71

SOME CONTRIBUTIONS TO DESCRIPTIVE SET THEORY

HAIMANTI SARBADHIKARI

INDIAN STATISTICAL INSTITUTE
CALCUTTA
1977

ACKNOWLEDGEMENTS

I am grateful to the Indian Statistical Institute for extending to me the facilities needed to carry out this work.

I am grateful to Professor Ashok Maitra for supervising this work, for bringing to my notice most of the problems discussed here and for many helpful suggestions.

I thank Dr. B. V. Rao for many discussions. The original versions of some of the proofs in this work were simplified by him.

I thank Dr. K.P.S.B.Rao and Dr. E.Grzegorek for allowing me to include some of our joint work in this thesis. I also thank Dr. K.P.S.B.Rao for some discussions.

I wish to thank Dr. S. C. Bagchi for spending much time in translating some papers needed for this work.

I thank Mr. Gour Mohon Das and Mr. Apurba Guha for typing and cyclostyling this thesis respectively.

Finally, I am grateful to all those members of the staff and research fellows of the Indian Statistical Institute who helped me in various ways.

H. Sarbadhikari

CONTENTS

p.t.o.

Page

INTRODUCTION AND SUMMARY

Current interest in the descriptive theory of sets stems largely from the many applications that the classical theory has found in probability theory, functional analysis, dynamic programming etc. As examples, we cite the fundamental paper of Blackwell [5] wherein he has shown that many of the pathologies of probability theory can be avoided if one takes the basic probability space to be an analytic set. As another example, we mention that several writers [7], [23] have shown that in Blackwell's model of dynamic programming [3] the existence of optimal strategies is related to the existence of measurable selectors.

The present thesis is partly motivated by problems relating to the theory of Borel and analytic sets that have their origin in the papers referred to above. However we wish to point out that the problems considered in the thesis are of independent interest in the descriptive theory of sets without reference to their applications in the fields mentioned above.

Chapters 1 - 3 deal with the existence of measurable selectors in a wide variety of situations. Chapter 6 deals with A-functions. Such functions arise naturally in problems of dynamic programming, see, for example, [7]. Chapter 4 deals with the class of measurable spaces that have come to be called Blackwell spaces. Such spaces are an abstraction of a property of analytic spaces noted by Blackwell in [5]. Chapter 5 deals with the complementation problem in the lattice of sub σ-fields of the Borel σ-field, a study which was initiated by D. Basu [2] in connection with problems of statistics and subfields.

Below we summarize the main results of the thesis.

The first chapter deals with selection theorems for multifunctions. The main result is the following:

Let X be any set and $\underline{\underline{H}}$ a family of subsets of X containing X and $\emptyset$ which is λ^+-additive, λ-multiplicative and satisfies the λ^+-WRP for some infinite cardinal λ. Let Y be a regular Hausdorff space of topological weight $\leq \lambda$. Suppose $F: X \longrightarrow \underline{\underline{C}}(Y)$ is such that $\{x: F(x) \cap C \neq \emptyset\} \in \underline{\underline{H}}$ for any closed subset C of Y. Then F admits a $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}$-measurable selector. The methods used can be imitated to prove a generalization of a result due to Sion [42]. They can also be used to give an alternative proof of the main result in [24].

The main result in the second chapter is a category analogue of a result of Blackwell and Ryll-Nardzewski in [6].

It is proved that a Borel subset of a product of Polish spaces with nonmeager sections admits a Borel uniformization.

The main result in Chapter 3 is the following: An α^{-}-partition of a complete metric space admits a selector of multiplicative Borel class α. This theorem generalizes the main result in [25] for Polish spaces.

In Chapter 4, it is shown that there exists a projective, non-analytic subset of the unit interval which is a strong Blackwell space. Some other properties of strong Blackwell spaces are also discussed.

In Chapter 5, it is shown that a countably generated sub σ-algebra of the Borel σ-algebra of an absolute Borel set has a relative minimal complement in the lattice of sub σ-fields of the Borel field.

In Chapter 6, we show that if f is an A-function which dominates a Borel function, then $f(x) = \sup_y g(x,y)$ for some Borel measurable function g. A similar result is proved about α^{-}-functions.

CHAPTER 1

SELECTION THEOREMS FOR MULTIFUNCTIONS

1. Introduction

In recent years the problem of the existence of nice selectors for multifunctions has received a great deal of attention. We single out the two following results which in some sense complement each other. The first result is due to Sion [42] the second to Kuratowski and Ryll-Nardzewski [20].

Theorem 1. Let X be an abstract set, $\underline{\underline{H}}$ a family of subsets of X. Suppose Y is a regular T_1 space of topological weight $\leq \aleph_1$ such that each family of open subsets of Y admits a countable subfamily with the same union. Let $F : X \to \underline{\underline{C}}(Y)$, the family of non-empty compact subsets of Y, such that $\{x : F(x) \cap C \neq \emptyset\} \in \underline{\underline{H}}$ for every closed set C in Y. Then there is a function $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$ and $f^{-1}(V) \in$ the σ-field generated by $\underline{\underline{H}}$ for every open set V in Y.

The result of Kuratowski and Ryll-Nardzewski is a sort of metrizable version of the above result.

Theorem 2. Let X be an abstract set, $\underline{\underline{L}}$ a field of subsets of X. Suppose Y is a Polish space and let $F : X \to 2^Y$, the family of non-empty closed subsets of Y, such that $\{x : F(x) \cap V \neq \emptyset\} \in \underline{\underline{L}}_\sigma$ for each open subset V of Y. Then there is a function $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$ and $f^{-1}(V) \in \underline{\underline{L}}_\sigma$ for each open set V in Y.

These results subsumed a large number of results on the existence of selectors which were scattered in the literature. While Theorems 1 and 2 could be applied to many interesting families of sets there were still certain familie[s] of sets, arising not infrequently in problems of Descriptive Set Theory, to which the theorems did not apply. To give just one such example in the context of Theorem 2, take X to be a Polish space and $\underline{\underline{H}}$ to be the family of coanalytic subsets of X. It is easily seen that the family $\underline{\underline{H}}$ cannot be expressed in the form $\underline{\underline{L}}_\sigma$, where $\underline{\underline{L}}$ is a field of subsets o[f] X. Furthermore in both results there are restrictions on the topological weight of the space Y.

In the context of Theorem 2 Maitra and Rao [24] have s[hown] recently that the family $\underline{\underline{L}}_\sigma$ can be replaced by families satisfying what they call the weak reduction principle for a certain cardinal which depends on the topological weight of th[e] space Y. The main purpose of this chapter is to show that an analogue of the result of Maitra and Rao can also be established in the non-metrizable framework in which Sion's result is set. Our methods are related to those used by Sion. And we show incidentally that these methods also apply to the metrizable situation and yield an alternative proof of the result of Maitra and Rao. Furthermore Sion's result falls out as a special case of ours.

2. Definitions and notation

Let X be any set, $\underline{\underline{H}}$ a family of subsets of X and τ any cardinal. We say that $\underline{\underline{H}}$ is τ-additive (τ-multiplicative) if whenever $\{A_\alpha : \alpha < \beta\} \subseteq \underline{\underline{H}}$ where $\beta < \tau$, $\bigcup_{\alpha<\beta} A_\alpha$ $(\bigcap_{\alpha<\beta} A_\alpha) \in \underline{\underline{H}}$. $\underline{\underline{H}}^c$ is the family of subsets of X whose complements belong to $\underline{\underline{H}}$ and $\underline{\underline{H}}_\tau$ is the smallest τ-additive family containing $\underline{\underline{H}}$. $\underline{\underline{H}}$ is said to satisfy the τ-weak reduction principle (τ-WRP) if given $\{A_\alpha : \alpha < \beta\} \subseteq \underline{\underline{H}}$, such that $\bigcup_{\alpha<\beta} A_\alpha = X$, where $\beta < \tau$, there exists a pairwise disjoint family of sets $\{B_\alpha : \alpha < \beta\} \subseteq \underline{\underline{H}}$ satisfying $B_\alpha \subseteq A_\alpha$ for all α and $\bigcup_{\alpha<\beta} B_\alpha = X$. $\aleph_0$ and $\aleph_1$ are used to denote the first infinite ordinal and the first uncountable ordinal respectively. (Note that cardinals are considered as initial ordinals).

If X is any set, $\underline{\underline{H}}$ a family of subsets of X and Y a topological space, then a function f on X into Y is called $\underline{\underline{H}}$-measurable if $f^{-1}(U) \in \underline{\underline{H}}$ for every open subset U of Y. f is called a selector for a multifunction F on X into the family of nonempty subsets of Y if $f(x) \in F(x)$ for all $x \in X$.

If $A \subseteq X \times Y$ for any sets X, Y, then A^x denotes the subset of Y given by $\{y : (x, y) \in A\}$. π_1 denotes the projection to the first co-ordinate on $X \times Y$.

For any subset A of a metric space, $\partial(A)$ stands for the diameter of A.

3. Main Results

Before proving the main theorem, we prove two propositions which will be used in the sequel.

Proposition 1. If Y is a complete metric space, F_n, $n = 1,2,\ldots$, a decreasing sequence of closed sets whose diameters tend to 0 as $n \to \infty$ and $\bigcap_n F_n \subseteq U$ where U is open, then there is an m such that $F_m \subseteq U$.

Proof. Let $F_n \not\subseteq U$ for any n. Then $F_n \cap U^c \neq \emptyset$ for $n = 1,2,\ldots$. Thus $F_n \cap U^c$, $n = 1,2,\ldots$, is a decreasing sequence of nonempty closed sets with diameters tending to 0 as $n \to \infty$. By completenessof Y, $\bigcap_n (F_n \cap U^c) \neq \emptyset$. Hence $\bigcap_n F_n \not\subseteq U$.

Proposition 2. Let Y be any topological space and K_n, $n = 1,2,\ldots$, a decreasing sequence of compact sets such that $\bigcap_n K_n \subseteq U$, U being an open set. Then $K_m \subseteq U$ for some m.

Proof is similar to that of proposition 1.

We now give an alternative proof of the following theorem of Maitra and Rao.

Theorem 1. Let X be any set and $\underline{\underline{H}}$ a family of subsets of X such that $\emptyset, X \in \underline{\underline{H}}$, $\underline{\underline{H}}$ is λ^+-additive, $\aleph_0$-multiplicative and satisfies the λ^+-WRP where λ is an infinite cardinal and λ^+ is its successor cardinal. If Y is a complete metric space of topological weight $\leq \lambda$ and $F: X \to 2^Y$ is a multifunction such that $\{x: F(x) \cap U \neq \emptyset\} \in \underline{\underline{H}}$ for any open $U \subseteq Y$, then F admits a $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}$-measurable selector.

Lemma. Let X, Y, $\underline{\underline{H}}$ and F be as in the theorem. Then there exists a sequence A_n, $n = 1,2,\ldots$, of subsets of $X \times Y$ satisfying the following for all n:

i) A_n^x is an open subset of Y for all x.

ii) For any $U \subseteq Y$, the sets $\{x: U \subseteq A_n^x\}$ and $\{x: \overline{A_n^x} \subseteq U\}$ belong to $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}$ and hence to $\underline{\underline{H}}$.

iii) $\overline{A_n^x} \subseteq A_{n-1}^x$ for all x and all $n > 1$.

iv) For all x, $\partial(A_n^x) < \frac{1}{n}$.

v) For all x, $A_n^x \cap F(x) \neq \emptyset$.

Proof. We construct the A_n's by induction as follows:

For each n, let $\{U_\alpha^n: \alpha < \lambda\}$ be an open base for Y such that $\emptyset \neq U_\alpha^n$ and $\partial(U_\alpha^n) < \frac{1}{n}$ for all α.

Put $C_\alpha^1 = \{x: F(x) \cap U_\alpha^1 \neq \emptyset\}$. Then $C_\alpha^1 \in \underline{\underline{H}}$ and $\bigcup_{\alpha<\lambda} C_\alpha^1 = X$. By λ^+-WRP of $\underline{\underline{H}}$, find a pairwise disjoint family of sets $\{B_\alpha^1: \alpha < \lambda\} \subseteq \underline{\underline{H}}$ such that $B_\alpha^1 \subseteq C_\alpha^1$ for all α and $\bigcup_{\alpha<\lambda} B_\alpha^1 = X$. As $\underline{\underline{H}}$ is λ^+-additive, $B_\alpha^1 \in \underline{\underline{H}} \cap \underline{\underline{H}}^c$ for all α. Put $A_1 = \bigcup_{\alpha<\lambda} (B_\alpha^1 \times U_\alpha^1)$. Clearly (i)-(v) are satisfied for $n = 1$.

Suppose A_n, $1 \leq n \leq m$ have been defined so that (i)-(v) are satisfied for $1 \leq n \leq m$. Put $C_\alpha^{m+1} = \{x: \overline{U_\alpha^{m+1}} \subseteq A_m^x$ and $F(x) \cap U_\alpha^{m+1} \neq \emptyset\}$. Now $\{x: \overline{U_\alpha^{m+1}} \subseteq A_m^x\} \in \underline{\underline{H}}$ by induction hypothesis and $\{x: F(x) \cap U_\alpha^{m+1} \neq \emptyset\} \in \underline{\underline{H}}$ by assumption, so since $\underline{\underline{H}}$ is $\aleph_0$-multiplicative, it follows that $C_\alpha^{m+1} \in \underline{\underline{H}}$ for all α. For any x, A_m^x is a nonempty open set by induction hypothesis. Hence $A_m^x = \bigcup_\alpha \{\overline{U_\alpha^{m+1}}: \overline{U_\alpha^{m+1}} \subseteq A_m^x\} = \bigcup_\alpha \{U_\alpha^{m+1}: \overline{U_\alpha^{m+1}} \subseteq A_m^x\}$. As $F(x) \cap A_m^x \neq \emptyset$ by induction hypothesis, there is some α such that $\overline{U_\alpha^{m+1}} \subseteq A_m^x$ and $F(x) \cap U_\alpha^{m+1} \neq \emptyset$. Hence $X = \bigcup_{\alpha < \lambda} C_\alpha^{m+1}$. By λ^+-WRP of $\underline{\underline{H}}$, there exists a pairwise disjoint family of sets $\{B_\alpha^{m+1}: \alpha < \lambda\} \subseteq \underline{\underline{H}}$ such that $\bigcup_{\alpha < \lambda} B_\alpha^{m+1} = X$ and $B_\alpha^{m+1} \subseteq C_\alpha^{m+1}$ for all α. Define $A_{m+1} = \bigcup_{\alpha < \lambda} (B_\alpha^{m+1} \times U_\alpha^{m+1})$. Clearly (i)-(v) are satisfied when $n = m+1$.

<u>Proof of the theorem</u>: Let A_n, $n = 1,2,\ldots$ be as in the lemma.

Put $G = \bigcap_n A_n$.

<u>Step 1</u>. G is the graph of a function f and f is a selector for F.

<u>Proof</u>. Let $x \in X$. Then $G^x = \bigcap_n A_n^x = \bigcap_n \overline{A_n^x}$ by (iii). As Y is complete, $\emptyset \neq \overline{A_{n+1}^x} \subseteq \overline{A_n^x}$ and $\partial(A_n^x) < \frac{1}{n}$ for all n, $\bigcap_n \overline{A_n^x}$ is a singleton. Define $f: X \to Y$ by $f(x) = y$ if $\{y\} = G^x$. Clearly, f is well defined.

As $G^x \cap F(x) = (\bigcap_n \overline{A_n^x}) \cap F(x) \neq \emptyset$ by (iii), (iv), (v), and the completeness of Y, $f(x) \in F(x)$.

Step 2. f is $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}$-measurable.

Let $U \subseteq Y$ be open. Then $f^{-1}(U) = \{x: \bigcap_n \overline{A_n^x} \subseteq U\}$ $= \{x: \overline{A_n^x} \subseteq U\}$ for some n. Thus $f^{-1}(U) = \bigcup_n \{x: \overline{A_n^x} \subseteq U\}$ $\in ((\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}) = (H \cap \underline{\underline{H}}^c)_{\lambda^+}$ as $\lambda \geq \aleph_0$

Corollary If X, Y, $\underline{\underline{H}}$ are as in theorem 1 and $F: X \to 2^Y$ is a multifunction such that $\{x: F(x) \cap C \neq \emptyset\} \in \underline{\underline{H}}$ for all closed $C \subseteq Y$, then F admits a $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}$ -measurable selector.

Proof. Let $U \subseteq Y$ be open. There exists a sequence C_n, $n = 1, 2, \ldots$, of closed subsets of Y such that $U = \bigcup_n C_n$. Now $\{x: F(x) \cap U \neq \emptyset\} = \bigcup_n \{x: F(x) \cap C_n \neq \emptyset\} \in \underline{\underline{H}}_{\aleph_1} = \underline{\underline{H}}$ as $\underline{\underline{H}}$ is λ^+-additive and hence $\aleph_1$-additive as λ is infinite.

Theorem 2. Let X be any set and $\underline{\underline{H}}$ a family of subsets of X, containing X and $\emptyset$, which is λ^+-additive, λ-multiplicative and satisfies the λ^+-WRP for some infinite cardinal λ. Let Y be a regular, Hausdorff space of topological weight $\leq \lambda$. Suppose $F: X \to \underline{\underline{C}}(Y)$ is a multifunction such that $\{x: F(x) \cap C \neq \emptyset\} \in \underline{\underline{H}}$ for any closed subset C of Y. Then F admits a $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}$ - measurable selector.

Note: Without loss of generality, we can take $\lambda > \aleph_0$ as otherwise, Y is metrizable and can be replaced by its completion so that the theorem can be deduced from the previous one.

Lemma. Let X, Y, $\underline{\underline{H}}$ and F be as above. Let $\{U_\alpha : \alpha$ is a successor ordinal and $\alpha < \lambda\}$ be an open base for Y such that $U_\alpha \neq \emptyset$ for any α. Then there exists a family $\{A_\alpha : \alpha < \lambda\}$ of subsets of $X \times Y$ satisfying the following:

i) For each α and x, $\emptyset \neq A^x_\alpha \subseteq F(x)$ and A^x_α is compact.

ii) For each α, $\{x : A^x_\alpha \cap C \neq \emptyset\} \in \underline{\underline{H}}$ if $C \subseteq Y$ is closed.

iii) If $\alpha_1 < \alpha$, $A_\alpha \subseteq A_{\alpha_1}$ for all α and α_1.

iv) If α is a successor ordinal, then there exists

$B_\alpha \in \underline{\underline{H}} \cap \underline{\underline{H}}^c$ such that $(X \times \overline{U}_\alpha) \cap A_\alpha = (B_\alpha \times \overline{U}_\alpha) \cap A_\alpha =$ $(B_\alpha \times Y) \cap A_\alpha$.

Proof. We define the A_α's by induction as follows.

$A_0 = \bigcup_x (\{x\} \times F(x))$. Suppose A_β is defined for all $\beta < \alpha$ such that (i) - (iv) are satisfied if α is replaced by β.

Case 1. $\alpha = \beta + 1$ for some β.

For any successor ordinal γ, let $D^\beta_\gamma = \{x : A^x_\beta \cap \overline{U}_\gamma \neq \emptyset\}$. By induction hypothesis, $D^\beta_\gamma \in \underline{\underline{H}}$.

If $x \notin D^\beta_{\beta+1}$, $A^x_\beta \cap \overline{U}_{\beta+1} = \emptyset$. As $A^x_\beta \neq \emptyset$, by induction hypothesis, there is some basic open set U_γ such that $A^x_\beta \cap \overline{U}_\gamma \neq \emptyset$ and $\overline{U}_\gamma \cap \overline{U}_{\beta+1} = \emptyset$. Hence $X = D^\beta_{\beta+1} \cup$ $\bigcup_\gamma \{D^\beta_\gamma : \overline{U}_\gamma \cap \overline{U}_{\beta+1} = \emptyset\}$. Using λ^+-WRP of $\underline{\underline{H}}$, find a pairwise disjoint family of sets $B^\beta_{\beta+1}$, $\{B^\beta_\gamma : \overline{U}_\gamma \cap \overline{U}_{\beta+1} = \emptyset\}$ in $\underline{\underline{H}}$

such that $B^{\beta}_{\beta+1} \subseteq D^{\beta}_{\beta+1}$, $B^{\beta}_{\gamma} \subseteq D^{\beta}_{\gamma}$ if $\bar{U}_{\gamma} \cap \bar{U}_{\beta+1} = \emptyset$ and $B^{\beta}_{\beta+1} \cup \bigcup_{\gamma} \{ B^{\beta}_{\gamma} : \bar{U}_{\gamma} \cap \bar{U}_{\beta+1} = \emptyset \} = X$. Clearly $B^{\beta}_{\beta+1}$, $B^{\beta}_{\gamma} \in \underline{\underline{H}} \cap \underline{\underline{H}}^c$.

Define

$$A_{\beta+1} = ((B^{\beta}_{\beta+1} \times \bar{U}_{\beta+1}) \cup \bigcup_{\gamma} \{(B^{\beta}_{\gamma} \times \bar{U}_{\gamma}) : \bar{U}_{\gamma} \cap \bar{U}_{\beta+1} = \emptyset \}) \cap A_{\beta}$$

and $B_{\beta+1} = B^{\beta}_{\beta+1}$. (i), (iii) and (iv) are clearly satisfied by the induction hypothesis. To check (ii), let $C \subseteq Y$ be closed. Then $\{x : A^{x}_{\beta+1} \cap C \neq \emptyset\} = \{x : x \in B^{\beta}_{\beta+1}$ and $A^{x}_{\beta} \cap \bar{U}_{\beta+1} \cap C \neq \emptyset\}$ $\cup \bigcup_{\gamma} \{\{x : x \in B^{\beta}_{\gamma}$ and $A^{x}_{\beta} \cap \bar{U}_{\gamma} \cap C \neq \emptyset\} : \bar{U}_{\gamma} \cap \bar{U}_{\beta+1} = \emptyset\}$.

As $\underline{\underline{H}}$ is λ-multiplicative (and hence $\aleph_0$-multiplicative) and λ^{+}-additive, using the induction hypothesis we see that

$$\{x : A^{x}_{\beta+1} \cap C \neq \emptyset\} \in \underline{\underline{H}}$$

<u>Case 2.</u> α is a limit ordinal.

Let $A_{\alpha} = \bigcap_{\beta < \alpha} A_{\beta}$. As $\emptyset \neq A^{x}_{\beta} \subseteq F(x)$ for $\beta < \alpha$, each A^{x}_{β} is compact and $\{A^{x}_{\beta} : \beta < \alpha\}$ has the finite intersection property by (iii), it follows that $\emptyset \neq A^{x}_{\alpha} \subseteq F(x)$ and A^{x}_{α} is compact. Clearly (iii) is satisfied and (iv) does not need any verification as α is not a successor ordinal. To check (ii), let $C \subseteq Y$ be closed. $\{x : A^{x}_{\alpha} \cap C \neq \emptyset\} = \{x : (\bigcap_{\beta < \alpha} A^{x}_{\beta}) \cap C \neq \emptyset\} =$ $\{x : \bigcap_{\beta < \alpha} (A^{x}_{\beta} \cap C) \neq \emptyset\} = \bigcap_{\beta < \alpha} \{x : A^{x}_{\beta} \cap C \neq \emptyset\}$. The last equality is obtained by using the compactness of $A^{x}_{\beta} \cap C$, $\beta < \alpha$.

As $\alpha < \lambda$ and $\{x: A^x_\beta \cap C \neq \emptyset\} \in \underline{\underline{H}}$ for $\beta < \alpha$ by induction hypothesis, $\{x: A^x_\alpha \cap C \neq \emptyset\} \in \underline{\underline{H}}$ by λ-multiplicativity of $\underline{\underline{H}}$.

This completes the proof of the lemma.

Proof of the theorem. Let U_α, B_α, α is a successor ordinal $< \lambda$ and A_α, $\alpha < \lambda$ be as in the lemma. Put $G = \bigcap_{\alpha < \lambda} A_\alpha$.

Step 1. G is the graph of a function f and f is a selector for F.

By (i) and (iii) $\emptyset \neq G^x \subseteq F(x)$ for all x. We show that for all x, G^x is a singleton. If not, let there exist points (x,y), (x,z) in G where $y \neq z$. Find basic open sets $U_\alpha, U_\beta \subseteq Y$, where α, β are successor ordinals, such that $y \in U_\alpha$, $z \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$. Then $y \in U_\alpha \subseteq \bar{U}_\alpha$ and $z \notin \bar{U}_\alpha$. As $(x,y) \in G \subseteq A_\alpha$ it follows that $(x,y) \in (X \times \bar{U}_\alpha) \cap A_\alpha =$

$= (B_\alpha \times \bar{U}_\alpha) \cap A_\alpha$. Thus $x \in B_\alpha$ and therefore $(x,z) \in (B_\alpha \times Y) \cap A_\alpha$

$= (B_\alpha \times \bar{U}_\alpha) \cap A_\alpha$. Hence $z \in \bar{U}_\alpha$ which is a contradiction.

Define $f(x) = y$ if $\{y\} = G^x$.

Step 2. We now have to show that the function $f: X \to Y$ is $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}$-measurable.

Let $V \subseteq Y$ be open. Express V as $V = \bigcup_\alpha \{\bar{U}_\alpha : \bar{U}_\alpha \subseteq V\}$ so that $f^{-1}(V) = \bigcup_\alpha \{f^{-1}(\bar{U}_\alpha): \bar{U}_\alpha \subseteq V\} = \bigcup_\alpha \{(\pi_1(X \times \bar{U}_\alpha) \cap G): \bar{U}_\alpha \subseteq V\}$

It is enough to show that $\pi_1((X \times \bar{U}_\alpha) \cap G) \in H \cap \underline{\underline{H}}^c$ for any successor ordinal α.

Fix a successor ordinal α. Then $\pi_1((X \times \bar{\bar{U}}_\alpha) \cap G) = \pi_1((X \times \bar{\bar{U}}_\alpha) \cap \bigcap_{\gamma<\lambda} A_\gamma)$. We first note that $\pi_1((X \times \bar{\bar{U}}_\alpha) \cap \bigcap_{\gamma<\lambda} A_\gamma) = \bigcap_{\gamma<\lambda} \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\gamma)$. Clearly, $\pi_1((X \times \bar{\bar{U}}_\alpha) \cap \bigcap_{\gamma<\lambda} A_\gamma) \subseteq \bigcap_{\gamma<\lambda} \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\gamma)$. Let $x \in \bigcap_{\gamma<\lambda} \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\gamma)$. Then for all $\gamma < \lambda$, $A^x_\gamma \cap \bar{\bar{U}}_\alpha \neq \emptyset$. As $A^x_\gamma \cap \bar{\bar{U}}_\alpha$ is compact, using (iii), we see that $\bigcap_{\gamma<\lambda} (A^x_\gamma \cap \bar{\bar{U}}_\alpha) \neq \emptyset$ so that $x \in \pi_1((X \times \bar{\bar{U}}_\alpha) \cap \bigcap_{\gamma<\lambda} A_\gamma)$.

Again using (iii), we obtain $\bigcap_{\gamma<\lambda} \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\gamma) = \bigcap_{\alpha \leq \gamma<\lambda} \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\gamma)$. Hence $\pi_1((X \times \bar{\bar{U}}_\alpha) \cap G) =$

$$\bigcap_{\alpha \leq \gamma<\lambda} \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\gamma)$$

We next prove that $\bigcap_{\alpha \leq \gamma<\lambda} \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\gamma) = \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\alpha)$. Clearly, $\bigcap_{\alpha \leq \gamma<\lambda} \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\gamma) \subseteq \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\alpha)$. Let $x \in \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\alpha) = \pi_1((B_\alpha \times Y) \cap A_\alpha)$ (by (iv)). Then $x \in B_\alpha$ and hence $A^x_\alpha \subseteq \bar{\bar{U}}_\alpha$. If $\alpha \leq \gamma < \lambda$, $\emptyset = A^x_\gamma \subseteq A^x_\alpha \subseteq \bar{\bar{U}}_\alpha$ so that $A^x_\gamma \cap \bar{\bar{U}}_\alpha \neq \emptyset$. Hence $x \in \pi_1((X \times \bar{\bar{U}}_\alpha) \cap$ for $\alpha \leq \gamma < \lambda$.

Thus $\pi_1((X \times \bar{\bar{U}}_\alpha) \cap G) = \pi_1((X \times \bar{\bar{U}}_\alpha) \cap A_\alpha) = \pi_1((B_\alpha \times Y) \cap A_\alpha) = B_\alpha \in \underline{\underline{H}} \cap \underline{\underline{H}}^c$.

<u>Remark</u>. We do not know if theorem 2 holds if in the condition '$\{x : F(x) \cap C \neq \emptyset\} \in \underline{\underline{H}}$ for any closed subset C of Y' 'closed' is replaced by 'open'. However, we have the following:

Theorem 3. If X, Y, $\underline{\underline{H}}$ are as in theorem 2 and moreover $\underline{\underline{H}}$ is λ^+-multiplicative and if $F: X \to \underline{\underline{C}}(Y)$ is such that $\{x: F(x) \cap U \neq \emptyset\} \in \underline{\underline{H}}$ for any open $U \subseteq Y$, then F admits a $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_{\lambda^+}$-measurable selector.

Proof. Let $\{U_\alpha: \alpha < \lambda\}$ be a base for Y consisting of nonempty open sets and let $C \subseteq Y$ be closed. Then $Y - C = \bigcup_\alpha \{\bar{U}_\alpha : \bar{U}_\alpha \subseteq Y - C\}$ $= \bigcup_\alpha \{U_\alpha : \bar{U}_\alpha \subseteq Y - C\}$. As F(x) is compact, $F(x) \subseteq Y - C$ if, and only if, for some n and some $\alpha_1, \ldots, \alpha_n < \lambda$, $F(x) \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ $\subseteq \bigcup_{i=1}^n \bar{U}_{\alpha_i} \subseteq Y - C$. Thus $\{x: F(x) \subseteq Y - C\}$

$$= \bigcup_n \bigcup_{(\alpha_1, \ldots, \alpha_n)} \{x: F(x) \subseteq \bigcup_{i=1}^n \bar{U}_{\alpha_i} \subseteq Y - C\} = \bigcup_n \bigcup_{(\alpha_1, \ldots, \alpha_n)} \{x: F(x) \cap \bigcap_{i=1}^n \bar{U}^c_{\alpha_i} = \emptyset \text{ and } \bigcup_{i=1}^n \bar{U}_{\alpha_i} \subseteq Y - C\} \in ((\underline{\underline{H}}^c)_{\lambda^+})_{\aleph_0} =$$

$(\underline{\underline{H}}^c)_{\lambda^+} = \underline{\underline{H}}^c$ by assumption. Thus $\{x: F(x) \cap C \neq \emptyset\} \in \underline{\underline{H}}$ and we can invoke theorem 2.

By a suitable modification of the proof of theorem 2 we can prove the following generalisation of Sion's theorem:

Theorem 4. Let X be any set, $\underline{\underline{H}}$ a family of subsets of X which is λ-additive, λ-multiplicative and satisfies the λ-WRP for some cardinal $\lambda > \aleph_0$. Suppose Y is a regular Hausdorff space of topological weight $\leq \lambda$ such that given any family of open sets in Y, there is a subfamily of cardinality $< \lambda$ with the same union. Let $F: X \to \underline{\underline{C}}(Y)$ be a multifunction such that $\{x: F(x) \cap C \neq \emptyset\}$ $\in \underline{\underline{H}}$ for any closed subset C of Y. Then F admits a $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_\lambda$-measurable selector.

By putting $\lambda = \aleph_1$, we can deduce Sion's theorem from the above.

Remark. Theorem 4 holds if in the condition $\{x: F(x) \cap C \neq \emptyset\} \in \underline{\underline{H}}$ for any closed subset C of Y 'closed' is replaced by 'open'. Thus we have the following:

Theorem 5. If X, Y, $\underline{\underline{H}}$ are as in theorem4 and if $F: X \to \underline{C}(Y)$ is such that $\{x: F(x) \cap U \neq \emptyset\} \in \underline{\underline{H}}$ for any open $U \subseteq Y$, then F admits a $(\underline{\underline{H}} \cap \underline{\underline{H}}^c)_\lambda$-measurable selector.

Proof. Let $\{U_\alpha: \alpha < \lambda\}$ be a base for Y consisting of non-empty open sets and let $C \subseteq Y$ be closed. Then $Y - C = \bigcup_\alpha \{\bar{U}_\alpha: \bar{U}_\alpha \subseteq Y - C\} = \bigcup_\alpha \{U_\alpha: \bar{U}_\alpha \subseteq Y - C\}$. By our assumption, there is a subfamily $\{V_\gamma: \gamma < \beta\}$ of $\{U_\alpha: \bar{U}_\alpha \subseteq Y - C\}$ such that $\beta < \lambda$ and $\bigcup_{\gamma < \beta} V_\gamma = \bigcup_\alpha \{U_\alpha: \bar{U}_\alpha \subseteq Y - C\}$. Clearly, $\bigcup_{\gamma < \beta} \bar{V}_\gamma = Y - C$. As before, $\{x: F(x) \subseteq Y - C\} = \bigcup_n \bigcup_{(\gamma_1, \dots, \gamma_n)} \{x: F(x) \cap \bigcap_{i=1}^{n} \bar{V}^c_{\gamma_i} = \emptyset\} \in ((\underline{\underline{H}}^c)_\lambda)_{\aleph_1} = (\underline{\underline{H}}^c)_\lambda$. As $\underline{\underline{H}}$ is λ-multiplicative $(\underline{\underline{H}}^c)_\lambda = \underline{\underline{H}}^c$. Hence $\{x: F(x) \cap C \neq \emptyset\} \in \underline{\underline{H}}$. Now we can invoke theorem 4.

CHAPTER 2

UNIFORMIZATION OF BOREL SETS

1. Introduction

If B is a subset of the plane $\mathbb{R} \times \mathbb{R}$, a set $G \subseteq B$ such that (i) the projections of G and B to the first coordinate are equal and (ii) each vertical line meets G in at most one point, is called a uniformization of the set B. Around the turn of the century, J. Hadamard (to whom is also due the term 'uniformization') posed the question whether a Borel set in the plane could be uniformized by a Borel set. It was not until around 1930 that this question received a satisfactory answer. Novikov [28] showed that there is a Borel set in the plane, indeed a planar G_δ subset, whose projection to the first coordinate is Borel but which does not admit a Borel uniformization. Somewhat earlier Luzin and Sierpinski had independently proved that a planar Borel set can always be uniformized by a coanalytic set. Furthermore it was shown by Sierpinski that if a Borel set in the plane has a Borel projection to the first coordinate then any analytic uniformization of the set is necessarily Borel [40]. And with this Hadamard's question could be said to have received a satisfactory answer.

Further efforts in the 1930's were concentrated on finding sufficient conditions on a Borel set in the plane in order that it admit a Borel uniformization. Lusin [21] proved in 1930 that a Borel subset of the plane, all of whose vertical sections are countable, can be uniformized by a Borel set. The independent efforts of Kunugui [16] in Japan and Novikov Arsenin and Shchegolkov in the Soviet Union led at the end of the thirties to the result that a planar Borel set, all of whose vertical sections are σ-compact, is uniformizable by a Borel set.

In 1963 Blackwell and Ryll-Nardzewski [6] proved a result which implies that a planar Borel set, all of whose (non-empty) vertical sections have positive Lebesgue measure, is uniformizable by a Borel set. The main result of the present chapter is a category analogue of this result. We prove that a planar Borel set, all of whose (non-empty) vertical sections are non-meagre, can be uniformized by a Borel set. Furthermore we give two more sufficient conditions which ensure that a planar Borel set is uniformizable by a Borel set.

2. Definitions and notation

In this chapter, X and Y always stand for uncountable Polish spaces and ∂ stands for the diameter of a set.

Let $B \subseteq X \times Y$. For any $x \in X$, B^x is the subset of Y given by $\{y : (x,y) \in B\}$. A set C is said to uniformize B if $C \subseteq B$ and for all x in X, $B^x \neq \emptyset$ implies C^x is a singleton.

A subset of a Polish space is called meager if it is a countable union of nowhere dense sets. It is called comeager if its complement is meager. For $A \subseteq X \times Y$ and $U \subseteq Y$, let A^*_U denote the subset of X given by $\{x : A^x \cap U$ is comeager in $U\}$.

Let f be a function defined on an absolute Borel set into a Polish space. Put $Z_f = \{y : f^{-1}(y)$ is a singleton$\}$,

$I_f = \{y . f^{-1}(y)$ contains an isolated point$\}$,

$D_f = \{y : f^{-1}(y)$ is countable and nonempty$\}$,

$C_f = \{y : f^{-1}(y)$ contains a point which is not its condensation point$\}$.

3. Main results

Lemma 1. If f is a Borel measurable function on an absolute Borel set into a Polish space, then Z_f, I_f, C_f and D_f are coanalytic [9], [17], [21].

Lemma 2. (Countable reduction principle for coanalytic sets). If $C_1, C_2, \ldots$ is a sequence of coanalytic subsets of a Polish space, then there exists a sequence $B_1, B_2, \ldots$ of pairwise disjoint coanalytic sets such that $\bigcup_n B_n = \bigcup_n C_n$ [17].

Theorem 1. A Borel set $B \subseteq X \times Y$ has a Borel uniformization if

(a) $x \in \pi_1 B$ implies B^x contains an isolated point or

(b) $x \in \pi_1 B$ implies B^x contains a point which is not its point of condensation,

where π_1 denotes the projection to the first co-ordinat

Proof. (a) Let V_n, $n = 1,2,\ldots$ be a countable open base for Y such that for all n $V_n \neq \emptyset$. For any n, define f_n on $B \cap (X \times V_n)$ by $f_n(x,y) = x$. Let $Z_n = \{x: B^x \cap V_n$ is a singleton$\}$

Then $Z_n = Z_{f_n}$. Thus Z_n is coanalytic by lemma 1. Also $\bigcup_n Z_n = \pi_1 B$. Hence $\bigcup_n Z_n$ is analytic and hence Borel. Let B_n, $n = 1,2,\ldots$ be disjoint coanalytic sets such that $B_n \subseteq$ for all n and $\pi_1 B = \bigcup_n Z_n = \bigcup_n B_n$. Clearly B_n, $n = 1,2,\ldots$ are Borel sets. Let $C = \bigcup_n ((B_n \times V_n) \cap B)$. Then C is a Bore uniformization of B.

(b) Let V_n, f_n, $n = 1,2,\ldots$ be as in case (a) and for any n, let $Z_n = \{x: B_x \cap V_n$ is countable and nonempty$\}$.

Then $Z_n = D_{f_n}$. Hence Z_n is coanalytic. Also $\bigcup_n Z_n = \pi_1 B$

and hence $\bigcup_n Z_n$ is Borel. As before, choose disjoint Borel sets B_n, $n = 1,2,\ldots$ such that $B_n \subseteq Z_n$ and $\bigcup_n B_n = \pi_1 B$. Put $D = \bigcup_n ((B_n \times V_n) \cap B)$. Then $D \subseteq B$ is a Borel set such that $\pi_1 D = \pi_1 B$ and for all x, D^x is countable. Hence D can be uniformized by a Borel set C (cf. [21]) Clearly, this C uniformizes B.

Corollary: Let X be absolute Borel and Z a separable metric space. Let $f: X \to Z$ be Borel measurable and $f(X) = Z$. If $Z = I_f$ or C_f, then f admits a Borel selector i.e. there is a Borel subset B of X such that f restricted to B is one-to-one and $f(B) = f(X) = Z$.

Before proving the next theorem, we prove some lemmas.

Lemma 1. ([43]). Let W be a subset of an uncountable Polish space X such that W has the Baire property and let U be a nonempty open subset of X. Let $V_1, V_2, \ldots$ be an open base of X such that $V_n \neq \emptyset$ for $n = 1,2,\ldots$. Then $W \cap U$ is meager if, and only if, for $V_n \subseteq U$, $n = 1,2,\ldots$, $W \cap V_n$ is not comeager in V_n.

The proof is simple.

Lemma 2. ([43]). Let $B \subseteq X \times Y$ and let $V_1, V_2, \ldots$ be an open base for Y such that for all n, $V_n \neq \emptyset$. Then for any nonempty open subset U of Y, $x \in ((X \times Y) - B)^*_U$ if, and only if, for all $V_n \subseteq U$, $x \notin B^*_{V_n}$.

Proof. Let $x \in ((X \times Y) - B)^*_U$. Then $((X \times Y) - B)^x \cap U$ is comeager in U. Hence $B^x \cap U$ is meager in U. Hence for all $V_n \subseteq U$, $B^x \cap V_n$ is not comeager in V_n i.e. $x \notin B^*_{V_n}$.

The converse follows by reversing the argument.

Lemma 3. Let $B \subseteq X \times Y$ be Borel and let $U \subseteq Y$ be a non-empty open set. Then B^*_U is Borel.

Proof. Let $\underline{\underline{C}} = \{B : B$ is Borel and B^*_U is Borel for all nonempty open $U \subseteq Y\}$.

We show that $\underline{\underline{C}}$ contains all Borel sets.

Step 1. If B_1, B_2 are Borel sets in X and Y respectively then $B_1 \times B_2 \in \underline{\underline{C}}$ since, for any U, either $B^*_U = \emptyset$, or $B^*_U = B_1$.

Step 2. $\underline{\underline{C}}$ is closed under countable intersections.

Let $B_1, B_2, \ldots \in \underline{\underline{C}}$ and $U \subseteq Y$ be open and nonempty. Now $(\bigcap_n B_n)^x \cap U$ is comeager in U if and only if $B_n^x \cap U$ is comeager in U for all n. Thus $(\bigcap_n B_n)^*_U = \bigcap_n B^*_{nU}$.

Thus $(\bigcap_n B_n)^*_U$ is Borel and hence $\bigcap_n B_n \in \underline{\underline{C}}$.

Step 3. $\underline{\underline{C}}$ is closed under complementation.

Let $V_1, V_2, \ldots$ be a countable base for Y consisting of nonempty open sets. For any B, $((X \times Y) - B)^*_U =$

$\bigcap_n \{X - B^*_{V_n} : V_n \subseteq U\}$.

Theorem 2. If $B \subseteq X \times Y$ is such that for all $x \in \pi_1 B$, B^x is a nonmeager subset of Y, then B has a Borel uniformization.

This result follows from:

Theorem 3. Let $B \subseteq X \times Y$ be a Borel set such that for all $x \in \pi_1 B$, B^x is a comeager subset of Y. Then B has a Borel uniformization.

Proof of theorem 2. Assume theorem 3. Let $V_1, V_2, \ldots$ be a countable base for Y consisting of nonempty open sets. Put $D_n = B^*_{V_n} - \bigcup_{m<n} B^*_{V_m}$. D_n is Borel by lemma 3. Now, by theorem 3, the Borel subset $B \cap (D_n \times V_n)$ of $X \times V_n$ has a Borel uniformization C_n for all n.

Again, by lemma 1, $\pi_1 B = \bigcup_n B^*_{V_n} = \bigcup_n D_n$, since B^x is not meager for $x \in \pi_1 B$. Hence $\bigcup_n C_n$ is a Borel uniformization of B.

To prove theorem 3, we need the following:

Lemma: Given any nonempty open subset U of Y and any Borel subset B of $X \times Y$, there exist a sequence $Z_1, Z_2, \ldots$ of Borel subsets of $X \times Y$ such that

(a) $Z_k \subseteq X \times U$ for all k.

(b) $\bigcap_k Z_k \subseteq B$

(c) Given any open W such that $\emptyset \neq W \subseteq U$, any k and any $\epsilon > 0$, we can find a Borel set $F \subseteq Z_k \cap (X \times W)$ such that for all x, F^x, considered as a subset of Y, is closed, has diameter $< \epsilon$ and if $x \in B^*_U$, F^x is not meager.

Proof. Let $\underline{\underline{M}} = B \subseteq X \times Y$: B is Borel and satisfies the above

We show that $\underline{\underline{M}}$ contains all Borel sets.

Step 1. $\underline{\underline{M}}$ contains closed sets.

Let W_m, $m = 1,2,\ldots$ and V_n, $n = 1,2\ldots$ be countable open bases for X and Y respectively such that for all m, n, $W_m \neq \emptyset$, $V_n \neq \emptyset$. Let $B \subseteq X \times Y$ be closed. There are open sets $U_k \subseteq X \times Y$ such that $B = \bigcap_k U_k$. Let $U \subseteq Y$ be nonempty and open. Put $Z_k = U_k \cap (X \times U)$, $k = 1,2,\ldots$. Clearly, (a) and (b) are satisfied.

To see that (c) is satisfied, we fix k, ϵ and W and construct F. Now $Z_k \cap (X \times W) = U_k \cap (X \times W)$ is open. Hence $Z_k \cap (X \times W) = \bigcup_{(m,n)} \{W_m \times V_n: W_m \times V_n \subseteq Z_k \cap (x \times W)\}$. Let $L = \{m: W_m \times V_n \subseteq Z_k \cap (X \times W)$ for some $n\}$. Corresponding to each m in L, choose n_m such that $W_m \times V_{n_m} \subseteq Z_k \cap (X \times W)$ and let $V_{\psi(m)}$ satisfy $\emptyset \neq V_{\psi(m)} \subset \overline{V_{\psi(m)}} \subseteq V_{n_m}$ and $\partial(V_{\psi(m)}) < \epsilon$. Put $F = \bigcup_{m \in L} ((W_m - \bigcup_{\substack{n < m \\ n \in L}} W_n) \times \overline{V_{\psi(m)}})$. Note that if $x \in B_U^*$, $U_k^x \cap W \neq \emptyset$ so that $x \in \pi_1(U_k \cap (X \times W)) = \bigcup_{m \in L} W_m$.

It is now easy to check (c).

Step 2. $\underline{\underline{M}}$ is closed under countable intersections.

Let $B_n \in \underline{\underline{M}}$, $n = 1,2,\ldots$ and let $U \subseteq Y$ be nonempty and open. For each n, let the sequence Z_{nk}, $k = 1,2,\ldots$ satisfy (a), (b) and (c) when B is replaced by B_n and Z_k by Z_{nk}.

Rearrange the double sequence Z_{nk}, $n = 1,2,\ldots$, $k = 1,2,\ldots$ in the form of a simple sequence, say Z_k, $k = 1,2,\ldots$. This new sequence satisfies (a), (b), (c) if $B = \bigcap_n B_n$.

Step 3. $\underline{\underline{M}}$ is closed under countable unions.

Let $B_n \in \underline{\underline{M}}$, $n = 1,2,\ldots$ and let $U \subseteq Y$ be nonempty open. We construct Z_k, $k = 1,2,\ldots$ such that (a), (b), (c) are satisfied if $B = \bigcup_n B_n$.

Let V_m, $m = 1,2,\ldots$ be a countable open base consisting of nonempty sets for U. For each fixed pair (n,m), let Z_{nmk}, $k = 1,2,\ldots$ satisfy (a), (b), (c) if B is replaced by B_n, U by V_m and Z_k by Z_{nmk}. For all n,m,k, put

$$D_{nm} = B^*_{nV_m} - \bigcup_{j<n} B^*_{jV_m},\quad E_m = \bigcup_n D_{nm} = \bigcup_n B^*_{nV_m},$$

$$Z_{mk} = \bigcup_n (Z_{nmk} \cap (D_{nm} \times Y)),\quad Z_k = \bigcup_m (Z_{mk} - \bigcup_{i<m} (E_i \times V_i)).$$

Clearly, Z_k is a Borel subset of $X \times U$ for each k.

$$\bigcap_k Z_k = \bigcap_k \bigcup_m (Z_{mk} - \bigcup_{i<m} (E_i \times V_i)) = \bigcup_m \bigcap_k (Z_{mk} - \bigcup_{i<m} (E_i \times V_i))$$

since $Z_{mk} \subseteq E_m \times V_m$ for all k and m.

For any m
$$\bigcap_k (Z_{mk} - \bigcup_{i<m} (E_i \times V_i)) = \bigcap_k \bigcup_n (Z_{nmk} \cap (D_{nm} \times Y) - \bigcup_{i<m} (E_i \times V_i))$$

$$= \bigcup_n \bigcap_k (Z_{nmk} \cap (D_{nm} \times Y) - \bigcup_{i<m} (E_i \times V_i))$$ since D_{nm}, $n = 1,2,\ldots$ is a disjoint family of sets.

Thus $\bigcap_k Z_k \subseteq \bigcup_m \bigcup_n \bigcap_k Z_{nmk} \subseteq \bigcup_n B_n$.

Fix a positive integer k, an $\epsilon > 0$ and an open W, $\emptyset \neq W \subseteq U$. For all m, n, put $H_m = E_m - \bigcup_{i<m} \{E_i : W \cap V_i \neq \emptyset\}$ and $G_{nm} = H_m \cap D_{nm}$. For all m such that $W \cap V_m \neq \emptyset$ and all n, choose Borel sets F_{nm} such that $F_{nm} \subseteq Z_{nmk} \cap (X \times (W \cap$ and for all x, F^x_{nm} is closed, $\partial(F^x_{nm}) < \epsilon$ and if $x \in B^*_{nV_m}$, then F^x_{nm} is not meager. Put $F = \bigcup_n \bigcup_m (G_{nm} \times Y) \cap F_{nm}$: $W \cap V_m \neq \emptyset$. Then F is clearly a Borel subset of $X \times W$. To see that $F \subseteq Z_k$, take $(x,y) \in F$; we show that $(x,y) \in Z_k$. There exists a unique ordered pair (n,m) such that $W \cap V_m \neq \emptyset$ and $(x,y) \in (G_{nm} \times Y) \cap F_{nm}$. As $F_{nm} \subseteq Z_{nmk}$ and $G_{nm} \subseteq D_{nm}$, $(x,y) \in Z_{nmk} \cap (D_{nm} \times Y) \subseteq Z_{mk}$. Let $i < m$. If $W \cap V_i \neq \emptyset$, $x \notin E_i$ and if $W \cap V_i = \emptyset$, $y \notin V_i$ since $F_{nm} \subseteq X \times W$ and hence $y \in W$. Thus $(x,y) \notin \bigcup_{i<m} (E_i \times V_i)$. Hence $(x,y) \in Z_k$.

Clearly, F^x is closed and $\partial(F^x) < \epsilon$ for all x. Let $x \in (\bigcup_n B_n)^*_U$ i.e. $(\bigcup_n B_n)^x \cap U$ is comeager in U. We show that there is some n,m such that $W \cap V_m \neq \emptyset$ and $x \in G_{nm}$. Then $F^x = F^x_{nm}$ and $x \in B^*_{nV_m}$. Hence F^x is not meager.

It is enough to show that there is an m satisfying $W \cap V_m \neq \emptyset$ and $x \in E_m$. Clearly $(\bigcup_n B_n)^x \cap U$ is comeager in U implies $\bigcup_n (B^x_n \cap W) = (\bigcup_n B_n)^x \cap W$ is comeager in W so that there is some n satisfying $B^x_n \cap W$ is not meager in W. Hence there is a $V_m \subseteq W$ such that $x \in B^*_{nV} \subseteq E_m$.

Proof of theorem 3. Let V_n, $n = 1,2,\ldots$ be a countable base of nonempty open sets for Y. Let Z_k, $k = 1,2,\ldots$ satisfy (a), (b), (c) of the lemma when U is taken to be Y.

We define, by induction, a sequence C_k, $k = 1,2,\ldots$ of Borel subsets of $X \times Y$ such that for all k

i) $C_k \subseteq C_{k-1}$ if $k > 1$

ii) $C_k \subseteq Z_k$

iii) For all x, C_k^x is closed in Y and $\partial(C_k^x) < \frac{1}{k}$

iv) If $x \in \pi_1 B$, C_k^x is not meager in Y.

Then $\bigcap_k C_k$ is the required Borel uniformization of B. By the lemma, taking $W = Y$, $k = 1$ and $\epsilon = 1$, find a Borel set $C_1 \subseteq Z_1$ satisfying (ii) - (iv).

Suppose $C_1,\ldots,C_m$ have been defined. Put $H_n = C^*_{mV_n} - \bigcup_{j<n} C^*_{mV_j}$. Choose a Borel set $F_{m+1,n} \subseteq Z_{m+1} \cap (X \times V_n)$ such that for all x, $F^x_{m+1,n}$ is closed, $\partial(F^x_{m+1,n}) < \frac{1}{m+1}$ and if $x \in \pi_1 B$, $F^x_{m+1,n}$ is not meager. Put $C_{m+1} = \bigcup_n ((H_n \times Y) \cap F_{m+1,n})$. Clearly, C_{m+1} is a Borel subset of Z_{m+1}. To show $C_{m+1} \subseteq C_m$, it is enough to show that if $C^x_{m+1} \neq \emptyset$, $C^x_{m+1} \subseteq C^x_m$. Let $C^x_{m+1} \neq \emptyset$. There is a unique n such that $x \in H_n$. Therefore $x \in C^*_{mV_n}$ and as C^x_m is closed this implies $V_n \subseteq C^x_m$. Thus $C^x_{m+1} = F^x_{m+1,n} \subseteq V_n \subseteq C^x_m$.

Clearly, for all x, C^x_{m+1} is closed and $\partial(C^x_{m+1}) < \frac{1}{m+1}$.

Let $x \in \pi_1 B$. By induction hypothesis, C^x_m is not meager and hence $x \in \bigcup_n C^*_{mV_n} = \bigcup_n H_n$. Therefore $C^x_{m+1} = F^x_{m+1,n}$ for some n so that C^x_{m+1} is not meager.

4. A related result.

In this section we prove the following main theorem.

Theorem 4. Let $B \subseteq X \times Y$ be such that for all $x \in \pi_1 B$, B^x is a comeager subset of Y. Then there exist Borel sets $Z_k \subseteq X \times Y$, $k = 1,2,\ldots$ such that $\bigcap_k Z_k \subseteq B$ and for all k and x, Z^x_k is open and if $x \in \pi_1 B$, Z^x_k is dense (and hence comeager) in Y.

Using this theorem we give an alternative proof of theor 3 which uses the countable reduction principle for coanalytic sets.

Theorem 4 follows from the next theorem by taking $U = Y$.

Theorem 5. Let $B \subseteq X \times Y$ be Borel. Given any nonempty ope subset U of Y, there is a sequence Z_k, $k = 1,2,\ldots$ of Bore sets in $X \times Y$ such that

a) $Z_k \subseteq X \times U$

b) $\bigcap_k Z_k \subseteq B$

c) For all k and x, Z^x_k is an open subset of Y and if $x \in B^*_U$, then Z^x_k is comeager in U.

Proof: Let $\underline{\underline{M}} = \{B \subseteq X \times Y: B$ is Borel and satisfies the above$\}$. We show that $\underline{\underline{M}}$ contains all Borel subsets of $X \times Y$.

Step 1. Clearly, $\underline{\underline{M}}$ contains all G_δ sets and hence all closed sets.

Step 2. It is easy to see that $\underline{\underline{M}}$ is closed under countable intersections.

Step 3. $\underline{\underline{M}}$ is closed under countable unions.

Let B_n, $n = 1,2,\ldots$ be in $\underline{\underline{M}}$ and let $U \subseteq Y$ be non-empty and open. Let $V_m, m = 1,2,\ldots$ be a countable hase of nonempty open sets for U. For any fixed m, n, let Z_{nmk}, $k = 1,2,\ldots$ satisfy (a), (b), (c) where B is replaced by B_n, U by V_m and Z_k by Z_{nmk}. For all m and k, define E_m and Z_{mk} as in the lemma used in the proof of theorem 3 and let $Z_k = \bigcup_m (Z_{mk} - \bigcup_{i<m} (E_i \times \overline{V_i}))$. It is easy to see that the sequence Z_k, $k = 1,2,\ldots$ satisfies (a) and (b) if $B = \bigcup_n B_n$ and that for all k and x, Z_k^x is open. Let $x \in (\bigcup_n B_n)^*_U$. To show that Z_k^x is comeager in U, it is enough to show that it is dense in U. We prove this in two steps.

Step 1. $V_x = \bigcup_m \{V_m : x \in E_m\}$ is dense in U

Step 2. $\overline{Z_k^x} \supseteq V_x$.

Proof of step 1. As $x \in (\bigcup_n B_n)^*_U$, $x \in (\bigcup_n B_n)^*_{V_m}$ for all m. Thus given m, there is some n such that $B_n^x \cap V_m$ is not meager in V_m and hence there is some $V_s \subseteq V_m$ such that $x \in B^*_{nV_s} \subseteq E_s$. Now $V_s \subseteq V_x \cap V_m$ so that $V_x \cap V_m \neq \emptyset$.

<u>Proof of step 2.</u> $Z^x_k = \bigcup_m (Z^x_{mk} - \bigcup_{i<m} \{ \bar{V}_i : x \in E_i \})$

$= \bigcup_m \{ (Z^x_{mk} - \bigcup_{i<m} \{ \bar{V}_i : x \in E_i \}) : x \in E_m \}$.

Fix m. If $x \in E_m$, there is some n for which $x \in D_{nm}$ so that $Z^x_{mk} = Z^x_{nmk}$. By hypothesis $\overline{Z^x_{nmk}} \supseteq V_m$ as $x \in B^*_{V_m}$. Hence $\overline{Z^x_{mk}} \supseteq V_m$ and therefore $\overline{Z^x_{mk}} \supseteq \overline{V_m}$. Hence

$\overline{(Z^x_{mk} - \bigcup_{i<m} \{ \bar{V}_i : x \in E_i \})} \supseteq \bar{V}_m - \bigcup_{i<m} \{ \bar{V}_i : x \in E_i \}$. Thus

$\overline{Z^x_k} \supseteq \bigcup_m \{ \bar{V}_m : x \in E_m \} \supseteq V_x$.

<u>An alternative proof of theorem 3.</u> Let Z_k, $k = 1,2,\ldots$ be defined as in theorem 4. We next define, by induction, a sequence C_k, $k = 1,2,\ldots$ of Borel subsets of $X \times Y$ such that for all k.

a) $C_k \subseteq Z_k$

b) for all x, C^x_k is an open subset of Y, $\partial(C^x_k) < \frac{1}{k}$ and $\overline{C^x_k} \subseteq C^x_{k-1}$ if $k > 1$

c) $x \in \pi_1 B$ implies $C^x_k \neq \emptyset$.

Let V_{1n}, $n = 1,2,\ldots$ be a countable base for Y consisting of nonempty open sets such that $\partial(V_{1n}) < 1$ for all n.

Let $D_{1n} = \{x: V_{1n} \subseteq Z_1^x\} = \{x: V_{1n} \cap ((X \times Y) - Z_1)^x = \emptyset\} =$ $X - \pi_1((X \times V_{1n}) \cap ((X \times Y) - Z_1))$. D_{1n} is coanalytic for all n and $\bigcup_n D_{1n} = \pi_1 Z_1$ is analytic and hence Borel. Find disjoint Borel sets B_{1n}, $n = 1,2,\ldots$ such that $B_{1n} \subseteq D_{1n}$ and $\bigcup_n B_{1n} = \pi_1 Z_1$. Put $C_1 = \bigcup_n (B_{1n} \times V_{1n})$.

Suppose $C_1, \ldots, C_m$ have been defined. Let $V_{m+1,n}$, $n = 1,2,\ldots$ be a countable base for Y consisting of nonempty open sets such that $\partial(V_{m+1,n}) < \frac{1}{m+1}$ for all n.

Let $D_{m+1,n} = \{x: \bar{V}_{m+1,n} \subseteq C_m^x \cap Z_{m+1}^x\}$

$= \{x: \bar{V}_{m+1,n} \cap ((X \times Y) - (C_m \cap Z_{m+1}))^x = \emptyset\}$

$= X - \pi_1((X \times \bar{V}_{m+1,n}) \cap ((X \times Y) - (C_m \cap Z_{m+1})))$. $D_{m+1,n}$ is coanalytic for all n and $\bigcup_n D_{m+1,n} = \pi_1(C_m \cap Z_{m+1})$ since $(C_m \cap Z_{m+1})^x$ is open by induction hypothesis. Hence $\bigcup_n D_{m+1,n}$ is Borel. Find disjoint Borel sets $B_{m+1,n}$, $n = 1,2,\ldots$ such that $B_{m+1,n} \subseteq D_{m+1,n}$ and $\bigcup_n B_{m+1,n} = \pi_1(C_m \cap Z_{m+1})$. Put $C_{m+1} = \bigcup_n (B_{m+1,n} \times V_{m+1,n})$. Only (c) needs checking, (a) and (b) being evident. Let $x \in \pi_1 B$. Now, Z_{m+1}^x is dense in Y and C_m^x is open by induction hypothesis. Hence $Z_{m+1}^x \cap C_m^x =$ $(Z_{m+1} \cap C_m)^x \neq \emptyset$. Therefore $x \in \pi_1(Z_{m+1} \cap C_m) = \bigcup_n B_{m+1,n}$. Hence $C_{m+1}^x \neq \emptyset$.

Put $C = \bigcap_k C_k$. C uniformizes B for $C \subseteq \bigcap_k Z_k \subseteq B$ and if $x \in \pi_1 B$, $C^x = \bigcap_k C_k^x = \bigcap_k \overline{C_k^x}$ is a singleton by (b) and (c).

CHAPTER 3

SELECTION THEOREMS FOR PARTITIONS OF COMPLETE METRIC SPACES

1. Introduction

If $\underline{\underline{Q}}$ is a partition of a set X, a set $S \subseteq X$ which meets each element of $\underline{\underline{Q}}$ in exactly one point is called a selector for $\underline{\underline{Q}}$ (also sometimes referred to as a cross-section of $\underline{\underline{Q}}$). If X is a complete metric space, elements of $\underline{\underline{Q}}$ are closed subsets of X and moreover there are definability conditions on $\underline{\underline{Q}}$, the question arises if a selector for $\underline{\underline{Q}}$ can be found such that it too satisfies certain definability conditions.

This problem has been considered in recent articles by Kuratowski and Maitra and Maitra and Rao when X is a Polish space [19],[25]. Much earlier Bourbaki [8] had proved that if $\underline{\underline{Q}}$ is a lower semi-continuous partition of X into closed sets then there is a G_δ selector for $\underline{\underline{Q}}$. Kuratowski and Maitra extended these results to the case of partitions $\underline{\underline{Q}}$ which are of class α^- or α^+ (these are defined in analogy with l.s.c. and u.s.c. partitions) for countable ordinals α. Using different methods, Maitra and Rao obtained somewhat more precise results.

The aim of this chapter is to show that the results of Maitra and Rao carry over to the non-separable case. In this situation the methods of Maitra and Rao, which are essentially of a countable nature, do not quite work and we have to use results from the theory of Borel sets in non-separable metric spaces due to D. Montgomery [27].

2. Definitions and notation

In this chapter, we take X to be a complete metric space and $\underline{\underline{Q}}$ to be a partition of X into closed subsets. For x, y in X, we write $x \sim y$ to denote that x and y belong to the same element of $\underline{\underline{Q}}$. If $A \subseteq X$, put

$A^* = \{x :$ there is some $Q \in \underline{\underline{Q}}$ such $x \in Q$ and $Q \cap A \neq \emptyset\}$

A^* is called the saturation of A with respect to $\underline{\underline{Q}}$.

$\underline{\underline{Q}}$ is called an α^--partition of X if the saturation of every open subset of X with respect to $\underline{\underline{Q}}$ is a Borel set of additive class α. It is called an α^+-partition if the saturation of every closed subset of X with respect to $\underline{\underline{Q}}$ is of multiplicative Borel class α.

Let τ be any infinite cardinal. Let $B_\tau = \{\sigma : \sigma < \tau\}^{\aleph_0}$ where $\{\sigma : \sigma < \tau\}$ is given the discrete topology. For any x in B_τ let x_i denote the i^{th} co-ordinate of x. For any ordinals $\sigma_1, \ldots, \sigma_n < \tau$, $\Sigma(\sigma_1 \cdots \sigma_n)$ stands for $\{x : x \in B_\tau$ and $x_i = \sigma_i$, $i = 1, \ldots n\}$.

3. The Main result

Theorem. If $\alpha > 0$ an $\underline{Q}$ is an α-partition of X, then $\underline{Q}$ admits a selector of multiplicative Borel class α.

We first prove some lemmas.

Lemma 1. If X has topological weight $\leq \tau$ where τ is any infinite cardinal, then there exists an open continuous map f from a closed subset E of B_{τ} onto X such that

a) $\{ f(\Sigma(\sigma_1) \cap E) : \sigma_1 < \tau \}$ is a locally finite family of sets,

b) for any $k > 1$ and any $\sigma_1, \dots, \sigma_{k-1} < \tau$, $\{ f(\Sigma(\sigma_1, \dots, \sigma_k) \cap E) : \sigma_k < \tau \}$ is a locally finite family of sets relative to $f(\Sigma(\sigma_1, \dots, \sigma_{k-1}) \cap E)$.

Proof. We first define, by induction, a system of open sets $U_{\sigma_1, \dots \sigma_k}$, $\sigma_1, \dots, \sigma_k < \tau$, $k = 1, 2, \dots$ such that

i) $\partial(U_{\sigma_1, \dots, \sigma_k}) < \frac{1}{k}$ for all $k \geq 1$ and all $\sigma_1, \dots, \sigma_k < \tau$ where ∂ denotes the diameter.

ii) For all $k > 1$ and all $\sigma_1, \dots, \sigma_k < \tau$, $\bar{U}_{\sigma_1, \dots, \sigma_{k-1}, \sigma_k} \subseteq U_{\sigma_1, \dots, \sigma_{k-1}}$.

iii) $\{ U_{\sigma_1} : \sigma_1 < \tau \}$ is a locally finite cover of X. For $k > 1$ and any $\sigma_1, \dots, \sigma_{k-1} < \tau$, $\{ U_{\sigma_1, \dots, \sigma_{k-1} \sigma_k} : \sigma_k < \tau \}$ is a locally finite family of sets relative to $U_{\sigma_1, \dots, \sigma_{k-1}}$ which covers $U_{\sigma_1, \dots, \sigma_{k-1}}$.

Let $\{V_i : i \in I\}$ be any open cover for X such that for all i, $\partial(V_i) < 1$. Take a locally finite open refinement of this cover, say $\{W_j : j \in J\}$. As topological weight of $X \leq \tau$, we can find a subcover of $\{W_j : j \in J\}$ which has cardinality $\leq \tau$. Let this new cover be $\{U_{\sigma_1} : \sigma_1 < \tau\}$. Clearly, $\{U_{\sigma_1} : \sigma_1 < \tau\}$ is locally finite and $\partial(U_{\sigma_1}) < 1$ for all σ_1.

Let $k > 1$. Suppose for any $\sigma_1, \ldots, \sigma_{k-1} < \tau$, $U_{\sigma_1, \ldots, \sigma_{k-1}}$ has been defined. Let $\{V_{i_1} : i_1 \in I_1\}$ be an open cover of $U_{\sigma_1, \ldots, \sigma_{k-1}}$ such that for all i_1, $\overline{V}_{i_1} \subseteq U_{\sigma_1, \ldots, \sigma_{k-1}}$ and $\partial(V_{i_1}) < \frac{1}{k}$. Since $U_{\sigma_1, \ldots, \sigma_{k-1}}$ is a metric space of topological weight $\leq \tau$ we can find an open refinement of $\{V_{i_1} : i_1 \in I_1\}$ which is locally finite relative to $U_{\sigma_1, \ldots, \sigma_{k-1}}$ and then take a subcover of this of cardinality $\leq \tau$. We thus arrive at an open cover $\{U_{\sigma_1, \ldots, \sigma_{k-1}\sigma_k} : \sigma_k < \tau\}$ of $U_{\sigma_1, \ldots, \sigma_{k-1}}$ which is locally finite relative to $U_{\sigma_1, \ldots, \sigma_{k-1}}$ and satisfies (i) and (ii).

Thus for any $\sigma_1, \ldots, \sigma_k < \tau$, $U_{\sigma_1, \ldots, \sigma_k}$ is defined satisfying (i), (ii) and (iii).

Let $E \subseteq B_\tau$ be defined as follows. $(\sigma_1, \sigma_2, \ldots) \in E$ if and only if $\bigcap_k \overline{U}_{\sigma_1, \ldots, \sigma_k} \neq \emptyset$. It is easy to verify that E is closed.

Define f on E by $f((\sigma_1, \sigma_2, \ldots))$ = the unique element of $\bigcap_k \bar{U}_{\sigma_1 \ldots \sigma_k}$ $(= \bigcap_k \dot{U}_{\sigma_1, \ldots, \sigma_k})$. Clearly, $f(E) = X$. Let $f((\sigma_1, \sigma_2, \ldots)) = x$ and let N be a neighbourhood of x. As $\{x\} = \bigcap_k \bar{U}_{\sigma_1, \ldots, \sigma_k} \subseteq N$ and $\partial(\bar{U}_{\sigma_1 \ldots \sigma_k}) < \frac{1}{k}$, there is a k such that $x \in \bar{U}_{\sigma_1 \ldots \sigma_k} \subseteq N$ (see Chapter 1). Now $f(\Sigma(\sigma_1 \ldots \sigma_k) \cap E) \subseteq \bar{U}_{\sigma_1 \ldots \sigma_k}$ and $\bar{U}_{\sigma_1 \ldots \sigma_k} \subseteq N$. Thus $f(\Sigma(\sigma_1 \ldots \sigma_k) \cap E) \subseteq N$. Hence f is continuous.

$\Sigma(\sigma_1 \ldots \sigma_k), \sigma_1, \ldots, \sigma_k < \tau$, $k = 1, 2, \ldots$ form an open base for the topology on B_τ. Hence, to show that f is open, it is enough to show that $f(\Sigma(\sigma_1 \ldots \sigma_k) \cap E)$ is open for any $\sigma_1, \ldots, \sigma_k$, $k \geq 1$. As a matter of fact, we show that $f(\Sigma(\sigma_1 \ldots \sigma_k) \cap E) = U_{\sigma_1 \ldots \sigma_k}$. Clearly, $f(\Sigma(\sigma_1 \ldots \sigma_k) \cap E) \subseteq U_{\sigma_1 \ldots \sigma_k}$. Let $x \in U_{\sigma_1 \ldots \sigma_k}$. Then there is a point $(\tau_1, \tau_2, \ldots) \in \Sigma(\sigma_1 \ldots \sigma_k)$ such that $x \in \bigcap_k \bar{U}_{\tau_1 \ldots \tau_k}$. Thus $(\tau_1, \tau_2, \ldots) \in \Sigma(\sigma_1 \ldots \sigma_k) \cap E$ and $f((\tau_1, \tau_2, \ldots)) = x$.

Since $f(\Sigma(\sigma_1 \ldots \sigma_k) \cap E) = U_{\sigma_1 \ldots \sigma_k}$, it is clear from (iii) that f satisfies (a) and (b).

Lemma 2. On X, there is a relation $<$ and relations $<_k$, $=_k$ for each positive integer k such that:

(a) $x =_k y$ if, and only if, neither $x <_k y$ nor $y <_k x$.

(b) $x < y$ if, and only if, $x <_k y$ for some k.

(c) $<$ is a linear order on X such that each nonempty closed subset has a first element.

(d) For any a in X, $\{x: x <_k a\}$ is open.

(e) $=_k$ is an equivalence relation with equivalence classes which are both F_σ and G_δ.

(f) $x =_k y$ and $y <_k z$ implies $x <_k z$

$x =_k y$ and $z <_k y$ implies $z <_k x$.

(g) $\{x : x =_1 a\}$, $a \in X$, is a locally finite cover of X; for each k and each b in X, $\{x : x =_{k+1} a\}$, $a =_k b$ is a relatively locally finite cover of $\{x : x =_k b\}$.

Proof: Let X have topological weight $\leq \tau$ where τ is an infinite cardinal. Let $E \subseteq B_\tau$ be as in lemma 1 and let f be the map on E defined there. For each x in X, $f^{-1}(x)$ is closed in E and hence in B_τ. Therefore $f^{-1}(x)$ contains a lexicographic minimum which we denote by min $f^{-1}(x)$.

Define the relation $<_k$ on X by $x <_k y$ if there exists some $r \leq k$ such that $(\min f^{-1}(x))_i = (\min f^{-1}(y))_i$ for $i < r$ and $(\min f^{-1}(x))_r < (\min f^{-1}(y))_r$. Define $=_k$ and $<$ as in (a) and (b). It is clear that $x < y$ if, and only if, min $f^{-1}(x)$ precedes min $f^{-1}(y)$ in the lexicographic order so that $<$ is a linear order on X. Let C be a nonempty closed subset of X. Then $f^{-1}(C)$ is closed in E and hence in B_τ. Let σ^0 be the first element of $f^{-1}(C)$ according to the lexicographic order and let $f(\sigma^0) = x_0$. Then x_0 is the first element of C.

To see this let $x \in C$ and $x \neq x_o$. Then $f^{-1}(x) \subseteq f^{-1}(C)$ and $\sigma_o \notin f^{-1}(x)$. Hence σ^o precedes $\min f^{-1}(x)$ in B . As $\sigma_o \in f^{-1}(x_o)$, $\min f^{-1}(x_o)$ precedes $\min f^{-1}(x)$. Hence $x_o < x$.

Let $a \in X$ and suppose $\min f^{-1}(a) = (\tau_1, \tau_2, \dots)$.

Then
$$\{x: x <_k a\} = \{x: (\min f^{-1}(x))_1 < \tau_1, \text{ or } ((\min f^{-1}(x))_1 = \tau_1 \text{ and } (\min f^{-1}(x))_2 < \tau_2) \text{ or } \dots ((\min f^{-1}(x))_i = \tau_i,\ i = 1, \dots, k-1 \text{ and } (\min f^{-1}(x))_k < \tau_k)\}$$

$$= \bigcup_{\sigma_1 < \tau_1} f(\Sigma(\sigma_1) \cap E) \cup \Big(\bigcup_{\sigma_2 < \tau_2} f(\Sigma(\tau_1, \sigma_2) \cap E) - \bigcup_{\sigma_1 < \tau_1} f(\Sigma(\sigma_1) \cap E)\Big) \cup \dots \cup \Big(\bigcup_{\sigma_k < \tau_k} f(\Sigma(\tau_1, \dots, \tau_{k-1}, \sigma_k) \cap E) - \Big(\bigcup_{\sigma_1 < \tau_1} f(\Sigma(\sigma_1) \cap E) \cup \dots \bigcup_{\sigma_{k-1} < \tau_{k-1}} f(\Sigma(\tau_1, \dots \tau_{k-2}, \sigma_{k-1}) \cap E)\Big)\Big)$$

$$= \bigcup_{\sigma_1 < \tau_1} f(\Sigma(\sigma_1) \cap E) \cup \bigcup_{\sigma_2 < \tau_2} f(\Sigma(\tau_1, \sigma_2) \cap E) \cup \dots \bigcup_{\sigma_k < \tau_k} f(\Sigma(\tau_1, \dots \tau_{k-1}, \sigma_k) \cap E).$$

This is an open set as f is an open mapping.

$=_k$ is clearly an equivalence relation. Let $a \in X$ and suppose $\min f^{-1}(a) = (\tau_1, \tau_2, \dots)$. Then
$$\{x : x =_k a\} = \{x: (\min f^{-1}(x))_i = \tau_i,\ i = 1, \dots, k\}$$

$$= f(\Sigma(\tau_1, \dots, \tau_k) \cap E) - \Big(\bigcup_{\sigma_1 < \tau_1} f(\Sigma(\sigma_1) \cap E) \cup \bigcup_{\sigma_2 < \tau_2} f(\Sigma(\tau_1, \sigma_2) \cap E) \cup \dots \cup \bigcup_{\sigma_k < \tau_k} f(\Sigma(\tau_1, \dots \tau_{k-1}, \sigma_k) \cap E)\Big)$$

Thus $\{x : x =_k a\}$ is both an F_σ and a G_δ set.

(f) is clearly true.

$\{x: x =_1 a\}$ evidently covers X as a runs over X. If $\min(f^{-1}(a))_1 = \tau_1$, $\{x: x =_1 a\} = f(\Sigma(\tau_1)\cap E) - \bigcup_{\sigma_1<\tau_1} f(\Sigma(\sigma_1)$

As $f(\Sigma(\tau_1)\cap E) = U_{\tau_1}$ and hence forms a locally finite famil τ_1 varies, so does $\{x: x =_1 a\}$ as a varies. Now fix $b \in X$. Let $a =_k b$. As $x =_{k+1} a$ implies $x =_k a$ and hence $x =_k$ we have $\{x: x =_{k+1} a\} \subseteq \{x: x =_k b\}$. Again $\{x: x =_{k+1} a\}$ covers X as a runs through X. Let $y \in \{x: x =_k b\}$. For some $y =_{k+1} a$, then $y =_k a$ for this a. Again, $y =_k b$. Hence Thus $\{x: x =_{k+1} a\}$, $a =_k b$ covers $\{x: x =_k b\}$. Let (min $= \mu_i$, $i = 1,\ldots,k$ and $(\min f^{-1}(a))_i = \tau_i$ $i = 1,2,\ldots,k+1$.

If $a =_k b$, then $\tau_i = \mu_i$ for $i = 1,\ldots,k$. Then $\{x: x =_{k+1}$

$$= f(\Sigma(\tau_1,\ldots,\tau_{k+1})\cap E) - \Big(\bigcup_{\sigma_1<\tau_1} f(\Sigma(\sigma_1)\cap E) \cup \bigcup_{\sigma_2<\tau_2} f(\Sigma(\tau_1,$$

$$\bigcup_{\sigma_{k+1}<\tau_{k+1}} f(\Sigma(\tau_1,\ldots,\tau_k,\sigma_{k+1})\cap E)\Big)$$

$$= U_{\tau_1\ldots\tau_{k+1}} - \Big(\bigcup_{\sigma_1<\tau_1} U_{\sigma_1} \cup \bigcup_{\sigma_2<\tau_2} U_{\tau_1\sigma_2} \cdots \cup \bigcup_{\sigma_{k+1}<\tau_{k+1}} U_{\tau_1\ldots}$$

$$\{x: x =_k b\} = U_{\tau_1\ldots\tau_k} - \Big(\bigcup_{\sigma_1<\tau_1} U_{\sigma_1} \cup \bigcup_{\sigma_2<\tau_2} U_{\tau_1\sigma_2} \cup \cdots$$

$$\cup \bigcup_{\sigma_k<\tau_k} U_{\tau_1\ldots\tau_{k-1}\sigma_k}\Big).$$

As $U_{\tau_1\ldots\tau_{k+1}}$ is a locally finite family relative to $U_{\tau_1\ldots}$ as τ_{k+1} varies, $\{x: x =_{k+1} a\}$, $a =_k b$, is a locally finit family relative to $\{x: x =_k b\}$.

Lemma 3. Let $<_k$, $=_k$ be defined on X as in lemma 2. For any $a \in X$ and $k \geq 1$, let $[a]_k$ denote the set $\{x: x =_k a\}$. Suppose for each a and k, a subset $Z[a]_k$ of $[a]_k$ is defined. Then for any $b \in X$ and $k > 1$, the family $\{Z[a]_k : a \in [b]_{k-1}\}$ is a family of subsets of $[b]_{k-1}$ which is locally finite relative to $[b]_{k-1}$.

Proof: If $a \in [b]_{k-1}$, then $a =_{k-1} b$, hence if $x =_k a$, then $x =_{k-1} b$. Thus $[a]_k \subseteq [b]_{k-1}$ so that $Z[a]_k \subseteq [b]_{k-1}$. By the previous lemma, $\{[a]_k : a \in [b]_{k-1}\}$ is locally finite relative to $[b]_{k-1}$. Hence $\{Z[a]_k : a \in [b]_{k-1}\}$ is also locally finite relative to $[b]_{k-1}$.

Proof of the theorem. Define $<_k$, $=_k$ and $<$ on X as in lemma 2. Let B consist of the first element of each Q in $\underline{Q}$. Then B is clearly a selector for $\underline{Q}$. It remains to show that B is of multiplicative class α.

Step 1. $B = \bigcap_k \bigcup_a \{x: x =_k a \text{ and } x \notin \{z: z <_k a\}^*\}$.

Proof. Fix k. Let $\omega \in B$. Now, $\omega =_k \omega$ and if $z <_k \omega$, then $z < \omega$ and hence $z \not\sim \omega$. Thus $\omega \notin \{z: z <_k \omega\}^*$. Hence we have $\omega \in \bigcup_a \{x: x =_k a \text{ and } x \notin \{z: z <_k a\}^*\}$ for all k. So ω belongs to the right hand side.

Let $\omega \notin B$. Then there exists some $\omega_1 < \omega$ such that $\omega_1 \sim \omega$.

Suppose $\omega_1 <_1 \omega$. If, for some a, $\omega =_1 a$, $\omega_1 <_1 a$ so that, as $\omega \sim \omega_1$, $\omega \in \{z : z <_1 a\}^*$. Thus $\omega \notin \{x: x =_1 a$ and $x \notin \{z : z <_1 a\}^*\}$ for any a. Hence ω does not belong to the right hand side.

Step 2. B is of multiplicative Borel class α.

Proof. Let $[a]_k$ be as in lemma 3 and let $X[a]_k$ be the set $\{x : x =_k a$ and $x \notin \{z: z <_k a\}^*\}$. Observe that if $b \in [a]_k$ i.e. if $[b]_k = [a]_k$, then

$\{x : x =_k a$ and $x \notin \{z : z <_k a\}^*\}$ is the same as

$\{x : x =_k b$ and $x \notin \{z : z <_k b\}^*\}$ Thus $X[a]_k$ is unambiguously defined. Clearly $X[a]_k$ is of multiplicative Borel class α.

Now $B = \bigcap_k \bigcup_a X[a]_k$. To show B is of multiplicative Borel class α, it is enough to show $\bigcup_a X[a]_k$ is of multiplicative Borel class α.

Fix $k > 1$. Now, $\bigcup\{X[a]_k : a \in X\} = \bigcup\{X[a]_k : a \in [b]_{k-1}, b \in X\}$

$= \bigcup_b\{\bigcup\{X[a]_k : a \in [b]_{k-1}\}\}$

Fix b and suppose $a \in [b]_{k-1}$. Note that $X[a]_k \subseteq [a]_k$. Hence by Lemma 3, $\{X[a]_k : a \in [b]_{k-1}\}$ is a locally finite family of sets in the relative topology of $[b]_{k-1}$. As $X[a]_k$ is of multiplicative Borel class α in X and hence in $[b]_{k-1}$, by a result in ([27]), $\bigcup\{X[a]_k : a \in [b]_{k-1}\}$ is of multiplicative class α in $[b]_{k-1}$ and hence in X. As $\cup\{X[a]_k : a \in [b]_{k-1}\}$ $\subseteq [b]_{k-1}$ and $\underset{a}{\cup} X[a]_k = \underset{b}{\cup}\{\cup\{X[a]_k : a \in [b]_{k-1}\}\}$, we repeat this process. After k-1 steps, we obtain an union of the form $\underset{a}{\cup} Z[a]_1$ where $Z[a]_1 \subseteq \{x. \ x =_1 a\}$ and is of multiplicative class α. Thus $\underset{a}{\cup} X[a]_k$ is a union of a locally finite family of sets of multiplicative class α and hence is itself of multiplicative class α (see [27]). The case when $k = 1$ can clearly be dealt with similarly.

Hence $B = \underset{k}{\cap} \underset{a}{\cup} X[a]_k$ is of multiplicative class α.

<u>An alternative proof.</u> For any k and any $\sigma_1, \ldots, \sigma_k < \tau$ define $U_{\sigma_1 \ldots \sigma_k}$ as in lemma 1. Define $H_{\sigma_1 \ldots \sigma_k}$ by induction as follows:

$$H_{\sigma_1} = U_{\sigma_1} - (\underset{\sigma < \sigma_1}{\cup} U_{\sigma})^* .$$

$$H_{\sigma_1 \ldots \sigma_k} = U_{\sigma_1 \ldots \sigma_k} \cap H_{\sigma_1 \ldots \sigma_{k-1}} - (\underset{\sigma < \sigma_k}{\cup} U_{\sigma_1 \ldots \sigma_{k-1} \sigma})^* \quad \text{if } k > 1.$$

Using induction, we can prove that $H_{\sigma_1 \ldots \sigma_k}$ is of multiplicative class α for all k and all $\sigma_1 \ldots \sigma_k < \tau$. $\{H_{\sigma_1} : \sigma_1 < \tau\}$ is a locally finite family of sets as $H_{\sigma_1} \subseteq U_{\sigma_1}$. For any $k > 1$ and any $\sigma_1, \ldots \sigma_{k-1} < \tau$, $\{H_{\sigma_1 \ldots \sigma_{k-1} \sigma_k} : \sigma_k < \tau\}$ is a locally finite family of sets relative to $U_{\sigma_1 \ldots \sigma_{k-1}}$ since $H_{\sigma_1 \ldots \sigma_{k-1} \sigma_k}$ $\subseteq U_{\sigma_1 \ldots \sigma_{k-1} \sigma_k}$. Let $B = \underset{k}{\cap} \underset{(\sigma_1, \ldots, \sigma_k)}{\cup} H_{\sigma_1 \ldots \sigma_k}$. We show that B is the required selector.

Step 1. B is a selector for $\underline{\underline{Q}}$.

Let $Q \in \underline{\underline{Q}}$ and let τ_1 = the smallest σ such that $Q \cap U_\sigma \neq \emptyset$. Then $Q \cap (\bigcup_{\sigma < \tau_1} U_\sigma) = \emptyset$ so that $Q \cap (\bigcup_{\sigma < \tau_1} U_\sigma)^* = \emptyset$. Hence $Q \cap U_{\tau_1} = Q \cap H_{\tau_1} \neq \emptyset$. Also if $\lambda < \tau_1$, $Q \cap H_\lambda \subseteq Q \cap U_\lambda = \emptyset$ and if $\lambda > \tau_1$, $Q \subseteq (\bigcup_{\sigma < \lambda} U_\sigma)^*$ so that $Q \cap H_\lambda = \emptyset$. Suppose for a given k, we have obtained $\tau_1, \ldots, \tau_k < \tau$ such that $Q \cap U_{\tau_1 \ldots \tau_k}$ $= Q \cap H_{\tau_1 \ldots \tau_k} \neq \emptyset$ and for $(\sigma_1, \ldots, \sigma_k) \neq (\tau_1, \ldots, \tau_k)$, $Q \cap U_{\sigma_1 \ldots \sigma_k} = \emptyset$. Let τ_{k+1} be the smallest σ for which $Q \cap U_{\tau_1 \ldots \tau_k \sigma} \neq \emptyset$. Then $Q \cap H_{\tau_1 \ldots \tau_k \tau_{k+1}} = Q \cap U_{\tau_1 \ldots \tau_{k+1}} \cap H_{\tau_1 \ldots \tau_k}$ $= Q \cap U_{\tau_1 \ldots \tau_k \tau_{k+1}} \cap Q \cap U_{\tau_1 \ldots \tau_k} = Q \cap U_{\tau_1 \ldots \tau_{k+1}} \neq \emptyset$ and for $(\sigma_1, \ldots, \sigma_{k+1}) \neq (\tau_1, \ldots, \tau_{k+1})$, $Q \cap H_{\sigma_1 \ldots \sigma_{k+1}} = \emptyset$. Thus we obtain a sequence $\tau_1, \tau_2, \ldots$ such that for all k, $Q \cap H_{\tau_1 \ldots \tau_k} =$ $Q \cap U_{\tau_1 \ldots \tau_k} \neq 0$ and for any $\sigma_1, \ldots, \sigma_k < \tau$ such that $(\sigma_1, \ldots, \sigma_k) \neq$ $(\tau_1, \ldots, \tau_k)$, $Q \cap H_{\sigma_1 \ldots \sigma_k} = \emptyset$. Hence $Q \cap B = Q \cap \bigcap \bigcup_{(\sigma_1, \ldots \sigma_k)} H_{\sigma_1 \ldots \sigma_k}$ $= Q \cap \bigcap H_{\tau_1 \ldots \tau_k} = Q \cap \bigcap_k U_{\tau_1 \ldots \tau_k} = Q \cap \bigcap_k \bar{U}_{\tau_1 \ldots \tau_k}$ which is clearly a singleton.

Step 2. B is a Borel set of multiplicative class α.

It is enough to prove that for any k, $\bigcup_{(\sigma_1, \ldots, \sigma_k)} H_{\sigma_1 \ldots \sigma_k}$ is of multiplicative class α. Hence by a result in ([27]), $\bigcup_{\sigma_1} H_{\sigma_1}$

is of multiplicative class α.

Let $k > 1$. $\bigcup_{(\sigma_1 \ldots \sigma_{k-1}\,\sigma_k)} H_{\sigma_1 \ldots \sigma_{k-1}\,\sigma_k} = \bigcup_{(\sigma_1 \ldots \sigma_{k-1})} \bigcup_{\sigma_k} H_{\sigma_1 \ldots \sigma_{k-1}\,\sigma_k}$

$\{H_{\sigma_1 \ldots \sigma_{k-1}\,\sigma_k} : \sigma_k < \tau\}$ is a locally finite family of sets of multiplicative class α in the relative topology of $U_{\sigma_1 \ldots \sigma_{k-1}}$. Hence as before $\bigcup_{\sigma_k} H_{\sigma_1 \ldots \sigma_{k-1}\,\sigma_k}$ is of multiplicative class α in $U_{\sigma_1 \ldots \sigma_{k-1}}$ and hence in X. As $\bigcup_{\sigma_k} H_{\sigma_1 \ldots \sigma_{k-1}\,\sigma_k} \subseteq U_{\sigma_1 \ldots \sigma_{k-1}}$ we can repeat this argument. After finitely many steps we arrive at $\bigcup_{(\sigma_1 \ldots \sigma_k)} H_{\sigma_1 \ldots \sigma_k}$ which is thus shown to be of multiplicative class α. Hence B is of multiplicative class α.

Corollary. If $\underline{\underline{Q}}$ is an α^+- partition of X where $\alpha \geq 0$, then $\underline{\underline{Q}}$ admits a selector of multiplicative Borel class $\alpha + 1$.

Proof. This follows from the fact that an α^+ partition is an $(\alpha + 1)^-$ partition.

4. Concluding remarks.

1. The result of the main theorem cannot be improved upon as far as the class of the selector is concerned. To see this consider the following example given in ([25]).

Let $X = [0,1]$, $\alpha \geq 0$. Choose $E \subseteq X$ such that $\frac{1}{2} \in E$, E is symmetric about $\frac{1}{2}$ and E is of multiplicative class α but not of additive class α.

Let $\underline{Q} = [\{x\} : x \in E] \cup [\{x, 1-x\} : x \in X - E]$. Then $\underline{Q}$ is an α^- partition of X and $\underline{Q}$ does not admit a selector of additive Borel class α.

2. If in this example, E is taken to be of additive class α but not of multiplicative class α then $\underline{Q}$ becomes an α^+ partition which does not admit a selector of multiplicative class α. Thus an α^+ partition need not admit a selector of multiplicative class α.

3. We do not know if an α^+ partition always admits a selector of additive Borel class α where $\alpha > 0$. For $\alpha = 0$, this is not true. To show this we give an example from ([25]).

Take X to be the unit circle with the usual topology. Let $\underline{Q} = [\{(x,y),(-x,-y)\} : (x,y) \in X]$. Then $\underline{Q}$ is a 0^+-partition of X which does not admit an open selector.

4. The $\underline{Q}$ given above is also a 0^--partition of X which does not admit a closed selector. Thus the theorem in this chapter does not hold for $\alpha = 0$.

5. If X is 0-dimensional, i.e. if it has a clopen base, then the theorem is true even for $\alpha = 0$.

<u>Proof</u>. A 0-dimensional space of topological weight is homeomorphic to a closed subspace of B_τ (see [25]). Thus we can take X to be a closed subspace of B_τ for some $\tau \geq \aleph_0$. Let B be obtained by taking the first element of each Q in $\underline{Q}$ according to the lexicographic order. Then

$$B = \bigcap_k \bigcup_{(\sigma_1 \dots \sigma_k)} \Big\{ x \in X : x \in \Sigma(\sigma_1, \dots, \sigma_k) \text{ and } x \notin$$

$$\Big(\bigcup_{\tau_1 < \sigma_1} \Sigma(\tau_1) \cup \dots \cup \bigcup_{\tau_k < \sigma_k} \Sigma(\sigma_1, \dots, \sigma_{k-1}, \tau_k) \Big)^* \Big\}$$

$$= \bigcap_k \bigcup_{(\sigma_1 \dots \sigma_k)} \Big(\Sigma(\sigma_1, \dots, \sigma_k) - \Big(\bigcup_{\tau_1 < \sigma_1} \Sigma(\tau_1) \cup \dots$$

$$\cup \bigcup_{\tau_k < \sigma_k} \Sigma(\sigma_1, \dots, \sigma_{k-1}, \tau_k) \Big)^* \Big) .$$

Fix k. Then $\bigcup_{(\sigma_1 \dots \sigma_k)} \Big(\Sigma(\sigma_1, \dots \sigma_k) - \Big(\bigcup_{\tau_1 < \sigma_1} \Sigma(\tau_1) \cup \dots$

$$\bigcup_{\tau_k < \sigma_k} \Sigma(\sigma_1, \dots \sigma_{k-1}, \tau_k) \Big)^* \Big)$$

is the union of a discrete family of closed sets and is therefore closed. Hence B is closed.

CHAPTER 4

BLACKWELL SPACES

1. Introduction

Suppose that X is an analytic set and $\underline{\underline{B}}$ the Borel σ-field of X. Blackwell [5] observed that the canonical Borel structure $(X, \underline{\underline{B}})$ of the analytic set X has the following property: if $\underline{\underline{C}}_1$, $\underline{\underline{C}}_2$ are countably generated sub σ-fields of $\underline{\underline{B}}$ which have identical atoms, then $\underline{\underline{C}}_1 = \underline{\underline{C}}_2$. Blackwell deduced the property from the First Principle of separation for analytic sets.

The above property can be formulated for any countably generated Borel structure and Borel structures with this property are called strong Blackwell spaces. Interest in such structures was mainly generated by the question, posed by Blackwell, whether a strong Blackwell space is Borel isomorphic to the canonical Borel structure of an analytic set. An affirmative answer to the question would have yielded an intrinsic characterization of analytic sets. However, as has been shown recently by Orkin [29] and Ryll-Nardzewski, there are non-analytic subsets of the real line, which when equipped with the relativized Borel σ-field, become strong Blackwell spaces.

The main result of this chapter is that if there is a projective well ordering of the real line of type $\aleph_1$ (which is true under the axiom of constructibility) then there are projective non-analytic subsets of the line which with the relativized Borel σ-field are strong Blackwell spaces. Furthermore, we show that the class $\underset{=}{C}$ of subsets of the line, which when endowed with the relativized Borel σ-field are strong Blackwell spaces, does not have pleasant closure properties. On the other hand the class $\underset{=}{C}$ is large. Indeed, we show that any subset of the line can be expressed as an intersection of two elements from $\underset{=}{C}$.

2. Definitions and notation

Let X be any set. A σ-algebra of subsets of X is called separable if it is countably generated and contains singletons. Let $\underset{=}{B}$ be a separable σ-algebra on X. Say that $(X, \underset{=}{B})$ is a Blackwell space if whenever a sub σ-algebra $\underset{=}{C}$ of $\underset{=}{B}$ is separable, $\underset{=}{C} = \underset{=}{B}$; $(X, \underset{=}{B})$ is called a strong Blackwell space if whenever $\underset{=}{C}_1$, $\underset{=}{C}_2$ are countably generated sub σ-algebras of $\underset{=}{B}$ with identical atoms, $\underset{=}{C}_1 = \underset{=}{C}_2$. Clearly, every strong Blackwell space is Blackwell.

If X is a metric space, $\underline{\underline{B}}_X$ denotes the Borel σ-algebra on X. X is called a (strong) Blackwell space if $(X, \underline{\underline{B}}_X)$ is a (strong) Blackwell space.

In a Polish space X, define the class of projective sets to be the smallest family containing Borel sets and closed under differences and continuous mappings. Continuous images of Borel sets are called analytic sets, complements of analytic sets are called coanalytic sets.

If X is any set and $B \subseteq X$, we use I_B to denote the indicator function of B i.e. $I_B(x) = 1$ if $x \in B$.

$= 0$ if $x \notin B$.

The characteristic function of a sequence B_n, $n = 1,2,\ldots$ of subsets of X is defined to be the function f where $f(x) = \sum_{n=1}^{\infty} \frac{2}{3^n} I_{B_n}(x)$.

We use I to denote $[0, 1]$ with the usual topology. We treat cardinals as initial ordinals.

3. Characterization of Blackwell and strong Blackwell spaces

In ([22]), we find the following characterization of Blackwell spaces:

Theorem 1. Let $\underline{\underline{B}}$ be a separable σ-algebra on a set X. Then the following are equivalent:

(a) $(X, \underline{\underline{B}})$ is a Blackwell space.

(b) If $\underline{\underline{C}}$ is a separable σ-algebra on a set Y and f a one-one mapping from X onto Y such that $f^{-1}(\underline{\underline{C}}) \subseteq \underline{\underline{B}}$, then $f(\underline{\underline{B}}) = \underline{\underline{C}}$.

(c) If f is a one-one mapping from X into a Polish space Y such that $f^{-1}(\underline{\underline{B}}_Y) \subseteq \underline{\underline{B}}$, then $f(\underline{\underline{B}}) = \underline{\underline{B}}_{f(X)}$.

(d) Every countable collection A_n, $n = 1,2,\ldots$ of sets in $\underline{\underline{B}}$ which separates points of X generates $\underline{\underline{B}}$.

We give a similar result about strong Blackwell spaces in theorem 2. Before proceeding to this, we prove some lemmas.

Lemma 1. Let X be any set and $\underline{\underline{B}}$ a σ-algebra on X generated by B_n, $n = 1,2,\ldots$ If f is the characteristic function of B_n, $n = 1,2,\ldots$, then $f^{-1}(\underline{\underline{B}}_{f(X)}) = \underline{\underline{B}}$.

Proof: Clearly, I_{B_n} is measurable with respect to $\underline{\underline{B}}$ and therefore so is $\frac{2}{3^n} I_{B_n}$. Hence f, being the limit of a series of $\underline{\underline{B}}$-measurable functions is itself $\underline{\underline{B}}$- measurable. Thus $f^{-1}(\underline{\underline{B}}_{f(X)}) \subseteq \underline{\underline{B}}$.

To see that $\underline{\underline{B}} \subseteq f^{-1}(\underline{\underline{B}}_{f(X)})$, note that $f^{-1}(\underline{\underline{B}}_X)$ is a σ-algebra containing B_n for $n = 1,2,\ldots$ Thus $\underline{\underline{B}} \subseteq f^{-1}(\underline{\underline{B}}_X)$.

Lemma 2. Let $\underline{\underline{B}}$ be a separable σ-algebra on a set X. Then the following are equivalent.

(a) $(X, \underline{\underline{B}})$ is a strong Blackwell space.

(b) For every pair $\underline{\underline{C}}_1$, $\underline{\underline{C}}_2$ of countably generated sub σ-algebras of $\underline{\underline{B}}$ such that $\underline{\underline{C}}_1 \subseteq \underline{\underline{C}}_2$ and $\underline{\underline{C}}_1$, $\underline{\underline{C}}_2$ have identical atoms, $\underline{\underline{C}}_1 = \underline{\underline{C}}_2$.

Proof. Clearly (a) implies (b).

Suppose (b) holds. To prove (a), let $\underline{\underline{A}}_1$, $\underline{\underline{A}}_2$ be countably generated sub σ-algebras of $\underline{\underline{B}}$ with identical atoms. Then $\underline{\underline{A}}_1 \subseteq \underline{\underline{A}}_1 \vee \underline{\underline{A}}_2 \subseteq \underline{\underline{B}}$ and $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ are countably generated and have identical atoms. Hence $\underline{\underline{A}}_1 = \underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$. Thus $\underline{\underline{A}}_2 \subseteq \underline{\underline{A}}_1$. Similarly, $\underline{\underline{A}}_1 \subseteq \underline{\underline{A}}_2$. Hence $\underline{\underline{A}}_1 = \underline{\underline{A}}_2$.

<u>Theorem 2.</u> Let $\underline{\underline{B}}$ be a separable σ-algebra on a set X. Then the following conditions are equivalent.

(a) $(X, \underline{\underline{B}})$ is a strong Blackwell space.

(b) If Y is any set, $\underline{\underline{A}}$ a separable σ-algebra on Y and f a function from X onto Y such that $f^{-1}(\underline{\underline{A}}) \subseteq \underline{\underline{B}}$ then $(Y, \underline{\underline{A}})$ is a strong Blackwell space.

(c) If $Y, \underline{\underline{A}}, f$ are as in (b), $(Y, \underline{\underline{A}})$ is a Blackwell space.

(d) If f is a function on X to the real line such that $f^{-1}(B) \in \underline{\underline{B}}$ for any Borel set B, then $(f(X), \underline{\underline{B}}_{f(X)})$ is a Blackwell space.

<u>Proof.</u>

To show (a) implies (b), suppose (a) holds. Let $Y, \underline{\underline{A}}, f$ be as in (b). Let $\underline{\underline{A}}_1, \underline{\underline{A}}_2$ be countably generated sub σ-algebras of $\underline{\underline{A}}$ with identical atoms. Then $f^{-1}(\underline{\underline{A}}_1), f^{-1}(\underline{\underline{A}}_2)$ are countably generated sub σ-algebras of $\underline{\underline{B}}$ with identical atoms. Hence $f^{-1}(\underline{\underline{A}}_1) = f^{-1}(\underline{\underline{A}}_2)$. Since f is onto, this implies $\underline{\underline{A}}_1 = \underline{\underline{A}}_2$.

(b) implies (c) is evident

(c) implies (d) is clear.

To show (d) implies (a), let $\underline{\underline{B}}_1, \underline{\underline{B}}_2$ be countably

generated sub σ-algebras of $\underline{\underline{B}}$ such that $\underline{\underline{B}}_1 \subseteq \underline{\underline{B}}_2$ and $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ have identical atoms. Let f be the characteristic function of $C_1, C_2, \ldots$ where $\underline{\underline{B}}_2$ is generated by C_n, $n = 1,2,\ldots$. If (d) holds, then $(f(X), \underline{\underline{B}}_{f(X)})$ is a Blackwell space. Now by lemma 1, $\underline{\underline{B}}_2 = f^{-1}(\underline{\underline{B}}_{f(X)})$, so in order to prove $\underline{\underline{B}}_1 = \underline{\underline{B}}_2$, it is enough to show $\underline{\underline{B}}_1 = f^{-1}(\underline{\underline{B}}_{f(X)})$. Let $\underline{\underline{B}}_1$ be generated by B_n, $n = 1,2,\ldots$. As $\underline{\underline{B}}_1 \subseteq f^{-1}(\underline{\underline{B}}_{f(X)})$ there exist A_n, $n = 1,2,\ldots$ in $\underline{\underline{B}}_{f(X)}$ such that $B_n = f^{-1}(A_n)$. Let $A_1, A_2, \ldots$ generate the sub σ-algebra $\underline{\underline{A}}$ of $\underline{\underline{B}}_{f(X)}$. We now show $\underline{\underline{A}} = \underline{\underline{B}}_{f(X)}$. Since $(f(X), \underline{\underline{B}}_{f(X)})$ is Blackwell, by theorem 1 it is enough to show that A_n, $n = 1,2,\ldots$ separate points of $f(X)$. Let $y_1, y_2 \in f(X)$ and $y_1 \neq y_2$. Then $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are distinct atoms of $\underline{\underline{B}}_2$ and hence of $\underline{\underline{B}}_1$. Consequently, there is some B_n such that $f^{-1}(y_1) \subseteq B_n$ and $f^{-1}(y_2) \cap B_n = \emptyset$. As $B_n = f^{-1}(A_n)$, it follows that $y_1 \in A_n$, $y_2 \notin A_n$.
As $f^{-1}(\underline{\underline{A}}) \subseteq \underline{\underline{B}}_1$, $f^{-1}(\underline{\underline{B}}_{f(X)}) \subseteq \underline{\underline{B}}_1$. Thus $\underline{\underline{B}}_1 = f^{-1}(\underline{\underline{B}}_{f(X)})$.

<u>Question:</u> Is there a Blackwell space which is not strong Blackwell ?

4. <u>Ryll-Nardzewski's construction</u>

Notation: If f, g are functions from I into I, put

$B_{f\,g} = \{(u, v)\colon f(u) = f(v) \text{ and } g(u) \neq g(v)\}$

$A_{f\,g} = \{y\colon \text{for some } u, v \text{ in I}, f(u) = f(v) = y \text{ and } g(u) \neq g(v$

= Projection to the third co-ordinate of

$\{(u, v, y): f(u) = f(v) = y$ and $g(u) \neq g(v)\}$.

Note that if f, g are Borel measurable, $B_{f\,g}$ is a Borel set and $A_{f\,g}$ is an analytic set. We say that a subset X of I has property (P) if for every pair (f,g) of Borel measurable functions from I into I such that $A_{f\,g}$ is uncountable, there exist (u, v) in $B_{f\,g}$ such that $u \in X$, $v \in X$.

Theorem (Ryll-Nardzewski). If a subset X of I has property (P), then X is a strong Blackwell space.

Proof. Let $\underline{\underline{B}}_1$, $\underline{\underline{B}}_2$ be countably generated sub σ-algebras of $\underline{\underline{B}}_X$ with identical atoms such that $\underline{\underline{B}}_1 \subseteq \underline{\underline{B}}_2$. Suppose $\underline{\underline{C}}_1 \subseteq \underline{\underline{C}}_2 \subseteq \underline{\underline{B}}_I$ be countably generated σ-algebras such that $\underline{\underline{B}}_1 = \underline{\underline{C}}_1 \cap X$, $\underline{\underline{B}}_2 = \underline{\underline{C}}_2 \cap X$. Let f, g be the characteristic functions of countable families of sets generating $\underline{\underline{C}}_1$ and $\underline{\underline{C}}_2$ respectively.

Step 1. We show that $A_{f\,g}$ is countable.

If $A_{f\,g}$ is uncountable, there exist u, v in X such that $f(u) = f(v)$ and $g(u) \neq g(v)$. Thus u, v are in different $\underline{\underline{C}}_2$ atoms but in the same $\underline{\underline{C}}_1$ atom. Hence they are in different $\underline{\underline{B}}_2$ atoms but the same $\underline{\underline{B}}_1$ atom. Contradiction.

Step 2. Put $Y = f^{-1}(I - A_{f\,g})$.

As $I - A_{f\,g}$ is Borel, $Y \in \underline{\underline{C}}_1$. Plainly, if $y \in I - A_{f\,g}$ then $f^{-1}(y)$ is an atom of both $\underline{\underline{C}}_1$ and $\underline{\underline{C}}_2$. Since Y is a Borel subset of I, $(Y, \underline{\underline{B}}_Y)$ is a strong Blackwell space and hence $\underline{\underline{C}}_1 \cap Y = \underline{\underline{C}}_2 \cap Y$.

Step 3. Since $Y \in \underline{\underline{C}}_1$, $X \cap Y$ and $X - Y$ are in $\underline{\underline{B}}_1$. As $I - Y$ is a countable union of $\underline{\underline{C}}_1$ atoms, $X - Y$ must be a countable union of $\underline{\underline{B}}_1$ atoms. Consequently, if $E \subseteq X - Y$ and E is a union of $\underline{\underline{B}}_1$ atoms, then $E \in \underline{\underline{B}}_1$.

Step 4. Now let $A \in \underline{\underline{B}}_2$. Find $C \in \underline{\underline{C}}_2$ such that $A = C \cap X$. Write $A = ((C \cap Y) \cap X) \cup (C \cap (X - Y))$. Now by step 2, $C \cap Y \in \underline{\underline{C}}_2 \cap Y = \underline{\underline{C}}_1 \cap Y \subseteq \underline{\underline{C}}_1$, so $(C \cap Y) \cap X \in \underline{\underline{B}}_1$. Moreover, $C \cap (X - Y)$ is a union of $\underline{\underline{B}}_2$ atoms, hence of $\underline{\underline{B}}_1$ atoms. So by step 3, $C \cap (X - Y) \in \underline{\underline{B}}_1$. This proves $A \in \underline{\underline{B}}_1$ so that $\underline{\underline{B}}_1 = \underline{\underline{B}}_2$.

Construction of a non-analytic subset of I with property (P) (Ryll-Nardzewski).

Let $\{P_\alpha : \alpha < c\}$ and $\{(f_\alpha, g_\alpha) : \alpha < c\}$ enumerate all nonempty perfect subsets of I and all ordered pairs of Borel measurable functions from I into I respectively. For each $\alpha < c$ define by transfinite induction, finite sets E_α, F_α, G_α as follows:

Let E_τ, F_τ, G_τ be defined for $\tau < \alpha$. Let a_α, b_α be distinct elements of $P_\alpha - \bigcup_{\tau < \alpha} (E_\tau \cup F_\tau \cup G_\tau)$. Such elements exist as P_α has cardinality c and $\bigcup_{\tau < \alpha} (E_\tau \cup F_\tau \cup G_\tau)$ has cardinality $\leq$ (maximum of $\aleph_0$ and card (α)) $< c$. Put $E_\alpha = \{a_\alpha\}$, $F_\alpha = \{b_\alpha\}$. If $A_{f_\alpha g_\alpha}$ is countable put $G_\alpha = \emptyset$. Otherwise,

$A_{f_\alpha g_\alpha}$ and hence $B_{f_\alpha g_\alpha}$ has cardinality c so that we can find $(u_\alpha v_\alpha) \in B_{f_\alpha g_\alpha}$ such that $u_\alpha, v_\alpha \notin E_\alpha \cup F_\alpha \cup (\bigcup_{\tau < \alpha} E_\tau \cup F_\tau \cup G_\tau))$. In this case, put $G_\alpha = \{u_\alpha, v_\alpha\}$.

Let $X = \bigcup_{\alpha < c} (E_\alpha \cup G_\alpha)$. Since $\bigcup_{\alpha < c} G_\alpha \subseteq X$, it follows that X has property (P). Again, neither X nor $I - X$ contains a non-empty perfect set so that X is not analytic.

We now modify the above to construct a non-analytic subset X of I such that both X and $I-X$ have property (P). For this purpose, let $\{P_\alpha : \alpha < c\}$ and $\{(f_\alpha, g_\alpha): \alpha < c\}$ be as in Ryll-Nardzewski's construction. Define, by transfinite induction, finite sets $E_\alpha, F_\alpha, G_\alpha, H_\alpha$ for all $\alpha < c$ as follows:

Suppose $E_\tau, F_\tau, G_\tau, H_\tau$ have been defined for all $\tau < \alpha$. Let a_α, b_α be distinct elements of $P_\alpha - \bigcup_{\tau < \alpha} (E_\tau \cup F_\tau \cup G_\tau \cup H_\tau)$. Let $E_\alpha = \{a_\alpha\}$, $F_\alpha = \{b_\alpha\}$. If $A_{f_\alpha g_\alpha}$ is countable, put $G_\alpha = H_\alpha = \emptyset$. Otherwise, choose distinct elements $u_\alpha, v_\alpha, s_\alpha, t_\alpha$ from $I - (\bigcup_{\tau < \alpha} (E_\tau \cup F_\tau \cup G_\tau \cup H_\tau) \cup E_\alpha \cup F_\alpha)$ such that $f_\alpha(u_\alpha) = f_\alpha(v_\alpha)$, $g_\alpha(u_\alpha) \neq g_\alpha(v_\alpha)$, $f_\alpha(s_\alpha) = f_\alpha(t_\alpha)$ and $g_\alpha(s_\alpha) \neq g_\alpha(t_\alpha)$. Put $G_\alpha = \{u_\alpha, v_\alpha\}$, $H_\alpha = \{s_\alpha, t_\alpha\}$. Let $X = \bigcup_{\alpha < c} (E_\alpha \cup G_\alpha)$, then $I - X \supseteq \bigcup_{\alpha < c} (F_\alpha \cup H_\alpha)$. Hence both X and $I - X$ have property (P). Again, since neither X nor $I - X$ contains a nonempty perfect set, X is not analytic.

5. Failure of closure properties for Blackwell and strong Blackwell spaces.

For the sake of definiteness, we work with subsets of the real line unless otherwise stated in this section.

Proposition 1. The union of two strong Blackwell spaces need not even be Blackwell.

Proof. Let $A \subseteq I$ be a non-analytic set such that both A and $I - A$ have the property (P) and hence are strong Blackwell spaces.

Let B be the subset of the real line given by $B = \{x+2 \, ; \, x \in I - A\}$. As B is homeomorphic to $I - A$, B is a strong Blackwell space.

Let $C = A \cup B$. Then C is not a Blackwell space. To see this, let $\underline{\underline{D}} = \{D : D = (E \cap A) \cup ((E+2) \cap B),\ E$ is a Borel subset of $I\}$, where $E+2 = \{x+2 : x \in E\}$.

Define f on C into I by
$$f(x) = x \quad \text{if } x \in A$$
$$= x-2 \quad \text{if } x \in B$$

As A, B are Borel subsets of C, f is Borel measurable. Again f is one-one and $f^{-1}(\underline{\underline{B}}_I) = \underline{\underline{D}}$. Hence $\underline{\underline{D}}$ is a separable sub σ-algebra of $\underline{\underline{B}}_C$. But $A \notin \underline{\underline{D}}$ while $A \in \underline{\underline{B}}_C$. Hence C is not a Blackwell space.

Proposition 2. If X is a Blackwell space and B an absolute Borel set such that $X \cap B = \emptyset$, then $X \cup B$ is a Blackwell space

Proof. Let $A = X \cup B$ and let $\underline{\underline{C}} \subseteq \underline{\underline{B}}_A$ be a separable σ-algebra. Let $C_1, C_2 \ldots$ generate $\underline{\underline{C}}$ and suppose f is the characteristic function of $C_1, C_2, \ldots$. f restricted to B is a one-one Borel measurable function and hence $f(B)$ is an absolute Borel set contained in $f(A)$. Since $f^{-1}(\underline{\underline{B}}_{f(A)}) = \underline{\underline{C}}$, $B = f^{-1}(f(B)) \in \underline{\underline{C}}$. Thus $\underline{\underline{C}} \cap B$ and $\underline{\underline{C}} \cap X \subseteq \underline{\underline{C}}$ i.e. $\underline{\underline{B}}_B$ and $\underline{\underline{B}}_X \subseteq \underline{\underline{C}}$. Hence $\underline{\underline{B}}_A \subseteq \underline{\underline{C}}$.

Proposition 3. Any subset of I can be written as the intersection of two strong Blackwell spaces.

Proof. Let $E \subseteq I$. Let A be chosen so that A and $I - A$ both have the property (P). Let $C = A \cup E$ and $D = (I - A) \cup E$. Note that C and D also have the property (P) and hence they are strong Blackwell spaces. Clearly, $E = C \cap D$.

Remark. As any uncountable Polish space is Borel isomorphic to I, the proposition is true if we take any Polish space instead of I.

Proposition 4. There exist two strong Blackwell spaces contained in I whose Cartesian product $(\subseteq I \times I)$ is not even Blackwell.

Lemma. Suppose $(X, \underline{\underline{B}})$ is a (strong) Blackwell space. Suppose $E \in \underline{\underline{B}}$. Then $(E, \underline{\underline{B}} \cap E)$ is (strong) Blackwell.

Proof. We prove it for Blackwell spaces, the proof for strong Blackwell spaces is similar.

Let $\underline{\underline{C}} \subseteq \underline{\underline{B}} \cap E$ be a separable σ-algebra on E. Let $\underline{A}$ be the σ-algebra generated by $\underline{C}$ on E and $\underline{B} \cap (X-E)$ on $X-E$. Then $\underline{A}$ is separable and as $E \in \underline{B}$, $\underline{A} \subseteq \underline{B}$. Thus $\underline{A} = \underline{B}$. But $\underline{C} = \underline{A} \cap E$. Hence $\underline{C} = \underline{B} \cap E$.

Proof of the proposition. We know there exist strong Blackwell spaces whose intersection is not even Blackwell. Hence it is enough to prove that if the product of two subsets of I is Blackwell, then so is their intersection.

Let $B, C \subseteq I$ be such that $B \times C$ is Blackwell. Let $D = \{(x, x); x \in I\}$. $(B \times C) \cap D$ is homeomorphic to $B \cap C$. Hence, it is enough to show that $(B \times C) \cap D$ is Blackwell. This follows from the lemma as $(B \times C) \cap D$ is a Borel subset of $B \times C$.

Proposition 5. The field generated by the strong Blackwell spaces in a Polish space X is the power set.

Proof. Follows from proposition 3.

Proposition 6. There are 2^c strong Blackwell spaces contained in any Polish space.

Proof. Follows from proposition 5.

6. Construction of a projective, non-analytic, strong Blackwell space.

This construction can be done under the following assumption: I can be well ordered in a transfinite sequence of type Ω_1 (the first uncountable ordinal) by a relation $\prec$ such that $\{(x, y) ; x \prec y\}$ is a projective subset of $I \times I$. Note that as I is Borel isomorphic to any uncountable Polish space X, this assumption is equivalent to the one that any uncountable Polish space can be well ordered in this way. Such a result follows from Godel's axiom of constructibility (V = L) which is consistent with the axioms of ZF set theory.

Let X be any set and $\underline{\underline{F}}$ a family of subsets of X. By an universal set for the family $\underline{\underline{F}}$, we mean a subset F of I x X such that $F^t = \{x : (t, x) \in F\}$ give the family $\underline{\underline{F}}$ as t runs through I.

<u>Lemma 1.</u> There exist projective subsets R and P of I x I which are universal for the families of countable subsets and uncountable Borel subsets of I respectively.

Let U be a Borel subset of I x I universal for the family of F_σ sets of I. The existence of such a U is proved in ([17]). Let $C = \{t : U^t$ is countable$\}$ C is a coanalytic set (see ([17]) and also chapter 3). Let $R = U \cap (C \times I)$. R is universal for the family of countable subsets of I.

Let A be an analytic subset of I x I universal for the family of Borel sets in I (see ([39])). Let $D = \{t : A^t$ is uncountable$\}$. Then $t \in D$ if, and only if, $A^t \not\subset R^{t_1}$ for any t_1 i.e. if, and only if, for any t_1, there is an x such that $(t, x) \in A$ and $(t_1, x) \notin R$. In symbols, $t \in D \iff$ $\forall t_1 \exists x ((t, x) \in A$ and $(t_1, x) \notin R)$ where $\forall$ stands for

for all, $\exists$ stands for there exists and $\Longleftrightarrow$ stands for if and only if. By a theorem in ([17]) D is projective. Let $P = (A \cap (D \times I)) \cup (D^c \times I)$. Then P is universal for the family of uncountable Borel subsets of I.

We now proceed to construct a set E which we then prove is non-analytic, projective and strong Blackwell.

Construction

Let F be an analytic set universal for the family of Borel sets in I^2. Let $T \subseteq I$ be defined by $t \in T$ if and only if $F^t = \{(x,y): (t,x,y) \in F\}$ is a graph. In symbols

$$t \in T \Longleftrightarrow \forall x \exists y\ ((x,y) \in F^t) \text{ and } \forall x \forall y \forall y_1 (((x,y) \in F^t \text{ and } (x,y_1) \in F^t) \Rightarrow y = y_1)$$

i.e. $\forall x \exists y ((t,x,y) \in F)$ and $\forall x \forall y \forall y_1 (((t,x,y) \in F \text{ and } (t,x,y_1) \in F) \Rightarrow y = y_1)$

Then T is a projective set (see [17]).

Let $Z \subseteq I$ be defined by $z \in Z$ if and only if there exist Borel measurable functions f, g from I into I such that $A_{f\,g}$ is uncountable and $B_{f\,g} = F^z$. Replacing f, g by their graphs, say F^{t_1} and F^{t_2}, we get the following.

$$z \in Z \Longleftrightarrow \exists t_1 \exists t_2 (t_1 \in T \text{ and } t_2 \in T \text{ and } \forall u \forall v (((z,u,v) \in F) \Longleftrightarrow (\forall y (((t_1,u,y) \in F) \Rightarrow ((t_1,v,y) \in F)) \text{ and } \forall y (((t_2,u,y) \in F) \Rightarrow ((t_2,v,y) \notin F)))) \text{ and } \forall t \exists y \exists u \exists v (((t_1,u,y) \in F) \text{ and } ((t_1,v,y) \in F) \text{ and } \forall y_1 (((t_2,u,y_1) \in F) \Rightarrow ((t_2,v,y_1) \notin F)) \text{ and } (t,y) \notin R)).$$

Plainly, Z is projective.

Let $Y = I \times I \cup \{(2,3)\}$. For countable $X \subseteq Y$ and $t \in I$, define $G(X,t)$ as follows:

$$G(X, t) = (P^{2t} \times P^{2t}) - (D \cup X^*) \text{ if } 0 \leq t \leq \tfrac{1}{2}$$
$$= F^{2t-1} - X^* \text{ if } \tfrac{1}{2} < t \leq 1 \text{ and } 2t-1 \in Z$$
$$= \{(2,3)\} \text{ otherwise}$$

where $D = \{(x,x): x \in I\}$ and $X^* = (\pi_1(X) \times I) \cup (\pi_2(X) \times I) \cup (I \times \pi_1(X)) \cup (I \times \pi_2(X))$ where π_1 and π_2 denote projection to the first and second co-ordinate respectively.

For $0 \leq t \leq \frac{1}{2}$, card $(P^{2t}) = c$. As card $(X) \leq \aleph_0$ $G(X,t) \neq \emptyset$. For $\frac{1}{2} < t \leq 1$ and $2t-1 \in Z$, there are Borel measurable functions f, g such that $A_{f\,g}$ is uncountable and $F^{2t-1} = B_{f\,g}$. Hence $G(X,t) \neq \emptyset$ as card $(X) \leq \aleph_0$. Clearly, if $\frac{1}{2} < t \leq 1$ and $2t-1 \notin Z$, $G(X, t) \neq \emptyset$.

Let $\prec$ and $\prec'$ well order I and Y respectively in a projective manner. Define a function g on I into Y by trans - finite induction as follows:

$g(t) = p\, G(g[A(t)], t)$ where $p\, W$ is the first element of a subset W of Y according to $\prec'$ and $A(t) = \{u ; u \prec t\} \subseteq I$ (see [18]).

Let $A = \{x: \exists y\ \exists t\ ((x, y) = g(t))$ and $x \neq 2\}$ and let $B = \{y: \exists x\ \exists t\ (((x, y) = g(t))$ and $\frac{1}{2} < t \leq 1$ and $2t-1 \in Z)\}$. Put $A \cup B = C$. If $0 \notin Z$, let $E = C$. If $0 \in Z$, choose $(x_0, y_0) \in F^0$ and put $E = C \cup \{x_0, y_0\}$.

The proof of the fact that E is the required set is given by the following propositions:

Proposition 1. There is a projective set R' universal for the family of countable subsets of Y such that $\{(x, y, s, t): (x,y) \in G(R'^s, t)\}$ is a projective set.

Proof. Let R' be a projective set universal for the family of countable subsets of Y.

Let $H = \{(x,y,s,t): (x,y) \in G(R'^s,t)$ and $0 \le t \le \frac{1}{2}\}$

$J = \{(x,y,s,t): (x,y) \in G(R'^s,t)$ and $\frac{1}{2} < t \le 1$ and $2t-1 \in Z\}$

$K = \{(x,y,s,t): (x,y) \in G(R'^s,t)$ and $\frac{1}{2} < t \le 1$ and $2t-1 \notin Z\}$.

It is enough to show that H, J and K are projective sets.

Now $(x,y,s,t) \in H$ if, and only if,

$0 \le t \le \frac{1}{2}$ and $(\exists q(q = 2t$ and $(q,x) \in P$ and $(q,y) \in P))$ and $x \ne y$ and $(\forall u \forall v (((s,u,v) \in R') \Rightarrow (x \ne u$ and $x \ne v$ and $y \ne u$ and $y \ne v)))$.

Thus H is a projective set.

$(x,y,s,t) \in J$ if, and only if,

$(\frac{1}{2} < t \le 1$ and $(\exists q(q = 2t - 1$ and $q \in Z$ and $(q,x,y) \in F))$ and $(\forall u \forall v (((s,u,v) \in R') \Rightarrow (x \ne u$ and $x \ne v$ and $y \ne u$ and $y \ne v)$

Thus J is a projective set.

$(x,y,s,t) \in K$ if, and only if,

$\frac{1}{2} < t \le 1$ and $(\exists q(q = 2t - 1$ and $q \notin Z))$ and $x = 2$ and $y = 3$.

Thus K is a projective set.

Proposition 2. E is a projective subset of I.

Proof. It is enough to show that C is a projective subset of I. This follows from the following theorem of Kuratowski: Let Y be an uncountable Polish space and for every countable $X \subseteq Y$ and $t \in I$, let $G(X,t)$ be a nonempty subset of Y. Let $\prec'$, $\prec$ be projective well orderings on Y and I respectively and define f on I into Y by $f(t) = p\, G(f[A(t)], t)$ where $A(t) = \{u: u \prec t\} \subseteq I$ and for $W \subseteq Y$, $p\, W$ is the first element of W according to $\prec'$. If there exists a universal set $R' \subseteq I \times Y$ for the family of countable subsets of Y such that $\{(t, s, y): y \in G(R'^{s}, t)\}$ is projective, then $\{(t,x): x = f(t)\}$ is also a projective set.

Using this and proposition 1, we see that $\{(t,x,y): (x,y) = g(t)\}$ is projective. Hence A,B are projective and therefore so is C. Clearly $C \subseteq I$.

Proposition 3. E is not analytic.

Proof. As before, it is enough to show that C is not analytic. For any t, put $g(t) = (a(t), b(t))$. Except in the case $\frac{1}{2} < t \leq 1$ and $2t-1 \notin Z$, when $a(t) = 2$ and $b(t) = 3$, all the $a(t)$'s and $b(t)$'s are distinct and lie in I. Any uncountable Borel set is P^{2t} for some t such that $0 \leq t \leq \frac{1}{2}$. For such a t, $a(t) \in P^{2t} \cap C$, $b(t) \in P^{2t} \cap (I-C)$. Hence neither C nor $I-C$ contains an uncountable Borel set. Thus C is not Lebesgue measurable and hence not analytic.

Proposition 4. E is strong Blackwell.

This follows from Ryll-Nardzewski's theorem as E has property (P) since for any z in Z, there is some $(x,y) \in F^z$ such that $x \in E$ and $y \in E$.

Propositions 1 to 4 show that E is the required set.

Proposition. There is a non-analytic, projective, strong Blackwell space which is Lebesgue measurable and even one with positive Lebesgue measure.

Proof. Let $C \subseteq I$ be the Cantor set and let E be the projective, non-analytic, strong Blackwell space constructed above. Let ψ be a Borel isomorphism from I onto C. Then $\psi(E)$ is a projective, non-analytic, strong Blackwell space. As $\psi(E) \subseteq C$, $\psi(E)$ has Lebesgue measure zero. Let $G = (I-C) \cup \psi(E)$. clearly G is non-analytic, projective and has Lebesgue measure. To show that G is strong Blackwell, it is enough to show that G has property (P). Let f,g be Borel measurable functions from I into I such that A_{fg} is uncountable.

Case 1. $\{y: \exists u \exists v (u \in C, v \in C, f(u) = f(v) = y, g(u) \neq g(v))\}$ is uncountable. In this case, there exist u, v in $\psi(E)$ such that $(u,v) \in B_{fg}$. This follows from the fact that ψ is a Borel isomorphism from I onto C and E has property (P).

Case 2. $\exists u \exists v (u \in I-C, v \in I-C, (u, v) \in B_{fg})$. In this case, clearly there exist u, v in E such that $(u,v) \in B_{fg}$.

Case 3. $\{y\colon \exists u \, \exists v \, (u \in C, v \in I - C, \; f(u) = f(v) = y, g(u) \neq g(v))\}$ is uncountable. Thus $\pi_1(B_{fg} \cap (C \times (I - C)))$ is an uncountable analytic set $\subseteq C$ where π_1 denotes projection to the first co-ordinate. As neither $\psi(E)$ nor $C - \psi(E)$ contains an uncountable analytic set, $\psi(E) \cap (\pi_1(B_{fg} \cap (C \times (I - C))))$ is not empty. Let $u \in \psi(E) \cap (\pi_1(B_{fg} \cap (C \times (I - C))))$. Then there exists v such that $(u, v) \in B_{fg} \cap (C \times (I - C))$. Thus $u \in \psi(E) \subseteq G$, $v \in I - C \subseteq G$ and $(u, v) \in B_{fg}$.

Thus G has property (P).

CHAPTER 5

COMPLEMENTATION IN THE LATTICE OF BOREL STRUCTURES

1. Introduction

In connection with his study of maximal and minimal elements of families of statistics, D. Basu [2] posed the follo problem. Let $(X, \underline{\underline{A}})$ be a Borel structure and let $\underline{\underline{B}}$ be a sub σ-field of $\underline{\underline{A}}$. Does there exist a complement of $\underline{\underline{B}}$ relative to $\underline{\underline{A}}$, i.e. is there a σ-field $\underline{\underline{C}}$ on X such that $\underline{\underline{B}} \vee \underline{\underline{C}} = \underline{\underline{A}}$ and $\underline{\underline{B}} \wedge \underline{\underline{C}} = \{X, \emptyset\}$? In other words, if $\underline{\underline{L}}$ is the lattice of sub σ-fields of $\underline{\underline{A}}$, then is $\underline{\underline{L}}$ a complemented lattice ?

B. V. Rao showed in his doctoral dissertation that if X is an abstract set, $\underline{\underline{L}}$ the lattice of all σ-fields on X, then $\underline{\underline{L}}$ is complemented if and only if X is countable. B. V. Rao moreover gave a partial solution to the problem of characterizing those countably generated sub σ-fields of a standard Borel space which admit complements relative to the parent Borel σ-field.

In this chapter, we present a complete solution. Indeed, we prove that all such sub σ-fields admit complements. K. P. S. Bhaskara Rao [33] had already shown this for countably generated sub σ-fields with countable atoms. We then completed the solution by proving that a countably generated sub σ-field with at least one uncountable atom admits a complement. And finally E Grzegorek proved that minimal complements exist for any countably generated sub σ-field of a standard Borel space [37].

These results are presented in this chapter, some in the more general context of X an analytic set and $\underline{\underline{A}}$ the Borel σ-field on X. However we are unable to solve the problem completely in this situation. In particular we do not know if a countably generated sub σ-field of $\underline{\underline{A}}$ with countable atoms admits a complement relative to $\underline{\underline{A}}$.

2. Definitions and notation

Let X be any set and $\underline{\underline{B}}$ a σ-algebra on X. Let $\underline{\underline{A}}$ and $\underline{\underline{C}}$ be substructures (i.e. sub σ-algebras) of $\underline{\underline{B}}$. $\underline{\underline{A}} \vee \underline{\underline{C}}$ denotes the σ-algebra generated by $\underline{\underline{A}} \cup \underline{\underline{C}}$, $\underline{\underline{A}} \wedge \underline{\underline{C}}$ denotes $\underline{\underline{A}} \cap \underline{\underline{C}}$. We say that $\underline{\underline{C}}$ is weak a complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}$ if $\underline{\underline{A}} \vee \underline{\underline{C}} = \underline{\underline{B}}$. $\underline{\underline{C}}$ is called a complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}$ if it is a relative weak complement and $\underline{\underline{A}} \wedge \underline{\underline{C}} = \{\emptyset, X\}$. A relative (weak) complement $\underline{\underline{C}}$ of $\underline{\underline{A}}$ is said to be minimal if no proper substructure of $\underline{\underline{C}}$ is a relative (weak) complement of $\underline{\underline{A}}$.

A set B is called a selector for an atomic σ-algebra $\underline{\underline{A}}$ if $A \cap B$ is a singleton for every atom A of $\underline{\underline{A}}$. B is called a partial selector for $\underline{\underline{A}}$ if $A \cap B$ is either empty or a singleton for every atom A of $\underline{\underline{A}}$. For any set A, A^c denotes the complement of A.

For a metric space X, we use $\underline{\underline{B}}_X$ to denote the Borel σ-algebra on X. In this chapter, we take X to be an analytic subset of some Polish space Y. I denotes the closed interval $[0, 1]$ with the usual topology.

3. Main results

We first prove some lemmas.

Lemma 1 ([31]). If $\underline{\underline{A}}$, $\underline{\underline{C}}$ are sub σ-algebras of $\underline{\underline{B}}_X$ on X such that $\underline{\underline{C}}$ is a minimal weak complement of $\underline{\underline{A}}$ relative to

$\underline{B}_X$, then $\underline{C}$ is a complement of $\underline{A}$ relative to $\underline{B}_X$. Hence $\underline{C}$ is a relative minimal complement of $\underline{A}$.

Proof. It is enough to show that $\underline{A} \wedge \underline{C} = \{\emptyset, X\}$. If possible let A be a nonempty set such that $A \neq X$ and $A \in \underline{A} \wedge \underline{C}$. Let $x \in A$, $y \in A^c$. Let $\underline{D} = \{C : C \in \underline{C}, \{x, y\} \subseteq C$ or $\{x, y\} \cap C = \emptyset\}$. $\underline{D}$ is a proper substructure of $\underline{C}$ since $A \notin \underline{D}$. We show $\underline{A} \vee \underline{D} = \underline{B}_X$ so that $\underline{C}$ is not a relative minimal weak complement of $\underline{A}$.

It is enough to show that $\underline{C} \subseteq \underline{A} \vee \underline{D}$. Let $Z \in \underline{C}$. If $Z \supseteq \{x, y\}$ or $Z \cap \{x, y\} = \emptyset$, then $Z \in \underline{D}$. Suppose $Z \notin \underline{D}$. Then either $x \in Z$, $y \notin Z$ or $x \in Z^c$, $y \notin Z^c$. Without loss of generality, suppose $x \in Z$, $y \notin Z$. Now $Z \cap A^c \in \underline{C}$ and $(Z \cap A^c) \cap \{x, y\} = \emptyset$. Hence $Z \cap A^c \in \underline{D}$. Also $(Z \cap A) \cup A^c \in \underline{C}$ and $\{x, y\} \subseteq (Z \cap A) \cup A^c$. Hence $(Z \cap A) \cup A^c \in \underline{D}$. As $A \in \underline{A}$, $Z \cap A = ((Z \cap A) \cup A^c) \cap A \in \underline{A} \vee \underline{D}$. Thus $Z = (Z \cap A) \cup (Z \cap A^c) \in \underline{A} \vee \underline{D}$. Hence $\underline{C} \subseteq \underline{A} \vee \underline{D}$.

Lemma 2.([31]). If $\underline{A}$ is a substructure of $\underline{B}_X$ and $\underline{C}$ is a (weak) complement of $\underline{A}$ relative to $\underline{B}_X$ then there is a countably generated substructure $\underline{D}$ of $\underline{C}$ which is also a relative (weak) complement of $\underline{A}$.

Proof. Let $\underline{G}$, $\underline{H}$ be generators for $\underline{A}$ and $\underline{C}$ respectively. Then $\underline{G} \cup \underline{H}$ generates $\underline{B}_X$. Let Z_n, $n = 1, 2, \ldots$ be a countable subfamily of $\underline{\underline{G}} \cup \underline{\underline{H}}$ which generates $\underline{\underline{B}}_X$. Such a family exists since $\underline{\underline{B}}_X$ is countably generated. Let $\underline{\underline{D}}$ be the σ-algebra generated by those Z_n's which are not in $\underline{\underline{G}}$. As these Z_n's are in $\underline{\underline{H}}$, $\underline{\underline{D}} \subseteq \underline{\underline{C}}$. Again $\underline{\underline{A}}$ contains $\underline{\underline{G}}$ and hence contains the remaining Z_n's. Thus $\underline{\underline{A}} \vee \underline{\underline{D}}$ contains all Z_n's and hence $\underline{\underline{A}} \vee \underline{\underline{D}} = \underline{\underline{B}}_X$.

Lemma 3. (see [5]).

Two countably generated sub σ-algebras of the Borel σ-algebra on an analytic set are equal if and only if they have the same atoms.

Theorem 1. (E. Grzegorek). Let $\underline{\underline{A}}$, $\underline{\underline{C}}$ be countably generated sub σ-algebras of $\underline{\underline{B}}_X$. Then $\underline{\underline{A}} \vee \underline{\underline{C}} = \underline{\underline{B}}_X$ if, and only if, every atom of $\underline{\underline{C}}$ is a partial selector for $\underline{\underline{A}}$.

Proof. $\underline{\underline{A}} \vee \underline{\underline{C}}$ is countably generated. Hence $\underline{\underline{A}} \vee \underline{\underline{C}} = \underline{\underline{B}}_X$ if, and only if, they have the same atoms i.e. if, and only if, $\underline{\underline{A}} \vee \underline{\underline{C}}$ separates points i.e. if and only if every atom of $\underline{\underline{C}}$ is a partial selector for $\underline{\underline{A}}$.

Theorem 2. (E. Grzegorek). Let $\underline{\underline{A}}$, $\underline{\underline{C}}$ be countably generated substructures of $\underline{\underline{B}}_X$. Then $\underline{\underline{C}}$ is a minimal complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}_X$ if and only if

(a) every atom of $\underline{\underline{C}}$ is a partial selector for $\underline{\underline{A}}$

(b) if C_1, C_2 are distinct atoms of $\underline{\underline{C}}$, then $C_1 \cup C_2$ is not a partial selector for $\underline{\underline{A}}$.

Proof. Let (a) and (b) hold. By theorem 1, $\underline{\underline{A}} \vee \underline{\underline{C}} = \underline{\underline{B}}_X$. To prove that $\underline{\underline{C}}$ is a minimal complement, it is enough to show that if $\underline{\underline{D}}$ is a countably generated substructure of $\underline{\underline{C}}$ such that $\underline{\underline{A}} \vee \underline{\underline{D}} = \underline{\underline{B}}_X$, then $\underline{\underline{D}} = \underline{\underline{C}}$. By lemma 3, it is enough to show that given such a $\underline{\underline{D}}$, it has the same atoms as $\underline{\underline{C}}$. Suppose not. Then there exists a countably generated $\underline{\underline{D}} \subseteq \underline{\underline{C}}$ such that $\underline{\underline{A}} \vee \underline{\underline{D}} = \underline{\underline{B}}_X$ and there is an atom D of $\underline{\underline{D}}$ containing two distinct atoms of $\underline{\underline{C}}$. But in that case D is not a partial selector for $\underline{\underline{A}}$ and hence $\underline{\underline{A}} \vee \underline{\underline{D}} \neq \underline{\underline{B}}_X$.

Conversely, let $\underline{\underline{C}}$ be a minimal complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}_X$. By theorem 1, (a) holds. Suppose (b) does not hold. Then let C_1, C_2 be distinct atoms of $\underline{\underline{C}}$ such that $C_1 \cup C_2$ is a partial selector for $\underline{\underline{A}}$. Let $\underline{\underline{D}}$ be generated by $C_1 \cup C_2$ and $\underline{\underline{C}} \cap (X - (C_1 \cup C_2))$. Then $\underline{\underline{D}}$ is countably generated and every atom of $\underline{\underline{D}}$ is a partial selector for $\underline{\underline{A}}$. Thus $\underline{\underline{A}} \vee \underline{\underline{D}} = \underline{\underline{B}}_X$. As $\underline{\underline{D}}$ is a proper substructure of $\underline{\underline{C}}$, this contradicts the fact that $\underline{\underline{C}}$ is a minimal complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}_X$.

Theorem 3 If $\underline{\underline{A}} \subseteq \underline{\underline{B}}_X$ has an uncountable atom, then $\underline{\underline{A}}$ has a minimal complement relative to $\underline{\underline{B}}_X$.

Proof. Let A be an uncountable atom of $\underline{\underline{A}}$. As A is analytic, we can find an absolute Borel set $B \subseteq A$ such that B and $Y - B$ are uncountable. Let g be a Borel isomorphism from B onto $Y - B$ (for proof of the existence of such a g see [17]).

Define f on X into Y by

$$f(x) = \begin{cases} x & \text{if } x \in X - B \\ g(x) & \text{if } x \in B \end{cases}$$

Then f is a Borel measurable function X.

Let $\underline{\underline{C}}$ be the σ-algebra on X generated by $f^{-1}(\underline{\underline{B}}_Y) \cup \{A - B\}$. Clearly $\underline{\underline{C}}$ is a countably generated sub σ-algebra of $\underline{\underline{B}}_X$.

Let $\underline{\underline{D}}$ be the σ-algebra on X generated by $\underline{\underline{C}} \cup \{A\}$. Clearly $\underline{\underline{D}} \subseteq \underline{\underline{A}} \vee \underline{\underline{C}} \subseteq \underline{\underline{B}}_X$. To show $\underline{\underline{A}} \vee \underline{\underline{C}} = \underline{\underline{B}}_X$, it is enough to show $\underline{\underline{D}} = \underline{\underline{B}}_X$. As $\underline{\underline{D}}$ is countably generated, by lemma 3, it is enough to show that $\underline{\underline{D}}$ contains singletons.

Now the atoms of $\underline{\underline{C}}$ are of the form

(a) $\{x, g(x)\} \cap (A - B)$, $x \in B$, $g(x) \in X - B$ if $\{x, g(x)\} \cap (A - B) \neq \emptyset$

(b) $\{x, g(x)\} \cap ((X - A) \cup B)$, $x \in B$, $g(x) \in X - B$,

(c) $\{x\}$, $x \in B$, $g(x) \in Y - X$.

Note that the atoms of the form (a) are just $\{g(x)\}$ where $x \in B$, $g(x) \in A$. Those of the form (b) are $\{x\}$ if $x \in B$, $g(x) \in A$ and $\{x, g(x)\}$ if $x \in B$, $g(x) \in X - A$.

The atoms of $\underline{\underline{D}}$ are of the form $C \cap A$ and $C \cap (X - A)$ where C is an atom of $\underline{\underline{C}}$. Thus they are of the form:

(a) $\{g(x)\}$, $x \in B$, $g(x) \in A$

(b) $\{x\}$, $x \in B$, $g(x) \in A$

(c) $\{x\}$, $x \in B$, $g(x) \in X - A$

(d) $\{g(x)\}$, $x \in B$, $g(x) \in X - A$

(e) $\{x\}$, $x \in B$, $g(x) \in Y - X$.

Thus $\underline{\underline{D}} = \underline{\underline{A}} \vee \underline{\underline{C}} = \underline{\underline{B}}_X$.

We now show that $\underline{\underline{C}}$ is a minimal complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}_X$. Let $\underline{\underline{E}}$ be a substructure of $\underline{\underline{C}}$ such that $\underline{\underline{A}} \vee \underline{\underline{E}} = \underline{\underline{B}}_X$. We can assume that $\underline{\underline{E}}$ is countably generated. As $\underline{\underline{C}}$ is countably generated, to show $\underline{\underline{C}} = \underline{\underline{E}}$, it is enough to show that they have the same atoms. If not, let C_1 and C_2 be distinct atoms of $\underline{\underline{C}}$ contained in the same atom E of $\underline{\underline{E}}$. As each atom of $\underline{\underline{C}}$ intersects A, E contains two distinct points a_1 and a_2 of A. Hence $\underline{\underline{A}} \vee \underline{\underline{E}}$ does not separate a_1 and a_2 which contradicts the fact that $\underline{\underline{A}} \vee \underline{\underline{E}} = \underline{\underline{B}}_X$.

Theorem 4. Let $\underline{\underline{A}} \subseteq \underline{\underline{B}}_X$ be countably generated. If there is some $B \in \underline{\underline{B}}_X$ which is a selector for $\underline{\underline{A}}$, then $\underline{\underline{A}}$ has a complement relative to $\underline{\underline{B}}_X$.

Proof. Let $\underline{\underline{C}}$ be generated by $B \cup \underline{\underline{B}}_{X-B}$. It is easy to see that $\underline{\underline{A}} \vee \underline{\underline{C}}$ is a separable substructure of $\underline{\underline{B}}_X$. Hence $\underline{\underline{A}} \vee \underline{\underline{C}} = \underline{\underline{B}}_X$. Let $\emptyset \neq D \in \underline{\underline{A}} \wedge \underline{\underline{C}}$. As $D \in \underline{\underline{A}}$, $D \supseteq$ an atom of A and therefore $D \cap B \neq \emptyset$. As B is an atom of $\underline{\underline{C}}$ and $D \in \underline{\underline{C}}$, $D \supseteq B$. Hence $D \cap A \neq \emptyset$ for every atom A of $\underline{\underline{A}}$. As $D \in \underline{\underline{A}}$, this implies

$D = X$. Thus $\underline{\underline{A}} \wedge \underline{\underline{C}} = \{\emptyset, X\}$.

Remark. There are analytic sets X and countably generated $\underline{\underline{A}} \subseteq \underline{\underline{B}}_X$ which do not have either an uncountable atom or a selector in $\underline{\underline{B}}_X$ but have complements (in fact minimal complements) relative to $\underline{\underline{B}}_X$ (see example 1). We do not know if there exist any analytic set X on which a countably generated substructure of $\underline{\underline{B}}_X$ without relative complement can be constructed. If, however, 'analytic' is replaced by 'absolute Borel' the answer to this question is in the negative as our next theorem shows.

Theorem 5. If X is absolute Borel, every countably generated substructure $\underline{\underline{A}}$ of $\underline{\underline{B}}_X$ has a minimal complement relative to $\underline{\underline{B}}_X$.

Proof. There are two cases to be considered.

Case 1. $\underline{\underline{A}}$ has an uncountable atom. The proof for this case is given in theorem 3.

Case 2. All atoms of $\underline{\underline{A}}$ are countable. In this case, there exists a countable family G_n, $n = 1,2,\ldots$ of disjoint Borel sets such that $\bigcup_n G_n = X$ and each G_n is a partial selector for $\underline{\underline{A}}$. (see [21]). It is easy to choose the G_n's in such a way that, for distinct G_n and G_m, $G_n \cup G_m$ is not a partial selector for $\underline{\underline{A}}$. Denote by $\underline{\underline{C}}$ the σ-algebra generated by G_n, $n = 1,2,\ldots$ The atoms of $\underline{\underline{C}}$ are G_n, $n = 1,2,\ldots$ whence, by theorems 1 and 2, $\underline{\underline{C}}$ is a minimal complement of $\underline{\underline{A}}$

relative to $\underline{\underline{B}}_X$.

<u>Remark</u>. We do not know the conditions under which $\underline{\underline{A}} \subseteq \underline{\underline{B}}_X$ has a relative complement (relative minimal complement) if $\underline{\underline{A}}$ is not countably generated even if X is taken to be an absolute Borel set. (see examples).

<u>4. Examples</u>

1. Let $Y = I \times I$ and let A_1, A_2 be analytic subsets of I such that $A_1 \cup A_2 = I$ and there does not exist any absolute Borel set B such that $B \subseteq A_1$, $I - B \subseteq A_2$. Let $X = (A_1 \times \{\frac{1}{4}\}) \cup (A_2 \times \{\frac{3}{4}\}) \subseteq I \times I$ and $\underline{\underline{A}} = \{D: D = (B \times I) \cap X$ where B is a Borel subset of $I\}$.

Then $\underline{\underline{A}}$ is countably generated and has atoms of the form $\{(x, \frac{1}{4})\}$, $x \in A_1 - A_2$; $\{(x, \frac{3}{4})\}$, $x \in A_2 - A_1$ and $\{(x, \frac{1}{4}), (x, \frac{3}{4})\}$, $x \in A_1 \cap A_2$. Thus $\underline{\underline{A}}$ does not have any uncountable atom.

If possible, let $\underline{\underline{A}}$ admit a selector $B \in \underline{\underline{B}}_X$. Then B is analytic and $\pi_1 B = I$ where π_1 is the projection to the first co-ordinate. As π_1 is one-to-one on B, B is absolute Borel. Let $C = \pi_1 (B \cap (I \times \{\frac{1}{4}\}))$. Then C is absolute Borel and $C = \{x: (x, \frac{1}{4}) \in B\}$. Thus $C \subseteq A_1$ and $I - C \subseteq A_2$ which is a contradiction. Thus $\underline{\underline{A}}$ does not admit a selector in $\underline{\underline{B}}_X$.

Now, let $\underline{\underline{C}} = \{E: E = (I \times B) \cap X$ where B is a Borel subset of $I\}$. By theorems 1 and 2, $\underline{\underline{C}}$ is a minimal complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}_X$.

2. This example shows that even if X is absolute Borel and $\underline{\underline{A}} \subseteq \underline{\underline{B}}_X$ is countably generated and has a minimal complement relative to $\underline{\underline{B}}_X$, $\underline{\underline{A}}$ may not admit a Borel selector. By theorem it is enough to exhibit an absolute Borel set X and a countably generated substructure of $\underline{\underline{B}}_X$ which does not admit a Borel selector.

Let $X = I$ and $A \subseteq I$ be analytic non Borel. Let f be a Borel measurable function on I with $f(I) = A$. (For existence of such an f, see [17]). Let $\underline{\underline{A}} = f^{-1}(\underline{\underline{B}}_I)$. Then $\underline{\underline{A}}$ is a countably generated substructure of $\underline{\underline{B}}_X$. Suppose $\underline{\underline{A}}$ admits a Borel selector B. Then f is one-to-one on B and $f(B) = A$ so that A is absolute Borel. Hence $\underline{\underline{A}}$ cannot admit a Borel selector.

3. This example gives a σ-algebra on I which is not countably generated and yet has a minimal complement relative to $\underline{\underline{B}}_I$.

Let $\underline{\underline{A}}$ be the σ-algebra on I generated by $[0, \frac{1}{2})$ and $\{\{x\} ; \frac{1}{2} \leq x \leq 1\}$. By theorem 3, $\underline{\underline{A}}$ has a minimal complement relative to $\underline{\underline{B}}_I$. Clearly, $\underline{\underline{A}}$ is not countably generated since, otherwise, $\underline{\underline{A}} \cap [\frac{1}{2}, 1]$ is separable and hence must be $\underline{\underline{B}}_{[\frac{1}{2},1]}$ which is not the case.

By a slight modification of this construction $\underline{\underline{A}}$ can ev be chosen so that it is not atomic.

4. This is again an example of a substructure $\underline{\underline{A}}$ of $\underline{\underline{B}}_I$ which is not countably generated and yet has a minimal complement relative to $\underline{\underline{B}}_I$. However, in this case, $\underline{\underline{A}}$ has no uncountable atom so that theorem 3 does not apply.

Let X be the real line. Fix a non Borel set S symmetric about 0 such that $0 \in S$. Let $\underline{\underline{A}} = \{B: B$ is Borel in X and if $x \in B$ and $-x \notin B$, then $x \in S\}$.

We first note that $\underline{\underline{A}}$ is a σ-algebra. For let $A_1, A_2, \dots \in \underline{\underline{A}}$, $x \in \bigcup_n A_n$ and $-x \notin \bigcup_n A_n$. Then for some m, $x \in A_m$ and $-x \notin A_m$. Hence $x \in S$. Thus $\bigcup_n A_n \in \underline{\underline{A}}$.

Let $A \in \underline{\underline{A}}$ and $x \in A^c$, $-x \notin A^c$. Then $-x \in A$ and $x \notin A$. Hence $-x \in S$ and therefore $x \in S$. Thus $A^c \in \underline{\underline{A}}$.

Clearly $\emptyset, X \in \underline{\underline{A}}$.

Note that the atoms of $\underline{\underline{A}}$ are $\{x\}$, $x \in S$ and $\{x, -x\}$, $x \notin S$. $\underline{\underline{A}}$ is not countably generated. To see this, let $A_1, A_2, \dots$ generate $\underline{\underline{A}}$ if possible. Let f be the characteristic function of $A_1, A_2, \dots$. Clearly f is Borel measurable and hence $\{x: f(x) = f(-x)\}$ is a Borel set. But $\{x : f(x) = f(-x)\} = S^c$ which is not Borel. This is a contradiction.

Let $\underline{\underline{C}} = \{\emptyset, X, (-\infty, 0], (0, \infty)\}$. We claim that $\underline{\underline{C}}$ is a minimal complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}_X$. Since $\underline{\underline{D}} \subsetneq \underline{\underline{C}}$

and $\mathbf{D} \neq \mathbf{C}$ implies $\mathbf{D} = \{\emptyset, X\}$, it is enough to show $\mathbf{A} \vee \mathbf{C} = \mathbf{B}_X$. Let $B \in \mathbf{B}_X$. Now let

$E_1 = \{x: x \in B, -x \notin B$ and $x > 0\}$, $E_2 = \{x: x \in B,$
$-x \notin B$ and $x < 0\}$.

$F_1 = \{x: -x \in E_1\}$, $F_2 = \{x: -x \in E_2\}$. Clearly, E_1, E_2, F_1, F_2 are Borel sets and $E_1 \cup F_1$, $E_2 \cup F_2$, $B - (E_1 \cup E_2) \in \mathbf{A}$. Also $E_1 = (E_1 \cup F_1) \cap (0, \infty)$ and $E_2 = (E_2 \cup F_2) \cap (-\infty, 0]$ so that $E_1, E_2 \in \mathbf{A} \vee \mathbf{C}$. Thus $B = E_1 \cup E_2 \cup (B - (E_1 \cup E_2)) \in \mathbf{A} \vee \mathbf{C}$.

The next two examples are those of substructures of $\mathbf{B}_I$ without relative complements. The first one is not atomic. The second one has singleton atoms.

5. Let $A \subseteq I$ be any non Borel set and $\mathbf{A} = \{B: B \in \mathbf{B}_I, B \cap A = \emptyset$ or $A \subseteq B\}$. Then $\mathbf{A}$ does not have a complement relative to $\mathbf{B}_I$. Otherwise, let $\mathbf{C}$ be a relative complement of $\mathbf{A}$. We can suppose $\mathbf{C}$ to be countably generated. Let $\mathbf{D} \subseteq \mathbf{A}$ be countably generated such that $\mathbf{D} \vee \mathbf{C} = \mathbf{B}_I$. As $\mathbf{D} \subseteq \mathbf{A}$, there is an atom D of $\mathbf{D}$ such that $A \subseteq D$. As $D \in \mathbf{B}_I$, $D \neq A$. Let $x \in D - A$. As $\{x\}$ is an atom of $\mathbf{D} \vee \mathbf{C}$, there is an atom C of $\mathbf{C}$ such that $\{x\} = D \cap C$. But this implies $C \cap A = \emptyset$ so that $C \in \mathbf{A}$. Thus $C \in \mathbf{A} \cap \mathbf{C}$ and hence $C = X$ which is clearly impossible as $C \cap A = \emptyset$.

6. ([31]) $\underline{\underline{A}}$ is the countable cocountable σ-algebra on I. If possible, let $\underline{\underline{C}}$ be a complement of $\underline{\underline{A}}$ relative to $\underline{\underline{B}}_I$. We can suppose $\underline{\underline{C}}$ to be countably generated. Let C be an atom of $\underline{\underline{C}}$. As $C \notin \underline{\underline{A}}$, C is uncountable. Also $\underline{\underline{A}} \cap C = \underline{\underline{B}}_C$ so that $\underline{\underline{B}}_C$ is the countable cocountable σ-algebra on C. This is clearly impossible.

5. Open Problems

1. We do not know if there is any analytic set X such that there exists $\underline{\underline{A}} \subseteq \underline{\underline{B}}_X$ which has a relative complement but no relative minimal complement.

2. The problem of characterising the atomic substructures of the Borel σ-algebra on an analytic (or even Borel) set X which have complements relative to $\underline{\underline{B}}_X$ remains unsolved.

3. If X is analytic and $\underline{\underline{A}} \subseteq \underline{\underline{B}}_X$ is countably generated and has countable atoms, does $\underline{\underline{A}}$ have a (minimal) complement relative to $\underline{\underline{B}}_X$?

Note: After this thesis was written, problem 3 was solved by Dr. K.P.S.Bhaskara Rao. The answer to this question is in the affirmative.

CHAPTER 6

SOME PROPERTIES OF A-FUNCTIONS AND α- FUNCTIONS

1. Introduction

Following Kuratowski [17], we say that a function f on the line into the line is an A-function if $\{x : f(x) > c\}$ is analytic for every real c. Plainly if g is a real-valued Borel measurable function defined on the plane such that $\sup_y g(x,y)$ is finite for every real x, then $f(x) = \sup_y g(x,y)$ is an A-function. The motivation for this chapter comes from our investigation of whether the converse of the last statement holds.

Characterization of $(\underline{\underline{M}}, *)$ functions, in the sense of Hausdorff [13], are also given when $\underline{\underline{M}}$ is the family of Borel sets of additive class α in a Polish space.

2. Definitions and notation

If X is any set and $\underline{\underline{M}}$ a class of subsets of X, then following Hausdorff, we call a real valued function f on X a $(\underline{\underline{M}}, *)$ function if $\{x : f(x) > c\}$ is in $\underline{\underline{M}}$.

If X is a metric space and $\underline{\underline{M}}$ the family of sets of additive Borel class α, then $(\underline{\underline{M}}, *)$ functions are called α^--functions. If X is a Polish space and $\underline{\underline{M}}$ the family of analytic sets, $(\underline{\underline{M}}, *)$ functions are called A-functions.

Let X be a metric space. Let $\underline{\underline{S}}_0$ be the family of open subsets of X. $\underline{\underline{B}}_0 = \sigma(\underline{\underline{S}}_0)$ and, for $0 < \alpha < \Omega_1$, $\underline{\underline{S}}_\alpha = \mathcal{A}(\sigma(\bigcup_{i<\alpha} \underline{\underline{S}}_i))$, $\underline{\underline{B}}_\alpha = \sigma(\underline{\underline{S}}_\alpha)$ where, for any family $\underline{\underline{G}}$ of subsets of X, $\sigma(\underline{\underline{G}})$ and $\mathcal{A}(\underline{\underline{G}})$ denote the σ-algebra generated by $\underline{\underline{G}}$ and the smallest family containing $\underline{\underline{G}}$ and closed under operation $\mathcal{A}$, respectively. We call functions of the class $(\underline{\underline{S}}_\alpha, *)$ S_α-functions. Note that if X is Polish, $\underline{\underline{S}}_1$ is the family of analytic sets so that S_1-functions are just A-functions.

A function h on a metric space X is said to be of class α if $h^{-1}(U)$ is of additive Borel class α for every open set U in the range space.

A complete ordinary function system on a set X is a class $\underline{\underline{F}}$ of real valued functions on X satisfying:

(a) Every constant function is in $\underline{\underline{F}}$.

(b) If $f, g \in \underline{\underline{F}}$, then max. (f, g), min (f, g), $f \pm g$, $f \cdot g \in \underline{\underline{F}}$.

If g does not vanish anywhere, $\frac{f}{g} \in \underline{\underline{F}}$.

(c) If f_n, $n = 1,2,\ldots$ is a sequence of functions in $\underline{\underline{F}}$ converging uniformly to a function f, then $f \in \underline{\underline{F}}$.

We use R to denote the real line with the usual topology.

3. α^{-} - functions

Theorem 1. Let $\underline{\underline{F}}$ be a complete ordinary function system on a set X. Let $\underline{\underline{P}}$ be the family of sets of the form $\{x : h(x) > c\}$ where $h \in \underline{\underline{F}}$ and c is real. Then $f \in (\underline{\underline{P}}, *)$ if, and only if, there is a real valued function g on $X \times R$ such that

(a) $g(x,y)$ is continuous in y for fixed x,

(b) $g(x,y)$ is in $\underline{\underline{F}}$ for fixed y,

(c) $\sup_y g(x,y) = f(x)$ for each x.

Lemma 1. Let $P \in \underline{\underline{P}}$. There is an $f \in \underline{\underline{F}}$ such that for all x $0 \leq f(x) \leq 1$ and $P = \{x : f(x) > 0\}$.

Proof. Let $P = \{x: g(x) > c\}$ where $g \in \underline{\underline{F}}$ and c is real. Put $g_1 = g - c$. Then $g_1 \in \underline{\underline{F}}$ and $P = \{x: g_1(x) > 0\}$. Let $g_2 = \max(g_1, 0)$. Then $g_2 \in \underline{\underline{F}}$, for all x $g_2(x) \geq 0$ and $g_2(x) > 0$ if, and only if, $g_1(x) > 0$. Put $f = \frac{g_2}{1 + g_2}$. Then $0 \leq f(x) \leq 1$ for all x, $f \in \underline{\underline{F}}$ and $P = \{x: f(x) > 0\}$.

Lemma 2. $\underline{\underline{P}}$ is closed under countable unions.

Proof. Let $P_1, P_2, \ldots, \in \underline{\underline{P}}$. By lemma 1, let $f_n \in \underline{\underline{F}}$ be such that $0 \leq f_n(x) \leq 1$ for all x and $P_n = \{x : f_n(x) > 0\}$, $n \geq 1$. Let $f = \sum_n \frac{1}{2^n} f_n$. Then $f \in \underline{\underline{F}}$ and $f(x) > 0$ if, and only if, $f_n(x) > 0$ for some n. Thus $\bigcup_n P_n = \{x: f(x) > 0\}$. Hence $\bigcup_n P_n \in \underline{\underline{P}}$.

Lemma 3. If $P \in \underline{\underline{P}}$, there is a sequence $f_1, f_2, \ldots$ in $\underline{\underline{F}}$ such that the function g given by $g(x) = \sup_n f_n(x)$ is 1 on P and 0 outside P.

Proof. Let $P = \{x: f(x) > 0\}$ where $f \in \underline{\underline{F}}$ and $f \geq 0$. Put $f_n(x) = \min (n f(x), 1)$, $n \geq 1$. This sequence answers our purpose.

Lemma 4. Let $f \in (\underline{\underline{P}}, *)$. Then there is an increasing sequence f_n, $n = 1, 2, \ldots$ in $\underline{\underline{F}}$ such that $f_n(x)$ converges to $f(x)$ for all x.

Proof. It is enough to find $f_1, f_2, \ldots$ in $\underline{\underline{F}}$ such that $f(x) = \sup_n f_n(x)$.

Case 1. $f(x) > -1$ for all x. Clearly, if $g = f + 1$, then $g \in (\underline{\underline{P}}, *)$. Hence, without loss of generality, we can take $f(x) > 0$ for all x.

Fix $m \geq 1$. Put $P_n = \{x: f(x) > \frac{n}{m}\}$. Then $P_n \in \underline{\underline{P}}$. For each $n \geq 1$, there is an increasing sequence $f_{n1}, f_{n2}, \ldots$ in $\underline{\underline{F}}$ such that if $g_n(x) = \lim_{k \to \infty} f_{nk}(x) = \sup_k f_{nk}(x)$ then $g_n(x) = 1$ on P_n and $g_n(x) = 0$ outside P_n. Put $h_m = \frac{1}{m}(g_1 + g_2 + \ldots)$. Then $h_m(x) = \lim_{n \to \infty} \frac{1}{m}(g_1(x) + \ldots + g_n(x))$ $= \sup_n \frac{1}{m}(g_1(x) + \ldots + g_n(x)) = \sup_n \frac{1}{m} \lim_{k \to \infty} (f_{1k}(x) + \ldots + f_{nk}(x))$ $= \frac{1}{m} \sup_{n,k} (f_{1k}(x) + \ldots + f_{nk}(x))$ since $f_{1k}, \ldots f_{nk}$ increase with k.

Now, if $\frac{n-1}{m} < f(x) \leq \frac{n}{m}$, then $g_1(x) = \dots = g_{n-1}(x) = 1$ and $g_n(x) = g_{n+1}(x) = \dots = 0$, so that $h_m(x) = \frac{n-1}{m}$. Thus, for all x, $h_m(x) < f(x) \leq h_m(x) + \frac{1}{m}$. Hence, $f(x) = \lim_{m \to \infty} h_m(x) = \sup_m h_m(x)$ for all x. Thus $f(x) = \sup_{m,n,k} \frac{1}{m}(f_{1k}(x) + \dots + f_{nk}(x))$ where $f_{1k}, \dots, f_{nk} \in \underline{\underline{F}}$.

Case 2. f is not bounded below by -1. Define g by $g = \frac{f}{1+|f|}$. Then $g \in (\underline{P}, *)$ and $|g(x)| < 1$ for all x. Let $g_1, g_2, \dots$ be an increasing sequence in $\underline{\underline{F}}$ such that $g(x) = \sup_n g_n(x)$. Clearly $g_n(x) < 1$ for all n and x. By replacing g_n by $\max.(g_n, -1)$ if necessary, we can suppose $g_n(x) \geq -1$ for all n and x. Put $h_n = \frac{1}{2} g_n + \frac{1}{2^2} g_{n+1} + \frac{1}{2^3} g_{n+2} + \dots$ Then $h_n \in \underline{\underline{F}}$ and, for all n and x, $g_n(x) \leq h_n(x) \leq h_{n+1}(x) \leq g(x)$. Hence $\sup_n h_n(x) = g(x)$. We now show that $|h_n(x)| < 1$ for all n and x. We know that $h_n(x) \leq g(x) < 1$. Enough to show $h_n(x) > -1$. If, for some n and x, $h_n(x) > g_n(x)$, then clearly $h_n(x) > -1$. Suppose $h_n(x) = g_n(x)$. Then $g_n(x) = g_{n+1}(x) = \dots = g(x) > -1$. Hence $h_n(x) > -1$. Thus we can define f_n by $f_n = \frac{h_n}{1 - |h_n|}$, $n \geq 1$. Then $f_n \in \underline{\underline{F}}$ and $f_n(x)$ increases to $\frac{g(x)}{1 - |g(x)|} = f(x)$.

Proof of the theorem. Suppose $g(x,y)$ satisfies (a), (b), (c). Let c be any real number. Then by (c) and (a)
$f(x) > c \iff \exists y\,(g(x,y) > c) \iff \exists r(r$ is rational and $g(x, r) > c)$ where $\exists$ stands for there exists and $\iff$ stands for if and only if.

Now by (b) $\{x: g(x,r) > c\} \in \underline{P}$ for any fixed r. Thus $\{x: f(x) > c\} = \bigcup_r \{x: g(x,r) > c\}$ the union being taken over all rationals r. As $\underline{P}$ is closed under countable unions, $\{x: f(x) > c\} \in \underline{P}$ or $f \in (\underline{P}, *)$.

Conversely, let $f \in (\underline{P}, *)$. Let f_n, $n = 1,2,\ldots$ be an increasing sequence of functions in $\underline{F}$ which converges to f. Define g on $X \times R$ by $g(x,y) = (f_{n+1}(x) - f_n(x))(|y| - n) + f_n(x)$ for $n \leq |y| \leq n+1$. It is easy to see that g is well defined and satisfies (a) and (b). As $f_n(x) \leq g(x,y) \leq f_{n+1}(x)$ for $n \leq |y| \leq n+1$ and $\sup_n f_n(x) = f(x)$, it follows that $\sup_y g(x,y) = f(x)$ for each x.

Proposition:

Let $\underline{F}$ be the family of all real valued functions of class α on a metric space X. Then $\underline{F}$ is a complete ordinary function system and the sets of the form $\{x: f(x) > c\}$, $f \in \underline{F}$, c real, are just the sets of additive Borel class α.

Lemma. Let X, Y be metric spaces and $f: X \to Y$ a function of class α. If g is a real valued continuous function on Y, then the function h on X defined by $h(x) = g(f(x))$ is of class α.

Proof. Let U be an open subset of the real line. Then $h^{-1}(U) = f^{-1}(g^{-1}(U))$. As g is continuous, $g^{-1}(U)$ is open and hence $h^{-1}(U)$ is of additive Borel class α.

Proof of the proposition.

Clearly constants are functions of class α. Let f, g be real valued functions of class α on X. Let $h : X \to R^2$ be defined by $h(x) = (f(x), g(x))$. Then h is of class α. To show this, it is enough to show that $h^{-1}(U_1 \times U_2)$ is of additive Borel class α where U_1, U_2 are open subsets of R. This is clearly true as $h^{-1}(U_1 \times U_2) = f^{-1}(U_1) \cap g^{-1}(U_2)$.

Now $(x,y) \to x \pm y$, $(x,y) \to x \cdot y$, $(x,y) \to \max(x,y)$, $(x,y) \to \min(x,y)$ are all continuous functions on R^2 and $(x,y) \to \frac{x}{y}$ is continuous on $R \times (R - \{0\})$. Hence $f \pm g$, $f \cdot g$, $\max(f,g)$, $\min(f,g)$ and $\frac{f}{g}$, provided g does not vanish anywhere, are all functions of class α.

Let $f_1, f_2, \ldots$ be a sequence of real valued functions of class α on X converging uniformly to a function f. We show that f is of class α. Note that there is a subsequence $f_{m_1}, f_{m_2}, \ldots$ such that for all n, $|f_{m_n+k}(x) - f(x)| < \frac{1}{n}$ for all x and k. Let F be a closed subset of R. We show that $f^{-1}(F) = \bigcap_n \bigcap_k \left\{x : \rho(f_{m_n+k}(x), F) \leq \frac{1}{n}\right\}$ where ρ is the usual distance in R.

Let $x \in f^{-1}(F)$. Then $f(x) \in F$ and $\varrho(f(x), f_{m_n+k}(x)) = |f(x) - f_{m_n+k}(x)| < \frac{1}{n}$ for all n and k.

Conversely, let $\varrho\ (f_{m_n+k}(x), F) \leq \frac{1}{n}$ for all n and k. For each n, $f_{m_n+1}, f_{m_n+2}, \ldots$ converges to $f(x)$ and hence $\varrho(f(x), F) \leq \frac{1}{n}$. As F is closed, this implies $f(x) \in F$.

Now $\left\{x: \varrho(f_{m_n+k}(x), F) \leq \frac{1}{n}\right\} = f^{-1}_{m_n+k}\left(\left\{y: \varrho(y,F) \leq \frac{1}{n}\right\}\right)$ and hence is of multiplicative Borel class α. Thus $f^{-1}(F)$ is of multiplicative Borel class α and hence f is a function of class α. Clearly, any set of the form $\left\{x: f(x) > c\right\}$, $f \in \underline{\underline{F}}$, c real is of additive Borel class α.

Let A be any set of additive Borel class α. If $\alpha = 0$, A is a cozero set and hence $A = \left\{x: f(x) > 0\right\}$ for some continuous function f. Let $\alpha > 0$. In this case, $A = \overset{\infty}{\underset{i=1}{U}} A_i$, where each A_i is ambiguous of class α. Define f on X into R by

$f(x) = \sum_i \frac{1}{2^i} I_{A_i}(x)$ where I_{A_i} is the indicator function of A_i as usual. As I_{A_i} is of class α and $\underline{\underline{F}}$ is closed under uniform convergence, f is of class α. Also $A = \left\{x: f(x) > 0\right\}$.

As a consequence of theorem 1 and the above proposition we have:

<u>Theorem 2</u>. Let f be a real valued function on a metric space X. Then f is an α^{-}-function if, and only if, there is a real valued function g defined on $X \times R$ such that $g(x,y)$ is a

continuous function of y for fixed x, of class α in x for fixed y and $f(x) = \sup_y g(x, y)$.

4. A-functions.

In this section we take X to be an uncountable Polish space.

Theorem 3. Let f be a real valued function on X which is bounded below. Then f is an A-function if, and only if, there is a real valued Borel measurable function g on X^2 such that $f(x) = \sup_y g(x,y)$.

Proof. Without loss of generality, we take $X = R$. For suppose the result is true for R. Let ψ be a Borel isomorphism from R onto X. If f is an A-function on X which is bounded below, then $f \circ \psi$ is an A-function on R and $f \circ \psi$ is bounded below. Hence $f \circ \psi(s) = \sup_t h(s,t)$ where h is a Borel measurable function on R^2. Let $g(x,y) = h(\psi^{-1}(x), \psi^{-1}(y))$ for $x, y \in S$. Clearly g is Borel measurable and $f(\psi(s))$ $= \sup_t g(\psi(s), \psi(t))$ or $f(x) = \sup_y g(x,y)$.

Let g be a Borel measurable function on R^2 and $f(x) = \sup_y g(x,y)$. Then given a real number c, $f(x) > c$ if and only if $\exists y\, (g(x,y) > c)$. Thus $\{x: f(x) > c\} =$ $\pi_1 \{(x,y): g(x,y) > c\}$ where π_1 denotes projection to the first co-ordinate. Hence $\{x: f(x) > c\}$ is analytic or f is an A-function. Note that in this part of the proof, the condition 'f is bounded below' is not used.

Conversely, let f be an A-function on R and a a fixed real number such that $f(x) > a$ for all x. Let $A = \{(x,y): f(x) > y\}$. Then $A = \{(x,y): \exists r\ (r$ is rational and $f(x) > r > y)\} = \bigcup_r \{(x,y): f(x) > r > y\}$ where the union is taken over all rational r. Clearly, A is analytic. Let $B \subseteq R^3$ be a Borel set such that A = projection of B (see[17]) i.e. $(x,y) \in A \iff \exists z\ ((x,y,z) \in B)$. Let k be a function defined on R^3 by $k(x,y,z) = y$ if $(x,y,z) \in B$

$= a$ otherwise.

Clearly, k is Borel measurable and $\sup_{(y,z)} k(x,y,z) =$ $\sup_{(y,z)} \{\{y: y < f(x)\} \cup \{a\}\} = f(x)$. Let ϕ be a Borel isomorphism on R onto R^2 and define h on R^2 onto R^3 by $h(x,y) = (x,\phi(y))$. Let $g(x,y) = k(h(x,y))$. Then g is Borel measurable and $f(x) = \sup_y k(x, \phi(y)) = \sup_y g(x,y)$.

<u>Remark 1.</u> It is easy to see that theorem 3 holds even if the condition 'f is bounded below' is replaced by 'f dominates a Borel function'. As a matter of fact, an A-function is of the form $\sup_y g(x,y)$ where g is Borel measurable if, and only if, it dominates a Borel function. Equivalently, every A-function on X is of the form $\sup_y g(x,y)$ where g is Borel measurable if, and only if, given an ascending sequence of analytic sets A_n, $n = 1,2,\ldots$ such that $\bigcup_n A_n = X$, there is an

ascending sequence B_n, $n = 1,2,\ldots$ of Borel sets such that $B_n \subseteq A_n$ and $\bigcup_n B_n = X$. To see this, suppose every A-function is of the form $\sup_y g(x,y)$ for some Borel measurable g. Then every A-function dominates a Borel function. Let A_n, $n = 1,2,\ldots$ be an increasing sequence of analytic sets such that $\bigcup_n A_n = X$. Define f on X by

$$f(x) = -1 \text{ if } x \in A_1$$
$$= -n \text{ if } x \in A_n - A_{n-1} \text{ for } n \geq 2.$$

Then f is an A-function. Suppose h is a Borel measurable function on X such that $f(x) \geq h(x)$ for all x. Let $B_n = \{x: h(x) \geq -n\}$, $n = 1,2,\ldots$

Then $B_n \subseteq A_n$ and B_n, $n = 1,2,\ldots$ is a sequence of Borel sets increasing to X.

Conversely, suppose for each sequence of analytic sets $A_1, A_2, \ldots$ increasing to X, there is a sequence $B_1, B_2, \ldots$ of Borel sets increasing to X such that $B_n \subseteq A_n$, $n = 1,2,\ldots$. Let f be an A-function on X. Let $A_n = \{f: f(x) > -n\}$. Let $B_n \subseteq A_n$, $n = 1,2,\ldots$ be Borel sets increasing to X. Let

$$h(x) = -1 \text{ if } x \in B_1$$
$$= -n \text{ if } x \in B_n - B_{n-1},\ n \geq 2.$$

Then $h(x)$ is Borel measurable and $f(x) \geq h(x)$ for all x. Hence $f(x) = \sup_y g(x,y)$ for some Borel measurable function g.

Another equivalent condition is the following. Given a sequence of analytic sets $A_n \subseteq X$, $n = 1,2,\ldots$, which increase to X, there exists an increasing sequence D_n, $n = 1,2,\ldots$, of Borel subsets of X^2 such that $\bigcup_n D_n = X^2$ and $\pi_1 D_n = A_n$ where π_1 denotes the projection to the first co-ordinate.

To see this, first suppose that given a sequence $A_1, A_2, \ldots$ of analytic sets increasing to X, there exists a sequence of Borel sets $B_1, B_2, \ldots$ increasing to X such that $B_n \subseteq A_n$. Further let $C_1, C_2, \ldots$ be a sequence of Borel subsets of X^2 such that $\pi_1 C_n = A_n$, $n = 1,2,\ldots$, (see [17]). Note that we can suppose the C_n's to be increasing. Let $D_n = C_n \cup (B_n \times X)$.

Conversely suppose given analytic sets $A_1, A_2, \ldots$ increasing to X, we can find Borel sets $D_1, D_2, \ldots$ increasing to X^2 such that $\pi_1 D_n = A_n$, $n \geq 1$. Fix $x_0 \in X$ and let $B_n = \{x : (x, x_0) \in D_n\}$, $n \geq 1$. Then $B_n \subseteq A_n$ for all n and $B_1, B_2, \ldots$ are Borel sets increasing to X.

The question of whether any of these conditions always hold remains unsolved.

<u>Remark 2:</u> An arbitrary real valued function on X need not dominate a Borel function. To see this, take $X = R$ and let $\{x_\alpha : \alpha < c\}$, $\{f_\alpha : \alpha < c\}$ enumerate the real numbers and the Borel functions on R into R, respectively. Define g on R by $g(x_\alpha) = f_\alpha(x_\alpha) - 1$. Then g does not dominate any Borel

Our next theorem answers in the negative the following question raised by D. Blackwell: If $\underline{\underline{A}}$ is the σ-algebra generated by analytic sets on an uncountable Polish space X and f is an A-function on X, is $f^{-1}(A) \in \underline{\underline{A}}$ for every analytic subset A of R where $\underline{\underline{A}}$ is the σ-algebra on X generated by the analytic sets?

<u>Theorem 4.</u> Let X be an uncountable Polish space and $\underline{\underline{A}}$ the σ-algebra on X generated by the analytic sets. There is an A-function f on X and an analytic subset C of R such that $f^{-1}(C) \notin \underline{\underline{A}}$.

This theorem is obtained from the next one by putting $\alpha = 1$

<u>Theorem 5.</u> If X is an uncountable Polish space, there is an S_α-function f on X and an analytic subset C of R such that $f^{-1}(C) \notin \underline{\underline{B}}_\alpha$.

<u>Proof.</u> It is a deep result of Kunugui that $\underline{\underline{B}}_\alpha$ is not closed under operation $\mathcal{A}$ (see [14]). Let $Z_{n_1 \ldots n_k}$, $n_1, \ldots, n_k$ are natural numbers and $k = 1,2,\ldots$, be elements of $\underline{\underline{B}}_\alpha$ such that $\bigcup_{n \in \mathcal{N}} \bigcap_k Z_{n_1 \ldots n_k} \notin \underline{\underline{B}}_\alpha$, where $\mathcal{N}$ denotes the family of all sequences of positive integers and $n = (n_1, n_2, \ldots)$. We can find countably many sets A_i, $i = 1,2,\ldots$ in $\underline{\underline{S}}_\alpha$ such that for all n and k, $Z_{n_1 \ldots n_k} \in \sigma(A_1, A_2, \ldots)$. Define a real valued function f on X by $f(x) = \sum_{i=1}^{\infty} \frac{2}{3^i} I_{A_i}(x)$, I_{A_i} being the

indicator function of A_i. As the sum of two S_α-functions, a positive constant multiple of an S_α-function and the limit of an increasing sequence of S_α-functions are all S_α-functions, it follows that f is an S_α-function. Since $f^{-1}(\underline{\underline{B}}_R) = \sigma(A_1, A_2, \ldots)$, where $\underline{\underline{B}}_R$ is the Borel σ-algebra on R, (see chapter 4) we can find, for all n and k, $B_{n_1 \ldots n_k} \in \underline{\underline{B}}_R$ such that $f^{-1}(B_{n_1 \ldots n_k}) = Z_{n_1 \ldots n_k}$. Let $C = \bigcup_{n \in \mathcal{N}} \bigcap_k B_{n_1 \ldots n_k}$. Then C is analytic and $f^{-1}(C) = \bigcup_{n \in \mathcal{N}} \bigcap_k Z_{n_1 \ldots n_k} \notin \underline{\underline{B}}_\alpha$.

<u>Remark</u>. Let X be any set and $\underline{\underline{L}}$ a σ-additive lattice on X, containing X and the null set, such that $\sigma(\underline{\underline{L}})$ is not closed under operation $\mathcal{A}$. For any function f of class $(\underline{\underline{L}}, *)$, $f^{-1}(\underline{\underline{B}}_R) \subseteq \sigma(\underline{\underline{L}})$. However, we can find an analytic set C and a function f of class $(\underline{\underline{L}}, *)$ such that $f^{-1}(C) \notin \sigma(\underline{\underline{L}})$. The proof is similar to that of theorem 5.

REFERENCES

[1] V. J. Arsenin and A. A. Ljapunov: Theory of A - sets, Upsehi Mat. Nauk, 5, (1950), No.5 (39), pp. 45 - 108 (Russian).

[2] D. Basu: Problems relating to the existence of maximal and minimal elements in some fields of statistics (subfields), Proc. V. Berkeley Symp., 1, (1965), pp. 41 - 50.

[3] D. Blackwell: Discounted dynamic programming, Ann. Math. Statist., 36, (1965), pp. 226 - 235.

[4] D. Blackwell: A Borel set not containing a graph, Ann. Math. Statist., 39, (1968), pp. 1345 - 1347.

[5] D. Blackwell: On a class of probability spaces, Proc. III Berkeley Symp., 2, (1956), pp. 1 - 6.

[6] D. Blackwell and C. Ryll - Nardzewski: Non-existence of everywhere proper conditional distributions, Ann. Math. Statist., 34, (1963), pp. 223 - 225.

[7] D. Blackwell, D. Freedman and M. Orkin: The optimal reward operator in dynamic programming, The Annals of Probability, 2, No.5, (Oct. 1974), pp. 926 - 941.

[8] N. Bourbaki: General Topology, part 2, Hermann, Paris, Addison - Wesley Publishing Co., Reading Mass, (1966).

[9] S. Braun: Quelques théoremès sur les cribles boreliens, Fund. Math., 20, (1933), pp. 168 - 172.

[10] R. Engelking: On Borel sets and B - measurable functions in metric spaces, Prace Matematyczne, 10, (1967), pp. 145 - 149.

[11] R. Engelking: Outline of General Topology, North Holland and PWN, (1968).

[12] R. Engelking, W. Heath and E. Michael: Topological well orderings and continuous selections, Inventiones Math., 6, (1968), pp. 150 - 158.

[13] F. Hausdorff: Set Theory, New York, (1957).

[14] K. Kunugui: Sur un théorème d'existence dans la théorie des ensembles projectifs, Fund. Math ., 29, (1937), pp. 167 - 181.

[15] K. Kunugui: Contribution à la théorie des ensembles boreliens et analytiques III, J. Fac. Sci. Hokkaido Imp. Univ. Ser. I., 8, (1940), pp. 79 - 108.

[16] K. Kunugui: Sur un problème de M.E. Szpilrajn, Proc. Imp. Acad. Sci. Japan, 16, (1940), pp. 73 - 78.

[17] K. Kuratowski: Topology, 1, Acad. Press (1966).

[18] K. Kuratowski: Ensembles projectifs et ensembles singuliers, Fund. Math., 35, (1948), pp. 131 - 140.

[19] K. Kuratowski and A. Maitra: Some theorems on selectors and their applications to semi-continuous decompositions, Bull. Acad. Polon. Sci., 22, No. 9 (1974), pp. 877 - 881.

[20] K. Kuratowski and C. Ryll - Nardzewski: A general theorem on selectors, Bull. Acad. Polon. Sci., 13, No. 6 (1965), pp. 397 - 403.

[21] N. Lusin: Leçons sur les ensembles analytiques et leurs applications, Gauthier Villars, Paris (1930).

[22] A. Maitra: Coanalytic sets that are not Blackwell spaces, Fund. Math., 67, (1970), pp. 251 - 254.

[23] A. Maitra: Discounted dynamic programming on compact metric spaces, Sankhya, Ser. A, 27, (1968), pp. 241 - 248.

[24] A. Maitra and B. V. Rao: Selection theorems and the reduction principles, Trans. Am. Math. Soc., 202, (1975), pp. 57 - 66.

[25] A. Maitra and B. V. Rao: Selection theorems for partitions of Polish spaces, Fund Math., 93, (1976), pp. 47 - 56

[26] E. Michael; A theorem on semi-continuous set valued functions, Duke Math. Journal, 26, (1959), pp. 647 - 652.

[27] D. Montgomery: Non separable metric spaces, Fund. Math., 25, (1935), pp. 527 - 533.

[28] P. Novikov: Sur les fonctions implicites measurables B, Fund. Math., 17, (1931), pp. 296 - 297.

[29] M. Orkin: A Blackwell space which is not analytic: Bull. Acad. Polon. Sci., 20, (1972), pp. 437 - 438.

[30] B. V. Rao: On coanalytic spaces and non isomorphic analytic sets, preprint.

[31] B. V. Rao: Studies in Borel structures, thesis, Indian Statistical Institute (1969).

[32] B. V. Rao: Lattice of Borel structures: Coll. Math., 23, (1971), pp. 213 - 216.

[33] K.P.S.B.Rao : Studies in Boolean Algebra and Measure Theory, thesis, Indian Statistical Institute, (1969).

[34] H. Sarbadhikari: A projective Blackwell space which is not analytic, Bull. Acad. Polon. Sci., 21, No.6, (1973), pp. 511 - 514.

[35] H. Sarbadhikari: A note on some properties of A - functions, Proc. Am. Math. Soc., 56, (April 1976), pp. 321 - 324.

[36] H. Sarbadhikari: Some uniformization results, Fund. Math., to appear.

[37] H. Sarbadhikari, K.P.S.B.Rao and E. Grzegorek: Complementation in the lattice of Borel structures, Coll. Math., 31, (1974), pp. 29-32.

[38] E. Selivanowski: Sur une classe d'ensembles definis par une infinité denombrable de conditions, C. R. des seances de l'Acad. des Sc. (Paris), 184, (1927), pp. 1311.

[39] W. Sierpinski: Sur un ensemble analytique plan universal pour les ensembles mesurables (B), Fund. Math., 12, (1928), pp. 75 - 77.

[40] W. Sierpinski: Sur l'uniformization des ensembles mesurables (B), Fund. Math., 16, (1930), pp. 136 - 139.

[41] J. R. Shoenfield: Mathematical Logic, Addison - Wesley publ., (1967).

[42] M. Sion: On uniformization of sets in topological spaces, Trans. Am. Math. Soc., 96, (1960), pp. 237 - 245.

[43] R. Vaught: Invariant sets in Topology and Logic, Fund. Math., 82, (1974), pp. 269 - 294.