been investigated by many authors, for example : Besicovitch [2], Anderson [1], Christensen [3] and Hoffmann - Jorgensen [12].

Utilizing the group structure of R^n, a general method for constructing determining classes in R^n was given by Sapogov [14]. Let $E \in B_{R^n}$ and let $D(E) = \{E+\underline{x} \mid \underline{x} \in R^n\}$. Sapogov has shown that $D(E)$ is a determining class for the class of all probability measures on R^n if, either E has positive finite Lebesgue measure or the support of the Fourier-transform of χ_E contains an open subset of R^n.

This thesis is concerned with the following situation:let (G,B_G) be a measurable group acting on a measurable space (X,B_X) and let μ and ν be two probability measures on (X, B_X). Do there exist sets $E \in B_X$ such that whenever $\mu(g\cdot E) = \nu(g\cdot E)$ for every $g \in G$, then $\mu = \nu$? That is, can the action of the group G on X be utilized to find determining classes for probability measures on (X, B_X)? Let a set $E \in B_X$ be called a <u>G-determining set</u> for a class P of probability measures on (X, B_X) if the class $\{g\cdot E \mid g \in G\}$ is a determining class for P. Our aim is to find G-determining sets and also to analyse the size of the class of all determining sets using the Baire category theorem appropriately. We give below a section-wise summary of the thesis.

§ 1. We introduce the concept of a G-determining set and give some examples to illustrate the idea.

§ 2. In this section we consider the translation action of a locally-compact second countable abelian group G on itself and study the following problem : do there exist determining sets for $M(G)$, the class of all probability measures on G? If yes, how big is the class of all determining sets? The main result of this section says that generically, a subset of G with finite positive Haar measure is a determining set for $M(G)$.

§ 3. The set up of this section is the same as of § 2. Let $E \in B_G$ be a fixed subset of G such that it has positive Haar measure and its closure is compact. We consider the question: for what kind of classes P of probability measures on G, is E a determining set? We show by an example that E need not be a determining set for $M(G)$. However, to every such set E one can associate a compact subgroup K of G (depending only on E) such that E becomes a determining set for the class $M(K\backslash G/K)$ of all K - invariant probability measures on G.

§ 4. In this section we consider the action of a locally-compact second-countable (not necessarily abelian) group G on itself by left multiplication. We show that every compact open subgroup K of G is a determining set for the class $M(G/K)$ of all right K - invariant probability measures on G. However, the problem of finding determining sets for $M(G)$ remains open.

§ 5. We consider the translation action of a locally-compact second-countable abelian group G on itself and introduce the concept of a weak-convergence determining set: a set $E \in B_G$ is called a weak-convergence determining set for a class P of probability measures on G if for every sequence $\{\mu_n\}_{n=0,1,2,\ldots}$ in P, $\mu_n(E+g) \to \mu_o(E+g)$ for every $g \in G$ implies that μ_n converges weakly to μ_o. We show that generically, a subset of G with finite positive Haar-measure is a weak-convergence determining set for $M(G)$. We also show that to every set $E \in B_G$ such that it has positive Haar measure and has compact closure, a compact subgroup K of G can be associated such that E becomes a weak-convergence determining set for $M(K \backslash G/K)$, the class of all K-invariant probability measures on G.

§ 6. Let G be a locally-compact second countable group and let K be a compact subgroup of G. In 6.1 we introduce the concept of a symmetric pair (G,K): a pair (G,K) is called symmetric if there exists an automorphism τ of G and Borel maps ψ_1 and ψ_2 of G into K such that $\tau(g) = \psi_1(g)g^{-1}\psi_2(g)$ for every $g \in G$.

In 6.2 and 6.3, we develop the theory of Fourier-transform for measures in $M(K \backslash G/K)$. Using this, in 6.4 we show that generically, a subset of G with finite positive Haar measure is a determining set for $M(K \backslash G/K)$. In 6.5, we consider standard symmetric pairs, i.e., the case when G is a non-compact semi-simple lie-group with finite center and K is a maximal compact subgroup of G. In this case it is shown that every $E \in B_G$ with positive Haar-measure and with compact closure is a determining set for $M(K \backslash G/K)$. Finally, in 6.6 , the implications of the above results are explicitly described in a particular case: the standard symmetric pair $(SL(2, \mathbb{C}), SU(2))$.

§ 7. In this last section, weak-convergence determining sets for the class $M(K \backslash G/K)$, in a symmetric space (G,K) are analysed. A generic result analogus to the one in § 5 is obtained.

§ 1. DETERMINING SETS : DEFINITION AND SOME EXAMPLES

Let $(X, \mathcal{B}_X)$ be a measurable space. Let $(G, \mathcal{B}_G)$ be a measurable group, i.e., G is a group and $\mathcal{B}_G$ is a σ - algebra of subsets of G such that

$$(g,h) \to gh^{-1}, \quad g,h \in G$$

is a measurable map from $(G \times G, \mathcal{B}_G \times \mathcal{B}_G)$ into $(G, \mathcal{B}_G)$.

By an action of $(G, \mathcal{B}_G)$ on $(X, \mathcal{B}_X)$ we mean a measurable map $\phi : G \times X \to X$, such that

(i) $\phi(e,x) = x$ for every $x \in X$, e being the identity element in G ;

(ii) $\phi(g_1, \phi(g_2, x)) = \phi(g_1g_2, x)$ for every $g_1, g_2 \in G$ and $x \in X$;

(iii) For every $g \in G$, the map $x \mapsto \phi(g, x)$ is a measurable map from X into itself. When such a map exists, we say that G acts on X. We shall sometimes denote $\phi(g, x)$ by $g \cdot x$. For $g \in G$ and $E \in \mathcal{B}_X$, we shall write $g \cdot E$ for the set

$$\{g \cdot x \mid x \in E\} .$$

1.1 Definition : Let $(X, \mathcal{B}_X)$ be a measurable space and let a measurable group $(G, \mathcal{B}_G)$ act on $(X, \mathcal{B}_X)$. Let ϕ denote this action. A set $E \in \mathcal{B}_X$ is called a (G, ϕ) - determining set for a class $\mathcal{P}$ of measures on $(X, \mathcal{B}_X)$ if E has the following property: for $\mu, \nu \in \mathcal{P}$, if

$$\mu(g \cdot E) = \nu(g \cdot E) \text{ for every } g \in G ,$$

then $\mu = \nu$.

Whenever, it is clear from the context what G and ϕ are, we shall call a (G, ϕ) - determining set simply a determining set.

Clearly, if E is a determining set for a class P of measures on (X, B_X), then so is the set $g \cdot E$ for every $g \in G$.

1.2. Example (Sapogov) : Let $G = X = R^n$ be the n-dimensional Euclidean space and let B_{R^n} be the σ - algebra of Borel subsets of R^n. Let λ_{R^n} denote the Lebesgue measure on R^n and let $M(R^n)$ denote the class of all probability measures on R^n. Let R^n act on itself as, $(x,y) \to x+y$ for every $x,y \in R^n$. Sapogov [14] has shown that a set $E \in B_{R^n}$ is a determining set for $M(R^n)$ whenever, either $0 < \lambda_{R^n}(E) < \infty$, or the support of the Fourier-transform of χ_E contains an open subset of R^n.

1.3. Example : Let X = G be any locally-compact second-countable group and let B_G be the σ - algebra of Borel subsets of G. Let λ_G be a Haar measure on G and let $M(G)$ denote the class of all probability measures on G. Let G act on itself by left-multiplication. In view of Example 1.2, one asks the question : is every set $E \in B_G$ with $0 < \lambda_G(E) < \infty$, a determining set for $M(G)$? We show by an example that the above question cannot be answered in the affirmative.

Let $G = Z \times K$, where Z is the integer group and K is any compact abelian group. Choose two probability measures μ_1 and μ_2 on K such that $\mu_1 \neq \mu_2$. Choose a non-zero probability measure λ on Z. Put

$$\mu = \lambda \times \mu_1, \quad \nu = \lambda \times \mu_2 \quad .$$

Let, $E = \{0\} \times K$.

Then E is a compact subset of $Z \times K = G$ and $\lambda_G(E) = 1$. Further, for any $g = (n,k) \in G$

$$\mu(E + g) = \mu(E + (n,k))$$
$$= \lambda\{n\} \cdot \mu_1(K)$$
$$= \lambda\{n\} .$$

Similarly,

$$\nu(E + g) = \lambda\{n\} , \text{ for every } g = (n,k) \in G.$$

Thus,

$$\mu(E + g) = \nu(E + g) \text{ for every } g \in G.$$

But $\mu \neq \nu$. Thus E is not a determining set for $M(G)$.

However, consider $\theta = \delta_o \times \lambda_K$, where δ_o denotes the probability measure degenerate at $0 \in Z$ and λ_K denotes the normalised Haar measure of K. Then it is easy to see that

$$\mu * \theta = \nu * \theta ,$$

where $*$ denotes the convolution operation.

1.4. Lemma : Let G be a locally-compact second countable abelian group and let $\hat{G}$ be the dual group of G. Let $E \in B_G$ be such that $0 < \lambda_G(E) < \infty$ and the set $\{\gamma \in \hat{G} | \hat{\chi}_E(\gamma) \neq 0\}$ is a dense subset of $\hat{G}$. Then E is a determining set for $M(G)$.

Proof : Let $\mu, \nu \in M(G)$ be such that

$$\mu(E + g) = \nu(E + g) \tag{1.1}$$

for every $g \in G$. We have to show that $\mu=\nu$. Condition (1.1) is equivalent to

$$\int \chi_E(g+h)\, \mu(dh) = \int \chi_E(g+h)\, \nu(dh)$$

for every $g \in G$.

Thus, for every character γ in $\hat{G}$, we have

$$\int \overline{\langle g,\gamma\rangle} \left(\int \chi_E(g+h)\, \mu(dh)\right)\lambda_G(dg) = \int \overline{\langle g,\gamma\rangle} \left(\int \chi_E(g+h)\, \nu(dh)\right)\lambda_G(dg)$$

Using Fubini's theorem and changing g to $g-h$, we have for every γ in $\hat{G}$,

$$\int\left(\int\chi_E(g)\, \overline{\langle g-h,\gamma\rangle}\, \lambda_G(dg)\right)\mu(dh) = \int\left(\int\chi_E(g)\, \overline{\langle g-h,\gamma\rangle}\, \lambda_G(dg)\right)\nu(dh) .$$

Since each $\gamma \in \hat{G}$ is a homomorphism, we have

$$\int\left(\int\chi_E(g)\, \overline{\langle g,\gamma\rangle}\, \overline{\langle -h,\gamma\rangle}\, \lambda_G(dg)\right)\mu(dh) = \int\left(\int\chi_E(g)\overline{\langle g,\gamma\rangle}\, \overline{\langle -h,\gamma\rangle}\, \lambda_G(dg)\right)\nu(dh).$$

Once again, using Fubini's theorem, we have for every γ in $\hat{G}$,

$$\left(\int\chi_E(g)\, \overline{\langle g,\gamma\rangle}\, \lambda_G(dg)\right)\cdot\left(\int\overline{\langle -h,\gamma\rangle}\, \mu(dh)\right)$$
$$= \left(\int\chi_E(g)\overline{\langle g,\gamma\rangle}\lambda_G(dg)\right)\cdot\left(\int\overline{\langle -h,\gamma\rangle}\nu(dh)\right) . \qquad (1.2)$$

For a probability measure η on G, let $\bar{\eta}$ denote the probability measure defined by

$$\bar{\eta}(A) = \eta(-A), \quad A \in \mathcal{B}_G ,$$

where $-A = \{-g \mid g \in A\}$. Then, equation (1.2) can be written as

$$\hat{\chi}_E(\gamma)\, \hat{\bar{\mu}}(\gamma) = \hat{\chi}_E(\gamma)\, \hat{\bar{\nu}}(\gamma) \qquad (1.3)$$

for every $\gamma \in \hat{G}$. Since the set $\{\gamma \in \hat{G} \mid \hat{\chi}_E(\gamma) \neq 0\}$ is dense in $\hat{G}$ and $\hat{\bar{\mu}}$, $\hat{\bar{\nu}}$ are continuous functions on $\hat{G}$, it follows from equation (1.3) that

$$\hat{\bar{\mu}}(\gamma) = \hat{\bar{\nu}}(\gamma) \quad \text{for every} \quad \gamma \in \hat{G}.$$

Hence $\mu = \nu$.

1.5 <u>Example</u>: Let T denote the circle group, i.e. the multiplicative group of all complex-numbers of absolute value one. Let $X = G = T^n$ be the n-fold product of T with itself. Let T^n act on itself by left-multiplication, For $0 < \delta < 2\pi$ and any real number a, let

$$A(a,a+\delta) = \{e^{i\theta} \varepsilon T \mid a \leq \theta < a+\delta\} ,$$

i.e., $A(a,a+\delta)$ denotes an arc of length δ in T with end points e^{ia} and $e^{i(a+\delta)}$. Let

$$E = A(0,\delta_1) \times A(0,\delta_2)\times \ldots \times A(0,\delta_n),$$

where for every $j = 1,2,\ldots,n$, $0 < \delta_j < 2\pi$ and δ_j is an irrational multiple of 2π. Then E is a determining set for the class of all totally-finite measures on T^n.

In view of Lemma 1.4, we have only to show that the set

$$\{\gamma \varepsilon (T^n)^\wedge \mid \hat{\chi}_E(\gamma) \neq 0\}$$

is a dense subset of $(T^n)^\wedge$. Let $\underline{m} = (m_1, m_2, \ldots, m_n) \varepsilon (T^n)^\wedge = Z^n$. Then,

$$\hat{\chi}_E(\underline{m}) = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \left(\frac{e^{-im_j \delta_j} - 1}{-im_j} \right)$$

Since, for each $j = 1,2,\ldots,n$, δ_j is an irrational multiple of 2π, it follows that

$$\hat{\chi}_E(\underline{m}) \neq 0 \quad \text{for every } \underline{m} \varepsilon Z^n .$$

Hence, E is a determining set for the class of all totally-finite measures on T^n.

1.6. <u>Example</u> Let $G = T$ be the circle group. Let T be regarded as the additive group of real numbers modulo 2π, i.e. $T = [0,2\pi)$, the group operation being addition, +, module 2π. Let the semi-circle $[0,\pi)$ be denoted by X and let the semi-circle $[\pi,2\pi)$ be denoted by Y. Define

$$\Phi : T \times X \to X$$

as follows: for $g \in T$ and $x \in X$,

$$\Phi(g,x) = g \circ x = g+x \quad \text{if} \quad g+x \in X ,$$
$$= g+x+g_o \text{ if } g+x \in Y ,$$

where g_o denotes the point $\pi \in T = [0,2\pi)$. It is easy to see that Φ is a well defined action of T on the semi-circle X.

Let $E = A(0,\delta) = \{g \in T \mid 0 \leq g < \delta\}$, where $0 < \delta < \pi$ is an irrational multiple of 2π. Then E is a (T,Φ) - determining set for the class of all totally-finite measures on X.

To see this, let μ and ν be two totally-finite measures on X such that

$$\mu(g \circ E) = \nu(g \circ E) \quad \text{for every} \quad g \in G.$$

We have to show that $\mu = \nu$.

Define $\tilde{\mu}$ and $\tilde{\nu}$ on T as follows: for a Borel set $A \subset T$,

$$\tilde{\mu}(A) = \mu(A) \quad \text{if} \quad A \subsetneq X$$
$$= \mu(A+g_o) \quad \text{if} \quad A \subsetneq Y .$$

$\tilde{\nu}$ is defined similarly.

Let $\tilde{\mu}$ and $\tilde{\nu}$ be extended by additivity for general Borel subsets of T. Then, $\tilde{\mu}$ and $\tilde{\nu}$ are totally-finite measures on T and it is easy to see that

$$\tilde{\mu}(E+g) = \tilde{\nu}(E+g) \quad \text{for every} \quad g \in T.$$

Since, $E = A(0,\delta)$ and δ is an irrational multiple of 2π, it follows from Example 1.5 that $\tilde{\mu} = \tilde{\nu}$. Hence $\mu = \nu$.

1.7. Example : Let $(G_i, \mathcal{B}_{G_i})$, $i = 1,2$ be measurable groups acting on measurable spaces $(X_i, \mathcal{B}_{X_i})$, $i = 1,2$ respectively. Let the respective actions be ϕ_1 and ϕ_2. Let $G = G_1 \times G_2$ and $\mathcal{B}_G = \mathcal{B}_{G_1} \times \mathcal{B}_{G_2}$. Then $(G,\mathcal{B}_G)$ is a measurable group. Let $(X, \mathcal{B}_X) = (X_1 \times X_2, \mathcal{B}_{X_1} \times \mathcal{B}_{X_2})$. Define

$$\Phi : G \times X \rightarrow X$$

as

$$\Phi((g_1,g_2), (x_1,x_2)) = (\phi_1(g_1,x_1),\phi_2(g_2,x_2)) ,$$

for $g_i \in G_i$, $x_i \in X_i$, $i = 1,2$. Then Φ is an action of G on X and let it be denoted by (ϕ_1, ϕ_2).

Let $P(X_i)$ be the class of all totally-finite measures on $(X_i,\mathcal{B}_{X_i})$,i=1,2 respectively, and let $P(X)$ denote the class of all totally-finite measures on $(X, \mathcal{B}_X)$.

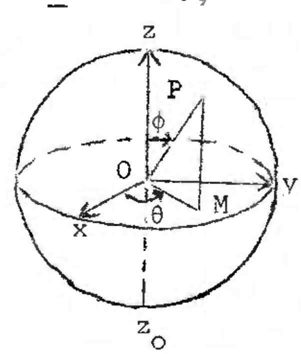
It is easy to see that if E_i is a (G_i, ϕ_i) - determining set for $P(X_i)$, $i = 1,2$ respectively, then the set

$$E = E_1 \times E_2$$

is a (G,Φ) - determining set for $P(X)$.

1.8. Example : Let $S^2 = \{(x,y,z) \in R^3 \mid x^2+y^2+z^2= 1\}$ be the surface of the unit sphere in R^3. In polar coordinates we can write

$$S^2 = \{(\phi, \theta)|0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi\},$$



i.e., a point P on S^2 has polar-coordinates (ϕ,θ) if ϕ is the angle between the radius vector OP and the z-axis, and θ is the angle between the projection OM of the radius vector OP on the xy-plane and the x-axis, θ being measured in the counter-clockwise direction. In otherwords, we consider S^2 as the cartesian product of the semi-circle $[0,\pi]$ and the circle $T = [0,2\pi)$. Let $X = [0,\pi) \times T$, i.e., $X = S^2 \setminus \{Z_o\}$, where the point Z_o has cartesian coordinates $(0,0,-1)$. Let T act on the semi-circle $[0,\pi)$ as described in Example 1.6, and let Φ_1 denote this action. Let the natural action of T on itself be denoted by Φ_2. Then $\Phi = (\Phi_1, \Phi_2)$ is an action of $T \times T$ on X as described in Example 1.7.

Let $0 < \delta_1 < \pi$ and $0 < \delta_2 < 2\pi$ be such that δ_1 and δ_2 are irrational multiples of 2π. Then we know from Example 1.6 that $A(0,\delta_1)$ is a (T,Φ_1) - determining set for the class of all totally-finite measures on $[0,\pi)$. We also know from Example 1.5 that $A(0,\delta_2)$ is a (T, Φ_2) - determining set for the class of all totally - finite measures on T. It follows from Example 1.7 that the set $E = A(0,\delta_1) \times A(0,\delta_2)$ is a $(T \times T, \Phi)$ - determining set for the class of all totally-finite measures on X. This fact can be explicitly stated as follows:

Let μ and ν be two totally-finite measures on S^2 such that $\mu\{Z_o\} = \nu\{Z_o\}$ and let μ,ν satisfy the following relations:

(i) $\mu(A(a,a+\delta_1) \times A(b, b+\delta_2)) = \nu(A(a,a+\delta_1) \times A(b,b+\delta_2))$,

whenever $0 \leq a < \pi - \delta_1$, $0 \leq b < 2\pi$;

(ii) $\mu(A(a,\pi) \times A(b,b+\delta_2)) + \mu(A(0,a+\delta_1-\pi) \times A(b,b+\delta_2))$

$= \nu(A(a,\pi)\times A(b,b+\delta_2)) + \nu(A(0,a+\delta_1-\pi)\times A(b,b+\delta_2))$,

whenever $\pi-\delta_1 \leq a < \pi$, $0 \leq b < 2\pi$.

Then $\mu = \nu$.

§ 2. DETERMINING SETS IN ABELIAN GROUPS

Let G be a locally-compact second-countable group and let $\mathcal{B}_G$ be the σ-algebra of Borel subsets of G. Let λ_G denote a Haar-measure on G and let $M(G)$ denote the class of all probability measures on G. When G is abelian and $\hat{G}$ denotes the dual group of G, then by Lemma 1.4 we know that every $E \in \mathcal{B}_G$ with $0 < \lambda_G(E) < \infty$ and such that the set $\{\gamma \in \hat{G} \mid \hat{\chi}_E(\gamma) \neq 0\}$ is dense in $\hat{G}$, is a determining set for $M(G)$. Also by Example 1.3, not every $E \in \mathcal{B}_G$ with $0 < \lambda_G(E) < \infty$ is a determining set for $M(G)$. This leads to the natural question: how big is the class of all determining sets for $M(G)$? In this section we will show that when G is abelian, then generically a set of finite positive Haar-measure in G is a determining set for $M(G)$. Let,

$$s(G) = \{E \in \mathcal{B}_G \mid \lambda_G(E) < \infty\} .$$

Let $A,B \in s(G)$. We say that A is <u>equivalent</u> to B, and write $A \sim B$, whenever $\lambda_G(A \Delta B) = 0$. The relation $\sim$ is an equivalence relation on $s(G)$. We denote the set of equivalence classes of $s(G)$ under this relation by $s(G)$ itself. For $A,B \in s(G)$, define

$$d(A,B) = \lambda_G(A \Delta B) .$$

Then d is a well-defined metric on $s(G)$ and $s(G)$ becomes a complete metric space under this metric. For any real number $\alpha > 0$, let

$$s_\alpha(G) = \{E \in s(G) \mid \lambda_G(E) < \alpha\} .$$

Then $s_\alpha(G)$ is an open subset of $(s(G),d)$. Let d_α denote the restriction of the metric d to $s_\alpha(G)$. Then, a metric d'_α can be defined on $s_\alpha(G)$ such that d'_α is equivalent to d_α and $(s_\alpha(G),d'_\alpha)$ is a complete metric space, (see Dieudonne [6], p. 55).

2.1. Lemma : Let G be a locally-compact second countable group and let $f: G \to \mathbb{C}$ be a bounded continuous function such that $f(e) = 1$. Then the set

$$D(\alpha;f) = \{E \in s_\alpha(G) \mid \int_E f(x)\lambda_G(dx) \neq 0\}$$

is an open dense subset of $(s_\alpha(G), d'_\alpha)$.

Proof : Clearly $D(\alpha ; f)$ is open in $s_\alpha(G)$. To show that $D(\alpha ; f)$ is dense in $s_\alpha(G)$, we have to show that for every $\delta > 0$ and $E \in s_\alpha(G)$, there exists an $A \in D(\alpha ; f)$ such that $\lambda_G(A \Delta E) < \delta$.

Let $E \in s_\alpha(G)$ be chosen arbitrarily and fixed. Let $\delta > 0$ be arbitrary. Without loss of generality, let $\delta < 1$. If $E \in D(\alpha ; f)$, then there is nothing to prove. Let $E \notin D(\alpha ; f)$, i.e., $\int_E f(x)\lambda_G(dx) = 0$.

Since, f is continuous and $f(e) = 1$, choose an open neighbourhood N of e in G such that

$$0 < \lambda_G(N) < \min(\delta, \alpha-\lambda_G(E)) ,$$

and

$$|1-f(x)| < \delta \text{ for every } x \in N.$$

Then,

$$\begin{aligned} \left| \int_N f(x)\lambda_G(dx) \right| &\geq \lambda_G(N) - \int_N |1-f(x)|\lambda_G(dx) \\ &> (1-\delta)\, \lambda_G(N) \\ &> 0 . \end{aligned}$$

Thus,

$$\int_N f(x)\lambda_G(dx) \neq 0 .$$

Now two cases can arise :

Case (i) $\int_{E \cup N} f(x)\, \lambda_G(dx) \neq 0$.

Put, $A = E \cup N$. Then $\int_A f(x)\lambda_G(dx) \neq 0$ and

$$0 < \lambda_G(A) = \lambda_G(E \cup N)$$
$$\leq \lambda_G(E) + \lambda_G(N)$$
$$< \alpha$$

Thus, $A \in D(\alpha\ ; f)$ and clearly $\lambda_G(A \,\Delta\, E) < \lambda_G(N) < \delta$.

Case (ii) $\int_{E \cup N} f(x)\lambda_G(dx) = 0$.

Put $A = E \setminus (E \cap N)$.

Then,

$$\int_A f(x)\lambda_G(dx) + \int_N f(x)\lambda_G(dx) = \int_{E \cup N} f(x)\lambda_G(dx) = 0.$$

i.e.

$$\int_A f(x)\lambda_G(dx) = - \int_N f(x)\lambda_G(dx) \neq 0 ,$$

Further, clearly $0 < \lambda_G(A) < \alpha$ and $\lambda_G(A \,\Delta\, E) < \delta$.

Thus, in either case there exists an $A \in D(\alpha\ ; f)$ such that $\lambda_G(A \,\Delta\, E) < \delta$.

This proves the lemma completely.

2.2. Theorem : Let G be a locally-compact second-countable abelian group. Then for every real number $\alpha > 0$, there exists $D(\alpha\ ; G) \subset \mathcal{B}_G$ such that the following hold:

(i) $D(\alpha\ ; G)$ is a dense G_δ - subset of $(s_\alpha(G), d'_\alpha)$;

(ii) Every $E \in D(\alpha\ ; G)$ is a determining set for $M(G)$.

Proof : Since G is second-countable, its dual group $\hat{G}$ is separable. Let $\gamma_1, \gamma_2, \ldots$ be a dense subset of $\hat{G}$. Since γ_j is a continuous complex-valued functic on G and $\gamma_j(e) = 1$, it follows from Lemma 2.1 that the set

$$D(\alpha;\gamma_j) = \{E \in s_\alpha(G) \mid \int_E \overline{\langle x,\gamma_j\rangle} \lambda_G(dx) \neq 0\}$$

is an open dense subset of $(s_\alpha(G), d'_\alpha)$. Since $(s_\alpha(G), d'_\alpha)$ is a complete metric space, by Baire's Category Theorem,

$$D(\alpha;G) = \bigcap_{j=1}^{\infty} D(\alpha;\gamma_j)$$

is a dense subset of $(s_\alpha(G), d'_\alpha)$. Obviously $D(\alpha ; G)$ is a G_δ - set.

Let $E \in D(\alpha ; G)$. Then, $0 < \lambda_G(E) < \alpha$ and the set

$$\{\gamma \in \hat{G} \mid \hat{\chi}_E(\gamma) \neq 0\}$$

is a dense subset of $\hat{G}$, since it contains γ_j for every $j = 1,2,\ldots$.
Thus by Lemma 1.4, E is a determining set for $M(G)$.

Q.E.D.

§ 3. SOME MORE DETERMINING SETS IN ABELIAN GROUPS

Let G be a locally-compact second-countable abelian group and let $\mathcal{B}_G$ be the σ - algebra of Borel subsets of G. Let λ_G denote the Haar measure on G and let $M(G)$ denote the class of all probability measures on G. Let G act on itself by translation. Let $E \in \mathcal{B}_G$ be a fixed subset of G such that $\lambda_G(E) > 0$ and its closure, $\bar{E}$, is compact. We showed in Example 1.3 that such a set E need not be a determining set for $M(G)$. For what kind of classes P of probability measures on G, is E a determining set? In this section we show that to any such set E one can associate a compact subgroup K of G such that E becomes a determining set for the class $M(K \backslash G/K)$ of all K-invariant probability measures on G. To prove this result, we need some elementary results which we prove in 3.1 and 3.2.

3.1

Let $\mu, \nu \in M(G)$. The convolution of μ with ν, denoted by $\mu * \nu$, is defined by

$$(\mu * \nu)(A) = \int \mu(A-x)\nu(dx), \qquad A \in \mathcal{B}_G .$$

Clearly, $\mu * \nu \in M(G)$ and $\mu * \nu = \nu * \mu$.

Let $\mu \in M(G)$ and let f be any measurable function on $(G, \mathcal{B}_G)$. Consider the function $x \mapsto \int_G f(x-y)\mu(dy)$, $x \in G$. Whenever, this is well defined, it will be called the convolution of f with μ and written as $f * \mu$. For example, if $f \in L_1(G)$ then $f * \mu$ is well-defined and $f * \mu \in L_1(G)$. If f is a bounded continuous function, then $f * \mu$ is well defined and is a bounded continuous function. It is easily seen that the following relations are true in the sense that whenever either side is well-defined, so is the other and they are equal:

$$\left.\begin{array}{ll}\text{(i)} & f * (g * \mu) = (f * g) * \mu = (f * \mu) * g; \\ \text{(ii)} & f * (\mu * \nu) = (f * \mu) * \nu \;;\end{array}\right\} \quad (3.1)$$

where f, g are measurable functions on G and $\mu, \nu \in M(G)$.

3.2

Let K be any compact subgroup of G. Consider the quotient group G/K with the quotient topology. Let the σ - algebra of Borel subsets of G/K be denoted by $B_{G/K}$. Let λ_K denote the normalised Haar-measure of K and let $\lambda_{G/K}$ denote the Haar-measure of G/K. Let $I(K)$ denote the space of all K-invariant Borel functions on G, i.e. $f \in I(K)$ iff f is a Borel function on G and

$$f(x+k) = f(x) \quad \text{for every} \quad x \in G,\ k \in K.$$

Let $B(G/K)$ denote the space of all Borel functions on G/K. Define the map

$$T : I(K) \to B(G/K)$$

as follows: for every $f \in I(K)$ and $x \in G$,

$$(Tf)(x+K) = f(x) .$$

Then T is a well-defined map. Some properties of T are given by the following proposition.

3.2.1. Proposition : (i) T is one-one and onto;

(ii) Tf is bounded whenever $f \in I(K)$ is bounded and Tf has compact support whenever $f \in I(K)$ has compact support;

(iii) $T(f \cdot g) = (Tf) \cdot (Tg)$ for every $f, g \in I(K)$;

(iv) If $f \in I(K) \cap L_1(G)$ and $g \in I(K)$ is bounded, then

$$T(f * g) = (Tf) * (Tg) .$$

Proof : (i), (ii) and (iii) are easy to prove. We shall prove (iv).

For $f \in L_1(G)$, the Haar-measures λ_G, λ_K and $\lambda_{G/K}$ satisfy the following relation :

$$\int_G f(x)\lambda_G(dx) = \int_{G/K} T\left(\int_K f(x+k)\lambda_K(dk)\right)\lambda_{G/K}(d(x+K)) .$$

(See, Dieudonne [7], p. 249). In particular, if $f \in \mathcal{I}(K) \cap L_1(G)$, we have

$$\int_G f(x)\lambda_G(dx) = \int_{G/K} (Tf)(x+K)\,\lambda_{G/K}(d(x+K)) \qquad (3.2)$$

Now, let $f \in \mathcal{I}(K) \cap L_1(G)$ and let $g \in \mathcal{I}(K)$ be bounded. Let $f_x(y)=f(x-y)$ for $x,y \in G$. Then for $x \in G$

$$\begin{aligned}
[T(f * g)](x+K) &= (f * g)(x) \\
&= \int_G f_x(y)\, g(y)\, \lambda_G(dy) \\
&= \int_{G/K} [T(f_x \cdot g)](y+K)\, \lambda_{G/K}(d(y+K)) \\
&= \int_{G/K} (Tf_x)(y+K)\cdot(Tg)(y+K)\, \lambda_{G/K}(d(y+K)) \\
&= \int_{G/K} (Tf)(x-y+K)(Tg)(y+K)\, \lambda_{G/K}(d(y+K)) \\
&= [Tf * Tg](x+K).
\end{aligned}$$

This proves the proposition completely.

Next we prove a result concerning the real-zeros of a function of several complex-variables which we need for proving the main result of this section.

3.2.2. Lemma : Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-zero analytic function. Then the set $\{\underline{x} \in \mathbb{R}^n \mid f(\underline{x}) \neq 0\}$ is an open dense subset of $\mathbb{R}^n$.

<u>Proof</u> : Let $D = \{\underline{x} \in R^n \mid f(\underline{x}) \neq 0\}$. Clearly D is an open subset of R^n. We prove, by induction on n, that D is dense in R^n.

Let n=1. Then f is a non-zero analytic function of one complex-variable. Thus the set $\{z \in \mathbb{C} \mid f(z) = 0\}$ is atmost countable and consists of isolated points only. In particular the set $\{x \in R \mid f(x) = 0\}$ is atmost countable and all its points are isolated points. Hence the set $\{x \in R \mid f(x) \neq 0\}$ is dense in R.

Now suppose the statement is true for n = k-1. Let

$$A = \{z' \in \mathbb{C} \mid f(z',\underline{z}) = 0 \quad \text{for every } \underline{z} \in \mathbb{C}^{k-1}\} \quad .$$

Then A is a closed subset of $\mathbb{C}$ and all its points are isolated points. In particular, the set

$$B = \{t \in R \mid f(t,\underline{z}) = 0 \quad \text{for every } \underline{z} \in \mathbb{C}^{k-1}\}$$

is a closed subset of R and all its points are isolated points. Thus the set $R \setminus B$ is a dense subset of R. For every $t \in R \setminus B$, let

$$B_t = \{\underline{s} \in R^{k-1} \mid f(t,\underline{s}) \neq 0\} .$$

By the induction hypothesis, the set B_t is dense in R^{k-1} for every $t \in R \setminus B$. Consider the set

$$\bigcup_{t \in R \setminus B} (\{t\} \times B_t) \subset R^k \quad .$$

Clearly, it is a dense subset of R^k and is contained in the set $\{\underline{x} \in R^k \mid f(\underline{x}) \neq 0\}$. Hence the statement is also true for n = k.

This proves the lemma.

3.2.3. <u>Corollary</u> : Let $f : \mathbb{C}^{r+k} \to \mathbb{C}$ be a non-zero analytic function. Then the set

$$\{(\underline{z},\underline{x}) \in T^r \times R^k \mid f(\underline{z},\underline{x}) \neq 0\}$$

is an open dense subset of $T^r \times R^k$.

Proof : Apply induction on r and proceed on the lines of the proof of Lemma 3.2.2.

3.2.4 Corollary: Let $G = Z^r \times R^k$ and let f be any non-zero bounded Borel function on G with compact support. Then the set

$$\{\gamma \in T^r \times R^k \mid \hat{f}(\gamma) \neq 0\}$$

is an open dense subset of $T^r \times R^k$.

Proof : Write $\sigma \in G$ as $(\underline{n}, \underline{x})$, where $\underline{n} \in Z^r$ and $\underline{x} \in R^k$. Also, write $\gamma \in T^r \times R^k$ as $(\underline{\theta}, \underline{y})$, where $\underline{\theta} \in T^r$ and $\underline{y} \in R^k$. Then by the definition of $\hat{f}$, we have

$$\hat{f}(\underline{\theta},\underline{y}) = \frac{1}{(2\pi)^r} \int_{Z^r \times R^k} \exp\{-i \sum_{\ell=1}^{r} \theta_\ell n_\ell - i \sum_{j=1}^{k} x_j y_j\} f(\underline{n},\underline{x}) d(\underline{n}) d(\underline{x}),$$

where $\underline{\theta} = (\theta_1, \ldots, \theta_r)$, $\underline{n} = (n_1,\ldots,n_r)$, $\underline{x} = (x_1, \ldots, x_k)$ and $\underline{y} = (y_1, \ldots, y_k)$.

Define

$$\phi : \mathbb{C}^{r+k} \to \mathbb{C}$$

as follows:

$$\phi(\underline{z}) = \frac{1}{(2\pi)^r} \int_{Z^r \times R^k} z_1^{n_1} \cdot z_2^{n_2} \cdots z_r^{n_r} \cdot \exp\{-i \sum_{j=1}^{k} x_j z_{j+r}\} f(\underline{n},\underline{x}) d(\underline{n}) d(\underline{x}),$$

for every $\underline{z} = (z_1,\ldots,z_{r+k}) \in \mathbb{C}^{r+k}$. Then ϕ is a well defined analytic function on $\mathbb{C}^{r+k}$. Further ϕ is non-zero. In fact, ϕ restricted to $T^r \times R^k$ is $\hat{f}$. Thus, it follows from Corollary 3.2.3 that the set

$$\{\gamma \in T^r \times R^k \mid \hat{f}(\gamma) \neq 0\}$$

is an open dense subset of $T^r \times R^k$.

3.2.5. <u>Proposition</u> : Let G be a locally-compact second-countable abelian group and let $E \in \mathcal{B}_G$ be such that $\lambda_G(E) > 0$ and $\bar{E}$ is compact. Let G_o be the subgroup of G generated by E and let K be the maximal compact subgroup of G_o. Let $f \in L_1(G)$ be a continuous function such that $(\chi_E * f)(x) = 0$ for every $x \in G$. Then $f * \lambda_K \equiv 0$, where λ_K denotes the normalised Haar measure of K.

<u>Proof</u> : First note that G_o is an open subgroup of G. Further, G_o is a locally-compact, compactly generated abelian group. Hence, by the Structure Theorem, G_o is isomorphic to $Z^r \times R^k \times K$, where Z is the integer group, K is some compact abelian group and r,k are non-negative integers (see Hewitt and Ross [10], p. 90). So we can write $G_o = Z^r \times R^k \times K$. Then K is the maximal compact subgroup of G_o.

To prove the proposition, first assume that $f \in I(K)$. We shall show that under the given hypothesis, $f \equiv 0$. Let $x_o \in G$ be chosen arbitrarily and fixed. Put

$$f_{x_o}(x) = f(x_o + x), \qquad x \in G.$$

Then $f_{x_o} \in I(K) \cap L_1(G)$ is a continuous function. Further, from the given condition on f, we have

$$\int_E f_{x_o}(x-y)\lambda_G(dy) = 0 \quad \text{for every} \quad x \in G .$$

Since, the integration is only over a subset of G_o, we have

$$\int_E f_{x_o}(x-y)\, \lambda_{G_o}(dy) = 0 \quad \text{for every} \quad x \in G_o ,$$

where λ_{G_o} denotes the Haar-measure of G_o. Equivalently,

$$(\chi_E * f_{x_o})(x) = 0 \quad \text{for every} \quad x \in G_o .$$

Thus

$$[(\chi_E * f_{x_0}) * \lambda_K](x) = 0 \quad \text{for every} \quad x \in G_0 ,$$

i.e.,

$$[f_{x_0} * (\chi_E * \lambda_K)](x) = 0 \quad \text{for every} \quad x \in G_0 . \qquad (3.3)$$

Let $\psi = \chi_E * \lambda_K$. Then $\psi \in I(K)$ and ψ is a non-zero bounded function with compact support. Applying the map T(defined in section 3.2) to both sides of equation (3.3) and using Proposition 3.2.1, we have

$$(Tf_{x_0}) * (T\psi) = 0 \quad \text{on} \quad G_0/K = Z^r \times R^k .$$

Taking Fourier-transforms , we have

$$(\widehat{Tf_{x_0}})(\gamma) \cdot (\widehat{T\psi})(\gamma) = 0 \qquad (3.4)$$

for every $\gamma \in T^r \times R^k$. Since $T\psi$ is a non-zero bounded function with compact support, it follows from Corollary 3.2.4 that the set $\{\gamma \in T^r \times R^k \mid (\widehat{T\psi})(\gamma) \neq 0\}$ is dense in $T^r \times R^k$. Also $(\widehat{T\psi})$ is a continuous function on $T^r \times R^k$. Thus it follows from equation (3.4) that

$$(\widehat{Tf_{x_0}})(\gamma) = 0 \quad \text{for every} \quad \gamma \in T^r \times R^k = (\widehat{G_0/K}).$$

Since f_{x_0} is continuous, so is Tf_{x_0}. Thus it follows that $Tf_{x_0} \equiv 0$ on G_0/K, and hence $f_{x_0} \equiv 0$ on G_0. Since $x_0 \in G$ was chosen arbitrarily, we have $f \equiv 0$ on G.

To prove the proposition in the general case, put

$$\tilde{f} = f * \lambda_K .$$

Then $\tilde{f}$ is a continuous function, $\tilde{f} \in I(K) \cap L_1(G)$ and $\int_E \tilde{f}(x-y)\lambda_G(dy) = 0$ for every $x \in G$. Thus it follows from the above discussion that $f * \lambda_K \equiv 0$ on G.

3.3.1 Definition : Let G be a locally-compact second countable group and let K a subgroup of G. A probability measure η on G is said to be K-invariant if

$$\eta(k_1 A k_2) = \eta(A) \quad \text{for every} \quad A \in \mathcal{B}_G, \; k_1, k_2 \in K .$$

We denote the set of all K-invariant probability measures on G by $\mathcal{M}(K \backslash G/K)$

3.3.2 Theorem : Let G be a locally-compact second countable abelian group and let $E \in \mathcal{B}_G$ be such that $\lambda_G(E) > 0$ and $\bar{E}$ is compact. Let G_o be the subgroup of G generated by E and let K be the maximal compact subgroup of G_o. Let $\mu, \nu \in \mathcal{M}(G)$ be such that

$$\mu(E+x) = \nu(E+x)$$

for every $x \in G$. Then

$$\mu * \lambda_K = \nu * \lambda_K \quad .$$

In particular, E is a determining set for $\mathcal{M}(K \backslash G/K)$.

Proof : Let $\mu, \nu \in \mathcal{M}(G)$ and let

$$\mu(E+x) = \nu(E+x) \quad \text{for every} \quad x \in G ,$$

Equivalently,

$$(\chi_{-E} * \mu)(x) = (\chi_{-E} * \nu)(x) \text{ for every } x \in G,$$

where -E denotes the set $\{-x \mid x \in E\}$. Let f be any continuous function on G with compact support. Then for every $x \in G$, we have

$$[f * (\chi_{-E} * \mu)](x) = [f * (\chi_{-E} * \nu)](x)$$

Using equation (3.1), we have for every $x \in G$

$$[(f*\mu) * \chi_{-E}](x) = [(f * \nu) * \chi_{-E}](x) \qquad (3.5)$$

Put

$$\tilde{f} = f * \mu - f * \nu \quad .$$

Then $\tilde{f}$ is a continuous function and $\tilde{f} \in L_1(G)$. Further,

$$(\tilde{f} * \chi_{-E})(x) = 0 \quad , \quad \text{for every } x \in G .$$

Since the groups generated by E and - E are the same, we have from Proposition 3.2.5,

$$(\tilde{f} * \lambda_K)(x) = 0 \quad , \quad \text{for every } x \in G .$$

Using equation (3.1), we get

$$[f * (\mu * \lambda_K)](x) = [f * (\nu * \lambda_K)](x)$$

for every $x \in G$. Since this holds for every continuous function f with compact support, we have

$$\mu * \lambda_K = \nu * \lambda_K .$$

Q.E.D.

§ 4. WEAK CONVERGENCE DETERMINING SETS IN ABELIAN GROUPS

4.1

Let G be a locally-compact second countable abelian group. Let $\mathcal{B}_G$ denote the σ-algebra of Borel subsets of G and let λ_G denote the Haar measure of G. Let $M(G)$ denote the set of all probability measures on G. We say a sequence $\{\mu_n\}$ in $M(G)$ converges weakly to μ in $M(G)$, and write $\mu_n \Rightarrow \mu$, if for every bounded continuous function f on G

$$\int f(x)\, \mu_n(dx) \rightarrow \int f(x)\, \mu(dx)$$

as $n \rightarrow \infty$. The topology induced by this convergence is called the weak-topology on $M(G)$. For a detailed discussion of the weak-convergence and the weak-topology, we refer to Parthasarathy [13], Chapter -II. Under the convolution operation $*$ and the weak-topology, $M(G)$ becomes a complete separable metric semi-group.

A set $\Gamma \subset M(G)$ is said to be conditionally compact if the closure of Γ in the weak-topology is compact. Let $\{\mu_n\}$ and $\{\nu_n\}$ be two sequences in $M(G)$. Let $\lambda_n = \mu_n * \nu_n$ for every n. If the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are conditionally compact, then so is the sequence $\{\nu_n\}$, (see Parthasarathy [13], p.58). In particular, let $f \in L_1(G)$ be any bounded function and let $\{\mu_n\}$ be a sequence in $M(G)$. If $f * \mu_n \rightarrow f * \mu$, $\mu \in M(G)$, then it follows that $\{\mu_n\}$ is conditionally compact.

For $\mu \in M(G)$, let $\bar{\mu}$ denote the probability measure on G defined by

$$\bar{\mu}(A) = \mu(-A), \quad A \in \mathcal{B}_G .$$

Where $-A = \{-x \mid x \in A\}$. The map $\mu \rightarrow \bar{\mu}$ is a homeomorphism of $M(G)$ onto itself.

4.2. Definition : Let $E \in \mathcal{B}_G$. We say that E is a weak-convergence determining set for a class $\mathcal{P}$ of probability measures on G if E has the following property:

if for a sequence $\{\mu_n\}_{n=0,1,2,\ldots\ldots}$ in $\mathcal{P}$, $\mu_n(E+x) \to \mu_o(E+x)$ for every $x \in G$, then $\mu_n \Rightarrow \mu_o$

4.3. Example (Sapogov [14]) : Let $G = R^n$ and $\mathcal{P} = M(R^n)$. Let $E \in \mathcal{B}_{R^n}$ be such that $0 < \lambda_{R^n}(E) < \infty$. Then E is a weak-convergence determining set for $M(R^n)$.

4.4. Lemma : Let G be a locally-compact second countable abelian group and let $E \in \mathcal{B}_G$ be such that $0 < \lambda_G(E) < \infty$. Let $\{\mu_n\}_{n=0,1,2\ldots}$ be a sequence in $M(G)$ such that

$$\mu_n(E+x) \to \mu_o(E+x) \text{ for every } x \in G.$$

Then,

(i) $\{\mu_n\}_{n=1,2,\ldots\ldots}$ is conditionally compact ;

(ii) for every limit point ν of $\{\mu_n\}_{n=1,2,\ldots\ldots}$,

$$(\chi_E * \mu_o)(x) = (\chi_E * \nu)(x) \text{ for almost all } x(\lambda_G).$$

Proof : From the given condition on $\{\mu_n\}_{n=0,1,2,\ldots\ldots}$ we have

$$\chi_E * \bar{\mu}_n \to \chi_E * \bar{\mu}_o .$$

Since, $\chi_E \in L_1(G)$, it follows that $\{\bar{\mu}_n\}_{n=1,2\ldots\ldots}$ is conditionally compact and hence $\{\mu_n\}_{n=1,2,\ldots\ldots}$ is conditionally compact. This proves (i).

To prove (ii), let ν be any limit point of $\{\mu_n\}_{n=1,2,\ldots\ldots}$. Let $\{\mu_{n_k}\}_{k=1,2\ldots}$ be a subsequence of $\{\mu_n\}_{n=1,2,\ldots}$ such that

$$\mu_{n_k} \Rightarrow \nu ,$$

Going to Fourier-transforms, we have

$$\hat{\mu}_{n_k} \to \hat{\nu} .$$

Thus,

$$\hat{\chi}_E \cdot \hat{\mu}_{n_k} \to \hat{\chi}_E \cdot \hat{\nu} .$$

i.e.,

$$\widehat{(\chi_E * \mu_{n_k})} \to \widehat{(\chi_E * \nu)} \qquad (4.1)$$

On the other hand, since $\mu_{n_k}(E+x) \to \mu_o(E+x)$ for every $x \in G$, we have

$$\chi_E * \mu_{n_k} \to \chi_E * \mu_o .$$

Going to Fourier-transforms we have

$$\widehat{(\chi_E * \mu_{n_k})} \to \widehat{(\chi_E * \mu_o)} \qquad (4.2)$$

From equations (4.1) and (4.2), we have

$$(\chi_E * \mu_o)(x) = (\chi_E * \nu)(x) \text{ for almost all } x(\lambda_G).$$

This proves the Lemma.

4.5 <u>Lemma</u> : Let G be a locally-compact second-countable abelian group and let $E \in B_G$ be such that $0 < \lambda_G(E) < \infty$ and the set

$$\{\gamma \in \hat{G} \mid \hat{\chi}_E(\gamma) \neq 0\} ,$$

is dense in $\hat{G}$. Then E is a weak-convergence determing set for $M(G)$.

<u>Proof</u> : Let $\{\mu_n\}_{n=0,1,2\ldots}$ be a sequence in $M(G)$ such that

$$\mu_n(E+x) \to \mu_o(E+x) \quad \text{for every} \quad x \in G.$$

Then it follows from Lemma 4.4, that $\{\mu_n\}_{n=1,2,\ldots}$ is conditionally compact. Further, for any limit point ν of $\{\mu_n\}_{n=1,2,3\ldots}$, we have

$$(\chi_E * \mu_o)(x) = (\chi_E * \nu)(x) \text{ for almost all } x(\lambda_G).$$

Going to Fourier-transforms, we have

$$\hat{\chi}_E(\gamma)\,\hat{\mu}_o(\gamma) = \hat{\chi}_E(\gamma)\,\hat{\nu}(\gamma) \text{ for every } \gamma \in \hat{G} \;.$$

Since, $\{\gamma \in \hat{G} \mid \hat{\chi}_E(\gamma) \neq 0\}$ is a dense subset of $\hat{G}$ and $\hat{\mu}_o, \hat{\nu}$ are continuous on $\hat{G}$, we have

$$\hat{\mu}_o(\gamma) = \hat{\nu}(\gamma) \quad \text{for every} \quad \gamma \in \hat{G} \;.$$

Hence $\mu_o = \nu$.

Thus the sequence $\{\mu_n\}_{n=1,2,\ldots}$ has only one limit point, namely μ_o.

This proves the Lemma.

4.6. <u>Theorem</u> : Let G be a locally-compact second countable abelian group. Then for every real number $\alpha > 0$, there exists $D(\alpha;G) \subset \mathcal{B}_G$ such that the following hold:

(i) $D(\alpha;G)$ is a dense G_δ - subset of the metric space $(s_\alpha(G), d'_\alpha)$.
(Recall, $s_\alpha(G) = \{E \in \mathcal{B}_G \mid \lambda_G(E) < \alpha\}$) ;

(ii) Every $E \in D(\alpha;G)$ is a weak-convergence determining set for $M(G)$.

<u>Proof</u> : Let $D(\alpha;G)$ be constructed as in Theorem 2.2. We have only to prove (ii).

Let $E \in D(\alpha;G)$. Then it follows from the construction of $D(\alpha;G)$ that $\alpha > \lambda_G(E) > 0$ and the set $\{\gamma \in \hat{G} \mid \hat{\chi}_E(\gamma) \neq 0\}$ is dense in $\hat{G}$. It follows from Lemma 4.5 that E is a determining set for $M(G)$.

This proves the theorem.

4.7. Remark : Theorem 4.6 tells us that generically, every subset of G with finite positive Haar measure is a weak-convergence determining set for $M(G)$.

4.8. Theorem : Let G be a locally-compact second countable abelian group and let $E \in \mathcal{B}_G$ be such that $\lambda_G(E) > 0$ and $\bar{E}$ is compact. Let G_o be the subgroup of G generated by E and let K be the maximal compact subgroup of G_o. Then E is a weak-convergence determining set for $M(K\backslash G/K)$, the class of all K - invariant probability measures on G.

Proof : Let $\{\mu_n\}_{n=0,1,2....}$ be a sequence in $M(G)$ such that

$$\mu_n (E + x) \rightarrow \mu_o(E + x)$$

for every $x \in G$. We shall show that

$$\mu_n * \lambda_K \Rightarrow \mu_o * \lambda_K$$

This will prove in particular that E is a weak convergence determining set for $M(K\backslash G/K)$. Since

$$\mu_n(E+x) \rightarrow \mu_o(E+x)$$

for every $x \in G$, it follows from Lemma 4.4, that the sequence $\{\mu_n\}_{n=1,2,...}$ is conditionally compact. Further, for any limit point ν of $\{\mu_n\}_{n=1,2,...}$ we have

$$(\chi_E * \mu_o)(x) = (\chi_E * \nu)(x) \text{ for almost all } x(\lambda_G).$$

Let f be any continuous function on G with compact support. Then

$$[f*(\chi_E*\mu_o)](x) = [f*(\chi_E*\nu)](x) \text{ for every } x \in G.$$

This equation is similar to the equation (3.5). Proceeding as in the proof of Theorem 3.3.2 from equation (3.5) onwards, we will get

$$\mu_o * \lambda_K = \nu * \lambda_K \quad .$$

Thus, for every limit point ν of the sequence $\{\mu_n\}_{n=1,2,\ldots,}$ $\mu_o * \lambda_K = \nu * \lambda_K$.

Hence $\mu_n * \lambda_K \Rightarrow \mu_o * \lambda_K$.

Q.E.D.

§ 5. DETERMINING SETS IN GROUPS WHICH ARE NOT NECESSARILY ABELIAN

Let G be a locally-compact second countable group (not necessarily abelian). Let $\mathcal{B}_G$ denote the σ - algebra of Borel subsets of G and let λ_G be a Haar measure of G. Let $E \in \mathcal{B}_G$ be such that $0 < \lambda_G(E) < \infty$. In this section we consider the following problem: for what families $\mathcal{P}$ of probability measures on G, is E a determining set?

5.1. Example : Let $G = A_4$ be the group of all even permutations on four symbols, say $\{1,2,3,4\}$. The group A_4 has 12 elements. Considering each $g \in A_4$ as a bijection from $\{1,2,3,4\}$ onto itself, g can be represented as

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ g(1) & g(2) & g(3) & g(4) \end{pmatrix}$$

In this notation, the elements of A_4 are :

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$g_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$g_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}, \quad g_9 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$$g_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}, \quad g_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \quad g_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

The group operation on A_4 is defined by the composition of maps, i.e.,

$$g_i \, g_j = \begin{pmatrix} 1 & 2 & 3 & 4 \\ g_i g_j(1) & g_i g_j(2) & g_i g_j(3) & g_i g_j(4) \end{pmatrix}$$

A_4 is a non-abelian group. It is a topological group with the discrete topology. The σ-algebra B_{A_4} is the class of all subsets of A_4 and the Haar measure λ of A_4 is the measure which gives unit mass to every point of A_4.

Let E = $\{g_1, g_4\}$. Then E is a subgroup of A_4 and λ(E) > 0. Let μ and ν be two probability measures on A_4 defined as follows:

$$\mu\{g_1\} = \nu\{g_4\} = 1/78, \quad \mu\{g_2\} = \nu\{g_3\} = 2/78,$$
$$\mu\{g_3\} = \nu\{g_2\} = 3/78, \quad \mu\{g_4\} = \nu\{g_1\} = 4/78,$$
$$\mu\{g_5\} = \nu\{g_8\} = 5/78, \quad \mu\{g_6\} = \nu\{g_{10}\} = 6/78,$$
$$\mu\{g_7\} = \nu\{g_{11}\} = 7/78, \quad \mu\{g_8\} = \nu\{g_5\} = 8/78,$$
$$\mu\{g_9\} = \nu\{g_{12}\} = 9/78, \quad \mu\{g_{10}\} = \nu\{g_6\} = 10/78,$$
$$\mu\{g_{11}\} = \nu\{g_7\} = 11/78, \quad \mu\{g_{12}\} = \nu\{g_9\} = 12/78.$$

It is easy to see that $\mu(g\cdot E) = \nu(g\cdot E)$ for every $g \in A_4$. However, $\mu \neq \nu$. Thus E is not a determining set for the class of all probability measures on A_4. Let λ_E denote the Haar measure of the subgroup E, i.e.,

$$\lambda_E\{g_1\} = \lambda_E\{g_4\} = \frac{1}{2}$$

Then it is easy to see that $\mu * \lambda_E = \nu * \lambda_E$.

Our next Theorem shows that more generally, this happenes to be true for any locally-compact second-countable group.

5.1.1. <u>Definition</u> : Let G be a locally-compact second countable group and let K be a subgroup of G. $\mu \in M(G)$ is said to be <u>right K-invariant</u> if

$$\mu(Ak) = \mu(A) \quad \text{for every } A \in B_G,\ k \in K.$$

Let $M(G \setminus K)$ denote the class of all right K-invariant probability measures on G.

5.2. <u>Theorem</u> : Let G be a locally-compact second countable group. Let K be a compact open subgroup of G. Let $\mu, \nu \in M(G)$ be such that

$$\mu(gK) = \nu(gK) \text{ for every } g \in K.$$

Then $\mu * \lambda_K = \nu * \lambda_K$.

In particular, K is a determining set for the family $M(G \setminus K)$.

<u>Proof</u> : Since K is open, the family $\{gK\}_{g \in G}$ forms an open covering of G. Since G is second-countable, there exist $g_1, g_2, \ldots$ in G such that

$$G = \bigcup_{i=1}^{\infty} g_i K .$$

Since for $i \neq j$, $g_i K \cap g_j K = \emptyset$, to show that $\mu * \lambda_K = \nu * \lambda_K$ on G, it is sufficient to show that $\mu * \lambda_K = \nu * \lambda_K$ on $g_i K$, for every $i = 1, 2, \ldots$.

Let g_i be chosen arbitrarily and fixed. Consider the probability measures $\delta_{g_i^{-1}} * \mu * \lambda_K$ and $\delta_{g_i^{-1}} * \nu * \lambda_K$. Showing that $\mu * \lambda_K = \nu * \lambda_K$ on $g_i K$ is equivalent to showing that $\delta_{g_i^{-1}} * \mu * \lambda_K = \delta_{g_i^{-1}} * \nu * \lambda_K$ on K.

Suppose, $(\delta_{g_i^{-1}} * \mu * \lambda_K)(K) = 0$.

Then,

$$\begin{aligned} 0 &= (\delta_{g_i^{-1}} * \mu * \lambda_K)(K) \\ &= \mu(g_i K) \\ &= \nu(g_i K) \\ &= (\delta_{g_i^{-1}} * \nu * \lambda_K)(K) \end{aligned}$$

Thus, if either of the measures $\delta_{g_i^{-1}} * \mu * \lambda_K$ and $\delta_{g_i^{-1}} * \nu * \lambda_K$ vanishes on K then so does the other and hence they are equal.

Now suppose that $(\delta_{g_i^{-1}} * \mu * \lambda_K)(K) \neq 0$ and $(\delta_{g_i^{-1}} * \nu * \lambda_K)(K) \neq 0$. Note that

$$(\delta_{g_i^{-1}} * \mu * \lambda_K)(K) = (\delta_{g_i^{-1}} * \nu * \lambda_K)(K) .$$

To complete the proof we show that both these measures are right K-invariant. For this, let $A \in \mathcal{B}_K$. Then,

$$(\delta_{g_i^{-1}} * \mu * \lambda_K)(A) = \int \mu(g_i A k^{-1})\, \lambda_K(dk)$$

Since K is compact, λ_K is both left and right K-invariant. Thus for every $k_1 \in K$,

$$\begin{aligned}(\delta_{g_i^{-1}} * \mu * \lambda_K)(A) &= \int \mu(g_i A (k k_1^{-1})^{-1})\lambda_K(dk) \\ &= \int \mu(g_i A k_1 k^{-1})\, \lambda_K(dk) \\ &= (\delta_{g_i^{-1}} * \mu * \lambda_K)(Ak_1)\end{aligned}$$

Similarly,

$$(\delta_{g_i^{-1}} * \nu * \lambda_K)(A) = (\delta_{g_i^{-1}} * \nu * \lambda_K)(Ak_1) .$$

Q.E.D.

5.3

The following questions on a locally compact second countable group G remain open:

(1) Does there exist a determining set for $\mathcal{M}(G)$?

(2) Let $E \in \mathcal{B}_G$ be such that $0 < \lambda_G(E) < \infty$. Can one associate a compact subgroup K to E such that E becomes a determining set for the family $\mathcal{M}(G \setminus K)$ of all right K-invariant probability measures on G?.

§ 6. DETERMINING SETS IN SYMMETRIC PAIRS

Unless otherwise stated, throughout this section, G will stand for a locally-compact second countable group and $\mathcal{B}_G$ will stand for the σ-algebra of Borel subsets of G. For any compact subgroup K of G, $\mathcal{B}_K$ will denote the σ-algebra of Borel subsets of K and λ_K will denote the normalised Haar measure of K. We choose and fix, once for all, a Haar measure λ_G of G.

6.1.1. Definition : The pair (G,K) is said to be symmetric if there exists an automorphism τ of G with the following property: there exist Borel maps ψ_1 and ψ_2 from G into K such that

$$\tau(x) = \psi_1(x)\cdot x^{-1}\cdot\psi_2(x), \text{ for every } x \in G.$$

Note that τ(K) = K. We shall refer to τ as the automorphism associated with the symmetric pair (G,K)

6.1.2 Examples :

(i)

Let G be any compact group and let

$$K = \{(x,x) \in G \times G \mid x \in G\} .$$

Then K is a compact subgroup of G × G. Let

$$\tau : G \times G \to G \times G$$

be defined by

$$\tau(x,y) = (y,x), \quad x,y \in G.$$

Then τ is an automorphism of G × G and for every x,y ε G,

$$\tau(x,y) = (x,x)\cdot(x^{-1},y^{-1})\cdot(y,y) .$$

Thus (G × G,K) is a symmetric pair.

(ii)

Let $G = SL(n, R)$ be the group of all $n \times n$ real matrices with determinant 1. Let $K = S\,O(n)$ be the group of all $n \times n$ orthogonal matrices. Then K is a compact subgroup of G. Let

$$\tau : SL(n,R) \to SL(n,R)$$

be defined by

$$\tau(g) = (g^t)^{-1}, \; g \in SL(n,R).$$

Where g^t denotes the transpose of the matrix g. Then τ is an automorphism of $SL(n,R)$. Further, since every $g \in SL(n,R)$ can be written as $g = k_g\, p_g$, where $k_g \in S\,O(n)$ and p_g is a symmetric matrix, we get

$$\tau(g) = k_g g^{-1} k_g \quad , \quad g \in SL(n,R) \; .$$

Thus $(SL(n,R), SO(n))$ is a symmetric pair.

Similarly, it can be shown that $(SL(n,\mathbb{C}), SU(n))$ is a symmetric pair, the associated automorphism τ being

$$\tau(g) = (g^*)^{-1}, \quad g \in SL(n,\mathbb{C}),$$

where g^* denotes the conjugate transpose of the matrix g.

(iii)

Let G be a non-compact semi-simple lie-group with finite center and let K be a maximal compact subgroup of G. Then there exists an (involutive analytic) automorphism τ of G such that

$$\tau(g) = k_g^{-1} g \cdot k_g^{-1} \; ,$$

for every $g \in G$, where $g \to k_g$ is continuous map from G into K (see Helgason [9]). Thus the pair (G,K) is symmetric. We shall refer to this pair as the <u>standard symmetric pair</u>.

6.1.3. <u>Definition</u> : Let $M(G)$ denote the set of all probability measures on G. We say that $\mu \in M(G)$ is <u>K-invariant</u> if for every $A \in B_G$ and $k_1, k_2 \in K$,

$$\mu(k_1 A k_2) = \mu(A) .$$

Let $M(K\backslash G/K)$ denote the set of all K-invariant probability measures on G.

The aim of this section is to obtain determining sets for $M(K\backslash G/K)$. In 6.2 and 6.3 we develop the theory of Fourier-transforms for measures in $M(K\backslash G/K)$ and use it in 6.4 to obtain determining sets in symmetric pairs.

6.2

Let G be a locally-compact second countable group and let K be a fixed compact subgroup of G.

A <u>continuous unitary representation</u> of G in a Hilbert space H is a mapping Π which assignes to each $g \in G$ a unitary operator $\Pi(g)$ on H such that

(i) $\Pi(e) = 1$, the identity operator on H ;

(ii) $\Pi(g_1 g_2) = \Pi(g_1)\Pi(g_2)$ for every $g_1, g_2 \in G$;

(iii) for each $v \in H$, the mapping $g \to \Pi(g)v$ is a continuous mapping of G into H.

One calls H the <u>representing Hilbert space</u> of Π .

Let Π_1 and Π_2 be two continuous unitary representations of G in Hilbert spaces H_1 and H_2 respectively. We say Π_1 is <u>equivalent</u> to Π_2 if there exists a Hilbert space isomorphism $\Gamma : H_1 \to H_2$ such that

$$\Gamma \, \Pi_1(g) \Gamma^{-1} = \Pi_2(g)$$

for every $g \in G$.

A representation Π of G in a Hilbert Space H is said to be <u>irreducible</u> if no proper closed subspace of H is invariant under $\Pi(g)$ for every $g \in G$. A fundamental result due to Gelfand and Raikov ensures that there are sufficiently many irreducible continuous unitary representations of G which separate points of G. More precisely, for every $g_o \in G$, there exists an irreducible continuous unitary representation Π of G in some Hilbert space H such that $\Pi(g_o) \neq 1$, (see Hewitt and Ross [10] p. 343).

Let $U(G)$ denote the set of equivalence classes of all irreducible continuous unitary representations of G.

6.2.1. <u>Definition</u> : Let $\mu \in M(G)$ and let $\Pi \in U(G)$ with the representing Hilbert space H. Let $\hat{\mu}(\Pi)$ denote the unique bounded operator on H defined by

$$< \hat{\mu}(\Pi)v,w > \; = \; \int < \Pi(g)v,w > \mu(dg) \; ,$$

for every $v,w \in H$, where $< \cdot \, , \cdot >$ denotes the inner-product on H. Thus for $\mu \in M(G)$, we get a map

$$\Pi \to \hat{\mu}(\Pi)$$

which assigns to every $\Pi \in U(G)$ a bounded operator $\hat{\mu}(\Pi)$ on H, the representing Hilbert Space of Π. This map is called the <u>Fourier-transform</u> of μ.

The following properties of the Fourier - transform are well known (see Heyer [11]):

6.2.2. <u>Proposition</u> : (i) For $\mu \in M(G)$, $\hat{\mu}(I) = 1$, where I denotes the identity representation of G in some Hilbert space H;

(ii) $\hat{\delta}_{g_o}(\Pi) = \Pi(g_o)$ for every $g_o \in G$ and $\Pi \in U(G)$;

(iii) For $\mu,\nu \in M(G)$, $\Pi \in U(G)$,

$$\widehat{(\mu * \nu)}(\Pi) = \hat{\mu}(\Pi)\,\hat{\nu}(\Pi) ;$$

(iv) If for $\mu, \nu \in M(G)$, $\hat{\mu}(\Pi) = \hat{\nu}(\Pi)$ for every $\Pi \in U(G)$, then $\mu = \nu$;

(v) If $\{\mu_n\}_{n=0,1,2,\ldots}$ is a sequence in $M(G)$ such that $\mu_n \Rightarrow \mu_o$, then $\hat{\mu}_n(\Pi) \to \hat{\mu}_o(\Pi)$ weakly for every $\Pi \in U(G)$.

6.2.3. Definition : Let $\Pi \in U(G)$ and let H be the representing Hilbert space of Π. We say that Π is of class -1 with respect to K if there exists $v_o \in H$ such that $v_o \neq 0$ and $\Pi(k)v_o = v_o$ for every $k \in K$.

Let $U_1(G;K) = \{\Pi \in U(G) \mid \Pi$ is of class-1 with respect to $K\}$.

Let $\Pi \in U_1(G;K)$ with representing Hilbert Space H. Let

$$H(\Pi;K) = \{v \in H \mid \Pi(k)v = v \text{ for every } k \in K\}.$$

Then $H(\Pi;K)$ is a closed subspace of H. Let P_Π denote the unique bounded operator on H defined by

$$< P_\Pi v,w > = \int < \Pi(k)v,w > \lambda_K(dk)$$

for every $v,w \in H$. Then P_Π is an orthogonal projection with range $H(\Pi;K)$. Note that if $\Pi \in U(G)$ but $\Pi \notin U_1(G;K)$, then $H(\Pi;K) = 0$ and hence $P_\Pi = 0$.

6.2.4 Lemma : Let $\Pi \in U_1(G;K)$ and let for every $g \in G$,

$$\Phi_\Pi(g) = P_\Pi\, \Pi(g) P_\Pi .$$

Then $\Phi_\Pi(g)$ is a bounded operator on $H(\Pi;K)$ and has the following properties:

(i) For every $v \in H(\Pi;K)$, the function

$$g \to < \Phi_\Pi(g)v,v >$$

is a continuous positive-definite function on G;

(ii)

$$(\Phi_\Pi(g))^* = \Phi_\Pi(g^{-1})$$

for every $g \in G$, where $*$ denotes the adjoint of an operator;

(iii) For every $g_1, g_2 \in G$

$$\int \Phi_\Pi(g_1 k g_2)\ \lambda_K(dk) = \Phi_\Pi(g_1)\Phi_\Pi(g_2) \quad .$$

Proof: Let $v \in H(\Pi;K)$. Then for every $g \in G$,

$$< \Phi_\Pi(g)v,v> = < \Pi(g)v,v>$$

Since Π is a continuous unitary representation, it follows that the function

$$g \to < \Phi_\Pi(g)v,v>$$

is a continuous positive-definite function on G.

To prove (ii), let $v,w \in H(\Pi;K)$. Then

$$\begin{aligned}<(\Phi_\Pi(g))^*v,w> &= <v,\Phi_\Pi(g)w> \\ &= <v,P_\Pi\Pi(g)P_\Pi w> \\ &= <P_\Pi\Pi(g^{-1})P_\Pi v,w> \\ &= <\Phi_\Pi(g^{-1})v,w>\end{aligned}$$

Hence $(\Phi_\Pi(g))^* = \Phi_\Pi(g^{-1})$.

Finally, for $g_1, g_2 \in G$,

$$\begin{aligned}\int\Phi_\Pi(g_1kg_2)\lambda_K(dk) &= \int P_\Pi\Pi(g_1kg_2)P_\Pi\ \lambda_K(dk) \\ &= P_\Pi\ (\int\Pi(g_1kg_2)\ \lambda_K(dk))P_\Pi \\ &= P_\Pi\Pi(g_1)(\int\Pi(k)\ \lambda_K(dk))\Pi(g_2)P_\Pi \\ &= P_\Pi\Pi(g_1)P_\Pi\Pi(g_2)P_\Pi \\ &= (P_\Pi\ \Pi(g_1)P_\Pi)(P_\Pi\ \Pi(g_2)P_\Pi) \\ &= \Phi_\Pi(g_1)\cdot\Phi_\Pi(g_2).\end{aligned}$$

This proves the Lemma completely.

6.2.5. Lemma : Let $\mu,\nu \in M(K\backslash G/K)$. If for every $\Pi \in U_1(G;K)$

$$\int \Phi_\Pi(g)\,\mu(dg) = \int \Phi_\Pi(g)\,\nu(dg),$$

then $\mu = \nu$.

Proof : Let $\Pi \in U(G)$. If $\Pi \notin U_1(G;K)$, then clearly $\hat{\mu}(\Pi) = 0 = \hat{\nu}(\Pi)$.

Now, let $\Pi \in U_1(G;K)$. Then for every $k_1, k_2 \in K$,

$$\begin{aligned}\hat{\mu}(\Pi) &= \int \Pi(g)\,\mu(dg)\\ &= \int \Pi(k_1 g k_2)\,\mu(dg)\\ &= \Pi(k_1)\left(\int \Pi(g)\,\mu(dg)\right)\Pi(k_2)\end{aligned}$$

Integrating both sides with respect to k_1 and k_2, we get

$$\begin{aligned}\hat{\mu}(\Pi) &= P_\Pi\left(\int \Pi(g)\,\mu(dg)\right)P_\Pi\\ &= \int \Phi_\Pi(g)\,\mu(dg).\end{aligned}$$

Similarly,

$$\hat{\nu}(\Pi) = \int \Phi_\Pi(g)\,\nu(dg) .$$

Thus, for every $\Pi \in U(G)$

$$\hat{\mu}(\Pi) = \hat{\nu}(\Pi) .$$

Hence, by Proposition 6.2.2, $\mu = \nu$.

6.2.6. Definition : Let G be a locally-compact second countable group and let K be a compact subgroup of G. Let H be a Hilbert space and let $B(H)$ be the space of all bounded operators on H. A function $\Phi : G \to B(H)$ is said to be operator-valued K-spherical on G if

(i) $\Phi(e) = 1$, the identity operator on H ;

(ii) $\int \Phi(g_1 k g_2)\,\lambda_K(dk) = \Phi(g_1)\Phi(g_2)$,

for every $g_1, g_2 \in G$.

Further Φ is said to be of <u>positive definite type</u> if for every $v \in H$, the function

$$g \rightarrow < \Phi(g)v, v >$$

is a positive-definite function on G.

By Lemma 6.2.4, for every $\Pi \in U_1(G;K)$, there exists an operator-valued K-spherical function Φ_Π. For $\mu \in M(K\backslash G/K)$, if we define

$$\hat{\mu}(\Phi) = \int \Phi(g)\,\mu(dg) \quad ,$$

where Φ runs over the class of all operator - valued K-spherical functions of positive-definite type, then $\hat{\mu}$ behaves like the Fourier-transform, in view of Lemma 6.2.5.

In the next section, we apply these ideas to symmetric pairs.

<u>6.3.</u>

Let (G,K) be a symmetric pair and let τ be the automorphism associated with it.

<u>6.3.1.</u> <u>Lemma</u> : $M(K\backslash G/K)$ is a commutative semigroup under the convolution operation.

<u>Proof</u> : Let $\sigma(g) = (\tau(g))^{-1}$, $g \in G$.

Then, $\sigma : G \rightarrow G$ satisfies the following properties:

$$\text{(i)} \quad \sigma(gh) = \sigma(h)\,\sigma(g) \text{ for every } g,h \in G \; ; \tag{6.1}$$

$$\text{(ii)} \quad \sigma(g) = \psi_2(g)^{-1} \cdot g \cdot \psi_1(g)^{-1} \text{ for every } g \in G; \tag{6.2}$$

where ψ_1 and ψ_2 are the Borel maps from G into K whose existence is ensured by the properties of τ.

It is easy to see that σ leaves λ_K, the normalised Haar-measure of K, invariant. Further, using the definition of convolution, equations (6.1) and (6.2), it is easy to see that for any $\mu \in M(G)$,

$$(\lambda_K * \mu * \lambda_K)\sigma^{-1} = \lambda_K * \mu * \lambda_K \tag{6.3}$$

Let X be a random-variable on some fixed probability space $(\Omega,\mathcal{B},P)$ with values in the group G. We say X is distributed as $\mu \in \mathcal{M}(G)$ if

$$PX^{-1} = \mu .$$

We write this as $X \sim \mu$. Further for two random-variables X and Y on $(\Omega,\mathcal{B},P)$, we write $X \sim Y$ if $PX^{-1} = PY^{-1}$.

With the above terminology, equation (6.3) can be restated as follows:

if ξ, η, X are independent random-variables on $(\Omega,\mathcal{B},P)$ such that $\xi \sim \eta \sim \lambda_K$ and $X \sim \mu$, then

$$\sigma(\xi\cdot X\cdot\eta) \sim \xi\cdot X\cdot\eta \tag{6.4}$$

Now, let ξ,η,ζ,X and Y be independent random-variables on $(\Omega,\mathcal{B},P)$ with values in G. Let $\xi \sim \eta \sim \zeta \sim \lambda_K$, $X \sim \mu$ and $Y \sim \nu$, where $\mu,\nu \in \mathcal{M}(G)$.

We claim that

$$\xi\cdot X\cdot\zeta\cdot Y\cdot\eta \sim \xi\cdot Y\cdot\zeta\cdot X\cdot\eta$$

To see this, first note that we have from (6.4)

$$\begin{aligned} \xi\cdot X\cdot\zeta\cdot Y\cdot\eta &\sim \sigma(\xi\cdot X\cdot\zeta\cdot Y\cdot\eta) \\ &= \sigma(\zeta\cdot Y\cdot\eta)\cdot\sigma(\xi\cdot X) \end{aligned} \tag{6.5}$$

Since $\sigma(\zeta\cdot Y\cdot\eta)$ and $\sigma(\xi\cdot X)$ are independent and $\sigma(\zeta\cdot Y\cdot\eta) \sim \zeta\cdot Y\cdot\eta$, we have

$$\begin{aligned} \sigma(\zeta\cdot Y\cdot\eta)\cdot\sigma(\xi\cdot X) &\sim \zeta\cdot Y\cdot\eta\cdot\sigma(\xi\cdot X) \\ &\sim \zeta\cdot Y\cdot\sigma(\eta)\sigma(\xi\cdot X) \\ &\sim \zeta\cdot Y\cdot\sigma(\xi\cdot X\cdot\eta) \end{aligned} \tag{6.6}$$

Once again, $\zeta\cdot Y$ and $\sigma(\xi\cdot X\cdot\eta)$ are independent and $\sigma(\xi\cdot X\cdot\eta) \sim \xi\cdot X\cdot\eta$. So, we have

$$\begin{aligned} \zeta\cdot Y\cdot\sigma(\xi\cdot X\cdot\eta) &\sim \zeta\cdot Y\cdot\xi\cdot X\cdot\eta \\ &\sim \xi\cdot Y\cdot\zeta\cdot X\cdot\eta \end{aligned} \tag{6.7}$$

It follows from equations (6.5), (6.6) and (6.7) that

$$\xi\cdot X\cdot\zeta\cdot Y\cdot\eta \sim \xi\cdot Y\cdot\zeta\cdot X\cdot\eta \quad .$$

Rewritting this in terms of measures, we have

$$\lambda_K * \mu * \lambda_K * \nu * \lambda_K = \lambda_K * \nu * \lambda_K * \mu * \lambda_K ,$$

for every $\mu,\nu \in \mathcal{M}(G)$.

In particular, if μ , $\nu \in \mathcal{M}(K\backslash G/K)$, then

$$\mu * \nu = \nu * \mu$$

This proves the Lemma.

6.3.2 Lemma : Let (G,K) be a symmetric pair and let $\Pi \in \mathcal{U}_1(G;K)$. Then the Hilbert space $\mathcal{H}(\Pi;K)$ is one-dimensional.

Proof : Consider the family $\{\Phi_\Pi(g) : g \in G\}$ of bounded operators on $\mathcal{H}(\Pi;K)$ as constructed in Lemma 6.2.4. Let g_1, $g_2 \in G$. Then

$$\begin{aligned} \Phi_\Pi(g_1)\cdot\Phi_\Pi(g_2) &= P_\Pi\, \Pi(g_1)P_\Pi\, \Pi(g_2)\, P_\Pi \\ &= \int\Pi(g)\, \rho'(dg) \end{aligned}$$

where $\rho' = \lambda_K * \delta_{g_1} * \lambda_K * \delta_{g_2} * \lambda_K$.

Similarly,

$$\Phi_\Pi(g_2)\Phi_\Pi(g_1) = \int\Pi(g)\, \rho''(dg)$$

where $\rho'' = \lambda_K * \delta_{g_2} * \lambda_K * \delta_{g_1} * \lambda_K$.

Since, by Lemma 6.3.1, $\rho' = \rho''$, we have

$$\Phi_{\Pi}(g_1)\Phi_{\Pi}(g_2) = \Phi_{\Pi}(g_2)\Phi_{\Pi}(g_1), \text{ for every } g_1, g_2 \varepsilon G.$$

Also from Lemma 6.2.4,

$$(\Phi_{\Pi}(g))^* = \Phi_{\Pi}(g^{-1}), \text{ for every } g \varepsilon G.$$

It follows that the family $\{\Phi_{\Pi}(g) : g \varepsilon G\}$ is a commuting family of normal operators on $H(\Pi;K)$.

Now suppose that the dimension of $H(\Pi;K)$ is greater than one. Then we can find proper closed subspaces N_1 and N_2 of $H(\Pi;K)$ such that both N_1, N_2 are invariant under $\Phi_{\Pi}(g)$ for every $g \varepsilon G$, N_1 and N_2 are mutually orthogonal and

$$H(\Pi;K) = N_1 \oplus N_2 .$$

Let M denote the closed linear span of $\{\Pi(g)v : v \varepsilon N_1, g \varepsilon G\}$. Then $M \neq 0$ is a closed subspace of H and M is invariant under $\Pi(g)$ for every $g \varepsilon G$. Further, for every $g \varepsilon G$, $v \varepsilon N_1$ and $w \varepsilon N_2$, we have

$$\begin{aligned} < \Pi(g)v,w > &= < \Pi(g)P_{\Pi}v, P_{\Pi} w> \\ &= < P_{\Pi} \Pi(g)P_{\Pi}v,w > \\ &= < \Phi_{\Pi}(g)v,w > \\ &= 0. \end{aligned}$$

Thus $M \perp N_2$. Hence M is a proper closed subspace of H and is invariant under $\Pi(g)$ for every $g \varepsilon G$. This contradicts the irreducibility of Π.

Hence $\dim(H(\Pi;K)) = 1$.

6.3.3. Definition : Let (G,K) be a symmetric pair. A continuous complex-valued function ϕ on G is said to be K-spherical, if

(i) $\phi(e) = 1$;

(ii) for every $g_1, g_2 \in G$,

$$\int \phi(g_1 k g_2)\, \lambda_K(dk) = \phi(g_1)\phi(g_2)\,. \qquad (6.8)$$

Further, ϕ is said to be of <u>positive-definite type</u> if ϕ is positive definite.

Let $S(G;K)$ denote the set of all K-spherical functions on G which are of the positive-definite type. In view of section 6.2 and Lemma 6.3.2, it follows that for every $\Pi \in \mathcal{U}_1(G;K)$, the function $\phi(g) = \Phi_\Pi(g)$, $g \in G$, is an element of $S(G;K)$.

Let $S(G;K)$ be given the topology of uniform convergence on compact sets. Then $S(G;K)$ becomes a second-countable space. Note that, since every $\phi \in S(G;K)$ is a continuous positive-definite function, we have

$$\phi(e) = 1,\ \phi(g^{-1}) = \overline{\phi(g)},\ |\phi(g)| \le 1,$$

for every $g \in G$.

For $\mu \in M(K\backslash G/K)$, let the <u>Fourier-transform</u> of μ be defined by

$$\hat{\mu}(\phi) = \int \phi(g)\mu(dg),\ \phi \in S(G;K)\,.$$

The following properties of the Fourier-transform follow from the discussion in section 6.2:

6.3.3. <u>Proposition</u> :

(i) Let $\mu, \nu \in M(K\backslash G/K)$. If $\hat{\mu}(\phi) = \hat{\nu}(\phi)$ for every $\phi \in S(G;K)$, then $\mu = \nu$;

(ii) Let $\{\mu_n\}_{n=0,1,2,\ldots}$ be a sequence in $M(K\backslash G/K)$. If $\mu_n \Rightarrow \mu_o$, then $\hat{\mu}_n(\phi) \to \hat{\mu}_o(\phi)$ for every $\phi \in S(G;K)$;

(iii) For every $\mu \in M(K\backslash G/K)$, the map

$$\phi \mapsto \hat{\mu}(\phi),\ \phi \in S(G;K)$$

is continuous.

6.4. Determining sets in symmetric pairs : Let (G,K) be a symmetric pair. Using the theory of Fourier-transform for measures in $M(K\backslash G/K)$, as developed in sections 6.2 and 6.3, we show in this section that generically, every subset of G with finite positive Haar-measure is a determining set for $M(K\backslash G/K)$.

6.4.1. Lemma : Let (G,K) be a symmetric pair. Let $E \in B_G$ be such that $0 < \lambda_G(E) < \infty$ and the set

$$\{\phi \in S(G;K) \mid \int_E \phi(g)\, \lambda_G(dg) \neq 0\}$$

is dense in S(G;K). Then E is a determining set for $M(K\backslash G/K)$.

Proof : Let $\mu,\nu \in M(K\backslash G/K)$ and let

$$\mu(gE) = \nu(gE) \tag{6.9}$$

for every $g \in G$. We have to show that $\mu=\nu$. From equation (6.9), we have for every $\phi \in S(G;K)$.

$$\int\phi(g)(\int\chi_E(gh)\ \mu(dh))\lambda_G(dg) = \int\phi(g)(\int\chi_E(gh)\ \nu(dh))\lambda_G(dg)\ .$$

Using Fubini's theorem and changing g to gh^{-1}, we have for every $\phi \in S(G;K)$

$$\int(\int\chi_E(g)\phi(gh^{-1})\lambda_G(dg))\mu(dh) = \int(\int\chi_E(g)\phi(gh^{-1})\lambda_G(dg))\nu(dh)\ .$$

Since $\mu,\nu \in M(K\backslash G/K)$, changing h to hk^{-1}, we have for every $\phi \in S(G;K)$ and $k \in K$

$$\int(\int\chi_E(g)\phi(gkh^{-1})\lambda_G(dg))\mu(dh) = \int(\int\chi_E(g)\phi(gkh^{-1})\lambda_G(dg))\nu(dh)\ .$$

Integrating with respect to k and using Fubini's Theorem, we have for every $\phi \in S(G;K)$

$$\int[\int\chi_E(g)(\int\phi(gkh^{-1})\lambda_K(dk))\lambda_G(dg)]\mu(dh)$$
$$= \int[\int\chi_E(g)(\int\phi(gkh^{-1})\lambda_K(dk))\lambda_G(dg)]\nu(dh).$$

Using property (6.8) of ϕ , we have

$$\int(\int\chi_E(g)\phi(g)\phi(h^{-1})\lambda_G(dg))\mu(dh) = \int(\int\chi_E(g)\phi(g)\phi(h^{-1})\lambda_G(dg))\nu(dh) ,$$

for every $\phi \in S(G;K)$. Once again, using Fubinis theorem, we have

$$(\int_E \phi(g)\lambda_G(dg))(\int\phi(h^{-1})\mu(dh))=(\int_E\phi(g)\lambda_G(dg))(\int\phi(h^{-1})\nu(dh)),$$

for every $\phi \in S(G;K)$. Let

$$D = \{\phi \in S(G;K) \mid \int_E \phi(g)\, \lambda_G(dg) \neq 0\} .$$

Then for every $\phi \in D$, we have

$$\int\phi(h^{-1})\mu(dh) = \int\phi(h^{-1})\nu(dh) .$$

Since $\phi(h^{-1}) = \overline{\phi(h)}$, we have

$$\hat{\mu}(\phi) = \hat{\nu}(\phi) \text{ for every } \phi \in D. \tag{6.10}$$

Since, $\hat{\mu}$ and $\hat{\nu}$ are continuous functions on $S(G;K)$ and D is dense in $S(G;K)$, from equation (6.10) we have

$$\hat{\mu}(\phi) = \hat{\nu}(\phi) \text{ for every } \phi \in S(G;K).$$

Hence $\mu=\nu$.

6.4.2. <u>Theorem</u> : Let (G,K) be a symmetric pair. Then for every real number $\alpha > 0$,there exists $D(\alpha;G) \subset \mathcal{B}_G$ such that the following hold:

(i) $D(\alpha;G)$ is a dense G_δ - subset of $(s_\alpha(G),d'_\alpha)$.

(Recall, $s_\alpha(G) = \{E \in \mathcal{B}_G \mid \lambda_G(E) < \alpha\}$).

(ii) Every $E \in D(\alpha;G)$ is a determining set for $\mathcal{M}(K\backslash G/K)$.

<u>Proof</u> : Since $S(G;K)$ is a separable space, we can choose a dense subset $\phi_1, \phi_2,\ldots$ in $S(G;K)$. For every $j = 1,2, \ldots$, let

$$D(\alpha;\phi_j) = \{E \in s_\alpha(G) \mid \int_E \phi(x)\lambda_G(dx) \neq 0\} .$$

Then it follows from Lemma 2.1, that $D(\alpha;\phi_j)$ is an open dense subset of $(s_\alpha(G),d'_\alpha)$. Let

$$D(\alpha;G) = \bigcap_{j=1}^{\infty} D(\alpha;\phi_j) .$$

Since $(s_\alpha(G), d'_\alpha)$ is a complete metric space (see § 2), it follows from Baire's category theorem that $D(\alpha;G)$ is dense in $(s_\alpha(G),d'_\alpha)$. Clearly, $D(\alpha;G)$ is a G_δ-set. This proves (i).

To prove (ii), let $E \in D(\alpha;G)$. Then $0 < \lambda_G(E) < \alpha$ and the set

$$\{\phi \in S(G;K) \mid \int_E \phi(x)\, \lambda_G(dx) \neq 0\}$$

contains the set $\{\phi_j ; j=1,2,\ldots\}$. Hence it is a dense subset of $S(G;K)$ and it follows from Lemma 6.4.1. that E is a determining set for $M(K\setminus G/K)$.

This proves the Theorem.

6.5. <u>Determining sets in standard symmetric pairs</u>: Let (G,K) be a standard symmetric pair, i.e., G is a non-compact semi-simple Lie-group with finite center and K is a maximal compact subgroup of G.

Let $G = KAN$ be the Iwasawa decomposition of G. Let $\mathfrak{A}$ be the Lie-algebra of A. Let E_c be the space of all complex-valued linear functionals on $\mathfrak{A}$. Then a fundamental result due to Harish Chandra says that K-spherical functions on G are in one-to-one correspondence with the space E_c. More precisely, let $a(g)$ denote the unique element of $\mathfrak{A}$ such that $g = k.\exp(a(g)).n$, $k \in K$, $n \in N$. Then for every $\nu \in E_c$

$$\phi_\nu(g) = \int_K \exp(i\nu-\rho)(a(gk))\lambda_K(dk) \qquad (6.11)$$

is a K-spherical function on G. Here ρ is a fixed element of E_c. Further, each K-spherical function arises in this way for some $\nu \in E_c$. Let E_R denote the space of all real-valued linear functionals on $\mathfrak{A}$ so that

$E_C = E_R + iE_R$. Then for every $\lambda \in E_R$, ϕ_λ is a K-spherical function of the positive definite-type on G.

For the above representation and other properties of K-spherical functions, we refer to Helgason [9], Chapter X, § 3.

For $\mu \in M(K\backslash G/K)$, the <u>Fourier-transform</u> of μ can be defined as

$$\hat{\mu}(\lambda) = \int \phi_\lambda(g)\mu(dg) , \quad \lambda \in E_R .$$

Then $\hat{\mu}(\lambda)$ is a bounded continuous function on E_R. Further, if $\mu,\nu \in M(K\backslash G/K)$ are such that $\hat{\mu}(\lambda) = \hat{\nu}(\lambda)$ for every $\lambda \in E_R$, then $\mu = \nu$. (See Gangolli [8], § 4).

6.5.1. <u>Theorem</u> : Let (G,K) be a standard symmetric pair and let $E \in \mathcal{B}_G$ be such that $\lambda_G(E) > 0$ and $\bar{E}$ is compact. Then E is a determining set for $M(K\backslash G/K)$.

<u>Proof</u> : Let

$$D = \{\lambda \in E_R \mid \int_E \phi_\lambda(g)\lambda_G(dg) \neq 0\} .$$

In view of Lemma 6.4.1, we have only to prove that D is dense in E_R. Let $\Phi : E_c \to \mathbb{C}$ be defined by

$$\Phi(\nu) = \int_E \phi_\nu(g)\, \lambda_G(dg), \quad \nu \in E_c .$$

Since $\bar{E}$ is compact, Φ is a well-defined function. Further, it follows from equation (6.11) that Φ is analytic. Let us consider it as a function of n complex-variables, where n is the dimension of E_c. Then D is precisely the set

$$\{\lambda \in E_R \mid \Phi(\lambda) \neq 0\}.$$

Since Φ is a non-zero analytic function, it follows from Lemma 3.2.2, that D is a dense subset of E_R.

This proves the theorem.

6.5.2. Remark : Let G be a locally-compact second countable group and let K be a compact subgroup of G. Let $\Pi : G \to G/K$ denote the cannonical homomorphism from G onto the space of all left-cosets of G by K. Let G act on G/K in the natural way. It is easy to see that if $\tilde{E} \in \mathcal{B}_{G/K}$ is such that the set $E = \Pi^{-1}(\tilde{E})$ is a determining set for $M(K \backslash G/K)$, then $\tilde{E}$ is a determining set for the class of all K-invariant probability measures on G/K.

6.6.

Let $G = SL(2, \mathbb{C})$ be the group of all 2×2 complex-matrices of determinent one. Let $m > 0$ be some-fixed real number. For any $\underline{x} = (x_1, x_2, x_3) \in R^3$, let

$$x_o = + \sqrt{m^2 + x_1^2 + x_2^2 + x_3^2}\ , \tag{6.12}$$

and let

$$\tilde{\underline{x}} = \begin{pmatrix} x_o + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_o - x_1 \end{pmatrix}$$

The map $\underline{x} \to \tilde{\underline{x}}$ identifies every element of R^3 with a 2×2 Hermetian matrix of determinant m^2. It is easy to see that it is a one-to-one correspondence.

Let $g \in SL(2, \mathbb{C})$ and let $\underline{x} \in R^3$. Consider the matrix $g\, \tilde{\underline{x}}\, g^*$. The matrix $g\, \tilde{\underline{x}}\, g^*$ is a 2×2 Hermetian matrix of determinant m^2. Thus there exists a unique element of R^3, which we denote by $g \cdot \underline{x}$, such that $g\, \tilde{\underline{x}}\, g^* = \widetilde{(g \cdot \underline{x})}$. This enables us to define a map

$$\phi : SL(2, \mathbb{C}) \times R^3 \to R^3,$$

$$\phi(g, \underline{x}) = g \cdot \underline{x}\ , \quad g \in SL(2, \mathbb{C}),\ \underline{x} \in R^3.$$

It is easy to see that ϕ is a transitive action of $SL(2, \mathbb{C})$ on R^3. Let G_o denote the isotropy subgroup of G at the point $\underline{0} = (0,0,0) \in R^3$, i.e.,

$$G_0 = \{g \in SL(2,\mathbb{C}) \mid g \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} g^* = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \} .$$

It is easy to see that $G_0 = SU(2)$, the group of all 2 × 2 unitary matrices. Thus, we have

$$SL(2,\mathcal{C})/_{SU(2)} \cong R^3 .$$

A $SL(2,\mathbb{C})$ - invariant measure λ on R^3 is given by

$$d\lambda(\underline{x}) = \frac{dx_1\ dx_2\ dx_3}{+\sqrt{m^2 + x_1^2 + x_2^2 + x_3^2}} , \quad \underline{x} = (x_1,x_2,x_3) .$$

It follows from Theorem 6.5.1 and Remark 6.5.2, that a set $E \in B_{R^3}$ with $\bar{E}$ compact and $\lambda_{R^3}(E) > 0$ is a determining set for the class of all SU(2) - invariant measures on R^3. Since, $g \in SU(2)$ acts on R^3 as a rotation, we get :

let μ and ν be two rotation-invariant measures on R^3 and let $E \in B_{R^3}$ be such that $\bar{E}$ is compact and $\lambda_{R^3}(E) > 0$. If

$$\mu(g\cdot E) = \nu(g\cdot E) \quad \text{for every} \quad g \in SL(2,\mathcal{C}) ,$$

then $\mu = \nu$.

In particular, let

$$E = \{\underline{x} \in R^3 \mid ||\underline{x}|| < r\}, \quad \text{where} \quad r > m.$$

Then, for $g \in SL(2,\mathbb{C})$,

$$g\cdot E = \{\underline{x} \in R^3 \mid ||g\ \tilde{\underline{x}}\ g^*|| < r\} .$$

Note that,

$$SL(2, \mathbb{C}) = SU(2)\ A\ SU(2),$$

where

$$A = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0\} .$$

Thus, if μ and ν are two rotation invariant probability measures on R^3 such that

$$\mu\{\underline{x} \in R^3 \mid \left\|\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tilde{\underline{x}} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\| < r\}$$

$$= \nu\{\underline{x} \in R^3 \mid \left\|\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tilde{\underline{x}} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\| < r\}$$

for every $a > 0$ and some $r > m$, then $\mu = \nu$.

An easy computation shows that for $a > 0$ and $\underline{x} = (x_1, x_2, x_3)$,

$$\left\|\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tilde{\underline{x}} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\| < r$$

iff

$$x_o(a^2+a^{-2}) + x_1(a^2-a^{-2}) < + \sqrt{4(r^2-m^2)},$$

where x_o is given by (6.12).

Thus, if we put $s = \sqrt{4(r^2-m^2)}$ and $\alpha = a^2$, then we have the following:

6.6.1. Proposition : Let μ and ν be two rotation invariant probability measures on R^3. Let for some $m,s > 0$ and for every $\alpha > 0$, μ and ν satisfy the following:

$$\mu\{(x_1,x_2,x_3) \in R^3 \mid x_o(\alpha+\alpha^{-1}) + x_1(\alpha-\alpha^{-1}) < s\}$$

$$= \nu\{(x_1,x_2,x_3) \in R^3 \mid x_o(\alpha+\alpha^{-1})+x_1(\alpha-\alpha^{-1}) < s\} ,$$

where, for $(x_1, x_2, x_3) \in R^3$, $x_o = \sqrt{m^2 + x_1^2 + x_2^2 + x_3^2}$.

Then $\mu = \nu$.

§ 7. WEAK CONVERGENCE DETERMINING SETS IN SYMMETRIC PAIRS

Throughout this section, (G,K) will denote a symmetric pair.

7.1. Definition : A set $E \in \mathcal{B}_G$ is said to be a weak-convergence determining set for $M(K\setminus G/K)$ if E has the following property:

if $\{\mu_n\}_{n=0,1,2...}$ is a sequence in $M(K\setminus G/K)$ such that $\mu_n(gE) \to \mu_o(gE)$ for every $g \in G$, then $\mu_n \Rightarrow \mu_o$.

7.2. Lemma : Let (G,K) be a symmetric pair and let $E \in \mathcal{B}_G$ be such that $\infty > \lambda_G(E) > 0$ and the set

$$\{\phi \in S(G;K) \mid \int_E \phi(g)\lambda_G(dg) \neq 0\}$$

is dense in S(G;K). Then E is a weak-convergence determining set for $M(K\setminus G/K)$.

Proof : Let $\{\mu_n\}_{n=0,1,2,...}$ be a sequence in $M(K\setminus G/K)$ such that

$$\mu_n(gE) \to \mu_o(gE) \text{ for every } g \in G . \tag{7.1}$$

We have to show that $\mu_n \Rightarrow \mu_o$.

First note that equation (7.1) is equivalent to

$$(\chi_E * \bar{\mu}_n)(g) \to (\chi_E * \bar{\mu}_o)(g) \text{ for every } g \in G,$$

where for $\eta \in M(K\setminus G/K)$, $\bar{\eta}$ is defined by

$$\bar{\eta}(A) = \eta(A^{-1}),\ A \in \mathcal{B}_G.$$

Since, $\chi_E \in L_1(G)$ and $M(K\setminus G/K)$ is a closed subset of $M(G)$, it follows that the sequence $\{\mu_n\}_{n=1,2,...}$ is conditionally compact in $M(K\setminus G/K)$.

Let $\nu \in M(K\backslash G/K)$ be any limit point of $\{\mu_n\}_{n=1,2...}$ and let $\{\mu_{n_k}\}_{k=1,2,...}$ be a subsequence of $\{\mu_n\}_{n=1,2...}$ such that

$$\mu_{n_k} \Rightarrow \nu \quad .$$

Going to Fourier-transforms, we have

$$\hat{\mu}_{n_k}(\phi) \to \hat{\nu}(\phi) \quad \text{for every} \quad \phi \in S(G;K).$$

Thus,

$$\left(\int_E \phi(g)\lambda_G(dg)\right)\hat{\mu}_{n_k}(\phi) \to \left(\int_E \phi(g)\lambda_G(dg)\right)\hat{\nu}(\phi) \qquad (7.2)$$

for every $\phi \in S(G;K)$. On the other hand,

$$\mu_{n_k}(gE) \to \mu_o(gE) \quad \text{for every} \quad g \in G.$$

Thus,

$$\int\phi(g)\left(\int\chi_E(gh)\mu_{n_k}(dh)\right)\lambda_G(dg) \to \int\phi(g)\left(\int\chi_E(gh)\mu_o(dh)\right)\lambda_G(dg)$$

for every $\phi \in S(G;K)$. Using the fact that $\mu_{n_k}, \mu_o \in M(K\backslash G/K)$. and using the property (6.8) of ϕ, it is easy to see that the above is equivalent to

$$\left(\int_E \phi(g)\lambda_G(dg)\right)\hat{\mu}_{n_k}(\phi) \to \left(\int\phi(g)\lambda_G(dg)\right)\hat{\mu}_o(\phi) \qquad (7.3)$$

for every $\phi \in S(G;K)$. From equations (7.2) and (7.3), we have

$$\left(\int_E\phi(g)\lambda_G(dg)\right)\hat{\mu}_o(\phi) = \left(\int\phi(g)\lambda_G(dg)\right)\hat{\nu}(\phi) \qquad (7.4)$$

for every $\phi \in S(G;K)$. Since the set $\{\phi \in S(G;K) \mid \int_E\phi(g)\lambda_G(dg) \neq 0\}$ is dense in $S(G;K)$ and $\hat{\mu}_o$, $\hat{\nu}$ are continuous on $S(G;K)$, it follows from (7.4) that

$$\hat{\mu}_o(\phi) = \hat{\nu}(\phi) \quad \text{for every } \phi \in S(G;K) \; .$$

Hence $\mu_o = \nu$.

Thus, the sequence $\{\mu_n\}_{n=1,2,...}$ has only one limit point namely μ_o. Hence $\mu_n \Rightarrow \mu_o$.

7.3. Corollary : Let (G,K) be a standard symmetric pair and let $E \in \mathcal{B}_G$ be such that $\lambda_G(E) > 0$ and $\bar{E}$ is compact. Then E is a determining set for $M(K \backslash G/K)$.

7.4. Theorem : Let (G,K) be a symmetric pair. Then for every $\alpha > 0$, there exists $D(\alpha;G) \subset \mathcal{B}_G$ such that

(i) $D(\alpha;G)$ is a dense G_δ -subset of $(s_\alpha(G), d'_\alpha)$

(Recall $s_\alpha(G) = \{E \in \mathcal{B}_G \mid \lambda_G(E) < \alpha\}$).

(ii) Every $E \in D(\alpha;G)$ is a weak convergence determining set for $M(K \backslash G/K)$.

Proof : Let $D(\alpha;G)$ be as constructed in the Theorem 6.4.2. Then, only (ii) remains to be proved.

Let $E \in D(\alpha;G)$. It follows from the construction of $D(\alpha;G)$ that the set

$$\{\phi \in S(G;K) \mid \int_E \phi(g)\lambda_G(dg) \neq 0\}$$

is a dense subset of $S(G;K)$. Thus, by Lemma 7.3, it follows that E is a weak-convergence determining set for $M(K \backslash G/K)$.

7.5. Remark : Theorem 7.4 tells us that generically, a subset of G with finite positive Haar-measure is a weak-convergence determining set for $M(K \backslash G/K)$.

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