ON THE ASYMPTOTIC THEORY OF QUANTILES AND L-STATISTICS

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ABBREVIATIONS AND NOTATIONS

Abbreviations

r.v.	random variable
i.i.d.	independent and identically distributed
i.i.d.r.v.s	independent and identically distributed random variables
e.d.f.	empirical distribution function
w.r.t.	with respect to
a.s.	almost surely
d.f.	distribution function
A E V	asymptotic effective variance
m.g.f.	moment generating function

Notations

$U[0,1]$	uniform distribution on $[0,1]$
$\Phi(x)$	$\int_{-\infty}^{x} (2\pi)^{-1/2} e^{-x^2/2}\, dx$
$X_n \xrightarrow{\mathcal{L}} X$	The sequence of r.v.s $\{X_n\}$ converges to the r.v. X in distribution.
$X \overset{\mathcal{L}}{=} Y$	X and Y have same distributions.

$X_n = O(a_n)$ a.s.	X_n is a sequence of r.v.s and $\{a_n\}$ is a sequence of positive numbers such that $\limsup_{n \to \infty} \lvert X_n \rvert / a_n \le K$ a.s. where K is some positive constant.
$X_n = o(a_n)$ a.s.	$\lim_{n \to \infty} \lvert X_n \rvert / a_n = 0$ a.s.
$X_n = O_p(a_n)$	For every $\varepsilon > 0$, there exists a constant $K = K(\varepsilon)$ such $P(\lvert X_i / a_i \rvert > K) < \varepsilon$ for all $i \ge 1$.
$X_n = o_p(a_n)$	For every $\varepsilon > 0$, $P(\lvert X_i / a_i \rvert > \varepsilon) \longrightarrow 0$ as $i \longrightarrow \infty$,
$\forall$	for every
$x \in A$	x belongs to A
f^+	$= f$ if $f \ge 0$ and $= 0$ if $f \le 0$
f^-	$= 0$ if $f \ge 0$ and $= -f$ if $f \le 0$
$I(A)$	indicator function of the set A

K, K_i are used for sufficiently large positive constants without mention. At some places the Vinogradov symbol $<<$ instead of O is used for convenience.

CHAPTER 1

INTRODUCTION, SUMMARY AND A SURVEY OF RELATED LITERATURE

1.1 VARIOUS METHODS OF STUDYING QUANTILES

Let $\{X_i\}$ be a sequence of r.v.s. At the n^{th} stage we define e.d.f. as

$$F_n(x) = (\# \ X_i \leq x ; \ 1 \leq i \leq n)/n$$

and the t^{th} sample quantile as

$$Q_{nt} = \inf \{x : F_n(x) \geq t\} \quad \text{for} \quad t > 0 \quad \text{and} = Q_{n(0+)} \text{ for } t=0.$$

Most of the techniques of studying the process $\{Q_{nt} : 0 \leq t \leq 1\}$ consist of relating $\{Q_{nt}\}$ with some suitable linear statistics. The following are some of the commonly used methods for studying quantiles :

(i) The Direct Methods In The Independent Case. Here, one can actually find the exact distribution of quantiles and investigate their properties. This method is commonly found in the older literature on order statistics. Recently, Reiss (1976) applied this method to get Edgeworth expansion of the distributions of quantiles. This procedure is not at all flexible if the underlying r.v.s are dependent.

(ii) Methods Using Set Inequalities. It is easy to see that

$$\{F_n(x) > t\} \subseteq \{Q_{nt} \leq x\} \subseteq \{F_n(x) \geq t\}.$$

These set inequalities are used to find weak and strong laws for quantiles in the independent as well as in the weakly dependent cases (see section 1.3 for definitions of weak-dependence structures that we will be dealing with). Reiss (1974) uses this technique and obtains the Berry-Esseen bound for quantiles in the i.i.d. case. The main drawback of this method lies in the fact that it does not work when we have a linear combination of quantiles.

(iii) Methods Using A Property of Uniform Distribution. Let $\{U_i\}$ be a sequence of i.i.d. U[0,1] r.v.s and $U_i' = \log(1/U_i)$. If U_{kn} denotes the k^{th} order statistics at the n^{th} stage, then

$$U_{k,n} \stackrel{\mathcal{L}}{=} \sum_{i=1}^{k} U_i' \Big/ \sum_{i=1}^{n+1} U_i', \quad k = 1, 2, \ldots, n. \tag{1.1.1}$$

The statistic in the r.h.s. behaves more or less like a linear statistic. Further, if $X_{k,n}$ denote the k^{th} order statistics(at the n^{th} stage) of i.i.d. observations $\{X_i\}$ with a continuous d.f. F as the underlying d.f., then

$$X_{k,n} \overset{\mathcal{L}}{=} F^{-1}(U_{k,n}) , \quad k = 1,2,\ldots,n$$

(we take the left continuous version of F^{-1}). Thus the relationship (1.1.1) provides a method for investigating the probabilistic behaviours of order statistics and their functions. Chernoff et al (1967), Bjerve (1977), Csörgo and Révész (1977), among others, use this technique. Just like the direct method, this also has the drawback in that it depends heavily on the independence of the underlying r.v.s.

(iv) Methods Using The Bahadur-Kiefer Representation of Quantiles. Let $\{X_i\}$ be an i.i.d. sequence of r.v.s with the underlying d.f. F. Bahadur (1966) proved that if F is twice differentiable in a neighbourhood of $F^{-1}(t)$, $0 < t < 1$, with the first derivative bounded away from zero and the second bounded, then

$$Q_{nt} - F^{-1}(t) = [t - F_n(F^{-1}(t))]/F'(F^{-1}(t)) + O(n^{-3/4} (\log n)^{1/2} (\log \log n)^{1/4} \text{ a.s.} \tag{1.1.2}$$

Many of the asymptotic properties of quantiles are immediate from (1.1.2). Given below is a survey of the literature on this area of the investigation.

Kiefer (1967) showed that $n^{-3/4}(\log\log n)^{3/4}$ is the exact order of the remainder of the representation (1.1.2). Sen (1968) extended the Bahadur's result for the non-stationary m-dependent processes. The next important step on this topic was taken in Kiefer (1970 a). Apart from other valuable results, this paper shows that in the i.i.d. set-up if the underlying d.f. has a bounded support [a, b] on which it is twice differentiable with the first derivative bounded away from zero and the second bounded, then

(1.1.3) $$\sup_{0 \le t \le 1} [Q_{nt} - F^{-1}(t) + [F_n(F^{-1}(t)) - t]/F'(F^{-1}(t))]$$
$$= 0(n^{-3/4} (\log n)^{1/2} (\log\log n)^{1/4}) \quad \text{a.s.}$$

and this order is exact. Eicker (1970) gives an alternative proof of the above mentioned result of Kiefer (1967). Ghosh (1971) showed that the remainder in (1.1.2) is at least $o_p(n^{-1/2})$ assuming only that $F'(F^{-1}(t))$ exists and is positive. Dutta and Sen (1971) investigated similar representation for some stationary multivariate autoregressive processes. Sen (1972) extended Bahadur's result for stationary ϕ-mixing processes with a weaker order. Recently, Chanda (1976) obtained a representation of quantiles for linear processes with the order of remainder as $o_p(n^{-1/2})$.

(v) Hajek's Projection Method. Stigler (1969, 1974) project quantiles and their linear combinations on the space of linear functions of the observations and produces very satisfactory results on the asymptotic normality of linear functions of order statistics. It appears that this technique also uses the independence of the underlying r.v.s rather crucially.

The above list is certainly not exhaustive and it mentions only those techniques which are popularly used in the literature.

Chapters 2,3 and 4 of this thesis deal with some current problems on the Bahadur-Kiefer representation of quantile processes. These are : finding sharp asymptotic bounds of the remainders of the represention in the case of mixing r.v.s, investigation of the behaviour of the asymptotic representation towards the extremes of the quantile processes and the possible extensions of these results in the cases where the usual conditions on the differentiability of the underlying d.f. do not necessarily hold. This thesis also contains results on the rates of convergence to normality of quantiles in the mixing cases, asymptotic effective variances of quantiles in the m-dependence cases and strong approximations of weighted quantile processes by Gaussian processes with appropriate covariance kernels. Our theorem on the strong approximations is an addition to the results in an unpublished work of Csórgo and Révész (1977).

1.2 L-STATISTICS

Linear combinations of order statistics are called L-statistics. Because of their robustness, L-statistics have been drawing the attentions of many of the statisticians during the past three decades. Most of the works devoted to this topic can be classified into two groups — those which establish asymptotic properties and those which study the robustness. We mention below some important papers in these two areas and the main themes of their contents.

Jung (1955), Govindarajulu (1965), Bickel (1967), Chernoff et al (1967), Moore (1968), Shorack (1972, 1973, 1974) and Stigler (1969, 1973 and 1974) are some of the more commonly quoted papers on the asymptotic normality of L-statistics. However, this list is by no means exhaustive. Some of the techniques used in these papers were mentioned in the previous section. Bickel(1973) properly defined and studied this type of statistics in regression models. Asymptotic normality of multivariate L-statistics (see Bickel (1965) for some definitions) does not seem to be known except in some simple cases. Rao and Mehra (1975) investigated the asymptotic normality of the statistics in the case of mixing r.v.s. Rosenkrantz and O'Reilly (1972) used Skorohod representation to get Berry-Esseen bound with the order $n^{-1/4}$ for L-statistics. In a recent paper, Bjerve (1977) showed that the rate of convergence to normality is $n^{-1/2}$ for the trimmed type L-statistics in the i.i.d. situation. The drawbacks of this paper are that the proofs break down for any kind of dependence structures and the statistics must necessarily be of the trimmed type. Coming to the strong convergences and the laws of iterated logarithm, Ghosh (1972) obtained an a.s. asymptotic representation of L-statistics utilizing the idea of Moore (1968), which yields the law of iterated logarithm as a corollary. In two recent papers Wellner (1977a, 1977b) studied the strong-convergences of L-statistics within a fairly general set-up in the independent case.

Tukey (1949) first suggested the use of trimmed and Winsorized means as location estimators, keeping the robustness in mind. Bickel (1965) studied the asymptotic properties of these means in detail and calculated the infima of their relative efficiencies over the class of symmetric unimodal distributions. The principal estimate proposed by Huber (1964) resembles the trimmed mean as far as their asymptotic properties are concerned. A formal asymptotic relation between Huber's M-estimators and the

L-statistics was established by Jaeckel (1971). Gastwirth and Rubin (1969) obtained some 'maximin' type results regarding asymptotic efficiencies of L-statistics. Gastwirth and Cohen (1970) studied small sample behaviour of L-statistics. Hampel (1974) defined robustness in term of influence curves and proved that L-statistics with weight functions which are light towards tails are insensitive for outliers or 'wild observations'. A detailed review and evaluations of some important statistics with simulated data is presented in Andrews et al (1972). Sacks (1975) produced an 'universally efficient' type estimator considering a linear combination of order statistics with estimated weight function. Gastwirth and Rubin (1975) studied the effect of serial dependence in the data on the efficiency of some commonly used L-statistics. In their recent works, Bickel and Lehmann (1975(a), 1975(b) and 1976) define location, scale and dispersion rigorously. Population analogues of L-statistics with suitable weight functions turn up as examples. These papers have brought out some neat and interesting results on the statistics. Lastly, we mention an important work of Stigler (1977). This paper consists mainly of an evaluation of modern robust estimators with real data and concludes that, a small amount of trimming is the best way to deal with outliers. The 10 per cent trimmed mean turns out to be the recommended estimator for use.

One chapter of this thesis is devoted to the problem of finding an asymptotic representation of L-statistics which seems to provide an unified way of studying the statistics, since many of the asymptotic properties are immediate from the representation. We also have results on asymptotic effective variances, probabilities of deviations, uniform and non-uniform rates of convergence to normality and matching of weight functions with underlying d.f.s.

1.3 SOME WEAK DEPENDENCE STRUCTURES

In this section, we define the processes that we will be concerned with in Chapters 2-7. We also mention the relevant papers on these processes.

Let $\{X_i : -\infty < i < \infty\}$ be a stochastic process with

$$\mathbb{B}_a^b = \sigma(X_i : i \leq a \leq b).$$

(i) m-Dependent Processes. The process $\{X_i\}$ is called a m-dependent process if, for all K,

$$\sup_{A \in \mathbb{B}_{-\infty}^{h}} \sup_{B \in \mathbb{B}_{k+n}^{\infty}} |P(A\ B) - P(A)\,P(B)| = 0$$

whenever $n \geq m$. Processes obtained by considering finite linear combinations of an i.i.d. sequence of r.v.s provide simple examples of m-dependent processes.

(ii) ϕ-Mixing Processes. The process $\{X_i\}$ is called a ϕ-mixing process if, for all K,

$$\sup_{\substack{A \in \mathbb{B}^{k}_{-\infty} \\ P(A) > 0}} \; \sup_{B \in \mathbb{B}^{\infty}_{k+n}} |P(B|A) - P(B)| \leq \phi(n)$$

where $\{\phi(n)\}$ is a sequence of non-negative real numbers such that

$$1 \geq \phi(1) \geq \phi(2) \geq \ldots\ldots$$

and

$$\lim_{n \to \infty} \phi(n) = 0.$$

The notion of ϕ-mixing was first introduced by Ibragimov (1959). m-dependent processes, certain Markov processes and certain chains of ininite order are well-known examples of ϕ-mixing processes (See Ibragimov (1962)). Also, see Billingsley (1968) and Kesteen (1977) for examples of ϕ-mixing processes.

Ibragimov (1962) explored the weak convergences of normalised sums of ϕ-mixing r.v.s. See Serfling (1968) for some generalisations. The strong convergences and the laws of iterated logarithm for such processes were investigated, among others, by Iosifescu (1968), Reznik (1968) (see Stout (1974) for closing up a gap in the Reznik's paper), Oodaira and Yoshihara ((1971), (1972)), Hyde and Scott (1973) and Mclish (1975). Billingsby (1968) derives weak invariance principles for sample sums and empirical processes of ϕ-mixing r.v.s (see also Sen (1971) for an improvement of one of Billingsby's results). Philipp and Stout (1975) gives strong approximations of normalised sample sums by standard Brownian motion. Statulevičius (1974, 1977 a, 1977 b) provides Berry-Esseen bounds and probabilities of deviations for normalised sample sums of such processes. Ghosh and Babu (1977) obtained exact asymptotic expressions for probabilities of moderate deviations for some stationary ϕ-mixing processes.

(iii) Strong Mixing Processes. The process $\{X_i\}$ is called a strong mixing process if, for all K,

$$\sup_{A \varepsilon \mathbb{B}^k_{-\infty}} \quad \sup_{B \varepsilon \mathbb{B}^{\infty}_{k+n}} |P(A \cap B) - P(A)\, P(B)| \leq \alpha(n)$$

where $\{\alpha(n)\}$ is a sequence of non-negative numbers such that

$$1 \geq \alpha(1) \geq \alpha(2) \geq \ldots\ldots$$

and

$$\lim \quad \alpha(n) = 0 .$$

The idea of strong mixing was first introduced by Rosenblat (1956). Ibragimov (1965) showed the strong mixing property of Gaussian processes with suitable kernels. Examples of processes which are strong mixing but not ϕ-mixing can be obtained using the results of Ibragimov (1961) and Kolmogorov and Rozanov (1960). Chanda (1974) claimed to have established strong-mixing property of the linear process $\{X_i\}$ defined by

$$(1.3.1) \qquad X_i = \sum_{j=0}^{\infty} a_j \, Y_{i-j}$$

where $\{Y_j \, ; \, -\infty < j < \infty\}$ is a sequence of i.i.d. r.v.s and $\{a_i : i \geq 0\}$ is a sequence of real numbers such that $\Sigma \, a_i^2 < \infty$. Later Chanda's result was found to be in error and the corrected result appeared in Goredtski (1977).

Most of the papers mentioned above in the context of ϕ-mixing processes also explore the corresponding properties of strong mixing processes. We would like to include the paper Deo (1973) in this list. Deo (1973) provides an interesting proof of Davydov inequality (see Lemma 2.3.1 of this thesis) and studies the weak convergence of empirical processes in the strong mixing cases.

(iv) <u>Functions of Mixing Processes</u>. Let $\{Y_i : -\infty < i < \infty\}$ be a ϕ-mixing (strong mixing) process and $\{X_i\}$ be defined by

$$X_i = f(\ldots Y_{i-1}, Y_i, Y_{i+1}, \ldots)$$

where f is a measurable function from $R^{[0,\infty]}$ to R. Then the process $\{X_i\}$ is termed as a function of the ϕ-mixing (strong mixing) process $\{Y_i\}$. It is usually assumed that

$$(1.3.2) \qquad \| X_i - E(X_i \mid \mathbb{B}_{-\infty}^{i-n}) \|_r \leq \rho(n)$$

where $\mathbb{B}_a^b = \sigma(Y_i : a \leq i \leq b)$, r is some positive number and $\rho(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

For example, if $\{Y_i\}$ is a sequence of mixing r.v.s and $\{X_i\}$ is defined as in (1.3.1), then the process $\{X_i\}$ comes in this category. If the condition (1.3.2) holds, then the studies of functions of mixing processes and of mixing processes proced along similar lines. As an alternative to Goredtski's result, one can study linear processes by considering them as functions of independent r.v.s satisfying (1.3.2) (under minor conditions on $\{a_i\}$ and $\{Y_i\}$) with the additional advantage that $\{Y_i\}$ can be allowed to be a m-dependent process. See Billingsby(1968) for some non-trivial examples of functions of ϕ-mixing r.v.s and some interesting results on them. Philipp and Stout (1975) and Philipp (1977) contain some strong invariance principles for functions of strong mixing processes.

1.4 A SUMMARY OF CHAPTERS 2 - 8

In Chapter 2, we conclude that in the ϕ-mixing case, when the mixing coefficients satisfy the condition $\Sigma\, \phi^{1/2}(i) < \infty$, the orders of the remainders in the r.h.s.s of (1.1.2) and (1.1.3) are the same as the exact orders in the i.i.d. case obtained by Kiefer (1967, 1970 a). Since, an i.i.d. sequence of r.v.s is a particular case of ϕ-mixing processes, the orders that we have obtained for ϕ-mixing processes cannot be improved upon further, in general. However, our results do not exclude the possibility of the existence of some ϕ-mixing processes (obviously not an i.i.d. sequence) for which the orders of the remainders are sharper than the exact orders for the independent case. This problem has not been settled yet. For both strong mixing processes and functions of ϕ-mixing processes, the orders are shown to be $n^{-3/4}(\log n)^{1/2} (\log \log n)^{1/4}$ and $n^{-3/4} (\log n)(\log \log n)^{1/4}$ a.s. in the r.h.s.s of (1.1.2) and (1.1.3), respectively.

Chapter 3 deals with the representation of quantiles with non-uniform error bounds. The problem was posed in Kiefer (1970 a, 1970 b). As was expected, the order of the remainders get sharper and sharper as one moves towards the sample extremes. In proving so, some stability results on weighted empirical and quantile processes for weakly dependent r.v.s are established which seem to have their own importance.

In Chapter 4, the results of Chapter 2 are extended to general distributions satisfying the same conditions as in Bahadur (1966) and Kiefer (1970 a). We also pay attention to the behaviour of Q_{nt_n} where $t_n \longrightarrow t$ as $n \longrightarrow \infty$, $t \in (0,1)$. Nextly, we take up the representation of quantiles for distributions which fail to satisfy the usual conditions on differentiability. Included in the domain of our results are the distributions which have only right derivative, only left derivative or have both the derivatives, but they are unequal. Our approach consists of reformulating the problems in terms of uniform distribution and then, extending the results to the cases desired. In the last section of this chapter, we have some results on strong approximations of weighted quantile processes which utilize the results of Chapter 3.

The asymptotic representation of L-statistics is studied in Chapter 5. The representation linearises the L-statistics except for negligible remainders. In the proofs, we use the idea of Moore (1968) and our results of Chapter 3.

We investigate the rates of convergence to normality of normalised quantiles of mixing observations in Chapter 6. Non-regular cases (i.e. when the underlying d.f. is not differentiable) have also been taken into consideration. When specialised to the regular cases, the rate is found to be $n^{-1/2} \log n$ for ϕ-mixing processes. The second problem taken up in this chapter is the asymptotic effective variance (as defined in Bahadur (1960)) of sample quantiles of m-dependent observations. The corresponding result for sample sum of m-dependent r.v.s which is needed in the proof is also obtained in this chapter. The results help in justifying in part the use of sample quantiles as estimators.

In Chapter 7, we study probabilities of deviations in the specific case of trimmed type L-statistics. The problems are tackled by utilizing the tools which lead to the asymptotic representation of quantile processes. In the independent case, these problems can be solved by using some other well-known methods also, but the present method enjoys the property of being flexible enough for weakly dependent structures also. Nextly, in this chapter, the arguments leading to the results of Chapter 5 have been employed to calculate uniform and non-uniform rates of convergence to normality of L-statistics. The results on non-uniform rates of convergence to normality lead to in particular the probabilities of moderate deviations, L_p versions of Berry – Esseen bounds and convergence of higher order moments.

Finally, Chapter 8 deals with the problems of matching weight function with the underlying d.f.s taking the asymptotic relative efficiencies as the measures of performances. A general result of this chapter asserts, roughly, that a L-statistics which gives less weight to the sample extremes performs better when the underlying d.f. F has heavy tails than in the opposite case. The various terms appearing the statement just made are defined in this chapter and the problems are formulated in precise mathematical language. When specialised to trimmed means, these results are also in accord with the recent data work of Stigler (1977).

The contents of Chapter 8 are published (Sankhyā, Ser. B, 39, 26-35). A weaker version of theorem 2.3.2 was presented in the joint meeting of the Institute of Mathematical Statistics and the Indian Statistical Institute at New Delhi (December 16 – 18, 1977) and the abstract appeared in the IMS Bulletin (November 1977 issue). The contents of Chapter 2, in the present form, have been submitted for publication.

Before concluding, we remark that the above summary is meant to provide only an outline of the contents. All chapters that follow have an introductary section where the problems and the nature of the solutions are described more explicitly.

CHAPTER 2

REPRESENTATION OF QUANTILES FOR MIXING PROCESSES WITH MARGINALS $U[0, 1]$

2.1 INTRODUCTION

Let $\{U_i, i \geq 1\}$ be a strictly stationary sequence of r.v.s. Let E_n denote the e.d.f., that is

$$E_n(x) = (\# U_i \leq x, \; 1 \leq i \leq n)/n.$$

We define the t^{th} sample quantile $E_n^{-1}(t)$ as

$$\begin{aligned} E_n^{-1}(t) &= \inf \{x : E_n(x) \geq t\} && \text{for } t > 0 \\ &= E_n^{-1}(t+) && \text{for } t = 0. \end{aligned}$$

Kiefer has shown (see Kiefer (1967) and Kiefer(1970 a)) that if $\{U_i\}$ is a sequence of i.i.d. r.v.s with U_1 uniformly distributed on the unit interval $[0, 1]$, then

(2.1.1) $$\limsup_{n \to \infty} a_n^{-1} |R_n(t)| = 2^{5/4}\, 3^{-3/4} (t(1-t))^{1/4} \quad \text{a.s.}$$

and

(2.1.2) $$\limsup_{n \to \infty} b_n^{-1} R_n = 2^{-1/4} \quad \text{a.s.}$$

where

$$R_n(t) = (E_n^{-1}(t) - t) - (t - E_n(t))$$

$$R_n = \sup_{0 \leq t \leq 1} |R_n(t)|$$

$$a_n = n^{-3/4} (\log \log n)^{3/4}$$

$$b_n = n^{-3/4} (\log n)^{1/2} (\log \log n)^{1/4}.$$

In this chapter we shall show that the a.s. asymptotic orders of $|R_n(t)|$ and R_n are still maintained for certain ϕ-mixing processes. We also establish slightly weaker orders for $|R_n(t)|$ and R_n for strong mixing processes and functions of ϕ-mixing processes under suitable mixing conditions.

It may not be out of place here to mention that Sen (1972) has obtained a weaker result, namely for $t \varepsilon (0, 1)$,

$$|R_n(t)| = 0(n^{-3/4} \log n) \quad \text{a.s.}$$

for ϕ-mixing r.v's satisfying the condition

$$\sum_{i=1}^{\infty} \phi(i) \exp(\lambda i) < \infty$$

for some $\lambda > 0$ which is a much stronger on the mixing coefficients than that we assume in the next section of this chapter.

2.2 QUANTILES FOR ϕ-MIXING PROCESSES

We first state our results for ϕ-mixing processes.

Theorem 2.2.1 Let $\{U_i\}$ be a strictly stationary ϕ-mixing process with $P(X_1 \leq t) = t$ for $0 \leq t \leq 1$ and satisfying

$$(2.2.1) \qquad \sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty.$$

Then, for $0 < t < 1$,

$$(2.2.2) \qquad |E_n^{-1}(t)-t+E_n(t)-t| = O(n^{-3/4}(\log\log n)^{3/4}) \quad \text{a.s.}$$

Theorem 2.2.2. Under the conditions of Theorem 2.2.1,

$$(2.2.3) \qquad \sup_{0\leq t\leq 1} |E_n^{-1}(t)-t+E_n(t)-t| = O(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}) \quad \text{.. a.s.}$$

In view of Kiefer's results for the independent case, it follows that the orders of $|R_n(t)|$ and R_n cannot be improved in general for ϕ-mixing r.v.s.

To prove the theorems, we start with some lemmas. The proofs are given in some detail as the same arguments will be used in the subsequent sections also.

Define, for $0 \leq \alpha, \beta \leq 1$, (2.2.4) $$x_i(\alpha,\beta) = I(\min(\alpha,\beta) \leq U_i \leq \max(\alpha,\beta)) - |\alpha-\beta| .$$

(Recall that $I(A)$ denotes the indicator function of set A)

Lemma 2.2.1. Let $\{X_i\}$ be a sequence of ϕ-mixing r.v's satisfying (2.2.1). If ξ is measurable $\mathbb{B}^K_{-\infty}$ and η is measurable $\mathbb{B}^{\infty}_{K+n}$ $(n \geq 0)$, then

$$E|\xi|^r < \infty, \quad E|\eta|^s < \infty, \quad r,s \geq 1, \quad r^{-1} + s^{-1} = 1$$

implies

$$|E(\xi\eta) - E(\xi)E(\eta)| \leq 2(\phi(n))^{1/r} \|\xi\|_r \|\eta\|_s$$

where $\| \ \|_r$ denotes usual L_r norm. Further, if $P(|\eta| > K) = 0$, then

$$|E(\xi\eta) - E(\xi)E(\eta)| \leq 2K\,\phi(n)\ E(|\xi|).$$

The lemma is due to Ibragimov (1962). See Billingsley (1968) pp. 170-171 for a proof.

We will use the following moment inequality in the proof of the next lemma.

Lemma 2.2.2. Under the conditions of Theorem 2.2.1,

$$E|\sum_{i=1}^{u} x_i(\alpha,\ \alpha+\beta)|^{\delta} \leq c(\delta)\ u^{\delta/2}$$

for $\delta \geq 2$, where the constant $c(\delta)$ is independent of α, β and u.

Proof. Lemma 1.9 of Ibraginov (1962) states that if $\{Y_i\}$ is a stationary sequence of ϕ-mixing r.v.s with $\Sigma\, \phi^{1/2}(i) < \infty$ and $E|Y_1|^\delta < \infty$ for some $\delta \geq 2$, then

$$E\left|\sum_{i=1}^{u} Y_i\right|^\delta \leq c(\delta)\, u^{\delta/2}$$

where the constant $c(\delta)$ does not depend on u. A careful examination of the proof supplied there acertains that $c(\delta)$ depends only on ϕ, δ and $M > 0$, where $E|Y_1|^\delta < M$. So our lemma follows immediately for, $|x_i(\alpha, \alpha+\beta)| \leq 1$ for all $i \geq 1$ and $0 \leq \alpha, \beta \leq 1$.

Lemma 2.2.3. Let $\{U_i\}$ be as in Theorem 2.2.1. Then, there exists $d > 0$ such that, whenever $0 < \alpha < 1$, $0 < B \leq 1-\alpha$, $-b \leq \beta \leq b$, $1 \leq u \leq N$, $H \geq 0$ and $0 < D \leq b\, N^{19/24}$, we have

$$(2.2.5) \quad P\left(\left|\sum_{i=H+1}^{H+u} x_i(\alpha, \alpha+\beta)\right| > 2dD\right) \leq K_1 N^{-4} + K_2 \exp(-8D^2 N^{-1} b^{-1}).$$

Remark 2.2.1. If we drop the assumption that U_1 has the distribution $U[0, 1]$ and assume only that there exists $0 \leq t_1 < t_2 \leq 1$ such that

$$P(U_1 \leq s) = s \quad \text{whenever} \quad s \in [t_1, t_2]$$

then the proof of the lemma acertains that the inequality (2.2.5) still holds if

$$t_1 \leq \min(\alpha, \alpha+\beta) \leq \max(\alpha, \alpha+\beta) \leq t_2$$

This remark comes in use in the subsequent chapters.

<u>Remark 2.2.2.</u> Proof of Lemma 2.2.3 also implies that the condition $0 < D \leq bN^{19/24}$ can be replaced by a slightly more general condition, namely

$$0 < D \leq bN^{(3/4)+\varepsilon}, \quad \varepsilon > 0 \text{ fixed,}$$

but the constant d appearing in the statement of the lemma depends on ε.

<u>Proof of lemma 2.2.3.</u> Without loss of generality, we may assume that $H = 0$, $b \geq \beta > 0$. We first present the blocking procedure which will be used repeatedly in what follows. Let $p = p(u)$, $1 \leq p \leq u$, $k = k(u) = [u/2p]$. We write

(2.2.6)
$$\sum_{i=1}^{u} x_i(\alpha, \alpha+\beta) = \sum_{j=1}^{k+1} \xi_j + \sum_{j=1}^{k} \eta_j = Y(u) + Y'(u)$$

where

$$\xi_j = \sum_{i=1}^{p} x_{2p(j-1)+i}(\alpha, \alpha+\beta), \quad \eta_j = \sum_{i=1}^{p} x_{(2j-1)p+i}(\alpha, \alpha+\beta)$$

for $j = 1, 2, \ldots, k$ and $\xi_{k+1} = \sum_{i=2kp+1}^{u} x_i(\alpha, \alpha+\beta)$ or 0 according as $u-2kp \geq 1$ or not.

For the present case, we take $p = [u^{1/3}]$. Clearly, the lemma will follow if we show

$$P(Z > dD) \leq \text{the desired bound in } (2.2.5)$$

for $Z = Y(u)$, $Y'(u)$, $-Y(u)$, and $-Y'(u)$. We consider only the case $Z = Y(u)$ since the proof of the other cases are similar.

On letting $z = DN^{-1}b^{-1}$ and $\xi_i^* = \xi_i I(|\xi_i| \leq z^{-1})$, we obtain

(2.2.7) $$P(Y(u) > dD) \leq P(\sum_{i=1}^{k+1} \xi_i^* > dD) + P(Y(u) \neq \sum_{i=1}^{k+1} \xi_i^*)$$

$$\leq P(\sum_{i=1}^{k+1} \xi_i^* > dD) + \sum_{i=1}^{k+1} P(|\xi_i| > z^{-1}).$$

Now, using Markov inequality and Lemma 2.2.2 with $\delta = 60$, we get

$$\sum_{i=1}^{k+1} P(|\xi_i| > z^{-1}) \leq (k+1)\, c(\delta)\, (p\, z^2)^{\delta} \leq d_1 N^{-4}$$

Another application of Markov inequality gives

(2.2.8) $$P(\sum_{i=1}^{k+1} \xi_i^* > dD) \leq \exp(-z\, d\, D)\, E(\exp z \sum_{i=1}^{k+1} \xi_i^*)).$$

Since $|z\, \xi_i^*| \leq 1$, $\phi(p) = O(p^{-2})$ and $(\xi_1^*, \ldots, \xi_k^*)$ is a stationary sequence, we have, by repeated application of the second part of Lemma 2.2.1 with $h = E(\exp(z\, \xi_1^*)) + 2e\, \phi(p)$, (2.2.9) $$E(\exp(z \sum_{1}^{k+1} \xi_i^*))$$

$$\leq e\, E(\exp(z \sum_{1}^{k} \xi_i^*))$$

$$\leq 3\, E(\exp(z\, \xi_k^*))\, E(\exp(z \sum_{1}^{k-1} \xi_i^*)) + 6e\,\phi(p)\, E(\exp(z \sum_{1}^{k-1} \xi_i^*))$$

$$\leq 3\, h\, E(\exp(z \sum_{1}^{k-1} \xi_i^*))$$

$$\leq \cdots \leq 3\, h^k.$$

Clearly

$$h = 1 + |E(z\, \xi_1^*)| + O(z^2\, E(\xi_1^2)) + O(k^{-1}).$$

It follows from Lemmas 2.2.1 and 2.2.2 that

$$E(\xi_1^2) = pV(x_1(\alpha, \alpha+\beta)) + 2 \sum_{i=1}^{p-1} (p-i)\, \mathrm{cov}(x_1(\alpha, \alpha+\beta), x_{1+i}(\alpha, \alpha+\beta))$$

$$\leq p\beta + 2p\beta \sum_{j=1}^{p} \phi^{1/2}(j)$$

$$= O(p\beta) = O(pb)$$

Since $\log(1+x) \leq x$ for $x > -1$ and $|E(\xi_1^*)| \leq zE(\xi_1^2)$,

we have

(2.2.10) $$h^k = \exp(k \log h) = \exp(k \log(1 + O(z^2 pb) + O(k^{-1})))$$

$$\leq \exp(O(u z^2 b) + O(1))$$

$$\leq K_3 \exp(K_4 D^2 N^{-1} b^{-1})$$

The desired result now follows from (2.2.8), (2.2.9) and (2.2.10) on choosing $d = 8 + K_4$. This completes the proof of this lemma.

Lemma 2.2.4. Under the conditions of Theorem 2.2.1, there exists a constant $c > 0$ such that

(2.2.11) $$\limsup_{n \to \infty} \lambda_n^{-1} \sup_{0 \le t \le 1} |E_n(t) - t| < c \quad \text{a.s.}$$

where $\lambda_n = n^{-1/2} (\log \log n)^{1/2}$

Proof. Let $h(n) = \max_{1 \le j \le n} |E_n(j/n) - (j/n)|$. Since,

$$\sup_{0 \le t \le 1} |E_n(t) - t| \le h(n) + n^{-1},$$

it suffices to show that

(2.2.12) $$\limsup_{n \to \infty} \lambda_n^{-1} h(n) < c \quad \text{a.s.}$$

To prove (2.2.12), let d be as in (2.2.5), $n_r = [\exp(\sqrt{r})]$, $A_r = \{n : n_r \le n \le n_{r+1}\}$ and let $\lambda(r) = \lambda_{n_r}$. Observe that

$$\frac{1}{2} n_r (r+1)^{-1/2} \le n_{r+1} - n_r \le n_r/\sqrt{r} .$$

Define, for $n \in A_r$,

$$B_n = \left\{ \max_{1 \leq j \leq r} \left| \sum_{i=n_r+1}^{n} x_i(0, \tfrac{j}{r}) \right| > 2d\, n_r^{1/2} \right\}$$

$$C_n = \left\{ \max_{1 \leq j \leq n} \left| \sum_{i=1}^{n} x_i\left(\frac{[jr/n]}{r}, \frac{j}{n} \right) \right| > 2dn\, \lambda_n \right\}$$

and

$$H_r = \left\{ \max_{1 \leq j \leq r} \left| E_{n_r}(\tfrac{j}{r}) - \tfrac{j}{r} \right| > 2d\, \lambda(r) \right\} .$$

Now, Bonferroni inequality and (2.2.5) with (i) $H = 0$, $N = n$, $b = 1$ and $D = n_r\, \lambda(r)$ gives that $P(H_r) = O(r^{-2})$ (ii) $H = 0$, $N = n$, $b = 1/r$ and $D = n\, \lambda_n$ gives $P(C_n) = O(n^{-2})$ and (iii) $H = n_r$, $N = n_{r+1} - n_r$, $b = 1$ and $D = \sqrt{n_r}$ gives $P(B_n) = O(n^{-2})$. It is immediate from these estimates that

(2.2.13) $$\sum_{r=1}^{\infty} P\Big(H_r \bigcup_{n \varepsilon A_r} (B_n \cup C_n)\Big) < \infty .$$

Notice that $\{n\, \lambda_n\}$ is a non-decreasing sequence and that for $0 \leq \alpha \leq \beta \leq 1$,

$$x_i(0, \beta) = x_i(0, \alpha) + x_i(\alpha, \beta) \quad \text{a.s.}$$

So, for $n \varepsilon A_r$, we have, outside $H_r \cup B_n \cup C_n$, that $h(n) \leq 6d\, \lambda_n$ and hence (2.2.12) follows from (2.2.13) and the Borel-Cantelli lemma.

<u>Lemma 2.2.5.</u> If (2.2.11) holds, then

$$\limsup_{n \to \infty} \lambda_n^{-1} \sup_{0 \leq t \leq 1} |E_n^{-1}(t) - t| \leq c \quad \text{a.s.}$$

<u>Proof.</u> From (2.2.11) we have

$$P(\, s - c\lambda_n < E_n(s) < s + c\lambda_n, \ \forall \, s \in [0,1],$$
$$\text{for all sufficiently large } n) = 1$$

and hence

$$P(E_n(t - c\lambda_n) < t < E_n(t + c\lambda_n), \ \forall \ s \in [0, 1],$$
$$\text{for all sufficiently large } n) = 1.$$

Consequently

$$P(t - c\lambda_n \leq E_n^{-1}(t) \leq t + c\lambda_n, \ \forall \ s \in [0, 1],$$
$$\text{for all sufficiently large } n) = 1.$$

This proves the lemma.

We are now ready to prove Theorems 2.2.1 and 2.2.2.

<u>Proof of Theorem 2.2.1.</u> Clearly, for $0 \leq t \leq 1$,

$$|R_n(t)| \leq |E_n E_n^{-1}(t) - t| + |E_n E_n^{-1}(t) - E_n(t) - E_n^{-1}(t) + t|,$$

and

$$|E_n E_n^{-1}(t) - t| \leq E_n E_n^{-1}(t) - E_n(E_n^{-1}(t) - 0))$$
$$\leq |E_n E_n^{-1}(t) - E_n(t) - E_n^{-1}(t) + t| + |E_n(t) + E_n^{-1}(t) - t - E_n(E^{-1}(t) - 0)|$$

Therefore, in view of Lemma 2.2.5, we have, with probability 1,

$$(2.2.14)\qquad |R_n(t)| \leq 3 \sup_{|x-t|\leq 2c\lambda_n} |E_n(x) - E_n(t) - x + t|$$

$$\leq \frac{3}{n} \max_{|j|\leq [2cn\lambda_n]+1} \left| \sum_{i=1}^{n} x_i(t,\ t+\tfrac{j}{n}) \right| + \frac{3}{n}$$

$$= 3\, R_n^*(t) + (3/n)$$

for all sufficiently large n. So it is enough to show that

$$(2.2.15)\qquad \limsup_{n\to\infty}\ a_n^{-1}\ R_n^*(t) \leq K \quad \text{a.s.}$$

where $a_n = n^{-3/4}(\log\log n)^{3/4}$. Proof of (2.2.15) is similar to that of Lemma 2.2.4. Let n_r, A_r, d, $\lambda(r)$ be as in the proof of Lemma 2.2.4., $u(r) = 2c\lambda(r)$ and let $a(r) = 2\sqrt{c}\, n_r\, a_{n_r}$. Define, for $n \in A_r$,

$$B_n' = \left\{ \max_{|j|\leq r} \left| \sum_{i=n_r+1}^{n} x_i(t,\ t+ju(r)/r) \right| > 2d\, a(r) \right\}$$

$$C_n' = \left\{ \max_{|j|\leq [2cn\lambda_n]+1} \left| \sum_{i=1}^{n} x_i\left(t + \left[\frac{jr}{nu(r)}\right] \frac{u(r)}{r},\ t+\frac{j}{n}\right) \right| > 4\sqrt{c}\ d\, n\, a_n \right\}$$

and

$$H_r' = \left\{ \max_{|j|\leq r} \left| \sum_{i=1}^{n_r} x_i(t,\ t+ju(r)/r) \right| > 2da(r) \right\}$$

Here also, we estimate the probabilities of these events using Bonferroni enequality and Lemma 2.2.3. More precisely, taking $H = 0$, $N = n_r$, $b = u(r)$ and $D = a(r)$ in (2.2.5.) we get $P(H_r') = O(r^{-2})$; putting $H = n_r$, $N = n_{r+1} - n_r$, $b = u(r)$ and $D = a(r)$ in (2.2.5) we obtain $P(B_n') = O(n^{-2})$, and finally, choosing $H = 0$, $N = n$, $b = u(r)/r$ and $D = 2\sqrt{c}\, na_n$ we see that $P(C_n') = O(n^{-2})$. These estimates imply that

$$\sum_{r=1}^{\infty} P(H_r' \bigcup_{n\varepsilon A_r} (B_n' \cup C_n')) < \infty.$$

From this and the Borel-Cantelli lemma, (2.2.15) follows.

Remark 2.2.3. In the independendent case, $\sup_t |E_n E_n^{-1}(t) - t| \leq n^{-1}$ a.s. But this does not hold, in general, for ϕ-mixing r.v.s.

Proof of Theorem 2.2.2. From (2.2.14) we have that, with probability 1,

(2.2.16) $$\sup_t |R_n(t)| \leq 3 \sup_{0\leq t\leq 1} \sup_{|x-t|\leq 2c\lambda_n} |E_n(x)-E_n(t)-x+t|$$

for all sufficiently large n. Let

$$b_n = n^{-3/4}(\log n)^{1/2} (\log\log n)^{1/4},$$

$$v_n = [1/b_n] + 1 \quad \text{and} \quad v_n' = [2c\,\lambda_n/b_n] + 1.$$

Using some elementary inequalities, it follows that

$$\sup_{0\le t\le 1} \sup_{|x-t|\le 2c\lambda_n} |E_n(x) - E_n(t) - x + t|$$

$$\le \sup_{0\le t\le 1} \Big\{ \max_{|j|\le v_n'} |E_n(t + jb_n) - E_n(t) - jb_n| + b_n \Big\}$$

$$\le \max_{|j|\le v_n'} \sup_{0\le t\le 1} |E_n(t + jb_n) - E_n(t) - jb_n| + b_n$$

$$\le \max_{|j|\le v_n'+1} \max_{\ell\le v_n} |E_n((\ell+j)b_n) - E_n(\ell b_n) - jb_n| + 2b_n$$

$$= R_n^* + 2b_n \quad \text{(say)}$$

So, it is enough to show that

(2.2.17) $$\limsup_{n\to\infty} b_n^{-1} R_n^* \le K_5 \quad \text{a.s.}$$

Applying (2.2.5) with $H = 0$, $N = n$, $b = 2c\lambda_n$ and $D = 2\sqrt{c}\, n\, b_n$ and Bonferroni inequality, we get

$$P(R_n^* > 4\sqrt{c}\, d\, b_n)$$

$$\le 2n \sup_{0\le s\le 1} \sup_{|t|\le 2c\lambda_n} P(|E_n(s+t) - E_n(s) - t| > 4\sqrt{c}\, d\, b_n)$$

$$= O(n^{-2}).$$

Thus (2.2.17) follows with $K_1 = 4\sqrt{c}\,d + 2$ from the Borel-cantelli lemma. This completes the proof of Theorem 2.2.2.

Remark 2.2.4. Note that

$$|E_n^{-1}(t) - s| \leq |E_n^{-1}(t) - t| + |t-s| = O(\lambda_n) \quad \text{a.s.}$$

whenever $|t - s| = O(n^{-1/2})$. So, from the above proofs it is clear that Theorems 2.2.1 and 2.2.2 still hold if $E_n^{-1}(t) - t_n$ is replaced by $E_n^{-1}(t_n) - t_n$, where $|t_n - t| = O(n^{-1/2})$.

Remark 2.2.5. In particular, theorem 2.2.1 sharpens the result of Sen (1968) also for the m-dependent case.

2.3. QUANTILES FOR STRONG MIXING PROCESSES

In this section, we study quantile processes in the strong mixing case. We prove the following theorems.

Theorem 2.3.1. Let $\{U_i\}$ be a strictly stationary strong mixing process with $P(X_1 \leq t) = t$ for $0 \leq t \leq 1$ and

(2.3.1) $$\alpha(n) = O(e^{-\gamma n})$$

for some $\gamma > 0$. Then, for every $0 < t < 1$,

$$(2.3.2) \quad |E_n^{-1}(t) - t + E_n(t)-t| = O(n^{-3/4}(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{3}{4}}) \quad \text{a.s.}$$

<u>Theorem 2.3.2.</u> Under the conditions of Theorem 2.3.1,

$$(2.3.3) \quad \sup_{0\le t\le 1} |E_n^{-1}(t)-t+E_n(t)-t| = O(n^{-\frac{3}{4}}(\log n)(\log\log n)^{\frac{1}{4}}) \quad \text{a.s.}$$

To establish these theorems, we obtain some lemmas similar to those given in the previous section and use them in the same way. The following lemma of Davydov (1970) is used repeatedly.

<u>Lemma 2.3.1.</u> Let X_n be a stationary sequence of strong-mixing r.v.s. Let X and Y be two r.v.s measurable w.r.t. the σ-fields $\mathbb{B}_1^m$ and $\mathbb{B}_{m+n}^{\infty}$, respectively. Let r, s, t $\ge$ 1 be such that $r^{-1} + s^{-1} + t^{-1} = 1$ and let $\|X\|_r$ and $\|Y\|_s$ be finite. Then

$$(2.3.4) \quad |E(XY) - E(X)\,E(Y)| \le 10(\alpha(n))^{1/t} \|X\|_r \|Y\|_s$$

Further, if $\|X\|_\infty < \infty$, $\|Y\|_\infty < \infty$, then

$$(2.3.5) \quad |E(XY) - E(X)\,E(Y)| \le 4\,\alpha(n) \|X\|_\infty \|Y\|_\infty$$

See Deo (1973) for a proof.

Lemma 2.3.2. Let $\{Y_i\}$ be a stationary strong-mixing sequence of bounded r.v.s with mean zero. If

(2.3.6) $$\sum_{n=1}^{\infty} n^{\delta-1} \alpha(n) < \infty,$$

then for $\delta = 1, 2, 3, \ldots$

(2.3.7) $$E\left|\sum_{i=1}^{n} Y_i\right|^{2\delta} \leq d(\delta)\, n^{\delta}$$

where $d(\delta)$ depends only on the bound of Y_1, δ and $\{\alpha(n)\}$.

Proof. It is easy to check that

$$E\left|\sum_{i=1}^{n} Y_i\right|^{2\delta} \leq (2\delta)!\, n\, \Sigma' \left|E(Y_0 Y_{i_1} \ldots Y_{i_1+i_2+\ldots+i_{2\delta-1}})\right|$$

where Σ' denotes sum over the set A defined by

$$A = \left\{(i_1,\ldots,i_{2\delta-1}) : i_j \geq 0 \text{ are integers and } \sum_{j=1}^{2\delta-1} i_j \leq n\right\}.$$

We now divide the set A into $\binom{2\delta-1}{\delta}$ parts as follows. Fix $1 \leq j_1 < \ldots < j_\delta \leq 2\delta-1$. Corresponding to this choice, define

$$A(j_1,\ldots, j_\delta) = \{ (i_1,\ldots,i_{2\delta-1}) \varepsilon A : \min (i_{j_1}, \ldots, i_{j_\delta}) \geq \text{ the rest of the } i_j\text{'s} \}.$$

We consider two cases.

Case 1. Let $j_1 = 1$ and let $|Y_o| \leq N_o$ a.s. Taking $X = Y_o$ and $Y = Y_{i_1} \cdots Y_{i_1+i_2+\cdots+i_{2\delta-1}}$ in Lemma 2.3.1, it follows from (2.3.5) that (taking $\alpha(0) = 1$)

$$\sum_{A(j_1,\ldots,j_\delta)} |E(Y_o\, Y_{i_1} \cdots Y_{i_1+\cdots+i_{2\delta-1}})|$$

$$\leq 4\, N_o^{2\delta} \cdot \sum_{A(j_1,\ldots,j_\delta)} \alpha(i_1)$$

$$\leq 4\, N_o^{2\delta} \left(\prod_{\ell=2}^{\delta} \left(\sum_{i_{j_\ell}=1}^{n} 1 \right)\right) \sum_{i_1=0}^{n} (i_1+1)^{\delta-1}\, \alpha(i_1)$$

$$= O(n^{\delta-1})$$

The last step follows from 2.3.6.

Case 2. Suppose $j_1 \neq 1$. Then there exists ℓ such that $\ell, \ell+1 \in (j_1,\ldots, j_\delta)$. So by (2.3.5),

$$|E(Y_o Y_{i_1} \cdots Y_{i_1+\cdots+i_{\ell-1}}\, Y_{i_1+\cdots+i_\ell} \cdots Y_{i_1+\cdots+i_{2\delta-1}})|$$

$$\leq 4N_o^{2\delta}\, \alpha(i_\ell) + N_o^{\ell} |E(Y_{i_1+\cdots+i_\ell}\, Y_{i_1+\cdots+i_{\ell+1}} \cdots Y_{i_1+\cdots+i_{2\delta-1}})$$

$$\leq 4N_o^{2\delta} \left[\alpha(i_\ell) + \alpha(i_{\ell+1})\right]$$

while summing we supply the arguments of case 1 for $\alpha(i_\ell)$ and $\alpha(i_{\ell+1})$ seperately. This establishes the lemma.

For technincal reasons, we present two lemmas for the strong mixing case, which are similar to Lemma 2.2.3. The exponential inequality of Lemma 2.3.3 provides us an upper bound for the fluctuations of empirical processes. It is also used in proving Lemma 2.3.5.

Lemma 2.3.3. Let $\{U_i\}$ be as in Theorem 2.3.1. Then there exists a $\rho > 0$ such that, whenever $0 \leq \alpha < 1$, $0 < b \leq 1-\alpha$, $-b \leq \beta \leq b$, $1 \leq u \leq N$, $H \geq 0$ and $0 < D^2 \leq b\, N^{22/15}$, we have

$$(2.3.8) \qquad P\left(\left| \sum_{i=H+1}^{H+u} x_i(\alpha, \alpha+\beta) \right| > 2\rho D\right)$$

$$\leq K_6 \exp(-8D^2 N^{-1} b^{-1/2}) + K_7 N^{-8}.$$

Proof. The proof is similar to that of Lemma 2.2.3. Without loss of generality, we may assume that $H = 0$ and $b \geq \beta > 0$. We split the sum $\sum_{i=1}^{u} x_i(\alpha, \alpha+\beta)$ as in (2.2.6) taking $p = p(u) = [\,2\sqrt{u/\gamma}\,]$ where γ is as in (2.3.1). We shall show that

$$P(Y(u) > \rho D) \leq \text{the desired bound in (2.3.8)}$$

and conclude the lemma as in the proof of Lemma 2.3.3

Let $z = D N^{-1} b^{-1/2}$ and $\xi_i^* = \xi_i I(|\xi_i| \leq z^{-1})$.
Once again as in (2.2.7),

$$P(Y(u) > \rho D) \leq P(\sum_{i=1}^{k+1} \xi_i^* > \rho D) + kP(|\xi_1| > z^{-1}) + P(\xi_{k+1} > z^{-1}).$$

Using Markov inequality, Lemma 2.3.2 and the condition $D^2 \leq b N^{22/15}$, we have

$$(2.3.9) \qquad kP(|\xi_1| > z^{-1}) + P(\xi_{k+1} > z^{-1})$$

$$\leq (k+1)\, d(250)\, z^{500}\, p^{250} = O(N^{-8}).$$

Let $h = E(\exp(z \xi_1^*))$. It follows from (2.3.1), (2.3.5) and the fact that $|z \xi_i^*| \leq 1$ (so that $0 \leq h \leq e$) that

$$(2.3.10) \qquad E(\exp(z \sum_1^{k+1} \xi_i^*))$$

$$\leq e E(\exp(z \sum_1^{k} \xi_i^*))$$

$$\leq e E(\exp(z \xi_k^*) \exp(z \sum_1^{k-1} \xi_i^*))$$

$$\leq e(h E(\exp(z \sum_1^{k-1} \xi_i^*)) + 4 e^k \alpha(p))$$

$$\leq \ldots \leq e h^k + 4 e \alpha(p)(e^k + he^{k-1} + \ldots + eh^{k-1})$$

$$\leq 3\ h^k + 12\ k\ e^k\ \alpha(p)$$

$$= 3\ h^k + O(1).$$

Using (2.3.4) with $r = 4$, $s = 4$ and $t = 2$, we conclude that

$$E(\xi_1^2) = pV(x_1(\alpha, \alpha+\beta) + 2 \sum_{i=1}^{p-1} (p-i)\ \mathrm{cov}(x_1(\alpha, \alpha+\beta),\ x_{1+i}(\alpha, \alpha+\beta))$$

$$\leq p\beta(1-\beta) + 2 \sum_{i=1}^{p-1} (p-i)\ \alpha^{1/2}(i)\ (\beta(1-\beta))^{1/2}$$

$$= O(b^{1/2}\ p).$$

So

(2.3.11) $$h \leq 1 + |E(z\ \xi_1^*)| + O(E(z^2\ \xi_1^2\ ))$$

$$\leq 1 + O(z^2\ E(\xi_1^2)) \leq 1 + O(p\ b^{1/2})$$

Finally, by Markov inequality, (2.3.10) and (2.3.11),

(2.3.12) $$P(\sum_{i=1}^{k+1} \xi_i^* > \rho D) \leq \exp(-z\rho D)\ E(\ \exp(z \sum_{i=1}^{k+1} \xi_i^*\ ))$$

$$\leq K_8\ \exp(-z\rho D)\ \exp(k\ \log\ h)$$

$$\leq K_8\ \exp(-z\rho D + O(z^2\ N\ b^{1/2}))$$

$$\leq K_9\ \exp(-8\ D^2\ N^{-1}\ b^{-1/2})$$

The result follows from (2.3.9) and (2.3.12) on choosing $\rho = 8+K_8$.

Lemma 2.3.4. Under the conditions of Theorem 2.3.1, there exists a constant $K_{10} > 0$ such that

$$(2.3.13) \qquad \limsup_{n \to \infty} \lambda_n^{-1} \sup_{0 \leq t \leq 1} |E_n(t) - t| \leq K_{10} \quad \text{a.s.}$$

where $\lambda_n = n^{-1/2} (\log \log n)^{1/2}$.

The proof is omitted, since it results by using Lemma 2.3.3 in the same way Lemma 2.2.4 does by using Lemma 2.2.3.

Lemma 2.3.5. Let $\{X_i\}$ be as in Theorem 2.3.1. Then there exists a ρ_o such that whenever $0 < \alpha < 1$, $0 < \varepsilon \leq 1-\alpha$, $-\varepsilon \leq \beta \leq \varepsilon$, $1 \leq u \leq M$, $\varepsilon \geq u^{-7/10}$, $H \geq 0$ and $\varepsilon M^{59/60} \leq Q^2 \leq \varepsilon^{3/2} M^{13/10}$, we have

$$(2.3.14) \qquad P\left(\left| \sum_{i=H+1}^{H+u} x_i(\alpha, \alpha+\beta) \right| > 2 \rho_o Q \right)$$

$$\leq K_{11} \exp(-8 Q^2 (M \varepsilon \log M)^{-1}) + K_{12} M^{-4} .$$

Proof. We take, without loss of generality, $H = 0$, $\varepsilon > 0$ and split the sum $\sum_{i=1}^{u} x_i(\alpha, \alpha+\beta)$ as in (2.2.6) taking $p = p(u) = [u^{7/10}]$. This time we take $z = Q(M\varepsilon \log M)^{-1}$ and define $\xi_i^* = \xi_i I(|\xi_i| \leq z^{-1})$. Here also, we shall show that

$$P\left(\sum_{j=1}^{k+1} \xi_i > \rho_o Q \right) \leq \text{the desired bound in (2.3.14)}$$

and conclude the lemma similarly. Once again we write

$$(2.3.15) \quad P\left(\sum_{i=1}^{k+1} \xi_i > \rho_o Q\right) \leq P\left(\sum_{i=1}^{k+1} \xi_i^*\right) + kP(|\xi_1| > z^{-1}) + P(\xi_{k+1} > z^{-1})$$

To estimate the last two terms in the r.h.s. of (2.3.15), we apply our Lemma (2.3.3) with $b = \varepsilon$, $N = [M^{7/10}]$ $(\geq p)$ and $D = (2\rho z)^{-1}$. The condition $D^2 \leq b\, N^{22/15}$ of Lemma 2.3.3 is met because of the restriction that $Q^2 \geq \varepsilon M^{59/60}$. Thus, we have using (2.3.8), that

$$k\, P(|\xi_1| > z^{-1}) + P(\xi_{k+1} > z^{-1})$$

$$\leq K_6(k+1) \exp(-8(2\rho z)^{-2} M^{-7/10} \varepsilon^{-1/2}) + O(M^{-4})$$

$$= O(M^{-4})$$

in view of the restriction that $Q^2 \leq \varepsilon^{3/2} M^{13/10}$. We estimate the probability $P\left(\sum_{i=1}^{k+1} \xi_i^* > \rho_o Q\right)$ as in the proof of Lemma 2.3.3. The only difference is that we use a sharper estimate for $E(\xi_1^2)$, namely,

$$E(\xi_1^2) = O(p\, \varepsilon \log M).$$

We obtain this as follows.

$$(2.3.15) \qquad E(\xi_1^2) = p\,V(x_1(\alpha,\ \alpha+\beta)) + p\sum_{i=1}^{p-1}(1-i/p)\mathrm{cov}(x_1(\alpha,\alpha+\beta),\ x_{1+i}(\alpha,\ \alpha+\beta))$$

$$\leq p\,\beta + p\Big(\sum_{i=1}^{2\gamma^{-1}\log M} + \sum_{i=[2\gamma^{-1}\log M]+1}^{p-1}\Big)\mathrm{cov}(x_1(\alpha,\alpha+\beta),x_{1+i}(\alpha,\alpha+\beta$$

(γ is defined in (2.3.1)).

We estimate the first term in the r.h.s. of (2.3.15) by Cauchy - Schwartz inequality and the second by (2.3.5). Thus we have

$$E(\xi_1^2) = O(p\,\beta) + O(p\,\beta\log M) + O(1)$$

$$= O(p\,\varepsilon\log M)$$

The proof of this lemma is complete.

It is worth noting that the only reason why it has not been possible to get the sharpest possible orders in the strong mixing case is the appearance of log M term in the estimate of $E(\xi_1^2)$.

<u>Proof of Theorems 2.3.1 and 2.3.2.</u> The proofs follow from the Lemmas 2.3.4 and 2.3.5 as in the previous section

2.4 QUANTILES FOR FUNCTIONS OF ϕ-MIXING PROCESSES

Let $\{Y_i : i \geq 1\}$ be a stationary sequence of r.v.s satisfying ϕ-mixing condition

(2.4.1) $$\phi(n) = O(n^{-3-\delta}) , \quad \delta > 0.$$

Let $\underset{\sim}{\mathbb{B}}_a^b$ denote σ-field generated by $Y_i (a \leq i \leq b)$. Define

$$U_n = f(Y_n, Y_{n-1}, \ldots) \qquad n \geq 1,$$

$$U_{nm} = E(U_n \mid \underset{\sim}{\mathbb{B}}_{n-m}^n)$$

and

$$E|X_n - X_{nm}| \leq \rho(m).$$

For technical reasons, we shall be assuming that

(2.4.2) $$\rho(m) = O(e^{-\lambda m}) , \qquad \lambda > 0.$$

With this dependence structure on the sequence $\{U_n\}$ we obtain the following theorems on quantiles.

<u>Theorem</u> 2.4.1. Let $\{U_i\}$ be a sequence of functions of ϕ-mixing r.v.s as described above satisfying (2.4.1) and (2.4.2) with $P(U_1 \leq t) = t$, $0 \leq t \leq 1$. Then, for all $0 < t < 1$,

$$|E_n^{-1}(t) - t + E_n(t) - t| = O(n^{-3/4}(\log n)^{1/2}(\log\log n)^{3/4}) \quad \text{a.s.}$$

<u>Theorem 2.4.2.</u> Under the conditions of Theorem 2.4.1,

$$\sup_{0\leq t\leq 1} |E_n^{-1}(t)-t+E_n(t)-t| = O(n^{-3/4}(\log n)(\log\log n)^{1/4}) \quad \text{a.s.}$$

<u>Proofs</u>. It is plain from the proofs in sections 2.2 and 2.3 that once we have the exponential inequality similar to that in Lemma 2.2.3 the representation results follow without using the dependence structure of the underlying process any more So, let us conclude the above stated theorems proving the following lemma.

<u>Lemma</u> 2.4.1. Let $\{U_i\}$ be as in Theorem 2.4.1. Then, there exists a constant $a > 0$ such that, whenever $0 \leq \alpha < 1$, $0 < b \leq 1-\alpha$, $-b \leq \beta \leq b$, $1 \leq u \leq N$, $b \geq N^{-1}$, $N^{1/8} \leq D \leq bN^{3/4+\delta_1}$ where $\delta_1 = \delta/4(4+\delta)$, we have

(2.4.3) $$P\left(\left| \sum_{i=H+1}^{H+u} x_i(\alpha, \alpha+\beta) \right| > 3aD \right)$$

$$\leq K_{11} \exp(-8D^2(Nb \log N)^{-1}) + K_{12} N^{-4}.$$

<u>Proof</u>. Once again we take $H = 0$ and $\beta > 0$. We define $q = [u^{(1/4)-\delta_1}]$ and

$$x_{ij}(\alpha, \alpha+\beta) = I(\alpha \leq U_{ij} \leq \alpha+\beta) - P(\alpha \leq U_{ij} \leq \alpha+\beta), \quad j \geq 1.$$

Clearly

(2.4.4) $$P\left(\left| \sum_{i=1}^{u} x_i(\alpha, \alpha+\beta) \right| > 3aD \right)$$

$$\leq P\left(\left| \sum_{i=1}^{u} x_{iq}(\alpha, \alpha+\beta) \right| > 2aD \right)$$

$$+ P\left(\left| \sum_{i=1}^{u} (x_{iq}(\alpha, \alpha+\beta) - x_i(\alpha, \alpha+\beta)) \right| > aD)\right)$$

To estimate the second term in the r.h.s. of (2.4.4), let us observe that for any $q \geq 1$

(2.4.5) $$E| x_{iq}(\alpha, \alpha+\beta) - x_i(\alpha, \alpha+\beta) |$$

$$\leq 2E| I(\alpha \leq x_i \leq \alpha+\beta) - I(\alpha \leq X_{iq} \leq \alpha+\beta) |$$

$$\leq 2 [P(A) + P(B)]$$

where

$$A = \{U_i \in (\alpha, \alpha+\beta], \; U_{iq} \notin (\alpha, \alpha+\beta]\}$$

$$B = \{U_i \notin (\alpha, \alpha+\beta], \; U_{iq} \in (\alpha, \alpha+\beta]\}$$

Now

$$P(A) \leq P(U_i \in (\alpha, \alpha+\rho^{1/2}(q)]) + P(U_i \in (\alpha+\beta-\rho^{1/2}(q), \alpha+\beta])$$

$$+ P(|U_i - U_{iq}| > \rho^{1/2}(q))$$

$$\leq 3\rho^{1/2}(q)$$

The last step follows from Chebychev inequality and the definition of $\rho(q)$.

Similarly one shows that $P(B) \leq 3\rho^{1/2}(q)$ and hence

(2.4.6) $\quad$ l.h.s. of (2.4.5) $\leq 12\rho^{1/2}(q)$.

This estimate along with Chebychev inequality, Minkowsky inequality, the condition $D \geq N^{1/8}$ and (2.4.2) yields that

$$P\left(\sum_{i=1}^{u} |x_i(\alpha, \alpha+\beta) - x_{iq}(\alpha, \alpha+\beta)| > aD\right) = O(N^{-4})$$

Therefore, if sufficies to show that

(2.4.7) $\quad P\left(\sum_{i=1}^{u} x_{iq}(\alpha, \alpha+\beta)| > 2aD\right) \leq K_{11} \exp(-8D^2(N\, b \log N)^{-1})$.

Towards this end, we split the sum $\sum_{i=1}^{u} x_{iq}(\alpha, \alpha+\beta)$ as in (2.2.6) taking $p = 2q$ and replacing $x_i(\alpha, \alpha+\beta)$ by

$x_{iq}(\alpha, \alpha+\beta)$ throughout. If we choose $y = D(N\ b \log N)^{-1}$, if follows from the condition $D \leq b\ N^{3/4+\delta_1}$ that $|y\xi_i| \leq 2$ so that we need not truncate ξ_i's unlike in the proofs of previous exponential inequalities.

Once again, by Markov inequality

$$P\left(\sum_{i=1}^{k+1} \xi_i > aD\right) \leq \exp(-y\ a\ D)\ E\left(\exp\left(y \sum_{1}^{k+1} \xi_i\right)\right)$$

Following the proof of (2.2.9), one has

$$E\left(\exp\left(y \sum_{1}^{k+1} \xi_i\right)\right) \leq 3\ h^k$$

where $h = E(\exp(y\ \xi_1)) + 2e\phi(p)$. The choice of p and the condition (2.4.1) imply that $k\phi(p) = O(1)$. Rest of the proof is similar to that of Lemma 2.2.3 except that we need to argue that

(2.4.8) $$k\ E(\xi_1^2) = O(N\ b \log N).$$

Making use of Minkowsky inequality and the estimate (2.4.6) it follows that

(2.4.9) $$\|x_{iq}(\alpha, \alpha+\beta)\|_2 \leq \|x_i(\alpha, \alpha+\beta)\|_2 + \|x_i(\alpha, \alpha+\beta) - x_{iq}(\alpha, \alpha+\beta)\|_2 \leq \beta^{1/2} + O(n^{-1}).$$

In the following statements, we express $x_{ij}(\alpha, \alpha+\beta)$ by x_{ij} and $x_i(\alpha, \alpha+\beta)$ by x_i for brievity.

$$E(\xi_1^2) = E(\sum_1^p x_{iq})^2 = pE\, x_{iq}^2 + 2p \sum_{i=2}^{p} (1 - \frac{i-1}{p})\, E\, x_{1q}\, x_{iq}$$

$$\leq p\beta + O(k^{-1}) + 2p \sum_{i=2}^{[4\lambda^{-1} \log N]} \|x_{1q}\|_2 \; \|x_{iq}\|_2$$

$$+ 2p \sum_{i=[4\lambda^{-1} \log N]+1}^{p} [|Ex_1x_i| + |E(x_1(x_i - x_{iq}) + x_{iq}(x_1 - x_{1q}))$$

$$= 2p \sum_{i=[4\lambda^{-1} \log N]+1}^{p} |E\, x_1x_i| + O(k^{-1}) + O(p\beta \log N)$$

$$= 2p \sum_{i=[4\lambda^{-1} \log N]+1}^{p} [\, |E\, x_1 x_{i\,[i/2]}|$$

$$+ E\, x_1(x_i - x_{i\,[i/2]})|] + O(p\beta \log N) + O(k^{-}$$

$$= 2p \sum_{i=[4\lambda^{-1} \log N]+1}^{p} \|x_1\|_2 \|x_{i\,[i/2]}\|_2 \phi^{1/2}([i/2$$

$$+ O(p\beta \log N) + O(k^{-1})$$

$$= O(p\beta \log N) + O(k^{-1})$$

Using the estimate (2.4.6) and the condition $b \geq N^{-1}$. This yields (2.4.8) completing the proof of this lemma.

Remark 2.4.1. A close examination of the proofs reveals that the conditions $\Sigma\ \phi^{1/2}(i) < \infty$ and $\rho(N) = O(N^{-16})$ suffice to show that $R_n = O(n^{(-1/2)-\varepsilon}), \varepsilon > 0$ a.s. which is enough for many statistical purposes.

An Example. If $\{ \xi_n ,\ n = 0, \pm 1, \pm 2 \ \ldots \}$ is a sequence of m-dependent process and

$$X_n = a_1\ \xi_n + a_2\ \xi_{n-1} + \ \ldots$$

where $\{a_i\}$ satisfies the condition $a_i \ll e^{-\lambda i}$ then the sequence $\{ X_n \}$ fulfils the requirements of the theorems of this section.

CHAPTER 3

REPRESENTATION OF QUANTILE PROCESSES WITH NON-UNIFORM BOUNDS

3.1 INTRODUCTION

Let $\{U_i : i \geq 1\}$ be a stationary sequence of r.v.s having marginals as $U[0, 1]$. As in the previous chapter, we define

$$(3.1.1) \qquad E_n(x) = (\# U_i \leq x, \ 1 \leq i \leq n)/n, \qquad 0 \leq x \leq 1,$$

and

$$(3.1.2) \qquad R_n(t) = E_n^{-1}(t) - t + E_n(t) - t$$

where $E_n^{-1}(t)$, $0 \leq t \leq 1$, is the quantile process. In the preceeding chapter we obtained a.s. asymptotic bounds for $|R_n(t)|$ which are uniform in $t \in [0, 1]$. It is natural to expect that the order of $|R_n(t)|$ would be sharper when t is near the extremes of the interval $[0, 1]$ than the uniform order obtained previously. This phenomenon was also noted in Kiefer (1970 a) and Kiefer (1970 b) but it was left there as an open problem. In the section 3 of this chapter, we present some representations with non uniform bounds.

This kind of representations appear to be helpful in studying the sample extremes, weighted quantile processes and linear functions of order statistics. These applications are presented in chapters 4 and 5.

In section 2, some stability results are proved for weighted empirical processes both in the independent and dependent cases. These results are needed in the proofs of section 3. The results of section 2 are also used in the proofs of asymptotic representations of L-statistics presented in chapter 5.

For simplicity, we use ℓ_n and $\ell\ell_n$ for $\log n$ and $\log\log n$, respectively, in this chapter.

3.2 SOME STABILITY RESULTS FOR WEIGHTED EMPIRICAL PROCESSES

With $E_n(t)$, as in (3.1.1), let

$$(3.2.1)\quad \begin{cases} V_n(t, \varepsilon) = [E_n(t)-t]/(t(1-t))^{\varepsilon}, \quad 0<t<1, \quad 0\leq\varepsilon\leq 1/2 \\ \text{and} \\ V_n(\varepsilon) = \sup_{0<t<1} |V_n(t, \varepsilon)| \end{cases}$$

We first establish the a.s. results for the asymptotic fluctuations of $V_n(\varepsilon)$ when $0 \leq \varepsilon < 1/2$.

Theorem 3.2.1. (i) If $\{U_i\}$ is a sequence of m-depedent r.v.s and $0 \leq \varepsilon < 1/2$, we have

(3.2.2) $$V_n(\varepsilon) \ll n^{-1/2} \, \ell\ell_n^{1/2} \quad \text{a.s.}$$

(ii) If $\{U_i\}$ is a sequence of ϕ-mixing r.v.s such that $\phi(n) = O(n^{-\gamma})$, $\gamma \geq 2$ (when $\gamma = 2$, under the additional condition that $\Sigma\, \phi^{1/2}(i) < \infty$) and

$$0 \leq \varepsilon < 1/2 - 1/2(\gamma + 1),$$

we have (3.2.2).

(iii) If $\{U_i\}$ is a sequence of strong mixing r.v.s such that $\alpha(n) \ll e^{-\lambda n}$, $\lambda > 0$ and $0 \leq \varepsilon < 1/4$, we have (3.2.2).

Since the proof proceeds along the same lines as in the last chapter, we omit the details. The proof hinges crucially on the following

Lemma 3.2.1. Irrespective of the dependence structure of the process $\{U_i\}$,

$$E_n(n^{-1}\ell_n^{-1}\ \ell\ell_n^{-3/2}) = 0 = 1 - E_n(1 - n^{-1}\ell_n^{-1}\ \ell\ell_n^{-3/2})$$

for all sufficiently large n, with probability one.

Proof. We shall prove only the first part of the lemma, since the second follows by symmetry. By Markov inequality,

$$(3.2.3) \qquad P(E_n(3\, n^{-1}\, \ell_n^{-1}\, \ell\ell_n^{-3/2}) > n^{-1}) \leq 3\, \ell_n^{-1}\, \ell\ell_n^{-3/2}.$$

Let $n(r) = 2^r$ and $U_n^{(1)} = \min\{U_i : 1 \leq i \leq n\}$. It follows from Borel-Cantelli lemma and (3.2.3) that $U_{n(r)}^{(1)} > 3(n(r)\, \ell_{n(r)}\, \ell\ell_{n(r)}^{3/2})^{-1}$ for all sufficiently large r with probability one. Further, for n such that $n(r) < n \leq n(r+1)$

$$U_n^{(1)} \geq U_{n(r+1)}^{(1)} \geq 3(n(r+1)\, \ell_{n(r+1)}\, \ell\ell_{n(r+1)}^{3/2})^{-1} \geq n^{-1}\, \ell_n^{-1}\, \ell\ell_n^{-3/2}$$

for all sufficiently large r with probability one. This proves the lemma.

Let us concentrate on the proof of (i) part of theorem 3.2.1. The following exponential inequality would be the main tool in the proof.

Define, for $0 \leq \alpha, \beta \leq 1$

$$x_i(\alpha, \beta) = I(\min(\alpha, \beta) \leq U_i \leq \max(\alpha, \beta)) - |\alpha - \beta|.$$

<u>Lemma 3.2.2.</u> Let $\{U_i\}$ be m-dependent r.v.s. Then, there exist $d > 0$ and $K_1 > 0$ such that whenever $0 \leq \alpha < 1$ $-\alpha \leq \beta \leq 1 - \alpha$, $|\beta| \leq b > 0$, $1 \leq u \leq N$, $H \geq 0$ and $0 \leq D \leq Nb^{\theta}$ for some $0 \leq \theta \leq 1/2$, one has

$$(3.2.4) \qquad P\left(\left| \sum_{i=H+1}^{H+u} x_i(\alpha, \alpha+\beta) \right| > 2dDb^{\theta} \right) \leq K_1 \exp(-8D^2N^{-1}).$$

<u>Proof.</u> Let us take, without loss of generality, $H = 0$, $b \geq \beta > 0$. We divide the sum $\sum_{i=1}^{u} x_i(\alpha, \alpha+\beta)$ as in (2.2.6) with $p = m$ so that the alternative blocks are independent. Now, by Markov inequality,

$$(3.2.5) \qquad P(Y(u) > d\,D\,b^{\theta}) \leq \exp(-z\,d\,D\,b^{\theta})\, E\left(\exp\left(z \sum_{1}^{k+1} \xi_i\right)\right)$$

where $z = b^{-\theta} N^{-1} D$.

Using the facts that $|z\xi_i| \leq m$ and $E(\xi_1^2) \leq K_2 p\beta$, we obtain

$$(3.2.6) \qquad \text{l.h.s of } (3.2.5) \leq 3\exp(-dD^2N^{-1})\,(E(\exp(z\,\xi_1))^k$$

$$\leq \exp(-dD^2N^{-1} + k\log(1 + O(z^2p\beta))$$

$$\leq K_3 \exp(-8\,D^2\,N^{-1})$$

by choosing d appropriately. Similarly, one obtains the inequalities for $-Y(u)$, $Y'(u)$ and $-Y'(u)$ to complete the proof of the lemma.

<u>Proof of Theorem 3.2.1 (i)</u> Let us fix $0 < \delta \leq 1/2$ and show that

(3.2.7) $$\sup_{0<t\leq 1/2} |E_n(t) - t|\, t^{-1/2+\delta} \ll n^{-1/2}\, \ell\ell_n^{1/2} \quad \text{a.s.}$$

which will then imply (by considering the r.v.s $\{1-U_i\}$ and noting that the proof works for the left continuous version of $E_n(t)$ also) that

$$\sup_{1/2\leq t<1} |E_n(t) - t|\, (1-t)^{-1/2+\delta} \ll n^{-1/2}\, \ell\ell_n^{1/2} \quad \text{a.s.}$$

and these two together will complete the proof of (3.2.2).

We write $a = 1/\delta$ in what follows. Let us divide the interval $(0, 1/2]$ into three parts as follows

$$(0, 1/2] = I_{n1} \cup I_{n2} \cup I_{n3}$$

where $I_{n1} = (0, n^{-1}\ell_n^{-2}]$, $I_{n2} = (n^{-1}\ell_n^{-2}, \ell_n^{-a}]$ and $I_{n3} = (\ell_n^{-a}, 1/2]$.

Lemma (3.2.1) implies trivially that

$$\sup \{ | V_n(t, 1/2-\delta) | : t \in I_{n1} \} \ll n^{-1/2}\, \ell\ell_n^{1/2} \quad \text{a.s.}$$

Therefore, it sufficies to show that

(3.2.8) $$\sup \{ |V_n(t, 1/2-\delta)| : t \in I_{n2} \} \ll n^{-1/2}\, \ell\ell_n^{1/2} \quad \text{a.s.}$$

and

(3.2.9) $$\sup \{ |V_n(t, 1/2-\delta)| : t \in I_{n3} \} \ll n^{-1/2}\, \ell\ell_n^{1/2} \quad \text{a.s.}$$

To prove (3.2.8), we divide the interval I_{n2} into subintervals of length n^{-3} and observe that

(3.2.10) $$\sup \{ |E_n(t)-t| \, t^{-1/2 + \delta/2} : t \varepsilon I_{n2} \}$$

$$\leq \max \{|E_n(s)-s| \, s^{-1/2 + \delta/2} : s = n^{-1} \ell_n^{-2}+n^{-3},$$

$$n^{-1} \ell_n^{-2}+2n^{-3}, \ldots \, n^{-1} \ell_n^{-2}+ [\ell_n^{-a} n^3+1] \, n^{-3} \} + n^{-1/2}$$

Fix some value of s as in the r.h.s. of (3.2.10). From Lemma 3.2.2 (with $D = (N\ell_n)^{1/2}$ $\theta = 1/2 - \delta/2$, $u = N = n$, $H = 0$ and $\beta = b = s$)

$$P(|E_n(s) - s| > 2d \, s^{1/2 - \delta/2} \, n^{-1/2} \, \ell_n^{1/2}) \ll n^{-8}.$$

From this, Bonferroni inequality and the Borel-Cantelli lemma, it follows that

$$\sup \{ V_n(t, \tfrac{1}{2} - \tfrac{\delta}{2}) : t \varepsilon I_{n2} \} \ll n^{-1/2} \, \ell_n^{1/2} \quad \text{a.s.}$$

Since $t^{\delta/2} \leq \ell_n^{-1/2}$ for all $t \varepsilon I_{n2}$, (3.2.8) follows.

Comming to (3.2.9), we divide the interval I_{n3} into subintervals of length n^{-1} and see that

(3.2.11) $$\sup_{t\varepsilon I_{n3}} |E_n(t)-t|\ t^{-1/2+\delta} \leq \{ \max |E_n(s)-s|/ s^{1/2-\delta} :$$

$$s = \ell_n^{-a},\ \ell_n^{-a}+n^{-1} + \ldots \ell_n^{-a}+([\tfrac{n}{2}]+1)n^{-1} \} + n^{-1/2}.$$

Now, we define three sequences of events as in Lemma 2.2.4. Let $n_r = \exp(\sqrt{r})$ (so that $\frac{n_r}{2}(r+1)^{-1/2} \leq n_{r+1}-n_r \leq n_r/r$), $A_r = \{n : n_r < n \leq n_{r+1}\}$. For $n \varepsilon A_r$, define

$$B(n) = \{ \max_{\substack{s=\ell_n^{-a}+j/r^{4a} \\ 1\leq j\leq [r^{4a}/2]+1}} | \sum_{i=n_r+1}^{n} x_i(0, s)|\ s^{-1/2+\delta} > dn_r^{1/2} \}$$

$$H(r) = \{ \max_{\substack{s=\ell_n^{-a}+j/r^{4a} \\ 1\leq j\leq [r^{4a}/2]+1}} | E_{n_r}(s) - s|\ s^{-1/2+\delta} > 2d\ n_r^{-1/2} \ell\ell_{n_r}^{1/2} \}$$

and

$$C(n) = \{ \max_{1\leq j\leq [n/2]+1} |V_n(z(j), \tfrac{1}{2}-\delta) - V_n(z'(j), \tfrac{1}{2}-\delta)| > 2dn_r^{-1/2} \}$$

where $z(j) = \ell_n^{-a} + j/n$ and $z'(j) = \ell_n^{-a} + [j\ r^{4a}/n]\ r^{-4a}$.

We show that all the three sequences of events occur only finitely often. This would imply that $\{$ r.h.s. of 3.2.11 $\geq 6d\ n^{-1/2} \ell\ell_n^{1/2} + n^{-1/2} \}$ occurs only finitely often a.s. which proves (3.2.9).

We estimate the probabilities of $B(n)$ and $H(r)$ using Bonferroni inequality and Lemma 3.2.2. In Lemma 3.2.2, we take $b = 1$, $N = n_{r+1} - n_r$, $H = n_r$, $D = n_r^{1/2}$, $\theta = 1/2 - \delta$ for $B(n)$ and $b = 1$, $N = n_r$, $H = 0$, $D = n_r^{1/2} \ll n_{n_r}^{1/2}$ and $\theta = 1/2 - \delta$ for $H(r)$ to conclude that $P(B(n)) = O(n^{-2})$ and $P(H(r)) = O(r^{-2})$. Finally, coming to $C(n)$, we note that for $1 \leq j \leq [n/2] + 1$,

$$|V_n(z(j), \tfrac{1}{2} - \delta) - V_n(z'(j), \tfrac{1}{2} - \delta)|$$

$$\leq 2|E_n(z(j)) - z(j) - E_n(z'(j)) + z'(j)|/(z'(j))^{1/2 - \delta}$$

$$+ 2|E_n(z(j)) - z(j)| \left[(z(j))^{-1/2 + \delta} - (z'(j))^{-1/2 + \delta}\right]$$

$$\leq K_4 \, |E_n(z(j)) - z(j) - E_n(z'(j)) + z'(j)| \,/\, r^{-2a+1}$$

$$+ K_5 \, |E_n(z(j)) - z(j)| \,/\, r$$

where the last ineqality follows from the facts that

$$|z(j) - z'(j)| \leq r^{-4a}, \quad |(z(j))^{(-1/2)+\delta} - (z'(j))^{(-1/2)+\delta}| \leq K_6 \, r^{-2a},$$

$z(j) \geq r^{-a}$, $z'(j) \geq r^{-a}$ and $a = 1/r \geq 2$.

In view of these inequalities,

$$P(\,C(n)) \leq n \max_{1 \leq j \leq [n/2]+1} \left[P(|E_n(z(j)) - z(j) - E_n(z'(j)) + z'(j)|\right.$$

$$\geq dK_4^{-1} \, r^{-2a} \, n_r^{-1/2} \, r + P(|E_n(z(j)) - z(j)| > dK_5^{-1} \, n_r^{-1/2} \, r)$$

The first term in the brackets can be shown to be $O(n^{-3})$ by taking $b = r^{-4a}$, $N = n$, $H = 0$, $D = K_4^{-1} n_r^{1/2} \cdot r/2$ and $\theta = 1/2$ in Lemma 3.2.2 and the second term can be shown to be $O(n^{-3})$ by taking $b = z(j)$, $N = n$, $H = 0$, $D = K_5^{-1} n_r^{1/2} r/2$ and $\theta = 0$.

Now, the stated assertion follows from the Borel Cantelli lemma and Bonferroni inequality.

Proof of Theorem 3.2.1 (ii). The proof is similar to the above proof. The probability inequality to be used is stated below.

Lemma 3.2.3. Under the conditions of Theorem 3.2.1.(ii), there exists a constant $d > 0$ such that whenever $0 \leq \alpha \leq 1$, $-\alpha < \beta \leq 1 - \alpha$, $|\beta| \leq b > 0$, $1 \leq u \leq N$, $H \geq 0$, $0 \leq \theta \leq 1/2$ and $0 < D \leq b^{\theta} N^{1-1/2(\gamma+1)-\bar{\gamma}}$ for some $\bar{\gamma} > 0$ (fixed), then

$$(3.2.11) \qquad P\left(\left|\sum_{i=H+1}^{H+u} x_i(\alpha, \alpha+\beta)\right| > 2d\, b^{\theta} D\right) \leq K_7 N^{-4} + K_8 \exp(-8D^2 N^{-1})$$

This lemma is proved by imitating the proof of Lemma 2.2.3 and by choosing $p = u^{1/(\gamma+1)}$. The condition $\varepsilon < 1/2 - 1/2(\gamma+1)$ comes in when considering the interval I_{2n} (one starts with $\varepsilon = \frac{1}{2} - \frac{1}{2(\gamma+1)} - \delta$).

<u>Proof of Theorem 3.2.1 (iii)</u>. Once again, the proof follows by imitating the proof of part (i) using the following exponential bound.

<u>Lemma 3.2.4</u>. Under the conditions of Theorem 3.2.1 (iii), there exists a constant $d > 0$ such that whenever $0 \leq \alpha \leq 1$, $-\alpha < \beta < 1 - \alpha$, $|\beta| \leq b > 0$, $1 \leq u \leq N$, $H \geq 0$, $0 \leq \theta < 1/4$ and $0 < D < b^{\theta} N^{3/4 - \gamma}$, $\gamma > 0$ (fixed), then (3.2.11) holds.

This lemma is proved by imitating the proof of Lemma 2.3.2.

Some more results on the stability of weighted empirical processes are presented in

<u>Theorem 3.2.2</u>. (i) If $\{U_n\}$ is a sequence of m-dependent $U[0, 1]$ r.v.s, then for any $\gamma > 0$,

(3.2.12) $$V_n(1/2) \ll n^{-1/2} \ell_n^{1/2 + \gamma} \quad \text{a.s.}$$

(ii) If $\{U_n\}$ is a sequence of ϕ-mixing r.v.'s with $\phi(n) = O(e^{-\varrho n})$, $\varrho > 0$, then for any $\gamma > 0$,

$$V_n(1/2) \ll n^{-1/2} \ell_n^{3/2 + \gamma} \quad \text{a.s.}$$

To prove (i) part, we use

<u>Lemma 3.2.5</u>. If $\{ U_n \}$ is a sequence of m-dependent $U[0, 1]$ r.v.s and c_n is a sequence of positive constants such that $c_n \downarrow 0$, nc_n is non-decreasing in n after certain n onwards and $n\, c_n \geq 1$, then

$$E_n(c_n) \leq c_n\, \ell\ell_n \quad \text{a.s.} \tag{3.2.13}$$

<u>Proof</u>. Define $n(r) = 2^r$. With $H = 0$, $N = n(r)$, $\beta = c_{n(r)}$, $b = c_{n(r)} \ell\ell_{n(r)}$, $\theta = 1/2$ and $D = n(r)\, c_{n(r)}^{1/2}\, \ell\ell_{n(r)}^{1/2}$ in Lemma 3.2.2, we get

$$P(\, n(r+1)\, E_n(c_{n(r)}) > (4d+2)\, n(r)\, c_{n(r)}\, \ell\ell_{n(r)} \,)$$

$$\leq P(\, n_r\, E_n(c_{n(r)}) - n_r\, c_{n(r)} > 2d\, n(r)\, c_{n(r)}\, \ell\ell_{n(r)} \,)$$

$$= O(r^{-2}).$$

so that $n(r+1)\, E_n(c_{n(r)})$ exceeds $(4d+2)\, n(r)\, c_{n(r)}\, \ell\ell_{n(r)}$ only finitely often a.s. Now (3.2.13) follows, since

$$nE_n(c_n) \leq n(r+1)\, E_n(c_{n(r)}) \leq (4d+2) n(r) c_{n(r)} \ell\ell_{n(r)}$$

$$\leq (4d+2) n\, c_n\, \ell\ell_n$$

for all $n(r) < n \leq n(r+1)$ and r sufficiently large a.s.

Proof of Theorem 3.2.2. (i) . We will obtain (3.2.12) with $\gamma = 1/2$ and indicate at the end how the same proof works for any $\gamma > 0$. As remarked in the previous theorem, it is enough to show that

$$\sup_{0 < t \leq 1/2} |E_n(t) - t|\, t^{-1/2} \ll n^{-1/2} \ell_n \quad \text{a.s.}$$

Once again the interval $(0, 1/2]$ is divided into three parts :

$$(0, 1/2] = (0, n^{-1}\ell_n^{-1}\, \ell\ell_n^{-3/2}] \cup (n^{-1}\ell_n^{-1}\, \ell\ell_n^{-3/2}, n^{-1}\ell_n] \cup (n^{-1}\ell_n, \tfrac{1}{2}]$$

That supremum of $|E_n(t) - t|\, t^{-1/2}$ over the first interval is of the desired order follows from Lemma 3.2.1 trivially. As for the interval $(n^{-1}\ell_n, 1/2]$, we divide it into subintervals of length n^{-1} and note that

$$(3.2.14) \qquad \sup_{n^{-1}\ell_n \leq t \leq 1/2} |E_n(t) - t|\, t^{-1/2}$$

$$\leq \max \left\{ |E_n(s) - s|\, s^{-1/2} : s = n^{-1}\ell_n,\ n^{-1}\ell_n + n^{-1}, \ldots n^{-1}\ell_n + \left[\tfrac{n}{2} + 1\right] n^{-1} \right\} + n^{-1/2}$$

Then, as before, we apply the exponential inequality of Lemma 3.2.2, Bonferroni inequality and Borel-Cantelli lemma to get that

$$\sup \{ |E_n(t)-t| : t \in (n^{-1}\ell_n, 1/2] \} \ll n^{-1/2} \ell_n^{1/2} \quad \text{a.s.}$$

Finally coming to the second interval, we break it further into three parts $(n^{-1} \ell_n^{-1} \ell\ell_n^{-3/2}, n^{-1}]$, $(n^{-1}, n^{-1} \ell_n^{1/2}]$, $(n^{-1} \ell_n^{1/2}, n^{-1}\ell_n]$ and use the Lemma 3.2.5. We consider only the interval $(n^{-1} \ell_n^{1/2}, n^{-1}\ell_n]$. The other two can be handled similarly.

$$\sup_{n^{-1}\ell^{1/2} \leq t \leq n^{-1}\ell_n} |E_n(t)-t| t^{-1/2} \leq n^{1/2}\ell_n^{-1/4} \sup_{n^{-1}\ell_n^{1/2} \leq t \leq n^{-1}\ell_n} |E_n(t)-t|$$

$$\leq n^{1/2} \ell_n^{-1/4} [E_n(n^{-1}\ell_n) + n^{-1}\ell_n] \ll n^{-1/2} \ell_n^{3/4} \ell\ell_n$$

in view of Lemma 3.2.5.

For general $0 < \gamma \leq 1$, the interval $(n^{-1}, n^{-1}\ell_n]$ has to be divided into the subintervals $(n^{-1}, \ell_n^{\gamma}]$, $(n^{-1}\ell_n^{\gamma}, n^{-1}\ell_n^{2\gamma}]$, $\ldots$ $(n^{-1} \ell_n^{[1/\gamma]\gamma}, n^{-1}\ell_n]$.

<u>Proof of Theorem 3.2.2. (ii)</u>. Once again, the proof is similar to that of the (i) part. Essential tools and the changes are mentioned below.

Lemma 3.2.6. Let $\{U_i\}$ be a sequence of ϕ-mixing $U[0, 1]$ r.v.s with $\phi(i) \ll e^{-\varrho i}$, $\varrho > 0$. Then, there exists a $d > 0$ such that whenever $0 \leq \alpha \leq 1$, $-\alpha < \beta \leq 1 - \alpha$, $|\beta| \leq b > 0$, $1 \leq u \leq N$ and $0 < D \leq N\, b^{1/2} (\log N)^{-1}$,

we have

$$P\left(\left| \sum_{i=1}^{u} x_i(\alpha, \alpha+\beta) \right| > 2d\, b^{1/2} D \right) \leq K_9 \exp(-8\, D^2 N^{-1}).$$

This lemma is proved following the proof of Lemma 2.2.3 with $p = \varrho^{-1} \log u$. One does not need to truncate the r.v.'s ξ_i in the proof of this lemma.

Lemma 3.2.7. Let $\{U_n\}$ be same as in the previous lemma. If c_n satisfies the conditions of Lemma 3.2.5, then

$$E_n(c_n) \ll c_n \ell_n \ell\ell_n \qquad \text{a.s.}$$

The proof is similar to Lemma 3.2.5. One uses the probability bound given by Lemma 3.2.6.

In the main arguments of the proof, we divide the interval $(0, 1/2]$ into three parts $(0, n^{-1} \ell_n^{-1} \ell\ell_n^{-3/2}]$, $(n^{-1}\ell_n^{-1} \ell\ell_n^{-3/2}, n^{-1}\ell_n$ and $(n^{-1} \ell_n, 1/2]$. We divide the internal $(n^{-1} \ell_n, 1/2]$ as in the proof of the (i) part and note the inequality (3.2.14). For any value of s as in the r.h.s. of (3.2..4)

$$P(n|E_n(s)-s| > 2d\, n^{1/2} s^{1/2} \ell_n^{3/2}) = O(n^{-3})$$

by using the Lemma 3.2.6 with $N = n$, $\beta = s$, $b = s\ell_n^2$, $D = n^{1/2} \ell_n^{1/2}$ (so that the condition $D \leq N\, b^{1/2} (\log N)^{-1}$ is met). For the rest of the proof, we mimic the arguments of the (i) part,

Remark 3.2.1. The proof given for Theorem 3.2.2 (i) also shows that under the same condition

$$(3.2.15) \qquad \sup_{n^{-1} \log n \leq t \leq 1-n^{-1} \log n} |E_n(t)-t| \, (t(1-t))^{-\frac{1}{2}} \ll n^{-\frac{1}{2}} \ell_n^{\frac{1}{2}}$$

We remark in passing that not all the results established in this section will be used in the next section. Some of these results are to be used in the subsequent chapters.

3.3 REPRESENTATION OF QUANTILE PROCESSES WITH NON-UNIFORM BOUNDS

As mentioned in the introduction to this chapter, we present here some a.s. asymptotic representations of quantile processes which show the behaviour of the remainder near the boundaries of the interval [0,1]. The result for m-dependent r.v.s is stated below.

<u>Theorem 3.3.1.</u> If $\{U_i\}$ be a sequence of m-dependent r.v's, then

$$(3.3.1) \qquad \sup_{\frac{1}{n} \leq t \leq \frac{n-1}{n}} [t(1-t)]^{-1/4} |E_n^{-1}(t) - t + E_n(t) - t| \ll n^{-\frac{3}{4}} \log n \quad \text{a.s.}$$

<u>Remarks 3.3.1.</u> It may be mentioned that this kind of representation is not possible in the intervals $(0, n^{-1}]$ and $[\frac{n-1}{n}, 1)$ since $E_n^{-1}(s) = E_n^{-1}(1/n)$ for $0 \leq s \leq n^{-1}$ and $E_n^{-1}(s) = E_n^{-1}(1-n^{-1})$ for $1-n^{-1} \leq s \leq 1$ and it is known that in the independent case

$$\limsup_{n \to \infty} \frac{nE_n^{-1}(n^{-1})}{\log \log n} = \limsup_{n \to \infty} \frac{n(1-E_n^{-1}(1-n^{-1}))}{\log \log n} = 1 \quad \text{a.s.}$$

(see Robbins and Siegmund (1970).

We start with a lemma which gives us the estimates of fluctuations of weighted quantile processes.

<u>Lemma 3.3.1.</u> If $\{U_i\}$ is a sequence of m-dependent. U $[0, 1]$ r.v's then

$$(3.3.2) \quad \limsup_{n \to \infty} \sup_{n^{-1}\ell_n \leq t \leq 1 - n^{-1}\ell_n} n^{1/2} \ell_n^{-1/2} \frac{|E_n^{-1}(t) - t|}{(t(1-t))^{1/2}} < K \quad \text{a.s.}$$

<u>Proof</u>. Let us prove that

$$\text{(3.3.3)} \qquad \sup\{ |E_n^{-1}(t) - t| t^{-1/2} : n^{-1} \log n \le t \le 1/2 \}$$

$$\ll \quad n^{-1/2} \ell_n^{1/2} \qquad \text{a.s.}$$

Then, a similar proof will give

$$\sup \{(1-t)^{-1/2} |E_n^{-1}(t) - t| : 1/2 \le t \le 1 - n^{-1} \log n\}$$

$$\ll \quad n^{-1/2} \ell_n^{1/2} \qquad \text{a.s.}$$

and (3.3.2) will follow at once from these two.

Remark 3.2.1. guarantees the existence of a constant $c > 0$ such that

$$\text{(3.3.4)} \qquad \limsup_{n \to \infty} \sup_{n^{-1}\ell_n \le t \le 1/2} n^{1/2} \ell_n^{-1/2} |E_n(t) - t| t^{-1/2} < c \qquad \text{a.s.}$$

Therefore, for all sufficiently large n and $s \in [n^{-1} \ell_n, 1/2]$,

$$s - cs^{1/2} n^{-1/2} \ell_n^{1/2} < E_n(s) < s + cs^{1/2} n^{-1/2} \ell_n^{1/2} \qquad \text{a.s.}$$

(the null set is same for all s). Hence, for $s \in [\underset{\sim}{c}^2 n^{-1} \ell_n, 1/2]$ and n sufficiently large, when $\underset{\sim}{c} = c+1$,

$$(3.3.5)\qquad E_n(s-c\, s^{1/2}\, n^{-1/2}\, \ell_n^{1/2}) < s - c\, s^{1/2}\, n^{-1/2}\, \ell_n^{1/2} + c(s-c\, s^{1/2}\, n^{-1/2}\, \ell_n^{1/2})\, n^{-1/2}\, \ell_n^{1/2} < s$$

and

$$(3.3.6)\qquad E_n(s + 2c\, s^{1/2}\, n^{-1/2}\, \ell_n^{1/2}) > s+2\, c\, s^{1/2}\, n^{-1/2}\, \ell_n^{1/2} - c(s + 2c\, n^{-1/2}\, \ell_n^{1/2})^{1/2}\, n^{-1/2}\, \ell_n^{1/2} > s.$$

with probability one (null set being independent of s). Obviously, (3.3.5) and (3.3.6) imply that

$$(3.3.7)\qquad \sup_{\underset{\sim}{c}^2 n^{-1}\ell_n \leq t \leq 1/2} |E_n^{-1}(t)-t|\, t^{-1/2} \ll n^{-1/2}\ell_n^{1/2} \quad \text{a.s.}$$

Furthermore

$$\sup_{n^{-1}\ell_n \leq t \leq \underset{\sim}{c}^2 n^{-1}\ell_n} |E_n^{-1}(t)-t|\, t^{-1/2} \leq \frac{E_n^{-1}(\underset{\sim}{c}^2 n^{-1}\ell_n) + \underset{\sim}{c}^2 n^{-1}\ell_n}{(n^{-1}\ell_n)^{1/2}} \ll n^{-1/2}\, \ell_n^{1/2}$$

since (3.3.7) implies that $E_n^{-1}(\underset{\sim}{c}^2 n^{-1}\ell_n) \ll n^{-1}\ell_n$ a.s. Thus (3.3.3) holds. This completes the proof.

Proof of Theorem 3.3.1. Let us observe that

$$(3.3.8) \quad \sup_{n^{-1} \le t \le n^{-1}\ell_n} |E_n^{-1}(t)-t+E_n(t)-t| \, t^{-1/4}$$

$$\le \sup_{n^{-1} \le t \le n^{-1}\ell_n} [t^{-1/4}(E_n^{-1}(t)+E_n(t)) + 2t^{3/4}]$$

$$\le n^{1/4} [E_n^{-1}(n^{-1}\ell_n) + E_n(n^{-1}\ell_n)] + 2\, n^{-3/4} \, \ell_n^{3/4}$$

$$\ll n^{-3/4} \log n \quad \text{a.s.} \quad \text{(using Remark 3.2.1 and Lemma 3.3.1)}$$

Next, following the first step of the proof of the Theorem 2.2.1 and using (3.3.2) we get

$$(3.3.9) \quad \sup_{n^{-1}\ell_n \le t \le 1/2} |E_n^{-1}(t) - t + E_n(t) - t| \, t^{-1/4}$$

$$\le 3 \sup_{n^{-1}\ell_n \le t \le 1/2} \; \sup_{|s-t| \le 2Kt^{1/2}n^{-1/2}\ell_n^{1/2}} |E_n(s)-E(t)-s+t| \, t^{-1/4}$$

(K is the same constant appearing in (3.3.2)).

Fix a $t \in [n^{-1}\ell_n, 1/2]$ and divide the interval

$[t - 2K\, t^{1/2}\, n^{-1/2}\, \ell_n^{1/2},\ t + 2K\, t^{1/2}\, n^{-1/2}\, \ell_n^{1/2}]$ into sub-intervals of length n^{-3}. Then, by the usual kind of approximations,

$$\sup_{|s-t| \leq 2K\, t^{1/2} n^{-1/2} \ell_n^{1/2}} |E_n(s) - E_n(t) - s + t|\ t^{-1/4}$$

$$\leq \max_{|j| \leq v_n(t)} |E_n(t + j\, n^{-3}) - E_n(t) - jn^{-3}|\ t^{-1/4} + n^{-1}$$

where $v_n(t) = [2K\, t^{1/2}\, n^{5/2}\, \ell_n^{1/2}] + 1$.

Further, let us divide the interval $[n^{-1}\ell_n,\ 1/2]$ into sub intervals of length n^{-3} and note that if $s \in [n^{-1}\ell_n + in^{-3},\ n^{-1}\ell_n + (i+1)n^{-3}]$, then

$$\max_{|j| \leq v_n(s)} |E_n(s + j\, n^{-3}) - E_n(s) - j\, n^{-3}|\ s^{-1/4}$$

$$\leq \max_{|j| \leq v_n(\underline{a})+1} \underline{a}^{-1/4} |E_n(\underline{a} + j\, n^{-3}) - E_n(\underline{a}) - j\, n^{-3}| + n^{-1}$$

where $\underline{a} = n^{-1}\ell_n + (i+1)n^{-3}$. In view of this, (3.3.10) $\quad$ l.h.s. of (3.3.9) $\leq 2n^{-\frac{1}{4}} \max_{r \leq [n^3/2]} \max_{|j| \leq v_n(n^{-1}\ell_n + rn^{-3})+1}$

$$(n^{-1}\ell_n + rn^{-3})^{-\frac{1}{4}} |E_n(n^{-1}\ell_n + (r+j)n^{-3}) - E_n(n^{-1}\ell_n + rn^{-3}) - jn^{-3}|.$$

Let us fix some r and j (j depending upon r) as in the r.h.s. of (3.3.10). We shall estimate

(3.3.11) $\quad P(n|E_n(n^{-1}\ell_n + (r+j)n^{-3}) - E_n(n^{-1}\ell_n + rn^{-3}) - jn^{-3}|$

$$\geq 2d\sqrt{3K}\,(n^{-1}\ell_n + r\,n^{-3})^{1/4}\, n^{1/4}\, \ell_n^{3/4})$$

Using Lemma 3.2.2. We take $\beta = j\,n^{-3}$, $u = N = n$, $H = 0$, $b = 3\,K(n^{-1}\ell_n + r\,n^{-3})^{1/2}\, n^{-1/2}\, \ell_n^{1/2}$, $\theta = 1/2$, and $D = n^{1/2}\ell_n^{1/2}$ in Lemma 3.2.2 for the present case. We have to check the two conditions, namely, $|\beta| \leq b$ and $D \leq N\,b^{1/2}$. Now,

$$|\beta| \leq (v_n(n^{-1}\ell_n + r\,n^{-3}) + 1)\, n^{-3}$$

$$\leq 2\,K(n^{-1}\ell_n + r\,n^{-3})^{1/2}\, n^{-1/2}\, \ell_n^{1/2} + 2\,n^{-3}$$

$$\leq b \qquad \text{(for all large } n).$$

The senond condition is immediate assuming, without loss of generality, that $K > 1$. Hence, Lemma (3.2.2) gives that the l.h.s. of (3.3.11) $= O(n^{-8})$.

Now, a simple application of Bonferroni inequality and Borel - Cantelli lemma shows that

(3.3.12) $\quad$ l.h.s. of (3.3.9) $\ll n^{-3/4} \ell_n^{3/4}$ a.s.

Combining (3.3.8) and (3.3.9), we have

$$\sup_{n^{-1} \leq t \leq 1/2} |E_n^{-1}(t) - t + E_n(t) - t| t^{-1/4} \ll n^{-1} \ell_n \quad \text{a.s.}$$

Similarly, one shows that

$$\sup_{1/2 \leq t \leq 1-n^{-1}} |E_n^{-1}(t) - t + E_n(t) - t| (1-t)^{-1/4} \ll n^{-1} \ell_n \quad \text{a.s.}$$

completing the proof of the theorem.

We now state below the corresponding results for mixing r.v.s without proofs because the proofs are quite similar using Theorem (3.2.1). (ii), (iii) and Theorem 3.2.2 (ii).

<u>Theorem 3.3.2 (i)</u>. If $\{U_i\}$ is a sequence of ϕ-mixing $U[0, 1]$ r.v.s with $\phi(n) \ll n^{-\gamma}$ (if $\gamma = 2$, we further assume that $\Sigma\, \phi^{1/2}(i) < \infty$) and $0 \leq 2\varepsilon < \frac{1}{2} - \frac{1}{2(\gamma+1)}$, then

$$\sup_{n^{-1} \leq t \leq 1-n^{-1}} (t(1-t))^{-\varepsilon} |E_n^{-1}(t) - t + E_n(t) - t| \ll n^{-3/4} \ell_n^{3/4} \quad \text{a.s}$$

(ii) If $\{U_i\}$ is a sequence of ϕ-mixing $U[0, 1]$ r.v.s $\phi(n) \ll e^{-\gamma n}$, $\gamma > 0$, then

$$\sup_{n^{-1} \leq t \leq 1-n^{-1}} (t(1-t))^{-1/4} |E_n^{-1}(t)-t+E_n(t)-t| \ll n^{-3/4} \ell_n^2 \quad \text{a.s.}$$

(iii) If $\{U_i\}$ is a sequence of strong-mixing r.v.s with $\alpha(n) \ll e^{-\gamma n}$, $\gamma > 0$, then for all $0 < 2\varepsilon < 1/4$

$$\sup_{n^{-1} \leq t \leq 1-n^{-1}} (t(1-t))^{-\varepsilon} |E_n^{-1}(t)-t+E_n(t)-t| \ll n^{-3/4} \ell_n^{3/4} \quad \text{a.s.}$$

<u>Remark 3.3.2</u>. (3.3.12) states that, in the m-dependent case,

$$\sup_{n^{-1}\ell_n \leq t \leq 1/2} t^{-1/4} |E_n^{-1}(t)-t+E_n(t)-t| \ll n^{-3/4} \ell_n^{3/4} \quad \text{a.s.}$$

This implies that

$$\frac{\sqrt{n}\,(E_n^{-1}(t_n)-t_n)}{(t_n(1-t_n))^{1/2}} = -\frac{\sqrt{n}\,(E_n(t_n)-t_n)}{(t_n(1-t_n))^{1/2}} + o(1) \quad \text{a.s.}$$

whenever $t_n \to 0$ and $t_n n \ell_n^{-3} \to \infty$. This representation trivially gives the asymptotic normality of the ℓ.h.s. of 3.3.12 if t_n satisfies the conditions as mentioned above. In general, for any sequence of t_n such that

$\limsup_{n \to \infty} t_n - 1 + n^{-1}(\log n)^3 \leq 0$ and $\liminf_{n \to \infty} t_n - n^{-1}(\log n)^3 \geq 0$,

one has

$$\frac{\sqrt{n}\,(E_n^{-1}(t_n)-t_n)}{(t_n(1-t_n))^{1/2}} = -\frac{\sqrt{n}\,(E_n(t_n)-t_n)}{(t_n(1-t_n))^{1/2}} + O((t_n(1-t_n))^{-\frac{1}{4}}\, n^{-\frac{1}{4}}\, \ell_n^{\frac{3}{4}}) \quad \text{a.s.}$$

CHAPTER 4

REPRESENTATION OF QUANTILES FOR MIXING PROCESS
GENERAL CASE

4.1 INTRODUCTION

Our main aim in this chapter is to extend the results of the previous chapters regarding the representation of quantiles for uniform distributions to general distributions satisfying certain smoothness conditions. Most of the results available in this direction assume the knowledge of second derivative of the underlying d.f. We shall refer to such d.f. s as regular cases. In this chapter, we consider some more general d.f. s, (to be referred to as non-regular cases) which in particular include cases where only right derivatives exist, only left derivatives exist or both the left and right derivatives exist, but are unequal.

The main technique of this chapter consists in using the known results for uniform distributions by means of a suitable transformation. To be formal, let $\{X_n\}$ be a stationary sequence of r.v.s having marginals F (by definition F is right continuous). Let $F_n(x)$, $-\infty < x < \infty$ be the e.d.f. at the n^{th} stage and $F_n^{-1}(t)$, $0 \leq t \leq 1$, be the corresponding quantile process. We define $U_i = F(X_i)$. $E_n(t)$ and $E_n^{-1}(t)$ denote the e.d.f. and the quantile process respectively for U_i s.

If the distribution of U_i s is uniform, which is true if and only if F is continuous, there is not much difficulty in transforming the representation results known for $E_n^{-1}(t)$ to $F_n^{-1}(t)$ even in the non-regular cases to be considered in the section 3 of this chapter. But to get an asymptotic result for $F_n^{-1}(t)$, it appears superflous to assume that F is continuou throughout. Lemma 1 of Kiefer (1967) suggests a method for the independent case by which one needs to assume continuity of F only in a neighbourhood of $F^{-1}(t)$. However, it appears that the technique cannot be easily extended to the dependent situations also. We employ here an alternative method which uses local uniformity. The main idea is given by the following.

Lemma 4.1.1. Let X be a r.v. with d.f. F and $U = F(X)$. If F is continuous in an interval (a,b), then

(4.1.1) $$P(U \leq u) = u \quad \text{for all } u \in (F(a), F(b)).$$

In section 4, we obtain the strong approximation of quantile processes as mentioned in Chapter 1.

Throughout this chapter, we confine to only ϕ - mixing r.v. s, although, the same proof works for other weak-dependence structures considered in the previous chapters.

4.2 REPRESENTATION OF QUANTILES IN REGULAR CASES

For some $t \epsilon (0,1)$, let us define t^{th} quantile of d.f. F as

(4.2.1) $$Q_t = \inf \{x : F(x) \geq t\} .$$

Our first theorem provides representation of quantiles for fixed $t \epsilon (0,1)$.

Theorem 4.2.1. Let $\{X_i\}$ be a stationary sequence of ϕ-mixing r.v. s with, $\Sigma \phi^{1/2}(i) < \infty$ and X_1 having d.f. F. For some $t \epsilon (0,1)$, let us assume that F is twice differentiable in a neighbourhood of Q_t with first derivative bounded away from zero and the second bounded. Let Q_{nt} denote the t^{th} sample quantile. We then have :

(i) If t_n is a sequence of positive numbers from (0,1) such that $t_n \longrightarrow t$ as $n \longrightarrow \infty$, then

(4.2.2) $$Q_{nt_n} - Q_{t_n} = \frac{t - F_n(Q_t)}{F'(Q_t)} + R_n(F, t_n)$$

where

(4.2.3) $$|R_n(F,t_n)| = 0_p(n^{-3/4}(\log n)^{1/2} (\log \log n)^{1/4} + n^{-1/2} |t_n - t|^{1/2})$$

and

$$(4.2.4) \qquad |R_n(F,t_n)| = O(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4} + n^{-1/2}(t(n))^{1/2}(\log\log n))^{1/2}$$

a.s. with $t(n) = \sup_{i \geq n} \{|t_i - t|\}$

(ii) If $|t_n - t| = O(n^{-1/2})$ as $n \longrightarrow \infty$, then

$$(4.2.5) \qquad |Q_{nt_n} - Q_t - [t - F_n(Q_t)]/F'(Q_t)|$$

$$= O(n^{-3/4}(\log\log n)^{3/4} + |t_n - t|) \text{ a.s.}$$

We shall derive the theorem from

Lemma 4.2.1. Let $\{X_i\}$ be same as in the above theorem. Define $U_i = F(X_i)$. Let $E_n(t)$ and $E_n^{-1}(t)$ denote e.d.f. and the quantile process respectively for $\{U_i\}$. Then we have :

(i) If $t_n \longrightarrow t$ as $n \longrightarrow \infty$, then

$$(4.2.6) \qquad E_n^{-1}(t_n) - t_n + E_n(t) - t = R_n(t_n)$$

such that asymptotic behaviours of $|R_n(t_n)|$ is given by the r.h.s.s of (4.2.3) and (4.2.4).

(ii) If $|t_n - t| = O(n^{-1/2})$ as $n \longrightarrow \infty$, then

(4.2.7) $$|E_n^{-1}(t_n) - t + E_n(t) - t|$$
$$= 0(n^{-3/4}(\log\log n)^{3/4} + |t_n - t|) \quad \text{a.s.}$$

<u>Proof of Lemma 4.2.1 (i)</u>. The regularity conditions assumed on F imply that there exists a $\delta > 0$ such that

$$P(U_1 \leq s) = s \quad \text{if} \quad s \in [t - 2\delta, t + 2\delta].$$

In view of Remark 2.2.1, all the proofs of section 2.2 hold if we confine ourselves to the interval $[t - \delta, t + \delta]$. Consequently,

$$\sup_{s \in [t-\delta, t+\delta]} |E_n^{-1}(s) - s + E_n(s) - s|$$
$$= 0(n^{-3/4}(\log n)^{1/2} (\log\log n)^{1/4}) \text{ a.s.}$$

In particular, if $t_n \longrightarrow t$ as $n \longrightarrow \infty$, then

(4.2.8) $$|E_n^{-1}(t_n) - t_n + E_n(t_n) - t_n|$$
$$= 0(n^{-3/4} (\log n)^{1/2} (\log\log n)^{1/4})$$

a.s. Since

$$|E_n^{-1}(t_n) - t_n + E_n(t) - t|$$
$$\leq |E_n^{-1}(t_n) - t_n + E_n(t_n) - t_n| + |E_n(t_n) - t_n - E_n(t) + t| ,$$

it follows from (4.2.8) that (4.2.6) holds if we show that

$$(4.2.9) \qquad |E_n(t_n) - t_n - E_n(t) + t| = 0_p(n^{-1/2}\,|t_n - t|^{1/2})$$

and

$$(4.2.10) \qquad |E_n(t_n) - t_n - E_n(t) + t|$$

$$= 0(n^{-1/2}(t(n))^{1/2}\,(\log\log n)^{1/2} + n^{-3/4})$$

a.s.

With the help of Lemma (2.2.1), it is easy to see that

$$V(E_n(t_n) - t_n + E_n(t) - t) = 0(n^{-1}\,|t_n - t|).$$

This yields (4.2.9) utilizing Chebychev inequality. Coming to (4.2.10), let us take $t'(n) = t(n) + n^{-1/2}$. Clearly it suffices to show that

$$\sup_{|s-t| \leq t'(n)} |E_n(s) - s - E_n(t) + t| =$$

$$= 0(n^{-1/2}(t'(n))^{1/2}\,(\log\log n)^{1/2} + n^{-3/4}) \text{ a.s.}$$

We prove this upper bound just imitating the proof of Theorem 2.2.1 (we make use of the facts that $t'(n)$ is non-increasing in n and $t'(n) \geq n^{-1/2}$). The proof is omitted since it does not require any significant change.

Proof of Lemma 4.2.1 (ii). In view of Remarks 2.2.1 and 2.2.4, if $|t_n - t| = 0(n^{-1/2})$,

$$|E_n^{-1}(t_n) - t_n - E_n(t) + t| = 0(n^{-3/4}(\log \log n)^{3/4}) \quad \text{a.s.}$$

This yields (4.2.7) writing

$$|E_n^{-1}(t_n) - t - E_n(t) + t| \leq |E_n^{-1}(t_n) - t_n - E_n(t)+t| + |t_n-t|.$$

Proof of Theorem 4.2.1 (i). Regularty conditions of the theorem imply that F is strictly increasing in a neighbourhood of Q_t and hence, for all large n, $F(Q_{nt_n}) = E_n^{-1}(t_n)$, $t_n = F(Q_{t_n})$, $F_n(Q_t) = E_n(t)$. Therefore, the conclusion (4.2.6) can be rewritten as

$$\text{(4.2.11)} \qquad F(Q_{nt_n}) - F(Q_{t_n}) + F_n(Q_t) - t = R_n(t_n).$$

Since, F is strictly increasing in a neighbourhood of Q_t, it follows from (4.2.11) and the strong law of large numbers that $|Q_{nt_n} - Q_{t_n}| \longrightarrow 0$ a.s. Hence, by expanding $F(Q_{nt_n})$ by Tailor's expansion, we have

$$\text{(4.2.12)} \qquad (Q_{nt_n} - Q_{t_n}) F'(Q_{t_n}) + (Q_{nt_n} - Q_{t_n})^2 F''(\omega_x)$$

$$= - F_n(Q_t) + t + R_n(t_n) .$$

where ω_x is a random point between Q_{nt_n} and Q_{t_n}. But, since F' is bounded away from zero and F'' is bounded in a neighbourhood of Q_t, the representation (4.2.12) guarantees that

$$(4.2.13) \qquad Q_{nt_n} - Q_{t_n} = 0_p(n^{-1/2}) .$$

Thus the second term in the l h.s. of (4.2.12) is negligible as compared to $R_n(t_n)$. Further, let us note that

$$(4.2.14) \qquad |(Q_{nt_n} - Q_{t_n}) F'(Q_{t_n}) - (Q_{nt_n} - Q_{t_n}) F'(Q_t)|$$

$$= 0((Q_{nt_n} - Q_{t_n}) (Q_{t_n} - Q_t))$$

$$= 0_p(n^{-1/2} |t_n - t|) \quad \text{a.s.}$$

Now, (4.2.12), (4.2.13) and (4.2.14) prove (4.2.3).

Similarly we see that

$$|Q_{nt_n} - Q_{t_n}| = 0(n^{-1/2} (\log \log n)^{1/2}) \quad \text{a.s.}$$

and

$$\text{l.h.s. of (4.2.14)} = 0(n^{-1/2}(\log \log n)^{1/2} |t_n-t|) \quad \text{a.s.}$$

which gives (4.2.4).

Theorem 4.2.1 (ii) is derived from Lemma 4.2.1 (ii) by a similar kind of substitution.

<u>Corollary 4.2.1</u>. If $t_n \longrightarrow t$, arbitrary slowly, as $n \longrightarrow \infty$, then the representation (4.2.2) holds with $\sqrt{n}\, R_n(F, t_n) \longrightarrow 0$ in probability and $\sqrt{n}\, R_n(F, t_n)\, (\log \log n)^{-1/2} \longrightarrow 0$ a.s.

We now turn to representation of quantile processes for the case of general distributions. We extend Theorem 2.2.2 for general d.f. s as given below.

Let us say that a d.f. satisfies the condition (*) if for some interval I, $F'(x) = 0$ if $x \notin I$, $\inf \{F'(x) : x \in I\} = K_1^{-1} > 0$ and $\sup \{F''(x) : x \in I\} < \infty$. We have the following theorem for distributions satisfying the condition (*).

<u>Theorem 4.2.2.</u> Let $\{X_i\}$ be a stationary sequence of ϕ-mixing r.v. s with $\Sigma\, \phi^{1/2}(i) < \infty$ and the underlying d.f. F satisfying the condition (*). Then

$$R_n^*(F) = 0(n^{-3/4} (\log n)^{1/2} (\log \log n)^{1/4}) \text{ a.s.}$$

where

$$R_n^*(F) = \sup_{0 \le t \le 1} |R_n(F, t)|$$

and

$$(4.2.15) \qquad R_n(F, t) = Q_{nt} - Q_t + [F_n(Q_t) - t]/F'(Q_t)$$

<u>Proof</u>. The proof is immediate from Theorem 2.2.2 and the following two lemmas.

Lemma 4.2.2. Let $\{X_i\}$ be same as in Theorem 4.2.2. Let E_n denote the e.d.f. of the sequence of r.v. s $\{U_i\}$ defined by $U_i = F(X_i)$. Then for every $t \in [0,1]$

$$R_n(t) = F'(Q_t)\, R_n(F,t) + F''(\omega_t)\, (Q_{nt} - Q_t)^2$$

where $R_n(t) = E_n^{-1}(t) + E_n(t) - 2t$, $R_n(F,t)$ is defined by (4.2.15) and ω_t is a random point between Q_{nt} and Q_t.

Proof. Since F is strictly increasing on I, $E_n^{-1}(t) = F(Q_{nt})$, $E_n(t) = F_n(Q_t)$ and $t = F(Q_t)$. Hence,

$$(4.2.16) \qquad E_n^{-1}(t) - t + E_n(t) - t = F(Q_{nt}) - F(Q_t) + F_n(Q_t) - t.$$

The lemma follows from (4.2.16) applying Taylor's expansion and the condition (*).

Lemma 4.2.3. Under the hypothesis of Theorem (4.2.3), one has

$$(4.2.17) \qquad \sup_{0 \le t \le 1} |Q_{nt} - Q_t| = O(n^{-1/2}(\log \log n)^{1/2}) \text{ a.s.}$$

Proof. Let $E_n(x)$ be same as defined in the previous lemma. Now, lemma 2.2.5 guarantees that

$$(4.2.18) \qquad \sup_{0 \le t \le 1} |E_n^{-1}(t) - t| = O(n^{-1/2}(\log \log n)^{1/2}) \text{ a.s.}$$

Again, due to the facts that $E_n^{-1}(t) = F(Q_{nt})$, $t = F(Q_t)$, we have, with the help of mean value theorem and the condition(*), that

(4.2.19) $$\sup_{0 \leq t \leq 1} |Q_{nt} - Q_t| \leq K_1 |E_n^{-1}(t) - t|$$

where K_1 is same as the positive constant used in the definition of the condition (*). Now, (4.2.17) is a consequence of (4.2.18) and (4.2.19).

Remark 4.2.1. For real numbers $0 \leq \alpha \leq \beta \leq 1$, suppose that the underlying d.f. F (which need not have bounded support) satisfies the conditions : (a) $F'(x)$ exists for $x \in [Q_\alpha - \varepsilon, Q_\beta + \varepsilon]$, $\varepsilon > 0$, and is bounded away from zero on this interval (b) $F''(x)$ exists and is bounded on the interval $[Q_\alpha - \varepsilon, Q_\beta + \varepsilon]$. Then the proof of the theorem can be easily modified to conclude that

$$\sup_{\alpha \leq t \leq \beta} |R_n(F,t)| = O(n^{-3/4}(\log n)^{1/2}(\log \log n)^{1/4}) \text{ a.s.}$$

We mention in passing that the strong approximation results of sample mean known in literature for both independent and dependent cases (see Komlos et al (1975)) and Philipp and Stout (1975) can be extended suitably to quantiles utilizing these representations.

4.3 REPRESENTATION OF QUANTILES IN NON-REGULAR CASES

Theorem 4.3.1. Let $\{X_i\}$ be a stationary sequence of ϕ-mixing r.v. s with $\Sigma\, \phi^{1/2}(i) < \infty$ and X_1 having d.f. F. Let $g : (0, \infty) \longrightarrow (0,\infty)$ be a function such that $g(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and $g(0) = 0$. Let Q_{nt} be as usual the t^{th} sample quantile. Then, we have the following:

(i) Let us assume that F satisfies the condition

$$(4.3.1) \qquad \lim_{\varepsilon \downarrow 0} \frac{F(x+\varepsilon) - F(x)}{g(\varepsilon)} = d_1 > 0$$

and is continuous on $(x, x + \varepsilon_o]$ for some real x and $\varepsilon_o > 0$. If $F(x) = t$ and t_n is a sequence of numbers from $(0,1)$ such that $t_n \geq t$ and $t_n - t = O(n^{-1/2})$ as $n \longrightarrow \infty$, then

$$(4.3.2) \qquad d_1\, g((Q_{nt_n} - x)^+) = (t - F_n(x))^+ + R_{n,t}$$

where $\sqrt{n}\, R_{n,t} \longrightarrow 0$ in probability and $\sqrt{n}\, R_{n,t}(\log \log n)^{-1/2} \to 0$ a.s.

(ii) Let F satisfy the condition

$$(4.3.3) \qquad \lim_{\varepsilon \downarrow 0} \frac{F(x-\varepsilon) - F(x)}{g(\varepsilon)} = - d_2 < 0$$

and be continuous on $[x - \varepsilon_o, x)$ for some real x and $\varepsilon_o > 0$. If $F(x) = t$ and t_n is a sequence of numbers from $(0,1)$ such that $t_n \leq t$ and $t - t_n = O(n^{-1/2})$ as $n \longrightarrow \infty$, then

$$d_2\, g((Q_{nt_n} - x)^-) = (t - F_n(x))^- + R_{n,t}$$

where $\sqrt{n}\; R_{n,t} \longrightarrow 0$ in probability and $\sqrt{n}\, R_{n,t}(\log\log n)^{-1/2} \longrightarrow 0$ a.s.

(iii) Let, for some x,

$$\lim_{\varepsilon \downarrow 0} \frac{F(x + \varepsilon) - F(x)}{g(\varepsilon)} = d_1 > 0$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{F(x - \varepsilon) - F(x)}{g(\varepsilon)} = - d_2 < 0$$

so that F is strictly increasing at x. Let $t = F(x)$ and t_n be a sequence of numbers such that $|t_n - t| = O(n^{-1/2})$ as $n \longrightarrow \infty$. Define $h_n(t) = d_1$ if $Q_{nt_n} \geq x$, $= -d_2$ if $Q_{nt_n} \leq x$. If F is continuous in a neighbourhood of x, then

$$g(|Q_{nt} - Q_t|)\, h_n(t) = (t - F_n(x)) + R_{n,t}$$

where $\sqrt{n}\, R_{n,t} \longrightarrow 0$ in probability and $\sqrt{n}\, R_{n,t}(\log\log n)^{-1/2} \longrightarrow 0$ a.s.

The theorem is derived from

<u>Lemma 4.3.1</u>. Let $\{U_i\}$ be a stationary sequence of r.v. s satisfying the mixing condition stated in the above theorem. Let $E_n(t)$ and $E_n^{-1}(t)$ denote the e.d.f. and the quantile process for this sequence at the n^{th} stage. Further assume that for some $0 < t < t' < 1$, $P(U_1 \leq s) = s$ for all $s \in [t, t']$.

(i) If $t_n \geq t$ and $t_n - t = O(n^{-1/2})$ as $n \longrightarrow \infty$, then

(4.3.4) $$|(E_n^{-1}(t_n)-t)^{+}-(t-E_n(t))^{+}|= O(n^{-3/4}(\log\log n)^{3/4}+|t_n-t|)\text{ a.s.}$$

(ii) If $t_n \leq t'$ and $t'-t_n = O(n^{-1/2})$ as $n \longrightarrow \infty$, then

(4.3.5) $$|(E_n^{-1}(t_n)-t')^{-}-(t'-E_n(t'))^{-}|=O(n^{-3/4}(\log\log n)^{3/4}+|t_n-t'|)$$
a.s.

(iii) If $t_o \in (t,t')$ and $|t_n-t_o) = O(n^{-1/2})$, then

(4.3.6) $$|E_n^{-1}(t_n)-t_o+E_n(t_o)-t_o|= O(n^{-3/4}(\log\log n)^{3/4}+|t_n-t_o|)\text{a.s.}$$

<u>Proof</u>. (iii) part of this lemma is essentially same as Lemma 4.2.1 (ii). We present below the proof for (i) part and the (ii) part is proved similarly.

In view of Remark 2.2.1, a proof similar to that of Lemma 2.2.4 yields that

$$(4.3.7)\quad \sup_{t \leq s \leq t'} |E_n(s) - s| = 0(n^{-1/2}(\log \log n)^{1/2}) \text{ a.s.}$$

and hence there exists a constant K such that, with probability one,

$$E_n(s) > s - K_3 n^{-1/2}(\log \log n)^{1/2}$$

for all $s \varepsilon [t,t']$, for all n sufficiently large. As a consequence,

$$(4.3.8)\quad E_n(t + (t_n - t) + K_3 n^{-1/2}(\log \log n)^{1/2}) > t_n$$

for all sufficiently large n, a.s. But (4.3.8) amounts to saying that

$$(4.3.9)\quad \limsup_{n \to \infty} n^{1/2}(E_n^{-1}(t_n) - t)^+ (\log \log n)^{-1/2} < K_4 \quad \text{a.s.}$$

due to the fact that $t_n - t = 0(n^{-1/2})$.

In view of this conclusion, we claim that

$$(4.3.10)\quad |(E_n^{-1}(t_n) - t)^+ - (t - E_n(t))^+|$$

$$\leq 3 \sup_{0 \leq s-t \leq 2K_4 n^{-1/2}(\log \log n)^{1/2}} |E_n(s) - E_n(t) - s + t|$$

$$+ |t_n - t|$$

for all sufficiently large n, a.s.

To justify the claim, consider two cases.

Case 1. If $E_n^{-1}(t_n) \leq t$, $E_n(t) \geq t_n \geq t$. In this case l.h.s. of (4.3.10) = 0.

Case 2. In case $E_n^{-1}(t_n) > t$ ($\Longrightarrow t_n \geq E_n(t)$), we write

$$|(E_n^{-1}(t_n)-t)^+ - (t - E_n(t))^+|$$

$$\leq |(E_n^{-1}(t_n)-t)^+ - (t_n-E_n(t))^+| + |(t-E_n(t))^+-(t_n-E_n(t))^+|$$

$$= |E_n^{-1}(t_n)-t-t_n+E_n(t)| + |(t-E_n(t))^+-(t_n-E_n(t))^+|.$$

By considering, two different cases $E_n(t) \leq t$ and $E_n(t) \varepsilon [t, t_n]$, we see that $|(t-E_n(t))^+-(t_n-E_n(t))^+| \leq |t_n-t|$. Thus, if $E_n^{-1}(t_n) > t$,

$$|(E_n^{-1}(t_n)-t)^+-(t-E_n(t))^+| \leq |E_n^{-1}(t_n)-t_n+E_n(t)-t|+|t-t_n|$$

$$\leq |E_n\, E_n^{-1}(t_n)-E_n(t)-E_n^{-1}(t_n)+t|+|E_n\, E_n^{-1}(t_n)-t_n|+|t-t_n|$$

$$\leq |E_n\, E_n^{-1}(t_n)-E_n(t)-E_n^{-1}(t_n)+t|+|E_n\, E_n^{-1}(t_n)-E_n(E_n^{-1}(t_n)-0)|+|t_n-t|$$

$$\leq 2|E_n\, E_n^{-1}(t_n)-E_n(t)-E_n^{-1}(t_n)+t|+|E_n(E_n^{-1}(t_n)-0)-E_n(t)-E_n^{-1}(t_n)+t|$$

$$+ |t_n-t|$$

Combining the conclusions of Case 1, Case 2 and (4.3.9) we have (4.3.10). Finally, we show

$$(4.3.11) \qquad \sup_{0 \le s-t \le 2K_4 n^{-1/2}(\log\log n)^{1/2}} |E_n(s)-E_n(t) - s+t| = 0(n^{-3/4}(\log\log n)^{3/4})$$

a.s., by just imitating the proof of Theorem 2.2.1 keeping the Remark 2.2.1 in mind. Now (4.3.10) and (4.3.11) yield the lemma.

<u>Proof of Theorem 4.3.1 (i)</u>. Let us define $U_i = F(X_i)$. Conditions of Theorem 4.3.1 (i) ensure that there exists a $\varepsilon_o > 0$ such that F is strictly increasing at x and continuous in $(x, x+\varepsilon_o]$. Hence $P(U_1 \le s) = s$ for all $s \varepsilon [t, F(x+\varepsilon_o)]$. Also we have $E_n^{-1}(t_n) = F(Q_{nt_n})$, $E_n(t) = F_n(x)$ and $t = F(x)$. Hence, by an appeal to Lemma 4.3.1 (i)

$$(4.3.12) \qquad (F(Q_{nt_n}) - F(x))^+ = (t-F_n(x))^+ + 0(n^{-1/2}) \quad \text{a.s.}$$

It follows from representation (4.3.12) and the fact that F is strictly increasing at x that $(Q_{nt_n} - x)^+ \longrightarrow 0$ a.s. Substituting the value of ℓ.h.s. of (4.3.12) from (4.3.1), we have

(4.3.13) $d_1 \, g((Q_{nt_n} - x)^+) \, [1+o(1)] = (t-F_n(x))^+ + O(n^{-1/2})$ a.s.

Since $\sqrt{n}\,(t-F_n(x))^+ = o_p(1)$ and $\sqrt{n}\,(t-F_n(x))^+ (\log\log n)^{-1/2}$ $= O(1)$ a.s., (4.3.2) follows immediately from (4.3.13).

Parts (ii) and (iii) of the theorem are derived from parts (ii) and (iii) of Lemma (4.3.1) respectively by similar kind of substitution. In part (ii) the equality $E_n(t) = F_n(x)$ is not true in general for all sample points but it is true with probability one. In proving this we use a simple fact that if for some set A, $P(X_1 \in A) = 0$, then none of the X_i's fall in A with probability one.

Remark 4.3.1. The continuity parts of the assumptions in the above theorem are redundent as for as the validity if the theorem is concerned however we have assumed it for the sake of neatness in the arguments. Moreover, the examples where (4.3.1) holds and the continuity does not hold in $(x, x+\varepsilon)$ for any $\varepsilon > 0$ appear to be quite artificial.

Examples. The above theorem includes the following very common non-regular cases : (i) F is discontinuous at x but it has right derivative at x. (ii) In a neighbourhood of x, the graph of F is made of two straight lines joined together at x. (iii) F has a cusp at x.

Corollary 4.3.1. Suppose (4.3.1) and (4.3.2) are satisfied with $d_1 = d_2$ and $g(\varepsilon) = \varepsilon^{\delta}$, $\varepsilon > 0$ and that

$$(4.3.13) \quad 0 < \sigma_t^2 = V(I(X_1 \le Q_t)) + 2 \sum_{i=1}^{\infty} \operatorname{cov}(I(X_1 \le Q_t), I(X_{1+i} \le Q_t)).$$

It follows from the above theorem, the central limit theorems of Ibragimov (1962) and the laws of iteratated logarithm of Reznik (1968) that

$$\sqrt{n}\ |Q_{nt} - Q_t|^{\delta} \operatorname{sign}(Q_{nt} - Q_t) \xrightarrow{\mathcal{L}} N(0, \sigma_t^2/d_1^2)$$

and

$$\lim_{n\to\infty} \sqrt{n}\ |Q_{nt} - Q_t|^{\delta} (\log\log n)^{-1/2} = \sigma_t/d_1 \quad \text{a.s.}$$

These results imply that Q_t can be estimated efficiently for large n if δ is small.

Some similar results have been obtained by Ghosh and Sukhatme (1977) for independent r.v. s.

Remark 4.3.2. Let (4.3.1) and (4.3.2) be satisfied with $g(\varepsilon) = \varepsilon$ and $d_1 \ne d_2$, that is, d_1 is the right derivative and d_2 is the left derivative of F at x. Then it follows from the theorem that the asymptotic distribution of $\sqrt{n}\ (Q_{nt} - x)$ is a distribution whose density is same as that of $N(0, \sigma_t^2/d_2^2)$ on $(-\infty, 0)$ and that of $N(0, \sigma_t^2/d_1^2)$ on $(0, \infty)$ where σ_t is defined by (4.3.13). Further

$$\limsup_{n \to \infty} (Q_{nt}-x) \sqrt{n} (\log \log n)^{-1/2} = \sigma_t/d_1 \quad \text{a.s.}$$

and

$$\liminf_{n \to \infty} (Q_{nt}-x) \sqrt{n} (\log \log n)^{-1/2} = -\sigma_t/d_2 \quad \text{a.s.}$$

In the next theorem we obtain stronger representations of quantiles under stronger smoothness conditions than assumed in the previous theorem. We state and prove below the stronger representation corresponding to the (i) part of the previous theorem. One can similarly obtain other versions which correspond to (ii) and (iii) parts.

Theorem 4.3.2. Let $\{X_i\}$ be same as in the previous theorem. Let g_1, g_2 be two functions from $(0,\infty)$ to $(0,\infty)$ such that $g_1(\varepsilon) \longrightarrow 0$, $g_2(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, g_1 is strictly increasing in a neighbourhood of zero and $g_1(0) = 0$. Assume that F satisfies

$$(4.3.14) \qquad \frac{F(x+\varepsilon) - F(x)}{g_1(\varepsilon)} = d_1(1 + O(g_2(\varepsilon))), \; d_1 > 0, \text{ as}$$

$\varepsilon \longrightarrow 0$ at some real x and it is continuous in $(x, x + \varepsilon_o]$ for some ε_o positive. If $F(x) = t$ and t_n is a sequence of numbers from (0,1) such that $t_n \geq t$ and $t_n - t = O(n^{-1/2})$ as $n \longrightarrow \infty$, then

(4.3.15) $$d_1\ g_1((Q_{nt_n} - x)^+) = (t - F_n(x))^+ + R_{n,t}$$

where

(4.3.16) $$R_{n,t} = O(n^{-3/4}(\log\log n)^{3/4} + |t_n - t| + \lambda_n g_2(K_5\ g_3(K_6\ \lambda_n))),\quad \text{a.s.},$$

where $\lambda_n = n^{-1/2}(\log\log n)^{1/2}$ and g_3 is inverse function of g_1 which is well defined in a neighbourhood of zero.

Proof. Once again we shall be applying the Lemma 4.3.1 (i) to derive this theorem. Following the proof of the previous theorem, we have, applying the (i) part of Lemma 4.3.1 , that

(4.3.17) $$(F(Q_{nt_n}) - F(x))^+ = (t - F_n(x))^+ + O(n^{-3/4}(\log\log n) + |t_n - t|).$$

Substituting the value of ℓ.h.s. of (4.3.17) from (4.3.14), we see that

(4.3.18) $$d_1\ g_1((Q_{nt_n} - x)^+)\ [1 + O(g_2(Q_{nt_n} - x)^+)] = (t - F_n(x))^+ + O(n^{-3/4}(\log\log n)^{3/4} + |t_n - t|)\ \text{a.s.}$$

As a consequence of (4.3.18),

(4.3.19) $g_1((Q_{nt_n}-x)^+) = O(\lambda_n)$ a.s. $(\lambda_n = n^{-1/2}(\log\log n)^{1/2})$

and hence

(4.3.20) $(Q_{nt_n}-x)^+ = O(g_3(K_6\ \lambda_n))$ a.s.

Feeding (4.3.19) and (4.3.20) back to (4.3.18), we obtain (4.3.15).

In the independent case, one can obtain the constant term implied by O - term using the results of Kiefer (1967).

4.4 APPROXIMATION OF WEIGHTED QUANTILE PROCESSES BY GAUSSIAN PROCESSES

A Brownian bridge $\underset{\sim}{B}(t)$, $0 \leq t \leq 1$, is a seperable Guassian process with the covariance kernel given by

$$\operatorname{cov}(\underset{\sim}{B}(t_1), \underset{\sim}{B}(t_2)) = t_1 \wedge t_2 - t_1 t_2 \ .$$

Let $\{U_i\}$ be a sequence of i.i.d. r.v. s with marginals as U[0,1]. Let $E_n(t)$ denote the e.d. f of $\{U_i\}$ at the n^{th} stage. Komlós et al (1975) established the following strong approximation of the process $E_n(t)$.

<u>Theorem 4.4.1</u>. Let $E_n(t)$ be as defined above and $\{B_i(t)\}$ be a sequence of independent Brownian bridges. Then, there exists a version of the processes $E_n(t)$ and $\{B_i(t)\}$ such that

$$(4.4.1) \qquad \sup_{0 \leq t \leq 1} \left| n(E_n(t)-t) - \sum_{i=1}^{n} B_i(t) \right| << (\log n)^2 \quad \text{a.s.}$$

We shall combine this theorem with the results of Chapter 3 to obtain some strong-approximations of weighted quantile processes with appropriate Gaussian processes.

Let us say that a d.f. F satisfies the condition (**) if the following are met :-

(i) F' and F'' exist throughout the support of F with positive F' and bounded F''. (ii) F' is bounded away from zero in the interval $[F^{-1}(\alpha) - \delta, F^{-1}(\beta) + \delta]$ for some $0 \leq \alpha \leq \beta \leq 1$ and $\delta > 0$. (iii) F' and F'' are non-decreasing in $(-\infty, F^{-1}(\alpha))$ and non-increasing in $(F^{-1}(\beta), \infty)$. (iv)

$$\limsup_{t \to 0} f_t/f_{4t} < \infty$$

$$\limsup_{t \to 1} f_{(1-t)}/f_{(1-4t)} < \infty$$

where $f_t = F'(F^{-1}(t))$.

Before coming to the approximation theorem, we would like to mention that the condition (iv), although looks rather restrictive, is met for most of the distributions which are of practical importance. As a matter of fact, if $E|X_1|^c < \infty$, $c > 0$ (X_1 having d.f. F), it is proved in Lemma 4.4.1 (see below) that $f_1 \geq K_7 t^{c^{-1}+1+\delta}$ for any $\delta > 0$ (K_7 depends on δ) and t in a neighbourhood of zero, and a similar inequality holds when t is near one. This means that $f_t \longrightarrow 0$ as $t(1 - t) \longrightarrow 0$ at a polynomial rate. This seems to be the intuitive reason for the condition to hold.

We now state our result of this section in

Theorem 4.4.2. Let $\{X_i\}$ be a sequence of i.i.d. r.v. s with the underlying d.f. F. We assume that F satisfies the condition (**). Let $F_n(x)$, Q_{nt} and Q_t denote e.d.f., t^{th} sample quantile and t^{th} population quantile respectively. $\omega(t)$ stands for some weight function on $[0,1]$ and $f_t = F'(F^{-1}(t))$. If $E|X_1|^{a+\delta} < \infty$ for some $a > 0$, $\delta > 0$ and $w(t) = O((t(1-t))^{\frac{1}{a}+\frac{1}{2}}$ as $t(1 - t) \longrightarrow 0$ then, there exists a version of the precesses $\{Q_{nt}\}$ and a sequence of independent Brownian bridges $\{B_i(t)\}$ such that

$$(4.4.2) \qquad \sup_{n^{-1}\leq t\leq 1-n^{-1}} \left|n\, w(t)(Q_{nt}-Q_t) - \frac{w(t)}{f_t} \sum_{i=1}^{n} B_i(t)\right| = O(n^{1/2-\gamma})$$

a.s. for some $\gamma > 0$.

The following lemma is of central importance in proving the theorem. It is also appealed to repeatedly in the next chapter.

<u>Lemma 4.4.1.</u> Let X be a r.v. with d.f. F and Q_u stands for any inverse of F. We then have :

(i) $\int_0^1 (u(1-u))^{a-\delta}\, d\, Q_u < \infty$ for some $a > 0$ and $0 < \delta < a$

$$\Longrightarrow \quad E|X|^{1/a} < \infty .$$

(ii) $E|X|^{a+\delta} < \infty$ for some $a, \delta > 0$

$$\Longrightarrow \quad \int_0^1 (u(1-u))^{1/a}\, d\, Q_u < \infty .$$

(iii) If $E|X|^{a+\delta} < \infty$ and F satisfies the condition (**), then

$$1/f_t = O((t(1-t))^{-1-1/a}) \quad \text{as} \quad t(1-t) \longrightarrow 0,$$

$$|\, Q_t \,| = O((t(1-t))^{-1/a}) \quad \text{as} \quad t(1-t) \longrightarrow 0,$$

$$\limsup_{t \to 0} \; t\, F''(Q_t)/f^2_{(4t)} < \infty$$

and

$$\limsup_{t \to 1} \; (1-t)\, F''(Q_{(1-t)})/f^2_{(1-4t)} < \infty .$$

<u>Proof of Lemma 4.4.4.(i).</u>

$$\int_{1/2}^1 (1-u)^{a-\delta}\, d\, Q_u < \infty$$

implies, by applying the transformation $x = Q_u$, that

$$\int_0^\infty (1 - F(x))^{a-\delta}\, dx < \infty . \tag{4.4.3}$$

Since (1-F(x)) is non-increasing, (4.4.3) implies that

$$(1-F(x))^{a-\delta} = O(x^{-1}) \quad \text{as} \quad x \longrightarrow \infty$$

$$\Longrightarrow \quad (1-F(x^a)) = O(x^{-a/(a-\delta)}) \quad \text{as} \quad x \longrightarrow \infty$$

$$\Longrightarrow \quad \int_0^\infty P(X^{a^{-1}} > x)\, dx < \infty$$

$$\Longrightarrow \quad E(X^+)^{1/a} < \infty .$$

Similarly one shows that

$$\int_0^{1/2} u^{a-\delta}\, d\, Q_u < \infty \Longrightarrow E(X^-)^{1/a} < \infty$$

and this completes the proof of the (i) part.

Proof of (ii).

$$E(X^+)^{a+\delta} < \infty$$

$$\Longrightarrow \quad \int_0^\infty P(X > x^{1/(a+\delta)})\, dx < \infty$$

$$\Longrightarrow \quad 1 - F(x^{(a+\delta)^{-1}}) = O(x^{-1})$$

$$\Longrightarrow \quad (1 - F(x))^{1/a} = O(x^{-(a+\delta)/a})$$

$$\Longrightarrow \int_0^{\infty} (1 - F(x))^{1/a}\, dx < \infty$$

$$\Longrightarrow \int_{1/2}^{1} (1 - u)^{1/a}\, d\, Q_u < \infty .$$

Similarly

$$E(X^-)^{a+\delta} < \infty \Longrightarrow \int_0^{1/2} u^{1/a}\, d\, Q_u < \infty .$$

<u>Proof of (iii).</u> Under the condition (**),

$$E|X|^{a+\delta} < \infty \Longrightarrow \int_0^1 \frac{(u(1-u))^{1/a}}{f_t}\, dt < \infty \quad \text{(by (i) part of this lemma)}$$

$$\Longrightarrow \int_{t/2}^{t} \frac{(U(1-u))^{1/a}}{f_t}\, dt \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 0$$

$$\Longrightarrow (t^{1/a}/f_t) \int_{t/2}^{t} dt \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 0$$

$$\Longrightarrow 1/f_t = 0(t^{-1-1/a}) \quad \text{as} \quad t \longrightarrow 0 .$$

Similarly, we see that

$$1/f_{(1-t)} = 0((1-t)^{-1-1/a}) \quad \text{as } t \longrightarrow 1.$$

The other statements of this part are derived using the order of $1/f_t$ and the mean value theorem. The proofs are simple and we omit them.

<u>Proof of Theorem 4.4.2.</u> Define $U_i = F(X_i)$. Let $E_n(t)$ denote the e.d.f. of $\{U_i\}$. Now, Theorem 4.4.1 is in force and we have the approximation (4.4.1) of $E_n(t)$ with a sequence of independent Brownian bridges $\{\underset{\sim}{B}_i(t)\}$. We obtain (4.4.2) with the sequence of independent Brownian bridges $\{\tilde{B}_i(t)\}$ defined by $\tilde{B}_i(t) = -\underset{\sim}{B}_i(t)$.

As in Remark (4.2.1), condition (**) implies that

$$\sup_{\alpha \le t \le \beta} \left|Q_{nt} - Q_t - \frac{t - E_n(t)}{f_t}\right| = O(n^{-3/4} \log n) \quad \text{a.s.}$$

This proves 4.4.2 if the supremum is taken on the interval $[\alpha, \beta]$ only.

Coming to the interval $[n^{-1}, \alpha]$, we shall show that, for some $\gamma > 0$,

$$(4.4.4) \qquad \sup_{n^{-1} \le t \le n^{-1}(\log n)^2} w(t)|Q_{nt} - Q_t| = O(n^{-1/2-\gamma}), \text{ a.s.}$$

$$(4.4.5) \qquad \sup_{n^{-1} \le t \le n^{-1}(\log n)^2} \frac{w(t)}{f_t} \left|\sum_{i=1}^{n} \tilde{B}_i(t)\right| = O(n^{1/2-\gamma}) \text{ a.s.}$$

$$(4.4.6) \quad \sup_{n^{-1}(\log n)^2 \leq t \leq \alpha} \left| w(t)(Q_{nt} - Q_t) - \frac{w(t)\,(t - E_n(t))}{f_t} \right|$$

$$= 0(n^{-1/2-\gamma}) \text{ a.s.}$$

and

$$(4.4.7) \quad \sup_{n^{-1}(\log n)^2 \leq t \leq \alpha} \left| \frac{w(t)(E_n(t)-t)}{f_t} - \frac{w(t)}{f_t} \sum_{i=1}^{n} \underset{\sim}{B}_i(t) \right|$$

$$= 0(n^{1/2-\gamma}) \text{ a.s.}$$

One obtains similar results on the interval $(\beta, 1-n^{-1})$. These results yield the desired approximation immediately. The proofs of the statements made above are presented below.

In view of Lemma 3.3.1 and Lemma 4.4.1 (iii),

$$\text{l.h.s. of } 4.4.4 << w(n \ (\log n)^2)\, Q_{(n^{-1}(\log n)^2)}$$

$$<< n^{-1/2-\gamma} \text{ a.s. for some } \gamma > 0 .$$

(4.4.7) is also immediate from 4.4.1, since

$$(4.4.8) \quad w\,(n^{-1}(\log n)^2)/f_{1/n} = 0(n^{1/2-\gamma}) \ \forall \ \gamma > 0.$$

To show (4.4.5), we note that

$$T(t) = n^{-1/2} \sum_{i=1}^{n} \tilde{B}_i(t) + n^{-1/2}(\sqrt{8} - 2)\, t \sum_{i=1}^{n} \tilde{B}_i(1/2)$$

is a standard Wiener process on $[0,1/2]$. Using maximal inequality for Wiener processes (see (5.3.10) of Stout (1974)),

$$P\Big(\sup_{n^{-1} \le t \le n^{-1}(\log n)^2} |T(t)| > n^{-1/2}(\log n)^2\Big)$$

$$\le 2\, P(|T(n^{-1}(\log n)^2)| > n^{-1/2}(\log n)^2) = O(n^{-2}).$$

Thus, we have, by Borel-Cantelli lemma, that

$$\sup_{n^{-1} \le t \le n^{-1}(\log n)^2} |T(t)| = O(n^{-1/2}(\log n)^2) \text{ a.s.}$$

Also, an application of Borel-Cantelli lemma yields that

$$\sup_{n^{-1} \le t \le n^{-1}(\log n)^2} n^{-1/2}\, t \Big|\sum_{i=1}^{n} \tilde{B}_i(1/2)\Big| = O(n^{-1+\gamma}) \text{ a.s. } \forall \gamma > 0 .$$

As a consequence

$$\sup_{n^{-1} \le t \le n^{-1}(\log n)^2} \Big|\sum_{i=1}^{n} \tilde{B}_i(t)\Big| = O((\log n)^2) \text{ a.s.}$$

From this and (4.4.8), (4.4.5) follows.

Lastly, we show 4.4.6. To this end, let us write

$$R_n(F,t) = Q_{nt} - Q_t + (F_n(Q_t) - t)/f_t$$

and

$$R_n(t) = E_n^{-1}(t) - t + E_n(t) - t .$$

Clearly, we only have to show that, for some $\gamma > 0$,

$$(4.4.9) \qquad \sup_{n^{-1}(\log n)^2 \le t \le \alpha} |R_n(F,t)| \, w(t) = o(n^{-1/2-\gamma}) \text{ a.s.}$$

It follows from Lemma 4.2.2 that

$$(4.4.10) \quad w(t) \, |R_n(F,t)|$$

$$\le w(t) \, |R_n(t)|/f_t + w(t) \, F''(\omega_t) \, (Q_{nt}-Q_t)^2/f_t$$

where ω_t is a random point between Q_{nt} and Q_t .

Theorem 3.3.1 and Lemma 4.4.1 guarantee that, for some $\gamma > 0$,

$$\sup_{n^{-1}(\log n)^2 \le t \le \alpha} w(t) \, |R_n(t)|/f_t = o(n^{-1/2-\gamma}) \text{ a.s.}$$

Now, we are left to estimate the second term in the r.h.s. of (4.4.10). By Lemma 3.3.1,

$$t/2 \leq E_n^{-1}(t) \leq 2t$$

in the range of t under consideration, for all sufficiently large n, a.s. Hence, $\omega_t \leq Q_{2t}$ for all sufficiently large n, a.s. Also,

$$(Q_{nt} - Q_t) = F^{-1}(E_n^{-1}(t) - F^{-1}(t)$$

$$= (E_n^{-1}(t) - t) / F'(\omega_t^*)$$

where ω_t^* is a random point between $E_n^{-1}(t)$ and t. Collecting all these observations, we see that

$$(4.4.11) \quad w(t)\, F''(\omega_t)\,(Q_{nt}-Q_t)^2/f_t << \frac{w(t)|F''(F^{-1}(2t))|(E^{-1}(t)-t)}{f_{t/2}^3}$$

$$<< \frac{w(t)}{f(t)} \cdot \frac{t|F''(F^{-1}(2t))|}{f_{t/2}^2} \cdot \left[\frac{E_n^{-1}(t)-t}{t^{1/2}}\right]^2 .$$

The term in the r.h.s. of (4.4.10) is seen to be $O(n^{-1/2-\gamma})$, a.s., using (4.4.11), (4.4.8) Lemma 4.4.1(iii) and Lemma 3.3.1. This completes the proof.

CHAPTER 5

REPRESENTATION OF L-STATISTICS

5.1 INTRODUCTION

In this chapter, we consider the problem of asymptotic representations of L-statistics. The representations linearise the statistics except for a negligible remainder. Central limit theorems, the laws of iterated logarithm and strong-approximations are immediate consequences of the representations. The idea of the representations can also be used to get rates of convergence to normality of such statistics. This problem is discussed in the next chapter.

In section 5.2, we consider L-statistics with smooth weight functions and employ the idea of Moore (1968) along with the stability results of Chapter 3. The problem is explicitly described in section 5.2.

In section 5.3, the results on quantile's representations, proved in Chapter 3, are utilized to establish the representations of L-statistics with more general weight functions. We also have a result which combines the Moore technique with the quantile representation technique and produces some representations which do not seem to be obtainable by either of these techniques alone.

5.2 APPLICATIONS OF THE MOORE'S TECHNIQUE

Let $\{X_n\}$ be a stationary sequence of r.v.s with X_1 having a continuous d.f. F on the real line and F_n is the corresponding e.d. f. For some weight function w on [0,1], consider the following linear combination of order statistics :

(5.2.1) $$L_n = \int_{-\infty}^{\infty} x\, w(F_n(x))\, d\, F_n(x)$$

and the corresponding parametric value

(5.2.2) $$L = \int_{-\infty}^{\infty} x\, w(F(x))\, dF(x)$$ (which we assume to exist). Let $E_n(u)$ denote the e.d.f. of $U_i = F(X_i)$, $1 \leq i \leq n$, (which are uniformly distributed on [0,1]) and Q be any inverse of F. Then, we can also write

(5.2.3) $$\begin{cases} L_n = \int_0^1 Q_{(u)}\, w(E_n(u))\, dE_n(u) \\ L = \int_0^1 Q_u\, w(u)\, du \end{cases}$$

Further, let us define

(5.2.4) $$Z_i = - \int_0^1 (I(U_i \leq u) - u)\, w(u)\, d\, Q_u \,.$$

Moore (1968) gives an elegant proof of the asymptotic normality of $\sqrt{n}\,(L_n - L)$, in the case when $\{X_n\}$ is a sequence of i.i.d. r.v.s by showing that, if w is sufficiently smooth and $E|X_1| < \infty$, then

$$RL_n = L_n - L - n^{-1} \sum_{i=1}^{n} Z_i = o_p(n^{-1/2}). \tag{5.2.5}$$

Later, Ghosh (1972) proves that if w has bounded second derivative (this condition is slightly stronger than that assumed by Moore (1968)) and

$\int_0^1 (u(1-u))^{1/(2+\delta_1)} \, d\, Q_u < \infty$, for some $\delta_1 > 0$ (which is equivalent to assuming that $E|X_1|^{2+\delta_2} < \infty$ for some $\delta_2 > 0$), then

$$RL_n = O(n^{-1} (\log n)^2) \text{ a.s.} \tag{5.2.6}$$

In Ghosh (1972), the author proposes to study the a.s. behaviour of RL_n under the more natural condition $E|X_1| < \infty$. We answer this question here by extending (5.2.6) under the condition $E|X_1| < \infty$. In fact, we obtain a slightly sharper order for the m-dependent case. If $E|X_1|^{1+\delta} < \infty$, $\delta > 0$, we sharpen the order by showing that

$$RL_n = O(n^{-1}(\log \log n)) \quad \text{a.s.}$$

We study the order of RL_n in the case of mixing r.v. s also. Recently, the problem for ϕ-mixing r.v.s has been attempted by Sotres and Ghosh (private communications) but the results obtained by these authors appear to be much weaker than those obtained here.

We conclude this section by relaxing the smoothness condition on w at finitely many points. In this case the proof turns out to be somewhat complicated rather unexpectedly.

Let us recall the definition of $V_n(\varepsilon)$ given by (3.2.1).

Theorem 5.2.1. Let w have bounded second derivative. We then have :

(i) If $\{X_i\}$ is a sequence of m-dependent r.v.s and $E|X_1| < \infty$, then, for any $\gamma > 0$,

$$RL_n = O(n^{-1} (\log n)^{1+\gamma}) \quad \text{a.s.}$$

(ii) If $\{X_i\}$ is a sequence of m-dependent r.v.s and $E|X_1|^{1+\delta} < \infty$, for some $\delta > 0$, then

$$RL_n = O(n^{-1} \log \log n) \quad \text{a.s.}$$

(iii) If $\{X_i\}$ is a sequence of ϕ-mixing r.v. with $\phi(n) = O(n^{-\theta})$, $\theta \geq 2$ (when $\theta = 2$, we further assume that $\Sigma\, \phi^{1/2}(i) < \infty$) and $E|X_1|^{1+\theta^{-1}+\delta} < \infty$, for some $\delta > 0$, then

$$RL_n = O(n^{-1} (\log \log n)) \text{ a.s.}$$

(iv) If $\{X_i\}$ is a sequence of ϕ-mixing r.v.s with $\phi(n) = O(e^{-\theta n})$, $\theta > 0$, and $E|X_1| < \infty$, then, $\forall\, \gamma > 0$,

$$RL_n = O(n^{-1} (\log n)^{2+\gamma}) \text{ a.s.}$$

(v) If $\{X_i\}$ is a sequence of strong-mixing r.v.s with $\alpha(n) = O(e^{-\theta n})$, $\theta > 0$, and $E|X_1|^{2+\delta} < \infty$, $\delta > 0$, then

$$RL_n << n^{-1} (\log \log n) \text{ a.s.}$$

<u>Proof</u>. Following Moore (1968) and Ghosh (1972), we write

$$(5.2.7) \qquad L_n - L = L_{n1} + L_{n2} + L_{n3} \quad (RL_n = L_{n2} + L_{n3})$$

where

$$L_{n1} = \int_0^1 Q_u \quad w'(u)(E_n(u)-u)du + \int_0^1 Q_u \quad w(u)\ d(E_n(u)-u),$$

$$L_{n2} = \int_0^1 Q_u \quad [w(E_n(u)) - w(u) - w'(u)(E_n(u)-u)] \; dE_n(u)$$

and

$$L_{n3} = \int_0^1 Q_u \quad w'(u) \; (E_n(u)-u) \; d(E_n(u)-u).$$

(In this representation we do not use the fact that w' is the derivative of w. Hence, in the proof of the next theorem, we replace w' by 0 whenever it does not exist.)

It follows by integration by parts that, under the condition $E|X_1| < \infty$ (which is the minimum moment condition assumed in this section),

(5.2.8) $$L_{n1} = n^{-1} \sum_{i=1}^{n} Z_i \quad (Z_i \text{ s are defined by (5.2.4)}).$$

Under the assumed conditions on w, taking

$$\bar{X}_n = n^{-1} \sum_{i=1}^{n} | \; Q_{U_i} \; |,$$

(5.2.9) $$L_{n2} = O(\bar{X}_n (V_n(0))^2) \quad \text{a.s.}$$

Also, with probability one,

(5.2.10) $$L_{n3} = \frac{1}{2} \int_0^1 Q_u \quad w'(u) \; d(E_n(u) - u)$$

$$+ \frac{1}{2n} \int_0^1 Q_u \quad w'(u) \; dE_n(u),$$

Clearly,

(5.2.11) the second term in the r.h.s. of (5.2.10) = $O(\bar{X}_n/n)$

and it follows by integration by parts and the condition $E|X_1| < \infty$ ($\Longleftrightarrow \int_0^1 |Q_u| du < \infty$) that

(5.2.12) $$\int_0^1 Q_u \; w'(u) \; d(E_n(u) - u)^2$$

$$= -[\int_0^1 w''(u) \; (E_n(u) - u)^2 \; Q_u \; du$$

$$+ \int_0^1 w'(u) \; (E_n(u) - u)^2 \; d \; Q_u \;]$$

$$= O((V_n(0))^2 + (V_n(\varepsilon))^2 \int_0^1 (u(1-u))^{2\varepsilon} \; d \; Q_u \quad , \frac{1}{2} \geq \varepsilon \geq 0 .$$

Now, all the statements of the above theorem are immediate using the estimates of $V_n(\varepsilon)$ proved in section 3.2, Lemma 4.4.1. (ii) and the above inequalities for L_{n2} and L_{n3}. Proof of Theorem 5.2.1 is complete.

To include some important statistics, like the trimmed means. in the domain of the present study, it would be worthwhile to relax the smoothness restrictions on w at finitely many points. In the theorem that follows, we extend the (i) part of Theorem 5.2.1 to the weight functions which need not even be continuous at finitely many points.

<u>Theorem 5.2.2</u>. Let $\{X_n\}$ be a sequence of m-dependent r.v.s such that $E|X_1| < \infty$. Let w have bounded second derivative everywhere except at finitely many jump points $\alpha_1, \alpha_2, \ldots, \alpha_r$ (by a jump point of w, we mean that both the left and right limits of w exist and the function is continuous at least from one side). If F admits density in a neighbourhood of each of these points $Q_{\alpha_1}, Q_{\alpha_2}, \ldots\ldots\ldots, Q_{\alpha_r}$ and the density is bounded away from zero in the corresponding neighbourhoods, then $\forall\ \gamma > 0$,

$$RL_n = O(n^{-1} (\log n)^{1+\gamma}) \quad \text{a.s.}$$

The following two lemmas are required in the proof.

<u>Lemma 5.2.1</u>. If $\{X_i\}$ satisfies the conditions of the above theorem and E_n is the e.d.f. of $\{U_i\}$ where $U_i = F(X_i)$, then

$$\text{(5.2.13)} \qquad \sup_{0 \le t \le 1} |E_n E_n^{-1}(t) - t| = O(n^{-1}) \quad \text{a.s.}$$

<u>Proof</u>. For any integer $1 \le j \le m$,

$$U_j,\ U_{m+j},\ U_{2m+j},\ \ldots\ldots$$

is a sequence of i.i.d. U[0,1] r.v.s, and therefore, these r.v.s take distinct values with probability one. This observation implies that, with probability one, at most m r.v.s out of the sequence $U_1, U_2, \ldots$ can take same value. This fact yields (5.2.13) trivially.

Lemma 5.2.2. For some fixed absolute constant c and $0 < a < 1$,

$$(5.2.14)\quad \#\{i : 1 \le i \le n \text{ and } U_i \in [a-c\, n^{-1/2}(\log n)^{1/2},\ a + c\, n^{-1/2}(\log n)^{1/2}]\} = O(n^{1/2}(\log n)^{1/2}) \quad \text{a.s.}$$

Proof. With probability one,

$$\text{l.h.s. of } (5.2.14) = n\, E_n(a+c\, n^{-1/2}(\log n)^{1/2}) - n\, E_n(a - c\, n^{-1/2}(\log n)^{1/2})$$

$$<< n\, V_n(0) + 2c\, n^{1/2}(\log n)^{1/2} = O(n^{1/2}(\log n)^{1/2})$$

(recall the definition of $V_n(0)$ given by (3.2.1))

a.s. Hence the lemma.

Proof of the Theorem. For the sake of simplicity in arguments, we assume that w satisfies smoothness conditions everywhere except at a jump point $\alpha \in (0,1)$. The proof can be easily extended to the case when there is more than one jump point. We argue in two steps as follows.

<u>Step 1</u>. Consider first the case when w is continuous at α but not differentiable. Define $w'(\alpha)$ to be zero. Since,

$$\int_0^1 Q_u \quad w'(u)(E_n(u)-u)du = \int_0^1 Q_u \quad (E_n(u)-u)dw(u)$$

still holds, we have $L_{n1} = n^{-1} \sum_{i=1}^{n} Z_i$ by integration by parts. We prove only $L_{n2} = O(n^{-1} \log n)$ a.s., since the proof for $L_{n3} = O(n^{-1} (\log n)^{1+\gamma})$, a.s. requires only minor changes.

From Lemma 2.2.5,

$$\limsup_{n \to \infty} n^{1/2}(\log n)^{-1/2} \sup_{0 \leq u \leq 1} |E_n^{-1}(u)-u| < 1 \quad \text{a.s.}$$

Writing $\gamma_n = n^{-1/2}(\log n)^{1/2}$, we express L_{n2} as follows :

$$L_{n2} = \Big(\int_0^{(\alpha-\gamma_n)^-} + \int_{\alpha-\gamma_n}^{\alpha+\gamma_n} + \int_{(\alpha+\gamma_n)^+}^{1} \Big) (Q_u \quad (w(E_n(u))-w(u)$$

$$- (E_n(u)-u) \; w'(u))) \; dE_n(u).$$

The desired orders of the three integrals one established easily using the choice of γ_n, Lemma 5.2.2, and the fact that, under the assumed conditions, w satisfies a Lipschitz condition of order 1.

<u>Step 2</u>. We finally relax the continuity of w at α. For definiteness, let w be right continuous at α and $w(\alpha) - w(\alpha -) = b \neq 0$. Define, a new continuous weight function w_1 as follows:

$$w_1(u) = w(u) \quad \text{for } u \in (0, \alpha)$$
$$= w(u) - b \quad \text{for } u \in (\alpha, 1].$$

Using the conclusion of step 1, if

$$L_n' = \int_0^1 Q_u \; w_1(E_n(u)) \, dE_n(u)$$

and

$$L' = \int_0^1 Q_u \; w(u) \, du$$

then, a.s.,

$$L_n' - L' = - \int_0^1 (E_n(u)-u) \; w_1(u) \; d \; Q_u \; + O(n^{-1}(\log n)^{1+\gamma})$$

We establish our claim by showing that, with probability one,

$$(L_n-L)-(L_n'-L') + \int_0^1 (E_n(u)-u)(w(u)-w_1(u)) \; Q_u \; = O(n^{-1}\log n)$$

To this end, we note, under the assumed conditions of the theorem, Q satisfies a Lipschitz condition of order 1 in a neighbourhood of the point α. The following are true up to the order $n^{-1} \log n$ a.s. :

$$(L_n - L_n') - (L - L') = b \int_{E_n^{-1}(\alpha)}^{1} Q_u \, dE_n(u) - b \int_{\alpha}^{1} Q_u \, du$$

$$= b \int_{E_n^{-1}(\alpha)}^{1} Q_u \, d(E_n(u) - u) - b \int_{\alpha}^{E_n^{-1}(\alpha)} Q_u \, du .$$

By integration by parts, we can write this as (upto $n^{-1} \log n$ a.s.

$$- b \, Q_{E_n^{-1}(\alpha)} (E_n(E_n^{-1}(\alpha)) - E_n^{-1}(\alpha))$$

$$- b \int_{E_n^{-1}(\alpha)}^{1} (E_n(u) - u) d \, Q_u - b \, Q_\alpha (E_n^{-1}(\alpha) - \alpha)$$

$$= - b \int_{\alpha}^{1} (E_n(u) - u) \, d \, Q_u + b(Q_{E_n^{-1}(\alpha)} - Q_\alpha) (E_n^{-1}(\alpha) - \alpha)$$

$$- b \, Q_{E_n^{-1}(\alpha)} (E_n \, E_n^{-1}(\alpha) - \alpha)$$

(since Q satisfies Lipschitz condition of order 1 in a neighbourhood of α)

$$= - \int_{0}^{1} (E_n(u) - u) \, (w(u) - w_1(u)) \, d \, Q_u .$$

This proves the theorem.

<u>Remark 5.2.1</u>. In particular, this result implies that the trimmed means can be linearised up to the order $n^{-1}(\log n)^{1+\gamma}$, $\gamma > 0$, a.s., which is impossible if one tries to do so using the asymptotic representations of quantiles (also, one requires the existence of density everywhere for the representations of quantiles).

To obtain possible extensions of Theorem 5.2.2 for mixing r.v.s, we rewrite our findings in an alternative form in

<u>Theorem 5.2.3.</u> Under the conditions of Theorem 5.2.2, $\forall\, \gamma > 0$,

$$(5.2.15) \qquad RL_n << \sum_{i=1}^{r} |E_n\, E_n^{-1}(\alpha_i) - \alpha_i| + n^{-1}(\log n)^{1+\gamma} \text{ a.s.}$$

Now, if the conditions of Theorem 5.2.1 (ii) holds, all the arguments of Theorem 5.2.2 go through except for the Lemma 5.2.1 (the result of Lemma 5.2.1 does not seem to be true for mixing r.v.s).

Following the proof of Theorem 2.2.2, it can be seen that under the mixing conditions of Theorem 5.2.1 (iii),

$$(5.2.16) \qquad \sum_{i=1}^{r} |E_n\, E_n^{-1}(\alpha_i) - \alpha_i|$$

$$<< \sum_{i=1}^{n} |E_n(E_n^{-1}(\alpha_i) - E_n(E_n^{-1}(\alpha_i) - 0)|$$

$$= O(n^{-1+(2\theta+2)^{-1}+\delta}), \quad \delta > 0 \quad \text{a.s.}$$

and if ϕ decays exponentially then

$$\ell.h.s. \text{ of } (5.2.16) = O(n^{-1} (\log n)^2) \quad a.s.$$

One gets similar results for the strong mixing cases also. These results provide us with slightly weaker extensions of Theorem 5.2.2 for the mxing cases. Obviously, if the bivariate distributions of (X_i, X_j) are absolutely contiuous w.r.t. Lebesgue measure on the two dimensional plane, for all $i \neq j$, then

$$\sup_{1 \leq t \leq 1} |E_n E_n^{-1}(a) - a| = O(n^{-1}) \quad a.s.$$

5.3. APPLICATIONS OF QUANTILE REPRESENTATION

Let $\{X_i\}$ be a stationary sequence of r.v.s with the underlying distribution F. Let Q_t be t^{th} population quantile and

$$Q_{nt} = \inf \{x : F_n(x) \geq t\} \quad \text{if } t > 0$$

(5.3.1)

$$= Q_{n(t+)} \quad \text{if } t = 0$$

where F_n denotes the e.d.f of $\{X_i\}$ at the n^{th} stage. For some d.f. W on $[0,1]$, we consider the following linear combination of order statistics :

(5.3.2) $$L_n^* = \int_0^1 Q_{nt} \, d\, W(t)$$

and the corresponding parametric value

$$(5.3.3) \qquad L^{*} = \int_{0}^{1} Q_t \, d\, W(t)$$

(If $W' = w$, w is bounded and $E|X_1| < \infty$, then $(L_n - L_n^{*}) = O(n^{-1})$ a.s. where L_n is defined by (5.2.1). If F is continuous and $W' = w$, then L defined by (5.2.2) and L^{*} defined by (5.3.3) coincide.

We present below a theorem on linearlisation of L-statistics which is proved by combining the Moore technique and the representation of quantile processes with uniform bounds. The result appears to be quite neat and strong enough to cover most of the practical situations.

<u>Theorem 5.3.1</u>. Let $\{X_i\}$ be a stationary sequence of m-dependent r.v.s with the underlying d.f. F. Assume that F is continuous and for some real numbers $0 \le \alpha \le \beta \le 1$, F satisfies the conditions :

(i) F' exists and is bounded away from zero on $[Q_\alpha - \varepsilon, Q_\beta + \varepsilon]$, for some $\varepsilon > 0$.

(ii) F'' exists and is bounded on the interval $[Q_\alpha - \varepsilon, Q_\beta + \varepsilon]$, $\varepsilon > 0$.

Let W be a d.f. on $[0,1]$ such that w' exists on $(0, \alpha + \delta) \cup (\beta - \delta, 1)$ for some $\delta > 0$ and has bounded second derivative. Let L_n^* and L^* be as defined by (5.3.2) and (5.3.3),

$$Z_i = \int_0^1 ([t - I(F(X_i)] / F'(Q_t)) \, dW(t)$$

and

$$RL_n = L_n^* - L^* - n^{-1} \sum_{i=1}^{n} Z_i .$$

Then, if $E|X_1| < \infty$,

$$RL_n = 0(n^{-3/4} (\log n)^{1/2} (\log \log n)^{1/4}) \quad \text{a.s.}$$

<u>Proof.</u> Using Remark 4.2.1, a.s.,

$$(5.3.4) \quad \int_\alpha^\beta (Q_{nt}-Q_t) \, dW(t) = \int_\alpha^\beta \frac{t-F_n(Q_t)}{F'(Q_t)} \, dW(t)$$

$$+ 0(n^{-3/4}(\log n)^{1/2}(\log \log n)^{1/4}$$

Considering the integral $\int_0^\alpha (Q_{nt}-Q_t) \, w(t)dt$, we observe that, since w satisfies a Lipschitz condition of order 1 in $(0, \alpha + \delta)$ and $E|X_1| < \infty$,

$$\int_0^{\alpha} Q_{nt}\ w(t)dt = \int_0^{Q_\alpha} x\ w(F_n(x))dF_n(x) + \int_{Q_\alpha}^{Q_{n\alpha}} x\ w(F_n(x))dF_n(x) + 0(n^{-1})$$

a.s.

Now, following the proof of Theorem 5.2.1, we show that

(5.3.5) $$\int_0^{Q_\alpha} x\ w(F_n(x))dF_n(x) - \int_0^{Q_\alpha} x\ w(F(x))dx$$

$$= \int_0^{\alpha} Q_u\ w'(u)(E_n(u)-u)du + \int_0^{\alpha} Q_u\ w(u)\ d(E_n(u)-u)$$

$$+ \ 0(n^{-1}(\log n)^2) \quad \text{a.s.}$$

($E_n(u)$ is e.d.f. of $\{F(X_i)\}$) .

By integration by parts

(5.3.6) $$\text{r.h.s. of (5.3.5)} = \int_0^{\alpha} \frac{t - F_n(Q_t)}{F'(Q_t)}\ w(t)dt$$

$$+ \ w(Q_\alpha)\ Q_\alpha(F_n(Q_\alpha) - \alpha) + 0(n^{-1}(\log n)^2) \quad \text{a.s.}$$

Therefore, if we show that, a.s.,

(5.3.7) $$\int_{Q_\alpha}^{Q_{n\alpha}} x\ w(F_n(x))dF_n(x) + w(Q_\alpha)\ Q_\alpha(F_n(Q_\alpha)-\alpha) = 0(n^{-1}(\log n)^2),$$

it would then follow

$$(5.3.8)\quad \int_0^{\alpha} (Q_{nt}-Q_t)\, w(t)dt = \int_0^{\alpha} \frac{t-F_n(Q_t)}{F'(Q_t)}\, w(t)dt + O(n^{-1}(\log n)^2)$$

a.s.

To prove (5.3.7), we use Lemmas 5.2.1 and 5.2.2. The following statements are true up to the order $n^{-1}(\log n)^2$ a.s.

$$\text{L.h.s. of (5.3.7)} = \int_{Q_\alpha}^{Q_{n\alpha}} [x\, w(F_n(x)) - Q_\alpha\, w(F_n(Q_\alpha)]\, dF_n(x)$$

$$\leq \int_{Q_\alpha}^{Q_{n\alpha}} |x\, w(F_n(x)) - Q_\alpha\, w(F_n(x))| + Q_\alpha\, w(F_n(x)) - Q_\alpha\, w(F_n(Q_\alpha))|)dF_n(x)$$

$$<< \int_{Q_\alpha}^{Q_{n\alpha}} |x-Q_\alpha|\, dF_n(x) + \int_{Q_\alpha}^{Q_{n\alpha}} |w(F_n(x)) - w(F_n(Q_\alpha))|\, dF_n(x)$$

$$<< |Q_{n\alpha} - Q_\alpha|\, |F_n(Q_\alpha) - F_n(Q_{n\alpha})| + \int_{Q_\alpha}^{Q_{n\alpha}} |F_n(x) - F_n(Q_\alpha)|\, dF_n(x)$$

$$<< n^{-1} (\log n)^2 \text{ a.s.}$$

proving (5.3.7) and hence (5.3.8).

Similarly, one shows that

$$(5.3.9)\quad \int_\beta^1 (Q_{nt}-Q_t)\, w(t)dt = \int_\beta^1 \frac{t - F_n(Q_t)}{F'(Q_t)}\, w(t)dt + O(n^{-1}(\log n)$$

a.s. The statements (5.3.4), (5.3.8) and (5.3.9) yield the theorem.

Remark 5.3.1. As in Theorem 5.2.2, here also we need to estimate $|F_n(Q_{n\alpha}) - \alpha|$ and $|F_n(Q_{n\beta}) - \beta|$ to extend the result for mixing r.v.s. As mentioned there, one can do it following the proof of Theorem 2.2.2. As an example, the above theorem holds in the same form if $\{X_n\}$ is a sequence of ϕ-mixing r.v.s with $\Sigma\, \phi^{1/2}(i) < \infty$ and $E|X_1|^{3/2+\delta} < \infty$, $\delta > 0$. If $\phi(n) = 0(e^{-\theta n})$, $\theta > 0$, then the theorem holds as stated.

We shall prove one more theorem on the representation of L-statistics, which assumes more about the underlying d.f. and less about the weight functions. The proof uses the results of Chapter 3.

Theorem 5.3.2. Let $\{X_n\}$ be a stationary sequence of r.v.s with the underlying d.f. as F. We assume that F satisfies the condition (**) of section (4.4). Let W be some d.f. on $[0,1]$ such that $w(t) = W'(t)$ exists on $(0,\alpha) \cup (\beta,1)$ where $0 \leq \alpha \leq \beta \leq 1$. We define L_n^*, L^* and Z_i s as in the previous theorem. We then have :

(i) If $\{X_n\}$ is a m-dependent process, $E|X_1|^{c+\delta} < \infty$, $0 < c \leq 2$, $\delta > 0$ and $w(t) = 0((t(1-t))^{\frac{1}{c}-\frac{1}{2}})$ as $t(1-t) \longrightarrow 0$, then

(5.3.10) $\quad L_n^* - L^* = n^{-1} \sum_{i=1}^{n} Z_i + O(n^{-1/2-\gamma})$ a.s., for some $\gamma > 0$.

(ii) If $\{X_n\}$ is a sequence of ϕ - mixing r.v.s with $\phi(n) = O(e^{-\theta n})$, $\theta > 0$, then (i) part of the theorem holds as stated.

(iii) If $\{X_n\}$ is a sequence of ϕ-mixing r.v.s with $\phi(n) = O(n^{-\theta})$, $\theta \geq 2$ (if $\theta = 2$, assume further that $\Sigma\, \phi^{1/2}(i) < \infty$). If

$$E|X_1|^{c+\delta} < \infty, \quad 0 < c \leq (4\theta + 4)/(2\theta + 1), \ \delta > 0$$

and

$$w(t) = O((t(1-t))^{\frac{1}{c} - \frac{1}{2} - \frac{1}{4(\theta+1)}}) \quad \text{as} \quad t(1-t) \to 0,$$

then (5.3.10) holds.

(iv) If $\{X_n\}$ is a strong mixing process with $\alpha(n) = O(e^{-\theta n})$, $\theta > 0$, and $E|X_1|^{c+\delta} < \infty$, $0 < c < 8/3$, $\delta > 0$ and $w(t) = O(t(1-t))^{1/c - 3/8})$ as $t(1-t) \to 0$, then (5.3.10) holds.

<u>Proof</u>. The proof of part (i) is presented below. Since, the tools required have been developed for mixing r.v.s also (in Chapter 3), we omit the proofs of the other parts.

The representation of $\int_{\alpha}^{\beta} (Q_{nt} - Q_t)\, dW(t)$ is the same as in (5.3.4) due to condition (ii) in (**).

Considering the interval $(0,\alpha)$, we shall show that, for some $\gamma > 0$,

$$(5.3.11) \qquad \int_0^{n^{-1}(\log n)^2} (Q_{nt}-Q_t)\, w(t)\, dt = o(n^{-1/2-\gamma}) \text{ a.s.},$$

$$(5.3.12) \qquad \int_0^{n^{-1}(\log n)^2} \frac{t-F_n(Q_t)}{F'(Q_t)}\, w(t)dt = o(n^{-1/2-\gamma}) \text{ a.s.}$$

and

$$(5.3.13) \qquad \int_{n^{-1}(\log n)^2}^{\alpha} (Q_{nt}-Q_t)\, w(t)dt = \int_{n^{-1}(\log n)^2}^{\alpha} \frac{t-F_n(Q_t)}{F'(Q_t)}\, w(t)dt$$

$$+ \; o(n^{-1/2-\gamma}) \text{ a.s.}$$

One obtains similar results on the interval $(\beta,1)$ to establish (5.3.10).

To show (5.3.11), we note that $E|X_1|^{c+\delta} < \infty$ implies that $Q_t = o(t^{-1/c})$ as $t \to 0$ (see Lemma 4.4.1(iii)). Also,

$$Q_{nt} \ll |Q_{n(1/n)}| \ll |Q_{(n^{-1}(\log n)^{-2})}|$$

by Lemma 3.3.1. Thus, ℓ.h.s. of (5.3.11)

$$\ll |Q_{(n^{-1}(\log n)^{-2})}| \int_0^{n^{-1}(\log n)^2} t^{1/c-1/2+\delta/2}\,dt$$

$$+ \int_0^{n^{-1}(\log n)^2} t^{-1/2+\delta/2}\,dt$$

$$= O(n^{-1/2-\gamma}) \text{ a.s. for some } \gamma > 0 .$$

By Lemmas 3.3.1, 3.2.5 and 4.4.1(iii),

$$F_n(Q_{(n^{-1}(\log n)^2)}) = O(n^{-1}(\log n)^3) \quad \text{a.s.,}$$

$$F_n(Q_{(n^{-1}(\log n)^{-2})}) = 0$$

for all sufficiently large n, a.s. and

$$(F'(Q_t))^{-1} = O(t^{-1/c-1}) \quad \text{as } t \longrightarrow 0 .$$

These facts imply (5.3.12) immediately.

To prove (5.3.13), we mimic the proof of (4.4.6).

CHAPTER 6

SOME ASYMPTOTIC RESULTS ON QUANTILES

6.1 INTRODUCTION

Recently, Reiss (1974, 1976) studied the rate of convergence to normality of sample quantiles of i.i.d. observations. The first problem tackled in this chapter is to obtain some Berry-Esseen bounds for quantiles in case of weakly dependent r.v.s. In section 6.2, a result of Statulevičius (1971a) regarding the Berry-Esseen bounds for sample sum of ϕ-mixing observations is extended to quantiles. It may be mentioned that the technique of this proof is limited to only quantiles and it breaks down, when we have a linear combination of quantiles.

The next problem considered in this chapter is that of asymptotic effective variances (A E V) of quantiles. It was realised in Basu (1956) and Bahadur (1960) that the classical asymptotic variance of an estimator has a very weak relation with the actual probability of concentration around the true value. Bahadur (1960) introduces A E V which is a more justifiable measure of asymptotic dispersion as for as the probability of concentration is concerned. The concept is explained briefly below.

Let $\hat{\mu}_n$ be any statistic with the corresponding parametric value μ. Consider the root of the equation :- For some $\varepsilon > 0$,

(6.1.1) $$P(|N(0,1)| > \varepsilon/\lambda_n(\hat{\mu}_n, \varepsilon)) = P(|\hat{\mu}_n - \mu| > \varepsilon)$$

where N(0,1) denotes a r.v. with standard normal distribution. Suppose there is a sequence $\nu(n)$ of positive numbers such that

(6.1.2) $$\underline{\lim}_{\varepsilon \to 0} \ \underline{\lim}_{n \to \infty} \frac{\lambda_n^2(\hat{\mu}_n, \varepsilon)}{\nu(n)} = 1 = \overline{\lim}_{\varepsilon \to 0} \ \overline{\lim}_{n \to \infty} \frac{\lambda_n^2(\hat{\mu}_n, \varepsilon)}{\nu(n)},$$

then $\nu(n)$ is called an A E V of $\hat{\mu}_n$. In particular, if

(6.1.3) $$P(|\hat{\mu}_n - \mu| > \varepsilon) = \exp(-\frac{n\,\varepsilon^2}{2\,\nu}[1 + \delta(n,\varepsilon)]), \ \nu > 0,$$

where $\lim_{\varepsilon \to 0} \ \overline{\lim}_{n \to \infty} |\delta(n, \varepsilon)| = 0$, then the root of the equation (6.1.1) satisfies (6.1.2) with $\nu(n) = \nu/n$ (see (1.4) of Bahadur (1960)), so that $\{\nu/n\}$ can be taken as A E V of $\hat{\mu}_n$.

In section 3, we obtain A E V of sample quantiles of m-dependent observations. To establish the result, we first obtain A E V of sample mean of m-dependent observations and then extend it to quantiles. Once again, the techniques of section 6.3 break down for L-statistics. The problem will be taken up again in Chapter 7 with different tools which resemble those for asymptotic representation of quantiles.

In what follows, $\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-x^2/2)dx.$

6.2 BERRY-ESSEEN BOUNDS FOR QUANTILES OF MIXING OBSERVATIONS

We start with the theorem of Statulevičius which we shall use.

Theorem 6.2.1. (see Statulevičius (1977 b)). Let $\{Y_i\}$ be a stationary sequence of ϕ - mixing r.v.s with zero mean and $\Sigma\, \phi^{1/2}(i) < \infty$. Then, there exists a constant c depending only upon $\sum_{1}^{\infty} \phi^{1/2}(i)$ such that

$$(6.2.1) \quad \sup_{x \in \mathbb{R}} |P(S_n \leq x\, \sigma(S_n)) - \Phi(x)| \leq c\, M_{3n} \log (1 + M_{3n}^{-1})$$

where $S_n = \sum_{i=1}^{n} Y_i$, $\sigma^2(S_n) = V(S_n)$ (assumed to be non-zero)

and $M_{3n} = \sum_{i=1}^{n} E|Y_i|^3/\sigma^3(S_n)$.

This theorem is also proved in Stalulevičius (1977a) under the stronger condition that ϕ is exponentially decaying.

Now, we state the Berry-Esseen bounds for quantiles which will be established in this section.

Theorem 6.2.2. Let $\{X_n\}$ be a stationary sequence of ϕ-mixing r.v.s with $\phi(n) = O(n^{-3})$ and the corresponding d.f. F. Let Q_{nt} denote the t^{th} sample quantile. We then have :

(i) At some real x_o, let F satisfy the condition

$$(6.2.2) \qquad \frac{F(x_o+\varepsilon)-F(x_o)}{\varepsilon^{\delta_1}} = d_1(1 + O(\varepsilon^{\delta_2})) \quad \text{as } \varepsilon \downarrow 0$$

where d_1, δ_1 and δ_2 are positive constants, $t = F(x_o)$ and

$$(6.2.3) \qquad 0 < \sigma_t^2 = V(I(F(X_1) \leq t) + 2 \sum_{i=1}^{\infty} \text{cov}(I(F(X_1) \leq t), I(F(X_{1+i}) \leq t).$$

If F is continuous in $(x_o, x_o+\varepsilon)$ for some $\varepsilon > 0$, then

$$(6.2.4) \qquad \sup_{x \geq 0} |P(\sqrt{n}\, d_1((Q_{nt}-x_o)^+)^{\delta_1} \leq x\,\sigma_t) - \Phi(x)| << n^{-1/2} \log n + n^{-\delta_2/2\delta_1} (\log n)^{1/2+\delta_2/2\delta_1}.$$

(ii) At some real x_o, let F satisfy

$$\frac{F(x_o-\varepsilon) - F(x_o)}{\varepsilon^{\delta_3}} = - d_2 \, (1 + 0(\varepsilon^{\delta_4})) \quad \text{as} \quad \varepsilon \downarrow 0$$

where d_2, δ_3 and δ_4 are positive constants and (6.2.3) holds. If F is continuous in $(x_o - \varepsilon, x_o)$, $\varepsilon > 0$, then

$$\sup_{x \leq 0} |P(-\sqrt{n}\, d_2 \, ((Q_{nt} - x_o)^-)^{\delta_3} \leq \leq x \, \sigma_t) - \Phi(x)|$$

$$< < n^{-1/2} \log n + n^{-\delta_4/2\delta_3} (\log n)^{1/2 + \delta_4/2\delta_3}$$

(iii) Suppose that at some $x_o \in \mathbb{R}$,

$$\frac{F(x_o + \varepsilon) - F(x_o)}{\varepsilon^{\delta_1}} = d_1(1 + 0(\varepsilon^{\delta_2})) \quad \text{as} \quad \varepsilon \downarrow 0$$

and

$$\frac{F(x_o - \varepsilon) - F(x_o)}{\varepsilon^{\delta_3}} - d_2 \, (1 + 0(\varepsilon^{\delta_4})) \quad \text{as} \quad \varepsilon \downarrow 0$$

where δ_1, δ_2, δ_3, δ_4, d_1 and d_2 are positive constants. If (6.2.3) holds with $t = F(x)$ and F is continuous in $(x_o - \varepsilon, x_o + \varepsilon)$, $\varepsilon > 0$, then

$$\sup_{x \geq 0} |P(\sqrt{n}\, d_1((Q_{nt}-x_o)^+)^{\delta_1} \leq x\,\sigma_t) - \Phi(x)|$$

$$+ \sup_{x \leq 0} |P(-\sqrt{n}((Q_{nt}-x_o)^-)^{\delta_3} \leq x\,\sigma_t) - \Phi(x)|$$

$$<< n^{-1/2}(\log n) + n^{-\delta_2/2\delta_1}(\log n)^{1/2+\delta_2/2\delta_1}$$

$$+ n^{-\delta_4/2\delta_3}(\log n)^{1/2+\delta_4/2\delta_3}$$

Theorem 6.2.2 is derived from the following

Lemma 6.2.1. Let $\{U_i\}$ be a stationary sequence of ϕ-mixing r.v.s with $\phi(n) = O(n^{-3})$ and $P(U_1 \leq s) = s$ for $s \in [t,t']$, $0 < t < t' < 1$. $E_n(s)$ denotes e.d.f. of $\{U_i\}$. Define

$$\sigma^2(s) = V(I(U_1 \leq s)) + 2\sum_{i=1}^{\infty} \text{cov}(I(U_1 \leq s), I(U_{1+i} \leq s)).$$

If $\sigma(t) > 0$, then

$$(6.2.5) \qquad \sup_{x \geq 0} |P(\sqrt{n}\,(E_n^{-1}(t)-t)^+ \leq x\,\sigma(t)) - \Phi(x)| << n^{-1/2} \log n$$

and if $\sigma(t') > 0$, then

$$(6.2.6) \quad \sup_{x \leq 0} |P(-\sqrt{n}\,(E_n^{-1}(t')-t')^{-} \leq x\,\sigma(t')) - \Phi(x))$$

$$<< n^{-1/2} \log n .$$

Proof of the lemma. The proof of (6.2.5) is given below and (6.2.6) can be proved similarly.

Step 1. For any constant $\gamma > 0$

$$P(\sqrt{n}\,(E_n^{-1}(t) - t)^{+} > \gamma\,\sigma(t)(\log n)^{1/2})$$

$$= P(E_n^{-1}(t) > t + \gamma\,\sigma(t)\,(\log n)^{1/2}\,n^{-1/2})$$

$$\leq P(t \geq E_n(t + \gamma\,\sigma(t)\,(\log n)^{1/2}\,n^{-1/2})$$

$$\leq P(-n\,E_n(t+ \gamma\,\sigma(t)\,(\log n)^{1/2}\,n^{-1/2}) + t$$

$$+ \gamma\,\sigma(t)(\log n)^{1/2}\,n^{-1/2} > \gamma\,n^{1/2}\sigma(t)(\log n)^{1/2})$$

These inequalities along with Lemma 2.2.3 (see also Remark 2.2.1) ensure the existence of a positive constant K such that

$$P(\sqrt{n}\,(E_n^{-1}(t)-t)^{+} > K\,\sigma(t)\,(\log n)^{1/2}) = O(n^{-1/2})$$

and

$$\Phi\,(K\,(\log n)^{1/2}) = O(n^{-1/2})$$

Therefore, it is enough to consider $x \in [0, K(\log n)^{1/2}]$ only in (6.2.5).

Step 2

Since $E_n(s) > t \Rightarrow s \geq E_n^{-1}(t)$ and $E_n^{-1}(t) \leq s \Longrightarrow t \leq E_n(s)$, it follows that

$$P(-\sqrt{n}\,(E_n(t + \frac{x\,\sigma(t)}{\sqrt{n}})) + t + \frac{x\,\sigma(t)}{\sqrt{n}} < x\,\sigma(t))$$

$$\leq P(\sqrt{n}\,(E_n^{-1}(t) - t) \leq x\,\sigma(t))$$

$$\leq P(-\sqrt{n}\,(E_n(t + \frac{x\,\sigma(t)}{\sqrt{n}})) + t + \frac{x\,\sigma(t)}{\sqrt{n}} \leq x\,\sigma(t)).$$

Also,

$$\text{l.h.s. of (6.2.7)} \geq P(-\sqrt{n}\,(E_n(t+\sigma(t)x/\sqrt{n}) + t+\sigma(t)x/\sqrt{n}) \leq x\,\sigma(t) - n^{-1/2}).$$

As a consequence, it is easy to conclude (6.2.5) if we show that

$$\text{(6.2.8)} \qquad \sup_{0 \leq x \leq 2K_1(\log n)^{1/2}} |P(-\sqrt{n}(E_n(t+\frac{x}{\sqrt{n}})+t+\frac{x}{\sqrt{n}}) \leq x\,\sigma(t)) - \Phi(x)|$$

$$= O(n^{-1/2} \log n)$$

where $K_1 = K\,\sigma(t)$.

<u>Step 3</u>. Define $U_{in} = U_{in}(x) = I(U_i \le t + \frac{x}{\sqrt{n}}) - t - \frac{x}{\sqrt{n}}$,

$0 \le x \le 3K_1(\log n)^{1/2}$. We claim that

$$(6.2.9) \quad |V(\sum_{i=1}^{n} U_{in}) - n\,\sigma^2(t)| = O(n^{1/2}\,(\log n)^{1/2})$$

where the constant involved in O term is free of x in $[0,\ 3K_1(\log n)^{1/2}]$.

The claim is verified as follows :

Since ,

$$n \sum_{i=n}^{\infty} \operatorname{cov}(I(U_1 \le t),\ I(U_{1+i} \le t)) \ll n \sum_{i=n}^{\infty} \phi^{1/2}(i) = O(n^{-1/2})$$

(if $\phi(n) = O(n^{-3})$), we have

$$n\,\sigma^2(t) = n[V(I(U_1 \le t)) + 2 \sum_{i=1}^{n-1} \operatorname{cov}(I(U_1 \le t), I(U_{1+i} \le t)) + O(n^{1/2}).$$

Also, since

$$\sum_{i=1}^{n} i \quad \operatorname{cov}(U_{1n}, U_{(1+i)n}) \ll \sum_{i=1}^{n} i\ \phi^{1/2}(i) = O(n^{1/2}) ,$$

it follows that

$$V(\sum_{i=1}^{n} U_{in}) = n[V(U_{1n}) + 2 \sum_{i=1}^{n-1} \operatorname{cov}(U_{1n}, U_{(1+i)n})] + O(n^{1/2}),$$

Now, to see (5.2.9), we show the following two inequalities.

(6.2.10) $n|V(U_{1n})-V(I(U_1\leq t)| = O(n^{1/2}(\log n)^{1/2})$

and

(6.2.11) $|\sum_{i=1}^{n-1} \mathrm{cov}(U_{1n}, U_{(1+i)n}) - \sum_{i=1}^{n} \mathrm{cov}(I(U_1\leq t), I(U_{1+i}\leq t))|$

$$= O(n^{-1/2}).$$

Proof of (6.2.10) is trivial. We rewrite the ℓ.h.s. of (6.2.11) as

(6.2.12) $|\sum_{i=1}^{n} E[U_{1n}(U_{(1+i)n} - I(U_{(1+i)} \leq t) + t)]$

$$+ E[(I(U_{(1+i)} \leq t) - t)(U_{1n} - I(U_1 \leq t) + t)]|$$

$$<< \sum_{i=1}^{n} n^{-1/2} (\log n)^{1/2} \phi(n) = O(n^{-1/2} (\log n)^{1/2}).$$

using the second part of Lemma 2.2.1. This completes the proof of (6.2.9).

<u>Step 4</u>. From Theorem 6.2.1, for any $x \in [0, 2K_1(\log n)^{1/2}]$,

$$(6.2.13) \quad \sup_{y \in \mathbb{R}} \left| P\left(-n\left(E_n\left(t+\frac{x}{\sqrt{n}}\right)+t+\frac{x}{\sqrt{n}}\right) \le y\left(V\left(\sum_{i=1}^{n} U_{in}(x)\right)\right)^{1/2}\right) - \Phi(y)\right|$$

$$\le c(\log n)\left(\sum_{i=1}^{n} E|U_{in}(x)|^3\right)/\left(V\left(\sum_{i=1}^{n} U_{in}(x)\right)\right)^{3/2}$$

$$\le K_2\, n^{-1/2} \log n$$

using the results of the step 3. But, since the r.h.s. of (6.2.13) is free from x, (6.2.13) also says that

$$(6.2.14) \quad \sup_{0 \le x \le 3K_1(\log n)^{1/2}} \left| P\left(-n\left(E_n\left(t+\frac{x}{\sqrt{n}}\right)+t+\frac{x}{\sqrt{n}}\right) \le x\sqrt{V\left(\sum_{i=1}^{n} U_{in}(x)\right)}\right) - \Phi(x)\right| \le K_2\, n^{-1/2} \log n\,.$$

Now, since

$$\frac{\left[V\left(\sum_{i=1}^{n} U_{in}(x)\right)\right]^{1/2}}{n^{1/2}\,\sigma(t)} = 1 + O\left(n^{-1/2}(\log n)^{1/2}\right)$$

and

$$\sup_{x \in \mathbb{R}} \left|\Phi(x) - \Phi\left(x\left(1+O\left(n^{-1/2}(\log n)^{1/2}\right)\right)\right)\right| = O\left(n^{-1/2} \log n\right)$$

the proof of this lemma is complete.

<u>Proof of Theorem 6.2.2</u>. We prove part (i) using (6.2.5). Part (ii) follows similarly from (6.2.6). Part (iii) is obtained by combining (i) and (ii).

Let $U_i = F(X_i)$. If the conditions of the part (i) of the theorem hold, then there exists a $t' > t$ such that $P(U_i \leq s) = s$ for $s \in [t,t')$, $\forall\, i \geq 1$. With this definition of the $\{U_i\}$, σ_t^2 defined by (6.2.3) is the same as $\sigma^2(t)$ defined in Lemma 6.2.1.

Now, (6.2.2) implies that

(6.2.15) $$(E_n^{-1}(t) - t)^+ = (F(Q_{nt}) - F(x_o))^+$$

$$= d_1((Q_{nt}-x_o)^+)^{\delta_1} + O(((Q_{nt} - x_o)^+)^{\delta_1+\delta_2}).$$

Consequently, there exist positive δ_5 and K_2 such that if $(E_n^{-1}(t) - t)^+ \leq \delta_5$, then

$$(E_n^{-1}(t)-t)^+ \geq K_2((Q_{nt} - x_o)^+)^{\delta_1}$$

$$\Rightarrow ((Q_{nt}-x_o)^+)^{\delta_1+\delta_2} \leq K_2((E_n^{-1}(t)-t)^+)^{(\delta_1+\delta_2)/\delta_1}$$

(6.2.16) $$\Longrightarrow |(E_n^{-1}(t)-t)^+ - d_1((Q_{nt}-x_o)^+)^{\delta_1}| \leq K_3((E_n^{-1}(t)-t)^+)^{(\delta_1+\delta_2)/\delta_1}$$

At this stage, we make use of the simple fact that, for any two r.v.s X and Y and $\theta > 0$,

$$(6.2.17) \quad \sup_{x \in \mathbb{R}} |P(X+Y \leq x) - \Phi(x)| \leq \sup_{x \in \mathbb{R}} |P(X \leq x) - \Phi(x)|$$

$$+ P(|Y|) > \theta) + \theta.(2\pi)^{-1/2} .$$

Applying (6.2.17) with

$$X = \sigma_t^{-1} \sqrt{n}\, d_1((Q_{nt} - x_o)^+)^{\delta_1}$$

$$Y = \sqrt{n}\, [(E_n^{-1}(t)-t)^+ - d_1((Q_{nt} - x_o)^+)^{\delta_1}]$$

and

$$\theta = K_4\, n^{-\delta_2/2\delta_1} (\log n)^{1/2 + \delta_2/2\delta_1}$$

and using (6.2.16), we see that

$$\ell.h.s. \text{ of } (6.2.4) \ll n^{-1/2}(\log n) + P(E_n^{-1}(t) - t > \delta_5)$$

$$+ P(E_n^{-1}(t)-t > (K_4/K_3)^{\delta_1/(\delta_1+\delta_2)} n^{-1/2}(\log n)^{1/2})$$

$$+ n^{-\delta_2/2\delta_1} (\log n)^{1/2 + \delta_2/2\delta_1}$$

$$\ll n^{-1/2} \log n + n^{-\delta_2/2\delta_1}(\log n)^{1/2 + \delta_2/2\delta_1}$$

as in the proof of the Step 1 of Lemma 6.2.1. Hence the theorem.

Remark 6.2.1. Making use of Theorem 2 of Statulevicius (1977 b), it can be shown that the above theorem holds in the strong mixing case if the $\{\alpha(i)\}$ is exponentially decaying. Similarly, the results of Stein (1972) can be used to show that in the m-dependent regular cases ($\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$) the rate of convergence to normality of sample quantiles is $n^{-1/2}$.

6.3 A E V OF QUANTILES OF m-DEPENDENT OBSERVATIONS

Our main aim in this section is to establish the following

Theorem 6.3.1. Let $\{X_n\}$ be a stationary sequence of m-dependent r.v.s with the underlying d.f. F. Let x_o be a point such that

(6.3.1)
$$\frac{F(x_o+\varepsilon) - F(x_o)}{\varepsilon} = d_1(1 + O(\varepsilon)) \text{ as } \varepsilon \downarrow 0$$

and

$$\frac{F(x_o-\varepsilon) - F(x_o)}{\varepsilon} = - d_2(1 + O(\varepsilon)) \text{ as } \varepsilon \downarrow 0$$

where d_1, d_2 are positive constants. Let $t = F(x_o)$,

$d_o = \min(d_1^2, d_2^2)$ and

$$0 < \sigma_t^2 = V(I(X_1 \le t)) + 2\sum_{i=1}^{m-1} \operatorname{cov}(I(X_1 \le t), I(X_{1+i} \le t))$$

if Q_{nt} denotes the t^{th} sample quantile, then

$$(6.3.2) \qquad P(|Q_{nt}-x_o| \geq \varepsilon) = \exp(-\frac{n\varepsilon^2 d_o}{2\sigma_t^2})\ [1+\delta_1(n,\varepsilon)]$$

where $\lim\limits_{\varepsilon \to 0} \ \overline{\lim\limits_{n \to \infty}} \ |\delta_1(n,\varepsilon)| = 0,$

and hence, A E V of Q_{nt} as an estimator of x can be taken as $\{\sigma_t^2/d_o\, n\}$.

To establish this theorem, we first obtain a similar result for sample mean of m-dependent observations in Lemma 6.3.1 (see below). It appears that the conclusion of the lemma cannot be easily derived from the known results on the probabilities of deviations for dependent r.v. s (see e.g. Statulevicius (1966, 1974)). In any case, the proof supplied here is self-contained, neat and interesting.

<u>Lemma 6.3.1</u>. Let $\{Y_i\}$ be a stationary sequence of m-dependent r.v.s. Let

$$v^2 = V(Y_1) + 2\sum_{i=1}^{m-1} \operatorname{cov}(Y_1, Y_{1+i}) > 0$$

and $\psi_t = E(\exp(t\,Y_1))$ exist for all $|t| \leq \theta$, $\theta > 0$. Then, for $\varepsilon > 0$,

$$(6.3.3) \qquad P(|n^{-1}\sum_{i=1}^{n} Y_i - E(Y_1)| > \varepsilon) = \exp(-\frac{n\varepsilon^2}{2v^2}[1+\delta_2(n,\varepsilon)])$$

where $\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} |\varepsilon_2(n,\varepsilon)| = 0$,

and hence, A E V of $n^{-1} \sum_1^n Y_i$ can be taken to be $\{v^2/n\}$.

Proof of Lemma 6.3.1. Without loss of generality, let $E\,Y_1 = 0$ and $v^2 = 1$. We show below that $P(\sum_1^n Y_i \geq n\,\varepsilon)$ can be expressed as the r.h.s. of (6.3.3) and a similar expression holds for $P(-\sum_{i=1}^n Y_i \geq \varepsilon)$ to complete the proof of (6.3.3). Since the proof is somewhat long we break it into several steps.

Step 1. If $\{Y_1, Y_2, \ldots, Y_r\}$ are identicall distributed r.v.s and $E(\exp(t\,Y_1))$ exists for $|t| \leq \theta$, $\theta > 0$, then, it follows by repeated application of Holder's inequality that (irrespective of the dependence structure)

$$E(\exp(t \sum_{i=1}^r Y_i)) < \infty$$

for $|t| \leq \theta / r$.

Step 2. Let $\varepsilon > 0$ be small enough so that $\varepsilon^{-1/2} \geq m$. Taking $p = [\varepsilon^{-1/2}]$ and $k = [n/(p+m)]$, we break the sum $\sum_1^n Y_i$ as follows ;

$$\sum_{i=1}^{n} Y_i = \sum_{j=1}^{k+1} \xi(j) + \sum_{j=1}^{k} \eta(j)$$

where

$$\xi(j) = \sum_{i=1}^{p} Y_{(p+m)(j-1)+i}, \quad \eta(j) = \sum_{i=1}^{m} Y_{(p+m)(j-1)+p+i}$$

for $j = 1, 2, \ldots, k$ and $\xi(k+1) = \sum_{i=(p+m)k+1}^{n} Y_i$ or 0

according as $n - (p+m)k > 0$ or not.

Step 3. Here, we show that $P(|\sum_{j=1}^{k} \eta(j)| > n\varepsilon^{9/8})$ is negligibly small compared to the r.h.s. of (6.3.3).

Since $\eta(1), \eta(2), \ldots$ is a sequence of i.i.d. r.v.s (m is fixed) we can apply Lemma 2.4 of Bahadur (1960). W. λ g. assume that $V(\eta(1)) > 0$ (otherwise, our effort for this step is trivial). Appealing to the results known for the independent case $(E \exp(\theta\, \eta(1)/m)) < \infty)$

$$(6.3.4) \quad P(|\sum_{i=1}^{k} \eta(j)| > n\varepsilon^{9/8}) = P(|\sum_{i=1}^{k} \eta(j)| > k(n/k)\varepsilon^{9/8})$$

$$= \exp\left(-\frac{k\,(n/k)^2\,\varepsilon^{9/4}}{2\,V(\eta(1))}\right)[1 + \delta_3(n,\varepsilon)]$$

where $\lim_{\varepsilon \to 0} \overline{\lim}_{n \to \infty} |\delta_3(n,\varepsilon)| = 0$. Substituting the value of k, we get

(6.3.5) $\quad$ ℓ.h.s. of (6.3.4) $= \exp(-\frac{n\,\varepsilon^{7/4}}{2\,V(\eta(1))}\,[1 + \delta_4(n,\varepsilon)]$

where $\lim_{\varepsilon \to 0} \overline{\lim}_{n \to \infty} |\delta_4(n,\varepsilon)| = 0$. The stated assertion follows immediately.

<u>Step 4</u>. We now estimate $P(|\xi(k+1)| > n\,\varepsilon^{9/8})$. By stationarity,

(6.3.6)
$$P(|\xi(k+1)| > n\,\varepsilon^{9/8}) \leq P(\sum_{i=1}^{p} |Y_i| > n\,\varepsilon^{9/8})$$

$$\leq m\,P(\sum_{i=1}^{[p/m]+1} |Y_{mi}| > m^{-1}\,n\,\varepsilon^{9/8})$$

$$\leq m\,\exp(-\theta\,m^{-1}\,n\,\varepsilon^{9/8})\,[E(\exp(\theta|Y_1|))]^{[p/m]+1}$$

$$\leq m\,\exp(-\theta\,m^{-1}\,n\,\varepsilon^{9/8} + 2\,\varepsilon^{-1/2}\log(\tilde{\psi}_\theta + 1))$$

$$= m\,\exp(-\theta\,m^{-1}\,n\,\varepsilon^{9/8}\,(1 + 2\,m\,\theta^{-1}\,\varepsilon^{-13/8}\,n^{-1}\log(\tilde{\psi}_\theta + 1))).$$

where $\tilde{\psi}_\theta = E(\exp(\theta|Y_1|))$. As a consequence, the ℓ.h.s.(6.3.6) is negligibly small compared to the r.h.s. (6.3.3).

<u>Step 5</u>. Let us write $\varepsilon_o = (\varepsilon \pm 2\,\varepsilon^{9/8})$. We now show that $P(\sum_{j=1}^{k} \xi(j) \geq n\,\varepsilon_o]$ can be expressed as the r.h.s. of (6.3.3). This result, combined with the conclusions of the previous two steps, completes the proof.

Although the $\xi(i)$ s are independent, the zone in which $\xi(1)$ has m.g.f. depends on ε. Consequently, the corresponding result for i.i.d.r.v.s cannot be used directly. We carry out the the proof as follows:

Let us write

(6.3.7) $$P(\sum_{1}^{k} \xi(j) > n\,\varepsilon_o) = P(\sum_{1}^{k} \xi'(j) > k\,\varepsilon_1)$$

where $\xi'(j) = \xi(j)/\varepsilon^{-1/4}$, $\varepsilon_1 = (n/k)\,\varepsilon^{1/4}\,\varepsilon_o = \varepsilon^{3/4}(1+0(\varepsilon^{1/8}))$.

For $|s| \leq \frac{\theta\,\varepsilon^{1/4}}{4}$, we define

$$\Psi_s^* = E(\exp(s\,\xi'(1))$$

and

$$M_s = (\Psi_s^*)^{-1} \int_{-\infty}^{\infty} x \exp(s\,x)\, d(P(\xi'(1) \leq x)).$$

By an appeal to the result mentioned in step 1, it follows that Ψ_s^* and M_s are well-defined in the region $|s| \leq \frac{\theta\,\varepsilon^{1/4}}{4}$.

Now, we claim that the equation $M_{\bar{\varepsilon}} = \varepsilon_1$ has a solution and that any solution has to satisfy

(6.3.8) $$\bar{\varepsilon} = \varepsilon_1 + O(\varepsilon_1^{5/4}).$$

To see (6.3.8), we first obtain some useful estimates of ψ_s^* and M_s

$$|\psi_s^* - 1 - \frac{s^2}{2} E(\xi'(1))^2| \leq \frac{s^3}{6} E(|\xi_1'(1)|^3 \exp(s|\xi'(1)|))$$

$$\leq \frac{s^3}{6} [(E(|\xi_1'(1)|^6) \quad [E(\exp(2s\, \varepsilon^{1/4} \sum_{i=1}^{p} |Y_1|))]]^{1/2}$$

$$= \frac{s^3}{6} [E(|\xi_1'(1)|^6) \, E(\exp(2\, s\, \varepsilon^{1/4}\, p|Y_1|))]^{1/2}$$

(by repeated applications of Holder's inequality)

The last step follows using Ibragimov's moment inequality mentioned in the proof of Lemma 2.2.2 and the fact that $|2\, s\, \varepsilon^{1/4}\, p| \leq \theta$.

Also,

$$E(\xi'(1))^2 = \varepsilon^{1/2} [\sum_{i=1}^{p} V(Y_1) + 2 \sum_{i=1}^{m-1} (p-i) \operatorname{cov}(Y_1, Y_{1+i})]$$

$$= \varepsilon^{1/2} p[V(Y_1) + 2 \sum_{i=1}^{m-1} (1-i/p) \operatorname{cov}(Y_1, Y_{1+i})]$$

$$= 1 + O(\varepsilon^{1/2}) ,$$

so that

$$|\psi_s^* - 1 - \frac{s^2}{2}| \leq K_6 s^2 \varepsilon^{1/4}.$$

By similar expansions, one finds that

(6.3.9) $$|M_s - s| \leq K_7 s \varepsilon^{1/4}$$

Obviously, (6.3.9) implies that

(6.3.10) $$\begin{cases} M_s \geq s - K_7 s \varepsilon^{1/4} = s[1 - K_7 \varepsilon^{1/4}] \\ \text{and} \quad M_s(1 + K_7 \varepsilon^{1/4})^{-1} \leq s \leq M_s(1 - K_7 \varepsilon^{1/4})^{-1} . \end{cases}$$

Evidently, if $s > 2\varepsilon_1(1 - K_7 \varepsilon^{1/4})^{-1}$, then $M_s > \varepsilon_1$ (start with sufficiently small ε). Also, it follows by the dominated covergence theorem that M_s is continuous in $[0, \frac{\theta \varepsilon^{1/4}}{4}]$ and right continuous at 0 (Clearly, $M_o = 0$). In view of these facts, it follows by intermediate value theorem that $M_{\bar{\varepsilon}} = \varepsilon_1$ has a solution.

If $\varepsilon \leq s \leq \frac{\theta \varepsilon^{1/4}}{4}$, (6.3.9) implies that

(6.3.11) $$M_s = s + O(s^{5/4}).$$

The inequalities in (6.3.10) ensure that, if ε is sufficiently small, then any solution to the equation $M_{\bar{\varepsilon}} = \varepsilon_1$ has to lie in $[\varepsilon, \frac{\theta \varepsilon^{1/4}}{4}]$. Therefore, (6.3.11) yields that

$$\varepsilon_1 = \bar{\varepsilon} + O(\bar{\varepsilon}^{5/4})$$

$$\Longrightarrow \quad \bar{\varepsilon} = \varepsilon_1 + O(\varepsilon_1^{5/4})$$

and hence we have the claim (6.3.8).

Now, define the d.f.

$$d\, F_n(x, \bar{\varepsilon}) = (\psi^*_{\bar{\varepsilon}})^{-k} \exp(\bar{\varepsilon}\, x)\, d(P(\sum_1^k \xi'(j) \leq x))$$

and

$$\sigma^2_{\bar{\varepsilon}} = (\psi^*_{\bar{\varepsilon}})^{-1} \int_{-\infty}^{\infty} x^2 \exp(\bar{\varepsilon}\, x)\, d(P(\xi'(1) \leq x) - M^2_{\bar{\varepsilon}} \; .$$

Then,

(6.3.12) $$\text{the r.h.s. of (6.3.7)} = \int_{k\,\varepsilon_1}^{\infty} (\psi^*_{\bar{\varepsilon}})^k \exp(-\bar{\varepsilon}\, x)\, d\, F_n(x, \bar{\varepsilon})$$

$$= A_{\bar{\varepsilon}} \int_0^{\infty} \exp(-D_{\bar{\varepsilon}}\, z)\, d\, F_n(\sigma_{\bar{\varepsilon}} \sqrt{k}\, z + k\, M_{\bar{\varepsilon}}, \bar{\varepsilon})$$

where

$$A_{\bar{\varepsilon}} = (\psi^*_{\bar{\varepsilon}})^k \exp(-\bar{\varepsilon}\, k\, \varepsilon_1) \qquad (\varepsilon_1 = M_{\bar{\varepsilon}})$$

$$D_{\bar{\varepsilon}} = \sigma_{\bar{\varepsilon}}\, \bar{\varepsilon} \sqrt{k} \; .$$

Using similar kind of expansions, as done above, we see that

$$\sigma^2_{\bar{\varepsilon}} = 1 + O(\bar{\varepsilon})$$

$$A_{\bar{\varepsilon}} = \exp\left(-k\left(\frac{\bar{\varepsilon}^2}{2} + O(\bar{\varepsilon}^{5/2})\right)\right)$$

$$D_{\bar{\varepsilon}} = \sqrt{k}\,(\bar{\varepsilon} + O(\bar{\varepsilon}^2)).$$

Further, since $F_n(\sigma_{\bar{\varepsilon}} \sqrt{k}\, z + k\, \varepsilon_1, \bar{\varepsilon})$ is d.f. of the normalised sum of k independent r.v.s, Katz's theorem (see Katz (1963)) implies that

$$\sup_{z \in \mathbb{R}} \left| F_n(\sigma_{\bar{\varepsilon}} \sqrt{k}\, z + k\, M_{\bar{\varepsilon}}, \bar{\varepsilon}) - \Phi(z) \right| \leq K_8\, k^{-1/2}.$$

Now, we are ready to estimate the desired probability :

$$\text{r.h.s. of } (6.3.12) = A_{\bar{\varepsilon}} \int_0^\infty \exp(-D_{\bar{\varepsilon}}\, z)\, d\,\Phi(z)$$

$$+ A_{\bar{\varepsilon}} \int_0^\infty \exp(-D_{\bar{\varepsilon}}\, z)\, d(F_n(\sigma_{\bar{\varepsilon}} \sqrt{k}\, z\, \varepsilon_1, \bar{\varepsilon}) - \Phi(z))$$

$$= A_{\bar{\varepsilon}} \exp(D^2_{\bar{\varepsilon}}/2)\,[1 - \Phi(D_{\bar{\varepsilon}})] + A_{\bar{\varepsilon}}\, O(k^{-1/2})$$

(using the integration by part)

= the expression desired in the r.h.s. of (6.3.3) by substituting the estimates of $A_{\bar{\varepsilon}}$, $D_{\bar{\varepsilon}}$ and $\bar{\varepsilon}$ (given by 6.3.8).

An Open Problem. It is not known to the author whether or not the content of the above lemma can be extended to mixing r.v.s even with exponentially decaying mixing coefficients.

Lemma 6.3.2. Let $\{U_i\}$ be a stationary sequence of m-dependent r.v.s such that

(6.3.13) $$P(U_1 \leq s) = s \quad \text{for} \quad s \in [t - \gamma, t + \gamma]$$

for some $\gamma > 0$ and

$$0 < \sigma^2(t) = V(I(U_1 \leq t)) + 2 \sum_{i=1}^{m-1} \text{cov}(I(U_1 \leq t), I(U_{1+i} \leq t)).$$

If $E_n(t)$ denotes the e.d.f. at the n^{th} stage, then

(6.3.14) $$P(E_n^{-1}(t) - t > \varepsilon) = \exp(-\frac{n\varepsilon^2}{2\sigma^2(t)}(1 + \delta_5(n, \varepsilon))$$

where $\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} |\delta_5(n, \varepsilon)| = 0$ and

(6.3.15) $$P(t - E_n^{-1}(t) > \varepsilon) = \exp(-\frac{n\varepsilon^2}{2\sigma^2(t)}(1 + \delta_6(n, \varepsilon))$$

where $\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} |\delta_6(n, \varepsilon)| = 0.$

Proof of the Lemma. By the usual arguments,

(6.3.16) $$P(-E_n(t+\varepsilon)+t+\varepsilon \geq \varepsilon+\varepsilon^2)$$

$$\leq P(E_n^{-1}(t)-t>\varepsilon) \leq P(-E_n(t+\varepsilon)+t+\varepsilon \geq \varepsilon).$$

Let us take $0<\varepsilon<\gamma$ and define

$$x_i(\alpha, \beta) = I(\alpha \leq U_i \leq \beta) - \beta + \alpha$$

where $0 \leq \alpha \leq 1$ and $\beta \geq \alpha$. Clearly,

(6.3.17) $$\text{r.h.s. of (6.3.16)} = P(-E_n(t)+t \geq (\varepsilon \pm \varepsilon^{5/4})$$

$$+ O(P(|\sum_{i=1}^{n} x_i(t, t+\varepsilon)| > n\,\varepsilon^{5/4})).$$

Our theorem of the previous section guarantees that

$$P(-E_n(t)+t \geq (\varepsilon \pm \varepsilon^{5/4})$$

can be expressed as the r.h.s. (6.3.14). Next, we claim that

$$P(|\sum_{i=1}^{n} x_i(t, t+\varepsilon)| > n\,\varepsilon^{5/4})$$

is negligibly small compared to the r.h.s. (6.3.14). We see this as follows :

$$P\left(\left|\sum_{i=1}^{n} x_i(t,t+\varepsilon)\right| > n\,\varepsilon^{5/4}\right)$$

$$= P\left(\left|\sum_{i=1}^{n} x_i(t,t+\varepsilon)\right| > (n\,\varepsilon^{3/4})\,\varepsilon^{1/2}\right) .$$

We use the estimate provided by Lemma 3.2.2 to see that this probability

$$\leq \exp(-K_8\, n\, \varepsilon^{3/2})$$

and hence, it is negligible. Similarly, we see that the ℓ.h.s. of (6.3.16) can be expressed as the r.h.s. of (6.3.14) to complete the proof of (6.3.14).

(6.3.15) is proved similarly.

<u>Proof of Theorem 6.3.1.</u> Define $U_i = F(X_i)$ so that (6.3.13) holds. With the definition of U_i's, σ_t^2 defined in the statement of this theorem is the same as $\sigma^2(t)$ defined in Lemma 6.3.2.

Let us prove using the first condition of (6.3.1) and (6.3.14) that

(6.3.18) $$P(Q_{nt} - x_o \geq \varepsilon) = \exp\left(-\frac{n\,\varepsilon^2\, d_1^2}{2\sigma_t^2}(1 + \delta_7(n,\varepsilon))\right)$$

where $\lim_{\varepsilon \to 0} \overline{\lim}_{n\to\infty} |\delta_7(n,\varepsilon)| = 0.$

Condition (6.3.1) implies that there exists a constant K_9 such that if

$$(E_n^{-1}(t) - t)^+ \leq K_9, \quad \text{then}$$

$$|(Q_{nt} - x_o)^+ - d_1^{-1}(E_n^{-1}(t) - t)^+| \leq K_{10} (E_n^{-1}(t) - t)^2$$

and hence,

$$P(E_n^{-1}(t) - t \geq \varepsilon\, d_1 + \varepsilon^{3/2}) - P(K_{10}(E_n^{-1}(t) - t)^2 \geq \varepsilon^{3/2})$$

$$- P(E_n^{-1}(t) \geq t + K_9)$$

$$\leq P(Q_{nt} - x_o \geq \varepsilon)$$

$$\leq P(d_1^{-1}(E_n^{-1}(t) - t) \geq \varepsilon - \varepsilon^{3/2}) + P(K_{10}(E_n^{-1}(t) - t)^2 \geq \varepsilon^{3/2})$$

$$+ P(E_n^{-1}(t) > t + K_9).$$

From these inequalities (6.3.18) follows by the usual kind of probability calculus as we have been doing.

Using the second condition of (6.3.1) and (6.3.15), one proves similarly that

(6.3.19) $$P(-Q_{nt} + Q_t \geq \varepsilon) = \exp\left(-\frac{n\,\varepsilon^2\, d_2^2}{2\sigma_t^2}(1 + \delta_8(n,\varepsilon))\right)$$

where $\lim_{\varepsilon \to 0} \overline{\lim}_{n \to \infty} |\delta_8(n, \varepsilon)| = 0.$

(6.3.18) and (6.3.19) yield (6.3.2) and hence, the theorem.

CHAPTER 7

ON ASYMPTOTIC EFFECTIVE VARIANCES AND APPROXIMATION TO TO NORMALITY OF L-STATISTICS

7.1 INTRODUCTION

In this chapter, we consider the problems of asymptotic effective variances (A E V), probabilities of moderate deviations and rates (uniform and non-uniform both) of convergence to normality of linear functions of order statistics.

As was seen in the last chapter, the problem of finding A E V reduces to that of finding expression for probabilities of certain type of deviations. In sections 7.2 and 7.3, this problem and the problem of finding probabilities of moderate deviations are tackled by exploiting the ideas used in obtaining the representation of quantile processes. This technique appears to be quite flexible for weakly dependent structures.

The above-mentioned method, based on quantile representation, is capable of producing Berry-Esseen bounds with the order of error as $n^{-1/4} \log n$ which is far away from being the best possible. In section 7.4, we shall make use of some of the representation results presented in Chapter 5 to get uniform and non-uniform Berry-Esseen bounds for L-statistics. This method produces very satisfactory results on Berry-Esseen bounds and the proof goes through for mixing r.v.s also. However, the technique suffers from the demerit that it demads smooth weight functions.

7.2 A E V OF L-STATISTICS

We shall investigate the A E V of trimmed type L-statistics in

Theorem 7.2.1. Let $\{X_n\}$ be a stationary sequence of m-dependent r.v.s with X_1 having d.f. F. Let Q_{nt} and Q_t, $0 < t < 1$, denote the t^{th} sample quantile and the t^{th} population quantile, respectively. W stands for some d.f. on $[\alpha, \beta]$, $0 \leq \alpha \leq \beta \leq 1$. Define

$$(7.2.1) \qquad L_n^* = \int_0^1 Q_{nt}\, dW(t) \quad \text{and} \quad L^* = \int_0^1 Q_t\, dW(t) .$$

Let us suppose that $F''(x)$ exists $\forall\, x \in [Q_\alpha - \delta, Q_\beta + \delta]$, for some $\delta > 0$, and

$$(7.2.2) \qquad \infty > K_1 \geq F'(x) \geq K_2^{-1} > 0 \quad \text{and} \quad |F''(x)| \leq K_3 < \infty$$

throughout the interval $[Q_\alpha - \delta, Q_\beta + \delta]$. We adopt the notations

$$F'(Q_t) = f_t, \quad Z_{it} = (t - I(X_i \leq Q_t))\, f_t^{-1} \quad \text{and} \quad Z_i = \int_0^1 Z_{it}\, dW(t).$$

If

$$0 < \sigma_L^2 = V(Z_1) + 2 \sum_{i=1}^{m-1} \text{cov}(Z_1, Z_{1+i}),$$

then

$$(7.2.3) \qquad P(|L_n^* - \overset{*}{L}| > \varepsilon) = \exp[-\frac{n\,\varepsilon^2}{2\,\sigma_L^2}(1 + \delta_1(n,\varepsilon))]$$

where $\lim_{\varepsilon \to 0} \overline{\lim}_{n \to \infty} |\delta_1(n,\varepsilon)| = 0$. Therefore, A E V of L_n^* can be taken to be $\{\sigma_L^2/n\}$.

Proof. Define $U_i = F(X_i)$. Let E_n denote the e.d.f. of $\{U_i\}$. Our arguments for the theorem go as follows:

Step 1. Let us prove first that, for any $\delta_1 > 0$ (fixed)

$$P(\sup_{\alpha-\delta/2 \leq t \leq \beta+\delta/2} |E_n^{-1}(t) - t| \geq \delta_1\, \varepsilon^{9/16})$$

is negligible compared to the r.h.s. of (7.2.3). By the usual kind of inversion, we see that, if $\delta_1\, \varepsilon^{9/16} \leq \delta/4$, this probability cannot exceed

$$(7.2.4) \qquad P(\sup_{\alpha - \frac{3\delta}{4} \leq t \leq \beta + 3\delta/4} |E_n(t)-t| \geq \delta_1\, \varepsilon^{9/16})$$

Splitting the interval $[\alpha - 3\delta/4,\ \beta + 3\delta/4]$ into subintervals of length n^{-1} and using Bonferroni inquality, let us note further that the expression in (7.2.4) is

$$\leq n \max_{\substack{s = \alpha - 3\delta/4 + j/n \\ j = 1,2,3 \ldots [(\beta-\alpha+3\delta/2)n]+1}} P(n|E_n(s)-s| \geq \frac{1}{2}\delta_1 \varepsilon^{9/16} n)$$

for all n sufficiently large. By an application of Lemma 3.2.2 (with $b = 1$, $D = 2\delta_1 \varepsilon^{9/16} n$), we have

$$P(n|E_n(s)-s| > \frac{1}{2}\delta_1 \varepsilon^{9/16} n) \leq K_4 \exp(- K_5 \varepsilon^{9/8} n)$$

for all s in the range of our interest. Thus the expression in (7.2.4) $\leq \exp(- K_5 \varepsilon^{9/8} n(1 + K_6 \frac{\log n}{n}))$

and this inequality justifies the claim made in this step.

Step 2. Define $R_n(t) = E_n^{-1}(t) + E_n(t) - 2t$. We claim here

(7.2.5) $$P(\sup_{\alpha \leq t \leq \beta} |R_n(t)| > \varepsilon^{9/8}) = P_1$$

is negligible compared to the r.h.s. of (7.2.3). Clearly,

(7.2.6) $$P_1 \leq P(\sup_{\alpha\leq t\leq\beta} |R_n(t)| > \varepsilon^{9/8}, \sup_{\alpha-\frac{\delta}{2} \leq t \leq \beta+\frac{\delta}{2}} |E_n^{-1}(t)-t| \leq \varepsilon^{9/16})$$

$$+ P(\sup_{\alpha - \frac{\delta}{2} \leq t \leq \beta + \frac{\delta}{2}} |E_n^{-1}(t) - t| > \varepsilon^{9/16})$$

It has already been noted in the first step that the second term of the r.h.s. of (7.2.6) is negligible. Next, let us observe that

(7.2.7) the first term in the r.h.s. of (7.2.6)

$$\leq P\Big(\sup_{\alpha \leq t \leq \beta} |E_n E_n^{-1}(t) - E_n(t) - E_n^{-1}(t) + t| > \frac{1}{2} \varepsilon^{9/8}, \sup_{\alpha - \delta/2 \leq t \leq \beta + \delta/2} |E_n^{-1}(t) - t| \leq \varepsilon^{9/16} \Big)$$

$$+ P\Big(\sup_{\alpha \leq t \leq \beta} |E_n E_n^{-1}(t) - t| \geq \frac{1}{2} \varepsilon^{9/8}, \sup_{\alpha-\delta/2 \leq t \leq \beta+\delta/2} |E_n^{-1}(t) - t| \leq \varepsilon^{9/16} \Big).$$

Since

$$|E_n E_n^{-1}(t) - t| \leq E_n E_n^{-1}(t) - E_n(E_n^{-1}(t) - 0)$$

$$\leq |E_n E_n^{-1}(t) - E_n^{-1}(t) - E_n(t) + t| + | E_n (E_n^{-1}(t) - 0) - E_n^{-1}(t) - E_n(t) + t|$$

it follows that

(7.2.8) r.h.s. of (7.2.7)

$$\leq 3 P\Big(\sup_{\alpha \leq s \leq \beta} \sup_{|s-t| \leq 2\varepsilon^{9/16}} |E_n(s+t) - E_n(s) - t| > \frac{1}{2} \varepsilon^{9/8} \Big)$$

$$\leq P\Big(\max_{1 \leq \lambda \leq [(\beta-\alpha)n+2]} \max_{|j| \leq [2\varepsilon^{9/16} n+1]} |E_n((\lambda+j)n^{-1}) - E_n(\lambda n^{-1}) - jn^{-1}| \geq \frac{1}{4}\varepsilon^{9/8}\Big)$$

for all n sufficiently large (which is good enough since the limit is to be taken w.r.t. n, first). We employed here arguments similar to that used in Theorem 2.2.2. By Bonferroni inequality, this probability can not exceed

$$\text{(7.2.9)} \quad n^2 \sup_{\alpha-\delta/2 \leq t \leq \beta+\delta/2} \sup_{|s-t| \leq 3\varepsilon^{9/16}} P(n|E_n(s+t)-E_n(s)-t| > \frac{n}{4}\varepsilon^{9/8})$$

Using Lemma 3.3.2 (with $N = n$, $b = 3\,\varepsilon^{9/16}$, $\theta = 1/2$ and $D = \frac{n\,\varepsilon^{27/32}}{8\,d\sqrt{3}}$) we find that, for s and t as in (7.2.9),

$$P(n|E_n(s+t)-E_n(s)-t| > \frac{n}{4}\varepsilon^{9/8}) \leq K_7 \exp(-K_8\,\varepsilon^{27/16}\,n).$$

so that the expression in (7.2.9) cannot exceed

$$K_7 \exp(-K_8\,\varepsilon^{27/16}\,n(1 + K_9 \frac{\log n}{n}))$$

which is negligible compared to the r.h.s. of 7.2.3.

Step 3. Since

$$\int_0^1 (Q_{nt} - Q_t)\, dW(t)$$

$$= \int_0^1 (Q_{E_n^{-1}(t)} - Q_t)\, dW(t)$$

we see, by Taylor's expansion and the conditions of the theorem that

$$\left| \int_0^1 (Q_{nt} - Q_t)\, dW(t) - \int_0^1 \frac{E_n^{-1}(t) - t}{f_t}\, dW(t) \right| \leq K_{10} \sup_{\alpha \leq t \leq \beta} (E_n^{-1}(t) - t)^2.$$

Also,

$$\left| \int_0^1 (E_n^{-1}(t) - t)\, f_t^{-1}\, dW(t) - \int_0^1 (t - E_n(t))\, f_t^{-1}\, dW(t) \right|$$

$$\leq K_{11} \sup_{\alpha \leq t \leq \beta} |E_n^{-1}(t) + E_n(t) - 2t|.$$

In view of these estimates

(7.2.10) $$\left| \int_0^1 (Q_{nt} - Q_t)\, dW(t) - n^{-1} \sum_{i=1}^{n} Z_i \right|$$

$$\leq K_{10} \sup_{\alpha \leq t \leq \beta} (E_n^{-1}(t) - t)^2 + K_{11} \sup_{\alpha \leq t \leq \beta} |E_n^{-1}(t) + E_n(t) - 2t|.$$

Converting (7.2.10) into probability inequality, we have

$$|P(\int_0^1 (Q_{nt}-Q_t)\, dW(t) > (\varepsilon) - P(n^{-1} \sum_{i=1}^{n} Z_i > \varepsilon \pm 2\, \varepsilon^{9/8})|$$

$$\leq P(K_{10} \sup_{\alpha \leq t \leq \beta} (E_n^{-1}(t) - t)^2 \geq \varepsilon^{9/8})$$

$$+ P(K_{11} \sup_{\alpha \leq t \leq \beta} |E_n^{-1}(t) + E_n(t) - 2t| \geq \varepsilon^{9/8}) .$$

This inequality, the results of the first two steps and Lemma 6.3.1 give the desired estimate for

$$P(\int_0^1 (Q_{nt} - Q_t)\, dW(t) > \varepsilon)$$

and similarly, we obtain the estimate for

$$P(\int_0^1 (Q_{nt} - Q_t)\, dW(t) < - \varepsilon)$$

to complete the proof of this theorem.

Using the idea of the representations given in Section 5.2, we can get A E V of L-statistics with smooth weight functions which need not necessarily be of the trimmed type, but, these result require the existence of m.g.f. of the underlying d.f. in a neighbourhood of zero.

7.3 ON PROBABILITIES OF MODERATE DEVIATIONS OF L-STATISTICS

In the last section, we displayed a technique by which one can transform the known results regarding probabilities of deviations of sample sum to L-statistics. In this section, we shall present some interesting results on probabilities of deviations of L-statistics. The details of the proofs are omitted since they are similar to that of the previous section.

Theorem 7.3.1. Let $\{X_i\}$ be a sequence of i.i.d.r.v.s and L_n^*, L^* and σ_L be the same as in the previous section. If $s_n \longrightarrow \infty$ and $s_n = o(n^{1/10})$ as $n \longrightarrow \infty$, then

$$P(\sqrt{n}\,(L_n^* - L^*) \geq \sigma_L\, s_n) \sim 1 - \Phi(s_n)$$

and

$$P(\sqrt{n}\,|L_n^* - L^*| \geq \sigma_L\, s_n) \sim 2(1 - \Phi(s_n)).$$

This result is derived from the following result of Cramer (1938).

Theorem 7.3.2. Let $\{Y_i\}$ be a sequence of i.i.d. r.v.s such that Y_1 has m.g.f. in a neighbourhood of zero. If $\sigma^2(Y_1) = V(Y_1) > 0$

$s_n \longrightarrow \infty$ as $n \longrightarrow \infty$
and
$s_n = o(n^{1/6})$ as $n \to \infty$, then

$$P(n^{-1/2} \sum_{i=1}^{n} (Y_i - E(Y_i)) \geq \sigma(Y_1) s_n) \sim 1 - \Phi(s_n)$$

and

$$P(n^{-1/2} | \sum_{i=1}^{n} (Y_i - E(Y_i))| \geq \sigma(Y_1) s_n) \sim 2(1 - \Phi(s_n)).$$

<u>Proof of Theorem 7.3.1</u>. Using (7.2.10),

$$|P(\sqrt{n}(L_n^* - L^*) \geq \sigma_L s_n) - P(n^{-1/2} \sum_{i=1}^{n} Z_i \geq \sigma_L s_n \pm 2n^{-1/10})$$

$$\leq P(K_{10} \sup_{\alpha \leq t \leq \beta} (E_n^{-1}(t) - t)^2 \geq n^{-6/10})$$

$$+ P(K_{11} \sup_{\alpha \leq t \leq \beta} |E_n^{-1}(t) + E_n(t) - 2t| \geq n^{-6/10})$$

The first statement of the theorem follows from the following three estimates :

(7.3.1) $$\Phi(s_n \pm (2/\sigma_L) n^{-1/10}) \sim \Phi(s_n)$$

(7.3.2) $$P(K_{10} \sup_{\alpha \leq t \leq \beta} (E_n^{-1}(t) - t)^2 \geq n^{-6/10}) = o(1 - \Phi(s_n))$$

and

$$(7.3.3) \qquad P(K_{11} \sup_{\alpha \leq t \leq \beta} |E_n^{-1}(t)+E_n(t)-2t| \geq n^{-6/10}) = o(1 - \Phi(s_n))$$

(7.3.1) is true because $s_n n^{-1/10} \to 0$. (7.3.2) is proved following the Step 1 of the proof of Theorem 7.2.1. To prove (7.3.3), we note that

$$\text{l.h.s. of (7.3.3)} \leq P(K_{11} \sup_{\alpha \leq t \leq \beta} |E_n^{-1}(t) + E_n(t) - 2t| \geq n^{-6/10}$$

$$\sup_{\alpha - \delta/2 \leq t \leq \beta + \delta/2} |E_n^{-1}(t)-t| \leq n^{-4/10})$$

$$+ P(\sup_{\alpha - \delta/2 \leq t \leq \beta + \delta/2} |E_n^{-1}(t)-t| \geq n^{-4/10}).$$

After this, rest of the estimation is similar to that of Step 2 of Theorem 7.2.1.

The second statement of the theorem is established by obtaining similar results for

$$P(-\sqrt{n}\,(L_n^* - L^*) \geq \sigma_L\, s_n) .$$

Proof of this theorem is complete.

Our next result of this section is on the probabilities of moderate deviations (deviations of order $(\log n)^{1/2}$) of L-statistics in mixing cases.

<u>Theorem 7.3.3.</u> Let the sequence of r.v.s $\{X_i\}$ satisfy either of the two conditions :

(i) $\{X_i\}$ is ϕ-mixing with $\Sigma\, \phi^{1/2}(i) < \infty$.

(ii) $\{X_i\}$ is strong-mixing with $\{\alpha(n)\}$ decaying exponentially.

If L_n^* and L^* are the same as in Theorem 7.2.1, and

$$0 < \sigma_L^2 = V(Z_1) + 2 \sum_{i=1}^{\infty} \operatorname{cov}(Z_1, Z_{1+i}),$$

then, for any $c > 0$

$$P(\sqrt{n}\,(L_n^* - L^*) > c\,\sigma_L (\log n)^{1/2}) \sim (2\pi\, c^2 \log n)^{-1/2}\, n^{-c^2/2}$$

and

$$P(\sqrt{n}\,|L_n^* - L^*| > c\,\sigma_L (\log n)^{1/2}) \sim 2(2\pi c^2 \log n)^{-1/2}\, n^{-c^2/2}.$$

This theorem follows from the following theorem, which in term follows from the results of Ghosh and Babu (1977) and Babu and Singh (1978 a).

<u>Theorem 7.3.4.</u> Let $\{Y_i\}$ be a stationary sequence of r.v.s satisfying either of the conditions

(i) $\{Y_i\}$ is ϕ-mixing with $\Sigma\, \phi^{1/2}(i) < \infty$.

(ii) $\{Y_i\}$ is strong-mixing with $\{\alpha(n)\}$ decaying exponentially.

Let

$$0 < \sigma^2(Y_1) = V(Y_1) + 2 \sum_{i=1}^{\infty} \operatorname{cov}(Y_1, Y_{1+i}).$$

If $E|Y_1|^{c+\delta} < \infty$, for some $c > 0$ and $\delta > 0$, then, for $\forall\, \gamma > 0$,

$$P(n^{-1/2} \sum_{i=1}^{n} (Y_i - E(Y_i)) \geq c\, \sigma(Y_1)\, (\log n)^{1/2} \pm n^{-\gamma})$$

$$\sim (2\pi c^2 \log n)^{-1/2}\, n^{-c^2/2}$$

and

$$P(n^{-1/2} \left| \sum_{i=1}^{n} (Y_i - E(Y_i))\right| \geq c\, \sigma(Y_1)\, (\log n)^{1/2} \pm n^{-\gamma})$$

$$\sim 2(2\pi c^2 \log n)^{-1/2}\, n^{-c^2/2}.$$

where γ is any positive constant.

<u>Proof</u>. This time, we use (7.2.10) to get the probability inequality

$$(7.3.4)\qquad \Big| P\big(\sqrt{n}(L_n^* - L^*) > c\,\sigma_L (\log n)^{1/2}\big) - P\Big(\sqrt{n}\sum_{i=1}^{n} Z_i \geq \sigma_L(\log n)^{1/2} \pm 2n^{-1/8}\Big)\Big|$$

$$\leq P\Big(\sup_{\alpha-\frac{\delta}{2}\leq t\leq \beta+\frac{\delta}{2}} |(E_n^{-1}(t) - t)^2 \geq n^{-1/8}\Big)$$

$$+ P\Big(\sup_{\alpha\leq t\leq\beta} |E_n^{-1}(t)+E_n(t)-2t| \geq n^{-\frac{1}{2}-\frac{1}{8}},\ \sup_{\alpha-\frac{\delta}{2}\leq t\leq\beta+\frac{\delta}{2}} |E_n^{-1}(t)-t| \leq n^{-\frac{1}{3}}\Big)$$

$$+ P\Big(\sup_{\alpha-\frac{\delta}{2}\leq t\leq\beta+\frac{\delta}{2}} |E_n^{-1}(t) - t| \geq n^{-\frac{1}{3}}\Big)$$

Rest of the arguments are similar to that of Theorem 7.2.1. We use here the probability bounds given by Lemmas 2.2.3 and 2.3.3. For the present use, we have to modify the statements of these lemmas slightly without altering the proofs. In the r.h.s. of inequality (2.2.5) we can replace $K_1 N^{-4}$ by $K_1(\lambda) N^{-\lambda}$ where $K_1(\lambda)$ is a constant depending on λ. A similar change can be made in Lemma 2.3.3. There changes are justified easily by looking into the proofs of these lemmas.

7.4 ON RATES OF CONVERGENCE TO NORMALITY OF L-STATISTICS

We obtain here uniform and non-uniform rates of convergence to normality of L-statistics utilizing the idea of the representations presented in section 5.3.

Our main tool for approximations is given in

Lemma 7.4.1. Let X, Y be two r.v.s (in general dependent). We have the following inequalities :

(i) For any $\bar{\theta} > 0$

$$(7.4.1) \qquad \sup_{x \in \mathbb{R}} |P(X+Y \leq x) - \Phi(x)| \leq \sup_{x \in \mathbb{R}} |P(X \leq x) - \Phi(x)| + P(|Y|) > \bar{\theta}) + (2\pi)^{-1/2} \bar{\theta}$$

(ii) For any $\bar{\theta} > 0$ and a real such that $\bar{\theta} < |a|$,

$$|P(X + Y) \leq a) - \Phi(a)|$$

$$\leq \max_{a^* = a \pm \bar{\theta}} |P(X \leq a^*) - \Phi(a^*)| + P(|Y| > \bar{\theta}) + (2\pi)^{-1/2} \bar{\theta} \exp(-(|a| + \bar{\theta})^2/2).$$

The proof of this lemma is trivial and we omit it.

We have the following easily derived uniform Berry-Esseen bound for L-statistics with smooth weight functions in the case of ϕ-mixing r.v.s.

<u>Theorem 7.4.1</u>. Let $\{X_i\}$ be a stationary sequence of ϕ-mixing r.v.s with $\Sigma\, \phi^{1/2}(i) < \infty$ and X_1 having a continuous d.f. F. Let F_n denote the e.d.f. of $\{X_i\}$. With w as some weight function on $[0,1]$, having bounded second derivative, define

$$L_n = \int_0^1 x\, w(F_n(x))\, dF_n(x),$$

$$L = \int_0^1 x\, w(x)\, dF(x)$$

$$Z_i = \int_0^1 [u - I(F(X_i) \leq u)]\, w(u)\, d\, Q_u \quad , \ i = 1, 2 \ldots$$

where Q denotes some inverse of F. If $E|Z_1^3| < \infty$, $E|X_1|^{9/4 + \delta} < \infty$ for some $\delta > 0$ and

$$0 < \sigma_L^2 = V(Z_1) + 2 \sum_{i=1}^{\infty} \operatorname{cov}(Z_1, Z_{1+i}),$$

then

$$\sup_{x \in \mathbb{R}} |P(\sqrt{n}(L_n - L) \leq x\, \sigma_L) - \Phi(x)| = O(n^{-1/2} \log n).$$

<u>Proof</u>. Rewriting the findings of the proof of Theorem 5.2.1,

$$(7.4.2) \quad |L_n - L - n^{-1} \sum_{i=1}^{n} Z_i| \leq K_{12}(V_n(0))^2 \, (n^{-1} \sum_{i=1}^{n} |\, Q_{U_i} \,|)$$

$$+ K_{13}\, n^{-2} \sum_{i=1}^{n} |Q_{U_i}| + K_{14}(V_n(a))^2 \int_0^1 (u(1-u))^{2a} d\, Q_u$$

$= E(L_n)$ (say)

where $U_i = F(X_i)$, $V_n(\varepsilon)$ is the same as defined by (3.2.1) and a is any positive real number. Since

$$E|X_1|^{\frac{9}{4}+\delta} < \infty \implies \int_0^1 (u(1-u))^{\frac{4}{9}-\frac{\delta}{2}} \, d\, Q_u \quad < \infty$$

(see Lemma 4.4.4(ii)),

$$K_{14}[V_n(\tfrac{2}{9} - \tfrac{\delta}{4})]^2 \int_0^1 (u(1-u))^{\frac{4}{9} - \delta/2} \, d\, Q_u$$

$$\leq K_{15}(V_n(\tfrac{2}{9} - \tfrac{\delta}{4}))^2.$$

Let us apply (7.4.1) with $X = \sqrt{n}\,(L_n - L)\,\sigma_L^{-1}$, $Y = \sqrt{n}\,(L_n - L - n^{-1} \sum_{i=1}^{n} Z_i)\,\sigma_L^{-1}$ and $\bar{\theta} = K_{16}\, n^{-1/2} \log n$ (K_{16} is a positive constant to be chosen later). In view of Theorem 6.2.1, our result would follow from the following :

There exist a constant K_{17} such that

(7.4.3) $\quad P((V_n(0))^2 \geq K_{17}\, n^{-1} \log n) = O(n^{-1/2})$

(7.4.4) $\quad P((V_n(0))^2\, n^{-1} \sum_{i=1}^{n} |\, Q_{U_i} \,| \geq K_{17}\, n^{-1} \log n) = O(n^{-1/2})$

(7.4.5) $\quad P(n^{-1} \sum_{i=1}^{n} |\, Q_{U_i} \,| > K_{17} \log n) = O(n^{-1/2})$

(7.4.6) $P((V_n(\frac{2}{9} - \frac{\delta}{4}))^2 \geq K_{17}\, n^{-1} \log n) = 0(n^{-1/2})$.

The assertion (7.4.3) is proved by the usual kind of partitioning of the interval [0,1] into subintervals and using Bonferroni-inquality. (7.4.5) follows from Chebyshev inequality

$$(E|X_1|^{9/4} < \infty \Longrightarrow E|Q_{U_1}|^{9/4} < \infty)$$

by writing

the ℓ.h.s. of (7.4.5) = $P(n^{-1}\sum_{i=1}^{n}|Q_{U_i}| - E|Q_{U_1}| > E|Q_{U_1}| + K_{17} \log n)$.

To see (7.4.4), we note

$$\ell.h.s. \text{ of } (7.4.4) \leq P(n^{-1}\sum_{i=1}^{n}|Q_{U_i}| - E|Q_{U_i}| \geq 1)$$

$$+ P((V_n(0))^2 \geq (1 + E|Q_{U_i}|)^{-1} K_{17}\, n^{-1} \log n)$$

and then follow the proofs of (7.4.3) and (7.4.5).

Finally, we prove 7.4.6 in

Lemma 7.4.1. For any δ such that $0 < \delta \leq 2/9$, there exists a constant K (depending upon δ) such that

(7.4.7) $P(V_n(2/9 - \delta) > K\, n^{-1/2}(\log n)^{1/2}) = 0(n^{-1/2})$.

In proving this lemma, we require a slightly different version of Lemma 3.2.3 given below.

<u>Lemma 7.4.2</u>. Let E_n denote e.d.f. of the ϕ-mixing sequence of r.v.s $\{U_i\}$ with $\phi(n) = O(n^{-\gamma})$, $\gamma \geq 2$(if $\gamma = 2$, assume further that $\Sigma\, \phi^{1/2}(i) < \infty$) and U_1 having distribution U [0,1]. If $0 \leq \rho \leq 1/2$, and $0 < D \leq [s(1-s)]^{\rho}\, n^{1-1/2(\gamma+1)-\bar{\gamma}}$ for some $\bar{\gamma} > 0$, then there exists a constant d such that

$$(7.4.8) \quad P(n|E_n(s)-s| > 2d(s(1-s))^{\rho}D) \leq K_{18}\, n^{-4} + K_{19} \exp(-8D^2\, n^{-1}).$$

This lemma is also proved initating the proof of Lemma 2.2.3 and by choosing $p = n^{1/(\gamma-1)}$.

<u>Proof of Lemma 7.4.1</u>. Write

$$(0,1) = (0, n^{-3/2}] \quad (n^{-3/2}, 1 - n^{-3/2}] \quad (1 - n^{-3/2}, 1).$$

Since after a certain n onwards

$$\sup_{0 < t \leq n^{-3/2}} |E_n(t) - t|\, t^{-2/9+\delta} > K\, n^{-1/2} (\log n)^{1/2}$$

$$\Longrightarrow E_n(n^{-3/2}) \geq 1/n$$

we have that

$$P(\sup_{0<t\leq n^{-3/2}} |E_n(t)-t|\ t^{-2/9+\delta} > K\,n^{-1/2}(\log n)^{1/2}) = O(n^{-1/2}).$$

By similar arguments

$$P(\sup_{1-n^{-3/2}<t<1} |E_n(t)-t|(1-t)^{-2/9+\delta} > K\,n^{-1/2}(\log n)^{1/2}) = O(n^{-1/2})$$

Now, we are left to show that

$$(7.4.9)\quad P(\sup_{n^{-3/2}\leq t\leq 1-n^{-3/2}} |E_n(t)-t|(t(1-t))^{-2/9+\delta} \geq K\,n^{-1/2}(\log n)^{1/2}) = O(n^{-1/2})$$

for some constant $K > 0$.

Dividing the interval $(n^{-3/2}, 1 - n^{-3/2}]$ into subintervals of length n^{-3}, it follows by some elementary approximations and Bonferroni inequality that

$$(7.4.10)\quad \text{l.h.s. of } (7.4.9) \leq n^3 \sup_{\frac{n^{-3/2}}{2}\leq t\leq 1-\frac{n^{-3/2}}{2}} P(\frac{|E_n(t)-t|}{(t(1-t))^{2/9-\delta}} > K\,n^{-1/2}(\log n^{1/2})).$$

To estimate the r.h.s. of (7.4.10) we use Lemma 7.4.2 with $\rho = 2/9 - \delta$, $D = n^{1/2}(\log n)^2$ and $\gamma = 2$. Then we see that, for some $K > 0$,

$$\text{r.h.s of (7.4.10)} = O(n^{-1})$$

and this proves Lemma 7.4.1.

Remark 7.4.1. Theorem 7.4.1 holds even if we relax the smoothness conditions on w of finitely many points, but we need the stronger condition $\phi(n) = O(e^{-\theta n})$ for some $\theta > 0$ because of the factors like $|E_n E_n^{-1}(\alpha) - \alpha|$. The result also holds in the strong mixing case with exponentially decaying $\{\alpha(n)\}$.

Finally, we state some non-uniform rates of convergence to normality of L-statistic in the i.i.d situation.

Theorem 7.4.2. Let $\{X_i\}$ be i.i.d. r.v.s and L_n, L and Z_i be the same as defined in the previous theorem. Let $\sigma_L = V(Z_1) > 0$. We then have :

(i) Let $E|Z_1|^{2+c} < \infty$ for $c > 0$. If $E|X_1|^{1+\frac{c}{2-c}} < \infty$ for $0 < c \leq 1$, $E X_1^2 < \infty$ for $0 \leq c \leq 2$ and $E|X_1|^c < \infty$ for $c \geq 2$, then, for all $b \in \mathbb{R}$ such that $b^2 \leq (c+1) \log n$,

$$(7.4.11) \qquad \left|P\left(\frac{\sqrt{n}}{\sigma_L}(L_n - L) \leq b\right) - \Phi(b)\right|$$

$$\leq K_{20}\, n^{-c^*}(\log n)\ \exp[-(1-\sigma)\, b^2/2] + n^{-c/2}(\log n)^{-1}$$

$$+ n\, P(|Z_1| > K_{21}\, n^{1/2}\, |b|)$$

where $c^* = \frac{1}{2}\min(c,1)$, $\sigma = c^*(c+1)^{-1}$ and K_{20}, K_{21} are constants not depending upon b and n.

(ii) Let $E|Z_1|^{2+c} < \infty$ and $E|X_1|^{2+c+\delta} < \infty$ for some $c > 0$ and $\delta > 0$. Then for all $b \in \mathbb{R}$ such that $b^2 \geq (c+1)(\log n)$,

$$(7.4.12) \qquad \left|P\left(\frac{n^{1/2}}{\sigma_L}(L_n - L) \leq b\right) - \Phi(b)\right|$$

$$\leq K_{22}\, n^{-c/2}\, b^{-2-c-\delta^*/4} + n\, P(|Z_1| > K_{23}|b|)$$

where $\delta^* = \min(\delta, 1)$.

The theorem is derived from the following results of Michel (1976).

<u>Theorem 7.4.3</u>. (See Michel (1976), Theorem 1). If $\{Y_i\}$ is a sequence of i.i.d.r.v.s with $E\, Y_1 = 0$, $E\, Y_1^2 = 1$ and $E|Y_1|^{2+c} < \infty$ for some $c > 0$ then, there exist constants

K_{23} and K_{24} (depending upon c) such that for all $b \in \mathbb{R}$ with $b^2 \leq (c+1) \log n$,

$$|P(\sum_1^n X_i \leq n^{1/2} b) - \Phi(b)|$$

$$\leq K_{23} n^{-c^*} \exp[-(1-\sigma) b^2/2] + n P(|Y_1| > K_{24} n^{1/2}|b|)$$

where $c^* = \frac{1}{2} \min(c,1)$ and $\sigma = c^*(c+1)^{-1}$.

Theorem 7.4.4. (See Michel (1976) Theorem 2). Under the set up of the previous theorem, there exist constants K_{25} and K_{26} (depending upon c) such that for all $b \in \mathbb{R}$ with $b^2 \geq (c+1) \log n$

$$|P(\sum_{i=1}^n Y_i \leq n^{1/2} b) - \Phi(b)| \leq K_{25} n^{-c/2} b^{-2(c+2)}$$

$$+ n P(|Y_1| > K_{26} n^{1/2}|b|)$$

The following lemmas are also used in proving the Theorem 7.4.2(i).

Lemma 7.4.3. Let E_n be the e.d.f. of i.i.d. r.v.s $\{U_i\}$ with U_1 having the distribution $U[0,1]$. If $0 \leq \rho \leq \frac{1}{2}$ and $0 < D \leq (t(1-t))^\rho n$, then there exists a constant d such that

$$P(n\,|E_n(t) - t| > 2d(t(1-t))^{\rho}D) \le K_{27}\exp(-8D^2 n^{-1}) .$$

This lemma is essentially Lemma 3.2.2 and the proof requires very minor modifications.

<u>Lemma 7.4.4.</u> If $\{Y_i\}$ is a sequence of i.i.d. r.v.s with $E|Y_1|^{1+c} < \infty$ where $0 \le c \le 1$, then

$$E|\sum_{i=1}^{n} Y_i| = O(n^{1/(c+1)}).$$

<u>Proof</u>. Let Y_{in} be r.v.s obtained by truncating Y_i at $n^{1/(c+1)}$ for all $i \ge 1$. With this definition of $\{Y_{in}\}$,

$$E|\sum_{i=1}^{n} Y_i| \le E\sum_{i=1}^{n} |Y_i - Y_{in}| + E|\sum_{i=1}^{n} Y_{in}|$$

$$\le n\,E(|Y_1|\,I(|Y_1| > n^{1/(c+1)})) + \|\sum_{i=1}^{n} Y_{in}\|_2$$

$$<< n.\,n^{-c/(c+1)} + [n\,E(Y_1^2) + n^2(E\ Y_{in})^2]^{1/2}$$

$$<< n^{1/(c+1)} + [n.\,n^{(1-c)/(1+c)} + n^2(E(|Y_1|I(|Y_1| > n^{1/(c+1)})))^2]^{1/2}$$

$$= O(n^{1/(c+1)}).$$

Proof of Theorem 7.4.2 (i). W.ℓ.g. assume $V(Z_1) = 1$. For $|b| \leq 1$, the result follows from Theorem 7.4.1. For $|b| \geq 1$, we apply Lemma 7.4.1(ii) with

$$X = \sqrt{n}\,(L_n - L)\,\sigma_L^{-1}$$

$$Y = \sqrt{n}\,(L_n - L - n^{-1} \sum_{i=1}^{n} Z_i)\,\sigma_L^{-1}$$

and

$$\bar{\theta} = K_{28}\, n^{-1/2} \log n$$

where K_{28} is fixed after taking into account the other requirements of the proof. The term

$$|P(n^{-1/2} \sum_{i=1}^{n} Z_i \leq b \pm \bar{\theta}) - \Phi(b \pm \bar{\theta})|$$

is estimated using the above mentioned theorems of Michel. If $(b \pm \bar{\theta})^2$ exceeds the zone of Theorem 7.4.3, we use Theorem 7.4.4. Thus, we essentially require to prove that

$$(7.4.13) \qquad P(R(L_n) > n^{-1} \log n) = O(n^{-c/2} (\log n)^{-1})$$

where $R(L_n)$ is defined by the r.h.s. of 7.4.2. The proof of (7.4.13) follows the same lines as Theorem 7.4.1. The moment conditions on X_1 have been adjusted suitably to handle the term $n^{-1} \sum_{i=1}^{n} |\bar{Q}_i|$. Lemma 7.4.4 comes in use when we have $0 \leq c \leq 1$.

Proof of Part (ii). We sketch the proof for $b > 0$. In case $b < 0$ the arguments are repeated for $-(L_n - L)$.

Since $b^2 \geq (c + 1) \log n$ implies that

$$1 - \Phi(b) \leq K_{29}\, n^{-c/2}\, b^{-c-2-\delta^*/4},$$

we only need to show that

$$(7.4.14) \qquad P(\sqrt{n}\,(L_n - L) > b) \leq K_{30}\, n^{-c/2}\, b^{-c-2-\delta^*/4} + n\, P(Z_1 > K_{31}\, b).$$

Let us choose $\delta_1 (> 0)$ small enough such that $((1 - \delta_1)\, b)^2 \geq (c + \frac{1}{2}) \log n$ and note that

$$\text{l.h.s. of (7.4.14)} \leq P(n^{-1/2} \sum_{i=1}^{n} Z_i \geq (1 - \delta_1) b) + P(R(L_n) > \delta_1 n^{-\frac{1}{2}}\, b).$$

The above mentioned theorems of Michel ensure that

$$P(n^{-1/2} \sum_{i=1}^{n} Z_i \geq (1 - \delta_1) b) \leq K_{31}\, n^{-c/2}\, b^{-c-2-\delta^*/4}$$

and hence the proof is completed by showing that

$$(7.4.15) \qquad P(R(L_n) \geq \delta_1\, n^{-1/2}\, b) \leq K_{32}\, n^{-c/2}\, b^{-c-2-\delta^*/4}.$$

The inequality (7.4.10) is proved passing on to the components constituting $R(L_n)$. The details are straightforward and resemble with the proof of Theorem 7.4.1 and are omitted.

Corollary 7.4.1. If $E|Z_1|^{2+c} < \infty$ and $E|X_1|^{2+c+\delta} < \infty$ for some $c > 0$ and $\delta > 0$, then

$$\left| E\left(\left|\frac{n^{1/2}(L_n - L)}{\sigma_L}\right|^{2+c}\right) - \pi^{-1/2}\, 2^{(2+c)}\, \Gamma((3+c)/2) \right|$$

$$= O(n^{-c^*} (\log n)^{1+c/2}).$$

where $c^* = \frac{1}{2}\min(c,1)$.

This corollary is proved along the lines of the proof of Theorem 6 of Michel (1976).

Theorem 7.4.2 also gives the probabilities of moderate deviations (similar to Theorem 7.3.4) and L_r version of Berry-Esseen bounds (similar to the results of Erickson (1973)) for the L-statistics under consideration.

Remark 7.4.1. Recently Babu, Ghosh and Singh (1978) extended the results of Michel (1976) for ϕ-mixing r.v.s. Using these extensions one can obtain possible versions of Theorem 7.4.2 in the ϕ-mixing case also.

CHAPTER 8

ON RELATIVE EFFICIENCIES OF L-STATISTICS

8.1 INTRODUCTION

Throughout this chapter, we restrict ourselves to the class $\mathcal{C}$ of strictly increasing absolutely continuous d.f. s. Let W be a d.f. on [0,1], symmetric about 1/2. Then following Bickel and Lehmann (1975 a).

$$\mu_W(F) = \int_0^1 F^{-1}(t)\, dW(t) \qquad \text{(F stands for a d.f.)}$$

is a location parameter. The most natural estimator of such a location parameter is

$$\hat{\mu}_W(F) = \int_0^1 F_n^{-1}(t)\, dW(t) ,$$

where F_n denotes e.d.f. at the n^{th} stage and $F_n^{-1}(t)$ is t^{th} sample quantile.

Asymptotic normality of such estimators has been established by several authors under various conditions as mentioned in Chapter 1. It also follows from our representation results of Chapter 5. In the independent case, the expression for asymptotic variances of these estimators are given by

$$\sigma^2(\hat{\mu}_W(F)) = V(Y)$$

where $Y = \int_0^1 ((t - I(F(X) \leq t))/f(F^{-1}(t))\, dW(t)$ and $f(x)=F'(x)$.

Using Lemma 3 of Shorack (1974), we can express the asymptotic variance as follows.

(8.1.1) $$\sigma^2(\hat{\mu}_W(F)) = V(Y) = V(\theta(W,F,\bar{U}))$$

where $\bar{U}$ is a r.v. having uniform distribution on [0,1] and

(81.2) $$\theta(W,F,t) = \int_{1/2}^{t} dW(t)/f(F^{-1}(t)).$$

Throughout this chapter, we assume that the formula (8.1.1) holds.

By efficiency of one estimator w.r.t. other we mean the ratio of asymptotic variances. Evidently, if the results of section 7.2 hold, then the same expression for relative efficiency also represents the ratio of asymptotic effective variances.

Whenever we replace $dW(t)$ by $w(t)dt$, it is assumed that W permits a density which is given by $w(t)$. In general, $e(W_2, W_1, F)$ denotes the relative efficiency of $\hat{\mu}_{W_2}(F)$ w.r.t. $\hat{\mu}_{W_1}(F)$ in the i.i.d. case when the underlying d.f. is F. For convenience, we will also adopt the following notations.

(i) $e(T_\alpha, F)$ denotes the efficiency of α-trimmed mean w.r.t. mean for d.f. F.

(ii) $e(WIN_\alpha, F)$ denotes the efficiency of α-Winsorized mean w.r.t. mean for F.

(iii) $e(T_\alpha, WIN_\alpha, F)$ denotes efficiency of α-trimmed mean w.r.t. α-Winsorised mean.

For the sake of convenience, unless the contrary is explicitly stated we shall be assuming that the underlying distributions are symmetric about zero which does not affect the generality of the results. Two distributions for which the efficiency is being compared are assumed to have the same location parameter. We shall denote by $\mathcal{D}$ the subset of consisting of all unimodel distributions.

Let $F, G \in \mathcal{C}$ and G has heavier tail than F (A precise definition is given later). Suppose W_1 and W_2 are two d.f.s on $[0,1]$ s.t. W_2 gives an estimator which is less sensitive for tails than that of W_1 (to be made precise later). Then, we shall obtain results of the form

$$e(W_2, W_1, F) \leq e(W_2, W_1, G)$$

In Bickel and Lehmann (1975 b)(also see Doksum (1969), G is defined to have heavier tail than F if

(8.1.3) $\quad G^{-1}(t)/F^{-1}(t)$ is non-decreasing in $(1/2, 1)$.

As long as the weight functions are modifications of the uniform distribution on $[0,1]$, (8.1.3) is enough. In order to achieve similar comparisons for a pair of estimators with general weight functions, a stronger condition (8.3.1) has been introduced for heaviness of tails. A consequence of lemma 8.3.1 enables us to get lower bounds for certain relative efficiencies over the class $\mathcal{D}$ defined above.

A particular result of this kind is contained in Bickel and Lehmann (1975b)(see Theorem 6) which proves that under the above set-up if (8.1.3) holds, then $E(T_\alpha, F) \leq e(T_\alpha, G)$.

8.2 LINEAR WEIGHT FUNCTIONS

The first result which we are going to prove is essentially a generalisation of Theorem 6 of Bickel and Lehmann (1975 b). Denote $G^{-1}(t)/F^{-1}(t)$ by $y(t)$, $(F^{-1}(t))^2$ by p_t and $(G^{-1}(t))^2$ by q_t .

Theorem 8.2.1. If G has heavier tail than F in the sense that, for some $0 < \alpha < 1/2$,

$$y(s) \leq y(1-\alpha) \leq y(t) \quad \text{for} \quad 1/2 < s \leq 1-\alpha \leq t < 1 ,$$

then

$$e(T_\alpha, F) \leq e(T_\alpha, G). \tag{8.2.1}$$

For the proof we need to strengthen Lemma 2b in Bickel and Lehmann (1975b) as follows :

Lemma 8.2.1. Let T_1 be a continuous distribution function on $(0, \infty)$ and T_2 be obtained by truncating T_1 at the point a. Let $\beta_1(x)$, $\beta_2(x)$ be positive functions, integrable w.r.t. T_1, s.t.

(i) $\beta_1(x)$ is non-decreasing and

(ii) $\dfrac{\beta_2(s)}{\beta_1(s)} \leq \dfrac{\beta_2(a)}{\beta_1(a)} \leq \dfrac{\beta_2(t)}{\beta_1(t)}$ for $0 < s \leq a \leq t < \infty$

then

$$\frac{\int \beta_2(x)\, d\, T_1(x)}{\int \beta_1(x)\, d\, T_1(x)} \geq \frac{\int \beta_2(x)\, d\, T_2(x)}{\int \beta_1(x)\, d\, T_2(x)} .$$

<u>Proof</u>. Define $T_o(x) = T_1(x)$ in $(0, a)$,

$$= \frac{1 + T_1(x)}{2} \quad \text{in } (a, \infty).$$

Let

(8.2.2) $$T_1^*(t) = \int_0^t \beta_1(x)\, d\, T_1(x) \Big/ \int_0^\infty \beta_1(x)\, d\, T_1(x).$$

Similarly, we define T_2^* and T_o^* replacing T_1 by T_2 and T_o respectively in (8.2.2) so that $\frac{d\,T_1^*}{d\,T_o^*}$ and $\frac{d\,T_2^*}{d\,T_o^*}$ exist (these expressions stand for Radon-Nikodym derivatives).

Set $\int_0^\infty \beta_1(x)\, d\, T_i(x) = \rho_i$ for $i = 0,1,2$. Then, the following are easy to verify

(8.2.3) $$\frac{d\,T_1^*}{d\,T_o^*} = \frac{\rho_o}{\rho_1} I(0, a) + \frac{2\,\rho_o}{\rho_1} I(a, \infty)$$

(8.2.4) $$\frac{d\,T_2^*}{d\,T_o^*} = \frac{\rho_o}{\rho_2} I(0, a) + \frac{2\,\rho_o}{\rho_2} I\{a\}.$$

Also, $\beta_1(x)$ is non-decreasing $\Longrightarrow$ $\rho_1 \geq \rho_2$. This fact, together with condition (ii) of the lemma, (8.2.3) and (8.2.4), implies that

$$(8.2.5) \qquad \left[\frac{\beta_2(x)}{\beta_1(x)} - \frac{\beta_2(a)}{\beta_1(a)} \right] \left[\frac{d\,T_1^*}{d\,T_o^*} - \frac{d\,T_2^*}{d\,T_o^*} \right] \geq 0 \quad \forall\, x \geq 0 .$$

Integrating (8.2.5) w.r.t. T_o^* we get the required result.

Proof of Theorem 8.2.1. By formula 8.1.1, we can write

$$(e(T_\alpha, F))^{-1} = \left[\int_{1/2}^{1-\alpha} p_t \, dt + \alpha\, p_t \right] / \left(\int_{1/2}^{1} p_t \, dt \right) (1 - 2\alpha)^2$$

(F is symmetric) and a similar expression holds for $e(T_\alpha, G)$. Now, the proof follows directly from the above lemma by putting $a = 1 - \alpha$, $\beta_1(t) = p_t$, $\beta_2(t) = q_t$ and T_1 as uniform distribution $[1/2, 1]$.

Corollary 8.2.1. Let $\alpha_o = \sup \{ \alpha \leq 1/2 : y(t)$ is non-decreasing in $[1 - \alpha, 1) \}$ and $y(s) \leq y(1 - \alpha_o) \;\forall\, 1/2 < s \leq 1 - \alpha_o$, then $\forall\, \alpha \leq \alpha_o$, (8.2.1) holds. For $\alpha_o = 1/2$, Theorem 6 of Bickel and Lehmann (1975) is a special case of this corollary.

Corollary 8.2.2. Let $t_o = \inf \{ t \leq 1/2 : y(t)$ is non-decreasing in $(\frac{1}{2}, 1-t)$ and $y(1 - t_o) \leq y(s)$ for $(1 - t_o) \leq s < 1$ then $\forall\, \alpha > t_o$, (8.2.1) holds.

Remark 8.2.1. The following result shows that symmetry of F and G can also be relaxed to a little extent.

Let F and G be such that $F^{-1}(t) = -F^{-1}(1-t)$, $G^{-1}(t) = -G^{-1}(1-t)$ for $t \le \alpha$. Then, under the conditions of theorem, an analogous condition in $(0, 1/2]$, and zero mean, (8.2.1) holds.

Proof. In effect, we have to show that

$$\frac{\int_{\alpha}^{1/2} p_t\, dt + \int_{1/2}^{1-\alpha} p_t\, dt + \alpha p_\alpha + \alpha p_{(1-\alpha)}}{\int_0^{1/2} p_t\, dt + \int_{1/2}^{1} p_t\, dt} \ge \text{c.e.G.}$$

$\Longleftrightarrow$

$$\frac{\left[\int_0^{\alpha} p_t\, dt - \alpha p_\alpha\right] + \left[\int_{1-\alpha}^{1} p_t\, dt - \alpha p_{(1-\alpha)}\right]}{\int_0^{1/2} p_t\, dt + \int_{1/2}^{1} p_t\, dt} \le \text{c.e.G.} \qquad (8.2.6)$$

where, here and elsewhere c.e.G means the corresponding expression for G.

The statement (8.2.6) follows from the following four inequalities :

(Use the fact that $\frac{a_1 + b_1}{c_1 + d_1} \le \frac{a_2 + b_2}{c_2 + d_2}$ if $\frac{a_1}{c_1} \le \frac{a_2}{c_2}$, $\frac{b_1}{d_1} \le \frac{b_2}{d_2}$, $\frac{a_1}{d_1} \le \frac{a_2}{d_2}$ and $\frac{b_1}{c_1} \le \frac{b_2}{c_2}$).

(i) $\left[\int_0^{\alpha} y(t)dt - \alpha\, y(\alpha) \right] / \int_0^{1/2} y(t)\, dt \leq$ c.e. G.

(ii) $\left[\int_{1-\alpha}^{1} y(t)dt - \alpha\, y(1 - \alpha) \right] / \int_{1/2}^{1} y(t)dt \leq$ c.e. G.

(iii) $\left[\int_0^{\alpha} y(t)dt - \alpha\, y(\alpha) \right] / \int_{1/2}^{1} y(t)dt \leq$ c.e. G.

(iv) $\left[\int_{1-\alpha}^{1} y(t)dt - \alpha\, y(1 - \alpha) \right] / \int_0^{1/2} y(t)dt \leq$ c.e. G.

The inequality (ii) is equivalent to

(8.2.7) $\left[\int_{1/2}^{1-\alpha} y(t)dt + \alpha\, y(1 - \alpha) \right] / \int_{1/2}^{1} y(t)\, dt \geq$ c.e. G

which is immediate from Lemma 8.2.1. The inequality (i) is proved analogously. (iii) is equivalent to (ii) and (iv) is equivalent to (i) due to the partial symmetry assumed in the remark.

The next theorem proved below studies the behaviour of efficiencies (w.r.t. mean) of a class of L-estimators which ignore tails to a lesser extent than trimmed means. The result can be obtained, under stronger assumptions, as a particular case of theorem 8.3.1 (see below) but the proof included here is of particular interest. In the statements which follow, $\bar{\bar{U}}$ stands for uniform distribution on [0,1] which, of course, leads to mean.

For convenience, write

$$\int_{1-\alpha}^{1} p_t \, dt = \mu_1, \quad \int_{1-\alpha}^{1} q_t \, dt = \mu_2, \quad \int_{1/2}^{1} p_t \, dt = \upsilon_1 \quad \text{and}$$

$$\int_{1/2}^{1} q_t \, dt = \upsilon_2 .$$

<u>Theorem 8.2.2</u> Let w be given by $w = c$ in $[\alpha, 1-\alpha]$ (with $1 < c < 1/(1-2\alpha)$), $= d$ otherwise s.t. $\int_0^1 w(t)dt = 1$. Assume $y(t)$ is non-decreasing in $[1-\alpha, 1)$ and for $1/2 < s \leq (1-\alpha)$, $y(s) \leq y(1-\alpha)$. Then, if W denotes d.f. corresponding to density function w,

$$e(W, \bar{U}, F) \leq c(W, \bar{U}, G).$$

<u>Proof</u>. After a little computation, it follows that a sufficient condition for the above result is

$$\text{(8.2.8)} \qquad \frac{1}{\upsilon_1}\left[\mu_1 - \int_{1-\alpha}^{1} (F^{-1}(1-\alpha) + \frac{d}{c}(F^{-1}(t) - F^{-1}(1-\alpha)))^2 dt\right]$$

$$\leq \text{c.e. } G.$$

Using the conditions of the above theorem, it follows easily that

$$\frac{\int_{1-\alpha}^{1} p_t \, dt}{\int_{1-\alpha}^{1} q_t \, dt} \leq \frac{\int_{1/2}^{1-\alpha} p_t \, dt}{\int_{1/2}^{1-\alpha} q_t \, dt}$$

which is equivalent to

(8.2.9) $$\mu_1/\mu_2 \leq \upsilon_1/\upsilon_2$$

It can be shown that the numerators in both sides of the inequality (8.2.8) are positive by proving that $e(W,U,F) > 1/c^2$ and $e(W,U,G) > 1/c^2$. Notice that without strict inequality it follows straightaway from Theorem 5 of Bickel and Lehmann(1975

In view of these facts, (8.2.8) follows if we show that

$$[\mu_1 - \int_{1-\alpha}^{1} (F^{-1}(1-\alpha) + \frac{d}{c}(F^{-1}(t) - F^{-1}(1-\alpha)))^2 \, dt]/\mu_1 \leq \text{c.e. } G$$

or equivalently,

$$\mu_1 / \int_{1-\alpha}^{1} (F^{-1}(t) + K\, F^{-1}(1-\alpha))^2 \, dt \leq \text{c.e.} G$$

since $0 < d/c < 1$ ($K > 0$ is a constant).

A sufficient condition for this is

(8.2.10) $$\mu_1/\mu_2 \leq \frac{F^{-1}(1-\alpha)}{G^{-1}(1-\alpha)} \; \frac{\int_{1-\alpha}^{1} F^{-1}(t)dt}{\int_{1-\alpha}^{1} G^{-1}(t)\, dt} .$$

To establish (8.2.10), note that

$$\int_{1-\alpha}^{1} p_t \, dt = \int_{1-\alpha}^{1} \frac{1}{y(t)} F^{-1}(t)\, G^{-1}(t) dt \le \frac{1}{y(1-\alpha)} \int_{1-\alpha}^{1} F^{-1}(t)\, G^{-1}(t) dt.$$

Hence, it is enough to show that

$$\frac{\int_{1-\alpha}^{1} F^{-1}(t)\, G^{-1}(t)\, dt}{\int_{1-\alpha}^{1} q_t \, dt} \le \frac{\int_{1-\alpha}^{1} F^{-1}(t) dt}{\int_{1-\alpha}^{1} G^{-1}(t)\, dt}$$

or

$$\text{(8.2.11)} \quad \frac{\int_{1-\alpha}^{1} [G^{-1}(t)]\, F^{-1}(t)\, dt}{\int_{1-\alpha}^{1} F^{-1}(t)\, dt} \le \frac{\int_{1-\alpha}^{1} [G^{-1}(t)]\, G^{-1}(t)\, dt}{\int_{1-\alpha}^{1} G^{-1}(t)\, dt}$$

Since $G^{-1}(t)$ is non-decreasing function, (8.2.11) follows from the arguments of stochastic ordering if we show that, $\forall\ (1-\alpha) \le t \le 1$, one has

$$\frac{\int_{1-\alpha}^{t} F^{-1}(t)\, dt}{\int_{1-\alpha}^{1} F^{-1}(t)\, dt} \ge \text{c.e. } G$$

or

$$\frac{\int_{1-\alpha}^{t} F^{-1}(t) dt}{\int_{1-\alpha}^{t} G^{-1}(t) dt} \ge \frac{\int_{1-\alpha}^{1} F^{-1}(t)\, dt}{\int_{1-\alpha}^{1} G^{-1}(t)\, dt}$$

But this is immediate from the fact that $y(t)$ is non-decreasing in $[1-\alpha, 1)$. This finishes the proof of this theorem.

Corollary 8.2.3. Let $\alpha_o = \sup\{\alpha \leq 1/2 : y(t)$ is non-decreasing in $[1-\alpha, 1)\}$ and $y(s) \leq y(1-\alpha_o)\ \forall\ 1/2 \leq s \leq (1-\alpha_o)$ then, the result of Theorem 8.2.2 holds for $\forall \alpha \leq \alpha_o$.

Remark 8.2.2. Here also symmetry of underlying d.f.s can be relaxed in the ~~central~~ part.

8.3 GENERAL WEIGHT FUNCTIONS PERMITTING DENSITIES

Let us write $f_t = f(F^{-1}(t))$, $g_t = g(G^{-1}(t))$ and $\bar{y}(t) = g_t/f_t$, where f and g denote densities of F and G, respectively. In this section, we introduce a new condition for the comparison of heaviness of tails, i.e., G has heavier tails than F if

(8.3.1) $\bar{y}(t)$ is non-increasing in $(1/2, 1)$.

It follows from corollary 8.3.1 that this condition is stronger than (8.1.3). One also observes that condition (8.3.1) is transitive and invariant under scale transformations of one distribution or both. The following distributions are arranged in increasing order of heaviness of tails according to to the criterion (8.3.1).

Example 8.3.1.

(i) Distribution with density $\frac{3}{2} x^2 e^{-|x|^3}$,

$$f(F^{-1}(t)) = 3[-\log 2(1-t)]^{2/3} (1-t)$$

(ii) Distribution with density $|x| e^{-x^2}$,

$$f(F^{-1}(t)) = [-\log 2(1-t)]^{1/2} 2(1-t)$$

(iii) Double exponential, $f(F^{-1}(t)) = (1-t)$.

Now, we prove a lemma which is of central importance in this section and also in the next section.

Lemma 8.3.1. Let us recall the definition of $\theta(W, F, t)$ given by (8.1.2). If F, G satisfy (8.3.1), then

$$\frac{\theta(W,G,t)}{\theta(W,F,t)} \quad \text{is non-decreasing in } (1/2, 1).$$

Proof. Set $1/2 < s < s' < 1$. Using (8.3.1), let us observe that

$$\frac{\theta(W,F,s)}{\theta(W,G,s)} = \frac{\int_{1/2}^{s} \bar{y}(t)\,(1/g_t)\,dW(t)}{\int_{1/2}^{s} (1/g_t)\,dW(t)}$$

$$\geq \bar{y}(s) \geq \frac{\int_s^{s'} \bar{y}(t)\,(1/g_t)\,dW(t)}{\int_s^{s'} (1/g_t)\,dW(t)}$$

From this the lemma follows easily.

Corollary 8.3.1. $\bar{y}(t)$ is non-increasing in $(1/2,1) \Longrightarrow y(t)$ is non-decreasing in $(1/2,1)$. Obviously, uniform distribution has the lightest tail in class $\mathcal{D}$ defined earlier according to criterion (8.3.1) and hence, according to criterion (8.1.3) also. This fact enables us to compute the infima of many relative efficiencies over the class $\mathcal{D}$. For example, $e(T_\alpha, U) = 1/(1+4\alpha) = \inf \{e(T_\alpha, F) : F \varepsilon \mathcal{D}\}$, a fact established by Bickel (1965) differently. Same procedure will work for $e(W, \bar{U}, F)$ of Theorem 8.2.2 also.

Using the lemma proved above, we establish a theorem which appears to be a quite general result of this kind.

Theorem 8.3.1. Let w_1, w_2 be densities of d.f.s W_1 and W_2 respectively which are symmetric about 1/2. Let $w_o = w_1/w_2$ be non-decreasing in $[1/2,1]$ and F, G satisfy (8.3.1). Then,

$$e(W_2, W_1, F) \leq e(W_2, W_1, G).$$

Proof. In effect, we want to prove

$$\frac{\int_{1/2}^{1} (\theta(W_1,F,t))^2\, dt}{\int_{1/2}^{1} (\theta(W_2, F, t))^2\, dt} \leq \text{ c.e. } G$$

or equivalently,

(8.3.2) $$\int_{1/2}^{1} (\theta(W_1,F,t)/\theta(W_2,F,t))^2 (\theta(W_2,F,t))^2 dt \Big/ \int_{1/2}^{1} (\theta(W_2,F,t))dt \leq \text{ c.e. } G.$$

We shall, first show that

(8.3.3) $$\theta(W_1,F,t)/\theta(W_2,F,t) \leq \text{ c.e. } G.$$

To see this, rewrite the inequality as

(8.3.4) $$\frac{\int_{1/2}^{t} w_o(t)\ (w_2(t)/f_t)dt}{\int_{1/2}^{t} (w_2(t)/f_t)\ dt} \leq \text{ c.e. } G.$$

It follows from the fact that $\theta(W_2,F,t)/\theta(W_2,G,t)$ is non-increasing that

$$(8.3.5)\qquad \frac{\int_{1/2}^{s} (w_2(x)/f_x)dx}{\int_{1/2}^{t} (w_2(x)/f_x)dx} \geq \frac{\int_{1/2}^{s} (w_2(x)/g_x)dx}{\int_{1/2}^{t} (w_2(x)/g_x)dx}$$

$\forall\ 1/2 \leq s < t$. The stochastic ordering given by (8.3.5) and the condition that $w_o(t)$ is non-decreasing in (1/2,1) yield (8.3.4).

In view of (8.3.3), (8.3.2) follows if we show that

$$(8.3.6)\qquad \int_{1/2}^{1} (\theta(W_1,F,t)/\theta(W_2,F,t))^2(\theta(W_2,F,t))^2dt/\int_{1/2}^{1}(\theta(W_2,F,t))^2dt$$

$$\leq \int_{1/2}^{1}(\theta(W_1,F,t)/\theta(W_2,F,t))^2(\theta(W_2,G,t))^2dt/\int_{1/2}^{1}(\theta(W_2,G,t))^2dt$$

Once again we employ the argument of stochastic ordering to see(8.3.6). We shall show that

$$(8.3.7)\qquad \theta(W_1,F,t)/\theta(W_2,F,t) \text{ is non-decreasing in } (1/2,1)$$

and

$$(8.3.8)\qquad \int_{1/2}^{z} (\theta(W_2,F,t))^2/\int_{1/2}^{1}(\theta(W_2,F,t))^2dt \geq \text{c.e. } G$$

$$\forall\ z \in (1/2,1).$$

(8.3.6) is concluded from (8.3.7) and (8.3.8) as in the proof of (8.3.3).

For (8.3.7), set $1/2 < s < s' < 1$. The inequality is a simple consequence of the following observation.

$$\frac{\int_{1/2}^{s} w_o(t)(w_2(t)/f_t)dt}{\int_{1/2}^{s} (w_2(t)/f_t)dt} \leq w_o(s) \leq \frac{\int_{s}^{s'} w_o(t)\,(w_2(t)/f_t)dt}{\int_{s}^{s'} (w_2(t)/f_t)dt}$$

Finally, (8.3.8) follows from the following inequalities. Writing

$$\theta^*(t) = (\theta(W_2,F,t)/\theta(W_2,G,t))^2$$

$\forall\, t \in (1/2,1)$, we see that $\theta^*(t)$ is non-increasing in $(1/2,1)$ (a consequence of lemma 8.3.1). Hence

$$\frac{\int_{1/2}^{z} \theta^*(t)\,(\theta(W_2,G,t))^2 dt}{\int_{1/2}^{z} (\theta(W_2,G,t))^2\, dt} \geq \theta^*(z) \geq \frac{\int_{z}^{1} \theta^*(t)(\theta(W_2,G,t))^2 dt}{\int_{z}^{1} (\theta(W_2,G,t))^2 dt}$$

This yields (8.3.8) completing the proof of the theorem.

With this theorem, we can derive various interesting results similar to Theorems 8.2.1 and 8.2.2 for general weight functions permitting densities and compute the corresspoding infima of relative efficiencies over the class $\mathcal{D}$ of d.f. s.

8.4 WEIGHT FUNCTIONS NOT PERMITTING DENSITIES

In this section, we shall study similar properties of L-statistics for the weight functions which have positive masses at few points.

Following the notations adopted in section 8.1

$$[\mu_1+\alpha p_{(1-\alpha)}]\ e(T_\alpha,\ WIN_\alpha, P) = (1-2\alpha)^2\ [\mu_1+\alpha(F^{-1}(1-\alpha)+\alpha/f_{(1-\alpha)})^2]$$

So, obviously, $\infty \geq e(T_\alpha,\ WIN_\alpha, F) \geq (1-2\alpha)^2$ with both ends sharp in (i.e., both the bounds can be approached controlling $f_{(1-\alpha)}$).

The theorem proved below throws some light on sensitivities of trimmed means and Winsorised means for the tails of underlying distributions. Also, it enables us to find $\inf\{e(T_\alpha,\ WIN_\alpha,\ F),\ F \in \mathcal{D}\}$.

Theorem 8.4.1. If for two symmetric distribution F and G, $y(t)$ is non-decreasing in $(1/2,\ 1-\alpha]$, then

$$e(T_\alpha,\ WIN_\alpha,\ F) \leq e(T_\alpha,\ WIN_\alpha,\ G).$$

Proof. Since the condition that $y(t)$ is non-decreasing and $e(T_\alpha,\ WIN_\alpha,\ G)$ are unaffected by scale transformations on F and G, it is enough to prove that $e(T_\alpha, WIN_\alpha, F) \leq e(T_\alpha, WIN_\alpha, G^*)$

where $G^*(x) = G(y(1-\alpha)x)$. Write $g^* = dF^*/dx$, $g^*_t = g^*(G^{*-1}(t))$ and $q^*_t = (G^{*-1}(t))^2$.

An easy calculation shows that the theorem is true if we show

$$\left[\left(\frac{\alpha}{f_{(1-\alpha)}}\right)^2 + \frac{2\alpha\, F^{-1}(1-\alpha)}{f_{(1-\alpha)}}\right] \Big/ \left(\int_{1/2}^{1-\alpha} p_t \, dt + \alpha\, p_{1-\alpha}\right)$$

$\leq$ the corressponding expression for G^*.

The above mentioned inequality is true under the light of following observations :

(i) G^{*-1}/F^{-1} is non-decreasing in $(1/2, 1-\alpha]$ and $G^{*-1}(1-\alpha) = F^{-1}(1-\alpha)$ imply that $G^{*-1} \leq F^{-1}$ in $(1/2, 1-\alpha]$

(ii) $\left[\frac{d(G^{-1}(t)/F^{-1}(t))}{dt}\right]_{t = 1-\alpha} \geq 0$ implies that $f_{(1-\alpha)} \geq g^*_{(1-\alpha)}$.

The proof of this theorem is complete.

The theorem which follows is analogous to Theorem 8.3.1 for Winsorized type estimators. It appears to be a general result of this kind.

Theorem 8.4.2. Let F and G satisfy (8.3.1) in $(1/2, 1-\alpha]$. Let w_1 and w_2 be densities of two symmetric d.f.s W_1 and W_2 respectively on $[0,1]$ s.t. $w_o = w_1/w_2$ be non-decreasing in $[1/2, 1]$. Define $w_2^* = c\, w_2$ in $[\alpha, 1-\alpha]$, $= 0$ otherwise, where c is a normalising constant. Let $w_1^* = w_1$ in $[\alpha, 1-\alpha]$, $= 0$ in $[\alpha, 1-\alpha]^c$. Let W_1^* be the d.f. on $[0,1]$ whose absolutely continuous has by the density w_1^* and have two atomic points α and $(1-\alpha)$ which share the rest of the mass equally. If W_2^* denotes the d.f. whose the density is w_2^*, then

$$e(W_2^*, W_1^*, F) \leq e(W_2^*, W_1^*, G).$$

We first prove a lemma which is quite similar to Theorem 8.4.1.

Lemma 8.4.1. Let $w_1^{**} = d\, w_1$ in $[\alpha, 1-\alpha]$, $= 0$ otherwise, where d is a normalising constant. Let W_1^{**} denote the d.f. corressponding to the density w_1^{**}, then

$$e(W_1^{**}, W_1^*, F) \leq e(W_1^{**}, W_1^*, G).$$

Proof. The arguments are parallel to that of Theorem 8.4.1. If we apply the transformation $G^*(x) = G(x/a)$, then

$$\theta(W_1, G^*, t) = a\,\theta(W_1, G, t) \quad \text{and} \quad \frac{d(\theta(W_1, G, t))}{dt} = \frac{w_1(t)}{g_t}.$$ For

this result, we apply the transformation

$$G^*(x) = G(\theta(W_1, G, 1-\alpha)\, x/\theta(W_1, F, 1-\alpha))$$

and mimic the proof of theorem 8.4.1.

Proof of Theorem 8.4.2. Notice that

$$e(W_2^*, W_1^*, F)$$
$$= e(W_2^*, W_1^{**}, F) \cdot e(W_1^{**}, W_1^*, F)$$
$$\leq e(W_2^*, W_1^{**}, G) \cdot e(W_1^{**}, W_1^*, G)$$
$$= e(W_2^*, W_1^*, G)$$ (using Lemma 8.4.1 and a consequence of Theorem 8.3.1).

This establishes the theorem.

Remark 8.4.1. $\infty \geq e(W_2^{**}, W_1^*, F) \geq 1/d^2$, with both ends sharp.

Remark 8.4.2. Theorem 8.4.1 gives us the following interesting numerical result.

$$\inf \{e(T_\alpha, WIN_\alpha, F) : F \in \mathcal{D}\} = e(T_\alpha, WIN_\alpha, \bar{U})$$

$$= (1 - 8\alpha^3 + 12\alpha^2)/(1 + 4\alpha).$$

For $\alpha = .05$, this quantity is about .86 while the universal lower bound is .81. In the light of these facts, one is safe in preferring trimmed means to Winsorized means.

Finally, we turn to the study of the behaviour of relative efficiencies of Winsorised means to means. Such nice comparisons are not available in this case and, as a consequence, uniform distribution is not the least favourable distribution for this efficiency (also see Bickel (1965)). The following result which is proved under a somewhat condition throws some light on the behaviour of this efficiency.

<u>Theorem 8.4.3</u>. If $y(t)$ is constant in $(1/2, 1-\alpha]$ and non-decreasing in $[1-\alpha_0, 1)$, then

$$e(WIN_\alpha, F) \leq e(WIN_\alpha, G) \quad \forall \alpha > \alpha_0 .$$

($e(WIN_\alpha, F)$ is defined in section 8.1).

<u>Proof</u>. In effect, we want to show that

$$(8.4.1) \qquad \frac{\mu_1 - \alpha[F^{-1}(1-\alpha) + \alpha/f_{(1-\alpha)}]^2}{\nu_1 - \mu_1 + \alpha[F^{-1}(1-\alpha) + \alpha/f_{(1-\alpha)}]^2} \leq \text{c.e. } G$$

($\mu_1, \mu_2, \nu_1, \nu_2$ are defined in section 8.2).

Let G^*, g_t^* and q_t^* be the same as defined in the proof of Theorem 8.4.1. Then, (8.4.1) can be concluded immediately after replacing G by G^* (which is again good enough, because everything is unaffected by scale transformations) from the following simple observations :

(i) $\int_{1/2}^{1-\alpha} p_t \, dt =$ the corresponding expression for G^*.

(ii) $\int_{1-\alpha}^{1} p_t \, dt \leq$ the corresponding expression for G^*.

(iii) $f_{(1-\alpha)} = g^*_{(1-\alpha)}$.

Example 8.4.1. Following is an example of the situation described in Theorem 8.4.3. Let $F(\sigma x)$ have MLR in σ. $G(x) = F(x/\sigma_o)$ for $|x| < 0$ and $= \frac{1}{2} F(x/\sigma_1) + \frac{1}{2} F(x/\sigma_2)$ for $|x| \geq z$, $z > 0$, where $\sigma_o, \sigma_1, \sigma_2$ are such that $F(z/\sigma_o) = \dfrac{F(z/\sigma_1) + F(z/\sigma_2)}{2}$. Here $1 - \alpha_o = F(z/\sigma_o) = G(z)$. The required property follows from a theorem stated in Bickel and Lehmann (1975b) p. 1062.

Remark 8.4.3. As $\alpha_o \longrightarrow 1/2$ in Theorem 8.4.3, the condition of constant $y(t)$ tends to become void. Heuristically, this explains $e(WIN_\alpha, U) \longrightarrow 1/3$ as $\alpha_o \longrightarrow 1/2$ where $1/3$ is $\inf \{e(WIN_\alpha, F) : F \in \mathcal{D}\}$ (see Bickel 1965).

Remark 8.4.4. Theorem 8.4.3 admits the following generalisation. Let $\bar{y}(t)$ be constant in $(1/2, 1 - \alpha_o)$ and non-increasing after that. Let W be a d.f. on $[0,1]$ (having density) and W^* be obtained by Winsorizing W at points α and $1 - \alpha$, where $\alpha \approx \alpha_o$. Then, $e(W^*, W, F) \leq e(W^*, W, G)$.

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