THE STATISTICAL BEHAVIOUR AND UNIVERSALITY PROPERTIES OF THE RIEMANN ZETA FUNCTION AND OTHER ALLIED DIRICHLET SERIES

By

BHASKAR BAGCHI

Thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy

CALCUTTA

APRIL, 1981

ACKNOWLEDGEMENT

I thank the department of theoretical statistics and mathematics of the Indian Statistical Institute for extending all the facilities for my work. The spirit of informality and of unhindered enquiry that prevails in the department has been a constant source of strength and inspiration to me.

I gratefully acknowledge my indebtedness to Professor G.J. Babu under whose supervision and constant encouragement this thesis was written. I thank him for the benefit of his time and attention which I could draw upon so liberally.

I am greatly indebted to Professor K. Ramachandra of the school of mathematics, Tata Institute of fundamental research, for having first brought to my notice the work of Professor S.M. Voronin. This played a crucial role in the evolution of the work reported here.

I owe my debts of gratitude to Professor A. Maitra who was so kind as to take time off from his own busy schedule of research, teaching and administration in order to translate a twenty-page long paper of Voronin from Russian into English. In as much as I have freely borrowed the techniques of this paper, the translation has been of paramount significance to my research.

My thanks are also due to Professors S.C. Bagchi and A. Sitaram who have been of great help to me whenever problems related to harmonic analysis cropped up in the course of my work. I thank Professor A. Reich of the university of Gottingen for making available to me his own unpublished work on the universality theorem.

I am also thankful to my colleagues and friends, of whom particular mention must be made of Mr. V.V. Srivatsa and Mr. C.K. Raju, for patiently listening to my endless ramblings on the Riemann hypothesis and recurrence even if (as I suspect) these failed to make much sense to them. I am grateful to Mr. C.K.Raju for some instructive (and occassionally amusing) discussions on the physical significance and historical origin of the notions of recurrence.

Finally, I thank Mr. [illegible].K. Bardhan for his efficient and painstaking typewriting and Mr. D. Chatterjee for his efficient cyclostyling.

Bhaskar Bagchi

NOTATIONS*

$\mathbb{N}$	The set of natural numbers.
$\mathbb{P}$	The set of primes.
$\mathbb{Z}$	The set of integers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Q}^+$	The set of positive rationals.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^+$	The set of positive reals.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{C}_\infty$	The Riemann sphere.
$\mathbb{T}$	The unit circle.
Ω_a^b	The strip $x+iy : a<x<b,\ y \in \mathbb{R}$.
W	The set of completely multiplicative functions from $\mathbb{N}$ to $\mathbb{T}$.
$C(\mathbb{R})$	The set of continuous functions from $\mathbb{R}$ to $\mathbb{C}$.
$C^*(\mathbb{R})$	The set of continuous functions from $\mathbb{R}$ to $\mathbb{C}_\infty$.
$H(\Omega)$	The set of analytic functions on the planar region Ω.
$M(\Omega)$	The set of meromorphic functions on the planar region Ω.
ζ	The Riemann Zeta function.
$L(\cdot,\chi)$	The Dirichlet L-series with character χ.

---

* Some of the symbols have been used in more senses than one. In each case, the intended meaning of such an ambiguous symbol should be clear from the context.

Γ	Euler's Gamma function.
exp	The exponential function.
Re	Real part.
Im	Imaginary part.
$\widehat{\ }$	Fourier transform.
$\lvert\cdot\rvert$	Modulus.
$\lVert\cdot\rVert$	Norm.
bd	Topological boundary.
$\perp$	Annihilator.
$(\cdot,\cdot)$	Greatest common divisor.
$(\cdot,\cdot)$	Inner product.
$(\cdot,\cdot)$	Open interval.
$[\cdot,\cdot]$	Closed interval.
Sup	Supremum.
limsup	Limit superior.
lim	Limit.
$\equiv$	Modular congruence.
$\equiv$	Identical equality.
$=$	Equality.
Σ	Summation.
Π	Product.
$\int$	Integral.
∂	Derivative.
$\overset{D}{=}$	Is identically distributed as.
$\overset{D}{\Longrightarrow}$	Converges in distribution to.
$\longrightarrow$	Converges to.

$\Rightarrow$	Is asymptotically distributed as.
$\downarrow$	Decreases to.
λ	Lebesgue measure.
$\#$	Number of elements of.
$<$	Less than.
$\leq$	Less than or equal to.
$<<$	Is dominated by.
$o(.)$	Landau's little " oh ".
$O(.)$	Landau's big "oh ".
∞	Infinity.
ε	Belongs to.
$\{ : \}$	Set formation.
$\bigcup$	Set theoretic union.
$\bigcap$	Set theoretic intersection.
$\triangle$	Set theoretic symmetric difference.
$\subseteq$	Set inclusion.
$E(.)$	Mathematical expectation.
$E(.\|.)$	Conditional expectation.

*****

TABLE OF CONTENTS

CHAPTER 0

INTRODUCTION AND SUMMARY

0.1 <u>A brief review of the literature</u> : The Riemann Zeta function $\zeta(z)$ is defined for complex z with $\mathrm{Re}(z) > 1$ by the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z},$$

and thence by analytic continuation it is defined as a meromorphic function on the entire complex plane with a simple pole at $z = 1$. In his famous paper ([44]) of 1859, Bernhard Riemann inaugurated the study of ζ as a function of a complex variable (Euler had already considered it for specific real values) by obtaining this analytic continuation. He also exhibited the intimate connection that obtains between the position of the complex zeros of the Zeta function and the distribution of the prime numbers. He also established the functional equation satisfied by this Zeta function. In view of this functional equation (in conjunction with the Euler product formala $\zeta(z) = \prod_p (1-p^{-z})^{-1}$ for $\mathrm{Re}(z) > 1$, where the product is over all primes p), Zeta vanishes at the points $z = -2, -4, \ldots$ (these being the so called trivial zeros) and all the other zeros are complex numbers lying in the " critical strip" $0 \leq \mathrm{Re}(z) \leq 1$, and they are symmetrically situated about the critical line $\mathrm{Re}(z) = \frac{1}{2}$. This observation led Riemann to conjecture that all the nontrivial zeros are indeed on the critical line (for a beautiful discussion of what exactly might have motivated this conjecture, see [23, pp.164-166]). This is the celebrated Riemann hypothesis.

Due to its paramount significance for the theory of the distribution of the primes, the Riemann hypothesis occupies a central position in pure mathematics. In 1896, J. Hadamard and de la Vallee Poussin independently established that the boundary of the critical strip is free from Zeta zeros. At the very onset of further investigation it was noticed that there is a close connection between the distribution of the Zeta zeros and the growth rate of the Zeta function with increasing imaginery part of the argument. Accordingly the main stream of research in analytic number theory has proceeded towards obtaining more and more refined growth estimates (of the Zeta function) and zero-density estimates (of the proportion of Zeta zeros lying to the right of the critical line). For a representative sample of the results obtained in this direction, see [42]. Though many prominent mathematicians continue to contribute to this line of enquiry, and though it has had profound implications for number theory, it must be admitted that this piecemeal approach has failed to make any qualitatively significant dent in the problem of the truth or falsity of the Riemann hypothesis itself. Today things remain more or less where Riemann, Hadamard and de la Vallee Poussin left them. For instance, it is not yet known if any proper substrip of the critical strip can be free of Zeta zeros.

A second line of enquiry into the Riemann Zeta function was opened up in a series of brilliant papers by Harald Bohr. The essence of this approach is to take a global view of the Zeta function and study its general value distribution in place of prematurely restricting oneself to a consideration of its zeros alone. In his paper [6] of 1912, Bohr initiated the study of the set $\{\zeta(x+iy) : y \in \mathbb{R}\}$ for arbitrary $x > 1$. This study was completed by Bohr and Jessen in [14] where they showed that there exists an $x_o > 1$ such that for $x > x_o$ the closure of this set is a ring - shaped region bounded by two convex curves one lying in the interior of the other; and for $1 < x \leq x_o$, this closure is a convex region bounded by a single convex closed curve. Indeed, $x_o = 1.764...$ is the unique real root $x > 1$ of the equation

$$\mathrm{Sin}^{-1}(p_1^{-x}) = \sum_{n=2}^{\infty} \mathrm{Sin}^{-1}(p_n^{-x}) .$$

(Here $\{p_n : n \geq 1\}$ is the sequence of primes in natural order) Further details regarding the geometry of the bounding closed curves were obtained by Kershner and Wintner in [35] and by Kershner in [34]. In his paper [7] of 1912, Bohr showed that a similar situation occurs in the value distribution of the function $\frac{\zeta'}{\zeta}$ on vertical lines $\mathrm{Re}(z) = x > 1$, in this case the corresponding regions are discs and circular rings respectively.

The problem of general value distribution of Zeta on vertical lines contained in the critical strip was considered by H. Bohr in [8]. There he established that the set $\{\zeta(x+iy) : y \in \mathbb{R}\}$ is dense in the complex plane for each x in $\frac{1}{2} < x \leq 1$ (certain function theoretic generalisations of this remarkable result of Bohr will constitute the focal point of this thesis.).

Once again it was Bohr who introduced the statistical approach in the study of the value distribution of the Zeta function. In [10], Bohr announced that for each $x > \frac{1}{2}$, the asymptotic behaviour of Zeta on the line $\{Re(z) = x\}$ is regulated by a probability law; this was proved in a series of three papers, (the last and most definitive of them being [14]) by Bohr and Jessen. In the language of weak convergence (convergence in distribution) of probability measures, this result may be described as follows. For any $Y > 0$, let μ_x^Y denote the probability measure defined on the complex plane by

$$\mu_x^Y(A) = \frac{1}{2Y} \lambda(\{y : -Y \leq y \leq Y \text{ and } \zeta(x+iy) \in A\})$$

for Borel subsets A of the plane. Then for each $x > \frac{1}{2}$, there exists a Borel probability μ_x on the plane such that μ_x^Y converges weakly to μ_x as $Y \to \infty$. Further, Bohr and Jessen showed that μ_x is absolutely continuous with respect to the Lebesgue measure on the plane, and that if $\overline{D}_x$ denote the density of μ_x with respect to Lebesgue measure, then $\overline{D}_x(w) \neq 0$ for all $w \neq 0$, provided $\frac{1}{2} < x \leq 1$. This last result clearly implies that for $\frac{1}{2} < x \leq 1$, the support of μ_x is the entire plane. In other words, for every nonempty open subset U of the plane, $\mu_x(U) > 0$. In consequence, for such sets U, and for $\frac{1}{2} < x \leq 1$, the set $\{t \in \mathbb{R} : \zeta(x+it) \in U\}$ has positive lower density (The notion of upper and lower density of a linear set, and more particularly their discrete versions, are well known in the applications of probability methods to the theory of value distribution of arithmetic functions; see, for example, the monograph [37] by Kubilius. The definitions are given in section 1.4 of this thesis), and a fortiori this set is nonempty. Thus, the knowledge of the support of μ_x implies the aforementioned denseness result of Bohr in a stronger form. Details regarding the behaviour of the density of μ_x were obtained by Wintner, Jessen and E.R. Van Kampen in the papers [31], [32] and [54]. Probability laws regulating the asymptotic behaviour of $\frac{\zeta'}{\zeta}$ on lines $\{Re(z) = x\}$ were discussed by Van Kampen and Wintner in [33] for $x > 1$ and by Kershner and Wintner in [36] for $\frac{1}{2} < x \leq 1$. In the latter paper, Kershner and Wintner had to give an involved argument to make allowance for possible Zeta zeros in the strip $\{\frac{1}{2} < Re(z) < 1\}$. In the paper [15], Borchsenius and Jessen related the aforementioned probability laws to the asymptotic relative frequency of a-points of the Zeta function in strips $(\frac{1}{2} <)$ $a < Re(z) < b$ $(\leq +\infty)$ (An α-point of Zeta is a complex number z such that $\zeta(z) = \alpha$). They also considered the asymptotic behaviour of the argument of Zeta. Their paper contains a succint account of the works briefly outlined above.

The statistical regularity in the asymptotic behaviour of a Dirichlet series may be traced back to the fact that a Dirichlet series is an infinite sum of periodic terms, and the sum inherits some of the characteristic features of periodicity. In the three papers [11], [12] and [13], H. Bohr introduced the notion of almost periodicity. In the third paper he defined an analytic almost periodic function. It turns out that any Dirichlet series defines an analytic almost periodic function in its half-plane of absolute convergence. In particular, Zeta is analytic almost periodic in $\{Re(z) > 1\}$. However, the half-plane of almost periodicity of a function given by a Dirichlet series can be shown to coincide with its half-plane of boundedness. Since the Zeta function is unbounded in every strip contained in the critical strip, it is not almost periodic in any region to the left of $Re(z) = 1$. Various generalisations of the notion of almost periodicity may be found in chapter 11 of [3]. One such generalisation is that of B^2-almost periodic functions. In [53], Wintner showed that the function $t \longrightarrow \frac{1}{\zeta(1+it)}$ is B^2-almost periodic. However, such generalisations do not appear to be very fruitful for the study of the Zeta function in the critical strip. Also, we are unaware of any prior work which relates the Riemann hypothesis itself to any such extension of almost periodicity. Yet it is of interest to note that the statistical theory of the Zeta function has been extended to arbitrary analytic almost periodic functions. For a detailed account of this extension, see the paper [30] by Jessen and Tornehave.

Bohr's line of enquiry appears to have been almost totally abandoned during the sixties. Then in 1972 S.M. Voronin obtained some significant generalisations ([51]) of Bohr's denseness result. He proved that if $z_1, \ldots, z_n$ are distinct points in the strip $\frac{1}{2} < \mathrm{Re}(z) < 1$ then the set

$$\{(\zeta(z_1+it), \ldots, \zeta(z_n+it)) : t \in \mathbb{R}\}$$

is dense in the n-dimensional unitary space $\mathbb{C}^n$. (Actually, Voronin's result is somewhat stronger in that he allows t to vary only over integral multiples of an arbitrarily fixed $h > 0$; further, he allows $\frac{1}{2} < \mathrm{Re}(z_j) \leq 1$.). He also proved that for any fixed z in $\frac{1}{2} < \mathrm{Re}(z) < 1$, the set

$$\{(\zeta(z+it), \zeta^{(1)}(z+it), \ldots, \zeta^{(n-1)}(z+it)) : t \in \mathbb{R}\}$$

is dense in $\mathbb{C}^n$.

In 1975, Voronin introduced ([52]) Hilbert space techniques in the study of the Zeta function to prove the following remarkable theorem. Let $0 < r < \frac{1}{4}$, and let $K = K_r$ be the closed disc $K = \{z \in \mathbb{C} : |z-\frac{3}{4}| \leq r\}$. Let f be a continuous non-vanishing function on K which is analytic in the interior of K. Let $\varepsilon > 0$. Then there exists $t \in \mathbb{R}$ such that

(i) $\quad \sup_{z \in K} |\zeta(z+it) - f(z)| < \varepsilon$.

This result has been dubbed the <u>universality theorem for the Zeta function</u>. In [43], Reich extended this theorem by showing that the set of all solutions t of (i) has positive lower density. To see that the universality theorem does indeed constitu a generalisation of the denseness result mentioned previously, let's note that it may be reformulated as follows. Let $X = X(K)$ be the space of all functions on K which are continuous on K and analytic in the interior of K. Let us equip X with the topology of uniform convergence. Let $S(K)$ denote the set of all $f \in X(K)$ such that $f \equiv 0$ or $f(z) \neq 0$ for all $z \in K$. For $t \in \mathbb{R}$ let $S^t(\zeta)$ denote the function given by

$$S^t(\zeta)(z) = \zeta(z + it);$$

then each $S^t(\zeta)$ may be regarded as a point in $X(K)$. Then the universality theorem says that the closure of the set $\{S^t(\zeta) : t \in \mathbb{R}\}$ contains $S(K)$. Notice that under Riemann hypothesis the said closure equals $S(K)$.

0.2 <u>Summary of the results in chapters 2-5</u>: In chapter 3 of this thesis we extend the statistical theory of the Zeta function and other Dirichlet series to a function space setting. For real t, let $S^t(\zeta)$ be as in the preceding paragraph, but now regarded as a point in the space of all meromorphic functions on the half-plane $\{\frac{1}{2} < \mathrm{Re}(z) < \infty\}$. For $T > 0$, let ν_T be the probability measure on this space given as the distribution of the random function $S^{T\theta}(\zeta)$ where θ is a random variable uniformly distributed on $[-1,1]$. It is shown that there is a probability measure υ on the said space such that as $T \longrightarrow \infty$, υ_T converges in distribution to υ. In the notation introduced in sections 1.5 and 2.2 below, we have $\zeta \Longrightarrow \upsilon$ on the half-plane $\{\mathrm{Re}(z) > \frac{1}{2}\}$. This clearly implies all the probabilistic results on the Zeta function mentioned above. But, unlike the method employed by Bohr et al, this result does not depend on the existence of an Euler product. On the contrary, it is obtained from general results on the class of analytic functions of finite order in a half-plane, represented by Dirichlet series and having finite mean-square value in the half-plane. It is show that each function in this class has an asymptotic distribution in the sense considered here. Further, the probability measure associated with its asymptotic behaviour is explicitly described. Let W denote the set of all completely multiplicative arithmetic functions (see [37, p. xii]) taking values in the unit circle. W can naturally be made into a compact topological group, and hence there is a Haar probability measure m on its Borel σ-field. In [9], Bohr introduced a notion of equivalence between Dirichlet series. Namely, two series $\sum_1^\infty a_n n^{-z}$ and $\sum_1^\infty b_n n^{-z}$ are said to be Bohr-equivalent in case there is a $w \in W$ such that $b_n = w(n)a_n$ for $n \geq 1$. For a fixed real number a, let $\mathfrak{F}_a$ denote the class of all functions f given by a Dirichlet series $f(z) = \sum a_n n^{-z}$ for sufficiently large $\mathrm{Re}(z)$, such that f is analytic and of finite order for $\mathrm{Re}(z) \geq a$

and $\int^{T}_{-T} |f(b+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$ for each $b > a$. For $f \in \mathcal{F}_a$, consider the class of all Dirichlet series which are Bohr-equivalent to f. The probability m on W induces a probability measure on the latter class relative to which almost all members of the class converges uniformaly on compact subsets of $\{Re(z) > a\}$, and hence defines an analytic function-valued random element. If μ_f denotes the distribution of this random function (so that μ_f is a probability on the space of analytic functions on $Re(z) > a$) then we show that $f \rightrightarrows \mu_f$ on $\{Re(z) > a\}$. (In [55], Wintner constructed a similar random function out of ± 1-valued completely multiplicative arithmetic functions and studied some of its almost sure properties. But, of course, this random function of Wintner does not correspond to the asymptotic behaviour of any given Dirichlet series.). In [9], Bohr showed that the two sets of values, assumed by two Bohr-equivalent Dirichlet series on a vertical line contained in their common half-plane of absolute convergence, have identical closure. The result mentioned above implies that in their half-plane of finite mean-square value two Bohr equivalent Dirichlet series have a common asymptotic distribution. This may be regarded as a qualitative extension of Bohr's equivalence theorem.

The above mentioned result of chapter 3 is shown to imply that any finite subclass of $\mathcal{F}_a$ have a joint asymptotic distribution on $\{Re(z) > a\}$. Since Zeta is not analytic in $\{Re(z) > \frac{1}{2}\}$, these results do not directly apply to the Zeta function. However, we use them to deduce that ζ, regarded as a point in the space of meromorphic functions on $\{\text{Re}(z) > \frac{1}{2}\}$, does have an asymptotic distribution. Indeed, if F is the random analytic function on $\{\text{Re}(z) > \frac{1}{2}\}$ defined by :

(ii) $$F(z,w) = \sum_{1}^{\infty} w(n)n^{-z}, \quad \text{Re}(z) > \frac{1}{2}, \; w \in W,$$

then it is shown that $\zeta \Longrightarrow F$ on $\text{Re}(z) > \frac{1}{2}$. Since differentiation is a continuous operator on our function space, the asymptotic behaviour of the higher derivatives of Zeta may readily be deduced from here. For example, we have $\zeta'/\zeta \Longrightarrow F'/F$ on $\text{Re}(z) > \frac{1}{2}$. It may also be of some interest to note that the random function F has the Euler product representation

(iii) $$F(z,w) = \prod_{p} (1-w(p)p^{-z})^{-1}, \quad \text{Re}(z) > \frac{1}{2},$$

where the product, which is over all primes p, is almost surely convergent, uniformly on compact subsets of $\{\text{Re}(z) > \frac{1}{2}\}$. In consequence, F is almost surely nonvanishing.

In chapter 4 we obtain discrete analogues of the results of chapter 3. For any fixed real number $h > 0$, let $\{X_N^h : N \geq 1\}$ be the sequence of random elements such that X_N^h assumes the 2N+1 values $S^{nh}(\zeta)$, $n = 0, \pm 1, \ldots, \pm N$, each with probability $\frac{1}{2N+1}$. It is shown that there is a random function F_h, taking its values in the space of all analytic functions on $\{\text{Re}(z) > \frac{1}{2}\}$, such that X_N^h converges in distribution to F_h as $N \longrightarrow \infty$. In our notation, we have $\zeta \Longrightarrow F_h$ mod h on $\{\text{Re}(z) > \frac{1}{2}\}$. Further, it is shown that for all but countably many values of h, $F_h = F$, where F is the random function of the previous paragraph. For the countably many exceptional values of h, the dependence of F_h on h, which will be fully described, is quite intricate. As in chapter 3, these results on the Zeta function will be deduced from analogous results on the asymptotic distribution modulo h of members of $\mathfrak{J}_a$.

We also employ the techniques of chapter 4 to obtain a result on the sequence of Dirichlet L-functions with increasing prime moduli. For any Dirichlet character χ, let $L(.,\chi)$ denote the associated Dirichlet L-function given by

$$L(z,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-z}.$$

We regard $L(.,\chi)$ as a meromorphic function on the half-plane $\{\operatorname{Re}(z) > \frac{1}{2}\}$. For each prime p, let X_p denote the random element taking the $p-1$ values $L(.,\chi)$, corresponding to the $p-1$ Dirichlet characters χ modulo p, each with probability $\frac{1}{p-1}$. Then it is shown that as $p \longrightarrow \infty$ through primes, X_p converges in distribution to the random function F defined by equation (ii) above. This may be paraphrased as : the sequence of L-functions with prime moduli mimics the asymptotic behaviour of the Zeta function. This result has its precursors in the paper [17] by Chowla and Erdos, and in [24] and [25] by Elliot

In chapter 5 <u>we generalise the universality theorem of Voronin in several directions</u>. Exactly as the denseness result of Bohr can be proved in a strengthened form by determining the support of an associated probability measure on the plane, we proceed to strengthen the universality theorem by determining the support of a probability measure on the space $H(\Omega)$ of all analytic functions on the strip $\Omega = \{\frac{1}{2} < Re(z) < 1\}$. It follows from the results of chapter 3, that the restriction of Zeta to Ω, regarded as a point in $H(\Omega)$, is asymptotically distributed like the restriction $F|\Omega$, of the random function F to Ω. We show that the support of $F|\Omega$ is the set $S = \{f \in H(\Omega) : f \equiv 0 \text{ or } \frac{1}{f} \in H(\Omega)\}$. From this we deduce that if K is <u>any simply connected and locally path connected compact subset of</u> Ω (and not necessarily a disc as in the generalisations of Voronin's theorem by Reich [43] and Laurincikas [38]), and f is any <u>continuous non-vanishing function</u> on K, analytic in he interior (if any) of K, then the inequality (i) above has a solution $t \in \mathbb{R}$. In fact, the set of all such solutions t has a positive lower density.

We also apply the results of chapter 4 to obtain <u>discrete versions of the above result</u>. Let $h > 0$ be an arbitrary but fixed real number. Let K, f be as in the preceding paragraph, and $\varepsilon > 0$. Then it is shown that the inequality

(iv) $$\sup_{z \in K} |\zeta(z+inh) - f(z)| < \varepsilon$$

has solutions in integers n. Indeed, the set of all such integer solutions n has positive lower density (discrete version). All three theorems of Voronin's paper [51], in their original discrete form, may be deduced from this result - although they can not be so deduced from Voronin's version of the universality theorem.

In [52], Voronin mentions that his methods can be used to derive an analogous universality theorem for an arbitrary but fixed L-function. The above-mentioned universality results of this thesis are particular cases of a more general universality theorem of chapter 5 on the joint approximation properties of finite sets of L-functions. This theorem, which is as follows, also implies the universality theorem for each fixed L-function. Let $k \geq 1$ be an integer, and let $n = \phi(k)$ be the number of integers in $[1,k]$ which are relatively prime to k. Let $\chi_1, \chi_2, \ldots, \chi_n$ be the n distinct Dirichlet characters modulo k. Let $K_1, \ldots, K_n$ be simply connected and locally path-connected compact subsets of $\underline{C}$. For each j, $1 \leq j \leq n$, let f_j be a nonvanishing continuous function on K_j which is analytic in the interior (if any) of K_j. Let $\varepsilon > 0$. Then the inequality

$$(v) \qquad \sup_{1 \leq j \leq n} \ \sup_{z \in K_j} |L(z+it, \chi_j) - f_j(z)| < \varepsilon$$

has a solution $t \in \mathbb{R}$. Indeed, the set of all such solutions t has positive lower density.

We also prove a discrete version of this joint universality theorem for an arbitrary increment $h > 0$. Moreover, from the asymptotic result of chapter 4 on the sequence of L-functions, we deduce a novel sort of universality theorem. Namely, we show that if K is a simply connected, locally path-connected subset of $\mathbb{C}$, f is a non-vanishing continuous function on K which is analytic in the interior (if any) of K, and $\varepsilon > 0$, then for each sufficiently large prime p, there exists a Dirichlet character χ modulo p such that

(vi) $$\sup_{z \in K} |L(z,\chi) - f(z)| < \varepsilon .$$

Indeed, there exists a constant $c > 0$ (depending on K, f, ε) such that (vi) holds for at least $c.p$ characters modulo p.

The joint universality theorem quoted above clearly implies there exists no non-trivial algebraic-differential identity relating the various L-functions. Many other consequences of these results have been discussed in chapter 5. In particular, it is shown that if $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, α is a rational number, then the Hurwitz Zeta function $\zeta(z,\alpha)$ satisfies the following unconstrained universality property. Let K be a simply connected, locally path connected compact subset of $\mathbb{C}$, let f be any (not necessarily non-vanishing) continuous function on K which is analytic in the interior (if any) of K. Let $\varepsilon > 0$. Then the set of all $t \in \mathbb{R}$ for which

(vii) $$\sup_{z \in K} |\zeta(z+it,\alpha) - f(z)| < \varepsilon$$

has positive lower density.

This result is true, and indeed is easier to prove, in case α is transcendental. It is definitely false for $\alpha = \frac{1}{2}$ or 1. Truth or falsity of this theorem for irrational algebraic α remains open. For rational $\alpha \neq \frac{1}{2}$ in $(0,1)$ and for transcendental $\alpha \in (0,1)$, we deduce that the set of real parts of the zeros of $\zeta(\cdot,\alpha)$ is dense in $[\frac{1}{2}, 1]$. Supplementary information on the zero-set of $\zeta(\cdot,\alpha)$ may be found in the works of Davenport and Heilbronn, [20], [21], of Cassel, [16], and of Spira, [48]. The unconstrained universality theorem also holds for the nth derivative $\zeta^{(n)}$ of the Zeta function $(n \geq 1)$. In consequence, the set of real parts of the zeros of $\zeta^{(n)}$ is dense in $[\frac{1}{2}, 1]$. For related work on the zero-set of $\zeta^{(n)}$, see Berndt, [2].

A partial converse of the universality theorems of chapter 5 is easy to establish. For example, if $K \subseteq \mathbb{C}$ is as before, and f is any function on K such that the set of all $t \in \mathbb{R}$ satisfying the inequality (i) above has positive lower density for each $\varepsilon > 0$, then f is continuous on K and non-vanishing analytic in the interior of K. This observation together with the universality theorems themselves, imply an important criterion for zero-free strips. Let us say that an analytic function f on a strip $\{a < \mathrm{Re}(z) < b\}$ is strongly recurrent on that strip in case for every compact set K contained in the strip, and for every $\varepsilon > 0$, the set of all $t \in \mathbb{R}$

for which $\sup_{z \in K} |f(z+it) - f(z)| < \varepsilon$ has positive lower density. Then a strip $\{a < \mathrm{Re}(z) < b\}$, where $\frac{1}{2} \leq a < b \leq 1$, is free of Zeta zeros if and only if Zeta is strongly recurrent on that strip. In particular, the Riemann hypothesis holds if and only if Zeta is strongly recurrent on the strip (*). The analogous statements for the L-functions are also valid. The examples $\zeta^{(n)}$ $(n \geq 1)$ and $\zeta(\cdot, \alpha)$ (with $0 < \alpha < 1$, α rational or transcendental, $\alpha \neq \frac{1}{2}$) show that for general Dirichlet series, strong recurrence on the strip (*) is consistent with the set of real parts of the zeros of the function being dense in $[\frac{1}{2}, 1]$. In chapter 5, we also give an example of an entire function ψ represented by a convergent Dirichlet series on $\mathrm{Re}(z) > 0$ which has a functional equation very similar to that of the Zeta function, such that ψ is strongly recurrent on (*), and the set of real parts of the zeros of ψ is dense in $[0,1]$. The relationship between strong recurrence and zero-free strips for the Zeta function is a result of the Euler product formula (iii) for the associated random function F. This in turn is a consequence of the Euler product formula for the Zeta function.

The universality theorems themselves are proved by exploiting the fact that in the product formula (iii), the factors are stochastically independent random elements, and that similar formulae exist for other L-functions. Following Voronin, [52], we base the proof on the introduction of suitable Hilbert spaces. But unlike Voronin, we do not use the result of Pecerskii on rearrangement of series in Hilbert spaces. The said result of Pecerskii is a real Hilbert space theorem which does not suffice for the proof of the joint universality theorem for L-functions. Instead, we base our proof on a new result (proposition 5.2.8) on conditionally convergent series in complex Hilbert spaces which we state and prove in chapter 5. This proposition may be of some interest of its own. The other major tool that we use is a theorem of V. Bernstein on entire functions of exponential type.

In chapter 1 we summarise some relevant notions and results from the theory of Topological transformation groups. This theory whose historical origin lies in the study of dynamical systems of Newtonian physics, is seen to provide the proper axiomatic basis for a unified study of both asymptotic distributions and strong recurrence. In particular, the notion of strong recurrence of points in an abstract flow (i.e., a topological transformation group with the additive group of reals or of integers as the phase group) is introduced following Gottschalk and Hedlund in [26]. Thus strong recurrence appears as a particular example of the more general concept of recursion. It may be noted that the classical definition of analytic almost periodicity due to Bohr has also been extended to a recursion notion on arbitrary topological transformation groups (see, e.g., [27, pp.31-48]). Thus strong recurrence appears as a close relative of almost periodicity. We also emphasise the inheritance theorem of

Gottschalk and Hedlund which relates the continuous and discrete versions of strong recurrence.

In chapter 2, we specialise the definitions of chapter 1 to flows on function spaces with shift as flow projection. It is shown that if f, g are analytic functions on a strip, such that f is strongly recurrent and g is given by an absolutely convergent Dirichlet series, then $f+g$ is strongly recurrent. The argument there may easily be generalised to show that if f is strongly recurrent and g is almost periodic in the sense of Bohr, then $f+g$ is strongly recurrent. (In particular, every almost periodic function is strongly recurrent.). This raises the question as to whether the sum of two strongly recurrent analytic functions on a strip is again strongly recurrent. In chapter 5 we represent the Zeta function (as also the other L-functions) as a finite sum of strongly recurrent functions on the strip ( ). Thus, an affirmative answer to this question would imply the Riemann hypothesis, and indeed the generalised Riemann hypothesis for L-functions. It may be mentioned that the sum of two analytic almost periodic functions is again almost periodic (theorem 5 of [3, p.143]).

On the other hand, it is shown in chapter 2 that subject to an extra condition of a technical nature on the asymptotic behaviour of f on the line $\{Re(z) = a\}$, if an $f \in \mathcal{F}_a$ is strongly recurrent on a strip $\{a < Re(z) < b\}$ then it is strongly recurrent on the entire half-plane $\{Re(z) > a\}$. This leads us to conjecture that this implication remains valid without the technical condition. We have called it the recurrence conjecture (2.4.10). In chapter 5 we show that the recurrence conjecture implies that the set of real parts of the Zeta zeros is dense in $[0,1]$ - the strongest possible negation of the Riemann hypothesis that is consistent with present knowledge. The conjecture also implies that the set of real parts of the zeros of $L(\cdot,\chi)$ is dense in the interval $[1-a(\chi), a(\chi)]$, where $a(\chi)$ is the supremum of this set.

It follows from the results of chapter 2 that the property of strong recurrence of a Dirichlet series remains unaffected if finitely many terms are added to or deleted from the series. Further, if P is a rational function of n variables $(n \geq 1)$ and f is strongly recurrent on a strip then so is $P(f,f^{(1)},\ldots,f^{(n-1)})$. Thus strong recurrence is a highly stable property. On the other hand, the Riemann hypothesis, even if true, is a highly unstable property of the Zeta function. Thus, for example, it is falsified if $\zeta(z)$ is replaced by $\zeta(z)-\alpha$ for any $\alpha \neq 0$ or by $\zeta^{(n)}$ for $n \geq 1$. Therefore further research on strong recurrence of analytic functions on a strip may be more fruitful than the attempts at direct study of the zero-free region of the Zeta function have so far been. The main objective of this thesis (apart from the more mundane one of procuring a degree for the author !) is to make a strong plea for the study of strong recurrence.

CHAPTER 1

STRONG RECURRENCE AND ASYMPTOTIC DISTRIBUTION IN FLOWS

1.1 Introduction and summary : In this chapter we present the basic definitions and results from Topological dynamics which will be used in the later chapters. Excepting for minor modifications to suit our purpose, the results and definitions in sections 1.2 and 1.3 are from Gottschalk and Hedlund [27]. In contradistinction to [27], we consider only Abelian phase groups, and accordingly use an additive notation. The inheritance theorem as well as the general notion of recursion in section 1.3 below first appeared in the paper [26] by Gottschalk and Hedlund, and were presented in a more refined form in [27]. Various recursion notions - particularly recurrence and almost periodicity, have been studied in great details. But the notion of strong recurrence that is presented in section 1.4 below appears to be relatively neglected. The definition of strong recurrence, as also the specialisation of the inheritance theorem to strong recurrence, is briefly mentioned in [26]. We elaborate on this theme in 1.4 since strong recurrence turns out to be of utmost importance to the theory of Dirichlet series in general and of the Zeta function in particular.

In section 1.5 we introduce the notion of asymptotic distribution of points in a flow. The intention is to apply probability theory to the study of recurrence. Although analogous notions have been used in Probabilistic number theory and the inspiration from Ergodic theory should be obvious (after all, abstract topological dynamics emerged in an attempt to free Ergodic theory from probabilistic considerations), this concept appears to be new in this setting. Its relationship with strong recurrence is explained in subsection 1.5.4. In 1.5.6 we give a sufficient condition for the existence of asymptotic distribution. This proposition has a well known analogue in Probability theory from which it has been deduced.

1.2 Topological transformation groups and flows :

1.2.1 Definitions : A topological transformation group is an ordered triple (X,G,π) consisting of a topological space X, an abelian topological group G, and a map $\pi: X \times G \longrightarrow X$ such that :

(a) (Identity axiom) $\pi(x,0) = x$, $(x \in X)$, where 0 is the identity element of G;

(b) (Homomorphism axiom) $\pi(\pi(x,t),s) = \pi(x,s+t)$, $(s,t \in G, x \in X)$;

(c) (Continuity axiom) π is continuous.

If (X,G,π) is a topological transformation group then X,G and π are called the phase space, the phase group, and the phase projection respectively.

1.2.2 Notation : If (X,G,π) is a topological transformation group then for each $t \in G$ we define the map $\pi^t : X \longrightarrow X$ by

$$\pi^t(x) = \pi(x,t) \quad (x \in X).$$

In this notation, the homomorphism axiom may be rewritten as

$$\pi^{t_1 + t_2} = \pi^{t_1} \circ \pi^{t_2}, \; (t_1, t_2 \in G).$$

1.2.3 <u>More definitions</u> (Homomorphism, invariance, restrictions, Cartesean product) : If (X,G,π) and $(Y,G,\hat{\pi})$ are two topological transformation groups having a common phase group G, then a map $\phi : X \longrightarrow Y$ is called a <u>homomorphism</u> in case

(a) ϕ is continuous,

and (b) $\phi \circ \pi^t = \hat{\pi}^t \circ \phi \; (t \in G)$.

In particular, if (X,G,π) is a topological transformation group, each π^t is a homomorphism of (X,G,π) into itself.
If (X,G,π) is a topological transformation group and Y is a subspace of X, then Y is said to be <u>invariant</u> in case

$$\pi^t(Y) \subseteq Y \; (t \in G).$$

If μ is a Borel probability measure on X, then μ is said to be <u>invariant</u> in case $\mu \circ \pi^t = \mu \; (t \in G)$.

If Y is an invariant subspace of X, then the restriction of π to $Y \times G$ maps $Y \times G$ into Y. This restriction (which we denote by π itself) satisfies all the axioms of 1.2.1. Thus (Y,G,π) is a topological transformation group in its own right. In this case we say that (Y,G,π) is a <u>subspace-restriction</u> of (X,G,π).

If H is a subgroup of G, then the restriction of π to $X \times H$ again satisfies the axioms of 1.2.1 for $t \in H$. Thus, (X,H,π) is again a topological transformation group. In this case, (X,H,π) is called a subgroup-restriction of (X,G,π).

Let I be an index set. For each $\alpha \varepsilon I$, let $(X_\alpha, G, \pi_\alpha)$ be a topological transformation group. The cartesean product

$$\prod_{\alpha \varepsilon I}(X_\alpha, G, \pi_\alpha) = (\prod_{\alpha \varepsilon I} X_\alpha, G, \prod_{\alpha \varepsilon I} \pi_\alpha)$$

is the topological transformation group with the cartesean product $\prod_{\alpha \varepsilon I} X_\alpha$ (with product topology) as its phase space, G as its phase group, and with the phase projection

$$\pi = \prod_{\alpha \varepsilon I} \pi_\alpha : \prod_{\alpha \varepsilon I} X_\alpha \times G \longrightarrow \prod_{\alpha \varepsilon I} X_\alpha \text{ given by}$$

$$\pi((x_\alpha : \alpha \varepsilon I), t) = (\pi_\alpha(x_\alpha, t) : \alpha \varepsilon I), (t \varepsilon G).$$

In case $I = 1, 2, \ldots, n$, we also denote the cartesean product by $(X_1 \times X_2 \times \ldots \times X_n, G, \pi_1 \times \ldots \pi_n)$. Further if $X_1 = X_2 = \ldots = X_n = X$ and $\pi_1 = \pi_2 = \ldots = \pi_n = \pi$ then the product is denoted by (X^n, G, π), or (if the phase projection is understood as given by the context) more simply by X^n.

1.2.4 Continuous and discrete flow : A continuous flow is a topological transformation group having the group $\mathbb{R}$ of reals (with usual addition) for its phase group. We shall speak of "the continuous flow (X,π)" : the phase group being implied to be $\mathbb{R}$.

A discrete flow is a topological transformation group having the group Z of integers (with usual addition) for its phase group. We shall speak of " the discrete flow (X,π)" : the phase group being implied to be Z.

If h is a positive real number, hZ is a closed subgroup of $\mathbb{R}$. In fact, these are the only proper closed subgroups of $\mathbb{R}$; they are necessarily syndetic. (Recall that a subgroup H of a topological group G is said to be syndetic in case $G = H + K$ for some compact K ($\subseteq G$.).

If (X,π) is a continuous flow, its subgroup restriction (X,hZ,π) is called " the <u>discrete subflow of (X,π) modulo h</u>".

1.3 <u>Recursion in topological transformation groups</u> :

1.3.1 <u>Definition</u> (recursion) : Let (X,G,π) be a topological transformation group. Let there be a distinguished class $\mathcal{A}$ of subsets of G. (Intuitively, $\mathcal{A}$ consists of subsets of G which are "large" in some suitable sense). A point $x \in X$ is said to be <u>$\mathcal{A}$-recursive under G</u> in case for each neighbourhood U of x, there is an $A \in \mathcal{A}$ such that $\pi(x,A) \subseteq U$.

1.3.2 <u>Proposition</u> : Let (X,G,π) and (Y,G,π) be two topological transformation groups and $\emptyset : X \longrightarrow Y$ a homomorphism between them. If $x \in X$ is $\mathcal{A}$-recursive then $\emptyset(x)$ is also $\mathcal{A}$-recursive.

Proof : Trivial.

1.3.3 <u>Definitions</u> : Let (X,G,π) be a topological transformation group, H a subgroup of G, and $\mathcal{A}$ a family of subsets of G.

Then we shall say that $x \in X$ is <u>$\mathcal{A}$-recursive under H</u> in case x is $\mathcal{A}_H$-recursive when regarded as a point in the subgroup-restriction (X,H,π). Here $\mathcal{A}_H = \{A \in \mathcal{A} : A \subseteq H\}$. In particular, if (X,π) is a continuous flow and $h > 0$, then we shall say that $x \in X$ is $\mathcal{A}$-recursive modulo h in case x is $\mathcal{A}$-recursive under $h\mathbb{Z}$.

1.3.4 Inheritance theorem (Gottshalk and Hedlund) : Let (X,G,π) be a topological transformation group such that G is locally compact, and let $\mathcal{A}$ be a family of subsets of G such that whenever $A,B,C \subseteq G$ satisfy (i) $A \subseteq B+C$, (ii) $A \in \mathcal{A}$ and (iii) C is compact, we also have $B \in \mathcal{A}$. Then for each closed syndetic subgroup H of G, and each $x \in X$, x is $\mathcal{A}$-recursive under G if and only if x is $\mathcal{A}$-recursive under H.

1.4 <u>Upper and lower densities; strong recurrence of points in a flow</u> :

1.4.1 <u>Definitions</u> (lower and upper density; continuous and discrete versions) : If $A \subseteq \mathbb{R}$ is Borel, the upper - and lower density of A (continuous version), denoted respectively by $\bar{d}(A)$ and $\underline{d}(A)$, are defined by :

$$\bar{d}(A) = \limsup_{T \to \infty} \frac{1}{2T} \lambda(A \cap [-T,T]),$$

$$\underline{d}(A) = \liminf_{T \to \infty} \frac{1}{2T} \lambda(A \cap [-T,T]).$$

Further, if $\underline{d}(A) = \bar{d}(A)$ then we write $d(A)$ for this common value and say that A has density (continous version) $d(A)$. If A is a subset of Z, the upper and lower density (discrete version) of A, denoted respectively by $\bar{\delta}(A)$ and $\underline{\delta}(A)$, are defined by :

$$\bar{\delta}(A) = \limsup_{N \to \infty} \frac{1}{2N+1} \# (A \cap [-N,N]) ,$$

$$\underline{\delta}(A) = \liminf_{N \to \infty} \frac{1}{2N+1} \# (A \cap [-N,N]) .$$

Further, if $\bar{\delta}(A) = \underline{\delta}(A)$ then we write $\delta(A)$ for the common value and say that A has density (discrete version) $\delta(A)$. Here, as also later, λ is Lebesgue measure on the real line, $\#(.)$ denotes the number of elements in $(.)$.

1.4.2 Definition : Let (X,π) be a continuous flow. Then $x \in X$ is said to be strongly recurrent if it is $\mathcal{A}$-recursive, where $\mathcal{A}$ is the family of all Borel subsets A of $\mathbb{R}$ for which $\bar{d}(A+I) > 0$ for each neighbourhood I of $0 \in \mathbb{R}$.

1.4.3 Proposition : Let (X,π) be a continuous flow, and $x \in X$. Then x is strongly recurrent if and only if for every neighbourhood U of x,

$$\bar{d}(\{t \in \mathbb{R} : \pi^t(x) \in U\}) > 0 .$$

Proof : The 'if' part is trivial since for every Borel set $B \subseteq \mathbb{R}$ and every nonempty $I \subseteq \mathbb{R}$, $\bar{d}(B+I) \geq \bar{d}(B)$.

To prove the 'only if' part, let x be strongly recurrent, and let's fix a neighbourhood U of x. Let $B = \{t \in \mathbb{R} : \pi^t(x) \in U\}$.

We have to show $\bar{d}(B) > 0$.

Since $\pi(x,0) = x$ and π is continuous, there exists neighbourhoods V of x and I of $0 \in \mathbb{R}$ such that $\pi(V \times I) \subseteq U$. Let $A = \{t \in \mathbb{R} : \pi^t(x) \in V\}$. Then for $t \in A$ and $h \in I$, $\pi^{t+h}(x) = \pi(\pi^t(x),h) \in \pi(V \times I) \subseteq U$. Hence $A + I \subseteq B$. Also, by assumption of strong recurrence of x, there is $C \subseteq A$ such that $\bar{d}(C+I) > 0$. Hence $\bar{d}(B) \geq \bar{d}(A+I) \geq \bar{d}(C+I) > 0$, so that $\bar{d}(B) > 0$ and the proof is complete.

1.4.4 <u>Proposition</u> : If (X,π) is a continuous flow and $x \in X$, then x is strongly recurrent mod h if and only if

$$\bar{\delta}(\{n \in \mathbb{N} : \pi^{nh}(x) \in U\}) > 0$$

for every neighbourhood U of x.

<u>Proof</u> : This may be proved exactly as the proposition in 1.4.3 once we notice that for subsets A of Z, $\bar{\delta}(A) > 0$ if and only if $\bar{d}(A+I) > 0$ for every neighbourhood I of 0.

1.4.5 <u>Lemma</u> : If A,B,C are Borel subsets of $\mathbb{R}$ such that $A \subseteq B+C$, $A \in \mathcal{A}$ and C is compact, then $B \in \mathcal{A}$. (Here $\mathcal{A}$ is as in definition 1.4.2).

<u>Proof</u> : Let I be any neighbourhood of 0. We have to show that $\bar{d}(B+I) > 0$. Take a neighbourhood J of 0 such that $J + J \subseteq I$. Since C is compact, there is a finite subset C_0 of C such that $C \subseteq C_0 + J$. Hence $C + J \subseteq C_0 + I$, and

therefore $A + J \subseteq B + C + J \subseteq B + C_o + I = B + I + C_o$. But $A \in \mathcal{A}$ and hence $\bar{d}(A+J) > 0$. Therefore $\bar{d}((B+I) + C_o) > 0$. Since C_o is finite, this implies $\bar{d}(B+I) > 0$ (In fact, for any Borel set $D \subseteq \mathbb{R}$, we have $\bar{d}(D) \geq \frac{1}{\# (C_o)} \bar{d}(D+C_o)$). This completes the proof as I was an arbitrary neighbourhood of 0.

1.4.6 <u>Proposition</u> (Inheritance theorem for strong recurrence): If (X,π) is a continuous flow, $x \in X$ and $h > 0$ is real, then x is strongly recurrent if and only if x is strongly recurrent modulo h.

<u>Proof</u>: In view of definition 1.4.2 and lemma 1.4.5, this is a special case of theorem 1.3.4.

1.4.7 <u>Definition</u>: Let (X,π) be a continuous (respectively discrete) flow. An $x \in X$ is said to be <u>periodic under π</u> (or simply " periodic " if the phase projection is given by the context) in case there is $h \in \mathbb{R}$ (respectively $h \in Z$), $h \neq 0$, such that $\pi^h(x) = x$. Equivalently, the requirement is $\pi^{t+h}(x) = \pi^t(x)$ for all $t \in \mathbb{R}$ (respectively all $t \in Z$). Any h satisfying this condition is said to be a <u>period</u> of x. The next proposition shows that strong recurrence (as also other recursion concepts found in the literature) is a generalisation of the notion of periodicity.

1.4.8 <u>Proposition</u>: If (X,π) is a flow and $x \in X$ is periodic then x is strongly recurrent.

Proof : We prove the proposition for continuous flows, the proof for discrete flows being entirely analogous.

Let $h \neq 0$ be a period of x. Then x is fixed under the subgroup hZ of $\mathbb{R}$ (i.e., $\pi^t(x) = x$ for $t \in hZ$). Hence, trivially, x is strongly recurrent modulo h. Now proposition 1.4.6 implies that x is strongly recurrent.

1.5 Asymptotic distribution of points in a flow :

1.5.1 Definition : Let (X, π) be a continuous flow, $x \in X$. Let μ be a Borel probability measure on X. We shall say that x has asymptotic distribution μ (or x is asymptotically distributed as μ) in case we have :

$$\underline{d}(\{t \in \mathbb{R} : \pi^t(x) \in U\}) \geq \mu(U)$$

for every open set $U \subseteq X$. In this case we shall write $x \Rightarrow \mu$. Similarly if (X, π) is a discrete flow, $x \in X$, and μ a Borel probability measure on X, then we shall say that x has asymptotic distribution μ (or x is asymptotically distributed as μ) in case we have :

$$\underline{\delta}(\{n \in Z : \pi^n(x) \in U\}) \geq \mu(U)$$

for every open set $U \subseteq X$. In this case also we write $x \Rightarrow \mu$.

If (X, π) is a flow, $x \in X$, μ a Borel probability measure on X, and ϕ is an X-valued random element whose distribution is μ, then we shall also write $x \Rightarrow \phi$ in place of $x \Rightarrow \mu$.

If (X,π) is a continuous flow, $x \in X$, $h > 0$, and μ is a Borel probability measure on X, then we shall write $x \rightrightarrows \mu$ modulo h in case $x \rightrightarrows \mu$ relative to the discrete subflow modulo h of (X,π). If in this situation, $\emptyset$ is an X-valued random element whose distribution is μ then we shall also write $x \rightrightarrows \emptyset$ mod h.

1.5.2 <u>Remarks</u> : Let (X,π) be a continuous (respectively discrete) flow. Let $x \in X$ and let μ be a Borel probability measure on X. For each $T > 0$, $T \in \mathbb{R}$ (respectively $N > 0$, $N \in Z$) let θ_T be a random variable which is uniformly distributed on $[-T,T]$ (respectively, let θ_N be a random variable such that $\theta_N = k$ with probability $\frac{1}{2N+1}$ for each $k \in [-N,N] \cap Z$), and let x_T (respectively x_N) be the X-valued random element defined by $x_T = \pi(x, \theta_T)$ (respectively $x_N = \pi(x, \theta_N)$). Then the definition in 1.5.1 may be rewritten as :

$x \rightrightarrows \mu$ if and only if $x_T \overset{D}{\Longrightarrow} \mu$ as $T \longrightarrow \infty$ through $\mathbb{R}$ (respectively $x_N \overset{D}{\Longrightarrow} \mu$ as $N \longrightarrow \infty$ through Z).

(See [4] for the definition of $\overset{D}{\Longrightarrow}$; convergence in distribution). Since a net of probability measures may have at most one limit in distribution, this also shows that the asymptotic distribution of an x in X, when it exists, is uniquely determined by x.

Lastly, it is easy to see that the asymptotic distribution of a point in a flow is necessarily invariant in the sense of 1.2.3.

1.5.3 Definitions (Support of a probability measure ; orbit, orbit closure, and spectrum of a point in a flow) : If X is a seperable topological space and μ a Borel probability measure on X then there exists a unique minimal closed set $\Lambda \subseteq X$ such that $\mu(\Lambda) = 1$. Λ is called the support of μ, and may be obtained as the set of all $y \in X$ such that for every neighbourhood U of y, we have $\mu(U) > 0$.

If (X,π) is a seperable flow (i.e., the phase space X is seperable) and $x \in X$ is asymptotically distributed as μ, then the spectrum of x is defined to be the support of μ.

If (X,G,π) is a topological transformation group, and $x \in X$, then the orbit of x is defined to be the set $\{\pi^t(x) : t \in G\}$. The orbit closure of x is, by definition, the closure of the orbit of x.

1.5.4 Proposition : Let (X,π) be a seperable flow. Suppose an $x \in X$ has asymptotic distribution. Let Λ and Γ be the spectrum and orbit closure, respectively, of x. Then :

(a) Λ and Γ are closed invariant subspaces of X,

(b) $\Lambda \subseteq \Gamma$,

and (c) If $x \in \Lambda$ then x is strongly recurrent.

Proof : These are trivial consequences of the definitions.

1.5.5 Remarks : If (X,π) is a flow, and Γ is the orbit closure of an $x \in X$, then by (a) of 1.5.4, Γ is an invariant closed set containing x (in fact, it is the " smallest " such set relative to set inclusion) and hence we may speak of the subspace restriction (Γ,π) of (X,π). Clearly an $x \in X$ is strongly recurrent / has asymptotic distribution relative to (X,π) if and only if it has this property relative to (Γ,π).

1.5.6 Proposition : Let (X,π) be a continuous flow on a seperable metric space X. Let ρ be a metric compatible with the topology of X. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence in X and let $\{\mu_n : n \in \mathbb{N}\}$ be a sequence of Borel probability measures on X. Let $x \in X$. We also assume :

(a) $\limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \rho(\pi^t(x), \pi^t(x_n))dt = o(1)$ as $n \to \infty$,

(b) $x_n \Rrightarrow \mu_n$ for each $n \in \mathbb{N}$,

and (c) The sequence $\{\mu_n\}$ is relatively compact in the topology of convergence in distribution.

Then there is a Borel probability measure μ on X such that :

(d) $x \Rrightarrow \mu$.

and (e) $\mu_n \overset{D}{\Longrightarrow} \mu$ as $n \to \infty$.

Proof : For each $T > 0$ let θ_T be a random variable which is uniformly distributed on $[-T,T]$. Suppose the θ_T's are defined on a common probability space $(\Omega, \ , P)$. For each $n \in \mathbb{N}$, let Y_n be an X-valued random element whose distribution is μ_n. We may (and do) assume that the Y_n's are also defined on the same space $(\Omega, \ , P)$. For each $n \in \mathbb{N}$ and each $T > 0$, let's define X-valued random elements X_T and $X_{n,T}$ by :

$$X_T = \pi(x, \theta_T) , \quad X_{n,T} = \pi(x_n, \theta_T) .$$

For each $\varepsilon > 0$, we have, by Chebychev's inequality [39,p.11],

$$P(\rho(X_{n,T}, X_T) > \varepsilon) \leq \frac{1}{\varepsilon} E(\rho(X_{n,T}, X_T))$$

$$= \frac{1}{\varepsilon} \cdot \frac{1}{2T} \int_{-T}^{T} \rho(\pi^t(x_n) , \pi^t(x))dt .$$

Hence, by assumption (a), we have :

(f) $\lim_{n \to \infty} \limsup_{T \to \infty} P(\rho(X_{n,T}, X_T) > \varepsilon) = 0$ for each $\varepsilon > 0$.

Also, for each n, $x_n \Rrightarrow \mu_n$, and hence, in view of 1.5.2 above, we have :

(g) $X_{n,T} \overset{D}{\Longrightarrow} Y_n$ as $T \longrightarrow \infty$ $(n \in \mathbb{N})$.

Also, if $\{\mu_{n_k}\}$ is a convergent subsequence of $\{\mu_n\}$ (let's say $\mu_{n_k} \overset{D}{\Longrightarrow} \mu$) then we also have :

(h) $Y_{n_k} \overset{D}{\Longrightarrow} \mu$ as $n_k \longrightarrow \infty$.

In view of theorem 4.2 of Billingslay [4,p.25], (f), (g) and (h) imply :

(i) $X_T \overset{D}{\Longrightarrow} \mu$.

In view of the remarks in 1.5.2, this means $x \Rrightarrow \mu$. Also, (i) shows that μ is independent of the choice of the subsequence $\{\mu_{n_k}\}$. Hence, due to assumption (c), we have $\mu_n \overset{D}{\Longrightarrow} \mu$ as $n \longrightarrow \infty$. This completes the proof.

1.5.7 <u>Proposition</u> : Let (X, π) and $(Y, \hat{\pi})$ be two flows (both continuous or both discrete). Let $\phi : X \longrightarrow Y$ be a homomorphism between them. Let $x \in X$ satisfy $x \rightrightarrows \mu$. Then $\phi(x) \rightrightarrows \mu o \phi^{-1}$.

<u>Proof</u> : We take the flows to be continuous (the discrete case is entirely analogous). Let U be an open subset of Y. Then $\phi^{-1}(Y)$ is an open subset of X, and we have :

$$\{t \in \mathbb{R} : \pi^t(\phi(x)) \in U\} = \{t \in \mathbb{R} : \phi(\pi^t(x)) \in U\}$$
$$= \{t \in \mathbb{R} : \pi^t(x) \in \phi^{-1}(U)\} .$$

Since $x \rightrightarrows \mu$ and $\phi^{-1}(U)$ is open, it follows that :

$$\underline{d}(\{t \in \mathbb{R} : \pi^t(\phi(x)) \in U\}) = \underline{d}(\{t \in \mathbb{R} : \pi^t(x) \in \phi^{-1}(U)\})$$
$$\geq \mu(\phi^{-1}(U))$$
$$= (\mu o \phi^{-1})(U) .$$

Since this holds for all open $U \subseteq Y$, it follows that $\phi(x) \rightrightarrows \mu o \phi^{-1}$. This completes the proof.

CHAPTER 2

FLOWS IN FUNCTION SPACES

2.1 Introduction and summary : In this chapter we allow shift transformations to act on spaces of analytic functions and of meromorphic functions as well as on the space of continuous functions on the line in order to study the flow structures that result. The emphasis is on the strongly recurrent points in them.

Section 2.2 contains the basic definitions and a short list of flow-homomorphisms between various function-space flows introduced there. In section 2.3 we concentrate on the set of strongly recurrent points in the flow $H(\Omega)$. By repeated use of the inheritance theorem of chapter 1, we are able to show that this set is closed under addition and multiplication by certain members of $H(\Omega)$, among them are the functions given by absolutely convergent Dirichlet series. In particular, the absolutely convergent Dirichlet series are themselves strongly recurrent. These observations lead us to ask (2.3.7) if the set of all strongly recurrent points form a subalgebra of $H(\Omega)$. In subsection 5.4.11 of chapter 5, we show that an affirmative answer to this question would imply the generalized Riemann hypothesis for Dirichlet L-functions.

One of the results of section 2.2 is that if an $f \in H(\Omega)$ is strongly recurrent then so are all its vertical sections. In 2.4 we prove an elementary result (proposition 2.4.2) to demonstrate the exact relationship that obtains here. We also prove a result (theorem 2.4.8) to the effect that if an analytic function represented by a Dirichlet series satisfies certain auxiliary conditions then strong recurrence of one vertical section of the function implies strong recurrence of the function in the entire half-plane to the right. This suggests that subject to suitable auxiliary conditions, a Dirichlet series will have a half-plane of of strong recurrence. This is the content of the recurrence conjecture of 2.4.10. In subsection 5.4.6 of chapter 5 it will be shown that this conjecture implies the strongest conceivable negation of the Riemann hypothesis.

In order to prepare for the proof of theorem 2.4.8, we decompose an analytic function given by a Dirichlet series into the sum of an absolutely convergent Dirichlet series and a residual term (lemma 2.4.6). This is the well known " Kernel method" of analytic number theory stated in its utmost generality.

Regarded as a study of the flow $H(\Omega)$, the results of this chapter are far from complete. However, they suffice as an ad-hoc basis for chapter 5, where the relationship between strong recurrence and zero-free strips of the Zeta function and other L-series will be established.

2.2 Shift transformation as phase projection on function spaces:

2.2.1 Standing notations: Throughout this thesis Ω will stand for a vertical half-plane or a vertical strip. Thus $\Omega = \Omega_a^b = \{z \in \mathbb{C} : a < \mathrm{Re}(z) < b\}$, where $-\infty \leq a < b \leq +\infty$.

By a <u>proper substrip</u> of Ω_a^b , we shall understand a subregion of the form Ω_c^d where $a < c < d < b$.

Further, if f is a function on Ω_a^b , and $a < c < b$, then f_c will denote the function on $\mathbb{R}$ defined by

$$f_c(t) = f(c + it) \quad (t \in \mathbb{R}).$$

2.2.2 <u>The function spaces $M(\Omega)$, $H(\Omega)$ and $C(\mathbb{R})$</u> ; Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere (i.e. the one point compactification of $\mathbb{C}$). The map $d : \mathbb{C}_\infty \times \mathbb{C}_\infty \longrightarrow \mathbb{R}^+$ given by

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{(1+|z_1|^2)^{1/2}(1+|z_2|^2)^{1/2}} , \quad d(z, \infty) = \frac{2}{(1+|z|^2)^{1/2}}$$

and $d(\infty, \infty) = 0$ $(z, z_1, z_2 \in \mathbb{C})$ is a metric on $\mathbb{C}_\infty$ compatible with the usual topology of $\mathbb{C}_\infty$. If Ω is as in 2.2.1, the space $C(\Omega, \mathbb{C}_\infty)$ of continuous functions from Ω into the metric space $(\mathbb{C}_\infty, d)$ may be equipped with the topology of uniform convergence on compacta. In this topology, a sequence f_n in $C(\Omega, \mathbb{C}_\infty)$ converges to the point f in the same space in case $d(f_n(z), f(z)) \longrightarrow 0$ as $n \longrightarrow \infty$ for each $z \in \Omega$, the convergence being uniform for z in compact subsets of Ω. This makes $C(\Omega, \mathbb{C}_\infty)$ into a complete metric space (for this and other unproved assertions of this subsection, see Conway [18]). The set of all meromorphic functions on Ω (i.e., the 'analytic' functions from Ω into $\mathbb{C}_\infty$) form a closed subset of $C(\Omega, \mathbb{C}_\infty)$, and hence is a complete metric space in its own right. This is denoted by $M(\Omega)$. Notice that the constant function ∞ is taken to be a member of $M(\Pi)$ (in this regard, our notation differs from that of [18]). The space of all analytic functions on Π form a subspace of $M(\Pi)$. This we denote by $H(\Pi)$. Finally, $C(\mathbb{R})$ will stand for the space of all complex-valued continuous functions on $\mathbb{R}$, equipped with the topology of uniform convergence on compacta.

2.2.3 The shift transformations on $M(\Pi)$, $H(\Pi)$ and $C(\mathbb{R})$:

Let $S : M(\Pi) \times \mathbb{R} \longrightarrow M(\Pi)$ be defined by : $S(f,t) = g$ where $g(z) = f(z+it)$ $(z \in \Pi,\ t \in \mathbb{R},\ f \in M(\Pi))$. It is easy to verify that $(M(\Pi), \mathbb{R}, S)$ satisfies the axioms (a), (b) and (c) of 1.2.1, and hence $(M(\Pi), S)$ is a continuous flow (see 1.2.4). $H(\Pi)$ is an invariant subspace of $M(\Pi)$, and hence the subspace restriction $(H(\Pi), S)$ is a continuous flow in its own right (see 1.2.3). We shall ordinarily be concerned with $(H(\Pi), S)$; but occasionall the embedding of $H(\Pi)$ in $M(\Pi)$ will be found useful.

Let $S : C(\mathbb{R}) \times \mathbb{R} \longrightarrow C(\mathbb{R})$ be defined by : $S(f,t) = g$ where $g(x) = f(x+t)$ $(x, t \in \mathbb{R},\ f \in C(\mathbb{R}))$. Then $(C(\mathbb{R}), S)$ is again a continuous flow. The use of the same letter S to denote two types of shifts in two different function spaces should not lead to any confusion.

From now on, our sole interest will be in the continuous flows introduced here. Thus, we shall speak of "the continuous flows $M(\Pi)$, $H(\Pi)$ and $C(\mathbb{R})$", the underlying phase projection in each case being understood to be as above.

2.2.4 <u>Some useful flow-homomorphisms on function spaces</u> :

(a) For each $n \geq 1$, D^n, the <u>nth</u> derivative operator, is clearly continuous on $M(\underline{(\overline{\ })})$ and commutes with all the S^t's. Therefore the D^n's are flow homomorphisms of $M(\underline{(\overline{\ })})$ into itself. $H(\underline{(\overline{\ })})$ is invariant under D^n, and therefore D^n is also a flow homomorphism of $H(\underline{(\overline{\ })})$ into itself.

(b) The metric **d** of 2.2.2 is easily seen to satisfy the identity $d(z_1,z_2) = d(\frac{1}{z_1},\frac{1}{z_2})$ $(z_1,z_2 \in \mathbb{C}_\infty)$. In consequence, the map $f \longrightarrow \frac{1}{f}$ is continuous on $M(\underline{(\overline{\ })})$. Since it clearly commutes with the S^t's, it follows that $f \longrightarrow \frac{1}{f}$ is a flow homomorphism (in fact a flow isomorphism) of $M(\underline{(\overline{\ })})$ into itself.

(c) The inclusion mapping from $H(\underline{(\overline{\ })})$ into $M(\underline{(\overline{\ })})$ is clearly a flow homomorphism.

(d) Let $-\infty \leq a < c < d < b \leq \infty$. Then the restriction map from $M(\underline{(\overline{\ })}_a^b)$ into $M(\underline{(\overline{\ })}_c^d)$ (i.e., the map sending $f \in M(\underline{(\overline{\ })}_a^b)$ into the restriction $f|\underline{(\overline{\ })}_c^d$ of f to $\underline{(\overline{\ })}_c^d$) is a flow homomorphism. Likewise, the restriction map from $H(\underline{(\overline{\ })}_a^b)$ into $H(\underline{(\overline{\ })}_c^d)$ is a flow homomorphism.

(e) Let $-\infty \leq a < b \leq \infty$. For $a < c < b$, the map $f \longrightarrow f_c$ (in the notation of 2.2.1) is a flow homomorphism of $H(\underline{(\overline{\ })})$ into $C(\mathbb{R})$.

(f) Let μ be a complex Borel measure on $\mathbb{R}$ with compact support. For $f \in C(\mathbb{R})$, the convolution $f*\mu$ of f and μ is the function on $\mathbb{R}$ given by $(f*\mu)(x) = \int f(x-t)d\mu(t)$ $(x \in \mathbb{R})$. The map $f \longrightarrow f*\mu$ is a flow homomorphism of $C(\mathbb{R})$ into itself.

2.2.5 <u>Remarks on strong recurrence in function spaces</u> : In view of the proposition in 1.3.2, each of the maps in 2.2.4 allow us to obtain new strongly recurrent " points " in the function spaces from given ones. (i) Thus, if $f \in H(\underline{(\overline{\ })})$ (or $f \in M(\underline{(\overline{\ })})$) is strongly recurrent then so is $f^{(n)} = D^n(f)$ for each $n \geq 1$. (ii) If a,b,c,d are as in (d) of 2.2.4, and $f \in H(\underline{(\overline{\ })}_a^b$ is strongly recurrent then so is the restriction of f to $\underline{(\overline{\ })}_c^d$, regarded as a point in $H(\underline{(\overline{\ })}_c^d)$. Notice that since we use the topology of uniform convergence on compacta, if the restriction of f to $\underline{(\overline{\ })}_c^d$ is strongly recurrent for all pairs c,d with $a < c < d < b$, then so is $f \in H(\underline{(\overline{\ })}_a^b)$. (iii) If $f \in H(\underline{(\overline{\ })}_a^b)$ is strongly recurrent and $a < c < b$, then, in view of (e) in 2.2.4, $f_c \in C(\mathbb{R})$ is strongly recurrent. Later (in 2.4.8) we shall prove a sort of converse to this statement. (iv) It follows from (c) of 2.2.4 that if an f in $H(\underline{(\overline{\ })})$ is strongly recurrent when viewed as a point in $H(\underline{(\overline{\ })})$ then it is strongly recurrent as a point in $M(\underline{(\overline{\ })})$; since $H(\underline{(\overline{\ })})$ is an invariant subspace of $M(\underline{(\overline{\ })})$ with the relativized topology, the converse is also true - as may be seen from definitions. (v) If $f \in H(\underline{(\overline{\ })})$ is non-vanishing throughout $\underline{(\overline{\ })}$ and if f is strongly recurrent, then so is $\frac{1}{f}$. This is because f, and therefore also $\frac{1}{f}$, is a strongly recurrent point of $M(\underline{(\overline{\ })})$, and as by hypothesis $\frac{1}{f} \in H(\underline{(\overline{\ })})$ the latter must also be a strongly recurrent point of

$H(\Omega)$. (vi) Finally, if μ is a complex Borel measure on $\mathbb{R}$ with compact support, $f \in C(\mathbb{R})$ is strongly recurrent, then so is $f*\mu$.

2.3 <u>Structure of the strongly recurrent points in $H(\Omega)$</u> :

2.3.1 <u>Proposition</u> : Let $\Omega = \Omega_a^b$. If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $H(\Omega)$, $f \in H(\Omega)$, and $f_n \longrightarrow f$ uniformly on each proper substrip of Ω, then f is strongly recurrent.

<u>Proof</u> : This is a trivial consequence of the definitions.

2.3.2 <u>Remarks</u> : Notice that $f \in H(\Omega)$ is periodic with period $t_o \in \mathbb{R}$ (in the sense of definition 1.4.7) as a point in the flow $H(\Omega)$ if and only if it has purely imaginary period it_o (in the usual sense) when regarded as a function on Ω. Conforming to definition 1.4.7, we shall continue to speak of t_o (rather than it_o) as a period of f. Thus, for example, we shall say that the exponential map $z \longrightarrow e^z$ $(z \in \mathbb{C})$ is a periodic point of $H(\mathbb{C})$ with period 2π.

2.3.3 <u>Proposition</u> : Let $f \in H(\Omega)$ be strongly recurrent, and let $g \in H(\Omega)$ be a finite sum of periodic points. Then $f+g$ and $f.g$ are strongly recurrent.

<u>Proof</u> : Let $g = h_1+h_2+\ldots+h_n$, where h_j is periodic with period t_j $(1 \leq j \leq n)$. Let $g_j = h_1+h_2+\ldots+h_j$ $(1 \leq j \leq n)$, $g_o = 0$. We shall prove, by induction on j, that $f+g_j$ is strongly recurrent for $0 \leq j \leq n$. It will then follow that in particular

$f+g = f+g_n$ is strongly recurrent. Let us put $\rho_j = f+g_j$ $(0 \leq j \leq n)$. By assumption, $\rho_o = f$ is strongly recurrent. Now suppose ρ_j is strongly recurrent for some j, $0 \leq j \leq n-1$. Then, by the inheritance theorem 1.4.6, ρ_j is strongly recurrent modulo t_{j+1}. Therefore $\rho_{j+1} = \rho_j + h_{j+1}$ is strongly recurrent modulo t_{j+1}. Another appeal to the inheritance theorem 1.4.6 yields that ρ_{j+1} is strongly recurrent, thus completing the inductive leap. The proof that $f.g$ is also strongly recurrent goes analogously.

2.3.4 <u>Corollary</u> : If $g \varepsilon H(\underline{(\,\overline{\ }\,)})$ is a finite sum of periodic points then g is strongly recurrent.

<u>Proof</u> : Follows from 2.3.3 on putting $f = 0$.

2.3.5 <u>Corollary</u> : If $f \varepsilon H(\underline{(\,\overline{\ }\,)})$ is strongly recurrent and $g \varepsilon H(\underline{(\,\overline{\ }\,)})$ is given by a Dirichlet series which is absolutely convergent throughout $\underline{(\,\overline{\ }\,)}$ then $f+g$ and $f.g$ are strongly recurrent. In particular g is strongly recurrent.

<u>Proof</u> : Let g_n be the <u>nth</u> partial sum of the Dirichlet series for g $(n \geq 1)$, Each g_n is a finite sum of periodic points (the typical term $a_k.k^{-z}$ is periodic with period $\frac{2\pi}{\log k}$ if $k \geq 2$ and 1 if $k = 1$). Hence by proposition 2.3.3, $f+g_n$ is strongly recurrent $(n \geq 1)$. Since the Dirichlet series for g is absolutely convergent throughout $\underline{(\,\overline{\ }\,)}$, $g_n \longrightarrow g$ uniformly on proper substrips of $\underline{(\,\overline{\ }\,)}$. Hence $f+g_n \longrightarrow f+g$ uniformly on proper substrips of $\underline{(\,\overline{\ }\,)}$. Therefore, by proposition 2.3.1,

$f+g$ is strongly recurrent. Similarly for $f.g$. The particular case follows on taking $f = 0$.

2.3.6 <u>Proposition</u>: The strongly recurrent points of $H(\Omega)$ are dense in $H(\Omega)$.

<u>Proof</u>: Let $D(\Omega)$ denote the set of all finite Dirichlet series, regarded as points in $H(\Omega)$. In view of 2.3.5, each point in $D(\Omega)$ is strongly recurrent. Therefore it suffices to show that $D(\Omega)$ is dense in $H(\Omega)$. Now, $H(\Omega)$ is a locally convex topological vector space over $\mathbb{C}$ (in fact it is a Frechet space), and $D(\Omega)$ is a linear subspace of $H(\Omega)$. Therefore, by Hahn-Banach theorem (see Rudin [46, p.59]), it suffices to show that each continuous linear functional L on $H(\Omega)$ which vanishes on $D(\Omega)$ vanishes identically. So let $L(f) = 0$ for $f \in D(\Omega)$. Since $H(\Omega)$ is a closed linear subspace of $C(\Omega)$, the spa e of complex-valued continuous functions on Ω with the compact open topology, another application of Hahn-Banach theorem shows L admits an extention to $C(\Omega)$. Therefore (Rudin [46, p.84]) there is a complex Borel measure μ with compact support $K \subseteq \Omega$, such that $L(f) = \int f d\mu$ (for $f \in C(\Omega)$ and in particular for $f \in H(\Omega)$). Since L vanishes on $D(\Omega)$, we have $\int n^{-z} d\mu(z) = 0$ $(n \geq 1)$. If we put $\phi(w) = \int e^{-zw} d\mu(z)$ $(w \in \mathbb{C})$ then this show ϕ vanishes on $\{\log n : n \geq 1\}$. But clearly ϕ is an entire function of finite order. Therefore (Theorem 2.5.12 of Boas [5, p.15]),

$\emptyset \equiv 0$. That is, $\int e^{-wz} d\mu(z) = 0 \quad (w \in \mathbb{C})$. Differentiating n times and setting $w = 0$, we get $\int z^n d\mu(z) = 0 \quad (n \geq 0)$. That is, L vanishes on polynomials. Since the polynomials are dense in $H(\mathbb{C})$, this implies $L \equiv 0$, thus completing the proof.

2.3.7 *Question* : In view of proposition 2.3.3, we may ask : *Is the set of strongly recurrent points in $H(\mathbb{C})$ a subalgebra of $H(\mathbb{C})$?* In other words, if $f, g \in H(\mathbb{C})$ are strongly recurrent, then does it follow that $f + g$ and $f \cdot g$ are also strongly recurrent ? Notice that $f \longrightarrow f^2$ and $f \longrightarrow \alpha f$ (for fixed $\alpha \in \mathbb{C}$) are flow homomorphisms of $H(\mathbb{C})$ into itself, so that if $f \in H(\mathbb{C})$ is strongly recurrent then so are f^2 and αf. Since $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$, the question above might be equivalently formulated as : *Is the set of strongly recurrent points of $H(\mathbb{C})$ closed under addition ?*

2.4 The relationship between strong recurrence of a point in $H(\mathbb{C})$ and of its vertical sections :

2.4.1 *Definition* : Let I be an index set and let $\{f^\alpha : \alpha \in I\}$ be a family of functions in $C(\mathbb{R})$. We shall say that $\{f^\alpha : \alpha \in I\}$ is *uniformly strongly recurrent* in case for every neighbourhood U of $0 \in C(\mathbb{R})$, we have

$$\bar{d}(\{t \in \mathbb{R} : S^t(f^\alpha) - f^\alpha \in U \text{ for all } \alpha \in I\}) > 0.$$

Notice that when I is singleton, the notion of uniformly strong recurrence reduces to the ordinary notion of strong recurrence.

2.4.2 <u>Proposion</u> : Let $\Omega = \Omega_a^b$ $(a < b)$, and $f \in H(\Omega)$. Then f is strongly recurrent if and only if $\{f_c : a < c < b\}$ $(\subseteq C(\mathbb{R}))$ is strongly recurrent, uniformly for c in compact subsets of (a,b).

<u>Proof</u> : Let f be strongly recurrent. Let $a < c_1 < c_2 < b$, and let U be a neighbourhood of $0 \in C(\mathbb{R})$. We have to show that there is a Borel set $A \subseteq \mathbb{R}$ with $\bar{d}(A) > 0$ such that whenever $t \in A$ and $c \in [c_1, c_2]$, $S^t f_c - f_c \in U$. There is compact $K \subseteq \mathbb{R}$ and $\varepsilon > 0$ such that when $\sup_{x \in K} |g(x)| < \varepsilon$, $g \in U$. Let $K_1 = [c_1, c_2] + iK$. Then K_1 is a compact subset of Ω, and hence the set V of all $h \in H(\Omega)$ for which $\sup_{z \in K_1} |h(z)| < \varepsilon$ is a neighbourhood of $0 \in H(\Omega)$. Since f is strongly recurrent, the set of all $t \in \mathbb{R}$ for which $S^t(f) - f \in V$ satisfies $\bar{d}(A) > 0$. Clearly, whenever $S^t(f) - f \in V$, we have :
$\sup_{x \in K} |S^t(f_c)(x) - f_c(x)| < \varepsilon$ for $c \in [c_1, c_2]$ and hence $S^t(f_c) - f_c \in U$ whenever $c \in [c_1, c_2]$ and $t \in A$. Thus this A works.

To prove the converse, we notice that every neighbourhood V of $0 \in H(\Omega)$ contains a set of the form

$$\{h \in H(\Omega) : \sup_{x \in K} |h_c(x)| < \varepsilon \text{ for } c \in [c_1, c_2]\}$$

where $a < c_1 < c_2 < b$ and $\varepsilon > 0$. Therefore the steps of the above proof can be reversed.

2.4.3 <u>Definitions</u> : (a) If μ is a complex Borel measure on $\mathbb{R}$, the <u>tail of μ</u> is defined to be the map $T : [0, \infty) \to [0, \infty)$

where $T(x) = |\mu|((-\infty, -x) \bigcup (x, \infty))$. Here $|\mu|$ is the total variation measure of μ (see [47, p.125] for definition). Clearly T is decreasing.

(b) If $f \in C(\mathbb{R})$, the <u>recurrence measure of</u> f is defined to be the function $\rho : (0, \infty) \longrightarrow [0,1]$ given by :

$$\rho(x) = \bar{d}(\{t \in \mathbb{R} : \sup_{|y| \le x} |f(y+t) - f(y)| \le \frac{1}{x}\}).$$

Clearly ρ is a decreasing function; <u>f is strongly recurrent if and only if $\rho(x) > 0$ for all $x > 0$</u>.

2.4.4 <u>Proposition</u> : Let $f \in C(\mathbb{R})$ and let μ be a complex Borel measure on $\mathbb{R}$. Let ρ be the recurrence measure of f and T be the tail of μ. Suppose we have :

(i) $\int_{-Y}^{Y} (\sup_{|x| \le \theta} |f(x+t)|)dt = O_\theta(Y)$ as $Y \longrightarrow \infty$, for each $\theta > 0$,

(ii) $\limsup_{x \longrightarrow \infty} \frac{\rho(\theta x)}{T(x)} = \infty$ for each $\theta > 0$,

(iii) $\int_{-\infty}^{\infty} |y| d|\mu|(y) < \infty$,

and (iv) $\int_{-\infty}^{\infty} |f(x-y)| d|\mu|(y) < \infty$, uniformly for x in compact subsets of $\mathbb{R}$.

<u>Then</u> $f * \mu \in C(\mathbb{R})$ is strongly recurrent.

<u>Proof</u> : By (iv) $f*\mu \in C(\mathbb{R})$. Let U be a neighbourhood of $0 \in C(\mathbb{R})$. We have to exhibit a Borel set $C \subseteq \mathbb{R}$ such that $\bar{d}(C) > 0$ and for $t \in C$, $S^t(f*\mu) - f*\mu \in U$. Let K be a compact subset of $\mathbb{R}$ and $\varepsilon > 0$ such that

$$\{ g \in C(\mathbb{R}) : \sup_{x \in K} |g(x)| < \varepsilon \} \subseteq U .$$

Then it suffices to show that for $t \in C$ and $x \in K$, we have $|(f*\mu)(x+t) - (f*\mu)(x)| < \varepsilon$. Without loss of generality we may take $0 < \varepsilon < 6$.

For $x \in K$, $t \in \mathbb{R}$,

$$|(f*\mu)(x+t) - (f*\mu)(x)| \leq \int_{-X}^{X} |f(x+t-y) - f(x-y)| \, d|\mu|(y)$$
$$+ \int_{|y| \geq X} |f(x+t-y)| \, d|\mu|(y)$$
$$+ \int_{|y| \geq X} |f(x-y)| \, d|\mu|(y)$$
$$= S_1(t) + S_2(t) + S_3 \quad \text{(say)}.$$

Here X is a "large" real number that remains to be specified. In the first place, in view of assumption (iv), there is $X_1 > 0$ such that for $X \geq X_1$, $S_3 \leq \frac{\varepsilon}{3}$ for all $x \in K$.

Let A be the set of all $t \in \mathbb{R}$ such that

$$|f(x+t) - f(x)| \leq \frac{\varepsilon}{6X} \quad \text{for} \quad |x| \leq \frac{6X}{\varepsilon} .$$

Since by choice $\varepsilon < 6$, there is $X_2 > 0$ such that for $X \geq X_2$, whenever $x \in K$ and $|y| \leq X$, $|x-y| \leq \frac{6X}{\varepsilon}$.

Hence, if $X \geq X_2$, $x \in K$ and $t \in A$, the integrand in $S_1(t)$ is $\leq \frac{\varepsilon}{6X}$ and hence $|S_1(t)| \leq \frac{\varepsilon}{3}$ (we assume, without loss of generality, that $\|\mu\| \leq 1$).

$$\frac{1}{2Y}\int_{-Y}^{Y} S_2(t)dt \leq \int_{|y|\geq X} d|\mu|(y)\ \frac{1}{2Y}\int_{-|y|-Y}^{|y|+Y} (\sup_{x\varepsilon K}|f(x+t)|)dt$$

$$\leq c_o \int_{|y|\geq X} (1+\frac{|y|}{Y})d|\mu|(y) \quad \text{for all large } Y.$$

(This is bacause of assumption (i). Here $c_o > 0$ depends only on K).

Due to assumption (iii), the integrand in the extreme right hand side is dominated by $(1+|y|)$ for $Y \geq 1$, and the latter is integrable with respect to $|\mu|$. Hence as $Y \longrightarrow \infty$, Lebesgue's dominated convergence theorem ([47,p.27]) yields :

$$\limsup_{Y\longrightarrow\infty} \frac{1}{2Y}\int_{-Y}^{Y} S_2(t)dt \leq c_o \int_{|y|\geq X} d|\mu|(y) = c_o T(X).$$

Hence, by a Chebychev type argument [39,p.11], if B is the set of all $t \varepsilon \mathbb{R}$ for which $S_2(t) < \frac{\varepsilon}{3}$ then we have :

$$\underline{d}(B) \geq 1 - \frac{3c_o}{\varepsilon} T(X).$$

Also notice that by definition of ρ, $\bar{d}(A) = \rho(\frac{6X}{\varepsilon})$.

Due to assumption (ii), there is $X \geq X_1, X_2$ such that

$$\rho(\frac{6X}{\varepsilon}) > \frac{3c_o}{\varepsilon} T(X).$$

Hence for this X, $\bar{d}(A) + \underline{d}(B) > 1$, so that if we put $C = A \cap B$ then $\bar{d}(C) > 0$. Also, for $x \varepsilon K$ and $t \varepsilon C$, we have $S_1(t) \leq \frac{\varepsilon}{3}$, $S_2(t) < \frac{\varepsilon}{3}$ and $S_3 \leq \frac{\varepsilon}{3}$. Hence for $x \varepsilon K$, $t \varepsilon C$, $|(f*\mu)(x+t) - (f*\mu)(x)| < \varepsilon$. This completes the proof.

2.4.5 <u>Proposition</u> : Let $f \in C(\mathbb{R})$ and let M be a family of complex Borel measures on $\mathbb{R}$. Let ρ be the recurrence measure of f and, for $\mu \in M$, let T_μ be the tail of μ. Suppose we have :

(i) $\int_{-Y}^{Y} (\sup_{|x| \leq \theta} |f(x+t)|)dt = O_\theta(Y)$ as $Y \longrightarrow \infty$, for each $\theta > 0$,

(ii) $\limsup_{x \to \infty} \frac{\rho(\theta x)}{T_\mu(x)} = \infty$, uniformly for $\mu \in M$, for each $\theta > 0$,

(iii) $\int_{-\infty}^{\infty} |y| \, d|\mu|(y) < \infty$, uniformly for $\mu \in M$,

(iv) $\int_{-\infty}^{\infty} |f(x-y)| \, d|\mu|(y) < \infty$, uniformly for $\mu \in M$ and for x in compact subsets of $\mathbb{R}$,

and (v) $\{ \|\mu\| : \mu \in M \}$ is bounded.

<u>Then</u> the family $\{ f*\mu : \mu \in M \} \subseteq C(\mathbb{R})$ is uniformly strongly recurrent.

<u>Proof</u> : Notice that in view of the uniform estimates in the hypothesis, the choices of X_1, X_2, X as in the proof of 2.4.4 may be made to depend only on K and ε and not on $\mu \in M$.

2.4.6 <u>Lemma</u> : Let b, c be real, $b < c$. Suppose f is a analytic in the half-plane Ω_b^∞, and is given by an absolutely convergent Dirichlet series in Ω_c^∞. Let $A > c-b$, and let η be analytic in the closure of Ω_{-A}^{A}. Suppose we are given :

(i) $\eta(0) = 1$,

(ii) $\eta(u+iv) = O(e^{-|v|})$ as $v \longrightarrow \pm\infty$, uniformly for u in $[-A, A]$, and (iii) $f(x+iy) = O(|y|^e)$, uniformly for x in compact subsets of (b,∞), for some $e > 0$ (i.e., f is of finite order in Ω_b^∞).

Then in Ω_b^{b+A} f has the decomposition $f = g+h$, where

$$g(z) = -\frac{1}{2\pi i}\int_{b-x-i\infty}^{b-x+i\infty} f(z+w)\eta(w)\,\frac{dw}{w}\ ;\ z = x+iy\ ,$$

and h is given by an absolutely convergent Dirichlet series in Ω_b^∞.

Proof : Define $h(z) = \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} f(z+w)\eta(w)\frac{dw}{w}$, $z \in \Omega_b^\infty$.

Due to hypothesis (ii), the integral exists.

Since $\mathrm{Re}(z) > b$ and $A > c-b$, $\mathrm{Re}(z+w) > c$. Hence $f(z+w)$ is given by its absolutely convergent Dirichlet series :

$f(z+w) = \sum_{n=1}^{\infty} a_n n^{-z-w}$, (say). Substituting this in the integral defining h, and interchanging sum and integral, we get :

$h(z) = \sum_{n=1}^{\infty} a_n \alpha_n n^{-z}$, $\mathrm{Re}(z) > b$, where $\alpha_n = \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} n^{-w}\eta(w)\frac{dw}{w}$.

Hence $|\alpha_n| \leq$ constant. $n^{-A}\int_{-\infty}^{\infty} |\eta(A+iv)|\,dv \leq$ constant. n^{-A} by (ii).

Therefore $|\alpha_n a_n n^{-z}| \leq |a_n| n^{-A-x}$, and $\sum_{n=1} |a_n| n^{-A-x} < \infty$ for $x > b$.

These estimates justify the interchange of sum and integral, and also show that h is given by an absolutely convergent Dirichlet series in Ω_b^∞.

Now the estimates (ii) and (iii) permit us to shift the contour of the integral defining h to the line $\mathrm{Re}(z) = b-x$. The only singularity of the integrand that is crossed is at $w = 0$ and the corresponding residue is $\rho(0)f(z) = f(z)$ by (i). Hence, by the residue theorem,

$$h(z) - \frac{1}{2\pi i}\int_{b-x-i\infty}^{b-x+i\infty} f(z+w)\eta(w)\frac{dw}{w} = f(z),$$ which is the required decomposition of f.

2.4.7 <u>Lemma</u> : Let f be analytic in the closure of Ω_a^{∞}, and of finite order there. Suppose f is given by an absolutely convergent Dirichlet series in Ω_c^{∞}. We also assume that

$$\int_{-Y}^{Y} |f(a+it)|dt = O(Y) \quad \text{as} \quad Y \longrightarrow \infty .$$

<u>Then</u> for any $b > a$ and any $\theta > 0$,

$$\int_{-Y}^{Y} \left(\sup_{|x|\leq\theta} |f_b(x+t)|\right)dt = O_\theta(Y) \quad \text{as} \quad Y \longrightarrow \infty .$$

<u>Proof</u> : Since f is bounded on $\mathrm{Re}(z) = b$ in case $b > c$, there is no loss of generality in assuming that $a < b \leq c$. Fix an $A > c-a$ and take an $\eta \in H(\Omega_{-A}^{A})$ satisfying the conditions of lemma 2.4.6. Then lemma 2.4.6 (with a in place of b) ensures that we have a decomposition $f = g+h$, $g,h \in H(\Omega_a^{a+A})$, where h is given by an absolutely convergent Dirichlet series in Ω_a^{∞},

and $$g(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} f_a(y+v)\,\frac{\eta(a-x+iv)}{a-x+iv}\,dv, \quad z = x+iy \quad (a < x < a+A).$$

Hence, if $a < x_o < a+A$ then we have, uniformly for x in $(x_o, a+A)$,

$$\int_{-Y}^{Y} |g(x+it)|dt \leq \text{constant.} \int_{-\infty}^{\infty} |\eta(a-x+iv)|dv \int_{-Y}^{Y} |f_a(t+v)|dt$$

$$\leq \text{constant.} \int_{-\infty}^{\infty} |\eta(a-x+iv)|dv \int_{-Y-|v|}^{Y+|v|} |f_a(t)|dt$$

$$\leq \text{constant.} \int_{-\infty}^{\infty} |\eta(a-x+iv)|(Y+|v|)dv$$

$$\leq \text{constant. } Y \text{ for large } Y.$$

Since h is uniformly bounded in $\Omega_{x_o}^{a+A}$, we have a similar estimate for h. As $f = g+h$ on Ω_a^{a+A}, we therefore get :

(i) $\int_{-Y}^{Y} |f_x(t)|dt = O(Y)$ as $Y \longrightarrow \infty$, uniformly for x in compact subsets of $(a, a+A)$.

Now let $\theta > 0$. Let γ be a simple closed curve lying in Ω_a^{a+A} and enclosing the line segment $[b-i\theta, b+i\theta]$. Such a γ may be chosen since by our assumption $a < b < a+A$. Then, for $x \in [-\theta, \theta]$, we have :

$$f_b(x+t) = f(b+it+ix) = \int_\gamma \frac{f(w+it)}{w-(b+ix)} dw / 2\pi i .$$

Hence it $\delta > 0$ is the distance of γ from the segment $[b-i\theta, b+i\theta]$ then $\sup_{|x| \leq \theta} |f_b(x+t)| \leq \frac{1}{2\pi\delta} \int_\gamma |f(w+it)| |dw|$.

$$\text{Hence, } \int_{-Y}^{Y} (\sup_{|x| \leq \theta} |f_b(x+t)|)dt \leq \frac{1}{2\pi\delta} \int_\gamma |dw| \int_{-Y}^{Y} |f(w+it)|dt$$

$$\leq \frac{1}{2\pi\delta} \int_\gamma |dw| \int_{-2Y}^{2Y} |f(u+it)|dt ,$$

where $u = \mathrm{Re}(w)$, provided Y is sufficiently large. Now, if u_0 is the infimum of $\{u : u+iv \in \gamma\}$ for some real v, then $u_0 > a$, and

$$\int_{-Y}^{Y} \left(\sup_{|x| \leq \theta} |f_b(x+t)| \right) dt \leq \frac{|\gamma|}{2\pi\delta} \sup_{u \geq u_0} \int_{-2Y}^{2Y} |f(u+it)|\, dt ,$$

where $|\gamma|$ is the length of γ.

Hence in view of the estimate (i), the proof is complete.

2.4.8 <u>Theorem</u> : Let f be analytic in the closure of the half-plane Ω_a^∞, and of finite order there. For some $c > a$, let f be represented by an absolutely convergent Dirichlet series in Ω_c^∞. Let $a < b < c$, and let ρ be the recurrence measure of f_b. We also assume :

(i) $$\int_{-Y}^{Y} |f(a+it)|\, dt = O(Y) \quad \text{as} \quad Y \longrightarrow \infty ,$$

and (ii) $$\limsup_{x \longrightarrow \infty} \rho(x) e_2(\theta x) = \infty \quad \text{for every} \quad \theta > 0 .$$

(Here $e_2(x) = \exp \exp x$).

<u>Then</u> f is strongly recurrent in Ω_b^∞.

<u>Proof</u> : Clearly it suffices to show that f is strongly recurrent in Ω_b^{b+A} for all large A. Accordingly we fix an $A > c-b$. Let $\eta \in H(\Omega_{-A}^{A})$ be defined by

$$\eta(z) = \frac{e^2}{e_2(i\alpha z)\, e_2(-i\alpha z)} , \quad |\mathrm{Re}(z)| \leq A .$$

Here $\alpha > 0$ is chosen so small that for some $\beta > 0$ the estimate

$$\eta(x+iy) = O\left(\frac{1}{e_2(\beta|y|)}\right) \quad \text{as} \quad y \longrightarrow \pm\infty$$

holds uniformly in $x \in [-A, A]$.

Clearly this function η satisfies the requirements of lemma 2.4.5. Hence, by that lemma, we have the decomposition $f = g+h$ on Ω_b^{b+A}, where h is given by an absolutely convergent Dirichlet series and

$$g(z) = -\frac{1}{2\pi i}\int_{b-x-i\infty}^{b-x+i\infty} f(z+w)\eta(w)\frac{dw}{w}.$$

For $b < x < b+A$, let μ_x be the complex Borel measure on $\mathbb{R}$ given by $d\mu_x(v) = -\frac{1}{2\pi}\frac{\eta(b-x-iv)}{b-x-iv}dv$. Then the integral representation of g may be rewritten as : $\underline{g_x = f_b * \mu_x}$, $b < x < b+A$.

If T_x is the tail of μ_x, then we have : for any β_1 in $(0,\beta)$, $T_x(y) = O\left(\frac{1}{e_2(\beta_1 y)}\right)$ as $y \longrightarrow \infty$, uniformly for x in compact subsets of $(b,b+A)$. Let K be any compact subset of $(b,b+A)$. Then, in view of hypothesis (ii) above, the family $\{M = \mu_x : x \in K\}$ satisfies the condition (ii) of proposition 2.4.5. It trivially satisfies conditions (iii), (iv), (v) of that proposition (with f_b in place of f). Finally, in view of hypothesis (i) above, lemma 2.4.7 shows that f_b satisfies the condition (i) of proposition 2.4.5. Hence that proposition yields :

$\{f_b * \mu_x : x \in K\}$ is uniformly strongly recurrent for each compact $K \subseteq (b,b+A)$. That is, $\{g_x : x \in (b,b+A)\}$ is strongly recurrent-uniformly for x in compact subsets of $(b,b+A)$. Therefore, by

2.4.2, g is strongly recurrent on Ω_b^{b+A}. Since h is given by a Dirichlet series which converges absolutely on Ω_b^{b+A}, it follows from 2.3.5 that $f = g+h$ is strongly recurrent on Ω_b^{b+A}. This completes the proof.

2.4.9 Remarks: In the proof of theorem 2.4.8, the choice of η is the best possible. In fact, by a Phragmen-Lindelof type argument it may be shown that if η is analytic and bounded in Ω_{-A}^{A} and satisfies $\liminf_{y \to \pm\infty} |\eta(iy)| e_2(\beta |y|) < \infty$ for every $\beta > 0$, then $\eta \equiv 0$. However, the introduction of an auxiliary analytic function η in order to obtain a suitable representation of f is a mere technicality, and therefore the above observation does not help to make the hypothesis (ii) of 2.4.8 look any less unnatural. This hypothesis clearly implies $\rho(x) > 0$ for all $x > 0$, hence it is stronger than the assumption (ii)' "f_b is strongly recurrent". This might lead one to conjecture that substituting (ii)' for (ii) in 2.4.8 yields a correct theorem. But we shall see later (5.4.10) that this conjecture is false. So we formulate the following weaker conjecture (notice that it is a partial converse to (ii) of 2.2.5, just as theorem 2.4.8 is a partial converse to (iii) of 2.2.5).

2.4.10 The recurrence conjecture: Let f be an analytic function of finite order on the closure of the half-plane Ω_a^{∞}. Let us assume that:

$$\int_{-Y}^{Y} |f(a+it)|dt = O(Y) \quad \text{as} \quad Y \longrightarrow \infty$$

and that for some $d > a$ f is represented on Ω_d^∞ by an absolutely convergent Dirichlet series. Let $a < b < c < d$. Under these assumptions, <u>if f is strongly recurrent on Ω_b^c then f is strongly recurrent on Ω_b^∞</u>.

CHAPTER 3

ASYMPTOTIC DISTRIBUTION OF POINTS IN THE FLOW $H(\underline{\bar{C}})$ (continuous version)

3.1 <u>Introduction and summary</u>: In this chapter we specialise the notions of the previous two chapters to the flow $H(\underline{\bar{C}})$. In section 3.2, we show that analytic functions which are uniformly bounded in the mean along vertical lines and which have Dirichlet series representation in some half-plane can be suitably approximated by absolutely convergent Dirichlet series. In later sections this device is repeatedly used to deduce results regarding the asymptotic behaviour of functions in this larger class from analogous results for absolutely convergent Dirichlet series. In particular, at the end of section 3.2 we show that every function in this class has an asymptotic distribution. In section 3.3 we show that if two functions of our class are representable by Bohr-equivalent Dirichlet series then they have identical asymptotic distribution. This result (theorem 3.3.2) may be regarded as an extention of Bohr's equivalence theorem, which, in its original form, pertains to the value distribution of two Bohr-equivalent Dirichlet series in their common half-plane of absolute convergence. In the first part of section 3.4 we introduce a probability space $(W,\mathbb{B},m)$ which is to play a crucial role in the rest of the thesis. In the later half-of section 3.4, we show that each function in our class is asymptotically distributed like a random function defined on $(W,\mathbb{B},m)$ and canonically associated with its Dirichlet series. This result (theorem 3.4.5) immediately implies theorem 3.3.2; however, the proof of the former depends on the latter. Theorem 3.4.10 generalises theorem 3.4.5 by consideration of the joint asymptotic behaviour of members of our class. These results are applied to the functions $\zeta(z)$ and $\frac{\zeta(2z)}{\zeta(z)}$ to show that they have a common asymptotic distribution F in the half-plane $\Omega_{1/2}^{\infty}$ (theorem 3.4.11). We conclude the chapter by making some remarks and raising some questions regarding applications and generalisations of these results.

3.2 <u>Asymptotic distribution of points in $H(\Omega)$</u>: In this section we specialize the results of section 1.5 to the flow $H(\Omega)$, with special emphasis on points of $H(\Omega)$ represented by Dirichlet series.

3.2.1 <u>Proposition</u>: Let $-\infty < a < b \leq \infty$, $\Omega = \Omega_a^b$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $H(\Omega)$ and f be in $H(\Omega)$. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Borel probability measures on $H(\Omega)$ such that $f_n \Rightarrow \mu_n$ for each $n \geq 1$. We also assume that

(a) $\int_{-T}^{T} |f(x+it)|dt = O(T)$ as $T \longrightarrow \infty$, uniformly for x in compact subsets of (a,b),

and (b) $\limsup_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} |f_n(x+it) - f(x+it)|dt = o(1)$ as $n \longrightarrow \infty$, uniformly for x in compact subsets of (a,b).

<u>Then</u> there exists a Borel probability measure μ on $H(\Omega)$ such that $\mu_n \overset{D}{\Longrightarrow} \mu$ as $n \longrightarrow \infty$ and $f \rightrightarrows \mu$.

<u>Proof</u>: Arguing as in the later part of the proof of lemma 2.4.7, we can deduce from assumptions (a) and (b) that for each compact $K \subseteq \Omega$, we have:

$$\text{(c)} \quad \int_{-T}^{T} (\sup_{z \varepsilon K} |f(z+it)|)dt = O(T) \quad \text{as} \quad T \longrightarrow \infty,$$

and (d) $\limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (\sup_{z \varepsilon K} |f_n(z+it) - f(z+it)|)dt = o(1)$ as $n \longrightarrow \infty$.

Let K_n be a sequence of compact subsets of Ω increasing to Ω such that each K_n is contained in the interior of K_{n+1}. For each $g \varepsilon H(\Omega)$, let $a_n(g) = \sup_{z \varepsilon K_n} |g(z)|$, and $||g||_n = \frac{a_n(g)}{1+a_n(g)}$. Finally, for $g_1, g_2 \varepsilon H(\Omega)$, let $\rho(g_1, g_2) = \sum_{n=1}^{\infty} 2^{-n} ||g_1 - g_2||_n$. Then, as is well known and easily established, ρ is a metric on $H(\Omega)$ which induces the usual topology. Also, (d) implies:

$$\text{(e)} \quad \limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \rho(S^t f_n, S^t f)dt = o(1) \quad \text{as} \quad n \longrightarrow \infty.$$

Thus $\{f_n\}$, f satisfy the condition (a) of proposition 1.5.6., while (b) of 1.5.6 is now part of our hypothesis. Hence in order to complete the proof by an appeal to that proposition, we need only show that $\{\mu_n\}$ is relatively compact.

Let $\varepsilon > 0$. Let K_n be as above. For $n \geq 1$ and $T > 0$ let $X_{n,T}$ be the $H(\Omega)$-valued random element defined by $X_{n,T}(z) = f_n(z+iT\theta)$, $z \in \Omega$, where the random variable θ is uniformly distributed on $[-1,1]$. Let X_n be a $H(\Omega)$-valued random variable such that the distribution of X_n is μ_n. Since $f_n \Rightarrow \mu_n$, the remarks in 1.5.2 imply that

$X_{n,T} \xrightarrow{D} X_n$ as $T \longrightarrow \infty$ $(n \geq 1)$.

For each $n \geq 1$, let M_n' be given by :

$$\frac{M_n' \varepsilon}{2^n} = \sup_{m \geq 1} \limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (\sup_{z \in K_n} |f_m(z+it)|)dt ,$$

and let $M_n = \max(M_n', 1)$. Due to (c) and (d), $0 < M_n < \infty$.

An application of Chebychev's inequality ([39,p.11]) yields :

$$P(\sup_{z \in K_n} |X_{m,T}(z)| > M_n) \leq \frac{1}{M_n} \frac{1}{2T} \int_{-T}^{T} (\sup_{z \in K_n} |f_m(z+it)|)dt .$$

Hence, letting $T \longrightarrow \infty$,

$$\limsup_{T \to \infty} P(\sup_{z \in K_n} |X_{m,T}(z)| > M_n) \leq \frac{M_n' \varepsilon}{M_n 2^n} \leq \frac{\varepsilon}{2^n} .$$

Since the map from $H(\Omega)$ to $\mathbb{R}$ sending $g \in H(\Omega)$ to $\sup_{z \in K_n} |g(z)|$ is continuous, and since $X_{m,T} \xrightarrow{D} X_m$ as $T \longrightarrow \infty$, the theorem 5.1 in [4, pp.30-31] gives :

$$\sup_{z \in K_n} |X_{m,T}(z)| \xrightarrow{D} \sup_{z \in K_n} |X_m(z)| \quad \text{as} \quad T \longrightarrow \infty .$$

Hence, letting $T \longrightarrow \infty$ in the above inequality, we get :

$$P(\sup_{z\varepsilon K_n} |X_m(z)| > M_n) \leq \frac{\varepsilon}{2^n} \quad (n \geq 1) .$$

Therefore, if we put $C_\varepsilon = \{g \varepsilon H(\Omega): \sup_{z\varepsilon K_n} |g(z)| \leq M_n \text{ for all } n \geq 1\}$ then $P(X_m \varepsilon C_\varepsilon) \geq 1-\varepsilon$, i.e., $\mu_m(C_\varepsilon) \geq 1-\varepsilon$ for all $m \geq 1$. Since by Montel's theorem (theorem 2.9 in [18,p.149]) C_ε is compact, and as $\varepsilon > 0$ was arbitrary, this shows that $\{\mu_n : n \geq 1\}$ is tight and hence by Prohorov's theorem (theorem 6.1 in [4,p.37]), it is relatively compact in the topology of weak convergence. So we are done.

3.2.2 <u>Proposition</u> : Let $-\infty < a < b < \infty$. Let f be analytic and of finite order in the closure of Ω_a^∞, and suppose f is represented by an absolutely convergent Dirichlet series in Ω_b^∞. We also assume that $\int_{-T}^{T} |f(a+it)|dt = O(T)$ as $T \longrightarrow \infty$.

<u>Then</u> (i) $\int_{-T}^{T} |f(x+it)|dt = O(T)$, uniformly for x in compact subsets of (a,∞),

and (ii) There is a sequence $\{f_n\}$ in $H(\Omega_a^\infty)$ such that each f_n is represented by an absolutely convergent Dirichlet series on Ω_a^∞, and $\limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f_n(x+it) - f(x+it)|dt = o(1)$ as $n \longrightarrow \infty$, uniformly for x in compact subsets of (a,∞).

<u>Proof</u> : The proof of (i) is contained in the first part of the proof of lemma 2.4.7.

To prove (ii) Let's fix an $A > b-a$, and consider the sequence $\{\eta_n\}_{n=1}^{\infty}$ of analytic functions on Ω_{-A}^{A} given by :

$$\eta_n(w) = \frac{w}{A}\,\Gamma\left(\frac{w}{A}\right)n^w,$$

where "Γ" is Euler's Gamma-function. Well known properties of the Gamma-function imply that we have :

(a) $\eta_n(0) = 1$,

(b) $\eta_n(u+iv) = O(e^{-\alpha|v|})$ uniformly for $|v| \geq 1$, $|u| \leq A$ for some $\alpha > 0$,

(c) $\lim\limits_{n \to \infty} \int_{-\infty}^{\infty} |\eta_n(u+iv)|(1+|v|)dv = 0$, uniformly for u in compact subsets of $(-A, 0)$,

and (d) $\frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} k^{-w}\eta_n(w)\frac{dw}{w} = \exp(-(\frac{k}{n})^A)$ $(k \geq 1,\ n \geq 1)$.

Hence as in lemma 2.4.6, we obtain a decomposition $f = f_n + g_n$, where $f_n(z) = \sum_{k=1}^{\infty} a_k \exp(-(\frac{k}{n})^A)k^{-z}$, $z \in \Omega_a^{\infty}$,

and $g_n(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} f_a(y+v)\,\frac{\eta_n(a-x+iv)}{a-x+iv}\,dv$, $(z = x+iy,\ a < x < a+A)$.

Here the a_k's are determined by $f(z) = \sum_{k=1}^{\infty} a_k k^{-z}$ $(z \in \Omega_b^{\infty})$, so that the Dirichlet series for f_n is absolutely convergent in Ω_a^{∞} (in fact throughout $\mathbb{C}$).

Let's fix $0 < \varepsilon < A-(b-a)$. Since the series for f converges uniformly in $\Omega_{a+A-\varepsilon}^{\infty}$, clearly we have $\lim\limits_{n\to\infty} f_n(z) = f(z)$ uniformly for $z \in \Omega_{a+A-\varepsilon}^{\infty}$. Hence we have,

$$\limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f_n(x+it) - f(x+it)|dt = o(1) \quad \text{as} \quad n \to \infty,$$

uniformly for $x \geq a+A-\varepsilon$.

Hence it suffices to show that with arbitrarily small $\varepsilon > 0$, the same holds uniformly for $x \varepsilon [a+\varepsilon, a+A-\varepsilon]$. From the integral representation of $g_n = f - f_n$, we get,

uniformly for $x \varepsilon [a+\varepsilon, a+A-\varepsilon]$,

$$|f_n(x+it) - f(x+it)| \leq c_o \int_{-\infty}^{\infty} |f_a(t+v)||\eta_n(a-x+iv)|dv.$$

Hence,

$$\frac{1}{2T}\int_{-T}^{T}|f_n(x+it) - f(x+it)|dt \leq c_o \int_{-\infty}^{\infty}|\eta_n(a-x+iv)|\frac{1}{2T}\int_{-|v|-T}^{|v|+T}|f_a(t)|dtdv$$

$$\leq c_1 \int_{-\infty}^{\infty} |\eta_n(a-x+iv)|(1+\frac{|v|}{T})dv$$

(by assumption on f_a).

Hence for $T \geq 1$, we have,

$$\frac{1}{2T}\int_{-T}^{T} |f_n(x+it) - f(x+it)|dt \leq c_1 \sup_{u\varepsilon(-A+\varepsilon,-\varepsilon)} \int_{-\infty}^{\infty} |\eta_n(u+iv)|(1+|v|)dv.$$

(c) shows that the right hand side above goes to zero as $n \to \infty$, so we are done.

3.2.3 <u>Lemma</u> : Let $-\infty \leq a < \infty$. Let $f \varepsilon H(\Omega_a^\infty)$ be represented by a Dirichlet series which converges absolutely in Ω_a^∞. <u>Then</u> there is a Borel probability measure μ on $H(\Omega_a^\infty)$ such that

μ ...

Proof : Let f_n be the nth partial sum of the series for f $(n \geq 1)$. Then $f = \lim_{n \to \infty} f_n$, and the convergence is uniform on proper substrips of Ω_a^∞. Hence $\{f_n\}$, f satisfy the hypothesis (a) and (b) of proposition 3.2.1. Hence, if we knew that for each $n \geq 1$ there is a Borel probability measure μ_n on $H(\Omega_a^\infty)$ such that $f_n \Rightarrow \mu_n$, then the result would follow from proposition 3.2.1. But each f_n is a Dirichlet polynomial. Hence it suffices to prove the special case of our lemma in which f is a Dirichlet polynomial.

So let $f(z) = \sum_{n=1}^{N} a_n n^{-z}$, $z \in \Omega_a^\infty$. Let $p_1, \ldots, p_k$ be the distinct primes which divide $\prod_{\substack{n=1 \\ a_n \neq 0}}^{N} n$. Clearly there is a continuous function Q from the k-dimensional torus Π^k into $H(\Omega_a^\infty)$ such that $S^t(f) = Q(p_1^{it}, \ldots, p_k^{it})$. Hence, if θ is a random variable which is uniformly distributed on $[-1,1]$, and if the $H(\Omega_a^\infty)$-valued random elements X_T and the Π^k-valued random elements Y_T $(T > 0)$ are defined by $X_T = S(f, T\theta)$, $Y_T = (p_1^{iT\theta}, \ldots, p_k^{iT\theta})$, then $X_T = Q(Y_T)$. Therefore, if we can exhibit a Borel probability measure υ on Π^k such that $Y_T \xrightarrow{D} \upsilon$ as $T \to \infty$, then it will follow that $X_T \xrightarrow{D} \mu = \upsilon \circ Q^{-1}$ as $T \to \infty$. That is, $f \Rightarrow \mu$.

The dual of the compact topological group Π^k is Z^k, where $\underline{n} = (n_1, \ldots, n_k) \in Z^k$ acts on Π^k by :

$\underline{z} = (z_1, \ldots, z_k) \longrightarrow \underline{z}^{\underline{n}} = \prod_{j=1}^{k} z_j^{n_j}$ (see, for example, W. Rudin, [45, pp.13, 37]). Therefore the Fourier transform ϕ_T of Y_T is the map $\phi_T : Z^k \longrightarrow \mathbb{C}$ given by :

$$\phi_T(\underline{n}) = E(Y_T^{\underline{n}}) = \frac{1}{2T} \int_{-T}^{T} \left(\prod_{j=1}^{k} p_j^{in_j t} \right) dt .$$

Since $\{\log p_1, \ldots, \log p_k\}$ is linearly independent over Z by virtue of the unique factorisation theorem, we have, after a trivial computation, $\lim_{T \to \infty} \phi_T = \phi$ where $\phi : Z^k \longrightarrow \mathbb{C}$ is given by $\phi(\underline{n}) = 0$ if $\underline{n} \neq \underline{0}$, and $= 1$ if $\underline{n} = \underline{0}$. But ϕ is the Fourier transform of the Haar measure υ on Π^k. Hence it follows from the theory of Fourier analysis on compact groups (see Rudin [45]) that $Y_T \overset{D}{\Longrightarrow} \upsilon$. So we are done.

3.2.4 <u>Theorem</u> : Let $-\infty < a < b < \infty$. Let f be an analytic function of finite order on the closure of Ω_a^∞, such that f is represented by an absolutely convergent Dirichlet series in Ω_b^∞. We also assume that $\int_{-T}^{T} |f(a+it)| dt = O(T)$ as $T \longrightarrow \infty$. <u>Then</u> there is a Borel probability measure μ on $H(\Omega_a^\infty)$ such that $f \Longrightarrow \mu$.

<u>Proof</u> : Let's get hold of a sequence $\{f_n\}$ as guranteed by proposition 3.2.2. Since each f_n is given by an absolutely convergent Dirichlet series on Ω_a^∞, by lemma 3.2.3 there is a Borel probability measure μ_n on $H(\Omega_a^\infty)$ such that $f_n \Longrightarrow \mu_n$ $(n \geq 1)$. Since by proposition 3.2.2 $\{f_n\}$, f, $\{\mu_n\}$

satisfy all the hypotheses of proposition 3.2.1, it follows that there exists a Borel probability measure μ on $H(\overline{(\ )}_a^{\infty})$ such that $\mu_n \overset{D}{\Longrightarrow} \mu$ as $n \longrightarrow \infty$ and $f \Rightarrow \mu$.

3.3 <u>Bohr equivalence and asymptotic distribution</u> :

3.3.1 <u>Definitions</u> :

(i) $f : \mathbb{N} \longrightarrow \mathbb{C}$ is said to be <u>multiplicative</u> in case $f(mn) = f(m)f(n)$ whenever $m, n \in \mathbb{N}$ are mutually prime.

(ii) $f : \mathbb{N} \longrightarrow \mathbb{C}$ is said to be <u>completely multiplicative</u> in case $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

(iii) $f : \mathbb{N} \longrightarrow \mathbb{C}$ is said to be <u>unimodular</u> in case $|f(n)| = 1$ for all n; i.e., if we actually have $f : \mathbb{N} \longrightarrow \mathbb{T}$.

(iv) Two Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$ and $\sum_{n=1}^{\infty} b_n n^{-z}$ are said to be <u>Bohr-equivalent</u> in case there exists a completely multiplicative unimodular function w such that $b_n = a_n w(n)$ for all $n \geq 1$.

3.3.2 <u>Theorem</u> : Let $-\infty < a < b < \infty$. Let f and f^* be two analytic functions of finite order on the closure of $\overline{(\ )}_a^{\infty}$ such that they are represented by two Bohr-equivalent and absolutely convergent Dirichlet series on $\overline{(\ )}_b^{\infty}$. We also assume that

$$\int_{-T}^{T} |f(a+it)|\,dt = O(T), \quad \int_{-T}^{T} |f^*(a+it)|\,dt = O(T), \text{ as } T \longrightarrow \infty.$$

<u>Then</u> f and f^* have the same asymptotic distribution on $\overline{(\ )}_a^{\infty}$. That is, if μ and μ^* are the Borel probability measures

(whose existence is guranteed by theorem 3.2.4) on $H(\Omega_a^\infty)$ such that $f \Rightarrow \mu$ and $f^* \Rightarrow \mu^*$ on Ω_a^∞ then $\mu = \mu^*$.

Proof: In the first place, let f be given by a Dirichlet polynomial: $f(z) = \sum_{n=1}^{N} a_n n^{-z}$. Then so is f^*, and we have $f^*(z) = \sum_{n=1}^{N} a_n^* n^{-z}$, where $a_n^* = w(n)a_n$, and w is a completely multiplicative unimodular function (the definition (iv), 3.3.1). Let $A = \{p_1, \ldots, p_k\}$ be the set of primes p such that $p|n$ for some $n \in [1,N]$. Let $\alpha_1, \ldots, \alpha_k \in \Pi$ be given by $\alpha_j = w(p_j)$. Let Q, Q^* be the maps from Π^k to $H(\Omega_a^\infty)$ such that $S^t(f) = Q(p_1^{it}, \ldots, p_k^{it})$, $S^t(f^*) = Q^*(p_1^{it}, \ldots, p_k^{it})$. Then it is easily seen that $Q^* = Q \circ T$ where $T : \Pi^k \to \Pi^k$ is given by $T(\underline{z}) = \underline{\bar\alpha} \cdot \underline{z} = (\bar\alpha_1 z_1, \ldots, \bar\alpha_k z_k)$. Let υ be the Haar probability measure on Π^k. Then we saw, in the course of proving lemma 3.2.3, that $f \Rightarrow \mu$ and $f^* \Rightarrow \mu^*$ where $\mu = \upsilon \circ Q^{-1}$, $\mu^* = \upsilon \circ Q^{*-1} = \upsilon \circ (Q \circ T)^{-1} = (\upsilon \circ T^{-1}) \circ Q^{-1}$. Since υ is Haar measure and T is multiplication by a fixed element of Π^k, $\upsilon = \upsilon \circ T^{-1}$. Hence $\mu^* = \upsilon \circ Q^{-1} = \mu$, as was to be shown.

Next, let f, and hence also f^*, be given by absolutely convergent Dirichlet series on Ω_a^∞: $f(z) = \sum_{n=1}^{\infty} a_n n^{-z}$, $f^*(z) = \sum_{n=1}^{\infty} a_n^* n^{-z}$. Let $f_N(z) = \sum_{n=1}^{N} a_n n^{-z}$, $f_N^*(z) = \sum_{n=1}^{N} a_n^* n^{-z}$ $(N \geq 1)$. By theorem 3.2.4, for each $N \geq 1$ there exist Borel probability measures μ_N, μ_N^* on $H(\Omega_a^\infty)$ such that $f_N \Rightarrow \mu_N$ and $f_N^* \Rightarrow \mu_N^*$.

As $N \longrightarrow \infty$, $f_N \longrightarrow f$ and $f_N^* \longrightarrow f^*$ uniformly on proper substrips of Ω_a^∞. Hence by proposition 3.2.1, there exist Borel probability measures μ, μ^* on $H(\Omega_a^\infty)$ such that $\mu_N \xrightarrow{D} \mu$, $\mu_N^* \xrightarrow{D} \mu^*$ as $N \longrightarrow \infty$, and $f \Rightarrow \mu$, $f^* \Rightarrow \mu^*$ on Ω_a^∞. Since $\sum_{n=1}^{\infty} a_n n^{-z}$ and $\sum_{n=1}^{\infty} a_n^* n^{-z}$ are assumed to be Bohr-equivalent, it follows that for each $N \geq 1$, $\sum_{n=1}^{N} a_n n^{-z}$ and $\sum_{n=1}^{N} a_n^* n^{-z}$ are also Bohr-equivalent. Hence by the preceding paragraph, we have $\mu_N^* = \mu_N$ for each $N \geq 1$. Letting $N \longrightarrow \infty$, we obtain $\mu^* = \mu$.

Finally, let f, f^* satisfy the hypotheses of this theorem. Let's say $f(z) = \sum_{n=1}^{\infty} a_n n^{-z}$, $f^*(z) = \sum_{n=1}^{\infty} a_n^* n^{-z}$, $z \in \Omega_b^\infty$. For each $N \geq 1$, let's put $f_N(z) = \sum_{n=1}^{\infty} a_n \exp(-(\frac{n}{N})^A) n^{-z}$, and $f_N^*(z) = \sum_{n=1}^{\infty} a_n^* \exp(-(\frac{n}{N})^A) n^{-z}$, where $A > b-a$ is fixed. Since f_N, f_N^* are absolutely convergent Dirichlet series, we again have $f_N \Rightarrow \mu_N$, $f_N^* \Rightarrow \mu_N^*$ on Ω_a^∞. As in the proof of theorem 3.2.4, we have $\mu_N \xrightarrow{D} \mu$, $\mu_N^* \xrightarrow{D} \mu^*$ as $N \longrightarrow \infty$ and $f \Rightarrow \mu$, $f^* \Rightarrow \mu^*$ on Ω_a^∞. Since the Dirichlet series of f and f^* are assumed to be Bohr-equivalent, so are those of f_N and f_N^* for each $N \geq 1$. Since the latter are absolutely convergent, it follows from the preceding paragraph that $\mu_N^* = \mu_N$ for each $N \geq 1$. Hence letting $N \longrightarrow \infty$, we obtain $\mu^* = \mu$.

3.3.3 Remarks : Harald Bohr's equivalence theorem asserts that if f and f^* are Bohr-equivalent Dirichlet series, then the images under f and f^* of vertical lines lying inside the common half-plane of absolute convergence have the same closure (see Apostol [1,pp.174-184]). Although often formally correct, this theorem usually loses its significance when pushed beyond the half-plane of absolute convergence. This is because in the typical situation, the image of a vertical line outside this half-plane is dense in the entire plane. However, theorem 3.3.2 shows that in the subtler sense of asymptotic distribution, functions given by Bohr-equivalent Dirichlet series continue to behave similarly even beyond the region of absolute convergence. Thus, this theorem may be looked upon as an extension of Bohr's equivalence theorem.

3.4 The structure of the asymptotic distribution of a Dirichlet series :

3.4.1 Some definitions and notations : W will stand for the infinite-dimensional torus. We shall index its co-ordinate spaces by the set $\mathbb{P}$ of primes. That is, we write : $W = \prod_{p \in \mathbb{P}} \Pi_p$, when each $\Pi_p = \Pi$, the unit circle. With the product topology and pointwise multiplication, W is a compact Abelian topological group. Let $\mathbb{B}$ stand for its Borel σ-field. Let m stand for the normalized Haar measure on $(W, \mathbb{B})$. That is m is the unique Borel probability measure on W such that $m(\alpha A) = m(A)$ for all $\alpha \in W$, $A \in \mathbb{B}$. The probability space $(W, \mathbb{B}, m)$ will play a crucial role in the sequel.

Each $w \in W$ may be regarded as a function from $\mathbb{P}$ into $\mathbb{T}$. Notice that w has a unique extension to $\mathbb{N}$ as a completely multiplicative unimodular function. Namely, for $w \in W$, we put $w(n) = \prod_{p \in \mathbb{P}} (w(p))^{\alpha(p,n)}$, $n \in \mathbb{N}$. Here we have put $\alpha(p,n) = \max\{k \geq 1 : p^k | n\}$. On the other hand, the restriction of each completely multiplicative unimodular function to $\mathbb{P}$ is a member of W. Thus we may (and do) identify W with the set of all unimodular completely multiplicative functions. We shall say that a property holds for almost all completely multiplicative unimodular functions in case the set A of all such functions having the property in question is Borel, with $m(A) = 1$.

Indeed, a $w \in W$ may be extended to the set Q^+ of all positive rationals. Namely, if $r \in Q^+$, $r = \frac{m}{n}$ $(m, n \in \mathbb{N})$ then we put $w(r) = \frac{w(m)}{w(n)}$. Notice that this does not depend on the particular representation $\frac{m}{n}$ of r. For each $r \in Q^+$, let $\chi_r : W \longrightarrow \mathbb{T}$ be defined by $\chi_r(w) = w(r)$ $(w \in W)$. For each $r \in Q^+$, χ_r is a character (i.e. a continuous homomorphism into the circle group $\mathbb{T}$) of W. On the other hand, each character of W corresponds to a uniquely determined $r \in Q^+$. All these claims may easily be verified.

For $t \in \mathbb{R}$, let $\alpha_t \in W$ be defined by $\alpha_t = (p^{-it} : p \in \mathbb{P})$. Clearly $\{\alpha_t : t \in \mathbb{R}\}$ is a one-parameter subgroup of W. That is, $t \longrightarrow \alpha_t$ is a continuous homomorphism of the additive group $\mathbb{R}$ into W.

Let us define $U_t : W \longrightarrow W$ by $U_t(w) = \alpha_t w$ $(w \in W)$. Then $\{U_t : t \in \mathbb{R}\}$ is a one-parameter group of measurable transformations on W. Further, since m is invariant under translation by points in W, $\{U_t : t \in \mathbb{R}\}$ is measure preserving. That is, for $t \in \mathbb{R}$, $A \in \mathbb{B}$, $m(U_t(A)) = m(A)$.

3.4.2 <u>Lemma</u> : The one-parameter group $\{U_t : t \in \mathbb{R}\}$ of measure preserving transformations of $(W, \mathbb{B}, m)$ is ergodic. That is, if $A \in \mathbb{B}$ satisfies $m(A \triangle U^t A) = 0$ for all $t \in \mathbb{R}$ then $m(A) = 0$ or 1.

<u>Proof</u> : Let $A \in \mathbb{B}$ be as in the statement and let $f : W \longrightarrow \mathbb{R}$ be the indicator of A. That is, $f(w) = 1$ if $w \in A$ and $= 0$ otherwise. By assumption on A, we have $f(\alpha_t w) = f(w)$ for almost all (m) $w \in W$ $(t \in \mathbb{R})$.

Let χ be any nontrivial character of W and let $\hat{f}$ be the Fourier transform of f. By 3.4.1, $\chi = \chi_r$ for some $r \in Q^+$. Hence $\chi(\alpha_t) = r^{-it}$. Therefore we can choose $t \in \mathbb{R}$ such that $\chi(\alpha_t) \neq 1$. Let's fix such a t. Then we have,

$$\hat{f}(\chi) = \int_W \chi(w) f(w) dm(w) \quad \text{(by definition of } f)$$

$$= \int \chi(\alpha_t w) f(\alpha_t w) dm(w) \quad \text{(since } m \text{ is Haar measure)}$$

$$= \chi(\alpha_t) \int \chi(w) f(w) dm(w) \quad \text{(since } \chi(\alpha_t w) = \chi(\alpha_t)\chi(w) \text{ and } f(\alpha_t w) = f(w) \text{ almost surely)}$$

$$= \chi(\alpha_t)\, \hat{f}(\chi).$$

But by choice of t, $\chi(\alpha_t) \neq 1$. Hence $\hat{f}(\chi) = 0$ for all nontrivial characters χ. Hence if $c = \hat{f}(\chi_0)$ (χ_0 being the trivial character) then $\hat{f}(\chi) = c\,\hat{1} = \hat{c}$ (where 1 is the constant function identically equal to 1, and c is the function identically equal to c). Since an integrable function is uniqualy determined (upto a set of probability zero) by its Fourier transform, it follows that $f(w) = c$ for almost all (m) $w \in W$. Since f takes just two values 0 and 1, we must have $c = 0$ or $c = 1$. That is $f = 0$ almost surely or $f = 1$ almost surely. Hence $m(A) = 0$ or 1 as was to be shown.

3.4.3 <u>Lemma</u> : Let $\alpha > 0$, and let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n \leq N} |a_n|^2 = O(N^{2\alpha})$ as $N \longrightarrow \infty$. <u>Then</u> for almost all $w \in W$, the series $\sum_{n=1}^{\infty} a_n w(n) n^{-z}$ converges uniformly for z in compact subsets of $\mathbb{C}_\alpha^\infty$. In consequence, the map $w \longrightarrow X(\cdot, w)$, where $X(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z}$ $(z \in \mathbb{C}_\alpha^\infty)$ defines an $H(\mathbb{C}_\alpha^\infty)$-valued random element on the probability space $(W, \mathbb{B}, m)$.

<u>Proof</u> : Let's fix $x_o > \alpha$. For $n \geq 1$, let's put $X_n(w) = a_n w(n) n^{-x_o}$. Then $\{X_n\}$ is a sequence of complex-valued random variables. One easily checks that $\int w(h)\overline{w(k)}\,dm(w) = 1$ if $h = k$ and $= 0$ otherwise. Therefore $\{X_n\}$ is a sequence of pairwise orthogonal random variables (i.e., $E(X_h \overline{X_k}) = \int X_h \overline{X_k}\,dm = 0$ for $h \neq k$) with $E(|X_n|^2) = |a_n|^2 n^{-2x_o}$. Also, by assumption on the a_n's, we have :

$$\sum_{n=1}^{\infty} (\log n)^2 E(|X_n|^2) = \sum_{n=1}^{\infty} (\log n)^2 |a_n|^2 n^{-2x_o} < \infty .$$

Therefore, by Rademacher's Theorem ([39,p.458]), the series

$\sum_{n=1}^{\infty} X_n$ converges almost surely. That is, for almost all $w \in W$, $\sum_{n=1}^{\infty} a_n w(n) n^{-x_0}$ converges. But from the general theory of Dirichlet series ([28,p.3]) we know that if a Dirichlet series converges at a point $z = x_0$ then it converges, uniformly on compacta, in the half-plane $\Omega_{x_0}^{\infty}$. Hence for almost all w, $\sum a_n w(n) n^{-z}$ converges, uniformly on compacta, in the half-plane $\Omega_{x_0}^{\infty}$. But $x_0 > \alpha$ was arbitrary. Hence, taking $x_0 = \alpha + \frac{1}{k}$ $(k \geq 1)$, we see that if A_k is the set of all $w \in W$ for which the series $\sum_{n=1}^{\infty} a_n w(n) n^{-z}$ converges uniformly on compact subsets of $\{ Re(z) > \alpha + \frac{1}{k} \}$ then $m(A_k) = 1$. Let $A = \bigcap_{k=1}^{\infty} A_k$. Then $m(A) = 1$, and for $w \in A$, the series converges uniformly on compact subsets of Ω_{α}^{∞}.

Finally, since the sequence of partial sums of the series defining $X(\cdot,w)$ converges uniformly on compact subsets of Ω_{α}^{∞} for $w \in A$, and since each term in this sequence is $H(\Omega_{\alpha}^{\infty})$-valued, it follows that the random element X thus defined is $H(\Omega_{\alpha}^{\infty})$-valued on the set A of probability one.

3.4.4 <u>Lemma</u> : Let $\alpha > 0$, $\{a_n\}$ a sequence of complex numbers such that $\sum_{n=1}^{N} |a_n|^2 = O(N^{2\alpha})$ as $N \longrightarrow \infty$. Let X be the corresponding $H(\Omega_{\alpha}^{\infty})$-valued random element as in the statement of lemma 3.4.3. <u>Then</u> for each $\beta > \alpha$, we have, for almost all $w \in W$, $\int_{1}^{T} |X(\beta+it,w)|^2 dt = O(T)$ as $T \longrightarrow \infty$.

<u>Proof</u> : Let Y be the (real-valued) random variable defined on $(W, \mathbb{B}, m)$ by $Y(w) = |X(\beta,w)|^2$. We have $X(\beta,.) = \sum_{n=1}^{\infty} X_n$, where $X_n = a_n n^{-\beta} w(n)$. As noticed in the proof of lemma 3.4.3, the X_n's are pairwise orthogonal random variables with $E(|X_n|^2) = |a_n|^2 n^{-2\beta}$. And by assumption on the a_n's, $\sum_{n=1}^{\infty} |a_n|^2 n^{-2\beta} < \infty$. Hence, by Perseval's relation we have $E(|X(\beta,.)|^2) = \sum_{n=1}^{\infty} E(|X_n|^2) < \infty$. That is, we have $Y \geq 0$ almost surely and $E(Y) = \int Y dm < \infty$. Since by lemma 3.4.2 the one-parameter group U_t of measure-preserving transformations is ergodic, it follows from the individual ergodic theorem of Birkhoff ([19,p.151]) that almost surely $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Y o U_t dt = E(Y)$. Since clearly $Y \circ U_t = |X(\beta+it,w)|^2$, we are done.

3.4.5 <u>Theorem</u> : Let $a \in \mathbb{R}$ and let f be analytic and of finite order in the closure of Ω_a^{∞}. We also assume that $\int_{-T}^{T} |f(a+it)|^2 dt = O(T)$ as $T \to \infty$ and that for some $b > a$, f is given on Ω_b^{∞} by an absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$. Let X be the $H(\Omega_a^{\infty})$-valued random element defined on W by $X(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z}$ $(z \in \Omega_a^{\infty}, w \in W)$. Then $f \Rrightarrow X$ on Ω_a^{∞}.

<u>Proof</u> : Let $\alpha > a$ and let $a < \alpha' < \alpha$. Since by assumption we have $\int_{-T}^{T} |f(a+it)|^2 dt = O(T)$, it follows from the mean-value

theorem of Carlson ([49, p.304]) that $\sum_{n=1}^{\infty} |a_n|^2 n^{-2\alpha'} < \infty$. Hence $\sum_{n=1}^{N} |a_n|^2 = O(N^{2\alpha})$ as $N \to \infty$. Therefore the random series $\sum_{n=1}^{\infty} a_n w(n) n^{-z}$ defines (lemma 3.4.3) an $H(\Omega^{\infty}_{\alpha})$-valued random element X such that for any $\beta > \alpha$, $\int_{-T}^{T} |X(\beta+it,.)|^2 dt = O(T)$ almost surely (lemma 3.4.4). Since $\alpha > a$ was arbitrary it follows that X, as defined above, is indeed an $H(\Omega^{\infty}_{a})$-valued random element such that for any $\alpha > a$, $\int_{-T}^{T} |X(\alpha+it,w)|^2 dt = O(T)$ for almost all $w \in W$.

By theorem 3.2.4, there exists a Borel probability measure μ on $H(\Omega^{\infty}_{a})$ such that $f \Rightarrow \mu$ on Ω^{∞}_{a} (the assumption $\int_{-T}^{T} |f(a+it)|^2 dt = O(T)$ clearly implies $\int_{-T}^{T} |f(a+it)| dt = O(T)$, and therefore all the hypotheses of theorem 3.2.4 are satisfied). So we have only to show that $\mu = \mu_X$, the distribution of X.

For any $\alpha > a$, let $\phi_\alpha : H(\Omega^{\infty}_{a}) \to H(\Omega^{\infty}_{\alpha})$ be the map which sends any $h \in H(\Omega^{\infty}_{a})$ to its restriction to Ω^{∞}_{α}. Let $\mathbb{B}_\alpha$ be the Borel σ-field of $H(\Omega^{\infty}_{\alpha})$. Since ϕ_α is a flow homomorphism from $H(\Omega^{\infty}_{a})$ to $H(\Omega^{\infty}_{\alpha})$ ((d) of 2.2.4) and $f \Rightarrow \mu$ on Ω^{∞}_{a}, it follows that $\phi_\alpha(f) \Rightarrow \mu \circ \phi_\alpha^{-1}$. That is, $f \Rightarrow \mu \circ \phi_\alpha^{-1}$ on Ω^{∞}_{α}.

If $w_o \in W$ is such that

(i) $X(.,w_o)$ is analytic on Ω^{∞}_{a} and $\int_{-T}^{T} |X(\alpha+it,w_o)| dt = O(T)$

then, as $X(\cdot,w_o)$ is clearly Bohr-equivalent to f, it follows from theorem 3.3.2 that

(ii) $X(\cdot,w_o) \Rightarrow \mu \circ \phi_\alpha^{-1}$ on Ω_α^∞.

Hence for any $A \in \mathbb{B}_\alpha$ which is a $\mu \circ \phi_\alpha^{-1}$-continuity set (i.e., such that $\mu \circ \phi_\alpha^{-1}(\mathrm{bd}A) = 0$ where bdA is the topological boundary of A), we have

(iii) $\underline{d}(\{t \in \mathbb{R} : S^t(X(\cdot,w_o)) \in A\}) = \mu \circ \phi_\alpha^{-1}(A)$.

(This may be seen by combining remarks 1.5.2 with the portmanteau theorem of [4]). So let us fix an $A \in \mathbb{B}_\alpha$ which is a $\mu \circ \phi_\alpha^{-1}$-continuity set. Let us define a (real-valued) random variable Y on $(W, \mathbb{B}, m)$ by : $Y(w) = 1$ if $X(\cdot,w) \in A$ and $= 0$ otherwise. Clearly $E(Y) = \int Y dm = m(X(\cdot,w) \in A) = \mu_X \circ \phi_\alpha^{-1}(A) < \infty$. Hence by the Individual Ergodic theorem ([19,p.151]) and lemma 3.4.2, we get: $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Y \circ U_t dt = E(Y)$ almost surely. Since the right hand side equals $\mu_X \circ \phi_\alpha^{-1}(A)$ and the left hand side equals $\underline{d}(t \in \mathbb{R} : S^t(X(\cdot,w)) \in A)$, we find that for almost all $w_o \in W$,

(iv) $\underline{d}(\{t \in \mathbb{R} : S^t(X(\cdot,w_o)) \in A\}) = \mu_X \circ \phi_\alpha^{-1}(A)$.

Since the set of all $w_o \in W$ satisfying (i) also has probability one, and as the intersection of two sets of probability one again has probability one, the set of all $w_o \in W$ satisfying both (i) and (iv) (and hence also (iii)) has probability one. Since a set of probability one is a fortiori nonempty, we may choose $w_o \in W$ satisfying both (iii) and (iv). For such a choice of w_o, we may combine (iii) and (iv) to obtain :

(v) $\mu \circ \phi_\alpha^{-1}(A) = \mu_X \circ \phi_\alpha^{-1}(A)$, whenever $A \in \mathbb{B}_\alpha$ is a $\mu \circ \phi_\alpha^{-1}$ - continuity set in $\mathbb{B}_\alpha$.

Since such sets A generate the σ-field $\mathbb{B}_\alpha$ (for any Borel probability measure υ on a metric space, the υ-continuity sets may easily be seen to generate the entire Borel σ-field), it follows that (v) holds for all $A \in \mathbb{B}_\alpha$. But $\alpha > a$ was arbitrary. Hence, if we put $S = \bigcup_{\alpha > a} \phi_\alpha^{-1}(\mathbb{B}_\alpha)$, then μ and μ_X agree on S. Since the family S clearly generates the Borel σ-field $\mathbb{B}_a$ of $H(\Omega_a^\infty)$, it follows that $\mu = \mu_X$, as was to be shown.

3.4.6 <u>Corollary</u> : Let ϕ be a multiplicative function such that $|\phi(n)| \leq 1$ $(n \in \mathbb{N})$. Let $f(z) = \sum_{n=1}^{\infty} \phi(n) n^{-z}$, $z \in \Omega_1^\infty$, and we also assume that f has an analytic continuation of finite order to the closure of Ω_α^∞ for some $\alpha \geq \frac{1}{2}$ and it satisfies $\int_{-T}^{T} |f(\alpha+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$. Let X be the $H(\Omega_\alpha^\infty)$-valued random element defined on $(W, \mathbb{B}, m)$ by :

$$X(z,w) = \sum_{n=1}^{\infty} \phi(n) w(n) n^{-z} = \prod_{p \in \mathbb{P}} \left(\sum_{n=0}^{\infty} \phi(p^n)(w(p)p^{-z})^n \right).$$

<u>Then</u> $f \Longrightarrow X$ on Ω_α^∞.

<u>Proof</u> : This is a special case of theorem 3.4.5. We need only check that the product defining X is almost surely convergent, uniformly on compact subsets of $\Omega_{1/2}^\infty$, and it equals the infinite sum representation of X. Since on Ω_1^∞ both the series and the product converge absolutely (for any $w \in W$), the

equality on Ω_1^∞ can easily be established by rearrangement of the product; hence, once we show that the product converges almost surely, the equality will extend to $\Omega_{1/2}^\infty$ by analytic continuation (recall that the sum does converge almost surely on $\Omega_{1/2}^\infty$ in view of lemma 3.4.3).

For $p \in \mathbb{P}$ let's put $X_p(z,w) = \sum_{n=1}^{\infty} \phi(p^n)(w(p)p^{-z})^n$ and $Y_p(z,w) = \phi(p)w(p)p^{-z}$. Clearly the sum converges uniformly on $\Omega_{1/2}^\infty$ and hence X_p is an $H(\Omega_{1/2}^\infty)$-valued random element. Also $|X_p(z)| \leq \frac{1}{p^x-1}$, so that $\sum_p |X_p(z)|^2$ converge uniformly on compact subsets of $\Omega_{1/2}^\infty$. Hence, in order to establish almost sure convergence of the product $\prod_p (1+X_p(\cdot,w))$, it suffices to show that $\sum_p X_p$ converges almost surely. But $|X_p(z,w) - Y_p(z,w)| \leq \frac{1}{p^{2x}-p}$, so that $\sum_p |X_p - Y_p|$ converges uniformly on compact subsets of $\Omega_{1/2}^\infty$ for each $w \in W$. Hence we need only show that ΣY_p converges almost surely. But this is a series of independent summands, and since for $z \in \Omega_{1/2}^\infty$, $E(Y_p(z,\cdot)) = 0$ and $E(|Y_p(z,\cdot)|^2) \leq p^{-2x}$ so that $\sum_p E(|Y_p(z,\cdot)|^2) < \infty$, the almost sure convergence of $\sum_p Y_p(z,\cdot)$ for each fixed $z \in \Omega_{1/2}^\infty$ follows from Kolmogorov's three series criterion (see [39,p.237]). The almost sure convergence of the series ΣY_p of $H(\Omega_{1/2}^\infty)$-valued random elements may now be deduced as in the proof of lemma 3.4.3.

3.4.7 <u>Notation</u> : The $H(\Omega^{\infty}_{1/2})$-valued random element F is defined on $(W, \mathbb{B}, m)$ by :

$$F(z,w) = \sum_{n=1}^{\infty} w(n)n^{-z} = \prod_{p\in\mathbb{P}} (1-w(p)p^{-z})^{-1}, \ \mathrm{Re}(z) > \tfrac{1}{2}.$$

By lemma 3.4.3, the series converges on $\Omega^{\infty}_{1/2}$ for almost all $w \in W$ and hence defines an $H(\Omega^{\infty}_{1/2})$-valued random element. As in corollary 3.4.6, the product also converges almost surely and equals the sum.

3.4.8 <u>Corollary</u> : Let $w_o \in W$ be such that $F(\cdot,w_o)$ has an analytic continuation of finite order to the closure of Ω^{∞}_{a} for some $a \geq \frac{1}{2}$ and satisfies $\int_{-T}^{T} |F(a+it,w_o)|^2 dt = O(T)$ as $T \longrightarrow \infty$. <u>Then</u> $F(\cdot,w_o) \Longrightarrow F$ on Ω^{∞}_{a}.

<u>Proof</u> : By corollary 3.4.6 (with w_o in place of ϕ) we have $F(\cdot,w_o) \Longrightarrow X$ where X is the $H(\Omega^{\infty}_{a})$-valued random element defined on $(W, \mathbb{B}, m)$ by $X(\cdot,w) = F(\cdot,ww_o)$. But since m is invariant under multiplication by w_o, $F(\cdot,ww_o)$ has the same distribution as $F(\cdot,w)$. So we are done.

3.4.9 <u>Remarks</u> : Let $e,\delta \in W$ be defined by $e(p) = 1$, $\delta(p) = -1$ $(p \in \mathbb{P})$. Thus e is the identity of W and δ is Liouville's function. In the notation of 3.4.7, we have $F(z,e) = \zeta(z)$, $F(z,\delta) = \dfrac{\zeta(2z)}{\zeta(z)}$. Since ζ has a pole at $z = 1$, $w_o = e$ does not satisfy the hypothesis of corollary 3.4.8 for $\frac{1}{2} \leq a \leq 1$. Since it is not known whether Zeta has any zero in $\Omega^{1}_{1/2}$ or not, a priori we only know that $F(.,\delta)$ is meromorphic in $\Omega_{1/2}^{\infty}$, and so corollary 3.4.8 can not be used for $w_o = \delta$ either. However, both $z \to \zeta(z)$ and $z \to \frac{\zeta(2z)}{\zeta(z)}$ are in $M(\Omega_{1/2}^{\infty})$, and since the notion of asymptotic distribution has been introduced for any flow, it is meaningful to ask if these two functions, regarded as points in the continuous flow $M(\Omega_{1/2}^{\infty})$, have asymptotic distributions. The next two theorems answer this question in the affirmative and shows that they too are asymptotically distributed like the random element F of 3.4.7 (since $H(\Omega_{1/2}^{\infty})$ is a subspace of $M(\Omega_{1/2}^{\infty})$, F may be regarded as a $M(\Omega_{1/2}^{\infty})$-valued random element; so this statement makes sense). But before that we need a proposition on joint asymptotic distribution.

3.4.10 <u>Theorem</u>: Let $a \in \mathbb{R}$, and let $f_1,\dots,f_n$ be analytic and of finite order in the closure of Ω_a^{∞}. We also assume that $\int_{-T}^{T} |f_j(a+it)|^2 dt = O(T)$ as $T \to \infty$, and for some $b > a$, f_j is given by an absolutely convergent Dirichlet series $(1 \leq j \leq n)$. Let X_j be the random element such that $f_j \Longrightarrow X_j$ on Ω_a^{∞}. <u>Then</u> $(f_1,\dots,f_n) \Longrightarrow (X_1,\dots,X_n)$.

(The conclusion is to be interpreted to mean that $(f_1,\dots,f_n)$, regarded as a point in the product flow $H(\Omega)^n$, is asymptotically distributed like the $H(\Omega)^n$-valued random element $(X_1,\dots,X_n)$.).

<u>Proof</u>: Let θ be a random variable uniformly distributed on $[-1,1]$. For any $T > 0$, let $(X_{1,T},\dots,X_{n,T})$ be the $H(\Omega)^n$-

valued random element given by $X_{j,T} = S(f_j, T\theta)$. We have to show that $(X_{1,T},\dots,X_{n,T}) \overset{D}{\Longrightarrow} (X_1,\dots,X_n)$ as $T \longrightarrow \infty$.

Since $f_j \Longrightarrow X_j$, the net $\{X_{j,T} : T > 0\}$ converges in distribution (to X_j) as $T \longrightarrow \infty$. A fortiori, $\{X_{j,T} : T > 0\}$ is relatively compact and hence (as $H(\Omega)$ is complete separable) tight. Hence for any $\varepsilon > 0$ there is a compact set $A_j \ (\subseteq H(\Omega))$ such that $P(X_{j,T} \notin A_j) < \frac{\varepsilon}{n}$. Let us put $A = A_1 \times \dots \times A_n$. Then $A \ (\subseteq H(\Omega)^n$ is compact and

$$P((X_{1,T},\dots,X_{n,T}) \notin A) = P(\bigcup_{j=1}^{n} (X_{j,T} \notin A_j)) \leq \sum_{j=1}^{n} P(X_{j,T} \notin A_j) < \varepsilon$$

for all $T > 0$. Since $\varepsilon > 0$ was arbitrary, it follows that the net $\{(X_{1,T},\dots,X_{n,T}) : T > 0\}$ is tight. Therefore we have :

(i) $\{(X_{1,T},\dots,X_{n,T}) : T > 0\}$ is relatively compact.

Let $\{T_r : r \in \mathbb{N}\}$ be any sequence of positive reals such that $T_r \longrightarrow \infty$ and $(X_{1,T_r},\dots,X_{n,T_r})$ converges weakly as $r \longrightarrow \infty$. Let us say

(ii) $(X_{1,T_r},\dots,X_{n,T_r}) \overset{D}{\Longrightarrow} (Y_1,\dots,Y_n)$ as $r \longrightarrow \infty$.

Let $z_1,\dots,z_m$ be arbitrary points in $\Omega = \Omega_a^\infty$. Let us put $d = \min_{1 \leq k \leq m} \operatorname{Re}(z_k)$. Then $d > a$. Let $c = a - d < 0$. Then $\phi : H(\Omega)^n \longrightarrow H(\Omega_c^\infty)$ defined by

$$\phi((g_1,\dots,g_n))(z) = \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{jk} g_j(z_k + z), \quad z \in \Omega_c^\infty$$

is clearly a continuous function (Here α_{jk} are arbitrary complex numbers). Hence (ii implies

$\phi((X_{1,T_r},\dots,X_{n,T_r})) \overset{D}{\Longrightarrow} \phi((Y_1,\dots,Y_n))$ as $r \longrightarrow \infty$. That is, if we put $g = \phi((f_1,\dots,f_n))$ then

(iii) $S(g,T_r\theta) \overset{D}{\Longrightarrow} \phi(Y_1,\dots,Y_n)$ as $r \longrightarrow \infty$.

By assumption, on Ω_b^∞ we have the representation $f_j(z) = \sum_{r=1}^{\infty} a_{r,j} r^{-z}$ $(1 \le j \le n)$. Therefore, on Ω_e^∞ (where $e = c + b - a > c$) we have the representation

$$g(z) = \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{jk} f_j(z + z_k) = \sum_{r=1}^{\infty} a_r r^{-z} ,$$

where $a_r = \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{jk} a_{rj} r^{-z_k}$.

Also g is clearly analytic on the closure of Ω_c^∞, and in view of assumptions on $f_1,\dots,f_n$, it is of finite order and satisfies $\int_{-T}^{T} |g(c+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$. Therefore, by theorem 3.4.5, we have $g \rightrightarrows \sum_{r=1}^{\infty} a_r w(r) r^{-z} = \phi(X_1,\dots,X_n)$ (after substituting the expressions for a_r's and noting that by theorem 3.4.5, $X_j(z,w) = \sum_{r=1}^{\infty} a_{r,j} w(r) r^{-z}$). That is $S(g,T\theta) \overset{D}{\Longrightarrow} \phi(X_1,\dots,X_n)$ as $T \longrightarrow \infty$. In particular, we have :

(iv) $S(g,T_r\theta) \overset{D}{\Longrightarrow} \phi(X_1,\dots,X_n)$ as $r \longrightarrow \infty$.

Combining (iii) and (iv) we obtain : $\phi(X_1,\dots,X_n) \overset{D}{=} \phi(Y_1,\dots,Y_n)$. Since the map from $H(\Omega_c^\infty)$ to $\mathbb{C}$ which sends h to $h(0)$ is measurable, it follows that $\phi(X_1,\dots,X_n)(0) \overset{D}{=} \phi(Y_1,\dots,Y_n)(0)$. That is, substituting for ϕ, we have :

(v) $$\sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{jk} X_j(z_k) \stackrel{D}{=} \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{jk} Y_j(z_k),$$

for $\alpha_{j,k} \varepsilon \psi$, $z_k \varepsilon \Omega_a^\infty$ $(1 \leq j \leq n, 1 \leq k \leq m)$.

Since the hyperplanes in $\mathbb{C}^{mn}$ form a distribution determining class (see [4, p.15], and also the proof of theorem 7.7 in [4,p.49]) and since (v) holds for arbitrary complex coefficients α_{jk}, it follows that :

(vi) $\{X_j(z_k) : 1 \leq j \leq n, 1 \leq k \leq m\} \stackrel{D}{=} \{Y_j(z_k) : 1 \leq j \leq n, 1 \leq k \leq m\}$.

(Here both sides are regarded as $\mathbb{C}^{mn}$-valued random elements).

Now let K be any compact subset of Ω_a^∞, $\varepsilon > 0$, and let $g_1, \dots, g_n \varepsilon H(\Omega_a^\infty)$. Let us put :

(vii) $U = \{(h_1, \dots, h_n) \varepsilon H(\Omega)^n : \sup_{z \varepsilon K} |h_j(z) - g_j(z)| \leq \varepsilon, 1 \leq j \leq n\}$.

Let $\{z_m : m \geq 1\}$ be a dense sequence in K. For $m \geq 1$, let

$U_m = \{(h_1, \dots, h_n) \varepsilon H(\Omega)^n : |h_j(z_k) - g_j(z_k)| \leq \varepsilon, 1 \leq j \leq n, 1 \leq k \leq m\}$.

From (vi) it clearly follows that

(viii) $m((X_1, \dots, X_n) \varepsilon U_m) = P((Y_1, \dots, Y_n) \varepsilon U_m)$, $m \geq 1$.

Since $\{z_m : m \geq 1\}$ is chosen to be dense in K, $U_m \downarrow U$ as $m \Longrightarrow \infty$.

Hence, letting $m \Longrightarrow \infty$ in (viii), we obtain :

(ix) $m((X_1, \dots, X_n) \varepsilon U) = P((Y_1, \dots, Y_n) \varepsilon U)$,

for any set U of the form (vii).

Since the class of all sets U of the form (vii) is easily seen to be a distribution determining class, it follows from (ix) that

(x) $(X_1,\ldots,X_n) \overset{D}{=} (Y_1,\ldots,Y_n)$.

(ii) and (x) together show that

(xi) $(X_{1,T_r}, X_{2,T_r}, \ldots, X_{n,T_r}) \overset{D}{\Longrightarrow} (X_1, X_2, \ldots, X_n)$ as $r \longrightarrow \infty$ whenever $T_r \longrightarrow \infty$ and $(X_{1,T_r}, \ldots, X_{n,T_r})$ converges weakly as $r \longrightarrow \infty$.

(i) and (xi) together imply that $(X_{1,T}, \ldots, X_{n,T}) \overset{D}{\Longrightarrow} (X_1, \ldots, X_n)$ as $T \longrightarrow \infty$. That is, $(f_1,\ldots,f_n) \Rrightarrow (X_1,\ldots,X_n)$, as was to be shown.

3.4.11 <u>Theorem</u> : (i) $\zeta \Rrightarrow F$ on $\Omega^{\infty}_{1/2}$,

and (ii) if η is defined by $\eta(z) = \dfrac{\zeta(2z)}{\zeta(z)}$, then

$$\eta \Rrightarrow F \text{ on } \Omega^{\infty}_{1/2}.$$

<u>Proof</u> : Let $f_1, f_2, f_3 \in H(\Omega^{\infty}_{1/2})$ be defined by :

$$f_1(z) = 1 - 2^{1-z},\ f_2(z) = (1 - 2^{1-z})\cdot\zeta(z) \text{ and } f_3(z) = \zeta(2z).$$

Each of f_1, f_2, f_3 have a representation by an absolutely convergent Dirichlet series in Ω^{∞}_{1} ; namely,

$$f_1(z) = 1.1^{-z} + (-2).2^{-z},\ f_2(z) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-z},\ f_3(z) = \sum_{n=1}^{\infty} a_n n^{-z},$$

where $a_n = 1$ if n is a perfect square and $= 0$ otherwise. Also, for any $a > \frac{1}{2}$, $\int_{-T}^{T} |f_j(a+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$. This is trivial for $j = 1,3$, and follows from theorem 7.2 of [50,p.117] for $j = 2$. Hence by theorem 3.4.5, we have $f_j \Rrightarrow X_j$ on $\Omega^{\infty}_{1/2}$, where

$$X_1(z,w) = w(1).1^{-z} + (-2).w(2).2^{-z} = 1 - w(2)2^{1-z}$$

$$X_2(z,w) = \sum_{n=1}^{\infty} (-1)^{n-1} w(n) n^{-z} = \sum_{n=1}^{\infty} w(n)n^{-z} - 2\sum_{n=1}^{\infty} \frac{w(2n)}{(2n)^z}$$

$$= (1-w(2)2^{1-z})F(z,w) ,$$

$$X_3(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z} = \sum_{n=1}^{\infty} w(n^2)(n^2)^{-z}$$

$$= \sum_{n=1}^{\infty} w^2(n) n^{-2z} = F(2z,w^2) .$$

By theorem 3.4.10, it follows that $(f_1,f_2,f_3) \overset{\longrightarrow}{\longrightarrow} (X_1,X_2,X_3)$. If $\emptyset_1 , \emptyset_2 : H(\Omega^{\infty}_{1/2})^3 \longrightarrow M(\Omega^{\infty}_{1/2})$ are defined by $\emptyset_1(g_1,g_2,g_3)) = \frac{g_2}{g_1}$, $\emptyset_2((g_1,g_2,g_3)) = \frac{g_1 g_3}{g_2}$, then clearly $\emptyset_1,\emptyset_2$ are flow-homomorphisms. Therefore, $(f_1 f_2,f_3) \overset{\longrightarrow}{\longrightarrow} (X_1,X_2,X_3)$ implies that $\emptyset_j(f_1,f_2,f_3) \overset{\longrightarrow}{\longrightarrow} \emptyset_j(X_1,X_2,X_3)$ $(j=1,2)$. That is, $\zeta \overset{\longrightarrow}{\longrightarrow} \frac{X_2}{X_1}$, $\eta \overset{\longrightarrow}{\longrightarrow} \frac{X_1 X_3}{X_2}$. But, from the computations above, $\frac{X_2}{X_1} = F$, so that $\zeta \overset{\longrightarrow}{\longrightarrow} F$ on $\Omega^{\infty}_{1/2}$. Also,

$$\left(\frac{X_1 X_3}{X_2}\right)(z,w) = F(2z,w^2)/F(z,w)$$

$$= \prod_{p\in\mathbb{P}} (1-w^2(p)p^{-2z})^{-1} \Big/ \prod_{p\in\mathbb{P}} (1-w(p)p^{-z})^{-1}$$

$$= \prod_{p\in\mathbb{P}} (1+w(p)p^{-z})^{-1} = F(z,w\delta) ,$$

where $\delta \in W$ is Liouville's function (see 3.4.9). Since $\delta \in W$, and m is invariant under multiplication by points in W, we have $F(\cdot,w\delta) \overset{D}{=} F(\cdot,w)$. That is, $\frac{X_1 X_3}{X_2} \overset{D}{=} F$. Therefore $\eta \overset{\longrightarrow}{\longrightarrow} F$ on $\Omega^{\infty}_{1/2}$.

3.4.12 <u>Some applications and comments</u> :

(a) Theorem 3.4.11 may be used to deduce the asymptotic behaviour of almost any property of the Riemann Zeta function according to the following scheme. Let $\emptyset$ be a measurable function of $M(\Omega^{\infty}_{1/2})$ into some metric space S ; we also assume that if D is the set of discontinuity points of $\emptyset$ then $m(F \in D) = 0$. Then for any open set $U \subseteq S$, (the "convergence in distribution" interpretation of the notion of asymptotic distribution together with theorem 5.1 of [4,p.30] implies that) we have :

$$\underline{d}(\{t \in \mathbb{R} : \emptyset(S^t(\zeta)) \in U\}) \geq m(\emptyset(F) \in U).$$

Also, if $U \subseteq S$ is a Borel set such that $m(\emptyset(F) \in \mathrm{bd}U) = 0$ ($\mathrm{bd}U$ being the boundary of U) then

$$d(\{t \in \mathbb{R} : \emptyset(S^t(\zeta)) \in U\}) = m(\emptyset(F) \in U).$$

For example if Q is any rational function of n variables ($n \geq 1$) then this method yields

$$Q(\zeta, \zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(n-1)}) \Rightarrow Q(F, F^{(1)}, F^{(2)}, \ldots, F^{(n-1)}) \quad \text{on } \Omega^{\infty}_{1/2}.$$

We give some more examples of this method in (b) and (c) below.

(b) Taking $\emptyset : M(\Omega^{\infty}_{1/2}) \longrightarrow \mathbb{R}$ to be $\emptyset(f) = f(x)$ where $x > \frac{1}{2}$ is fixed, we get : if $I \subseteq \mathbb{C}$ is a Borel set such that $m(F(x,\cdot) \in \mathrm{bd}I) = 0$ then

(i) $d(\{t \in \mathbb{R} : \zeta(x+it) \in I\}) = m(F(x) \in I)$.

In [31] Jessen and Wintner established (in our terminology) that the distribution of $F(x)$ is absolutely continuous with respect to Lebesgue measure (in that paper they also study the density of this distribution with respect to Lebesgue measure). Therefore (i) holds whenever $\lambda(\mathrm{bd}I) = 0$.

(c) Let $\arg : \mathbb{C} \longrightarrow (-\pi, \pi]$ be the function that sends any point in $\mathbb{C}$ to the unique value of its argument that lies in $(-\pi, \pi]$. From the product representation of the random function F, we obtain, for $x > \frac{1}{2}$,

$$\arg(F(x)) = \sum_p \arg(1-w(p)p^{-x})^{-1} \quad \text{(addition modulo } 2\pi).$$

Let μ_x be the probability distribution of $\arg F(x)$. Then the Fourier series $\{\hat{\mu}_x(n) : n \in Z\}$ of μ_x is given by :

$$\hat{\mu}_x(n) = E(e^{2\pi i n \arg F(x)}) = \int_W e^{2\pi i n \arg F(x,w)} dm(w)$$

$$= \prod_{p \in \mathbb{P}} \int_W \exp(2\pi i n \arg((1-w(p)p^{-x})^{-1}))dm(w)$$

$$= \prod_{p \in \mathbb{P}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(2\pi i n \arg(1-e^{i\theta}p^{-x})^{-1})d\theta$$

since $w(p)$ is uniformly distributed on $\mathbb{T}$.

An involved computation yields that :

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2\pi i n \arg(1-e^{i\theta}p^{-x})^{-1}} d\theta = \sum_{k=0}^{\infty} \binom{-n/2}{k} \binom{n/2}{k} p^{-2kx}.$$

Therefore we obtain :

$$\text{(ii)} \quad \hat{\mu}_x(n) = \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} \binom{-n/2}{k}\binom{n/2}{k} p^{-2kx} \quad (n \in Z,\ x > \tfrac{1}{2}).$$

Since a probability measure on $(-\pi, \pi]$ is uniquely determined by the sequence of its Fourier coefficients, the formula (ii) uniquely determines μ_x. Comparing this formula with the formula in the main theorem of Elliott's paper [24], we see that μ_x is precisely the probability measure that Elliott obtains in connection with his study of the asymptotic behaviour of the argument of $L(x+iy,\chi)$ as χ varies over the Dirichlet characters modulo a prime p with $p \longrightarrow \infty$. The mystery of the reappearance of this probability distribution will be clarified in 4.6.2. In [24] Elliott shows that μ_x is a continuous distribution for each $x > \frac{1}{2}$. Hence $\mu_x(\{\pi\}) = 0$ so that the map " arg " is almost surely continuous. Hence we obtain : if $I \subseteq (-\pi, \pi]$ is a Borel set such that $\mu_x(\mathrm{bd}I) = 0$ (in particular if I is an interval) then

(iii) $d(\{t \in \mathbb{R} : \arg(\zeta(x+it)) \in I\}) = \mu_x(I), \; x > \frac{1}{2}$.

(d) From the functional equation for the Riemann Zeta function (see for example [23,p.14]) we obtain that for each $x < \frac{1}{2}$ $\zeta_x(t) = f_x(t)\zeta_{1-x}(t)$ where $|f_x(t)| \longrightarrow \infty$ as $t \longrightarrow \pm\infty$. Since for $x < \frac{1}{2}$ ζ_{1-x} has a non-degenerate asymptotic distribution, it follows that :

(iv) $d(\{t \in \mathbb{R} : |\zeta_x(t)| > M\}) = 1$ for any $x < \frac{1}{2}$ and $M > 0$.

Thus, for $x < \frac{1}{2}$, ζ_x does not have an asymptotic distribution. Or, we might say that ζ_x is asymptotically degenerate at infinity. The question of existence of asymptotic distribution of $\zeta_{1/2}$ is left open by this discussion.

If μ is normalised Lebesgue measure on $[-\pi, \pi]$, then $\hat{\mu}(n) = 1$ if $n = 0$, $= 0$ if $n \neq 0$ $(n \in Z)$. The formula (ii) above, coupled with an elementary estimate of the expression in its right hand side, shows that $\hat{\mu}_x(n) \longrightarrow \hat{\mu}(n)$ as $x \downarrow \frac{1}{2}$ for each $n \in Z$. Therefore $\mu_x \xrightarrow{D} \mu$ as $x \downarrow \frac{1}{2}$. This suggests, but does not establish, that $\arg \zeta_{1/2}$ is asymptotically uniformly distributed.

(e) The situation described in (iv) above is reflected in the easily proved fact that <u>almost surely (m) the random function F (of 3.4.7) has the line $\{Re(z) = \frac{1}{2}\}$ as its natural boundary.</u> Indeed, in view of the product representation of F, we can write $\log F = G_1 + G_2 + G_3$ where the random functions G_1, G_2 are given by

$$G_1(z,w) = \sum_{p \in \mathbb{P}} w(p)p^{-z}, \quad G_2(z,w) = \frac{1}{2} \sum_{p \in \mathbb{P}} w(p)p^{-2z} \quad (Re(z) > 1)$$

and G_3 is given by a Dirichlet series which converges absolutely (for every w) in $\Omega^{\infty}_{1/3}$, and hence defines an analytic function there. An application of Kolmogorov's three series test (see [39, p.237]) shows that almost surely, the series for G_2 converges, uniformly on compact, on $\Omega^{\infty}_{1/4}$. Hence $G_2 + G_3$ admits, almost surely, an analytic continuation to $\Omega^{\infty}_{1/3}$, and in order to prove our assertion regarding F, it suffices to show that $\{Re(z) = \frac{1}{2}\}$ is the almost sure natural boundary of G_1. Now let z_0 be an arbitrary point on $\{Re(z) = \frac{1}{2}\}$. Another application of the three series criterion shows $\Sigma\, w(p)p^{-z_0}$ does not " converge almost surely" and hence it " diverges almost surely"

Also the law of iterated logarithm ([39,p.260]) in conjunction with the prime number theorem ([50,p.49]) show that

$$\limsup_{n\to\infty} \left(\left|\sum_{p\leq x} w(p)\right|(2x(\log x)^{-1}\log\log x)^{-1/2}\right) = 1$$ for almost all w. A fortiori, $\sum_{p\leq x} w(p) = o(x^{1/2})$ as $x \to \infty$ for almost all w. Hence, by a Tauberian theorem due to Landau ([28,p.47]), z_0 is a singular point of G_1, and hence of log F, for almost all w. Since the family of sets of probability one is closed under countable union, it follows that if D is any dense countable subset of $\{Re(z) = \frac{1}{2}\}$, then almost surely, all the points of D are singular points of log F. Since the set of singular points is closed, $\{Re(z) = \frac{1}{2}\}$ is the almost sure natural boundary of log F, and hence of F.

This result, together with theorem 3.4.11 above, suggests that $\zeta_{1/2}$ does not have asymptotic distribution. A resolution of this question will presumably depend on a closer study of the behaviour of F near the natural boundary.

(f) The proof of theorem 3.4.5 depends critically on the fact that $\{\log n : n \in \mathbb{N}\}$ is a module over $\mathbb{N}$ and $\{\log p : p \in \mathbb{P}\}$ is a basis for it. For generalized Dirichlet series of the form $\sum a_n \lambda_n^{-z}$ $(\lambda_n \uparrow \infty)$, the analogue will involve fixing a basis for the module over $\mathbb{N}$ generated by $\{\log \lambda_n : n \geq 1\}$. The situation is particularly simple when $\{\log \lambda_n : n \geq 1\}$ is linearly independent over Z. In this case, if $\sum a_n \lambda_n^{-z}$ is absolutely convergent in a half-plane Π, then it is asymptotically distributed like the $H(\Omega)$-valued random element $Y(\cdot,\underline{u})$ given by $Y(z,\underline{u}) = \sum_{n=1}^{\infty} a_n u_n \lambda_n^{-z}$, where $\underline{u} = \{u_n\}$ is a sequence of independent random variables each uniformly distributed on $\mathbb{T}$. But even in this case the transition to mean bounded analytic functions given by a series of the form $\Sigma\, a_n \lambda_n^{-z}$ may not be feasible because of absence of a suitable integral representation analogous to that of proposition 3.2.2. But in case $\lambda_n = n+\alpha$, where $0 < \alpha < 1$, such a representation is easily obtained by a trivial modification of the formula in 3.2.2. Also, if α is transcendental then $\{\log(n+\alpha) : n \in \mathbb{N}\}$ **is linearly independent** over $\mathbb{Z}$. Thus, suitable modification of the arguments of this chapter yields the following result :

Let $\alpha \in (0,1)$ be transcendental. Let $\zeta(\cdot,\alpha)$ be the Hurwitz Zeta function with parameter α (i.e., $\zeta(z,\alpha)$ is given by the series $\sum_{n=1}^{\infty} (n+\alpha)^{-z}$ on Ω_1^{∞}, and then by analytic continuation to the entire complex plane except for a simple pole at $z = 1$). Let Y_α be the $H(\Omega_{1/2}^{\infty})$-valued random element given by $Y_\alpha(z,\underline{u}) = \sum_{n=1}^{\infty} u_n (n+\alpha)^{-z}$, where $\underline{u} = \{u_n\}$ is a sequence of independent random variables each uniformly distributed on $\mathbb{T}$. Then

(v) $\zeta(\cdot,\alpha) \Longrightarrow Y_\alpha$ on $\Omega_{1/2}^{\infty}$.

(The difficulty caused by the pole of $\zeta(\cdot,\alpha)$ is easily circumvented). When α is rational, $\zeta(\cdot,\alpha)$ is given by the ratio of two ordinary Dirichlet series, and in this case the asymptotic distribution of $\zeta(\cdot,\alpha)$ is deducible from theorem 3.4.10. It is the random element defined on $(W, \mathbb{B}, m)$ by $\Sigma\, w(n+\alpha)(n+\alpha)^{-z}$ (in the notation of 3.4.1). In case α is irrational algebraic, the structure of the associated random element is much more complicated.

3.4.13 Two questions :

(a) Does the function $\zeta_{1/2} \in C(\mathbb{R})$ (given by $\zeta_{1/2}(x) = \zeta(\frac{1}{2} + ix)$, $x \in \mathbb{R}$) possess an asymptotic distribution ?

(b) Is it true that $\arg \zeta_{1/2}$ is asymptotically uniformly distributed in the sense that whenever a Borel set $A \subset (-\pi, \pi]$ satisfies $\lambda(\mathrm{bd}A) = 0$, does it follow that :

$$d(\, t \in \mathbb{R} : \arg \zeta(\tfrac{1}{2} + it) \in A \,) = \frac{1}{2\pi}\, \lambda(A) \quad ?$$

CHAPTER 4

ASYMPTOTIC DISTRIBUTION OF POINTS IN THE FLOW $H(\Omega)$ (Discrete version).

4.1 <u>Introduction and summary</u> : The principal topic of this chapter is the asymptotic distribution modulo $h > 0$ of the analytic functions considered in the previous chapter. A curious find is that in general the asymptotic behaviour depends on the " unit of time" $h > 0$ chosen. In fact, there is a cocountable class of values of h (those of type I in the terminology of 4.2.1 below) for which the asymptotic behaviour modulo h is the same as the continuous asymptotic behaviour described in chapter 3. But, for the countably many exceptional values of h (those of type II), the asymptotic behaviour depends on the algebraic nature of h. The dependence, however, is very slight in the sense that in the description of the corresponding random elements, the contribution due to only finitely many primes is affected by the value of h. All this is contained in theorem 4.5.1.

From theorem 4.5.1 we deduce that in case h is of type I, the asymptotic distributions modulo h of two Bohr-equivalent Dirichlet series are again identical. However, when h is of type II, the Bohr-equivalence classes get partitioned into smaller equivalence classes (which we may call the Bohr-equivalence classes modulo h) such that the asymptotic distribution modulo h of functions in the same class are identical, but those in distinct classes have different asymptotic behaviour modulo h even if they are Bohr-equivalent. The Bohr-equivalence classes modulo h correspond to the cosets of the closed subgroups W_h (of the group W) introduced in 4.2.1. This phenomenon, which should be contrasted with the result of theorem 3.3.2, is described in 4.5.

The distinction between type I and type II occurs, in a veiled form, in Voronin's paper [51] (h is of type II if and only if the equation (14) of [51] is solvable.). However, there the distinction appears as a technical detail, and its fundamental impact on the discrete version asymptotics of the Zeta function does not seem to have been realized.

From theorem 4.5.1, it is easy to deduce a discrete analogue of theorem 3.4.10 on joint asymptotic distribution. We have omitted the deduction since it would be entirely parallel to that of chapter 3. From this result (theorem 4.5.7) we have deduced the asymptotic behaviour modulo h of the functions $\zeta(z)$ and $\frac{\zeta(2z)}{\zeta(z)}$. This is theorem 4.5.8.

We have obtained the principal theorem of this chapter (theorem 4.5.1) from a very general result, of a somewhat technical nature, on uniformly distributed sequences of finite subsets of W_h (proposition 4.4.1). We also use this result to prove the theorem 4.6.1 on the asymptotic behaviour of Dirichlet L-functions with large prime moduli. Here it is shown that as the prime modulus p tends to infinity, the $p-1$ function $L(\cdot, \chi)$, with χ modulo p, mimic the (continuous version) asymptotic behaviour of the Riemann Zeta function. This theorem may be used to deduce and even generalize a result of Elliott on the asymptotic behaviour of the arguments of the L-functions at a point in $\Omega^{\infty}_{1/2}$ as the prime modulus tends to infinity (see remarks in 4.6.2 (b) and (c)).

4.2 <u>The subgroups W_h</u> : In this section we introduce, corresponding to each $h \in \mathbb{R}^+$, a closed subgroup W_h of the topological group W of 3.4.1. We study canonical representations and the structure of the dual group of each of these subgroups. The groups W_h will play a central role in the study of asymptotic distribution modulo h of Dirichlet series.

4.2.1 <u>Standing notations</u> : In this chapter $h > 0$ will denote a fixed but arbitrary positive real. It is to serve as the " unit of time" . We shall say that h is of <u>type I</u> in case

(i) $\exp(\frac{2\pi n}{h})$ is irrational for each $n \in \mathbb{N}$.

We shall say that h is of <u>type II</u> in case (i) is not satisfied. Notice that all but countably many values of h are of type I.

If h is of type II then n_o will stand for the smallest integer $n \geq 1$ such that $\exp(\frac{2\pi n}{h})$ is rational. In this case, the positive integers r, s and integers $\theta(p)$ $(p \in \mathbb{P})$ will be given by :

(ii) $\exp(\frac{2\pi n_o}{h}) = \frac{r}{s} = \prod_{p \in \mathbb{P}} p^{\theta(p)}$ (r, s are relatively prime).

$\mathbb{P}_o$ will denote the finite subset of $\mathbb{P}$ given by :

(iii) $\mathbb{P}_o = \{p \in \mathbb{P} : \theta(p) \neq 0\}$.

In either case α_h will denote (as in 3.4.1) the member of W given by $\alpha_h(p) = p^{-ih}$ $(p \in \mathbb{P})$. W_h will be the closed subgroup of W generated by α_h; $\mathbb{B}_h$ the Borel σ-field of W_h, μ_h the Haar probability measure of W_h, μ_h will often be regarded as a probability on $(W, \mathbb{B})$ supported in W_h.

4.2.2 <u>Lemma</u> : If h is of type I then $W_h = W$. If h is of type II then $W_h = \{w \in W : w(r) = w(s)\}$.
(Here r, s are as in 4.2.1).

<u>Proof</u> : Let W^* be the dual of W. That is, W^* is the group of characters of W (i.e., continuous homomorphisms of W into $\mathbb{T}$) with pointwise multiplication. As remarked in 3.4.1, a typical element of W^* is of the form χ_x, $x = \frac{a}{b} \in Q^+$, where $\chi_x(w) = w(x) = \frac{w(a)}{w(b)}$ $(a, b$ positive integers). Look at the annihilator $W_h^{\perp}$ of W_h. That is, $W_h^{\perp}$ is the subgroup $\{\chi \in W^* : \chi(w) = 1$ for $w \in W_h\}$.

Clearly $\chi_x \in W_h$ if and only if $\chi_x(\alpha_h) = 1$, i.e., if and only if $x^{-ih} = 1$, so that $x = \exp(\frac{2\pi n}{h})$ for some integer n. If h is of type I, this implies $n = 0$ and hence $x = 1$. If h is of type II, we may put $n = \alpha n_o + \beta$ where $0 \leq \beta < n_o$, α, β are integers. By definition of n_o, $\exp(\frac{2\pi n_o}{h})$ is rational, and hence $\exp(\frac{2\pi \alpha n_o}{h})$ is also rational. Since $\exp(\frac{2\pi n}{h}) = \exp(\frac{2\pi \alpha n_o}{h}) \exp(\frac{2\pi\beta}{h})$ is rational, on division we find that $\exp(\frac{2\pi\beta}{h})$ is also rational. Since $0 \leq \beta < n_o$, by virtue of the minimality of n_o, we must have $\beta = 0$, and hence $n = \alpha n_o$, $x = (\frac{r}{s})^{\alpha}$, $\alpha \in Z$. On the other hand, if $x = (\frac{r}{s})^{\alpha}$ for some integer α then clearly $\chi_x(\alpha_h) = 1$. Thus we have shown : If h is of type I then $W_h^{\perp} = \{\chi_1\}$; if h is of type II then $W_h^{\perp} = \{\chi_{r/s}^{\alpha} : \alpha \in Z\}$.

Now let's look at the annihilator $W_h^{\perp\perp}$ of $W_h^{\perp}$. That is, $W_h^{\perp\perp} = \{w \in W : \chi(w) = 1 \text{ for all } \chi \in W_h^{\perp}\}$. In view of the structure of $W_h^{\perp}$ seen above, we clearly have :

If h is of type I then $W_h^{\perp\perp} = W$.

If h is of type II then

$$W_h^{\perp\perp} = \{w \in W : \chi_{r/s}(w) = 1\}$$
$$= \{w \in W : w(r) = w(s)\}.$$

Since each W_h is a closed subgroup of W, we have $W_h^{\perp\perp} = W_h$ (see lemma 2.13 of [45,p.36]). This observation completes the proof.

4.3 <u>Some random elements on $(W_h, \mathbb{B}_h, \mu_h)$ and the structure of their distributions</u> :

4.3.1 <u>Lemma</u> : Let $\alpha \in \mathbb{R}^+$, and let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n \leq N} |a_n|^2 = O(N^{2\alpha})$ as $N \longrightarrow \infty$. <u>Then</u> for almost all (μ_h) $w \in W_h$, the series $\sum_{n=1}^{\infty} a_n w(n) n^{-z} = X(z,w)$ converges uniformly for z in compact subsets of $\mathbb{C}_{\alpha}^{\infty}$. In consequence, the map $w \longrightarrow X(\cdot,w)$ defines an $H(\mathbb{C}_{\alpha}^{\infty})$-valued random element on the probability space $(W_h, \mathbb{B}_h, \mu_h)$.

<u>Proof</u> : If h is of type I then by proposition 4.2.2, we have $(W_h, \mathbb{B}_h, \mu_h) = (W, \mathbb{B}, m)$, so that the statement reduces to that of lemma 3.4.3. So let's suppose h is of type II. As in the proof of lemma 3.4.3, it suffices to show that for an arbitrarily fixed $x_o > \alpha$, the series $\sum_{n=1}^{\infty} a_n w(n) n^{-x_c}$, of random variables defined on $(W_h, \mathbb{B}_h, \mu_h)$ converges almost surely. Accordingly let X_n be the random variable defined on $(W_h, \mathbb{B}_h, \mu_h)$ by $X_n(w) = a_n w(n) n^{-x_o}$. We have to show that $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Let's define the random variable X_n^* on $(W, \mathbb{B}, m)$ by $X_n^*(w) = a_n w(n) n^{-x_o}$. As in lemma 3.4.3, we see that X_n^* is a sequence of pairwise orthogonal random variables such that $\sum_{n=1}^{\infty} (\log n)^2 E(|X_n^*|^2) < \infty$. Hence by a minor modification in the proof of Rademacher's theorem (as given in [39, pp.457-458]) shows that $\sum_{n=1}^{\infty} E(X_n^* | I)$ converges almost surely for any subσfield I of $\mathbb{B}$. But if we take $I = I_h$ to be the subσ-field of $\mathbb{B}$ generated by W_h, then clearly $\{X_n : n \geq 1\} \overset{D}{=} \{E(X_n^* | I) : n \geq 1\}$. Hence $\sum_{n=1}^{\infty} X_n$ converges almost surely and we are done.

4.3.2 <u>Notations</u> : We define a sequence $\{z_{n,h} : n \geq 1\}$ of Π-valued random variables (upto its joint distribution) as follows.

If h is of type I, then $\{z_{p,h} : p \in \mathbb{P}\}$ is a sequence of independent random variables each uniformly distributed on Π. If h is of type II, we fix $p \in \mathbb{P}_o$ and let $\{z_{p,h} : p \in \mathbb{P} - \{p_o\}\}$ be a sequence of independent random variables each uniformly distributed on Π. Further, given the sequence $\{z_{p,h} : p \in \mathbb{P} - \{p_o\}\}$, z_{p_o} assumes the $|\theta(p_o)|$ values

$$\exp\left[-\frac{1}{\theta(p_o)}(2\pi ik + \sum_{p \in \mathbb{P}_o - p_o} \theta(p) \log z_{p,h})\right], \quad 0 \leq k < |\theta(p_o)|$$

each with probability $\frac{1}{|\theta(p_o)|}$. Here $\theta(p)$'s are as in 4.2.1.

Notice that if $\{z_p : p \in \mathbb{P}\}$ is a sequence of independent random variables each uniformly distributed on Π, then in the latter case the distribution of $\{z_{p,h} : p \in \mathbb{P}\}$ specified above is precisely the conditional distribution of $\{z_p : p \in \mathbb{P}\}$ given the event $\prod_{p \in \mathbb{P}_o} z_p^{\theta(p)} = 1$. Thus it is independent of the particular choice of p_o. Finally if the prime factorization of $n \in \mathbb{N}$ is

$$n = \prod_{p \in \mathbb{P}} p^{\alpha(p)}, \text{ we put } z_{n,h} = \prod_{p \in \mathbb{P}} z_{p,h}^{\alpha(p)} .$$

4.3.3 <u>Lemma</u> : Let $\alpha > 0$, and a sequence of complex numbers $\{a_n\}$ such that $\sum_{n=1}^{N} |a_n|^2 = O(N^{2\alpha})$ as $N \longrightarrow \infty$.

If $X(.,.)$ is the $H(\underline{\bigcap}_\alpha^\infty)$-valued random element defined on $(W_h, \mathbb{B}_h, \mu_h)$ by $X(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z}$, and X_h is the random element given by $X_h(z) = \sum_{n=1}^{\infty} a_n z_{n,h} n^{-z}$ then $X \overset{D}{=} X_h$.

<u>Proof</u> : Here $\{z_{n,h} : n \geq 1\}$ is the sequence of Π-valued random variables defined in 4.3.2 above. By 4.3.1, the series defining X converges almost surely (μ_h) and hence defines $H((\underline{\ \ })^{\infty}_{\alpha})$-valued random element. Hence if $\{w(n) : n \in \mathbb{N}\}$, regarded as a sequence of random variables on $(W_h, \mathbb{B}_h, \mu_h)$, is shown to have the same distribution as $\{z_{n,h} : n \in \mathbb{N}\}$ then it will immediately follow that the series defining X_h also converges almost surely and that $X \stackrel{D}{=} X_h$. Since for $n = \prod p^{\alpha(p)}$, $w(n) = \prod w(p)^{\alpha(p)}$ and $z_{n,h} = \prod z_{p,h}^{\alpha(p)}$, it suffices to show that $\{w(p) : p \in \mathbb{P}\} \stackrel{D}{=} \{z_{p,h} : p \in \mathbb{P}\}$ when the left hand side is regarded as a sequence of random variables on $(W_h, \mathbb{B}_h, \mu_h)$.

If h is of type I, this is self-evident in view of lemma 4.2.2. So let's assume h is of type II.

Let $K_1 = \prod_{p \in \mathbb{P} - \{p_o\}} \Pi_p$, where $\Pi_p \equiv \Pi$. That is, K_1 is the infinite-dimensional torus with its co-ordinate spaces indexed by $\mathbb{P} - \{p_o\}$. Let K_2 be the multiplicative group of size $|\theta(p_o)|$ consisting of the $\theta(p_o)$-<u>th</u> roots of unity. To each sequence $\{z_p : p \in \mathbb{P} - \{p_o\}\}$ in K_1 and each ρ in K_2 there corresponds a unique w in W_h given by $w(p) = z_p$ for $p \in \mathbb{P} - \{p_o\}$ and $w(p_o) = \rho \exp(-\sum_{p \in \mathbb{P}_o - p_o} \frac{\theta(p)}{\theta(p_o)} \log z_p)$. This sets up an isomorphism (algebraic and topological) ψ between $K_1 \times K_2$ and W_h. The Haar probability measure on K_1 is the product of the uniform distributions on its co-ordinate circle groups, the Haar probability measure on K_2 assigns $1/|\theta(p_0)|$ to each of its points, and the Haar probability measure on $K_1 \times K_2$ is the product of these two probabilities. We can use the map ψ to transform this description of the Haar probability measure of $K_1 \times K_2$ into a description of the Haar probability measure μ_h of W_h. This gives the requisite description of the joint distributio of $w(p) : p \in \mathbb{P}$ under the probability measure μ_h – thus completi the proof.

4.4 Proposition on uniformly distributed sequences of finite subsets of W_h:

4.4.1 Proposition : Let $\alpha \in \mathbb{R}^+$, and let $\{a_n : n \in \mathbb{N}\}$ be a sequenc of complex numbers such that $\sum_{n=1}^{N} |a_n|^2 = O(N^{2\alpha})$ as $N \longrightarrow \infty$. For $w \in W$, $z \in \Omega^\infty_{\alpha+1/2}$, let

(i) $X(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z}$.

Let X_h be the $H(\Omega^\infty_\alpha)$-valued random element defined by

(ii) $X_h(z) = \sum_{n=1}^{\infty} a_n z_{n,h} n^{-z}$, $z \in \Omega^\infty_\alpha$,

where $\{z_{n,h} : n \in \mathbb{N}\}$ is as in 4.3.2.

Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of finite subsets of W_h such that for each $w \in \bigcup_{n=1}^{\infty} A_n$, $X(\cdot,w)$ has an analytic continuation to Ω^∞_α satisfying :

(iii) $\frac{1}{\#(A_n)} \sum_{w \in A_n} |X(x+iy,w)|^2 = O(|y|^A)$ as $y \longrightarrow \pm\infty$,

uniformly for $n \geq 1$ and x in compact subsets of (α,∞) (here A is some positive real).

and (iv) $\sum_{w \in A_n} |X(z,w)|^2 = O(\#(A_n))$ as $n \longrightarrow \infty$,

uniformly for z in compact subsets of $\bigcirc_\alpha^\infty$. We also assume that for any open set $U \subseteq W_h$,

(v) $\liminf_{n \to \infty} \frac{\#(U \cap A_n)}{\#(A_n)} \geq \mu_h(U)$.

Then it follows that for any open set $M \subseteq H(\bigcirc_\alpha^\infty)$,

(vi) $\liminf_{n \to \infty} \frac{1}{\#(A_n)} \#(\{w \in A_n : X(\cdot,w) \in M\}) \geq P(X_h \in M)$.

Proof : For $m \geq 1$, $n \geq 1$, let us define the $H(\bigcirc_\alpha^\infty)$-valued random elements X_m, Y_n, $X_{m,n}$ by : $X_m(z,w) = \sum_{k=1}^{m} a_k w(k) k^{-z}$, $w \in W_h$ (regarded as a random element on $(W_h, \mathbb{B}_h, \mu_h)$), Y_n is the random element taking the $\#(A_n)$ values $X(\cdot,w)$ $(w \in A_n)$, each with probability $\frac{1}{\#(A_n)}$, $X_{m,n}$ is the random element taking the $\#(A_n)$ values $X_m(\cdot,w)$ $(w \in A_n)$ each with probability $\frac{1}{\#(A_n)}$. In view of lemma 4.3.1, X_m converges almost surely to X (defined by (i) and regarded as a random element on $(W_h \mathbb{B}_h, \mu_h)$). Hence it follows that

(vii) $X_m \overset{D}{\Longrightarrow} X$ as $m \longrightarrow \infty$.

Let U_n be the W_h-valued random variable taking the $\#(A_n)$ values w $(w \in A_n)$ each with probability $\frac{1}{\#(A_n)}$. Then, in view of the portmanteau theorem, our hypothesis (v) may be rewritten as :

(viii) $U_n \xrightarrow{D} \mu_h$ as $n \longrightarrow \infty$.

Such the map from W_h into $H(\underline{C}_\alpha^\infty)$ sending w to $X_m(\cdot,w)$ is continuous for each fixed $m \geq 1$, in view of theorem 5.1 of [4,p.30] we can deduce from (viii) that $X_m(\cdot,U_n) \xrightarrow{D} \mu_h \circ X_m^{-1}$ as $n \longrightarrow \infty$. But $X_m(\cdot,U_n) \stackrel{D}{=} X_{m,n}$, and $\mu_h \circ X_m^{-1}$ is the distribution of X_m. Hence we obtain :

(ix) $X_{m,n} \xrightarrow{D} X_m$ as $n \longrightarrow \infty$.

The conclusion (vi) of our proposition may be rewritten as $Y_n \xrightarrow{D} X_h$ as $n \longrightarrow \infty$. Since by lemma 4.3.3, we have $X_h \stackrel{D}{=} X$, we need to show that :

(x) $Y_n \xrightarrow{D} X$ as $n \longrightarrow \infty$.

In view of theorem 4.2 in [4, p.25], (x) can be deduced from (vii) and (ix), provided we show that :

(xi) $\lim\limits_{n\to\infty} \limsup\limits_{n\to\infty} P(\sup\limits_{z\in K} |X_{m,n}(z) - Y_n(z)| > \varepsilon) = 0$

for each compact $K \subseteq \underline{C}_\alpha^\infty$ and $\varepsilon > 0$.

By virtue of Chebychev's inequality, (xi) would follow from

$$\lim_{n\to\infty} \limsup_{n\to\infty} E(\sup_{z\in K} |X_{m,n}(z) - Y_n(z)|^2) = 0.$$

Computing the expectation, we see that we need only show that

(xii) $\lim\limits_{m\to\infty} \limsup\limits_{n\to\infty} \frac{1}{\#(A_n)} \sum\limits_{w\in A_n} \sup\limits_{z\in K} |X(z,w) - X_m(z,w)|^2 = 0$

for each compact $K \subseteq \underline{C}_\alpha^\infty$.

Accordingly, let's fix a compact set $K \subseteq \underline{C}_\alpha^\infty$. Let $a > \alpha$ and $A > 0$ be such that $K \subseteq \underline{C}_a^{a+A}$. Let $\delta > 0$ be a small positive real. By assumption (iii) $X(\cdot,w)$ is of finite order in $\bigcap_a^\infty$; $X_m(\cdot,w)$, being a Dirichlet polynomial, is also of finite order there. Hence the same is true of $X-X_m$. Hence, arguing as in proof of proposition 3.2.2, we obtain the representation

$X(\cdot,w) - X_m(\cdot,w) = f_m(\cdot,w) + g_m(\cdot,w)$ $\quad (w \in \prod_{n=1}^{\infty} A_n)$ where

$$f_m(z,w) = \sum_{k=m+1}^{\infty} a_k w(k) \exp(-(k\delta)^A) k^{-z} \quad \text{and}$$

$$g_m(z,w) = -\frac{1}{2\pi i A} \int_{a-x-i\infty}^{a-x+i\infty} (X(z+v,w) - X_m(z+v,w)) \Gamma(\tfrac{v}{A}) \delta^{-v} dv \quad (x = \mathrm{Re}(z) \in (a, a+$$

Now $\frac{1}{\#(A_n)} \sum_{w \in A_n} |f_m(z,w)|^2 = \sum_{k_1=m+1}^{\infty} \sum_{k_2=m+1}^{\infty} a_{k_1} \bar{a}_{k_2} \exp(-(k_1\delta)^A - (k_2\delta)^A)$

$$\times \hat{U}_n(\chi_{k_1/k_2}).$$

And due to (viii) above, we have

$$\lim_{n \to \infty} \hat{U}_n(\chi_{k_1/k_2}) = \hat{\mu}_h(\chi_{k_1/k_2}) = 1 \quad \text{if} \quad \chi_{k_1/k_2} \in W_h^{\perp}$$
$$= 0 \quad \text{otherwise.}$$

This observation, together with the description of $W_h^{\perp}$ obtained in 4.2.2 yields :

$$\lim_{n \to \infty} \frac{1}{\#(A_n)} \sum_{w \in A_n} |f_m(z,w)|^2 = \sum_{k=m+1}^{\infty} |a_k|^2 \exp(-2(k\delta)^A) k^{-2x}$$

in case h is of type I. And we get a more complicated expression for the limit in case h is of type II.

But, in either case, it is easy to deduce that uniformly for $z \in K$, we have :

$$\lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{\#(A_n)} \sum_{w \in A_n} |f_m(z,w)|^2 = 0.$$

Therefore, $\lim\limits_{m \to \infty} \limsup\limits_{n \to \infty} \frac{1}{\#(A_n)} \sum\limits_{w \in A_n} \sup\limits_{z \in K} |X(z,w) - X_m(z,w)|^2$

$$\leq \lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{z \in K} |g_m(z,w)|^2 .$$

This holds for an arbitrary $\delta > 0$.

Now, the integral formula for g_m shows that :

$$\frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{z \in K} |g_m(z,w)|^2$$

$$\ll \delta^c \int_{-\infty}^{\infty} \Big(\frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{|y| \leq y_o} |X_m(a+i(y+v),w)|^2\Big) \rho(v)\,dv$$

$$+ \delta^c \int_{-\infty}^{\infty} \Big(\frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{|y| \leq y_o} |X(a+i(y+v),w)|^2\Big) \rho(v)\,dv ,$$

where $c > 0$, $y_o > 0$ are such that for $z = x+iy \in K$, $x > a+c$, $|y| \leq y_o$; $\rho(v) = \sup\limits_{x \in Re(K)} |\Gamma(\frac{a-x+iv}{A})|^2$. Thus $\rho(v)$ decreases exponentially as $v \to \infty$. Also, by assumption (iii), $\frac{1}{\#(A_n)} \sum\limits_{w \in A_n} \sup\limits_{|y| \leq y_o} |X(a+i(y+v),w|^2$ is at most of polynomial growth, uniformly in n. Hence the second integrand above decays exponentially as $v \to \infty$. Therefore we can take $V > 0$ so large that

$$\frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{z \in K} |g_m(z,w)|^2$$

$$\ll \delta^c \int_{-V}^{V} \Big(\frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{|y| \leq y_o} |X(a+i(y+v),w)|^2\Big)\,dv$$

$$+ \delta^c \int_{-\infty}^{\infty} \Big(\frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{|y| \leq y_o} |X_m(a+i(y+v),w)|^2 \rho(v)\,dv .$$

But by assumption (iv), the first integrand is $O(1)$ as $n \to \infty$;

also since X_m is a Dirichlet polynomial, we can easily obtain the estimate $\limsup_{n \to \infty} \frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{|y| \leq y_0} |X_m(a+i(y+v),w)|^2 = O(1)$ as $m \to \infty$, uniformly in v (this may be estimated exactly as we obtained above the estimate for a similar quantity with f_m in place of X_m by exploiting the absolute convergence of the Dirichlet series for f_m). Hence we get :

$$\limsup_{n \to \infty} \frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{z \in K} |g_m(z,w)|^2 \ll \delta^c \text{, uniformly for } m \geq 1 .$$

Combining this with our previously obtained estimate for f_m, we get (since $X - X_m = f_m + g_m$) :

$$\limsup_{m \to \infty} \limsup_{n \to \infty} \frac{1}{\#(A_n)} \sum_{w \in A_n} \sup_{z \in K} |X(z,w) - X_m(z,w)|^2 \ll \delta^c .$$

Since $\delta > 0$ was arbitrary, and the left hand side above does not depend on δ (only f_m, g_m were defined in terms of δ), letting $\delta \downarrow 0$ in this estimate we get (xii) and hence (xi). Thus we are done.

4.4.2 Remarks : In the sequel we shall deduce results on the asymptotic distribution modulo h of Dirichlet series (theorem 4.5.1 and its corollaries) from the proposition 4.4.1 above. A proof of theorem 4.5.1 could easily have been modelled on the proof of theorem 3.4.5. However, we have chosen the alternate course of deducing 4.5.1 from the rather involved result of 4.4.1 beacuse this allows us a unified treatment of the " asymptotic distribution modulo h " results and a result on the asymptotic behaviour of the sequence of Dirichlet L-functions with large prime moduli (theorem 4.6.1). Moreover, proposition 4.4.1 should prove useful in future studies of the asymptotic behaviour of Dirichlet series along more general sequences of vertical shifts.

4.5 Asymptotic distribution modulo h :

4.5.1 Theorem : Let $a \in \mathbb{R}$ and let f be analytic and of finite order in the closure of $\underline{(\,)}_a^\infty$. We also assume that $\int_{-T}^{T} |f(a+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$ and that for some $b > a$, f is given on $\underline{(\,)}_b^\infty$ by an absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$. Let X_h be the $H(\underline{(\,)}_a^\infty)$-valued random element defined on $\underline{(\,)}_a^\infty$ by $X_h(z) = \sum_{n=1}^{\infty} a_n z_{n,h} n^{-z}$. Then $f \Longrightarrow X_h$ mod h on $\underline{(\,)}_a^\infty$.

Proof : Let $\alpha > a$ be arbitrary. Then arguments as in the proof of theorem 3.4.5 shows that it sufficies to prove the result with α in place of a. These arguments also show that $\sum_{n=1}^{N} |a_n|^2 = O(N^{2\alpha})$ as $N \longrightarrow \infty$. Further, if we put $X(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z}$, then $X(z, \alpha_h^n) = f(z+inh)$, which has an analytic continuation to the closure of $\underline{(\,)}_\alpha^\infty$.

If we put $A_n = \{ \alpha_h^m : m \in Z \cap [-n,n] \}$, then clearly $A_n \subseteq W_h$, $\#(A_n) = 2n+1$. Further if χ is a nontrivial character on W_h then $\chi(\alpha_h) \neq 1$, so that

$$\frac{1}{2n+1} \sum_{w \in A_n} \chi(w) = \frac{1}{2n+1} \sum_{m=-n}^{n} (\chi(\alpha_h))^m \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

Hence by Weil's criterion, the W_h-valued random variable U_n taking the $2n+1$ values in A_n each with probability $\frac{1}{2n+1}$ goes to the Haar measure μ_h of W_h as $n \longrightarrow \infty$ (i.e., $U_n \overset{D}{\Rightarrow} \mu_h$ as $n \longrightarrow \infty$). Thus, in order to apply proposition 4.4.1, we need only verify that

(i) $\frac{1}{2n+1} \sum_{m=-n}^{n} |f(\beta+iy+imh)|^2 = O((|y|)^A)$ as $y \longrightarrow \infty$,

uniformly for $\beta \geq \alpha$,

and (ii) $\frac{1}{2n+1} \sum_{m=-n}^{n} |f(z+imh)|^2 = O(1)$ as $n \longrightarrow \infty$

uniformly for z in compact subsets of Ω_α^∞.

Applying Gallagher's lemma (lemma 1.4 of [42,p.3]) to the function $t \longrightarrow f(z+it)$, $t \in [-(n+1)h, (n+1)h]$, we obtain :

(iii) $$\sum_{m=-n}^{n} |f(z+imh)|^2 \leq \frac{1}{h} \int_{-(n+1)h}^{(n+1)h} |f(z+it)|^2 dt + \left(\int_{-(n+1)h}^{(n+1)h} |f(z+it)|^2 dt \right. \times \left. \int_{-(n+1)h}^{(n+1)h} |f'(z+it)|^2 dt \right)$$

In particular, taking $z = \beta+iy$ for $\beta \geq \alpha$, we get :

$$\frac{1}{2n+1} \sum_{m=-n}^{n} |f(\beta+iy+imh)|^2 \leq \frac{1}{(2n+1)h} \int_{-(n+1)h-|y|}^{(n+1)h+|y|} |f(\beta+it)|^2 dt + \left(\frac{1}{(2n+1)h} \int_{-(n+1)h-|y|}^{(n+1)h+|y|} |f(\beta+it)|^2 dt \right)^{\frac{1}{2}} \times \left(\frac{1}{(2n+1)h} \int_{-(n+1)h-|y|}^{(n+1)h+|y|} |f'(\beta+it)|^2 dt \right)^{\frac{1}{2}} .$$

But arguing as in proof of lemma 2.4.7 and utilising the Cauchy integral representation of f and f', one obtains

$$\int_{-T}^{T} |f(\beta+it)|^2 dt \leq CT, \quad \int_{-T}^{T} |f'(\beta+it)|^2 dt \leq CT \quad \text{as } T \longrightarrow \infty,$$

uniformly for $\beta \geq \alpha$. Hence the above inequality yields :

$$\frac{1}{(2n+1)} \sum_{m=-n}^{n} |f(\beta+iy+imh)|^2 \leq \frac{4C(n+1)h+|y|)}{(2n+1)h} \leq C_1|y| \quad \text{as} \quad y \longrightarrow \infty,$$

uniformly in $n \geq 1$ and $\beta \geq \alpha$. Thus we have (i) with $A = 1$.

But we can actually prove that

$$\int_{-T}^{T} |f(z+it)|^2 dt \leq CT, \int_{-T}^{T} |f'(z+it)|^2 dt \leq CT \quad \text{as } T \longrightarrow \infty,$$

uniformly for z in compact subsets of $\bigcap_{\alpha}^{\infty}$ (C depending on the compact set). This observation, together with the inequality (iii), readily implies (ii) also.

Thus the proposition 4.4.1 is applicable, and therefore we have, for any open set $M \subseteq \bigcap_{\alpha}^{\infty}$,

$$\liminf_{n \to \infty} \frac{1}{2n+1} \#(\{m \in Z \cap [-n,n] : S^{mh}(f) \in M\}) \geq P(X_h | \bigcap_{\alpha}^{\infty} \in M).$$

That is, $f \Longrightarrow X_h$ on $\bigcap_{\alpha}^{\infty}$. Since this holds for any $\alpha > a$, the required result follows.

4.5.2 <u>Corollary</u> : Let f be analytic and of finite order on the closure of $\bigcap_{a}^{\infty}$. Let us suppose f is given by an absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$ on $\bigcap_{b}^{\infty}$ for some $b > a$. We also assume that $\int_{-T}^{T} |f(a+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$. Let X be the $H(\bigcap_{a}^{\infty})$-valued random element defined on $(W, \mathbb{B}, m)$ by

$$X(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z}.$$

If h is of type I <u>then</u> $f \Longrightarrow X$ modulo h on $\bigcap_{a}^{\infty}$.

Proof : By theorem 4.5.1, $f \Rightarrow X_h$ modulo h on Ω_a^∞. But lemmas 4.3.1 and 4.3.2 together show that if h is of type I then $X_h \overset{D}{=} X$. Thus we are done. Notice that X is also the continuous asymptotic distribution of f (theore 3.4.5).

4.5.3 Corollary (Bohr's equivalence principle modulo h) : Let f, f^* be analytic functions of finite order on the closure of Ω_a^∞. Let's suppose that

$$\int_{-T}^{T} |f(a+it)|^2 dt = O(T), \quad \int_{-T}^{T} |f^*(a+it)|^2 dt = O(T).$$

We also assume that for some $b > a$, f and f^* are given by two absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$, $\sum_{n=1}^{\infty} a_n^* n^{-z}$ which are Bohr-equivalent through an $w_o \in W_h$. That is, there exists $w_o \in W_h$ for which $a_n^* = w_o(n) a_n$ $(n \in \mathbb{N})$.

Then f, f^* have identical asymptotic distributions modulo h.

Proof : Let X_h, X_h^* be the random elements corresponding to f, f^* as given by theorem 4.5.1. Let X, X^* be the $H(\Omega_a^\infty)$-valued random elements defined on $(W_h, \mathbb{B}_h, \mu_h)$ by

$$X(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z}, \quad X^*(z,w) = \sum_{n=1}^{\infty} a_n w(n) n^{-z}.$$

By lemma 4.3.2, we have $X \overset{D}{=} X_h$, $X^* \overset{D}{=} X_h^*$. Hence $f \Rightarrow X$ mod h and $f^* \Rightarrow X^*$ mod h on Ω_a^∞. But notice that X, X^* are connected by the formula $X^*(\cdot, w) = X(\cdot, w_o w)$. Since $w_o \in W_h$, the Haar measure μ_h of W_h is invariant under multiplication by w_o. Thus $X \overset{D}{=} X^*$. So we are done.

4.5.4 <u>Corollary</u> (to theorem 4.5.1) : Let $\emptyset$ be a multiplicative function such that $|\emptyset(n)| \leq 1$ $(n \in \mathbb{N})$. Let $f(z) = \sum_{n=1}^{\infty} \emptyset(n) n^{-z}$, $(\mathrm{Re}(z) > 1)$ and we assume that f has an analytic continuation of finite order to the closure of $(\overline{\ })_a^{\infty}$ for some $a \geq \frac{1}{2}$, and it satisfies $\int_{-T}^{T} |f(a+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$. Let X_h be the $H((\overline{\ })_a^{\infty})$-valued random element given by :

$$X_h(z) = \sum_{n=1}^{\infty} \emptyset(n) z_{n,h} n^{-z} = \prod_{p \in \mathbb{P}} \left(\sum_{n=o}^{\infty} \emptyset(p^n)(z_{p,h} p^{-z})^n \right)$$

<u>Then</u> $f \Longrightarrow X_h$ modulo h on $(\overline{\ })_a^{\infty}$.

<u>Proof</u> : Since by definition of the $z_{n,h}$'s, $\emptyset(n) z_{n,h}$ is a multiplicative function of n, this can be deduced from theorem 4.5.1 exactly as corollary 3.4.6 was deduced from theorem 3.4.5, once we show that the product over $\mathbb{P}$ converges almost surely. But the product over $\mathbb{P}$ may be decomposed into a product over the finite set $\mathbb{P}_o$ and a product over $\mathbb{P} - \mathbb{P}_o$. By the description of the joint distribution of $\{z_{p,h} : p \in \mathbb{P}\}$ (4.3.2), these two products are stochastically independent, so it suffices to prove the almost sure convergence of the two products separately. Since the first product is over a finite index set $\mathbb{P}_o$, its convergence is trivial. Since the joint distribution of $\{z_{p,h} : p \in \mathbb{P} - \mathbb{P}_o\}$ is the same as that of the sequence $\{w(p) : p \in \mathbb{P} - \mathbb{P}_o\}$ of independent and identically distributed random variables on $(W, \mathbb{B}, m)$, the almost sure convergence of the second product follows from that of

$\prod_{p \in \mathbb{P} - \mathbb{P}_o} \sum_{n=o}^{\infty} \emptyset(p^n)(w(p)p^{-z})^n$, and this last fact may be established by an appeal to Kolmogorov's three series criterion excatly as in 3.4.6. So we are done.

4.5.5 Notation : The $H(\Omega^{\infty}_{1/2})$-valued random element F_h is defined by $F_h(z) = \sum_{n=1}^{\infty} z_{n,h} n^{-z} = \prod_{p \in \mathbb{P}} (1-z_{p,h}p^{-z})^{-1}$.

By the preceding analysis, both the sum and the product converge almost surely on $\Omega^{\infty}_{1/2}$. As a particular case of 4.5.4, we get

4.5.6 Corollary : Let $w_o \in W$ be such that the function $F(z,w_o) = \sum_{n=1}^{\infty} w_o(n) n^{-z}$ has an analytic continuation of finite order to the closure of Ω^{∞}_{a} for some $a \geq \frac{1}{2}$ and it satisfies

$$\int_{-T}^{T} |F(a+it,w_o)|^2 dt = O(T) \quad \text{as} \quad T \longrightarrow \infty .$$

Then $F(\cdot,w_o) \Longrightarrow F_h$ modulo h on Ω^{∞}_{a}.

(Notice that $F_h \overset{D}{=} F$ if h is of type I).

4.5.7 Theorem : Let $f_1, f_2, \ldots, f_n$ be analytic functions of finite order in the closure of Ω^{∞}_{a}. We also assume that $\int_{-T}^{T} |f_j(a+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$ $(1 \leq j \leq n)$ and that for some $b > a$, f_j is given by an absolutely convergent Dirichlet series on Ω^{∞}_{b} $(1 \leq j \leq n)$. Let $X_{j,h}$ be the $H(\Omega^{\infty}_{a})$-valued random element such that $f_j \Longrightarrow X_{j,h}$ modulo h on Ω^{∞}_{a} $(1 \leq j \leq n)$.

Then $(f_1, \ldots, f_n) \Longrightarrow (X_{1,h}, \ldots, X_{n,h})$ modulo h .

Proof : This may be deduced from theorem 4.5.1 exactly as theorem 3.4.10 was deduced from theorem 3.4.5.

4.5.8 <u>Theorem</u> : (i) $\zeta \Longrightarrow F_h$ modulo h on $\Omega^{\infty}_{1/2}$,

and (ii) If η is defined by $\eta(z) = \frac{\zeta(2z)}{\zeta(z)}$, then

$$\eta \Longrightarrow F_h \text{ modulo } h \text{ on } \Omega^{\infty}_{1/2} .$$

<u>Proof</u> : This may be deduced from theorem 4.5.7 exactly as theorem 3.4.11 was deduced from theorem 3.4.10.

4.6 <u>Asymptotic behaviour of Dirichlet L-functions for large prime moduli</u> :

4.6.1 <u>Theorem</u> : Let M be any open subset of $H(\Omega^{\infty}_{1/2})$. Then

$$\liminf_{p \to \infty} \frac{1}{p} \#(\{\chi : \chi \text{ is a Dirichlet character mod } p \text{ and } L(\cdot,\chi) \in M\}) \geq m(F \in M).$$

(Here $p \longrightarrow \infty$ through the sequence of primes).

<u>Proof</u> : We shall apply proposition 4.4.1 with a fixed h of type I (say $h = 1$ or $h = 2\pi$), so that $F_h \overset{D}{=} F$. Corresponding to any Dirichlet character χ of prime modulus p, we define an element χ^* of W by :

$$\chi^*(q) = \chi(q) \text{ if } q \in \mathbb{P} - \{p\}, \quad \chi^*(p) = 1 .$$

Then, in the notation of 3.4.7, we have $L(z,\chi) = (1-p^{-z})F(z,\chi^*)$, for any Dirichlet character χ modulo p. Let $\{p_n : n \geq 1\}$ be the sequence of primes in increasing order. Let A_n be the set of the $p_n - 2$ non-principal Dirichlet characters modulo p_n. Let $A^*_n = \{\chi^* : \chi \in A_n\}$. Let X_n, X^*_n be the $H(\Omega^{\infty}_{1/2})$-valued random elements such that X_n takes the $p_n - 2$ values $L(\cdot,\chi), \chi \in A_n$, each with probability $\frac{1}{p_n-2}$; X_n^* takes the p_n-2 values $F(.,\chi^*)$, $\chi^* \varepsilon A_n^*$, each with probability $\frac{1}{p_n-2}$ $(n \geq 2)$. Let f_n be the point in $H(\Omega_{1/2}^{\infty})$ given by $f_n(z) = (1-p_n^{-z})$. Then we have $X_n = f_n X_n^*$. We need to show that $X_n \xRightarrow{D} F$ as $n \longrightarrow \infty$. Since $f_n \longrightarrow 1$ as $n \longrightarrow \infty$, it suffices to show that $X_n^* \xRightarrow{D} F$.

But "$X_n^* \xRightarrow{D} F$" follows from proposition 4.4.1 with $a_n \equiv 1$ and the sequence $\{A_n^* : n \geq 1\}$ in place of $\{A_n : n \geq 1\}$ (and h of type I as already mentioned, so that $W_h = W$), and $\alpha = 1/2$, once we check the subsidiary hypotheses. Specifically, we have to check that $\frac{1}{p_n-2} \sum_{\chi \varepsilon A_n} |F(a+iy,\chi^*)|^2 = O(|y|^A)$ as $y \longrightarrow \infty$, uniformly for $n \geq 2$ and a in compact subsets of $(1/2,\infty)$, and $\frac{1}{p_n-2} \sum_{\chi \varepsilon A_n} |F(z,\chi^*)|^2 = O(1)$ as $n \longrightarrow \infty$, uniformly for z in compact subsets of $\Omega_{1/2}^{\infty}$.

Since the functions f_n are uniformly bounded in $\Omega_{1/2}^{\infty}$, it suffices to show that $\frac{1}{p} \sum'_{\chi \bmod p} |L(a+iy,\chi)|^2 = O(|y|^A)$ as $y \longrightarrow \infty$, uniformly for primes $p \geq 3$, a in compact subsets of $(1/2,\infty)$, and that $\sum'_{\chi \bmod p} |L(z,\chi)|^2 = O(p)$ as $p \longrightarrow \infty$ through primes, uniformly for z in compact subsets of $\Omega_{1/2}^{\infty}$. Here Σ' denotes the sum over all non-principal characters modulo p.

Both of these estimates are classical results on L-functions. For example, they are trivially contained (modulo a kernel argument) in the very much more powerful theorem 10.1 of [42,p.75].

Thus it only remains to verify hypothesis (v) of proposition 4.4.1. In view of Weil's criterion for uniform distribution in compact groups, it suffices to show that for each non-trivial character χ on W (this should be carefully distinguished from Dirichlet characters), we have :

(i) $\frac{1}{\#(A_n^*)} \sum_{w \in A_n^*} \chi(w) \longrightarrow 0$ as $n \longrightarrow \infty$.

Since χ is a non-trivial chacter on W, by 3.4.1, there exist m_1, m_2 $\mathbb{N}$, $(m_1, m_2) = 1$, $m_1 \neq m_2$, such that $\chi = \chi_{m_1/m_2}$. That is, $\chi(w) = w(m_1/m_2) = w(m_1)\overline{w(m_2)}$ $(w \in W)$. Hence

$$\frac{1}{\#(A_n^*)} \sum_{w \in A_n^*} \chi(w) = \frac{1}{\#(A_n)} \sum_{w \in A_n^*} w(m_1)\overline{w(m_2)} = \frac{1}{p_n - 2} \sum_{\chi \bmod p_n}{}' \chi^*(m_1)\overline{\chi^*(m_2)}.$$

But if n is sufficiently large then $m_1 \not\equiv 0$, $m_2 \not\equiv 0$ and $m_1 \not\equiv m_2$ modulo p_n. Hence

$$\frac{1}{\#(A_n^*)} \sum_{w \in A_n^*} \chi_{m_1/m_2}(w) = \frac{1}{p_n - 2} \sum_{\chi \bmod p_n}{}' \chi(m_1)\overline{\chi(m_2)}$$

$$= -\frac{1}{p_n - 2} + \frac{1}{p_n - 2} \sum_{\chi \bmod p_n} \chi(m_1)\overline{\chi(m_2)}$$

$$= -\frac{1}{p_n - 2}$$ (by the orthogonality of the Dirichlet characters). Therefore (i) holds.

Thus proposition 4.4.1 is applicable, and we have $X_n^* \xrightarrow{D} F$ as $n \longrightarrow \infty$. Therefore $X_n \xrightarrow{D} F$ as $n \longrightarrow \infty$. That is, for any open set $M \subseteq H(\Omega^{\infty}_{1/2})$,

$$\liminf_{\substack{p \to \infty \\ p \in \mathbb{P}}} \frac{1}{p} \# (\{ \chi : \chi \text{ is a non-principal Dirichlet character modulo } p \text{ and } L(\cdot, \chi) \in M \}) \geq m(F \in M) .$$

Since to each modulus p there is exactly one principal character, this inequality remains unchanged even if the principal characters are allowed in.

4.6.2 <u>Remarks</u> : (a) A comparison of theorems 3.4.11 and 4.6.1 show that the asymptotic behaviour of the Dirichlet L-functions with large prime modulus duplicates the asymptotic behaviour of the Riemann Zeta function for large vertical shifts. At an informal level this similarity of asymptotic behaviour is well known (so much so that the literature abounds with references to "p-analogues" of classical estimates for the Zeta function).

(b) We have proved the theorem 4.6.1 for the sequence $\{p_n\}$ of primes. An examination of the proof shows that the theorem goes through if $\{p_n\}$ is replaced by a sequence $\{a_n\}$ of positive integers such that the least prime divisor of a_n goes to infinity with n.

(c) Theorem 4.6.2 implies that for any $z_o \in \Omega^{\infty}_{1/2}$, the asymptotic distribution of the sequence of sets

$$\arg L(z_o, \chi) : \chi \text{ a Dirichlet character mod } p$$

is given by the distribution of $\arg F(z_o)$. If $z_o = x_o + iy_o$, then $F(z_o, w) = F(x_o, \alpha_{y_o} . w) \overset{D}{=} F(x_o, w)$. Hence the asymptotic distribution is as of $\arg F(x_o)$. But the probability distribution of $\arg F(x_o)$ has been identified in (c) of 3.4.12 with the probability measure μ_{x_o} of Elliott. Hence we have, for any μ_{x_o}-continuity subset A of $[-\pi, \pi]$,

$$\lim_{\substack{p \to \infty \\ p \in \mathbb{P}}} \frac{1}{p} \# (\{\chi : \chi \text{ is a Dirichlet character modulo } p \text{ and } \arg L(z_o, \chi) \in A\}) = \mu_{x_o}(A),$$

for $z_o = x_o + iy_o$, $x_o > 1/2$.

This is the qualitative content of the theorem in [24]. The remark (b) above shows that this theorem of Elliott remains valid if the sequence of primes is replaced by the sequences $\{a_n\}$ of the sort described in (b).

CHAPTER 5

UNIVERSALITY THEOREMS AND STRONG RECURRENCE

5.1 Introduction and summary : It follows from the results of chapter 3 that the $\phi(k)$ L-functions modulo k have a joint asymptotic distribution. In this chapter we determine the spectrum of this joint distribution restricted to $\Omega_{1/2}^{1}$. The result is contained in theorem 5.3.1 ; the spectrum turns out to be very large -- as large as is consistent with the existence of Euler products for the L-functions. In 5.3.3 we reformulate this result as a theorem on simultaneous approximation of $\phi(k)$ non-vanishing continuous functions by translates of the $\phi(k)$ L-functions. It will be seen that thus viewed, the main result of this thesis is a generalization of Voronin's universality theorem ([52]) in several directions. In the first place, it is a multi-dimensional result unlike Voronin's theorem which considers approximation by a single L-function (or, more specifically, by the Zeta function alone). Secondly, the functions admitting such approximation are defined on a fairly large class of compact subsets of the strip $\left\{ \frac{1}{2} < \mathrm{Re}(z) < 1 \right\}$, whereas Voronin's theorem (and later generalizations of that theorem in [38], [43]) considers functions defined on compact discs alone. This is a genuine extension since an arbitrary compact subset of the strip can not be covered by a single disc contained in the strip, whereas arbitrarily good approximation in the function space requires approximation on arbitrarily large compact sets. Thirdly, unlike

Voronin's result, we prove more than mere existence of a translat of the L-functions which is a good approximation to the given functions -- we show that the set of translates realizing a given degree of approximation is a set of positive lower density. Extensions in this direction have been anticipated by Reich ([43]) -- although only in the case of a single L-function approximating a given nonvanishing analytic function on a disc.

The discrete version results of chapter 4 lead to a similar discrete version universality theorem (5.3.4). This implies, in particular, that the set of translates realizing a given degree of accuracy intersects any given arithmetic progression on the line. Thus it is a large set in more senses than one.

Several implications of these results have been considered in this chapter. One important example (5.3.5) shows that a large class of Dirichlet series which closely resemble the L-functions (but do not actually equal any L-function) have an even larger spectrum -- namely the whole of $H(\overline{\Omega}_{1/2}^{1})$. Thus <u>arbitrary analytic functions</u> (and not merely the nonvanishing ones) can be approximated by translates of any given member of this class. In 5.3.11 we use the theorem 4.6.1 to prove a universality theorem of a novel sort -- here we consider approximation by L-functions belonging to a large prime modulus (not by its translates !). We include in the section several remarks (5.3.9) designed to bring out the full power of the universality theorems, and by a couple of questions (5.3.10) that present themselves.

In section 5.4 we show that the results of section 5.3 imply a strange relationship between strong recurrence and zero-free strips of the L-functions. This result is contained in theorem 5.4.1 which says that an L-function (and in particular the Zeta function) is zero-free in a substrip of the strip $\{\frac{1}{2} < Re(z) < 1\}$ if and only if it is strongly recurrent in that substrip. In 5.4.4 and 5.4.5 we give several examples to show that such a relationship does not obtain in the absence of Euler products. In particular, the example in 5.4.5 (which is due to Titchmarsh) shows that the existence of a functional equation (analogous to that of the L-functions) for a Dirichlet series is consistent with the existence of lots of zeroes in the critical strip and outside the critical line. In 5.4.6 we show that the recurrence hypothesis of chapter 2 implies a very strong negation of the Riemann hypothesis for the Zeta function. (The deduction is dependent on the existence of a pole of the Zeta function, and therefore does not go through for L-functions with nonprincipal characters.). On the other hand, in 5.4.11 we show that an affirmative answer to the question posed in 2.3.7 would imply the generalized Riemann hypothesis for L-functions.

In section 5.2 we prove a number of lemmas preparatory to the proof of the main results in 5.3. Following Voronin, we base the proof of the universality theorem on a Hilbert space result. However, Voronin uses a theorem of Pecerskii on rearrangement of series in Hilbert spaces. This is found to be unsuitable for our purposes since Pecerskii's theorem is a real Hilbert space result. Instead, we have proved an alternative complex Hilbert space result (5.2.8) which is better adapted to our needs. Indeed, even the proof of corollary 5.3.6, which could be based on Pecerskii's theorem, is simpler when based on our proposition 5.2.8.

The other important ingredient in the proof of the universality theorems is a theorem of V. Berstein on the behaviour of an entire function of exponential type along a sequence. In 5.2.3 we have restated this theorem in a form which we found convenient.

5.2 Some preparatory lemmas :

5.2.1 Definitions and notations : Let $0 < \theta_0 \leq \pi$. Let's recall that a function analytic in the closed angular region $|\arg(z)| \leq \theta_0$ is said to be of exponential type in case

$$\limsup_{r \to \infty} \frac{\log|f(re^{i\theta})|}{r} < \quad , \text{ uniformly for } |\theta| \leq \theta_0 .$$

In this case the indicator function of f is the function $h : [-\theta_0, \theta_0] \to \mathbb{R}$ defined by :

$$h(\theta) = \limsup_{r \to \infty} \frac{\log|f(re^{i\theta})|}{r} , \quad \theta \in [-\theta_0, \theta_0] .$$

If K is a nonempty compact convex subset of the plane, the supporting function h_K of K is defined by :

$$h_K(\theta) = \max_{z \in K} \operatorname{Re}(e^{-i\theta}.z) , \quad |\theta| \leq \pi .$$

If f is an entire function of exponential type, given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$, then the Borel transform F of f is defined by $F(z) = \sum_{n=0}^{\infty} n!\, a_n z^{n-1}$. F is analytic except in a neighbourhood of zero. The <u>conjugate indicator diagram</u> of a function f of exponential type is defined to be the closed convex hull of the set of singularity points of the Borel transform of f. Thus, if $f \not\equiv 0$, then its conjugate indicator diagram is a nonempty compact convex set. If D is the conjugate indicator diagram of f, then its <u>indicator diagram</u> is by definition the set $D^* = \{\bar{z} : z \in D\}$. <u>The indicator function of an entire function of exponential type equals the supporting function of its indicator diagram</u> (theorem 5.3.7 of [5,p.74]). In particular, if the conjugate indicator diagram of an entire function $f \not\equiv 0$ of exponential type is contained in Ω_a^∞ $(a \in \mathbb{R})$

then $\limsup\limits_{\substack{x \to \infty \\ x \in \mathbb{R}}} \frac{\log|f(x)|}{x} > a$.

5.2.2 <u>Lemma</u> : Let μ be a complex Borel measure on the plane with compact support contained in Ω_a^∞. Let f be given by : $f(z) = \int e^{sz} d\mu(z)$, $z \in \mathbb{C}$. Let's assume that $f \not\equiv 0$.

<u>Then</u> $\limsup\limits_{\substack{x \to \infty \\ x \in \mathbb{R}}} \frac{\log|f(x)|}{x} > a$.

<u>Proof</u> : Clearly f is an entire function of exponential type. A simple computation shows that its Borel transform is given by $F(z) = \int \frac{d\mu(s)}{z-s}$ for z outside the support of μ. Therefore the conjugate indicator diagram of f is contained in the convex hull of the support of μ, and hence is contained in Ω_a^∞. Since by

assumption $f \not\equiv 0$, the concluding remark of 5.2.1 now completes the proof.

5.2.3 Lemma (V. Bernstein) : Let f be an entire function of exponential type. Let $\{\lambda_n : n \geq 1\}$ be a sequence of complex numbers. Let α, β, δ be positive real numbers such that :

(i) $\limsup\limits_{\substack{y \to \infty \\ y \in \mathbb{R}}} \dfrac{\log|f(\pm iy)|}{y} \leq \alpha$,

(ii) $|\lambda_m - \lambda_n| \geq \delta|m-n| \quad (m, n \in \mathbb{N})$,

(iii) $\lim\limits_{n \to \infty} \dfrac{\lambda_n}{n} = \beta$,

(iv) $\alpha\beta < \pi$.

Then $$\limsup_{\substack{n \to \infty \\ n \in \mathbb{N}}} \frac{\log|f(\lambda_n)|}{|\lambda_n|} = \limsup_{\substack{x \to \infty \\ x \in \mathbb{R}}} \frac{\log|f(x)|}{x}.$$

Proof : In the first place, let's suppose $\beta = 1$. Therefore $0 < \alpha < \pi$. Let h be the indicator function of f. By hypothesis (i), $h(\frac{\pi}{2}) \leq \alpha$; $h(-\frac{\pi}{2}) \leq \alpha$. Hence, by theorem 5.1.2 of [5,p.66], $h(\theta) \leq h(0)\cos\theta + \alpha|\sin\theta|$ for $|\theta| \leq \frac{\pi}{2}$; $\alpha < \pi$. Also, as $\beta = 1$, we have $\lim\limits_{n \to \infty} \dfrac{\lambda_n}{n} = 1$. Hence by V. Bernstein's theorem, we have the stated conclusion in this case (see [5,p.185] and also the remarks in 5.2.4 below).

In order to deduce the general result, let's define the entire function g by : $g(z) = f(\beta z)$; let $\{\lambda_n'\}$ be the sequence given by $\lambda_n' = \dfrac{\lambda_n}{\beta}$. Then clearly g is of exponential type, and we have $\limsup_{\substack{y \to \infty \\ y \in \mathbb{R}}} \frac{\log|g(\pm iy)|}{y} \leq \alpha' = \alpha\beta < \pi$. Also $|\lambda'_m - \lambda'_n| \geq \delta' |m-n|$ with $\delta' = \frac{\delta}{\beta} > 0$, and $\lim_{n \to \infty} \frac{\lambda'_n}{n} = 1$. Hence by the case $\beta = 1$ proved above, we have : $\limsup_{\substack{n \to \infty \\ n \in \mathbb{N}}} \frac{\log|g(\lambda'_n)|}{|\lambda'_n|} = \limsup_{\substack{x \to \infty \\ x \in \mathbb{R}}} \frac{\log|g(x)|}{x}$.

Hence the required equality follows on substituting for g and λ'_n.

5.2.4 Remarks : In [5, p.185], Boas quotes V. Bernstein's theorem as follows :- " Let f be regular in $|\arg z| \leq \alpha \leq \frac{\pi}{2}$, and let $h(\theta) \leq a \cos\theta + b|\sin\theta|$, $|\theta| \leq \alpha$, where a, b are finite. If $b < \pi$, we have

$$\limsup_{n \to \infty} |\lambda_n|^{-1} \log|f(\lambda_n)| = \limsup_{r \to \infty} r^{-1} \log|f(r)| = h(0),$$ provided that $\{\lambda_n\}$ is a (complex) sequence such that $n/\lambda_n \to 1$ and $|\lambda_m - \lambda_n| \geq \delta|n-m|$, $n \neq m$, $\delta > 0$."

This is false. A counter example is given by $f(z) = \sin(\pi z)$. $\alpha = \frac{\pi}{3}$, $a = 2$, $b = 0$, $\lambda_n = n$, $\delta = 1$. An examination of the proof shows that the theorem has been rigorously established under the extra hypothesis that $h(0) = a = 0$, and it has been wrongly asserted that this results in no loss of generality. In actuality, from this particular case we can deduce the general theorem provided the hypothesis " $h(\theta) \leq a \cos\theta + b \sin\theta$ " is replaced by " $h(\theta) \leq h(0) \cos\theta + b \sin\theta$ " (then we may apply the particular case to $f(z)e^{-h(0)z}$). This is the rectified version of the theorem that we have used in 5.2.3 above.

5.2.5 <u>Lemma</u> : Let f be an entire function of exponential type. Let h,k be two relatively prime positive integers. We assume that $\limsup_{x \to \infty} \frac{\log|f(x)|}{x} > -1$. <u>Then</u> $\sum_{\substack{p \equiv h \bmod k \\ p \in \mathbb{P}}} |f(\log p)| = \infty$.

<u>Proof</u> : Since f is of exponential type, we may choose a finite $\alpha > 0$ such that $\limsup_{y \to \infty} \frac{\log|f(\pm iy)|}{y} \leq \alpha$. Let's now choose $\beta > 0$ so small that $\alpha\beta < \pi$. Let's <u>suppose</u> we have :

$$\sum_{\substack{p \equiv h \bmod k \\ p \in \mathbb{P}}} |f(\log p)| < \infty .$$

Let $A = \{n \in \mathbb{N} : \exists\, x \in ((n-\frac{1}{4})\beta,\ (n+\frac{1}{4})\beta) \text{ with } |f(x)| \leq e^{-x}\}$.

Then
$$\sum_{\substack{p \equiv h \bmod k \\ p \in \mathbb{P}}} |f(\log p)| \geq \sum_{n \notin A} \sum_{\substack{p \equiv h \bmod k \\ p \in \mathbb{P} \\ \log p \in ((n-\frac{1}{4})\beta, (n+\frac{1}{4})\beta)}} |f(\log p)|$$

$$\geq \sum_{n \notin A} \sum_{\substack{p \equiv h \bmod k \\ p \in \mathbb{P} \\ \log p \in ((n-\frac{1}{4})\beta, (n+\frac{1}{4})\beta)}} \frac{1}{p} \quad \text{(by definition of } A).$$

Now, the well known estimate

$$\sum_{\substack{p \leq x \\ p \equiv h \bmod k}} \frac{1}{p} = \frac{1}{\emptyset(k)} \log\log x + c + O((\log x)^{-2}) \quad \text{as } x \to \infty$$

shows that
$$\sum_{\substack{p \equiv h \bmod k \\ p \in \mathbb{P} \\ \log p \in ((n-\frac{1}{4})\beta, (n+\frac{1}{4})\beta)}} \frac{1}{p} = \frac{1}{2\emptyset(k)} \frac{1}{n} + O(\frac{1}{n^2}) \quad \text{as } n \to \infty .$$

Therefore $\sum_{n \notin A} \left(\frac{1}{2\phi(k)} \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \leq \sum_{\substack{p \equiv h \bmod k \\ p \in \mathbb{P}}} |f(\log p)| < \infty$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, this implies that $\sum_{n \notin A} \frac{1}{n} < \infty$.

A fortiori, the set A has asymptotic density = 1. That is, if we write $A = \{a_n : n \geq 1\}$, $1 \leq a_1 < a_2 < \ldots$ then $\lim_{n \to \infty} \frac{a_n}{n} = 1$.

Now, by definition of A, for each $n \geq 1$ there is a real number λ_n such that $\beta(a_n - \frac{1}{4}) < \lambda_n < \beta(a_n + \frac{1}{4})$ and $|f(\lambda_n)| \leq e^{-\lambda_n}$.

Therefore $\lim_{n \to \infty} \frac{\lambda_n}{n} = \beta$ and $\limsup_{n \to \infty} \frac{\log|f(\lambda_n)|}{\lambda_n} \leq -1$.

Hence by lemma 5.2.3, we get

$\limsup_{\substack{x \to \infty \\ x \in \mathbb{R}}} \frac{\log|f(x)|}{x} \leq -1$. This contradicts our assumption on f.

So we must have $\sum_{\substack{p \equiv h \bmod k \\ p \in \mathbb{P}}} |f(\log p)| = \infty$.

5.2.6 <u>Lemma</u> : Let $\underline{x}_1, \ldots, \underline{x}_n$ be <u>linearly dependent</u> vectors in an arbitrary complex vector space. Whenever $a_1, \ldots, a_n$ are complex numbers with $|a_j| \leq 1$ $(1 \leq j \leq n)$, there exist complex numbers $b_1, \ldots, b_n$ with $|b_j| \leq 1$ $(1 \leq j \leq n)$ and at least one $|b_j| = 1$ such that $\sum_{j=1}^{n} a_j \underline{x}_j = \sum_{j=1}^{n} b_j \underline{x}_j$.

<u>Proof</u> : By assumption there exist complex numbers $c_1, \ldots, c_n$, not all of them zero, such that $\sum_{j=1}^{n} c_j \underline{x}_j = 0$.

Let $K = \{ \underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : |\alpha_j| \leq 1 \text{ for } 1 \leq j \leq n \}$ and let

$I = \{ t \in \mathbb{R} : \underline{a} + t.\underline{c} \in K \}$.

Here $\underline{a} = (a_1, \ldots, a_n)$, $\underline{c} = (c_1, \ldots, c_n)$.

Since $\underline{a} \in K$, it follows that $0 \in I$. Thus I is nonempty. Since K is convex, I is a convex subset of the real line and hence is an interval. Since K is compact and $\underline{c} \neq \underline{0}$, I is bounded. Thus I is a nonempty bounded interval. Let t_o be one of the end points of I, and let $\underline{b} = (b_1, \ldots, b_n) = \underline{a} + t_o.\underline{c}$. Then clearly $\underline{b} \in \mathrm{bd}(K)$, the boundary of K. That is, $|b_j| \leq 1$ and at least one $|b_j| = 1$. Also,

$$\sum_{j=1}^{n} b_j \underline{x}_j = \sum_{j=1}^{n} a_j \underline{x}_j + t_o \sum_{j=1}^{n} c_j \underline{x}_j = \sum_{j=1}^{n} a_j \underline{x}_j .$$

So we are done.

5.2.7 Lemma : Let $\underline{x}_1, \ldots, \underline{x}_n$ be points in a complex Hilbert space and let $a_1, \ldots, a_n$ be complex number with $|a_j| \leq 1$ $(1 \leq j \leq n)$. Then there exist complex numbers $b_1, \ldots, b_n$ with $|b_j| = 1$ $(1 \leq j \leq n)$ such that

$$\| \sum_{j=1}^{n} a_j \underline{x}_j - \sum_{j=1}^{n} b_j \underline{x}_j \|^2 \leq 4 \sum_{j=1}^{n} \| \underline{x}_j \|^2 .$$

Proof : We shall prove the result by induction on n. It is trivial for $n = 1$. So let us assume its validity for n, and prove it for $n + 1$.

Let $\underline{x}_1, \ldots, \underline{x}_{n+1}$ be points in our Hilbert space, and let $a_1, \ldots, a_{n+1}$ be complex numbers with $|a_j| \leq 1$ $(1 \leq j \leq n+1)$. Let

y_{n+1} be the orthogonal projection of x_{n+1} into the span of $x_1,\ldots,x_n$. Thus, y_{n+1} is a linear combination of $x_1,\ldots,x_n$; and $z_{n+1} = x_{n+1} - y_{n+1}$ is orthogonal to $x_1,\ldots,x_n$ and hence also to y_{n+1}. In particular, $\{x_1,\ldots,x_n,y_{n+1}\}$ is a set of linearly dependent vectors. Hence by lemma 5.2.6, there exist complex numbers $c_1,\ldots,c_{n+1}$ such that $|c_j| \leq 1$ $(1 \leq j \leq n+1)$, $|c_{j_o}| = 1$ for some j_o and $\sum_{j=1}^{n} a_j x_j + a_{n+1} y_{n+1} = \sum_{j=1}^{n} c_j x_j + c_{n+1} y_{n+1}$.

Now we have to handle two cases separately :

Case I. $j_o = n+1$. That is, $|c_{n+1}| = 1$. By induction hypothesis there exist $b_1,\ldots,b_n$ with $|b_j| = 1$ $(1 \leq j \leq n)$ such that :

$$\Big\| \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} b_j x_j \Big\|^2 \leq 4 \sum_{j=1}^{n} \|x_j\|^2 .$$

Let us put $b_{n+1} = c_{n+1}$. We have :

$$\sum_{j=1}^{n+1} a_j x_j - \sum_{j=1}^{n+1} b_j x_j = \Big(\sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} b_j x_j\Big) + (a_{n+1} - c_{n+1}) z_{n+1} .$$

Since z_{n+1} is orthogonal to $x_1,\ldots,x_n$, this implies :

$$\Big\| \sum_{j=1}^{n+1} a_j x_j - \sum_{j=1}^{n+1} b_j x_j \Big\|^2 = \Big\| \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} b_j x_j \Big\|^2 + |a_{n+1} - c_{n+1}|^2 \, \|z_{n+1}\|^2$$

$$\leq 4 \sum_{j=1}^{n} \|x_j\|^2 + 4 \|z_{n+1}\|^2 \leq 4 \sum_{j=1}^{n+1} \|x_j\|^2$$

$$\Big(\text{since } \|x_{n+1}\|^2 = \|y_{n+1}\|^2 + \|z_{n+1}\|^2 \geq \|z_{n+1}\|^2\Big).$$

Since $|b_j| = 1$ for $1 \leq j \leq n+1$, we are done.

<u>Case II</u>. $1 < j_0 \leq n$. So, without loss of generality, we may assume $j_0 = 1$. That is, $|c_1| = 1$. By inductive hypothesis, there exist complex numbers $b_2, \ldots, b_{n+1}$ with $|b_j| = 1$ $(2 \leq j \leq n+1)$ such that :

$$\left\| \sum_{j=2}^{n} c_j \underline{x}_j + c_{n+1} \underline{y}_{n+1} - \sum_{j=2}^{n} b_j \underline{x}_j - b_{n+1} \underline{y}_{n+1} \right\|^2 \leq 4 \sum_{j=2}^{n} \|\underline{x}_j\|^2 + 4 \|\underline{y}_{n+1}\|^2$$

Let us put $b_1 = c_1$. In this case we have :

$$\sum_{j=1}^{n+1} a_j \underline{x}_j - \sum_{j=1}^{n+1} b_j \underline{x}_j = \left(\sum_{j=2}^{n} c_j \underline{x}_j + c_{n+1} \underline{y}_{n+1} - \sum_{j=2}^{n} b_j \underline{x}_j - b_{n+1} \underline{y}_{n+1} \right)$$
$$+ (a_{n+1} - b_{n+1}) \underline{z}_{n+1} .$$

Hence, as before, we deduce :

$$\left\| \sum_{j=1}^{n+1} a_j \underline{x}_j - \sum_{j=1}^{n+1} b_j \underline{x}_j \right\|^2 = \left\| \sum_{j=2}^{n} c_j \underline{x}_j + c_{n+1} \underline{y}_{n+1} - \sum_{j=2}^{n} b_j \underline{x}_j - b_{n+1} \underline{y}_{n+1} \right\|^2$$
$$+ |a_{n+1} - b_{n+1}|^2 \|\underline{z}_{n+1}\|^2$$
$$\leq 4 \sum_{j=2}^{n} \|\underline{x}_j\|^2 + 4 \|\underline{y}_{n+1}\|^2 + 4 \|\underline{z}_{n+1}\|^2$$
$$= 4 \sum_{j=2}^{n+1} \|\underline{x}_j\|^2 \leq 4 \sum_{j=1}^{n+1} \|\underline{x}_j\|^2 .$$

Thus we are done.

5.2.8 <u>Proposition</u> : Let $\{x_n : n \geq 1\}$ be a sequence in a complex Hilbert space X satisfying :

(i) $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, and (ii) $\sum_{n=1}^{\infty} |(x_n, x)| = \infty$ for $x \neq 0, x \in X$.

<u>Then</u> the set of all points which can be written as a convergent sum $\sum_{n=1}^{\infty} a_n x_n$ with $a_n \in \Pi$ $(n \geq 1)$ is dense in X.

<u>Proof</u> : Let $\{\varepsilon_n : n \geq 1\}$ be a sequence of independent random variables such that $P(\varepsilon_n = +1) = \frac{1}{2} = P(\varepsilon_n = -1)$. Let $X_n = \varepsilon_n x_n$, $n \geq 1$. Then $\{X_n : n \geq 1\}$ is a sequence of independent X-valued random elements which are uniformly bounded in norm, and which satisfy $E(X_n) = 0$, $\sum_{n=1}^{\infty} E(\|X_n\|^2) = \sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. Also, (ii) shows that no non-null vector is orthogonal to all the x_n's, so that the span of $\{x_n : n \geq 1\}$ is dense in X, so that X is seperable. Hence Kolmogorov's three-series criterion applies to X-valued random elements (see [29]) and hence $\sum_{n=1}^{\infty} X_n$ converges almost surely. That is, the series $\Sigma\, \varepsilon_n x_n$ converges for almost all sign sequences $\{\varepsilon_n\}$. Let us fix one such sequence. For this fixed sequence $\{\varepsilon_n\}$, we have :

(iii) $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges, $\varepsilon_n = \pm 1$ $(n \geq 1)$.

Now let $x_0 \varepsilon X$ be arbitrary and $\varepsilon > 0$. We have to exhibit a sequence $\{a_n\}$ in the unit circle such that $\sum_{n=1}^{\infty} a_n x_n$ converges and $\| x_0 - \sum_{n=1}^{\infty} a_n x_n \| < \varepsilon$.

Let us fix a positive integer N so large that

(iv) $\sum_{n=N}^{\infty} \|x_n\|^2 < \frac{\varepsilon^2}{36}$, and (v) $\| \sum_{n=m}^{\infty} \varepsilon_n x_n \| < \frac{\varepsilon}{3}$ for $m \geq N$.

(such a choice is possible because of (i) and (iii) above).

Let $A = \left\{ \sum_{m=N}^{N+k} \alpha_m x_m : k \geq 1, |\alpha_m| \leq 1 \text{ for } N \leq m \leq N+k \right\}$. We shall first show that A is dense in X. Suppose it is not. Then there exists $y_0 \not\varepsilon \bar{A}$. Notice that A is convex ; hence y_0

and $\overline{A}$ are disjoint closed convex subsets of X, the first of them is compact. Hence by theorem 3.4(b) of [46,p.58], there exists a continuous linear functional L on X such that $\mathrm{Re}\, L(x) > \mathrm{Re}\, L(y_o)$ for $x \in A$. Thus $L \neq 0$, and therefore by theorem 12.5 of [46,p.294], there exists an $y_1 \in X$, $y_1 \neq 0$ such that $L(x) = (x,y_1)$ $(x \in X)$. There exists a real number c such that $\mathrm{Re}(x,y_1) = \mathrm{Re}\, L(x) > -c$ for $x \in A$ (namely, we take $c = -\mathrm{Re}\, L(y_o)$). In particular if we choose α_m complex, $|\alpha_m| = 1$, such that $\alpha_m(x_m,y_1) = -|(x_m,y_1)|$ then $y_k = \sum_{m=N}^{N+k} \alpha_m x_m$ is in A for $k \geq 2$, and therefore

$$\mathrm{Re}(y_k,y_1) = -\sum_{m=N}^{N+k} |(x_m,y_1)| > -c \quad \text{for} \quad k \geq 2 .$$

That is $\sum_{n=N}^{N+k} |(x_m,y_1)| \leq c$ for $k \geq 2$. Hence $\sum_{n=N}^{\infty} |(x_m,y_1)| \leq c < \infty$.

Since $y_1 \neq 0$, this contradicts (ii). Hence the set A must be dense. Therefore we can choose an $x' \in A$ such that :

$$\text{(vi)} \qquad \|x_o - \sum_{j=1}^{N} x_j - x'\| < \frac{\varepsilon}{3} .$$

By definition of A, there exists $k \geq 1$ and α_m $(N \leq m \leq N+k)$ in the unit disc such that $x' = \sum_{m=N}^{N+k} \alpha_m x_m$. Hence by lemma 5.2.7, there exists $x'' \in X$ give by $x'' = \sum_{m=N}^{N+k} a_m x_m$, where $|a_m| = 1$ $(N \leq m \leq N+k)$ and $\|x'-x''\|^2 \leq 4 \sum_{m=N}^{N+k} \|x_m\|^2 \leq \frac{\varepsilon^2}{9}$ (due to (iv)). Hence

$$\text{(vii)} \qquad \|x' - x''\| < \frac{\varepsilon}{3} .$$

Let us put $x''' = \sum_{m=N+k+1}^{\infty} \varepsilon_m x_m$. Then by (v), we get :

(viii) $\|x'''\| < \frac{\varepsilon}{3}$.

Now let us put $y = \sum_{j=1}^{N} x_j + x' + x'' = \sum_{j=1}^{\infty} a_j x_j$ where

$a_j = 1$ if $1 \leq j < N$, $a_j = \varepsilon_j$ for $j \geq N+k+1$, and a_j is as above for $N \leq j \leq N+k$. Combining (vi), (vii) and (viii), we get $\| x_0 - y \| < \varepsilon$. Thus we are done.

5.2.9 Lemma : Let U be a simply connected planar region. Consider $H(U)^n$, the Cartesean product of n copies of the space $H(U)$ of analytic functions on U with compact open topology. Let $\{ \underline{f}_m : m \in \mathbb{N} \}$ be a sequence in $H(U)^n$ $(\underline{f}_m = (f_m^1, \ldots, f_m^n))$ which satisfies :

(i) whenever $\mu_1, \ldots, \mu_n$ are complex Borel measures with compact support contained in U such that

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int f_m^j d\mu_j \right| < \infty, \quad \text{we have} \quad \int z^r d\mu_j(z) = 0$$

for $1 \leq j \leq n$, $r = 0, 1, 2, \ldots,$

(ii) $\sum_{m=1}^{\infty} \underline{f}_m$ converges in $H(U)^n$,

and (iii) $\sum_{m=1}^{\infty} \sup_{z \in K} |\underline{f}_m(z)|^2 < \infty$ for any compact $K \subseteq U$

(here $|f_m(z)|^2 \equiv \sum_{j=1}^{n} |f_m^j(z)|^2$).

Then the set of all convergent series $\sum_{m=1}^{\infty} a_m \underline{f}_m$ with $a_m \in \mathbb{C}$, $|a_m| = 1$ $(m \geq 1)$ is dense in $H(U)^n$.

Proof : Let $\underline{g} = (g^1, \ldots, g^n) \in H(U)^n$, K a compact subset of U and $\varepsilon > 0$. We have to exhibit a sequence $\{a_m : m \geq 1\}$ in $\prod$ such that $\sum_{m=1}^{\infty} a_m \underline{f}_m$ converges and $\sup_{z \in K} |\underline{g}(z) - \sum_{m=1}^{\infty} a_n \underline{f}_n(z)| < \varepsilon$.

Let us choose and fix a simply connected region V such that $K \subseteq V$, the closure $\bar{V}$ of V is a compact subset of U and the boundary bd(V) of V is an analytic simple closed curve. Let us consider the Hardy space $H^2(V)$ on V (see [22, p.168] for definition). Since V is a simply connected proper subregion of the plane, according to [22 ,p.169], a conformal map of V onto the unit disc induces an isometric isomorphism between $H^2(V)$ and the classical H^2-space on the unit disc ; accordingly all the well known metrical properties of the latter space carries over to $H^2(V)$. In particular, we have the following :

(a) $H^2(V)$ is a complex Hilbert space. Let's denote its inner product by $\langle \cdot, \cdot \rangle$.

(b) Let $h \in H^2(V)$. Then there exists a complex Borel measure $\mu = \mu_h$ with its support contained in bd(V) such that whenever $h_1 \in H^2(V)$ has a continuous extension to $\bar{V}$, we have $\langle h_1, h \rangle = \int h_1 d\mu$. (In fact μ is absolutely continuous with respect to arc length, and its density is given by the complex conjugate of the almost sure boundary value of h. But we shall not need this fact).

(c) If $\{h_n : n \geq 0\}$ is a sequence in $H^2(V)$ such that $\lim_{n \to \infty} h_n = h_0$

in the topology of $H^2(V)$, then $\lim_{n\to\infty} h_n = h_o$ uniformly on compact subsets of V.

Further, since we have assumed that $bd(V)$ is an analytic simple closed curve, in view of theorem 10.2 of [22, p.169] and theorem 10.7 of [22, p.174], we also have :

(d) The polynomials are dense in the topology of $H^2(V)$.

For $\underline{h}_j = (h_j^1, \ldots, h_j^n)$, $j = 1,2$, in $H^2(V)^n$ (the Cartesean product of n copies of $H^2(V)$, with product topology), let us define $(\underline{h}_1, \underline{h}_2) = \sum_{j=1}^{n} \langle h_1^j, h_2^j \rangle$. This is an inner product that converts $H^2(V)^n$ into a complex Hilbert space; let $\|\cdot\|$ denote the corresponding norm. That is $\|\underline{h}\|^2 = (\underline{h}, \underline{h})$, $\underline{h} \in H^2(V)^n$. In view of assumption (iii) and observation (b) above, we readily get $\sum_{m=1}^{\infty} \|\underline{f}_m\|^2 < \infty$. Let $\underline{h} \in H^2(V)^n$ be such that $\sum_{m=1}^{\infty} |(\underline{f}_m, \underline{h})| < \infty$. By (b) above, there exist complex Borel measures with supports contained in $bd(V)$ such that $(\underline{f}_m, \underline{h}) = \sum_{j=1}^{n} \int f_m^j d\mu_j$. Thus we have $\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int f_m^j d\mu_j \right| < \infty$. Hence by assumption (i), $\int z^k d\mu_j(z) = 0$ for $1 \leq j \leq n$, $k = 0,1,2,\ldots$ That is, h^j is orthogonal to all the polynomials and hence by (d) above, $h_j = 0$ $(1 \leq j \leq n)$. Thus $\underline{h} = 0$.

Therefore all the assumptions of proposition 5.2.8 hold, so that we may conclude that the set of all convergent series (in $H^2(V)^n$) $\sum_{m=1}^{\infty} \alpha_m \underline{f}_m$ with $\alpha_m \in \Pi$, $m \geq 1$, is dense in $H^2(V)^n$.

Therefore by (c), there exists a sequence $\{\alpha_n\}$ in $\prod$ such that $\sum_{n=1}^{\infty} \alpha_n \underline{f_n}$ converges uniformly on K (as K is a compact subset of V) and $|\sum_{n=1}^{\infty} \alpha_n \underline{f_n} - \underline{g}| < \frac{\varepsilon}{2}$ on K. This, together with assumption (ii) show that we may choose an integer M so large that $|\sum_{n=1}^{M} \alpha_n \underline{f_n} - \underline{g}| < \frac{\varepsilon}{2}$ on K and $|\sum_{n=M+1}^{\infty} \underline{f_n}| < \frac{\varepsilon}{2}$ on K. Hence, putting $a_n = \alpha_n$ if $1 \leq n \leq M$ and $a_n = 1$ if $n > M$, we get $|\sum_{n=1}^{\infty} a_n \underline{f_n} - \underline{g}| < \varepsilon$ on K, as was to be established.

5.2.10 <u>Lemma</u> : Let $k \geq 1$ and $n = \phi(k)$ be the number of integers in $[1,k]$ which are relatively prime to k. Let $\chi_1, \ldots, \chi_n$ be the Dirichlet characters modulo k. For any prime p, let $\underline{f_p} = (f_p^1, \ldots, f_p^n) \in H(\underline{\Omega}^1_{1/2})^n$ be given by $f_p^j(z) \equiv -\log(1-\chi_j(p)p^{-z})$. <u>Then</u> the set of all convergent sums (in compact open topology) $\sum_{p \in \mathbb{P}} a_p \underline{f_p}$ with $a_p \in \prod$ is dense in $H(\underline{\Omega}^1_{1/2})^n$.

<u>Proof</u> : Let $\{\alpha_p : p \in \mathbb{P}\}$ be a fixed sequence in $\prod$ such that $\sum_{p \in \mathbb{P}} \alpha_p \underline{f_p}$ converges (for example, we may take $\alpha_p = (-1)^m$ if p is the <u>mth</u> prime). Let us put $\underline{g_p} = \alpha_p \underline{f_p}$. Then in order to prove the lemma, it clearly suffices to show that the set of all convergent sums $\sum_{p \in \mathbb{P}} a_p \underline{g_p}$ is dense in $H(\underline{\Omega}^1_{1/2})^n$.

Now by choice of α_p's, $\sum_{p \in \mathbb{P}} \underline{g_p}$ converges. Also, it is easy to see that $\sum_{p \in \mathbb{P}} \sup_{z \in K} |g_p(z)|^2 < \infty$ for any compact $K \subseteq \Omega^1_{1/2}$.

Therefore, by lemma 5.2.9, it sufficies to show that whenever $\mu_1, \ldots, \mu_n$ are complex Borel measures, with compact supports contained in $\Omega^1_{1/2}$, such that $\sum_{p \in \mathbb{P}} |\sum_{j=1}^{n} \int g_p^j d\mu_j| < \infty$, we necessarily have $\int s^q d\mu_j(s) = 0$ for $1 \leq j \leq n$, $q = 0, 1, 2, \ldots$

So let us fix $\mu_1, \ldots, \mu_n$ satisfying this hypothesis.

Let $h_p \in H(\Omega^1_{1/2})^n$ be given by $h_p^j(z) = \alpha_p \chi_j(p) p^{-z}$. Clearly $\sum_{p \in \mathbb{P}} |g_p(z) - h_p(z)| < \infty$ uniformly on compact subsets of $\Omega^1_{1/2}$.

Hence $\sum_{p \in \mathbb{P}} |\sum_{j=1}^{n} \int h_p^j(z) d\mu_j(z)| < \infty$. That is we have :

$$\sum_{p \in \mathbb{P}} |\sum_{j=1}^{n} \chi_j(p) \int p^{-z} d\mu_j(z)| < \infty .$$

Since the Dirichlet characters modulo k are periodic with period k this may be rewrit en as :

$$\sum_{\substack{p \in \mathbb{P} \\ p \equiv r \bmod k}} |\sum_{j=1}^{n} \chi_j(r) \int p^{-z} d\mu_j(z)| < \infty \quad \text{for} \quad 1 \leq r \leq k, \quad (r,k) = 1 .$$

Or, if we define the complex Borel measures υ_r by

$d\upsilon_r(z) = \sum_{j=1}^{n} \chi_j(r) d\mu_j(z)$, <u>then</u> :

(i) $\sum_{\substack{p \in \mathbb{P} \\ p \equiv r \bmod k}} |\int p^{-z} d\upsilon_r(z)| < \infty$ for $1 \leq r \leq k$, $(r,k) = 1$.

Since all the μ_j's have compact support contained in $\Omega^1_{1/2}$, the same is true of the υ_r's. Hence if we put

$\rho_r(z) = \int e^{-sz} d\upsilon_r(s)$, $1 \leq r \leq k$, $(r,k) = 1$, then by lemma 5.2.2, we have either $\rho_r \equiv 0$ or else

$$\text{(ii)} \quad \limsup_{\substack{x \to \infty \\ x \in \mathbb{R}}} \frac{\log|\rho_r(x)|}{x} > -1 .$$

But (i) may be rewitten as : $\sum_{\substack{p \equiv r \bmod k \\ p \in \mathbb{P}}} |\rho_r(\log p)| < \infty$.

Therefore by lemma 5.2.5, (ii) can not hold. Hence (ii) is false for each r, so that $\rho_r(z) = \int e^{-sz} d\upsilon_r(s) \equiv 0$. Let's fix an arbitrary integer $q \geq 0$. Differentiating the above equation q times, and then putting $z = 0$, we obtain $\int s^q d\upsilon_r(s) = 0$. Or, going back to the definition of υ_r's and putting $b_j = b_j(q) = \int s^q d\mu_j(s)$, $1 \leq j \leq n$, we obtain :

$$\sum_{j=1}^{n} b_j \chi_j(r) = 0 \quad \text{for} \quad 1 \leq r \leq k, \quad (r,k) = 1.$$

Since $\chi_j(r) = 0$ if $(r,k) > 1$, this implies $\sum_{j=1}^{n} b_j \chi_j \equiv 0$.

But the orthogonality relation for the Dirichlet characters implies that $\chi_1, \ldots, \chi_n$ are linearly independent over $\mathbb{C}$. Hence we must have $b_1 = b_2 = \ldots = b_n = 0$. That is $\int s^q d\mu_j(s) = 0$ for $1 \leq j \leq n$. Since $q \geq 0$ was arbitrary, this completes the verification of the hypothesis (i) of lemma 5.2.9. So we are done.

5.2.11 <u>Lemma</u> : Let G be a complete seperable topological group. Let $\{X_n : n \geq 1\}$ be a sequence of independent G-valued random elements. Let us suppose that $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Let A_n be the support of X_n. Then the support of $\sum_{n=1}^{\infty} X_n$ is the closure of the set of all $x \in G$ which may be written as a convergent sum $x = \sum_{n=1}^{\infty} x_n$, $x_n \in A_n$ $(n \geq 1)$.

Proof : This may be established by minor modifications of the proof of theorem 3.7.5 of [40,p.62] which gives a similar result for real valued random variables.

5.3 Universality theorems : In this section we shall determine the spectra of the joint distributions of the Dirichlet L-functions in the strip $\Omega^{1}_{1/2}$ and formulate the results thus obtained as universality theorems.

5.3.1 Theorem : Let $k \geq 1$, $n = \phi(k)$. Let $\chi_1, \ldots, \chi_n$ be the distinct Dirichlet characters modulo k. Let

$S = \left\{ f \in H(\Omega^{1}_{1/2}) : f \equiv 0 \text{ or } \frac{1}{f} \in H(\Omega^{1}_{1/2}) \right\}$. Regard

$(L(\cdot,\chi_1), \ldots, L(\cdot,\chi_n))$ as a p int in the continuous flow $H(\Omega^{1}_{1/2})^n$.

Then its spectrum is S^n.

Proof : By corollary 3.4.6 to theorem 3.4.5, and theorems 3.4.10 and 3.4.11, we get :

$$(L(\cdot,\chi_1), \ldots, L(\cdot,\chi_n)) \Longrightarrow (F^{\chi_1}, \ldots, F^{\chi_n}) \text{ on } \Omega^{\infty}_{1/2}$$

where F^{χ_j} is the $H(\Omega^{\infty}_{1/2})$-valued random element defined on $(W, \mathbb{B}, m)$ by

(i) $F^{\chi_j}(z,w) = \prod_{p \in \mathbb{P}} (1 - \chi_j(p) w(p) p^{-z})^{-1}$.

Therefore by proposition 1.5.7 (applied to the flow homomorphism from $H(\Omega^{\infty}_{1/2})^n$ to $H(\Omega^{1}_{1/2})^n$ defined by co-ordinatewise restriction) we have :

(ii) $(L(.,\chi_1),\ldots,L(.,\chi_n)) \Rightarrow (F^{\chi_1},\ldots,F^{\chi_n})$ on $\Omega^1_{1/2}$.

So let us regard $(F^{\chi_1},\ldots,F^{\chi_n})$ as a $H(\Omega^1_{1/2})^n$-valued random element. We have $(\log F^{\chi_1},\ldots,\log F^{\chi_n}) = \sum_{p\in \mathbb{P}} w(p)\underline{f}_p$, $w \in W$, where $\underline{f}_p$'s are as in lemma 5.2.10.

Since $\{w(p) : p \in \mathbb{P}\}$ is a sequence of independent random variables and the support of $w(p)$ is Π, $\{w(p)\underline{f}_p : p \in \mathbb{P}\}$ is a sequence of independent $H(\Omega^1_{1/2})^n$-valued random elements and the support of $w(p)\underline{f}_p$ is the set $\{a\underline{f}_p : a \in \Pi\}$, therefore, by lemma 5.2.11, the support of $(\log F^{\chi_1},\ldots,\log F^{\chi_n})$ is the closure of the set of all convergent sums $\sum_{p\in \mathbb{P}} a_p\underline{f}_p$ with $a_p \in \Pi$. But by lemma 5.2.10, this closure is the whole of $H(\Omega^1_{1/2})^n$. Thus the support of $(\log F^{\chi_1},\ldots,\log F^{\chi_n})$ is $H(\Omega^1_{1/2})^n$. Since the map from $H(\Omega^1_{1/2})^n$ into itself sending $(f_1,\ldots,f_n)$ to $(\exp f_1,\ldots,\exp f_n)$ is a continuous function sending $(\log F^{\chi_1},\ldots,\log F^{\chi_n})$ into $(F^{\chi_1},\ldots,F^{\chi_n})$ and sending $H(\Omega^1_{1/2})^n$ onto S_o^n (where $S_o = S - \{0\}$, S being as in the statement of this theorem), it immediately follows that the support of $(F^{\chi_1},\ldots,F^{\chi_n})$ contains S_o^n. But $H(\Omega^1_{1/2})^n$ is a seperable metric space, and therefore the support of a $H(\Omega^1_{1/2})^n$-valued random element is a closed set (see 1.5.3). Since by Hurwitz' theorem (theorem 2.5 of [18, p.148]), the closure of S_o is S, the closure of S_o^n is S^n. Hence the support of $(F^{\chi_1},\ldots,F^{\chi_n})$ contains S^n. On the other hand, (i) gives each F^{χ_j} as an almost sure convergent product of nonvanishing factors ; hence by Hurwitz' theorem, $(F^{\chi_1},\ldots,F^{\chi_n}) \in S^n$ almost surely ; thus the support of $(F^{\chi_1},\ldots,F^{\chi_n})$ is contained in S^n. Hence we have :

(iii) The support of $(F^{\chi_1},\ldots,F^{\chi_n})$ is S^n.

Combining (ii) and (iii), we get, from the definition of spectrum (1.5.3) that the spectrum of $(L(\cdot,\chi_1),\ldots,L(\cdot,\chi_n))$ is S^n.

5.3.2 <u>Theorem</u> : Let $h > 0$ be an arbitrary but fixed real number, $k \geq 1$ an integer, $n = \emptyset(k)$, $\chi_1,\ldots,\chi_n$ the distinct Dirichlet characters modulo k, $L(\cdot,\chi_1),\ldots,L(\cdot,\chi_n)$ the corresponding Dirichlet L-functions Regard $(L(\cdot,\chi_1),\ldots,L(\cdot,\chi_n))$ as a point in the discrete flow $H(\Omega^1_{1/2})^n$ modulo h. <u>Then</u> its spectrum is S^n where $S = \left\{ f \in H(\Omega^1_{1/2}) : f \equiv 0 \text{ or } \frac{1}{f} \in H(\Omega^1_{1/2}) \right\}$.

<u>Proof</u> : If h is of type I then by the results of chapter 4, we have $(L(\cdot,\chi_1),\ldots,L(\cdot,\chi_n)) \Rightarrow (F^{\chi_1},\ldots,F^{\chi_n})$ mod h on $\Omega^1_{1/2}$, hence the result follows from the description of the support of $(F^{\chi_1},\ldots,F^{\chi_n})$ on $\Omega^1_{1/2}$ obtained in 5.3.1.

If h is of type II, let $\{z_{p,h} : p \in \mathbb{P}\}$ be the sequence of Π-valued random variables introduced in 4.3.2. Let us define

$F_h^{\chi_j}$, $1 \leq j \leq n$, by $F_h^{\chi_j}(z) = \prod_{p \in \mathbb{P}} (1-z_{p,h}\chi_j(p)p^{-z})^{-1}$. Then, as before, we may deduce from 4.5.4, 4.5.7 and 4.5.8 that

(i) $(L(\cdot,\chi_1),\ldots,L(\cdot,\chi_n)) \overset{\Longrightarrow}{\Longrightarrow} (F_h^{\chi_1},\ldots,F_h^{\chi_n}) \bmod h$ on $\underline{\Omega}^1_{1/2}$.

Let $\mathbb{P}_o$ be the finite set of primes associated with h as in 4.2.1. We can write $(F_h^{\chi_1},\ldots,F_h^{\chi_n}) = X.Y$ (co-ordinatewise product) where $X = (X_1,\ldots,X_n)$, $Y = (Y_1,\ldots,Y_n)$ are the $H(\underline{\Omega}^1_{1/2})^n$-valued random elements defined by

$$X_j(z) = \prod_{p \in \mathbb{P}_o} (1-z_{p,h}\chi_j(p)p^{-z})^{-1}, \quad Y_j(z) = \prod_{p \in \mathbb{P}-\mathbb{P}_o} (1-z_{p,h}\chi_j(p)p^{-z})^{-1}$$

$1 \leq j \leq n$.

From the description of the joint distribution of $\{z_{p,h} : p \in \mathbb{P}\}$ given in 4.3.2, it readily follows that X and Y are stochastically independent. Also, the joint distribution of $\{z_{p,h} : p \in \mathbb{P} - \mathbb{P}_o\}$ is the same as of $\{w(p) : p \in \mathbb{P}\}$ on $(W, \mathbb{B}, m)$. Therefore, $Y \overset{D}{=} (F^{\chi_1},\ldots,F^{\chi_n})$. Hence from 5.3.1, we obtain that the support of Y is S^n. Since X,Y are independent, X is not degenerate at 0, the support of X lies in S^n, and S^n is closed under (co-ordinatewise) multiplication, it follows that the support of $(F_h^{\chi_1},\ldots,F_h^{\chi_n}) = X.Y$ is S^n. Hence by (i) above, we get the result.

The theorems 5.3.1 and 5.3.2 may be reformulated as follows :

5.3.3 <u>Theorem</u> (Joint universality of L-functions - continuous version) : Let $k \geq 1$, $n = \phi(k)$, $\chi_1, \ldots, \chi_n$ the distrinct Dirichlet characters modulo k. Let $K_1, \ldots, K_n$ be compact, simply connected and locally path connected subsets of $\mathbb{C}_{1/2}^{1}$. For $1 \leq j \leq n$, let f_j be a continuous function on K_j which is <u>non-vanishing</u> on K_j and analytic in the interior (if any) of K_j. Let $\varepsilon > 0$. <u>Then</u> the set A of all real numbers t which satisfies $\sup_{1 \leq j \leq n} \sup_{z \in K_j} |L(z+it, \chi_j) - f_j(z)| < \varepsilon$ has positive lower density (i.e., $\underline{d}(A) > 0$).

<u>Proof</u> : First let us assume that $f_1, \ldots, f_n$ admit <u>nonvanishing</u> analytic continuations to $\mathbb{C}_{1/2}^{1}$. Let U be the set of all $\underline{g} = (g_1, \ldots, g_n)$ in $H(\mathbb{C}_{1/2}^{1})^n$ such that

$\sup_{1 \leq j \leq n} \sup_{z \in K_j} |g_j(z) - f_j(z)| < \varepsilon$. By assumption, $\underline{f} = (f_1, \ldots, f_n)$ is in the spectrum S^n of $(L(\cdot, \chi_1), \ldots, L(\cdot, \chi_n))$ (theorem 5.3.1), also U is clearly an open neighbourhood of $\underline{f}$. Going back to the definition of spectrum, we obtain

$\underline{d}(\{t \in \mathbb{R} : (S^t L(\cdot, \chi_1), \ldots, S^t L(\cdot, \chi_n)) \in U\}) > 0$. That is, $\underline{d}(A) > 0$. (Notice that in this case the connectedness assumptions on $K_1, \ldots, K_n$ are unnecessary).

Next let $f_1, \ldots, f_n$ be as in the statement of the theorem. Since f_j is continuous and nonvanishing on K_j, and K_j is simply connected and locally pathwise connected, in consequence of theorem 5.1 of [41,p.156], there is a continuous function g_j on K_j such that $f_j = \exp(g_j)$; since f_j is analytic in the interior of K_j, so is g_j $(1 \leq j \leq n)$. Therefore by Mergelyan's theorem (theorem 20.5 of [47,p.423]), there is a sequence $\{P_m^j : m \geq 1\}$ of polynomials such that $P_m^j \longrightarrow g_j$ as $m \longrightarrow \infty$, uniformly on K_j. Hence if we put $f_j^* = \exp(P_m^j)$ for a sufficiently large m, then

(i) $\sup_{1 \leq j \leq n} \sup_{z \in K_j} |f_j(z) - f_j^*(z)| < \frac{\varepsilon}{2}$; f_j^* is entire and non-vanishing. Let B be the set of all $t \in \mathbb{R}$ for which

(ii) $\sup_{1 \leq j \leq n} \sup_{z \in K_j} |L(z+it,\chi_j) - f_j^*(z)| < \frac{\varepsilon}{2}$.

(i) and (ii) together show that $B \subseteq A$. Also, by the first part of this proof $\underline{d}(B) > 0$. Hence $\underline{d}(A) > 0$.

5.3.4 <u>Theorem</u> (Joint universality of L-functions - discrete version) : Let $h > 0$ be an arbitrary but fixed real number. Let $k \geq 1$ be an integer and $n = \phi(k)$. Let $\chi_1, \ldots, \chi_n$ be the distinct Dirichlet characters modulo k. Let $K_1, \ldots, K_n$ be compact, simply connected and locally path connected subsets of $\mathbb{C}^1_{1/2}$. For $1 \leq j \leq n$, let f_j be a continuous and <u>nonvanishing</u> function on K_j which is analytic in the interior of K_j. Let $\varepsilon > 0$. <u>Then</u> the set B of all <u>integers</u> m which satisfy $\sup_{1 \leq j \leq n} \sup_{z \in K_j} |L(z+imh,\chi_j) - f_j(z)| < \varepsilon$ has positive lower density (discrete version) (i.e., $\underline{\delta}(B) > 0$).

<u>Proof</u> : This may be deduced from theorem 5.3.2 exactly as theorem 5.3.3 was deduced from theore 5.3.1.

5.3.5 <u>Corollary</u> : Let $\{a_n : n \in \mathbb{N}\}$ be a periodic sequence of complex numbers with period $k \geq 1$ (i.e., $a_{n+k} \equiv a_n$). Let $\psi \in H(\Omega^1_{1/2})$ be defined by $\psi(z) = \sum_{\substack{n=1 \\ (n,k)=1}}^{\infty} a_n n^{-z}$. Then one of the following two alternatives hold :

(a) There is a constant $\alpha \in \mathbb{C}$ and a Dirichlet character χ modulo k such that $\psi = \alpha L(\cdot, \chi)$.

or (b) If K is a compact, simply connected and locally path connected subset of $\Omega^1_{1/2}$, f is <u>any</u> continuous function on K which is analytic in the interior (if any) of K, and if $\varepsilon > 0$, then the set of all $t \in \mathbb{R}$ for which $\sup_{z \in K} |\psi(z+it) - f(z)| < \varepsilon$ has positive lower density.

<u>Proof</u> : Let $n = \phi(k)$ and let $\chi_1, \ldots, \chi_n$ be the Dirichlet characters modulo k. From the orthogonality relation of the characters, we can deduce that there are $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $a_m = \sum_{j=1}^{n} \alpha_j \chi_j(m)$ for $1 \leq m < k$, $(m,k) = 1$. By periodicity, this holds for all $m \geq 1$ with $(m,k) = 1$. Hence

$$\psi(z) \equiv \sum_{j=1}^{n} \alpha_j L(z, \chi_j) .$$

(This shows, incidentally, that $\psi \in H(\Omega^1_{1/2})$). The map from $H(\Omega^1_{1/2})^n$ to $H(\Omega^1_{1/2})$ sending $(f_1, \ldots, f_n)$ to $\sum_{j=1}^{n} \alpha_j f_j$ is a flow homomorphism; hence by 1.5.8 and theorem 5.3.1, the spectrum of the image ψ of $(L(\cdot,\chi_1),\ldots,L(\cdot,\chi_n))$ under this map contains the closure of the image of the spectrum S^n of $(L(\cdot,\chi_1),\ldots,L(\cdot,\chi_n))$ In case (a) does not hold, at least two of the α_j's must be nonzero. Let's say $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. Since $\underline{0} \in S$ and S is closed under multiplication by nonzero scalars, the image of S^n under the said homomorphism clearly contains the set $S_1 = \{ f_1+f_2 : f_1 \in S,\ f_2 \in S \}$. Hence the spectrum of ψ contains $\overline{S}_1$. But S_1 contains all bounded analytic functions on $\Omega^1_{1/2}$ (if $f \in H(\Omega^1_{1/2})$ is bounded then there exist $z_0 \in \mathbb{C}$ such that $f(z) \neq z_0$ for all $z \in \Omega^1_{1/2}$; putting $f_1(z) \equiv f(z) - z_0$, $f_2(z) \equiv z_0$, we see that $f = f_1+f_2$, $f_1, f_2 \in S$), and the latter set is dense in $H(\Omega^1_{1/2})$ (e.g., see the proof of proposition 2.3.6). Hence the spectrum of ψ is the whole of $H(\Omega^1_{1/2})$.

From this the statement (b) may be deduced exactly as theorem 5.3.3 was deduced from 5.3.1.

5.3.6 <u>Corollary</u>: Let χ be an arbitrary Dirichlet character. Let K be a compact, simply connected and locally path connected subset of $\Omega^1_{1/2}$. Let f be a <u>nonvanishing</u> continuous function on K which is analytic in the interior (if any) of K; let $\varepsilon > 0$, $h > 0$. Then

(a) The set A of all $t \in \mathbb{R}$ for which $\sup_{z \in K} |L(z+it,\chi)-f(z)| < \varepsilon$ has positive lower dencity. A fortiori the set A is nonempty.

(b) The set B of all integers m for which

$\sup_{z \in K} |L(z+imh) - f(z)| < \varepsilon$ has positive lower density (discrete version). A fortiori the set B is nonempty.

In particular the above statements remain valid if we take the Riemann Zeta function in place of $L(\cdot, \chi)$.

<u>Proof</u> : These are immediate consequences of theorems 5.3.3 and 5.3.4. The last part follows since $\zeta = L(\cdot, \chi_0)$ where χ_0 is the unique (principal) Dirichlet character modulo one.

5.3.7 <u>Corollary</u> : Let $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$ be a rational number. Consider the Hurwitz Zeta function $\zeta(\cdot, \alpha)$ on $\Omega^1_{1/2}$. Let K be a compact, simply connected and locally path connected subset of $\Omega^1_{1/2}$. Let f be <u>any</u> continuous function on K which is analytic in the interior (if any) of K. Let $\varepsilon > 0$, $h > 0$. <u>Then</u>

(a) The set of all $t \in \mathbb{R}$ for which $\sup_{z \in K} |\zeta(z+it, \alpha) - f(z)| < \varepsilon$ has positive lower density (continuous version).

(b) The set of all $m \in \mathbb{Z}$ for which $\sup_{z \in K} |\zeta(z+imh, \alpha) - f(z)| < \varepsilon$ has positive lower density (discrete version).

<u>Proof</u> : By assumptions on α, there exist integers l, k, $1 \leq l < k$, $(l,k) = 1$, $k \geq 3$ such that $\alpha = \frac{l}{k}$.

$$\zeta(z,\alpha) = k^z \sum_{\substack{n \equiv l \\ \text{mod } k}} n^{-z} = f(z)\psi(z) \quad \text{for } \operatorname{Re}(z) > 1 \text{ (let's say)}.$$

Using theorems 3.4.5 and 3.4.10, we can easily show that $(f, \psi) \Rightarrow (X_1, X_2)$ on $\Omega^1_{1/2}$, where X_1, X_2 are random elements defined on $(W, \mathbb{B}, m)$ by $X_1(z,w) = \overline{w(k)}k^z, X_2(z,w) = \sum_{\substack{n \equiv 1 \\ \text{mod } k}} w(n)n^{-z}$.

Since $w(k)$ is stochastically independent of each $w(n)$ with $n \equiv 1$ modulo k (as $(k,1) = 1$), X_1, X_2 are stochastically independent. Also, by corollary 5.3.5 (with $a_n = 1$ if $n \equiv 1 \bmod k$, and $= 0$ otherwise) the spectrum of ψ is $H(\Omega^1_{1/2})$. That is, the support of X_2 is $H(\Omega^1_{1/2})$. Since X_1 is independent of X_2 and X_1 is not degenerate at 0, this implies that the support of X_1X_2 is $H(\Omega^1_{1/2})$. But $\zeta(\cdot,\alpha) \Rightarrow X_1X_2$ on $\Omega^1_{1/2}$. Hence the spectrum of $\zeta(\cdot,\alpha)$ is $H(\Omega^1_{1/2})$. This proves (a). (b) can be proved similarly by using the corresponding discrete version results.

5.3.8 <u>Theorem</u> : For a real σ, let ζ_σ be the restriction of ζ to the line $\{Re(z) = \sigma\}$ (i.e., ζ_σ is defined by $\zeta_\sigma(t) = \zeta(\sigma+it)$, $t \in \mathbb{R}$).

(a) If $\sigma < \frac{1}{2}$ then the spectrum of ζ_σ is empty.

(b) If $\frac{1}{2} < \sigma < 1$ then the spectrum of ζ_σ is $C(\mathbb{R})$. That is, for any continuous complex valued function f on $\mathbb{R}$, $T > 0, h > 0$ and $\varepsilon > 0$, then (i) the set of all $t \in \mathbb{R}$ for which $\sup_{|x| \leq T} |\zeta_\sigma(x+t) - f(x)| < \varepsilon$ has positive lower density (continuous version) and (ii) the set of all $m \in Z$ for which $\sup_{|x| \leq T} |\zeta_\sigma(x+mh) - f(x)| < \varepsilon$ has positive lower density (discrete version).

Finally, (c) the spectrum of ζ_1 is a proper subset of $C(\mathbb{R})$. (Analogous results hold for all the L-functions).

Proof : (a) is an immediate consequence of the equation (iv) of 3.4.12 (b).

To prove (b), notice that by corollary 5.3.6, the spectrum of ζ on $\Pi^1_{1/2}$ is $S = \{ f \in H(\Pi^1_{1/2}) : f \equiv 0 \text{ or } \frac{1}{f} \in H(\Pi^1_{1/2}) \}$. Let $\phi_\sigma : H(\Pi^1_{1/2}) \longrightarrow C(\mathbb{R})$ be the map which sends $f \in H(\Pi^1_{1/2})$ to the function f_σ where $f_\sigma(x) = f(\sigma+ix)$, $x \in \mathbb{R}$. ϕ_σ is a flow homomorphism. Hence by proposition 1.5.8, the spectrum of $\zeta_\sigma = \phi_\sigma(\zeta)$ contains $\overline{\phi_\sigma(S)}$. So it suffices to show that $\overline{\phi_\sigma(S)} = C(\mathbb{R})$. Clearly for any polynomial $P : \mathbb{R} \longrightarrow \mathbb{C}$, $f = e^P$ is in $\phi_\sigma(S)$. Since any $g \in C(\mathbb{R})$ can be approximated, uniformly on compacta, by polynomials, it follows that $\exp g \in \overline{\phi_\sigma(S)}$ for any $g \in C(\mathbb{R})$. But any nonvanishing $f \in C(\mathbb{R})$ may be written as $\exp g$ with $g \in C(\mathbb{R})$. Thus it sufficies to show that the set $A(\mathbb{R})$ of nowhere vanishing members of $C(\mathbb{R})$ is dense in $C(\mathbb{R})$. Clearly any linear function (i.e., function of the form $x \longrightarrow ax+b$, where $a,b \in \mathbb{C}$) can be approximated by members of $A(\mathbb{R})$, and since $A(\mathbb{R})$, and therefore $\overline{A(\mathbb{R})}$, is closed under pointwise multiplication, it follows that the pointwise product of finitely many linear functions is in $\overline{A(\mathbb{R})}$. Since any polynomial may be represented as such a product, it follows that all polynomials belong to $\overline{A(\mathbb{R})}$. Hence $\overline{A(\mathbb{R})} = C(\mathbb{R})$. Hence $\overline{\phi_\sigma(S)} = C(\mathbb{R})$, as was to be shown.

Proof of (c). Because of the pole of ζ at $z=1$, ζ_1 is not a point in $C(\mathbb{R})$. Therefore, as formulated, the statement in (c) does not really make any sense. However, ζ_1 may be regarded as a point in $C^*(\mathbb{R})$, the space of $\mathbb{C}_\infty$-valued continuous function on $\mathbb{R}$, with the topology of uniform convergence on compacta (here $\mathbb{C}_\infty$ is the Riemann sphere with the metric d of 2.2.2). The space $C^*(\mathbb{R})$ may be made into a continuous flow under shift transformation. $\phi_1 : M(\underline{\bigcap}^{\infty}_{1/2}) \longrightarrow C^*(\mathbb{R})$ defined by $\phi_1(f) = f_1$, where $f_1(x) = f(1+ix)$, defines a flow homomorphism. Since $\zeta \Rightarrow F$ when regarded as a point in $M(\underline{\bigcap}^{\infty}_{1/2})$, it follows that $\zeta_1 \Rightarrow F_1$ when regarded as a point in $C^*(\mathbb{R})$. Here F_1 is the $C(\mathbb{R})$-valued random element defined on $(W,\mathbb{B},m)$ by $F_1(x,w) = \sum_{n=1}^{\infty} w(n)n^{-1-ix}$. Therefore, in order to prove (c) above, it suffices to show that the <u>support of F_1 is a proper subset of $C(\mathbb{R})$</u>. This will follow once we show that the support of $\log F_1$ is a proper subset of $C(\mathbb{R})$. We have :

$$\log F_1(x,w) = \sum_{p\in\mathbb{P}} -\log(1-w(p)p^{-1-ix}) = G_1(x,w) + G_2(x,w), \quad x \in \mathbb{R},\ w \in W,$$

where $G_1(x,w) = \sum_{p\in\mathbb{P}} p^{-1-ix}$, $x \in \mathbb{R}$, $w \in W$.

Clearly there is a constant c_1 such that $|G_2(x,w)| \leq c_1$ for all $x \in \mathbb{R}$, $w \in W$. Let $f : R \longrightarrow \mathbb{C}$ be a twice continuously differentiable map with compact support, $f \neq 0$ (for example, we may take $f(x) = \exp(-\frac{1}{1-x^2})$ if $|x| \leq 1$, $= 0$ otherwise). We have $|\int f(x)G_2(x,w)dx| \leq c_1 \int |f(x)|dx < c_2$. Also, since for almost all

$w \varepsilon W$, the series for G_1 converges uniformly on compacta,

$\int G_1(x,w)f(x)dx = \sum_{p\varepsilon \mathbb{P}} \int p^{-1-ix}f(x)dx$, almost surely.

But, integrating by parts twice, one gets :

$$\int p^{-ix}f(x)dx = -\frac{1}{(\log p)^2}\int p^{-ix}f''(x)dx .$$

Hence $|\int G_1(x,w)f(x)dx| \leq \int |f''(x)|dx \sum_{p\varepsilon \mathbb{P}} \frac{1}{p(\log p)^2} = c_3 < \infty$.

Therefore $|\int \log F_1(x,w)f(x)dx| \leq c = c_1+c_3$ almost surely.

Here $0 < c < \infty$ and c depends only on f. Therefore for almost all $w \varepsilon W$, $\log F_1(.,w) \varepsilon A$ where $A = \{g \varepsilon C(\mathbb{R}) : |\int g(x)f(x)dx| \leq c\}$. Clearly A is a closed subset of $C(\mathbb{R})$; since $f \neq 0$, A is a proper subset of $C(\mathbb{R})$. Since the support of $\log F_1$ is contained in A, we are done.

5.3.9 Remarks : (a) The theorems 5.3.3 and 5.3.4 appear to be the uttermost limit to which the universality theorem of Voronin ([52]) may be extended. Indeed, Voronin's theorem is properly contained in the corollary 5.3.6. The theorems 5.3.3 and 5.3.4 may be used to conclude that no nontrivial algebraic-differential equation connects the Dirichlet L-functions. They also may be used to produce denseness results of all sorts. For example, from the corollary 5.3.6, we may deduce the following two :

(i) Let $z_1, \ldots, z_n$ be distinct points in $\Omega_{1/2}^{1}$. Let $h > 0$ be a real number, $k \geq 1$ an integer. For any $m \varepsilon Z$, let A_m be the $n \times k$ matrix whose (r,s)th entry is $\zeta^{(s-1)}(z_r+imh)$, $1 \leq r \leq n$, $1 \leq s \leq k$. Then the sequence $\{A_m : m \varepsilon Z\}$ is dense in the space $\mathbb{C}_{n,k}$ of $n \times k$ complex matrices. Indeed, for any nonempty open set $U \subseteq \mathbb{C}_{n,k}$, the set of all $m \in Z$ for which $A_m \in U$ has positive lower density.

(ii) Again, let n,k,h be as above. Let $A = ((\alpha_{r,s}))$ be a fixed complex matrix of order $n \times k$ such that $\alpha_{r,1} \neq 0$ $(1 \leq r \leq n)$ and the entries in each column of A are distinct. Let $B = ((N_{r,s}))$ be an $n \times k$ matrix with non-negative integral entries. Let U be a Jordan region with compact closure contained in $\Omega_{1/2}^{1}$. Then for infinitely many $m \in Z$, the equation

$$\zeta^{(s-1)}(z) = \alpha_{r,s} \text{ has } \underline{\text{exactly}} \; N_{r,s} \text{ solutions } z \text{ in}$$

$U + imh$ (counting multiplicity) for $1 \leq r \leq n$, $1 \leq s \leq k$.

Indeed the set of all $m \in Z$ for which this holds has positive lower density. This may be readily deduced from corollary 5.3.6 and Rouche's theorem (theorem 3.42 in [49, p.116]).

Notice that the results (i) and (ii) above clearly include all the theorems in the paper [51] of Voronin. A continuous analogous of these discrete version results could easily be proved. They hold for all L-functions ; further in view of corollary 5.3.7, (b) holds for $\zeta(\cdot,\alpha)$ without the restriction $\alpha_{r,1} \neq 0$ provided $\alpha \neq \frac{1}{2}$ is a rational number in $(0,1)$.

(b) Let $0 < \alpha < 1$, α transcendental. In view of the remarks in 3.4.12 (g), and the lemma 5.2.10, the spectrum of the Hurwitz Zeta function $\zeta(\cdot,\alpha)$ on $\Omega_{1/2}^{1}$ is the closure of the set of all convergent (in $H(\Omega^1_{1/2})$) series $\sum_{n=1}^{\infty} a_n(n+\alpha)^{-z}$, $a_n \in \Pi$. From lemmas 5.2.2 and 5.2.5 we see that whenever μ is a complex Borel measure with compact support contained in $\Omega^1_{1/2}$ is such that $\sum_{n=1}^{\infty} |\int (n+\alpha)^{-z} d\mu(z)| < \infty$, we have $\int z^{\rho} d\mu(z) = 0$ for $\rho = 0,1,2,\ldots$ Hence from lemma 5.2.9 we get :

(iii) If $\alpha \in (0,1)$ is transcendetal then the spectrum of $\zeta(\cdot,\alpha)$ on $\Omega^1_{1/2}$ is the whole of $H(\Omega^1_{1/2})$. This should be compared with corollary 5.3.7 above, which says that if $\alpha \in (0,1)$ is rational, $\alpha \neq \frac{1}{2}$ then the spectrum of $\zeta(\cdot,\alpha)$ on $\Omega^1_{1/2}$ is $H(\Omega^1_{1/2})$. Also, if $\alpha = \frac{1}{2}$, we have $\zeta(\cdot,\frac{1}{2}) = (2^z-1)\zeta(z)$. Hence from corollary 5.3.6, it can easily be deduced that the spectrum of $\zeta(\cdot,\frac{1}{2})$ on $\Omega^1_{1/2}$ is the set S of theorem 5.3.1. These observations together leave open only the case of an irrational algebraic number α.

5.3.10 Questions :

(a) What is the spectrum of $\zeta_{1/2}$? This is the only case left open after theorem 5.3.8. Notice that we do not know if $\zeta_{1/2}$ has an asymptotic distribution. So, rigorously speaking, this question does not make sense. However, in this context by the spectrum of $\zeta_{1/2}$ we understand the set of all $f \in C(\mathbb{R})$ for which every neighbourhood U of f satisfies

$$\underline{d}(\{t \in \mathbb{R} : S^t(\zeta_{1/2}) \in U\}) > 0 .$$

This appears to be a very difficult question. We do not even know if the set $\left\{ \zeta(\frac{1}{2} + ix) : x \in \mathbb{R} \right\}$ is dense in $\mathbb{C}$.

(b) What is the spectrum of $\zeta(\cdot, \alpha)$ on $\Omega^{1}_{1/2}$ if $\alpha \in (0,1)$ is an irrational algebraic number ? We expect the spectrum to be the whole of $H(\Omega^{1}_{1/2})$. But the proof is bound to involve many technical complications.

5.3.11 <u>Theorem</u> (Universality of the sequence of L-functions with prime moduli) : Let K be a compact, simply connected and locally path connected subset of $\Omega^{1}_{1/2}$. Let f be a <u>non-vanishing</u> continous function on K which is analytic in the interior (if any) of K. Let $\varepsilon > 0$. <u>Then</u> there is a constant $c > 0$ such that for all sufficiently large primes p, at least cp of the Dirichlet characters χ modulo p satisfy -

$$\sup_{z \in K} |L(z,\chi) - f(z)| < \varepsilon .$$

<u>Proof</u> : As in the proof of theorem 5.3.7, it suffices to prove the result in case f admits a nowhere vanishing analytic continuation to $\Omega^{1}_{1/2}$. In this case let

$$U = \left\{ g \in H(\Omega^{1}_{1/2}) : \sup_{z \in K} |g(z) - f(z)| < \varepsilon \right\} .$$

Let F be the random element introduced in 3.4.7. After theorem 5.3.1, we know that f belongs to the support of F. Since U is an open neighbourhood of f, it follows that $m(F \in U) > 0$. Hence by theorem 4.6.1 we have the required result.

5.3.12 Remarks : (a) From the theorem 5.3.12 we can deduce the following analogues of the results (i) and (ii) of 5.3.10 :

(i) Let $z_1, z_2, \ldots, z_n$ be distinct points in $\Omega^1_{1/2}$. For any Dirichlet character χ let $A(\chi)$ be the $n \times k$ matrix whose (r,s) th entry is $L^{(s-1)}(z_r, \chi)$. Then, for any open subset U of the space $\mathbb{C}_{n,k}$ of $n \times k$ complex matrices, the set $\{A(\chi) : \chi$ is a Dirichlet character modulo $p\}$ intersects U for all large prime p. Indeed, there is a $c > 0$ such that at least cp of the elements of this set belong to U for all large primes p.

(ii) Let n, k be positive integers. Let $A = ((\alpha_{r,s}))$ be a fixed complex matrix of order $n \times k$ such that $\alpha_{r,1} \neq 0$ $(1 \leq r \leq n)$ and the entries in each column of A are distinct. Let $B = ((N_{r,s}))$ be an $n \times k$ matrix with non-negative integral entries. Let U be a Jordan domain with compact closure contained in $\Omega^1_{1/2}$. Then for infinitely many Dirichlet characters χ with prime moduli, the equation $L^{(s-1)}(z) = \alpha_{r,s}$ has exactly $N_{r,s}$ solutions (counting multiplicity) z in U for $1 \leq r \leq n$ and $1 \leq s \leq k$. Indeed, there is a constant $c > 0$ such that at least cp of the $p-1$ Dirichlet characters χ midulo p satisfy this condition for all large primes p.

(b) In view of the remarks in 4.6.2(b), the theorem 5.3.12 and its consequences discussed above continue to hold if the sequence $\{p_n\}$ of primes is replaced by any sequence $\{a_n\}$ of positive integers such that the smallest prime divisor of a_n goes to infinity with n.

5.4 Strong recurrence and zero free regions :

5.4.1 Theorem : Let $\frac{1}{2} \leq a < b \leq 1$, and let χ be a Dirichlet character. Then the Dirichlet L-function $L(\cdot,\chi)$ is non-vanishing on Ω_a^b if and only if $L(\cdot,\chi)$ is strongly recurrent on Ω_a^b.

Proof : " Only if " : Consider the restriction of $L(\cdot,\chi)$ to Ω_a^b as a point in the continuous flow $H(\Omega_a^b)$. It follows from corollary 5.3.6 that its spectrum is $\{f \in H(\Omega_a^b) : f \equiv 0 \text{ or } \frac{1}{f} \in H(\Omega_a^b)\}$. If $L(\cdot,\chi)$ is nonvanishing on Ω_a^b then it belongs to its own spectrum. Hence by proposition 1.5.4(c) it follows that $L(\cdot,\chi)$ is strongly recurrent on Ω_a^b.

" If " : Let us suppose that $L(\cdot,\chi)$ is strongly recurrent on Ω_a^b and there is a $\rho \in \Omega_a^b$ such that $L(\rho,\chi) = 0$. Let γ be a simple closed curve contained in Ω_a^b such that if U is the region bounded by γ (so that $\overline{U} \subset \Omega_a^b$) then ρ is the only zero of $L(\cdot,\chi)$ in $\overline{U}$. Let us choose ε such that $0 < \varepsilon < \min_{z\in\gamma} |L(z,\chi)|$. Let $A = \{t \in \mathbb{R} : \sup_{z\in\overline{U}} |L(z+it,\chi) - L(z,\chi)| < \frac{\varepsilon}{2}\}$.

Since $L(\cdot,\chi)$ is assumed to be strongly recurrent, it follows that

(i) $\underline{d}(A) > 0$.

On the other hand, let V be the set of all $f \in H(\Omega_a^b)$ such that f has a zero inside U and $|f(z)| \geq \frac{\varepsilon}{2}$ for $z \in \gamma$. By Hurwitz' theorem ([18, p.148]) the set V is closed. Since by corollary 3.4.6, $L(\cdot,\chi) \Rightarrow F^{\chi}$ on Ω_a^b when F^{χ} is as given in 5.3.1, it follows from the portmanteau theorem that $\bar{d}(B) \leq m(F^{\chi} \varepsilon V)$ where $B = \{ t \varepsilon \mathbb{R} : S^t(L(\cdot,\chi)) \varepsilon V \}$. But the product representation of F^{χ} shows that F_{χ} is almost surely nonvanishing on Ω_a^b. Therefore $m(F^{\chi} \varepsilon V) = 0$. Hence

(ii) $\bar{d}(B) = 0$.

But Rouche's theorem [49, p.116] implies that $A \subseteq B$. Hence

(iii) $\underline{d}(A) \leq \bar{d}(B)$.

(i), (ii) and (iii) together yield a contradiction. So we are done.

5.4.2 <u>Corollary</u> : Let χ be a Dirichlet character. Then the Riemann hypothesis for $L(\cdot,\chi)$ is valid if and only if $L(\cdot,\chi)$ is strongly recurrent on $\Omega_{1/2}^1$. In particular, the classical Riemann hypothesis is valid if and only if Zeta is strongly recurrent on $\Omega_{1/2}^1$.

<u>Proof</u> : This follows from theorem 5.4.1 since, as is well known, all the nontrivial zeros of $L(\cdot,\chi)$ lie on the critical line $\{Re(z) = \frac{1}{2}\}$ if and only if $\Omega_{1/2}^1$ is free from zeros of $L(\cdot,\chi)$.

5.4.3 <u>Remarks</u> : The relationship between strips of strong recurrence and zero free regions of an L-function enunciated in 5.4.1 can be traced back to the existence of an Euler product for such a function. No such relationship need exist in the absence of an Euler product. This can be seen from the examples that follow.

5.4.4 <u>Examples</u> : Let $f \in H(\Omega^1_{1/2})$ be one of the following -

(i) $f = L^{(n)}(\cdot,\chi)$ where $n \geq 1$, χ a Dirichlet character.

(ii) $f = \zeta(\cdot,\alpha)$ where $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, α is rational or transcendental.

(iii) $f(z) = \sum_{\substack{n=1 \\ (n,k)=1}}^{\infty} a_n n^{-z}$, where $k \geq 1$, $\{a_n\}$ is a periodic sequence of complex numbers with period k, and f is not of the form $c.L(\cdot,\chi)$ for any constant c and Dirichlet character χ.

<u>Then</u> (a) f is strongly recurrent on $\Omega^1_{1/2}$,

and (b) The set of real parts of the zeros of f is dense in $[\frac{1}{2},1]$.

<u>Proof</u> : We first note that in each case the spectrum of f on $\Omega^1_{1/2}$ is the whole of $H(\Omega^1_{1/2})$. In cases (ii) and (iii) this follows from 5.3.7, 5.3. 9(b) and 5.3.5. If f is the <u>nth</u> derivative of $L(\cdot,\chi)$, then, since the spectrum of $L(\cdot,\chi)$ is $S = \{g \in H(\Omega^1_{1/2}) : g \equiv 0$ or $\frac{1}{g} \in H(\Omega^1_{1/2})\}$ it follows that the spectrum of f contains the closure of the set $\{g^{(n)} : g \in S\}$. So it suffices to notice that this set is dense (indeed, any bounded member of $H(\Omega^1_{1/2})$ differs from a member of S by a constant, hence this set contains the <u>nth</u> derivatives of all bounded members of $H(\Omega^1_{1/2})$. Since the bounded members constitute a dense set, the same must be true of their image under the onto operation of n-times differentiation).

In particular, f belongs to its own spectrum, and therefore by 1.5.4(c), f is strongly recurrent. This establishes (a). To prove (b), let $\frac{1}{2} < \alpha < \beta < 1$. Let $g \in H(\Omega_{1/2}^{1})$ have a zero inside Ω_α^β. Let K be a compact convex subset of Ω_α^β which contains this zero of g and which is such that the boundary γ of K is a simple closed curve on which g is non-zero. Let $0 < \varepsilon < \min_{z \in \gamma} |g(z)|$. Since g lies in the spectrum of f, there exists $t \in \mathbb{R}$ such that $\sup_{z \in K} |f(z+it) - g(z)| < \varepsilon$. Hence by Rouche's theorem f has a zero in $K - it$. Hence f has a zero in Ω_α^β. Since $\frac{1}{2} < \alpha < \beta < 1$ and α, β were otherwise arbitrary, this proves (b).

5.4.5 <u>Example</u> : Let ρ be the entire function defined, for $\mathrm{Re}(z) > 0$, by the series $\rho(z) = \sum_{n=1}^{\infty} a_n n^{-z}$, where -

$$a_n = \begin{cases} 0 & \text{if } n \equiv 0 \text{ modulo } 5 \\ 1 & \text{if } n \equiv 1 \quad " \quad " \\ -1 & \text{if } n \equiv -1 \quad " \quad " \\ a & \text{if } n \equiv 2 \quad " \quad " \\ -a & \text{if } n \equiv -2 \quad " \quad " \end{cases}$$

with $a = \dfrac{\sqrt{10 - 2\sqrt{5}}}{\sqrt{5} - 1} = 0.284079\ldots$

Titchmarsh has shown ([50, pp.240-241]) that ρ satisfies the functional equation

$$\left(\frac{5}{\pi}\right)^{z/2} \Gamma\left(\frac{1}{2} + \frac{z}{2}\right)\rho(z) = \left(\frac{5}{\pi}\right)^{\frac{1-z}{2}} \Gamma\left(\frac{1}{2} + \frac{1-z}{2}\right)\rho(1-z).$$

This, together with the arguments of 5.4.4, show that ρ is strongly recurrent in $\Omega_{1/2}^{1}$ and the set of real parts of the zeros of ρ is dense in $[0,1]$. We also know that ρ has zeros to the right of the line $\{Re(z) = 1\}$ ([50,pp.242-243]).

5.4.6 Theorem : If the recurrence conjecture (2.4.10) holds, then the set of real parts of the zeros of the Riemann Zeta function is dense in $[0,1]$.

Proof : Let $\emptyset$ denote the supremum of the real parts of the Zeta zeros. It is known that $\frac{1}{2} \leq \emptyset \leq 1$. Let us suppose that $\emptyset < 1$. Then $\Omega_{\emptyset}^{1}$ is free of Zeta zeros. Hence by theorem 5.3.1, ζ is strongly recurrent on $\Omega_{\emptyset}^{1}$. Hence by 2.2.5 (v), $\frac{1}{\zeta}$ is strongly recurrent on $\Omega_{\emptyset}^{1}$. Also, it is known that the Dirichlet series representation of $\frac{1}{\zeta}$ converges in $\Omega_{\emptyset}^{1}$ ([50, p.315]) and hence $\frac{1}{\zeta}$ is of finite order in $\Omega_{\emptyset}^{1}$ ([49,p.297]). Finally, for any a such that $\emptyset < a < 1$, it is known that $\frac{1}{\zeta}$ has finite mean square value on the line $Re(z) = a$ ([50,p.284]). Thus $\frac{1}{\zeta}$ satisfies all the hypotheses of 2.4.9 on Ω_{a}^{1}. Therefore by the recurrence conjecture, $\frac{1}{\zeta}$ is strongly recurent on Ω_{a}^{∞}. Since $\emptyset < a < 1$, and a was otherwise arbitrary, $\frac{1}{\zeta}$ is strongly recurrent on $\Omega_{\emptyset}^{\infty}$. But $\frac{1}{\zeta}$ has a zero inside $\Omega_{\emptyset}^{\infty}$; namely at $z = 1$. Arguing as in the proof of the " if " part of theorem 5.4.1, we can deduce that $\frac{1}{\zeta}$ has infinitely many zeros in $\Omega_{\emptyset}^{1}$. Thus ζ has infinitely many poles. But this is false. The only pole of ζ is at $z = 1$. This contradiction shows that $\emptyset = 1$.

Now let us suppose that the set of real parts of the Zeta zeros is not dense in $[0,1]$. Since the functional equation for the Zeta function [50, p.13] shows that the nontrivial Zeta zeros are symmetrically placed about the line $\{Re(z)=\frac{1}{2}\}$, it follows that there exist a,b, $\frac{1}{2}<a<b<1$, such that Ω_a^b is free of Zeta zeros. Hence by theorem 5.4.1, ζ is strongly recurrent in Ω_a^b. Let us put $f(z)=(1-2^{1-z})\cdot\zeta(z)$. Since the factor $(1-2^{1-z})$ is periodic, it follows from corollary 2.3.5 (or proposition 2.3.3 itself) that f is strongly recurrent. Now, on Ω_a^b f satisfies all the hypotheses of the recurrence conjecture. Hence in view of the assumed validity of that conjecture, f is strongly recurrent on Ω_a^∞. A fortiori, f is strongly recurrent on Ω_a^1. Since $(1-2^{1-z})^{-1}$ is also a periodic point in $H(\Omega_a^1)$, another application of proposition 2.3.3 shows that ζ is strongly recurrent in Ω_a^1. Hence by theorem 5.4.1, ζ has no zero in Ω_a^1. That is, $\emptyset \leq a<1$. But this contradicts our observation in the first part of this proof that $\emptyset = 1$ under recurrence conjecture. This contradiction proves the result.

5.4.7 <u>Theorem</u> : Let χ be a nonprincipal Dirichlet character and let $a(\chi)$ be the supremum of the real parts of the zeros of $L(\cdot,\chi)$. If the recurrence conjecture (2.4.10) holds then the set of real parts of the zeros of $L(\cdot,\chi)$ is dense in $[1-a(\chi),a(\chi)]$.

<u>Proof</u> : Since χ is nonprincipal, $L(.,\chi)$ has no pole, and therefore the first part of the proof of theorem 5.4.5 does not go through. However, the second part of that proof does go through, and this yields the above result.

5.4.8 <u>Theorem</u> : Let $\frac{1}{2} < \sigma < 1$. Then ζ_σ is strongly recurrent. The same holds for the vertical sections $L_\sigma(.,\chi)$ of a Dirichlet L-function $L(.,\chi)$.

<u>Proof</u> : Recall that $\zeta_\sigma \in C(\mathbb{R})$ is defined by $\zeta_\sigma(x) = \zeta(\sigma+ix)(x\in\mathbb{R})$. By theorem 5.3.8 the spectrum of ζ_σ is the whole of $C(\mathbb{R})$, and hence ζ_σ belongs to its own spectrum. Hence by proposition 1.5.4(c), ζ_σ is strongly recurrent. An analogue of theorem 5.3.8 for $L(.,\chi)$ can be deduced from corollary 5.3.6, and hence we can deduce the above result for $L(.,\chi)$.

5.4.9 <u>Question</u> : Is ζ_1 a strongly recurrent point of $C^*(\mathbb{R})$? What about $\zeta_{1/2}$ as a point of $C(\mathbb{R})$?

5.4.10 <u>Example</u> (on the recurrences conjecture) : There exists a real number a and a function f which is <u>analytic</u> and <u>of finite order</u> in the closure of Ω_a^∞ and which satisfies :

(i) f is represented by an absolutely convergent Dirichlet series in Ω_b^∞ for some $b > a$,

(ii) $\int_{-T}^{T} |f(a+it)|^2 dt = O(T)$ as $T \longrightarrow \infty$,

and (iii) f is strongly recurrent on the line $\{Re(z) = a\}$,

but (iv) f is <u>not</u> strongly recurrent on Ω_a^∞.

<u>Proof</u> : We have to consider two possibilities separately.

<u>Case I</u>. The Riemann hypothesis holds. In this case we take any a such that $\frac{1}{2} < a < 1$, and $f = \frac{1}{\zeta}$. By theorem 5.4.1, ζ and hence also $f = \frac{1}{\zeta}$, is strongly recurrent on $\Omega_{1/2}^{1}$. Hence f is strongly recurrent on the line $\{Re(z) = a\}$. However, f is not strongly recurrent in Ω_a^{∞} : if it were then one could deduce (as in the proof of theorem 5.4.1) that the zero of f at $z = 1$ would " reproduce itself" infinitely often, which is not the case. That f satisfies the other conditions under Riemann hypothesis is well known.

<u>Case II</u>. The Riemann hypothesis does not hold. Let $\emptyset$ denote, as before, the supremum of the real parts of the zeros of Zeta. In this case $\frac{1}{2} < \emptyset \leq 1$. Take any a such that $\frac{1}{2} < a < \emptyset$, and take $f(z) = (1-2^{1-z})\ \zeta(z)$. Since by theorem 5.3.8 ζ is strongly recurrent on the line $Re(z) = a$, and since $(1-2^{1-z})$ is periodic, the analogue of proposition 2.3.3 for $C(\mathbb{R})$ (which can be proved similarly) shows that f is strongly recurrent on the line $\{Re(z) = a\}$. But f is not strongly recurrent on Ω_a^{∞}. If it were, then it would in particular be strongly recurrent on Ω_a^{1}, and hence by proposition 2.3.3 Zeta would be strongly recurrent on Ω_a^{1}. By theorem 5.4.1, this would imply that ζ is nonvanishing on Ω_a^{1}, and hence $\emptyset \leq a$. But this contradicts the choice of a. That f satisfies the other conditions is well known.

5.4.11 <u>Theorem</u> : If the set of strong recurrent points of $H(\underline{(\overline{\ })}^{1}_{1/2})$ is closed under addition then the generalized Riemann hypothesis for Dirichlet L-functions hold.

<u>Proof</u> : As a particular case of 5.4.4 (iii), we have the following : Let $1 \leq h < k$, $(h,k) = 1$, $k \geq 3$. Then the function $\psi_{h,k}$ given by $\psi_{h,k}(z) = \sum_{\substack{n=1 \\ n \equiv h \bmod k}}^{\infty} n^{-z}$ is strongly recurrent on $\underline{(\overline{\ })}^{1}_{1/2}$.

If χ is a Dirichlet character modulo k, where $k \geq 3$, then $L(\cdot,\chi) = \sum_{\substack{h=1 \\ (h,k)=1}}^{k-1} \chi(h)\psi_{h,k}$ is an expression of $L(\cdot,\chi)$ as a finite sum of strongly recurrent members of $H(\underline{(\overline{\ })}^{1}_{1/2})$. Therefore under the given hypothesis, $L(\cdot,\chi)$ is strongly recurrent on $\underline{(\overline{\ })}^{1}_{1/2}$, and hence by theorem 5.4.1, $L(\cdot,\chi)$ has no zero in $\underline{(\overline{\ })}^{1}_{1/2}$. In view of the functional equation and the Euler product for $L(\cdot,\chi)$, this implies that all the nontrivial zeros of $L(\cdot,\chi)$ lie on the critical line $\{\mathrm{Re}(z) = \frac{1}{2}\}$.

If χ is the unique Dirichlet character modulo k, where $k = 1$ or 2, then $L(z,\chi) = \zeta(z)$ or $= (1-2^{-z})\zeta(z)$. In this case we use the representation

$$\zeta(z) = (1-3^{-z})^{-1}\psi_{1,3}(z) + (1-3^{-z})^{-1}\psi_{2,3}(z) .$$

Due to proposition 2.3.3, the two summands are again strongly recurrent on $\underline{(\overline{\ })}^{1}_{1/2}$. Hence we can complete the proof as before.

5.5 Concluding remarks :

(a) In our opinion, the chief merit of theorem 5.4.1 lies in the fact that it establishes the equivalence of a local property (namely absence of zeros) of the Riemann Zeta function (and the L-functions in general) and a global property (viz, strong recurrence) of the same function. In this regard the result of 5.4.1 differs from the countless equivalent formulations of the Riemann hypothesis, available in the literature, which are either trite rewordings (using, perhaps, different integral representations of the Zeta function) or else relate the Riemann hypothesis to the inscrutable growth rates of various arithmetic functions. A detailed study of $H(\Omega^{1}_{1/2})$ as a flow, with particular reference to its strongly recurrent points, should throw useful light on the question.

(b) In view of the inheritance theorem 1.4.8 (or as consequences of theorem 4.5.8) the criterion of theorem 5.4.1 could also be stated in terms of strong recurrence modulo h for an arbitrary real $h > 0$.

(c) It must be admitted that the combined effect of theorems 5.4.6, 5.4.8 and 5.4.11 is to leave the question of plausibility of the Riemann hypothesis in utter confusion. In view of theorem 2.4.8, we are inclined to favour the recurrence conjecture as against the possibility that the strongly recurrent points of $H(\Omega)$ form a subalgebra. But it should be noted that for the comparable notion of almost periodicity due to Bohr (which is a recursion notion for the space $H_0(\Omega)$ of bounded analytic functions on the strip Ω with the topology of uniform convergence and with shift as flow projection), the almost periodic points do form a subalgebra (theorem 5 of [3, p.143]).

(d) Theorem 5.4.8 says that each member of the class $\{\zeta_\sigma : \frac{1}{2} < \sigma < 1\}$ is a strongly recurrent point of $C(\mathbb{R})$. Notice that in view of the proposition 2.4.1 and theorem 5.4.1, the Riemann hypothesis is equivalent to the statement that this class is uniformly strongly recurrent.

(e) The corollary 5.3.6 may be interpreted as follows. Any statement regarding the local behaviour of $L(\cdot,\chi)$ in the strip $\Omega^1_{1/2}$ is either deducible from the Riemann hypothesis for $L(\cdot,\chi)$ (and the fact that $L(\cdot,\chi)$ is analytic in $\Omega^1_{1/2}$) or else it is untrue. The theorem 5.3.3 itself admits of a similar interpretation as to the collective local behaviour of a finite class of Dirichlet L-functions.

REFERENCES

[1] Apostol, T.M. (1976) : Modular functions and Dirichlet series in number theory. Springer-Verlag, New York.

[2] Berndt, B. (1970) : The number of zeros of $\zeta^{(k)}(s)$. J.Lond. math. soc. (series 2), 2, 577-580.

[3] Besicovitch, A.S. (1954) : Almost periodic functions. Dover Inc., New York.

[4] Billingsley, P. (1968) : Convergence of probability measures. John Wiley and sons, London.

[5] Boas, R.P. (Jr.) (1954) : Entire functions. Academic press, New York.

[6] Bohr, H. (1912) : Sur la function $\zeta(s)$ dans le demi-plan $\sigma > 1$. Comptes rendus, 154, 1078-1081.

[7] Bohr, H. (1912) : Uber die Funktion $\frac{\zeta'}{\zeta}(s)$. J. reine angew., 141, 217-234.

[8] Bohr, H. (1915) : Zur Theorie der Riemannschen Zetafunktion im kritischen streifen. Acta Math., 40, 67-100.

[9] Bohr, H. (1919) : Zur theorie der allgemeinen Dirichletschen Reihen. Math. Annalen, 79, 136-156.

[10] Bohr, H. (1922) : Uber diophantische Approximationen und ihre Anwendungen auf Dirichletsche Reihen, besonders auf die Riemannsche Zetafunktion. Proc. 5th congress of Scand. math., Helsingfors, 131-154.

11] Bohr, H. (1924) : Zur theorie der fastperiodischen FunktionenI. Acta Math., 45, 29-127.

12] Bohr, H. (1925) : Zur theorie der fastperiodischen Funktionen II. Acta Math., 46, 101-214.

13] Bohr, H. (1926) : Zur theoric der fastperiodischen Funktionen III. Acta Math., 47, 237-281.

14] Bohr, H. and Jessen, B. (1936) : On the distribution of the values of the Riemann Zeta function. Amer.J. Math., 58, 35-45.

15] Borchsenius, V. and Jessen, B. (1945) : Mean motions and values of the Riemann Zeta function. Acta Math., 80, 97-166.

16] Cassel, J.W.S. (1961) : Footnote to a note of Davenport and Heilbronn, J. Lond. math. soc., 36, 177-184.

17] Chowla, S. and Erdos, P. (1951) : A theorem on the distribution of values of L-series. J. Indian math. soc., 15A, 11-18.

18] Conway, J.B. (1973) : Functions of one complex variable. Springer-Verlag, New York.

19] Cramer, H. and Leadbetter, M.R. (1966) : Stationery and related processes. John Wiley and sons, London.

20] Davenport, H. and Heilbronn, H. (1936) : On the zeros of certain Dirichlet series I. J.Lond.math.soc. 11, 181-185.

21] Davenport, H. and Heilbronn, H. (1936) : On the zeros of certain Dirichlet series II. J.Lond.math.soc. 11, 307-312.

22] Duren, P.L. (1970) : Theory of H^p spaces. Academic press, New York and London.

23] Edwards, H.M. (1974) : Riemann's Zeta function. Academic press, New York.

24] Elliott, P.D.T.A. (1972) : On the distribution of $\arg L(s,\chi)$ in the half-plane $\sigma > \frac{1}{2}$. Acta Math., 20, 155-169.

25] Elliott, P.D.T.A. (1973) : On the distribution of the values of quadratic L-series in the half-plane $\sigma > \frac{1}{2}$. Inventiones math., 21, 319-338.

26] Gottschalk, W.H. and Hedlund, G.A. (1946) : Recursive properties of transformation groups. Bull. Amer. math.soc., 52, 488-489.

27] Gottschalk, W.H. and Hedlund, G.A. (1955) : Topological dynamics. Amer. math. soc. colloquium publications, 36.

28] Hardy, G.H. and Riesz, M. (1955) : The general theory of Dirichlet series. Cambridge univ. press, London.

29] Jain, N.C. (1975) : Central limit theorem in Banach space. Proc. 1st international conference on probability in Banach spaces, Oberwolfach, 113-130.

30] Jessen, B. and Tornehave, H. (1945) : Mean notions and zeros of almost periodic functions. Acta Math., 77, 137-279.

31] Jessen, B. and Wintner, A. (1935) : Distribution functions and the Riemann Zeta function. Trans. Amer. math. soc., 38, 48-88.

32] Kampen, E.R. Van (1937) : On the addition of convex curves and the densities of certain infinite convolutions. Amer.J.math., 59, 679-695.

33] Kampen, E.R. Van and Wintner, A. (1937) : Convolutions of distributions on convex curves and the Riemann Zeta function. Amer. j. math., 59, 175-204.

34] Kershner, R. (1937) : On the values of the Riemann Zeta function on fixed lines $\sigma > 1$. Amer.j.math., 59, 167-174.

35] Kershner, R. and Wintner, A. (1936) : On the boundary of the range of values of $\zeta(s)$. Amer.j.math., 58, 421-425.

36] Kershner, R. and Wintner, A. (1937) : On the asymptotic distribution of $\frac{\zeta'}{\zeta}(s)$ in the critical strip. Amer.j.math., 59, 673-678.

37] Kubilius, J. (1964) : Probabilistic methods in the theory of numbers. A.M.S. translation of math. monographs, 11.

38] Laurincikas, A. (1979) : Distribution des valeurs de certaines series de Dirichlet. Comptes rendus ser.A, 289, 43-45.

39] Loeve, M. (1955) : Probability theory, foundation, random sequences. Van Nostrand, Toronto.

40] Lukacs, E. (1970) : Characteristic functions. Griffin, London.

41] Massey, W.S. (1967) : Algebraic topology : an introduction. Harcourt, Brace and World Inc., New York.

42] Montgomery, H.L. (1971) : Topics in multiplicative number theory. Springer-Verlag, Berlin.

43] Reich, A. (1977) : Universelle werteverteilung von Eulerprodukten. Personal communication.

44] Riemann, B. (1892) : Uber die Anzahl der Primzahlen unter einer gegebenen Grosse. Gesammelte Werke (p.145). Teubner, Leipzig.

45] Rudin, W. (1962) : Fourier analysis on groups. Interscience, London.

46] Rudin, W. (1973) : Functional analysis. McGraw Hill, New York.

47] Rudin, W. (1974) : Real and complex analysis Tata-McGraw Hill, New Delhi.

48] Spira, R. (1976) : Zeros of Hurwitz Zeta functions. Math.comp., 30, 863-866.

49] Titchmarsh, E.C. (1958) : The theory of functions. Oxford univ. press, London.

50] Titchmarsh, E.C. (1951) : The theory of the Riemann Zeta function. Oxford univ. press, London.

51] Voronin, S.M. (1972) : The distribution of the non-zero values of the Riemann Zeta function. Trudy Mat. Inst. Steklov, 128, 131-150.

52] Voronin, S.M. (1975) : A theorem on the values of the Riemann Zeta function. Doklady Akad. Nauk S.S.S.R., 221, 771.

53] Wintner, A. (1936) : The almost periodic behaviour of the function $\frac{1}{\zeta(1+it)}$. Duke math. j., 2, 443-446.

54] Wintner, A. (1941) : On the asymptotic behaviour of the Riemann Zeta function on the line $\sigma = 1$. Amer.j.math., 63, 575-580.

55] Wintner, A. (1944) : Random factorisation and Riemann's hypothesis. Duke math.j., 11, 267-275.