SUFFICIENCY, PAIRWISE SUFFICIENCY AND BAYES SUFFICIENCY IN UNDOMINATED EXPERIMENTS

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INTRODUCTION

An experiment or a statistical structure consists of a set X and a family of probability measures $\mathbb{P}$ on X, indexed by a set Θ. The set X together with a σ-algebra $\underline{\underline{A}}$ of subsets of X is the sample space and the set Θ with a σ-algebra $\underline{\underline{C}}$ of subsets of Θ is the parameter space. To avoid trivialities we consider only situations where the parametrization is one-one, i.e. if θ_1 and θ_2 are distinct then so are P_{θ_1} and P_{θ_2}. Various notions of sufficiency of a σ-algebra $\underline{\underline{B}}$ is considered in statistical literature. Among these are the following.

(i) Classical. Conditional probability of $(X, \underline{\underline{A}})$ given $\underline{\underline{B}}$ is independent of θ.

(ii) Decision Theoretic. For every decision problem given any decision procedure there is an equivalent decision procedure based on $\underline{\underline{B}}$.

(iii) Bayes. Given any prior ξ on Θ, the posterior distribution of θ given $(X, \underline{\underline{A}})$ is the same as the posterior distribution given $(X, \underline{\underline{B}})$.

In what follows we will refer to (i) simply as sufficiency (ii) and (iii) will be called D-sufficiency and Bayes sufficiency respectively.

(i), (ii) and (iii) are known to be equivalent when $\{P_\theta : \theta \in (\underline{H})\}$ is dominated by a σ-finite measure. Burkholder's example of a non-sufficient σ-algebra containing a sufficient σ-algebra shows that in the undominated case neither (ii) nor (iii) is equivalent to (i). In this thesis we investigate the relationship between (i), (ii) and (iii) when all the σ-algebras involved are countably generated. Interest in countably generated σ-algebras stems from the fact that these and only these arise out of real valued functions.

Our attempts center around a conjecture of Blackwell. During a conversation, in the winter of '77, Blackwell conjectured that, when both the sample space and the parameter space are standard Borel, then for countably generated σ-algebras (i), (ii) and (iii) would be equivalent. (i) and (ii) turn out to be equivalent without any Standard Borel assumptions. However the situation is different in case of (i) and (iii). Examples show that without Standard Borel assumptions (iii) need not imply (i). We can show, and do so in this thesis, that for a class of Standard Borel experiments (iii) and (i) are indeed equivalent. The general case still remains unsolved.

In chapter I, we study the first part of Blackwell's conjecture, viz. (i) <==> (ii). It is shown there that a

D-sufficient σ-algebra always contains a sufficient σ-algebra. A theorem of Burkholder then establishes the equivalence of (i) and (ii) in the countably generated case.

Chapter II is devoted to a study of Bayes sufficiency in the countably generated case. The first theorem relates Bayes sufficiency to sufficiency on sets of measure 1. More precisely a σ-algebra $\underline{\underline{B}}$ is Bayes sufficient iff for every prior ξ on $(\underline{\underline{H}})$, $\underline{\underline{B}}$ is sufficient for P_θ s on a set of ξ-measure 1. In the countably generated situation Bayes sufficiency is also shown to be equivalent to pair-wise sufficiency for an enlarged class of probability measures. Using this result we show that " Test sufficiency" which is weaker than D-sufficiency, itself implies Bayes sufficiency. We then give examples which show that without Standard Borel assumptions on both the Sample space and the Parameter space Bayes sufficiency may fail to imply sufficiency. While we have not been able to establish the equivalence of Bayes sufficiency and sufficiency in the Standard Borel case, we show that " Bayes sufficiency implies sufficiency" is equivalent to the apparently weaker " Bayes sufficiency implies Test sufficiency".

In chapter III, we take up the class of Standard Borel Discrete experiments. In this case, that is when the sample space and the parameter space are both Standard Borel and the P_θ s are all discrete, pairwise sufficiency itself implies sufficiency. Since pairwise sufficiency is weaker than Bayes sufficiency, this result settles Blackwell's Conjecture in the discrete, Standard Borel Case. In the later part of Chapter III, we study some questions related to the existence of minimal sufficient σ-algebras. P_θ s being discrete, there is a natural minimal pairwise sufficient partition. The existence of minimal sufficient σ-algebras is shown to be equivalent to some set theoretic conditions on this pairwise sufficient partition. An example shows that even if the P_θ s are discrete, minimal sufficient σ-algebras may not exist.

The first part of Chapter IV, is essentially a generalisation of Chapter III. Here we seek conditions under which pairwise sufficiency would imply sufficiency. We introduce the notion of weak coherence and Borel localizable measures, both Standard Borel adaptations of known concepts. It is then shown that for experiments dominated by a Borel localizable measure, a completion of any pairwise sufficient σ-algebra yields a sufficient σ-algebra.

The last part of Chapter IV is also concerned with pairwise sufficiency, except that unlike the rest of this thesis, in this section non-countably generated σ-algebras are of prime interest. In this part we give two examples. The first describes a situation where a minimal pairwise sufficient σ-algebra does not exist and the second shows that a theorem in [12] cannot be improved. Ghosh, Morimoto and Yamada show in [12] that, in what they call weakly dominated experiments " a sub σ-algebra $\underline{\underline{B}}$ is pairwise sufficient and contains 'supports of P_{θ}' if and only if the probability densities admit a factorisation with respect to $\underline{\underline{B}}$". Thus in these situations, for the statistician interested only in the Likelihood function, pairwise sufficiency provides the most natural reduction of the sample space. This fact together with easy verifiability makes pairwise sufficiency, a concept of statistical interest. Our example shows that a factorization theorem of the sort mentioned above is not valid, if the experiment is not weakly dominated.

Parts of this thesis is in the process of publication. Chapter I and parts of Chapter II constitutes [28]. [26] is essentially Chapter III. The example on non-existence of minimal pairwise sufficient σ-algebras will appear in [12]. The rest is not yet submitted for publication.

CHAPTER I

INTRODUCTION

Let $\mathbb{E} = (X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\textcircled{H}})$ and $\mathbb{F} = (Y, \underline{\underline{B}}, P'_\theta : \theta \in \underline{\textcircled{H}})$ be two experiments. Blackwell [3] defined $\mathbb{F}$ to be sufficient for $\mathbb{E}$, if there exists a transition function $Q(.,.)$ from $(Y, \underline{\underline{B}})$ to $(X, \underline{\underline{A}})$, such that for all θ in $\underline{\textcircled{H}}$ and A in $\underline{\underline{A}}$.

$$\int_Y Q(y,A)\, dP'_\theta(y) = P_\theta(A) .$$

Later in [4] he showed that, if $\underline{\textcircled{H}}$ is finite then, $\mathbb{F}$ is sufficient for $\mathbb{E}$ iff for a certain class of decision problems given any decision rule δ in $\mathbb{E}$ there is a decision rule δ' in $\mathbb{F}$ which is equivalent to δ.

Suppose $\underline{\underline{B}}$ is a sub-σ-algebra of $\underline{\underline{A}}$, then by considering the experiments $\mathbb{E} = (X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\textcircled{H}})$ and $\mathbb{F} = (X, \underline{\underline{B}}, P_\theta : \theta \in \underline{\textcircled{H}})$ we get a notion of sufficiency of $\underline{\underline{B}}$. Decision theoretic interpretation of this notion of sufficiency, when $\underline{\textcircled{H}}$ is infinite was given by Brown. Brown [8] defined a stronger equivalence relation between decision rules and introduced the concept of adequate (D-sufficient in our terminology) sub σ-algebras. Brown further showed that adequacy (D-sufficiency) is equivalent to sufficiency in the sense of Blackwell.

In this chapter we study the relationship between sufficiency and D-sufficiency. The main theorem states that any D-sufficient σ-algebra always contains a sufficient σ-algebra.

Section 1

Let $E = (X, \underline{\underline{A}}, P_\theta : \theta \varepsilon (\underline{\underline{H}}))$ be an experiment

Definition 1.1. A sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ is sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \varepsilon (\underline{\underline{H}}))$ if given any bounded real valued $\underline{\underline{A}}$ - measurable function f, there is a $\underline{\underline{B}}$ - measurable function f^* such that f^* is a version of $E_\theta(f/\underline{\underline{B}})$ for all θ in $(\underline{\underline{H}})$.

We will now consider a weaker notion of sufficiency Let $(D, \underline{\underline{D}})$ be a set D equipped with a σ-algebra $\underline{\underline{D}}$. $(D, \underline{\underline{D}})$ will be referred to as the Decision Space. By a decision rule $\delta(.,.)$, we mean a transition function from $(X, \underline{\underline{A}})$ to $(D, \underline{\underline{D}})$, i.e. for x in X $\delta(x,.)$ is a probability measure on $\underline{\underline{D}}$ and for every E in $\underline{\underline{D}}$, $\delta(x, E)$ is as a function of x, $\underline{\underline{A}}$ - measurable. A decision rule $\delta(.,.)$ is said to be $\underline{\underline{B}}$ - measurable if, for all $E \varepsilon \underline{\underline{D}}$, $\delta(x, E)$ is $\underline{\underline{B}}$ - measurable. Two decision rules δ and δ' are equivalent if for E in $\underline{\underline{D}}$ and θ in $(\underline{\underline{H}})$.

$$\int_X \delta (x, E)\, dP_\theta = \int_X \delta'(x, E)\, dP_\theta \qquad \ldots (1.1)$$

Definition 1.2. A sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ is D-sufficient if given any decision space $(D, \underline{\underline{D}})$ and a decision rule $\delta(.,.)$ there is a $\underline{\underline{B}}$ - measurable $\delta'(.,.)$ which is equivalent to $\delta(.,.)$ i.e. for all E in $\underline{\underline{D}}$ and θ in $(\underline{\underline{H}})$

$$\int_X \delta (x, E)\, dP_\theta = \int_X \delta'(x, E)\, dP_\theta .$$

This notion of D-sufficiency, we believe was introduced by Brown in [8], where it is called adequacy. However, since the term adequacy has gained currency in statistical literature to denote a similar but distinct concept, we prefer using the word D-sufficiency (for Decision Theoretic Sufficiency). The following easy but interesting proposition also appears in Brown [8].

Proposition 1.1. $\underline{\underline{B}}$ is D-sufficient for $(X, A, P_\theta : \theta \in (\underline{\underline{H}}))$ iff there is a $\underline{\underline{B}}$ - measurable transition function $Q(.,.)$ from $(X, \underline{\underline{B}})$ to $(X, \underline{\underline{A}})$ such that for $A \in \underline{\underline{A}}$ and $\theta \in (\underline{\underline{H}})$

$$\int_X Q (x, A)\, dP_\theta = P_\theta(A) \qquad \ldots (1.2)$$

Proof. Suppose $Q(.,.)$ satisfying (1.2) is given. Then for any decision rule $\delta(.,.)$ the rule $\delta'(.,.)$ defined by

$$\delta'(x, E) = \int_X \delta(y, E)\, Q(x, dy)$$

is $\underline{\underline{B}}$ - measurable and equivalent to δ.

For the converse choose $(D, \underline{\underline{D}})$ to be $(X, \underline{\underline{A}})$ and set $\delta(x, A) = I_A(x)$. Then the $\underline{\underline{B}}$ - measurable $\delta'(.,.)$ which is equivalent to $\delta(.,.)$ satisfies (1.2).

D-sufficiency implies sufficiency if $\{P_\theta : \theta \in \underline{\underline{H}}\}$, is dominated by a σ-finite measure. Any σ-algebra containing a D-sufficient σ-algebra is itself D-sufficient. Consequently Burkholder's [9] example of a non sufficient σ-algebra containing a sufficient σ-algebra, shows that in the undominated case D-sufficiency does not in general imply sufficiency. From the above remarks it is clear that, rather than implication, a more meaningful question would be " Does a D-sufficient σ-algebra always contain a sufficient σ-algebra ?" The main theorem in this section answers the question in the affirmative.

Let $\underline{\underline{B}}$ be D-sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\underline{H}})$ and $Q(.,.)$ be the $\underline{\underline{B}}$- measurable transition function satisfying (1.2). It is tempting to conjecture (see for instance [8]) that $\underline{\underline{B}}'$, the σ-algebra generated by $\{ Q(x, A) : A \in \underline{\underline{A}} \}$ is sufficient and $Q(.,.)$ is an omnibus version of the regular conditional probability given $\underline{\underline{B}}'$. It turns out that if $\underline{\underline{A}}$ is countably generated then $\underline{\underline{B}}'$ is indeed sufficient. However even when $\underline{\underline{A}}$ is countably generated $Q(.,.)$ may fail to be the required version of the conditional probability. This is shown by the following example.

EXAMPLE 1.1

$$X = \{1, 2, 3, 4\}$$

$$\underline{\underline{A}} = \underline{\underline{B}} \quad \text{Power set of } X$$

$$P_1 = \left[\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right]$$

$$P_2 = \left[\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}\right]$$

$Q(i, \{j\})$ is the element $Q_{i,j}$ in Q given by

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then $\underline{\underline{B}}' = \underline{\underline{A}}$. But $Q(i,j)$ is not a version of the conditional probability.

We now come to the main theorem of this section.

THEOREM 1.1. Suppose $\underline{\underline{B}}$ is D-suffieient for $(X, \underline{\underline{A}}, P_\theta : \theta \varepsilon \underline{\underline{\Theta}})$. Then $\underline{\underline{B}}$ contains a sufficient σ-algebra.

Proof. Let $Q(.,.)$ be a $\underline{\underline{B}}$ - measurable transition function satisfying (1.2). For each bounded $\underline{\underline{A}}$ - measurable function f, define

$$T f(x) = \int_X f(y)\, Q(x,dy) .$$

Associate with each bounded $\underline{\underline{A}}$ - measurable function f, a $\underline{\underline{B}}$ - measurable function f^* as follows

$$f^*(x) = \begin{cases} \lim\limits_{n \to \infty} \frac{1}{n} \sum\limits_{k=1}^{n} T^k f(x) & \text{whenever the limit exists,} \\ 0 & \text{otherwise .} \end{cases}$$

Let $\underline{\underline{B}}_0 = \sigma \left\{ f^* : f \text{ bounded } \underline{\underline{A}} \text{ measurable} \right\}$

$\underline{\underline{B}}_0 \subset \underline{\underline{B}}$ and we shall show that $\underline{\underline{B}}_0$ is sufficient.

By Hopf's Ergodic Theorem (see [24]) for all $\theta \varepsilon \underline{\underline{\Theta}}$

$$f^*(x) = E_\theta (f | \underline{\underline{B}}_\theta) \qquad \ldots (1.3)$$

where $\underline{\underline{B}}_\theta = \left\{ A \varepsilon \underline{\underline{A}} : T I_A = I_A \; [P_\theta] \right\}$.

By (1.3) $\underline{\underline{B}}_o = \underline{\underline{B}}_\theta \ [P_\theta]$. Hence again using (1.3)

$$f^*(x) = E_\theta (f \mid \underline{\underline{B}}_o) \quad [P_\theta]$$

establishing sufficiency of $\underline{\underline{B}}_o$.

<u>THEOREM 1.2</u>. If $\underline{\underline{B}}$ is countably generated and D-sufficient then $\underline{\underline{B}}$ is itself sufficient.

<u>Proof</u>. By theorem 5 of Burkolder [9] any countably generated σ-algebra containing a sufficient σ-algebra is itself sufficient.

<u>COROLLARY</u>. If $\underline{\underline{A}}$ is countably generated then the σ-algebra $\underline{\underline{B}}'$ generated by $\{ Q(x, A) : A \in \underline{\underline{A}} \}$ is itself sufficient.

<u>Proof</u>. Since $\underline{\underline{A}}$ is countably generated so is $\underline{\underline{B}}'$. Further since $Q(.,.)$ is $\underline{\underline{B}}'$ measurable, $\underline{\underline{B}}'$ is D-sufficient. Theorem 1.2 now completes the proof.

Weaker forms of D-sufficiency can be obtained by considering restricted class of decision spaces such as

(D_1) Compact metric decision spaces

(D_2) Finite decision spaces

(D_3) Two point decision spaces.

When the sample space is Standard Borel D_1 would be equivalent to D [8]. D_3 is known in the literature as 'Test sufficiency'. We do not know the relationship between D_1, D_2, D_3 in the undominated case. For a discussion of these problems in the general case we refer to Brown [8] and Morimoto [22].

CHAPTER II

INTRODUCTION

This chapter is devoted to an elucidation of the concept of Bayes sufficiency. A statistic T. (or equivalently a sub σ-algebra $\underline{\underline{B}}$) is Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \varepsilon \Theta)$ if, for every prior ξ on (Θ), the posterior distribution of θ given $\underline{\underline{A}}$ depends on x only through T ($\underline{\underline{B}}$ - measurable). In the first part of this chapter we show that Bayes sufficiency is equivalent to sufficiency on sets of measure 1. In the later sections we investigate the relationship between Bayes sufficiency and sufficiency. Our attempts in this direction center around a conjecture of Blackwell. Blackwell had conjectured to us that in the Standard Borel case, Bayes sufficiency and sufficiency are equivalent for countably generated sub σ-algebras. While we have not been able to settle Blackwell's conjecture, our examples show that without standard Borel assumptions Bayes sufficiency need not imply sufficiency. We also show that " Bayes sufficiency implies sufficiency is equivalent to the apparently weaker " Bayes sufficiency implies test sufficiency".

Assumptions and Notations : The following assumptions and notations will be valid throughout this chapter.

$(\Theta, \mathcal{C})$ is a measurable space and $\theta \longrightarrow P_\theta(.)$ is a transition function from $(\Theta, \mathcal{C})$ to $(X, \mathcal{A})$. $(\hat{\Theta})$ stands for the set of probability measures on $(\Theta, \mathcal{C})$. For each probability measure ξ on $(\hat{\Theta})$, λ_ξ is the probability measure on $(X \times \Theta, \mathcal{A} \times \mathcal{C})$ defined by

$$\lambda_\xi \, (A \times C) = \int_C P_\theta \, (A) \, d \, \xi \, (\theta)$$

and P_ξ is the marginal of λ_ξ on $(X, \mathcal{A})$. $X \times \mathcal{C}$ denotes the σ-algebra of sets of the form $X \times C$, C in $\mathcal{C}$. $\mathcal{A} \times \Theta$ and $\mathcal{B} \times \Theta$ are similarly defined. Functions defined on X, are sometimes looked upon as functions of x and θ. More precisely, if f is a function on X, then by $\bar{f}$ we shall mean the function on $X \times \Theta$ defined as $\bar{f}\,(x, \theta) = f\,(x)$.

Section 1.

Definition 2.1.1. A sub σ-algebra $\mathcal{B}$ of $\mathcal{A}$ is Bayes sufficient for $(X, \mathcal{A}, P_\theta : \theta \in \Theta)$, if for all C in $\mathcal{C}$ and ξ in $(\hat{\Theta})$

$$E_{\lambda_\xi}(I_{X \times C} \mid \mathcal{B} \times \Theta) = E_{\lambda_\xi}(I_{X \times C} \mid \mathcal{A} \times \Theta).$$

The following proposition is essentially a restatement of the definition.

Proposition 2.1.1. The following are equivalent.

(i) $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\Theta : \Theta \varepsilon (\underline{\underline{H}}))$.

(ii) The σ-algebras $\underline{\underline{A}} \times (\underline{\underline{H}})$ and $X \times \underline{\underline{C}}$ are conditionally independent given $\underline{\underline{B}} \times (\underline{\underline{H}})$ on the probability space $(X \times (\underline{\underline{H}}), \underline{\underline{A}} \times \underline{\underline{C}}, \lambda_\xi)$ for all ξ in $\widehat{(\underline{\underline{H}})}$.

(iii) For every bounded $\underline{\underline{A}}$ - measurable function f on X, there is a $\underline{\underline{B}}$ - measurable function f^* such that

$$\bar{f}^* = E_{\lambda_\xi}(\bar{f} \mid \underline{\underline{B}} \times \underline{\underline{C}}) \text{ for all } \xi \text{ in } \widehat{(\underline{\underline{H}})}.$$

Proof. Immediate from proposition 25.3 A of [20]

Theorem 2.1.1. Suppose $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are countably generated. Then $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\Theta : \Theta \varepsilon (\underline{\underline{H}}))$ iff for every ξ in $\widehat{(\underline{\underline{H}})}$, there is a set E_ξ in $\underline{\underline{C}}$ of ξ- measure 1, such that $\underline{\underline{B}}$ is sufficient for $(X, \underline{\underline{A}}, P_\Theta : \Theta \varepsilon E_\xi)$.

Proof. 'If part'.

Given ξ there is a E_ξ of ξ measure 1 such that $\underline{\underline{B}}$ is sufficient for $(X, \underline{\underline{A}}, P_\Theta : \Theta \varepsilon E_\xi)$. Now for any bounded $\underline{\underline{A}}$ - measurable function f, choose an f^*, $\underline{\underline{B}}$ - measurable such that

$$f^*(x) = E_\theta(f \mid \underline{\underline{B}}) \quad \text{for} \quad \theta \in E_\xi .$$

Now for C in $\underline{\underline{C}}$ and B in $\underline{\underline{B}}$

$$\int_C \int_B \bar{f}^* \, d\lambda_\xi = \int_C \int_B \bar{f}^* \, dP_\theta(x) \, d\xi(\theta) = \int_{C \cap E_\xi} \int_B \bar{f}^* \, dP_\theta(x) \, d\xi(\theta)$$

$$= \int_{C \cap E_\xi} \int_B f \, dP_\theta(x) \, d\xi(\theta) = \int_C \int_B \bar{f} \, d\lambda_\xi .$$

Therefore, $E_{\lambda_\xi}(\bar{f} \mid \underline{\underline{B}} \times \underline{\underline{C}}) = \bar{f}^* = E_{\lambda_\xi}(\bar{f} \mid \underline{\underline{B}} \times (\underline{\underline{H}})$

and by Proposition 2.1.1. $\underline{\underline{B}}$ is Bayes sufficient.

'Only if part'.

Fix $\xi \in (\hat{\underline{\underline{H}}})$. Given f bounded $\underline{\underline{A}}$ - measurable there is, by Bayes sufficiency of $\underline{\underline{B}}$, a $\underline{\underline{B}}$ - measurable f^* such that

$$\bar{f}^* = E_{\lambda_\xi}(\bar{f} \mid \underline{\underline{B}} \times \underline{\underline{C}}).$$

Now $\int_C \int_B \bar{f}^*(x) \, dP_\theta(x) \, d\xi(\theta) = \int_C \int_B f(x) \, dP_\theta(x) \, d\xi(\theta)$ for all C in $\underline{\underline{C}}$.

Therefore $\int_B f^*(x) \, dP_\theta(x) = \int_B f(x) \, dP_\theta(x)$ a.e. ξ for each B in $\underline{\underline{B}}$.

By running B through a countable algebra generating $\underline{\underline{B}}$, we get

$$\int_B f^*(x) \, dP_\theta(x) = \int_B f(x) \, dP_\theta(x) \quad \text{for all } B \in \underline{\underline{B}} \text{ outside}$$

a ξ - null set N_ξ. Thus $f^*(x) = E_\theta(f \mid \underline{\underline{B}})$ for $\theta \notin N_\xi$.

Now, taking a countable union of null sets with f running through indicators of sets in a countable algebra generating $\underline{\underline{A}}$, the theorem is proved.

The next proposition is repeatedly used in the sequel.

Proposition 2.12. Assume that $\underline{\underline{B}}$ is countably generated. Let f be a bounded $\underline{\underline{A}}$ - measurable function. There is then a version of $E_\theta (f \mid \underline{\underline{B}})$ which is jointly measurable in x and θ, with respect to $\underline{\underline{B}} \times \underline{\underline{C}}$.

Proof. Let $\{ B_1 , B_2 , \ldots \}$ generate $\underline{\underline{B}}$, $\underline{\underline{B}}_n$ be the σ-algebra generated by $B_1 , B_2 \ldots B_n$ and $B_n^1 , B_n^2 , \ldots , B_n^{k(n)}$ denote a the atoms of $\underline{\underline{B}}_n$.

For $A \in \underline{\underline{A}}$,

$$f_\theta^n (x) = \sum_1^{k(n)} \frac{P_\theta(A \cap B_n^i)}{P_\theta(B_n^i)} I_{B_n^i}(x)$$

is a version of $E_\theta (I_A \mid \underline{\underline{B}}_n)$ which is jointly measurable with respect to $\underline{\underline{B}}_n \times \underline{\underline{C}}$.

Define $f_\theta^*(x) = \lim_n f_\theta^n(x)$ whenever the limit exists,

$\qquad 0$ otherwise.

By Levy's theorem [21] for each θ, $f_\theta^n(x)$ converges almost everywhere to $E_\theta(f \mid \underline{\underline{B}})$. Therefore $f_\theta^*(x) = E_\theta(f \mid \underline{\underline{B}})$. The proof can now be completed by considering simple functions and their limits.

Our next theorem relates Bayes sufficiency to pairwise sufficiency for an enlarged class of probability measures. For each ξ in $\widehat{(\underline{\underline{H}})}$. Let P_ξ be the measure on $(X, \underline{\underline{A}})$ defined by

$$P_\xi(A) = \int_{(\underline{\underline{H}})} P_\theta(A)\, d\xi(\theta).$$

P_ξ is really the marginal of λ_ξ on $(X, \underline{\underline{A}})$.

<u>Theorem 2.1.2</u>. Assume $\underline{\underline{A}}$, $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are countably generated. Then $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{\underline{H}}))$ iff $\underline{\underline{B}}$ is pairwise sufficient for $(X, \underline{\underline{A}}, P_\xi : \xi \in \widehat{(\underline{\underline{H}})})$.

<u>Proof</u>. Suppose $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{\underline{H}}))$. Then, given P_{ξ_1} and P_{ξ_2}, consider the prior $\xi = \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2$ $\underline{\underline{B}}$ is then by Theorem 2.1.1 sufficient for $\{P_\theta : \theta \in E_\xi\}$ where E_ξ is a set of ξ - measure 1. Therefore given any bounded $\underline{\underline{A}}$ - measurable function f, there is a $\underline{\underline{B}}$ - measurable function f^* such that

$$f^*(x) = E_\theta(f \mid \underline{\underline{B}}) \quad \text{for } \theta \in E_\xi .$$

It is easy to see(since E_ξ has also ξ_1 and ξ_2 measure 1) that

$$f^*(x) = E_{\xi_1}(f \mid \underline{\underline{B}}) \quad \text{and} \quad f^*(x) = E_{\xi_2}(f \mid \underline{\underline{B}})$$

For the other part, given any ξ in $\hat{(\underline{\bar{H}})}$, we shall show that for any bounded $\underline{\underline{A}}$ - measurable function $f(x)$, there is a $\underline{\underline{B}}$ - measurable function $f^*(x)$ which is a version of $E_{\lambda_\xi}(\bar{f} \mid \underline{\underline{B}} \times \underline{\underline{C}})$. This will estabish that $\underline{\underline{A}} \times (\underline{\bar{H}})$ and $X \times \underline{\underline{C}}$ are conditionally independent given λ_ξ $\underline{\underline{B}} \times (\underline{\bar{H}})$ under λ_ξ.

Let $\underline{\underline{C}}_n$ be a sequence of increasing finite algebras generating $\underline{\underline{C}}$. Let $C_n^1, \ldots, C_n^{k(n)}$ denote the atoms of $\underline{\underline{C}}_n$. Fix n. Consider the k(n) measures $\xi_1, \xi_2, \ldots, \xi_{k(n)}$ on $(\underline{\bar{H}})$ defined by

$$\xi_i(G) = \frac{\xi(C \cap C_n^i)}{\xi(C_n^i)}$$

If $\xi(C_n^i)$ is zero then we do not define ξ_i.

Now $\xi_1, \xi_2, \ldots, \xi_{k(n)} \in \hat{(\underline{\bar{H}})}$. Further $\underline{\underline{B}}$ is pairwise sufficient for $(X, \underline{\underline{A}}, P_\xi : \xi \in \hat{(\underline{\bar{H}})})$ and pairwise sufficiency implies sufficiency for every finite collection of P_ξ s, in particular for $\{P_{\xi_1}, P_{\xi_2}, \ldots, P_{\xi_{k(n)}}\}$. Therefore, for any bounded $\underline{\underline{A}}$ - measurable function f, there is a $\underline{\underline{B}}$ - measurable function f_n^* such that

$$f_n^*(x) = E_{p_{\xi_i}}(f_n \mid \underline{\underline{B}}) \quad \text{for} \quad i = 1,2,\ldots,k(n) .$$

From the definition of $\xi_1, \xi_2, \ldots, \xi_{k(n)}$ it is easy to see that

$$\bar{f}_n^* = E_{\lambda_\xi}(\bar{f} \mid \underline{\underline{B}} \times \underline{\underline{C}}_n) .$$

Since $\underline{\underline{B}} \times \underline{\underline{C}}_n \uparrow \underline{\underline{B}} \times \underline{\underline{C}}$ and $E_{\lambda_\xi}(\bar{f} \mid \underline{\underline{B}} \times \underline{\underline{C}}_n)$ forms a martingale

$$f^*(x) = \lim_n f_n^*(x) = E_{\lambda_\xi}(\bar{f} \mid \underline{\underline{B}} \times \underline{\underline{C}}) .$$

Since $\bar{f}^*$ are all $\underline{\underline{B}} \times (\underline{\underline{H}})$ measurable so is $\bar{f}^*$ and this proves the theorem.

Section 2.

In this section we investigate the relationship between Bayes sufficiency and other notions of sufficiency. It is immediate from the definition that sufficiency implies Bayes sufficiency. Theorem 1.2 establishes that D-sufficiency implies sufficiency. In fact, even the weaker notion of test sufficiency implies Bayes sufficiency. Before we prove this we recall the definition of test sufficiency.

<u>Definition 2.2.1</u>. A sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ is test sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{H}))$ if, given any bounded $\underline{\underline{A}}$ - measurable function f lying between 0 and 1, there is a $\underline{\underline{B}}$ - measurable function f^*, $0 \leq f^* \leq 1$, such that

$$\int_X f(x)\, dP_\theta = \int_X f^*(x)\, dP_\theta \quad \text{for all } \theta \text{ in } (\underline{H}).$$

It is known [22] and can be proved via the Neyman-Pearson lemma, that if $\underline{\underline{B}}$ is test sufficient then it is pairwise sufficient.

<u>Theorem 2.2.1</u>. Assume $\underline{\underline{A}}$, $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are countably generated. If $\underline{\underline{B}}$ is test sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{H}))$ then $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{H}))$.

<u>Proof</u>. Note that since $\underline{\underline{B}}$ is test sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{H}))$ it is also test sufficient for $(X, \underline{\underline{A}}, P_\xi : \xi \in (\hat{\underline{H}}))$. To see this, given f, $0 \leq f \leq 1$ and $\underline{\underline{A}}$ - measurable, get $\underline{\underline{B}}$ - measurable f^* satisfying $\int f\, dP_\theta = \int f^* dP_\theta$ for θ in $(\underline{H})$. Now for the same f^*

$$\int_X f(x)\, dP_\xi = \int \int f(x)\, dP_\theta(x)\, d\xi(\theta) = \int \int f^*(x)\, dP_\theta(x)\, d\xi(\theta)$$

$$= \int_{(\underline{H})} f^*(x)\, dP_\xi(x).$$

Consequently $\underline{\underline{B}}$ is pairwise sufficient for $(X, \underline{\underline{A}}, P_\xi : \xi \in (\hat{\underline{H}}))$.
By Theorem 2.1.2 $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\hat{\underline{H}}))$.

We now turn to the converse. Does Bayes sufficiency always imply sufficiency or even test sufficiency ? It is known that if $\{P_\theta : \theta \in (\underline{H})\}$ is dominated by a σ-finite measure then Bayes sufficiency indeed implies sufficiency. It is also clear from Burkholder's example of a non-sufficient σ-algebra containing a sufficient σ-algebra, that in the general undominated case Bayes sufficient σ-algebras need not always be sufficient.

We give below two examples to show that even with some additional assumptions on the σ-algebras involved, Bayes sufficiency need not imply sufficiency. Both these examples are strongly set theoretic and for the results in set theory we refer to [16] and [17].

Example 2.2.1

This is an example to show that even if all the σ-algebras considered, namely $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}$, are countably generated still Bayes sufficiency may fail to imply sufficiency.

$X = (\underline{H}) = [0,1]$

$D =$ A non-Borel universally measurable subset of $[0,1]$

$\underline{\underline{B}}$ = Borel σ-algebra on $[0,1]$

$\underline{\underline{A}}$ = $\underline{\underline{C}}$: σ-algebra generated by $\{\underline{\underline{B}}, D\}$.

$P_\theta(A) = I_A(\theta)$ i.e. P_θ is the measure degenerate at θ.

<u>Claim</u>. $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{\underline{H}}))$ but not sufficient.

To see that $\underline{\underline{B}}$ is Bayes sufficient, given any $\xi \in (\underline{\underline{H}})$ there is a B_ξ in $\underline{\underline{B}}$ such that $B_\xi \cap D$ and $B_\xi \cap D^c \in \underline{\underline{B}}$ and $\xi(B_\xi) = 1$. $\underline{\underline{B}}$ is then clearly sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\underline{B}}_\xi)$. By considering $I_D(x)$ it is easy to see that $\underline{\underline{B}}$ is not sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{\underline{H}}))$.

<u>Example 2.2.2</u>.

This example shows that even the additional requirement that the sample space be Standard Borel is inadequate to ensure that countably generated Bayes sufficient σ-algebras are sufficient. This example heavily depends on a theorem of Blackwell and Ryll-Nardzewski [6].

Let X be a Borel subset of $[0,1] \times [0,1]$ such that

(i) Projection of X to the first coordinate is whole of $[0,1]$

(ii) X does not contain any Borel graph

(iii) X contains a Co-analytic graph, that is a Co-analytic set $(\underline{\underline{H}})$, which is the graph of a function on $[0,1]$.

For examples of such sets see [7]. Now let

$\underline{\underline{A}}$ be the Borel σ-algebra on X

$\underline{\underline{B}}$ be the vertical σ-algebra i.e. sets of the form $B \times [0,1]$ B - Borel in $[0,1]$, and

$(\underline{\underline{H}})$ be the Co-analytic set in X which is also a graph i.e. $(\underline{\underline{H}}) = \{(t, \emptyset(t)) : t \in [0,1]\}$. Set

$\underline{\underline{C}} = \underline{\underline{A}} \cap (\underline{\underline{H}})$. Finally let

P_θ be the measure on X which is degenerate at θ

i.e. $P_{(t,\emptyset(t))}(A) = I_A(t,\emptyset(t))$

Then P_θ is a measurable transition function from $(\underline{\underline{H}})$ to X. $\underline{\underline{B}}$ is Bayes sufficient. To see this we argue as follows.

Given ξ on $(\underline{\underline{H}})$ there is a set E_ξ Borel in X, contained in $(\underline{\underline{H}})$ and of ξ measure 1. Let B_ξ be the projection of E_ξ to the first co-ordinate. B_ξ is clearly Borel in $[0,1]$. Since $\{(t, \emptyset(t)) : t \in B_\xi\} = E_\xi$ is a Borel set, $\emptyset$ restricted to B_ξ is a Borel measurable function.

Define a transition function Q_ξ as

$$Q_\xi (t,y, A) = I_A (t, \emptyset(t)) \quad \text{for} \quad t \in B_\xi$$
$$= I_A (t_o,y_o) \qquad t \notin B_\xi$$

Then for all A, $Q_\xi (t,y, A)$ is $\underline{\underline{B}}$ - measurable and is further a regular conditional probability given $\underline{\underline{B}}$ for all θ in E_ξ. Therefore by theorem 2.2.1. $\underline{\underline{B}}$ is Bayes sufficient.

On the other hand $\underline{\underline{B}}$ is not sufficient. For if $\underline{\underline{B}}$ were sufficient, there will be a $\underline{\underline{B}}$ measurable transition function which is a version of the regular conditional probability given $\underline{\underline{B}}$ for all θ in $(\underline{\underline{H}})$ [18]. Since $\{ P_\theta : \theta \in (\underline{\underline{H}}) \}$ contains all point measures on $\underline{\underline{B}}$ this version must necessarily be everywhere proper. The existence of an everywhere proper $\underline{\underline{B}}$ measurable transition function would imply by a theorem of Blackwell and Ryll-Nardzewski that X contains a Borel graph.

Unlike the other notions of sufficiency considered earlier, Bayes sufficiency involves the structure of the parameter space. The priors to be considered depends on the σ-algebra on $(\underline{\underline{H}})$, which should be rich enough to ensure that $\theta \longrightarrow P_\theta$ is a measurable transition function. Example 2.2.2 makes this dependence on the structure of the prameter space evident. Thus any theorem of the kind " Bayes sufficiency implies sufficiency" will need restrictions on both the sample space and the parameter space. In this context we mention below a conjecture due to Blackwell.

CONJECTURE (Blackwell). If $(X, \underline{\underline{A}})$ and $((\underline{\underline{H}})\ \underline{\underline{C}})$ are both Standard Borel then any countably generated σ-algebra $\underline{\underline{B}}$ which is Bayes sufficient is also sufficient.

We have not been able to prove the conjecture completely. In the next chapter we will prove the conjecture when the P_θ s are discrete.

The last theorem in this section is a kind of meta theorem. We show below that " Bayes sufficiency implies sufficiency" is equivalent to the apparently weaker " Bayes sufficiency implies test sufficiency" . Towards this we will first prove a lemma.

Let $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{\underline{H}}))$ be an experiment. Let $\underline{\underline{B}}$ be a countably generated sub σ-algebra of $\underline{\underline{A}}$ and let $\{B_i ; i \in \mathbb{N}\}$ be a countable algebra generating $\underline{\underline{B}}$.

Let $\underline{\underline{H}}_o = \{ (\theta, n) : P_\theta(B_n) > 0 \}$.

for $(\theta, n) \in \underline{\underline{H}}_o$ define $P_{\theta,n}$ by $P_{\theta,n}(A) = \dfrac{P_\theta(A \cap B_n)}{P_\theta(B_n)}$.

We note that if $\underline{\underline{H}}$ is standard Borel so is $\underline{\underline{H}}_o$. Further $(\theta, n) \longrightarrow P_{\theta,n}$ is a measurable transition function if $\theta \longrightarrow P_\theta$ is. Finally test sufficiency of $\underline{\underline{B}}$ for $\{ P_{\theta,n} : (\theta, n) \in \underline{\underline{H}}_o \}$ is equivalent to sufficiency of $\underline{\underline{B}}$ for $\{ P_\theta : \theta \in \underline{\underline{H}} \}$.

<u>Lemma</u>. If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are countably generated and if $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\underline{H}})$ then $\underline{\underline{B}}$ is also Bayes sufficient for $(X, \underline{\underline{A}}, P_{\theta,n} : (\theta, n) \in \underline{\underline{H}}_o)$.

<u>Proof</u>. Let ξ be a prior on $\underline{\underline{H}}_o$. Consider ξ' the marginal of ξ on $\underline{\underline{H}}$. Since $\underline{\underline{B}}$ is Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\underline{H}})$ there is a subset $E_{\xi'}$ of $\underline{\underline{H}}$ of ξ' measure 1 for which $\underline{\underline{B}}$ is sufficient. Define E_ξ as $E_{\xi'} \times \mathbb{N} \cap \underline{\underline{H}}_o$. Then $\xi(E_\xi) = 1$ and $\underline{\underline{B}}$ is sufficient for E_ξ. Hence $\underline{\underline{B}}$ is Bayes sufficient.

" <u>Theorem</u>" The following propositions are equivalent.

<u>Proposition A</u> Suppose $(X, \underline{\underline{A}})$ and $(\underline{\underline{H}}, \underline{\underline{C}})$ are Standard Borel. If $\underline{\underline{B}}$ is a countably generated sub σ-algebra of $\underline{\underline{A}}$ which is Bayes sufficient then $\underline{\underline{B}}$ is sufficient.

<u>Proposition B</u> Suppose $(X, \underline{\underline{A}})$ and $((\overline{\underline{H}}), \underline{\underline{C}})$ are Standard Borel. If $\underline{\underline{B}}$ is a countably generated sub σ-algebra of $\underline{\underline{A}}$ which is Bayes sufficient then $\underline{\underline{B}}$ is test sufficient.

<u>Proof</u>. Proposition (A) trivially entails Proposition (B).

Now suppose proposition B is true. To see the validity of proposition (A), let $(X, \underline{\underline{A}}, P_\theta : \theta \in (\overline{\underline{H}}))$ be any experiment where $(X, \underline{\underline{A}})$ and $((\overline{\underline{H}}), \underline{\underline{C}})$ are Standard Borel. Suppose $\underline{\underline{B}} \subset \underline{\underline{A}}$ is countably generated and Bayes sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in (\overline{\underline{H}}))$.

Consider the experiment $(X, \underline{\underline{A}}, P_{\theta,n} :(\theta,n) \in (\overline{\underline{H}})_0)$. It follows from the lemma that $\underline{\underline{B}}'$ satisfies the assumptions of Proposition $\underline{B}$. Consequently $\underline{\underline{B}}$ is test sufficient for $(X, \underline{\underline{A}}, P_{\theta,n} :(\theta,n) \in (\overline{\underline{H}})_0)$ and hence sufficient for $(X,\underline{\underline{A}}, P_\theta :\theta \in (\overline{\underline{H}}))$

<u>Section 3</u>.

In this section we show that, under certain conditions a completion of a Bayes sufficient σ-algebra is sufficient. These results are analogous to those in pairwise sufficiency proved by Hasegawa and Perlman.[14]

Let $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{H}))$ be an experiment. Let $f_\theta(x)$ be a family of functions satisfying

(i) $f_\theta(x)$ is bounded and measurable in (θ, x),

(ii) for every ξ in $\widehat{(\underline{H})}$, there is an $E_\xi \subset (\underline{H})$ of ξ measure 1 and an $\underline{\underline{A}}$ - measurable $f_\xi(x)$ such that, for $\theta \in E_\xi$

$$f_\theta(x) = f_\xi(x) \; [P_\theta] .$$

In what follows we consider experiments in which every family of functions satisfying (i) and (ii) above, has an equivalent $\underline{\underline{A}}$ - measurable function. More precisely given any $f_\theta(x)$ satisfying (i) and (ii) there is an $\underline{\underline{A}}$ - measurable function f such that

$$\text{for all } \theta \text{ in } (\underline{H}), \quad f_\theta(x) = f(x) \; [P_\theta] .$$

For these experiments we state the following theorem.
Let N_ξ denote the class of $\underline{\underline{A}}$ - measurable P_ξ null sets and $N = \bigcap_{\theta \in (\underline{H})} N_\theta$.

<u>Theorem 2.3.1</u> If a countably generated σ-algebra $\underline{\underline{B}}$ is Bayes sufficient then $\hat{\underline{\underline{B}}} = \bigcap_\xi \underline{\underline{B}} \vee N_\xi$ is sufficient. Consequently if $\hat{\underline{\underline{B}}} = \underline{\underline{B}} \vee N$ then $\underline{\underline{B}}$ is itself sufficient.

Let $f(x)$ be bounded $\underline{\underline{A}}$ - measurable. Since $\underline{\underline{B}}$ is countably generated there is by ~~Theorem~~ 2.1.2 a jointly measurable version $f^*_\Theta(x)$ of $E_\Theta(f \mid \underline{\underline{B}})$. We will now show that $f^*_\Theta(x)$ satisfies (ii). Since $\underline{\underline{B}}$ is Bayes sufficient there is by Theorem 2.1.1 a set E_ξ of ξ measure 1 such that $\underline{\underline{B}}$ is sufficient for $(X, \underline{\underline{A}}, P_\Theta : \Theta \varepsilon E_\xi)$. Consequently, there is an $f^*_\xi(x)$, $\underline{\underline{B}}$ - measurable, such that

$$f^*_\xi(x) = f_\Theta(x) \; [P_\Theta] \text{ for } \Theta \text{ in } E_\xi .$$

There is then an $f^*(x)$, $\underline{\underline{A}}$ - measurable satisfying

$$f^*(x) = f_\Theta(x) \; [P_\Theta] \text{ for all } \Theta \text{ in } \underline{\underline{H}} .$$

Since $f^*_\xi(x)$ is $\underline{\underline{B}}$ - measurable and

$$P_\xi \left\{ x : f^*_\xi(x) \neq f^*(x) \right\} =$$

$$\int_{E_\xi} P_\Theta \left\{ x : f^*_\xi(x) \neq f^*(x) \right\} d\xi(\Theta) + \int_{E^c_\xi} P_\Theta \left\{ x : f^*_\xi(x) \neq f^*(x) \right\} d\xi(\Theta)$$

$$= 0 .$$

$f^*(x)$ is $\underline{\underline{B}} \vee N_\xi$ measurable for each ξ. And this completes the proof.

CHAPTER III

INTRODUCTION

In this chapter we study Discrete Standard Borel experiments. Specifically we consider experiments in which both the sample space and the parameter space are Standard Borel and further the P_θ s are discrete measures. The main theorem in section 1 asserts that in this set up pairwise sufficiency is equivalent to sufficiency. Since Bayes sufficiency always implies pairwise sufficiency, the theorem settles Blackwell's conjecture on the equivalence of Bayes sufficiency and sufficiency in the Standard Borel case for discrete probabilities. In the later part of this chapter we study questions related to the existence of minimal sufficient sub σ-algebras. The existence of minimal sufficient sub σ-algebras is shown to be equivalent to some set theoretic conditions on the minimal pairwise sufficient partition. We also show, by an example, that even in the discrete case minimal sufficient σ-algebras need not exist.

ASSUMPTIONS.

Throughout this chapter, unless otherwise mentioned, we assume

(i) $(X, \underline{\underline{A}})$ and $(\textcircled{H}, \underline{\underline{C}})$ are Standard Borel.

(ii) For each θ in $\textcircled{H}$, P_θ is a discrete probability on $(X, \underline{\underline{A}})$. Further $\theta \longrightarrow P_\theta(.)$ is a transition function from $(\textcircled{H}, \underline{\underline{C}})$ to $(X, \underline{\underline{A}})$.

(iii) If $A \in \underline{\underline{A}}$ and $P_\theta(A) = 0$ for all θ in $\textcircled{H}$ then $A = \emptyset$.

Section 1.

We begin by defining pairwise sufficiency.

Definition 3.1.1 A sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ is pairwise sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \in \textcircled{H})$ if for every pair θ, θ' in $\textcircled{H}$, $\underline{\underline{B}}$ is sufficient for $(X, \underline{\underline{A}}, P_\theta, P_{\theta'})$.

It is easily noted that sufficiency as well as Bayes sufficiency always implies pairwise sufficiency. It is also easy to construct examples of pairwise sufficient σ-algebra which are not sufficient. In fact example 2.2.1 of the last chapter is one such. Note that in example 2.2.1, the P_θ s considered are all discrete. We show in this section that such situations cannot arise in the Standard Borel case.

Only the " only if " part of the following theorem is needed in the sequel. The " if " part, we feel, provides a justification for considering the Borel σ-algebra $\underline{\underline{A}}$ on X rather than the power set.

<u>Theorem 3.1.1</u> Under the assumptions $P_\theta(x)$ is a transition function iff $P_\theta(x)$ is jointly measurable in θ and x.

<u>Proof</u>. Suppose $P_\theta(x)$ is a transition function.

Let $M = \left\{ A \subseteq X \times X : P_\theta(A^x) \text{ is measurable in } \theta \text{ and } x \right\}$

M is closed under finite disjoint unions and is a monotone class containing rectangles and hence contains the product σ-algebra. Now since D, the diagonal in X x X, belongs to M, $P_\theta(x) = P_\theta(D^x)$ is jointly measurable in θ and x.

For the converse note that $S = \left\{ (\theta, x) : P_\theta(x) > 0 \right\}$ is a Borel set in $(\underline{H}) \times X$. Further the θ sections of S are atmost countable. Therefore there is, by Lusin's Theorem [17], a sequence of measurable functions f_n defined on $(\underline{\underline{H}})$ and taking values in X such that $S = \bigcup_n (\theta, f_n(\theta))$.

Define a sequence of functions $\emptyset_n(\theta)$ as

$$\emptyset_1(\theta) = P_\theta(f_1(\theta))$$

$$\emptyset_2(\theta) = P_\theta(f_2(\theta)) \quad \text{if} \quad f_1(\theta) \neq f_2(\theta)$$

$$= 0 \qquad \text{Otherwise}$$

.. ..

.. ..

$$\emptyset_n(\theta) = P_\theta(f_n(\theta)) \quad \text{if} \quad f_n(\theta) \neq f_i(\theta) \quad \text{for any } i=1,2,\ldots,n-1$$

$$= 0 \qquad \text{Otherwise}$$

Then for any Borel set A in $\underline{\underline{A}}$

$$P_\theta(A) = \sum_{n=1}^{\infty} I_A(f_n(\theta)) \emptyset_n(\theta) .$$

Since for each n, $\emptyset_n(\theta)$ and $I_A(f_n(\theta))$ are $\underline{\underline{C}}$ - measurable as functions of θ , $P_\theta(A)$ is also $\underline{\underline{C}}$ - measurable.

<u>Theorem 3.1.2</u>. A countably generated sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ is pairwise sufficient iff it is sufficient.

<u>Proof</u>. Given any bounded $\underline{\underline{A}}$ - measurable function f, since $\underline{\underline{B}}$ is countably generated (see Proposition 2.1.2), there is a function $f_\theta(x)$ jointly measurable in θ and x such that for all θ, $f_\theta(x) = E_\theta\ (f \mid \underline{\underline{B}})$.

We will now construct a $\underline{\underline{B}}$ - measurable function f^* such that for all θ, $f^*(x) = f_\theta(x)\ [P_\theta]$. Towards this, first note that $S = \{ (\theta, x) : P_\theta(x) > 0 \}$ is by Theorem 3.1.1 Borel in $(\underline{H}) \times X$.

Define

$$f^*(x) = \sup_{\theta \in S^x} f_\theta(x)$$

$$(S^x = \{ \theta : (\theta, x) \in S \}).$$

Since $f_\theta(x) = E_\theta\ (f \mid \underline{\underline{B}})$ and $\underline{\underline{B}}$ is pairwise sufficient, for each x, $f_\theta(x)$ is constant on S^x. Thus for each θ in $(\underline{H})$ $f^*(x) = f_\theta(x)\ [P_\theta]$. We now show that f^* is $\underline{\underline{A}}$ - measurable.

$$\{ x : f^*(x) > a \} = \{ x : \sup_{S^x} f_\theta(x) > a \} = P_X [\{ (\theta, x) : f_\theta(x) > a \} \cap S]$$

where P_X denotes the projection to the X co-ordinate.

Since for each x, $f_\theta(x)$ is constant on the x-section of S,

$$\{x : f^*(x) \leq a\} = \{x : \sup_{S^x} f_\theta(x) \leq a\} = P_X[\{(\theta,x): f_\theta(x) \leq a\} \cap S].$$

Thus being projections of Borel sets $\{x : f^*(x) > a\}$ and its complement $\{x : f^*(x) \leq a\}$ are both Analytic and consequently Borel [16]. Hence $f^*(x)$ is $\underline{\underline{A}}$ - measurable.

That $f^*(x)$ is also $\underline{\underline{B}}$ - measurable follows from Blackwell's Theorem [15]. To see this note that if x any y belong to the same atom of $\underline{\underline{B}}$ then $f_\theta(x) = f_\theta(y)$ and hence $f^*(x)$ is constant on $\underline{\underline{B}}$ - atoms. Therefore $\{x : f^*(x) > a\}$ is a $\underline{\underline{A}}$ - measurable set which is a union of $\underline{\underline{B}}$ atoms and so belongs to $\underline{\underline{B}}$.

<u>Remarks</u> 1) Assuption (iii) namely " if $P_\theta(A) = 0$ for all θ then $A = \emptyset$ " is not necessary for the validity of Theorem 3.2.2. To see this, note that $X' = \{x : P_\theta(x) > 0$ for some $\theta\}$ being the projection of $S = \{(\theta, x) : P_\theta(x) > 0\}$ is analytic. Therefore given any f bounded, $\underline{\underline{A}}$-measurable, by restricting f to X' and imitating the proof of Theorem 3.2.2 we can get an f^* defined on X'. This f^* can now be extended as a $\underline{\underline{B}}$ - measurable function to whole of X.

2) Proof of Theorem 3.2.2 shows in essence that the discrete Standard Borel experiment is weakly coherent in the the sense of Chapter IV.

Pairwise sufficiency being weaker than Bayes sufficiency the following corollary is immediate.

Corollary. A countably generated σ-algebra is Bayes sufficient iff it is sufficient.

Certain other notions of sufficiency in terms of decision spaces, namely 2-point decision spaces and finite decision spaces were discussed in Chapter I. Pairwise sufficiency being the weakest of these it follows that, in the set up of this chapter all these notions are equivalent.

Section 2.

We now turn to the existence of minimal sufficient sub σ-algebras. Since P_θ s are discrete there is a canonical pairwise sufficient partition. Theorem 3.1.2 leads us to beleive that the minimal sufficient σ-algebra should be describable in terms of this pairwise sufficient partition. The Theorems in this section substantiate this belief.

We begin by defining the canonical pairwise sufficient partition. For θ in $(\underline{\underline{H}})$, S_θ will denote the support $\{x : P_\theta(x) > 0\}$ of P_θ. For θ, θ' in $(\underline{\underline{H}})$ define $\frac{P_\theta}{P_\theta + P_{\theta'}}$ as

$$\frac{P_\theta}{P_\theta + P_{\theta'}}(x) = \frac{P_\theta(x)}{P_\theta(x) + P_{\theta'}(x)} \quad \text{for } x \text{ in } S_\theta$$

$$= 0 \quad \text{for } x \text{ not in } S_\theta .$$

Let $\underline{\underline{B}}$ be the smallest σ-algebra generated by $\left\{ \frac{P_\theta}{P_\theta + P_{\theta'}} : \theta, \theta' \in (\underline{\underline{H}}) \right\}$. The σ-algebra $\underline{\underline{B}}$ is atomic, in fact each atom is atmost countable, and this gives rise to a partition of X. This partition of X will be denoted by $\mathbb{P}(\underline{\underline{B}})$. It is noted in [12] and can, in fact, be easily checked that $\underline{\underline{B}}$ is pairwise sufficient, contains supports of P_θ, and is also minimal with respect to these properties. That is if $\underline{\underline{B}}'$ is any other pairwise sufficient σ-algebra containing S_θ s then $\underline{\underline{B}} \subseteq \underline{\underline{B}}'$.

Theorem 3.2.1. The following are equivalent for a sub σ-algebra $\underline{\underline{C}}$ of $\underline{\underline{A}}$

(i) $\underline{\underline{C}}$ is minimal sufficient

(ii) $\underline{\underline{C}}$ is the smallest countably generated σ-algebra containing $\underline{\underline{B}}$

(iii) $\underline{\underline{C}}$ is sufficient and $\underline{\underline{C}} = \bigcap_{\theta} \underline{\underline{B}} \vee N_{\theta}$.

Proof. (i) ⟹ (ii) : By Theorem 3.1.2 any countably generated σ-algebra containing $\underline{\underline{B}}$ is sufficient. Further, by Burkholder's theorem [9] $\underline{\underline{C}}$ is itself countably generated. Hence $\underline{\underline{C}}$ is contained in any countably generated σ-algebra containing $\underline{\underline{B}}$.

(ii) ⟹ (iii) That $\underline{\underline{C}}$ is sufficient follows from Theorem 3.1.2.

We will show that $\underline{\underline{C}} = \bigcap_{\theta} \underline{\underline{B}} \vee N_{\theta}$. Towards this we will first establish that $\underline{\underline{C}} = \bigcap_{\theta} \underline{\underline{C}} \vee N_{\theta}$.

Let $A \in \bigcap_{\theta} \underline{\underline{C}} \vee N_{\theta}$. We claim that A is a union of $\underline{\underline{C}}$ - atoms. For, if there is an atom C of $\underline{\underline{C}}$ such that $C \cap A \neq \emptyset$ and $C \cap A^c \neq \emptyset$, choose θ_1, θ_2 such that $P_{\theta_1}(C \cap A) > 0$ and $P_{\theta_2}(C \cap A^c) > 0$.

Since $\underline{\underline{B}}$ and consequently $\underline{\underline{C}}$ is pairwise sufficient

$$\bigcap_{\theta} \underline{\underline{C}} \vee N_{\theta} = \bigcap_{\theta_i, \theta_j} \underline{\underline{C}} \vee N_{\theta_i, \theta_j}.$$

Therefore, given A, there is A' in $\underline{\underline{C}}$ such that

$P_{\theta_1}(A \triangle A') = 0$ and $P_{\theta_2}(A \triangle A') = 0$. But this is not possible. Blackwell's Theorem [5] now yields $\underline{\underline{C}} = \bigcap_{\theta} \underline{\underline{C}} \vee N_{\theta}$.

$$\underline{\underline{B}} \subseteq \underline{\underline{C}} \text{ implies } \bigcap_{\theta} \underline{\underline{B}} \vee N_{\theta} \subseteq \bigcap_{\theta} \underline{\underline{C}} \vee N_{\theta} = \underline{\underline{C}}$$

The other inclusion will follow from the following two facts.

a) Atoms of $\underline{\underline{B}}$ are same as the atoms of $\underline{\underline{C}}$.

Suppose not. Since $\mathbb{P}(\underline{\underline{B}}) \subseteq \underline{\underline{C}}$, let E be an atom of $\underline{\underline{B}}$ containing more than one atom of $\underline{\underline{C}}$. Then the σ-algebra

$\underline{\underline{C}}' = (\underline{\underline{C}} \cap E^c) \vee \{E\}$ is a countably generated σ-algebra containing $\underline{\underline{B}}$ and strictly contained in $\underline{\underline{C}}$. This contradicts the minimality of $\underline{\underline{C}}$ stated in (ii).

b) $E \in \underline{\underline{A}}$, and E is a union of $\underline{\underline{B}}$ atoms, then, $E \in \bigcap_{\theta} \underline{\underline{B}} \vee N_{\theta}$.

$$I_E(x) = I_E(x) \cdot I_{S_\theta}(x) \; [P_\theta] \text{ for all } \theta \text{ in } \underline{\underline{H}}$$

Since for each θ, S_θ is a countable set in $\underline{\underline{B}}$ and E is a union of $\underline{\underline{B}}$ atoms, $I_E(x) \cdot I_{S_\theta}(x)$ is for each θ in $\underline{\underline{H}}$ $\underline{\underline{B}}$ - measurable. Therefore $E \in \bigcap_{\theta} \underline{\underline{B}} \vee N_{\theta}$.

(iii) $\Rightarrow$ (i) $\bigcap_{\theta} \underline{\underline{B}} \vee N_{\theta} = \underline{\underline{C}}$ is sufficient. Hence $\underline{\underline{C}}$ is countably generated [9]. That $\underline{\underline{C}}$ is minimal now follows from minimality of $\underline{\underline{B}}$, and from the fact that every sufficient sub σ-algebra of $\underline{\underline{A}}$ is necessarily countably generated.

We need the following before stating the next proposition. Any atomic σ-algebra on X induces an equivalence relation on X, namely $x \sim y$ iff x and y belong to the same atom. We say that an equivalence relation is Borel if the set $\{(x,y) : x \sim y\}$ is a Borel set in the product. A partition is said to be induced by a real valued measurable function f, if f is Borel measurable and $f(x) = f(y)$ iff x and y belong to the same atom.

Theorem 3.2.2. The following are equivalent :

(i) $\underline{\underline{B}}$ is induced by a real valued measurable function.

(ii) The relation induced by $\underline{\underline{B}}$ is Borel.

(iii) $\bigcap_{\theta} \underline{\underline{B}} \vee N_{\theta}$ is sufficient.

Proof. (i) $\Rightarrow$ (ii) is immediate. For if T is a function inducing B then $x \sim y \Leftrightarrow T(x) = T(y)$.

(ii) ⟹ (iii). We shall show that if $\underline{\underline{B}}$ induces a Borel relation then given any set $A \in \underline{\underline{A}}$ there is a jointly measurable function $f_\theta(x)$ such that $f_\theta(x) = E_\theta(I_A \mid \underline{\underline{B}})$ $[P_\theta]$ for all $\theta \in \textcircled{H}$.

$$\text{Define} \quad f_\theta(x) = \frac{P_\theta(A \cap R^x)}{P_\theta(R^x)} \, I_{S_\theta}(x)$$

where $R = \{(x,y) : x \sim y\}$.

An argument similar to the proof of Prpn. 2.1.2 shows that $f_\theta(x)$ is jointly measurable in θ and x. Proof of sufficiency of $\bigcap_\theta \underline{\underline{B}} \vee N_\theta$ is in the same lines as that of theorem 3.1.2.

(iii) ⟹ (i) is clear since sufficiency of $\bigcap_\theta \underline{\underline{B}} \vee N_\theta$ implies it is countably generated and hence given by a real valued measurable function.

Theorem 3.2.2 yields some sufficient conditions for the existence of a minimal sufficient sub σ-algebras. The relation induced by $\underline{\underline{B}}$ can be easily characterised as the intersection of the two relations

i) $(x,y) \in R_1$ iff $P_\theta(x) > 0 \iff P_\theta(y) > 0$

ii) $(x,y) \in R_2$ iff for all θ_1, θ_2 such that $P_{\theta_1}(x)\, P_{\theta_1}(y) > 0$,

$$P_{\theta_2}(x)\, P_{\theta_2}(y) > 0 \quad \text{and} \quad \frac{P_{\theta_1}(x)}{P_{\theta_1}(y)} = \frac{P_{\theta_2}(x)}{P_{\theta_2}(y)}$$

R_1^c can be written as

$$P\left[\left\{(\theta,x,y) : P_\theta(x) > 0 , P_\theta(y)=0\right\} \bigcup \left\{(\theta,xy) : P_\theta(x)=0, P_\theta(y) > 0\right\}\right]$$

R_2^c can be written as

$$P\left[\left\{(\theta_1,\theta_2,x,y) : P_{\theta_1}(x)\, P_{\theta_1}(y)\, P_{\theta_2}(x)\, P_{\theta_2}(y) > 0\right\} \bigcap \left\{(\theta_1,\theta_2,x,y) : \frac{P_{\theta_1}(x)}{P_{\theta_1}(y)} \neq \frac{P_{\theta_2}(x)}{P_{\theta_2}(y)}\right\}\right]$$

Where P denotes projection to the $X \times X$ Co-ordinate space.

Clearly both R_1^c and R_2^c are analytic or R_1 and R_2 are coanalytic and the intersection $R_1 \bigcap R_2$ is also in general coanalytic.

In general $\underline{\underline{B}}$ will not induce a Borel relation, equivalently there will not in general exist a minimal sufficient sub σ-algebra even in the discrete case. Below we give an example of this. We note that in the example $\theta_1 \neq \theta_2 \Longrightarrow P_{\theta_1} \neq P_{\theta_2}$.

<u>Example</u> $\Theta = [0, 2]$, $X = [-1, 2]$.

A is a Symmetric non Borel analytic subset of $[-1, 1]$.

g is a measurable map from $[1, 2]$ onto A.

Define ϕ_1 and ϕ_2 two measurable functions on Θ to X as

$$\phi_1(\theta) = \theta$$

$$\phi_2(\theta) = g(\theta) \text{ on } [1, 2].$$

$$= -\theta \text{ on } [0, 1]$$

$$P_\theta(E) = \frac{1}{2} I_E(\phi_1(\theta)) + \frac{1}{2} I_E(\phi_2(\theta))$$

$\underline{\underline{B}}$ then has atoms $(x, -x)$ for x in $A^c \cap [-1, 1]$ and singletons otherwise and this is not induced by a Borel function.

CHAPTER IV

INTRODUCTION

Before giving a summary of this chapter we will briefly describe the relevant work in this area.

Let $(X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\underline{H}})$ be an experiment, $\underline{\underline{C}}$ the σ-algebra on $\underline{\underline{H}}$ and $\underline{\underline{B}}$ a sub σ-algebra of $\underline{\underline{A}}$. It is known and can be proved easily(see [13]) that if $\{ P_\theta : \theta \in \underline{\underline{H}} \}$ is dominated by a σ-finite measure, then pairwise sufficiency implies sufficiency. There has been attempts to generalise this result and show that even in the undominated case pairwise sufficiency is related to sufficiency. Pitcher [25] introduced compact statistical structures, Basu and Ghosh discrete statistical structures and finally,Hasegawa and Perlman Coherent experiments. It is now known that coherence is equivalent to compactness and the discrete structure a special case of both. That these concepts are natural generalisation of domination by σ-finite measures was established by Dipenbrock [10], who showed that compactness and coherence are both equivalent to domination by a localizable measure. Their theorems connecting pairwise sufficiency with sufficiency is of the form " if $\underline{\underline{B}}$ is pairwise sufficient then $\bigcap_{\theta_1,\theta_2} \underline{\underline{B}} \vee N_{\theta_1,\theta_2}$ is sufficient".

While experiments dominated by a σ-finite measure are coherent, Rogge [27] showed that if $\underline{\underline{A}}$ is countably generated then any coherent experiment is necessarily dominated by a σ-finite measure. Thus in countably generated situations, in particular in the Standard Borel case, compactness is not more general than domination by a σ-finite measure. However, in view of Theorem 3.1.2, which asserts that in the Standard Borel case if the P_θ s are discrete, then pairwise sufficiency is equivalent to sufficiency, and since $P_\theta(x)$ can be thought of as densities with respect to the counting measure, a similar generalisation seems possible. The first part of this chapter centers on such a generalisation.

The first part of this chapter is motivated by the work of Hasegawa - Perlman and the theorem of Dipenbrock. We define the notion of weak coherence, Borel localizable and Borel decomposable measures - all Standard Borel adaptations of known concepts, and then show the experiments dominated by Borel localizable measures satisfying an additional measurability property are weakly coherent.

In the second part of this chapter we give two examples obtained in response to questions raised by J.K. Ghosh. In [12] it is shown that in dominated (by a localizable measure) experiments there always exists a minimal pairwise sufficient σ-algebra. We give an example to show that minimal pairwise sufficient σ-algebras need not exist in the general undominated case. The second example is in the context of Neyman factorisation theorem. It is shown in [12], that for experiments dominated by a localizable (locally localizable) measure, a sub σ-algebra is pairwise sufficient and contains " supports" iff the densities admit a factorisation with respect to it. The question then arises as to whether such a theorem is true without the assumption of localizability of the dominating measure. Our example answers the question in the negative.

Section 1

Let $(X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\underline{H}})$ be an experiment. Assume further that $\underline{\underline{H}}$ is equipped with a σ-algebra $\underline{\underline{C}}$ and that $\theta \longrightarrow P_\theta$ is a measurable transition function from $(\underline{\underline{H}}, \underline{\underline{C}})$ to $(X, \underline{\underline{A}})$. Throughout this section $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}$ are assumed to be countably generated.

Definition 4.1.1. A family of functions $f_\theta(x)$, jointly measurable in θ and x, is weakly pairwise coherent if given θ_1, θ_2 there is an $\underline{\underline{A}}$ - measurable function $f_{\theta_1 \theta_2}(x)$ such that

$$f_{\theta_1 \theta_2}(x) = f_{\theta_1}(x) \quad [P_{\theta_1}]$$
$$= f_{\theta_2}(x) \quad [P_{\theta_2}] .$$

$f_\theta(x)$ is weakly coherent if there is an $\underline{\underline{A}}$ - measurable function $f(x)$ such that

$$f(x) = f_\theta(x) \quad [P_\theta] \quad \text{for all } \theta \text{ in } (\underline{\underline{H}}) .$$

Definition 4.1.2. An experiment $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{\underline{H}}))$ is weakly coherent if every family of weakly pairwise coherent functions is weakly coherent.

Remark. Coherence in the sense of Hasegawa - Perlman can be obtained from weak coherence by taking $\underline{\underline{C}}$ to be the power set of $(\underline{\underline{H}})$.

Weakly coherent experiments form a subclass of experiments considered in section 3 of Chapter II. Even in the

Standard Borel case, there are experiments covered by Theorem 2.3.1 of Chapter II, which do not fall in the ambit of weak coherence. Below is an example of one such experiment.

<u>Example</u>. $X = (\underline{\underline{H}}) = [0,1]$.

$\underline{\underline{A}} = \underline{\underline{C}}$ Borel σ-algebra on $[0,1]$.

$P_\theta = \frac{1}{2}\delta_\theta + \frac{1}{2}\lambda$ where δ_θ is the point mass at θ and λ is the Lebesgue measure on $[0,1]$.

Considering the diagonal in $[0,1] \times [0,1]$ it can be seen that $(X, \underline{\underline{A}}, P_\theta : \theta \in (\underline{\underline{H}}))$ is not weakly coherent. On the other hand taking ξ to be the Lebesgue measure on $(\underline{\underline{H}})$, for any function $f_\theta(x)$ satisfying (i) and (ii) of section 3 (Chapter II) it is easily seen that

$$f_\theta(x) = f(x) \quad [P_\theta] \quad \text{for all } \theta \text{ in } (\underline{\underline{H}})$$

where $f(x) = f_x(x)$.

We now introduce Borel localizable and Borel decomposable measures. These notions correspond to the well known (see for instance [29]) localizable and strictly localizable measures and unlike them Borel localizability turns out to be equivalent to Borel decomposability.

Definition 4.1.3. Let $(X, \underline{\underline{A}})$ be a Standard Borel space. A measure m on $(X, \underline{\underline{A}})$ is Borel localizable if there is a Standard Borel space $(T, \underline{\underline{T}})$ and a Borel subset E of $T \times X$ satisfying

(i) $0 < m(E^t) < \infty$

(ii) $t_1 \neq t_2$ then $m(E^{t_1} \cap E^{t_2}) = 0$

(iii) for all A in $\underline{\underline{A}}$, $m(A) = \sum_{t \in T} m(A \cap E^t)$

(iv) If B is a Borel subset of E, then $\{B^t : t \in T\}$ has an m essential supremum in $\underline{\underline{A}}$.

Definition 4.1.4. A Borel localizable measure m on a Standard Borel space is Borel decomposable if there is an E satisfying (i), (ii) and (iii) of Definition 4.1.3 and also

(ii)' $t_1 \neq t_2$ then $E^{t_1} \cap E^{t_2} = \emptyset$.

Any set E satisfying conditions (i), (ii), (iii) and (ii)' will be referred to as a Borel decomposition of $(X, \underline{\underline{A}}, m)$.

We note that in case of Borel decomposability condition (iv) is automatically satisfied. For, if E is a Borel decomposition of $(X, \underline{\underline{A}}, m)$ then for any Borel set $B \subseteq E$, $\bigcup_t B^t$ is itself Borel and acts as an essential supremum of $\{B^t : t \in T\}$.

Theorem 4.1.1. If $(X, \underline{\underline{A}}, m)$ is Borel localizable then it is Borel decomposable.

Proof. Since m is Borel localizable get a Borel decomposition satisfying

(i) $0 < m(E^t) < \infty$

(ii) for $t_1 \neq t_2$ $m(E^{t_1} \cap E^{t_2}) = 0$

(iii) $m(A) = \sum_{t \in T} m(A \cap E^t)$

(iv) for every Borel set $B \subseteq E$, $\{B^t : t \in T\}$ has an m essential supremum in $\underline{\underline{A}}$.

We will construct an E^*, Borel subset of $T \times X$, such that

(i) for all $t \in T$, $E^t = E^{*t}$ $[m]$

(ii) $E^{*t_1} \cap E^{*t_2} = \emptyset$.

Let $\{C_1, C_2, \ldots\ldots\}$ be a countable algebra generating $\underline{\underline{T}}$ for each i define $F_i = \operatorname{ess\,sup}_{t \in C^i} E^t$.

Now define E^* by

$$E^{*t} = \bigcap_{\substack{i \\ t \in C_i}} F_i - \bigcup_{\substack{j \\ t \notin C_j}} F_j$$

Then E^* is Borel in $T \times X$, for $E^* = \{(t,x) : \emptyset(t,x) = 1\}$

where $\emptyset(t,x) = \prod_i [I_{C_i}(t) I_{F_i}(x) + (1 - I_{C_i}(t)(1 - I_{F_i}(x)]$

It can be easily verified that E^* satisfies the required properties.

Examples of Borel Decomposable measures

(i) $(X, \underline{\underline{A}})$ Standard Borel and m a σ-finite measure on $(X, \underline{\underline{A}})$. Choose $T = \mathbb{N}$ and $\{E^n : n \in \mathbb{N}\}$ any decomposition of $(X, \underline{\underline{A}})$ into sets of positive finite measure.

(ii) $(X, \underline{\underline{A}})$ Standard Borel, m counting measure.
Choose $T = X$ and E to be the diagonal in $X \times X$.

(iii) $X = [0,1] \times [0,1]$, $\underline{\underline{A}}$ Borel σ-algebra on X.
$(T, \underline{\underline{T}}) = ([0,1]$, Borel σ-algebra).
$m(A) = \sum_t \lambda(A^t)$ where λ is the Lebesgue measure on $[0,1]$.

Let m be a Borel decomposable measure on $(X, \underline{\underline{A}})$ and E be a Borel decomposition of $(X, \underline{\underline{A}}, m)$. For each t let m_t be the measure m restricted to E^t.

Definition 4.1.5. We say that m is strongly Borel decomposable if there is a Borel decomposition E of $(X, \underline{\underline{A}}, m)$ such that for all Borel subset B of E

$$t \longrightarrow m_t(B) = m(B \cap E)^t \text{ is measurable in } t .$$

Note that examples (i), (ii) and (iii) above are indeed strongly decomposable. Example (ii) can be modified to get a decomposable but not strongly decomposable measure. For this choose a non measurable positive function $\emptyset$ on X and set $m(x) = \emptyset(x)$.

An experiment $(X, \underline{\underline{A}}, P_\theta : \theta \in \underline{\underline{H}})$ where $(X, \underline{\underline{A}})$ and $(\underline{\underline{H}}, \underline{\underline{C}})$ are Standard Borel is dominated by a strongly Borel decomposable measure m, if

(i) for each θ in $\underline{\underline{H}}$, P_θ is dominated by m and $\frac{dP_\theta}{dm}$ exists

(ii) $\{ P_\theta : \theta \in \underline{\underline{H}} \} \equiv m$ i.e. $P_\theta(A) = 0$ for all $\theta \in \underline{\underline{H}}$ iff $m(A) = 0$.

We have assumed "strong" Borel decomposability rather than Borel decomposability to ensure the measurability of certain functions. This is exemplified in the following lemma.

Lemma 1. Assume that $(X, \underline{\underline{A}}, P_\theta : \theta \in (\overline{H}))$ is dominated by a "strongly" Borel decomposable measure m and let E be a strong Borel decomposition of $(X, \underline{\underline{A}}, m)$. Then for each Borel subset of $T \times (\overline{H}) \times X$, the following functions are measurable in (θ, t).

(i) $(\theta, t) \longrightarrow P_\theta (B^{\theta, t})$ where $B^{\theta, t} = \{ x : (\theta, t, x) \in B \}$

(ii) $(\theta, t) \longrightarrow m (B^{\theta, t} \cap E^t)$

Proof. (i) Let $\underline{\underline{M}} = \{ B \subset (H) \times T \times X : P_\theta (B^{\theta, t})$ is measurable in $(\theta, t) \}$

$\underline{\underline{M}}$ contains all rectangles, is closed under finite disjoint unions and is further a monotone class. Consequently $\underline{\underline{M}}$ contains all Borel sets in $(\overline{H}) \times T \times X$.

(ii) $\underline{\underline{M}}' = \{ B \subset (H) \times T \times X : m (B^{\theta, t} \cap E^t)$ is measurable in $(\theta, t) \}$

That $\underline{\underline{M}}'$ contains all rectangles follows from 'strong' decomposability of m. $\underline{\underline{M}}'$ is closed under finite disjoint unions. Further, since for all t, $m(E^t) < \infty$, $\underline{\underline{M}}'$ is also a monotone class.

<u>Lemma 2</u>. Let $D = \{(\theta, t, x) : P_\theta(E^t) > 0$ and $x \in E^t\}$ and D_1 be the projection of D to the $\underline{H}$ x X space. Then the function $\theta \longrightarrow m(D_1^\theta \cap A)$ is measurable in θ for every Borel subset A of X.

<u>Proof</u>. D is Borel in $\underline{H} \times T \times X$ (by lemma 1). Further for each (θ, x) there is at most one t such that $(\theta, t, x) \in D$. Therefore D_1 is Borel in $\underline{H} \times X$.

$$\text{Let } D_2 = \{(\theta, t) : P_\theta(E^t) > \theta\}.$$

D_2 is a Borel set in $\underline{H} \times T$ such that each θ section of D_2 is atomost countable. Therefore by Lusin's Theorem [17] there are measurable functions $g_1, g_2, \ldots$ defined on $\underline{H}$ taking values in T such that

$$D_2 = \bigcup_{i=1}^{\infty} (\theta, g_i(\theta)).$$

Fix any A in $\underline{\underline{A}}$. Define a sequence of functions $\emptyset_1, (\theta), \emptyset_2(\theta), \ldots$ by

$$
\begin{aligned}
\phi_1(\theta) &= m\,(E \cap A)^{g_1(\theta)} \\
\phi_2(\theta) &= m\,(E \cap A)^{g_2(\theta)} && \text{if } g_1(\theta) \neq g_2(\theta) \\
&= 0 && \text{if } g_1(\theta) = g_2(\theta) \\
\phi_n(\theta) &= m\,(E \cap A)^{g_n(\theta)} && \text{if } g_n(\theta) \neq g_i(\theta) \\
& && \text{for } i = 1,2,\ldots,n-1 \\
& \quad 0 && \text{otherwise.}
\end{aligned}
$$

Then $m\,(A \cap D_1^{\theta}) = \sum_{n=1}^{\infty} \phi_n(\theta)$ which is measurable in θ .

<u>Theorem 4.1.2</u>. If $(X\,,\underline{\underline{A}}\,,P_\theta : \theta \,\varepsilon\, \underline{(H)})$ is dominated by a strongly Borel Decomposable measure m , then there is a jointly measurable version of $\dfrac{dP_\theta}{dm}$.

<u>Proof</u>. Let $D = \left\{ (\theta,t,x) : P_0\,(E^t) > 0 \text{ and } x \,\varepsilon\, E^t \right\}$ and D_1 be the projection of D to the $\underline{(H)} \times X$ space . Then D_1 has the following properties

(i) $P_\theta\,(D_1^{\theta}) = 1$ for all θ in $\underline{(H)}$

(ii) $m\,(D_1^{\theta})$ is σ-finite for all θ in $\underline{(H)}$.

To see (ii) note that $D_1^{\theta} = \bigcup_{\substack{t \\ P_\theta(E^t) > 0}} E^t$ and $\left\{ t : P_\theta(E^t) > 0 \right\}$ is at most countable.

Now fix finite algebras $\underline{\underline{A}}_n$ generating $\underline{\underline{A}}$ and denote the atoms by, $A_n^{'}, \ldots, A_n^{k(n)}$

Define
$$f_n^{\theta}(x) = \sum_{i=1}^{k(n)} \sum_{j=1}^{\infty} \frac{P_{\theta}(A_n^i \cap E^{\phi_j^{'}(\theta)})}{m(A_n^i \cap E^{\phi_j^{'}(\theta)})} I_{A_n^i \cap E^{\phi_j^{'}(\theta)}}(x)$$

where $\phi_j^{'}(\theta)$ are obtained from the $g_1, g_2, \ldots$ of lemma 2 as follows. Fix some ξ outside T and declare $E^{\xi} = \emptyset$.

$$\phi_1^{'}(\theta) = g_1(\theta)$$

$$\begin{aligned} \phi_2^{'}(\theta) &= g_2(\theta) && \text{if } g_1(\theta) \neq g_2(\theta) \\ &= \xi && \text{if } g_1(\theta) = g_2(\theta) \end{aligned}$$

$$\begin{aligned} \phi_n^{'}(\theta) &= g_n(\theta) && \text{if } g_n(\theta) \neq g_i(\theta) \text{ for } i = 1, \ldots, n-1 \\ &= \xi && \text{Otherwise.} \end{aligned}$$

Then by a well known theorem (see [21]), since m is finite on $E^{\phi_i(\theta)}$, f_{θ}^n converges to $\frac{dP_{\theta}}{dm}$. Since for each n, $f_{\theta}^n(x)$ is jointly measurable, $f_{\theta}(x)$ defined by

$$\begin{aligned} f_{\theta}(x) &= \lim_n f_{\theta}^n(x) && \text{if it exists} \\ & \quad 0 && \text{Otherwise} \end{aligned}$$

is the required version.

Theorem 4.1.3. If $(X, \underline{\underline{A}}, P_\theta : \theta \varepsilon (\underline{\underline{H}}))$ is dominated by a strongly Borel decomposable measure then it is weakly coherent.

Proof. Let m be the dominating measure and E be a strong Borel decomposition. By theorem 1, we choose a jointly measurable version of $\frac{dP_\theta}{dm}$.

Denote by $S = \left\{ (\theta, x) : \frac{dP_\theta}{dm} > 0 \right\}$

Suppose $f_\theta(x)$ is weakly pairwise coherent. Then by letting $f_\theta(x)$ to be zero outside S, it is possible to extend $f_\theta(x)$ as a weakly pairwise coherent family of functions on $(X, \underline{\underline{A}}, P_{\bar{\theta}} : \bar{\theta} \varepsilon (\bar{\underline{\underline{H}}}))$

where $(\bar{\underline{\underline{H}}}) = \left\{ (a_1) \quad a_i \geq 0 \quad \Sigma\, a_i = 1 \right\} x\ (\underline{\underline{H}})\ x\ (\underline{\underline{H}})\ x \ldots$

and $P_{\bar{\theta}} = \sum_{1}^{\infty} a_i P_{\theta_i}$

Therefore we will assume without loss of generality that $\left\{ P_\theta : \theta \varepsilon (\underline{\underline{H}}) \right\}$ is closed under countable convex combinations. We will also assume for simplicity that $f_\theta(x) = I_{B_\theta}(x)$.

We will briefly describe the idea of the proof. On each E^t, P_θ is a family of measures dominated, in fact, equivalent to the finite measure $m|E^t$. Therefore there is some θ' such that $P_{\theta'} \equiv m$ on E^t. Also since $B_\theta(x)\, I_{E^t}$ is coherent, $B_{\theta'}\, I_{E^t}$ will be a P_θ equivalent version of $B_\theta\, I_{E^t}$ for all θ. Our proof shows that $B_{\theta'}$ on E^t can be defined independendently of θ' and also can be done measurably in t. Having got B_t s we piece them together to get a B.

Define h on $D_2 = \left\{(\theta,t) : P_\theta(E^t) > 0\right\}$ by

$$h(\theta,t) = \frac{m(E^t \cap S^\theta)}{m(E^t)} \quad \text{is measurable in } (\theta,t), \text{ and}$$

therefore $D_o = \left\{(\theta,t) : h(\theta,t) = 1\right\}$ is Borel in $(\underline{H}) \times T$.

Note that $(\theta,t) \varepsilon D_o$ iff P_θ is equivalent to m on E_t. By the theorem of Halmos and Savage [13] for every t, there is atleast one θ such that $(\theta,t) \varepsilon D_o$. It can be seen that $I_{E^t} I_{B_\theta} = I_{B_{\theta'}} I_{E^t}$ $[P_\theta]$ for all θ, if $(\theta',t) \varepsilon D_o$.

As before choose $\underline{\underline{A}}_n$ finite algebras generating $\underline{\underline{A}}$. Let $A_n^1, A_n^2, \ldots, A_n^{k(n)}$ denote the atoms of $\underline{\underline{A}}_n$. For fixed (θ,t) in D_o

$$I_{B_\theta}(x) \; I_{E^t}(x) = \lim_{n \to \infty} \sum_{i=1}^{k(n)} \frac{m(A_n^i \cap B_\theta \cap E^t)}{m(A_n^i \cap E^t)} \; I_{E^t \cap A_n^i}(x) \quad [P_\theta].$$

We will show that for each i and n, $\dfrac{m(A_n^i \cap B_\theta \cap E^t)}{m(A_n^i \cap E^t)}$ is independent of θ and is further a measurable function of t. First note that $(\theta_1, t), (\theta_2, t) \in D_0 \Longrightarrow B_{\theta_1} I_{E^t} = B_{\theta_2} I_{E^t}$ [m] and hence

$$\frac{m(A_n^i \cap B_{\theta_1} \cap E^t)}{m(A_n^i \cap E^t)} = \frac{m(A_n^i \cap B_{\theta_2} \cap E^t)}{m(A_n^i \cap E^t)}.$$

On D_0 look at the function $g(\theta, t) = \dfrac{m(A_n^i \cap B_\theta \cap E^t)}{m(A_n^i \cap E^t)}$. Then $g(\theta,t)$ is measurable in (θ, t) and is constant on each t section. By arguing in the same lines as in the proof of Theorem 3.1.2 $g^*(t) = \sup_{D_0^t} g(\theta,t)$ is measurable in t and further $g^*(t) = g(\theta,t)$ if $(\theta,t) \in D_0$.

Therefore for each (i,n) $\dfrac{m(A_n^i \cap B_\theta \cap E^t)}{m(A_n^i \cap E^t)} \; I_{E^t \cap A_n^i}$ is a measurable function of only t and x, and therefore the function $f_t(x)$ defined by

$$f_t(x) = \begin{cases} \lim\limits_{n} \sum\limits_{1}^{k(n)} \dfrac{m(A_n^i \cap B_\theta \cap E^t)}{m(A_n^i \cap E^t)} I_{E^t \cap A_n^i} & \text{if the limt exists} \\ 0 & \text{Otherwise} \end{cases}$$

is also measurable in (t,x). Further since for each t_o there is some θ_o, $(\theta_o, t_o) \varepsilon D_o$ and $f_t(x) = I_{B_{\theta_o}} I_{E^t}$ $[m]$,

$$f_t(x) = I_{B_\theta}(x)\, I_{E^t}(x) \quad [P_\theta] \quad \text{for all } \theta \text{ in } (\underline{H}).$$

Now we can define $f(x) = \sum\limits_t f_t(x) \; I_{E^t}(x)$ and then $f(x) = I_{B_\theta}(x)$ $[P_\theta]$ for all θ.

This completes the proof of the theorem.

Theorem 4.1.4. Suppose $(X, \underline{\underline{A}}, P_\theta : \theta \varepsilon (\underline{H}))$ is weakly coherent. If $\underline{\underline{B}}$ is a countably generated sub σ-algebra of $\underline{\underline{A}}$ which is pairwise sufficient then $\hat{\underline{\underline{B}}} = \bigcap\limits_\theta \underline{\underline{B}} \vee N_\theta$ is sufficient.

Proof. Let f be any bounded $\underline{\underline{A}}$ - measurable function. Get $f_\theta(x)$ a jointly measurable version of $E_\theta(f \mid \underline{\underline{B}})$. Since $\underline{\underline{B}}$ is pairwise sufficient, $f_\theta(x)$ is weakly pairwise coherent. Now since $(X, \underline{\underline{A}}, P_\theta : \theta \varepsilon (\underline{H}))$ is coherent, there is an f^* such that

$$f^*(x) = f_\theta(x) \quad [P_\theta] \quad \text{for all } \theta.$$

Since each $f_\Theta(x)$ is $\underline{\underline{B}}$ measurable, $f^*(x)$ is $\underline{\underline{B}} \vee N_\Theta$ measurable for each Θ.

Remark Since $\underline{\underline{B}}$ is pairwise sufficient

$$\bigcap_{\Theta} \underline{\underline{B}} \vee N_\Theta = \bigcap_{\Theta_1, \Theta_2} \underline{\underline{B}} \vee N_{\Theta_1, \Theta_2} .$$

Therefore in the above theorem one can assert that

$\bigcap_{\Theta_1, \Theta_2} \underline{\underline{B}} \vee N_{\Theta_1, \Theta_2}$ is itself sufficient. Combining Theorem 4.1.3 and 4.1.4 we get.

Theorem 4.1.5. If $(X, \underline{\underline{A}}, P_\Theta : \Theta \varepsilon \text{ (H)})$ is dominated by a strongly Borel decomposable measure, then for any countably generated σ-algebra $\underline{\underline{B}}$, the completion $\hat{\underline{\underline{B}}} = \bigcap_{\Theta_1, \Theta_2} \underline{\underline{B}} \vee N_{\Theta_1, \Theta_2}$ is sufficient.

Remark. Suppose $\underline{\underline{B}}$ is countably generated and pairwise sufficient and further if m admits a decomposition E such that for each t, E^t is $\underline{\underline{B}}$ - measurable, then $\underline{\underline{B}}$ is itself sufficient. This is immediate from the construction of $f^t(x)$. In fact this is precisely what happens in the discrete case.

For, in the discrete case given a countably generated pairwise sufficient σ-algebra $\underline{\underline{B}}$, it is easy to see that the atoms of $\underline{\underline{B}}$ are countable. Hence for T one can take the space of atoms of $\underline{\underline{B}}$, (Quotient space) and for each t take E^t to be the t - atom. T is in general analytic, however Theorem 4.1.3 goes through even when T is analytic.

We will now give an example to show that Theorem 4.1.5 cannot be improved in the sense that while $\hat{\underline{\underline{B}}}$ is sufficient $\underline{\underline{B}}$ itself need not be.

<u>Example 4.1.2.</u> $X = [0,1] \times [0,1]$, $(\underline{\underline{H}}) = [0,1] \cup \{2\}$

$\underline{\underline{A}}$ = Borel σ-algebra on $[0,1] \times [0,1]$

$\underline{\underline{C}}$ = Borel σ-algebra on $[0,1] \cup \{2\}$

for $\theta \in [0,1]$ P_θ = Lebesgue measure on $\theta \times [0,1]$

P_2 = Lebesgue measure on the diagonal in X.

To construct m, take $T = [0,1] \cup \{2\}$. For $t \in [0,1]$ define $E^t = t \times [0,1] - \{t,t\}$ and for $t = 2$ define E^t = diagonal in $[0,1] \times [0,1]$.

We now define m by $m(A) = \sum_{t \in [0,1]} \lambda(A^t) + \lambda'(A \cap D)$ where λ is the Lebesgue measure on $[0,1]$ and λ' Lebesgue measure on the diagonal.

In this example, the σ-algebra of vertical Borel sets, i.e. sets of the form $B \times [0,1]$, is pairwise sufficient but not sufficient.

Section 2

In this section we give two examples both in response to questions raised by J.K. Ghosh. These examples are in undominated situations and unlike the earlier chapters can occur only when $\underline{\underline{A}}$ is not countably generated.

Example 4.2.1. This examples describes a situation where a minimal pairwise sufficient σ-algebra does not exist.

Let I be an uncountable set

$X = (\underline{H}) = \{0,1\}^I$

$\underline{\underline{A}}$: Product of discrete σ-algebra on $\{0,1\}$

P_θ is for each θ the point mass at θ.

It is easy to see that a sub σ-algebra of $\underline{\underline{A}}$ is pairwise sufficient iff it separates points in X. We will show that $\underline{\underline{A}}$ does not contain a minimal separating σ-algebra. More precisely we will show that any sub σ-algebra of $\underline{\underline{A}}$, which separates points in X, contains another separating σ-algebra.

Let $\underline{\underline{B}}$ be any separating sub σ-algebra of $\underline{\underline{A}}$. For every B in $\underline{\underline{A}}$, consequently for every B in $\underline{\underline{B}}$, there is a countable set $K(B) \subseteq I$ which determines B. K(B) will be called a support of B. Note that any countable set containing a support of B is also a support of B. In what follows ε_i s take the value 0 or 1.

Let $x_o = \underline{0}$ $\qquad x_1 = \underline{1}$

Then there is a B_o and B_1 in $\underline{\underline{B}}$ such that $x_o \in B_o$, $x_1 \in B_1$ and $B_o \cap B_1 = \emptyset$, $B_o \cup B_1 = X$. Let K(1) be a common support of B_o and B_1. We now define four points $x_{00}, x_{01}, x_{10}, x_{11}$ by

$$x_{\varepsilon_1 \varepsilon_2} = \varepsilon_1 \text{ on } K(1)$$

$$= \varepsilon_2 \text{ on } I - K(1)$$

We now obtain sets $B_{\varepsilon_1 \varepsilon_2}$ in $\underline{\underline{B}}$, pairwise disjoint covering X' and $x_{\varepsilon_1 \varepsilon_2} \varepsilon B_{\varepsilon_1 \varepsilon_2}$.

Let $K(2) \supset K(1)$ denote a common suport of $\left\{ B_{\varepsilon_1, \varepsilon_2} : \varepsilon_i = 0 \text{ or } 1 \right\}$. We now define eight points $x_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$ by

$$\begin{aligned} x_{\varepsilon_1 \varepsilon_2 \varepsilon_3} &= \varepsilon_1 \quad \text{on} \quad K(1) \\ &= \varepsilon_2 \quad \text{on} \quad K(2) - K(1) \\ &= \varepsilon_3 \quad \text{on} \quad I - K(2) \end{aligned}$$

Corresponding to these points we get $B_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$.

Continuing we get a Branching sequence of sets $B_{\varepsilon_1 \varepsilon_2, \ldots, \varepsilon_n}$ and corresponding supports $K(n)$. Since $\bigcup_{n=1}^{\infty} K(n)$ is countable, for any sequence $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots.)$, $B(\underline{\varepsilon}) = \bigcap_{n=1}^{\infty} B_{\varepsilon_1 \varepsilon_2, \ldots, \varepsilon_n}$ is in $\underline{\underline{B}}$ and is also non-empty.

Define $\underline{\underline{C}} = \left\{ C \varepsilon \underline{\underline{B}} : C \text{ or } C^c \text{ is contained in countably many } B(\underline{\varepsilon})\text{'s} \right\}$. Now observe that

(i) $\underline{\underline{C}}$ separates points in X.

If given x, y there is a B in $\underline{\underline{B}}$ such that $x \in B$, $y \notin B$. Now choose ε such that $x \in B(\varepsilon)$. Then $x \in B \cap B(\varepsilon)$ and $B \cap B(\varepsilon)$ is in $\underline{\underline{C}}$. However $y \notin B \cap B(\varepsilon)$

ii) $\underline{\underline{C}}$ is property contained in $\underline{\underline{B}}$.

$B_o \in \underline{\underline{B}}$, but B_o does not belong to $\underline{\underline{C}}$.

Before giving the next example we start with some preliminaries, mostly taken from [12].(See also [18])

Let $(X, \underline{\underline{A}})$ be a measurable space. We recall that $\underline{\underline{A}}$ is not assumed to be countably generated. In fact such an assumption would make most of the following trivial. Let m be a measure on $(X, \underline{\underline{A}})$, with the finite subset property, i.e. for every A in $\underline{\underline{A}}$, with $m(A) > 0$, there is a $B \in \underline{\underline{A}}$, $B \subset A$ such that $0 < m(B) < \infty$.

Let $\underline{\underline{A}}(F) = \{ A \in \underline{\underline{A}} \quad m(A) < \infty \}$

$$\underline{\underline{A}}(1) = \{ E \subset X : E \cap A \in \underline{\underline{A}} \text{ for all } A \text{ in } \underline{\underline{A}}(F) \}$$

The measure m can be extended as a measure $\bar{m}$ to $\underline{\underline{A}}(1)$ by

$$\bar{m}(E) = \sup_{A \in \underline{\underline{A}}(F)} m(A \cap E)$$

$(X, \underline{\underline{A}}, m)$ will be called locally determined if $\underline{\underline{A}}(1) = \underline{\underline{A}}$.

Definition 4.2.1. m on $\underline{\underline{A}}$ is called localizable (respectively locally localizable) if it satisfies the following condition.

Suppose that $\underline{\underline{F}}$ is any subfamily of $\underline{\underline{A}}(F)$. Then there exists an essential supremum of $\underline{\underline{F}}$ with respect to m in $\underline{\underline{A}}$ (respectively $\underline{\underline{A}}(1)$) ; i.e. there is a F_o in $\underline{\underline{A}}$ (respectively $\underline{\underline{A}}(1)$) such that

a) $m(F - F_o) = 0$ (respectively $\bar{m}(F - F_o) = 0$) for all F in $\underline{\underline{F}}$

b) $m(F_o - A) = 0$ (respectively $\bar{m}(F_o - A) = 0$) for all A in $\underline{\underline{A}}$ which satisfies " $m(F - A) = 0$ for all F in $\underline{\underline{F}}$ ".

Note that m is locally localizable iff $\bar{m}$ on $\underline{\underline{A}}(1)$ is localizable.

We will need the following theorem regarding localizability.

Definition 4.2.2 A family $\{ f(x,A) : A \in \underline{\underline{A}}(F) \}$ is called an m-cross section if it has the following property

(i) $f(x,A) = 0$ outside A

(ii) for any A, B in $\underline{\underline{A}}(F)$ it holds that

$$f(x,A)\, I_{A \cap B}(x) = f(x,B)\, I_{A \cap B}(x) \quad [m]$$

Theorem 4.2.1. Suppose m is a measure with the finite subset property, then m is localizable (respectively locally localizable) iff to each m - cross section $\{f(x,A) : A \in \underline{\underline{A}}(F)\}$ there is an $\underline{\underline{A}}$ - measurable (respectively $\underline{\underline{A}}(l)$ measurable) function $f(x)$ such that

$$f(x) \cdot I_A(x) = f(x,A) \quad [m] \quad \text{for all } A \text{ in } \underline{\underline{A}}(F).$$

An experiment $(X, \underline{\underline{A}}, P_\theta : \theta \in (H))$ is said to be weakly (locally weakly) dominated if there is a localizable (locally localizable) measure m on $(X,\underline{\underline{A}})$ such that

(i) for each θ, $P_\theta << m$ and $\frac{dP_\theta}{dm}$ exists

(ii) $\{P_\theta : \theta \in (\underline{\underline{H}})\} \equiv m$ i.e. $P_\theta(A) = 0$ for all θ iff $m(A) = 0$.

In the next theorem by a support of P_θ we mean the set $\left\{ x : \frac{dP_\theta}{dm} > 0 \right\}$ for some version of $\frac{dP_\theta}{dm}$.

Theorem 4.2.1. (Ghosh, Morimoto, Yamada)

Assume that $(X, \underline{\underline{A}}, P_\theta : \theta \varepsilon (\underline{\underline{H}}))$ is locally weakly dominated by m and let $\underline{\underline{B}}$ be a sub σ-algebra of $\underline{\underline{A}}$.

Then $\underline{\underline{B}}$ is pairwise sufficient for $(X, \underline{\underline{A}}, P_\theta : \theta \varepsilon (\underline{\underline{H}}))$ and contains the support of $\{P_\theta : \theta \varepsilon (\underline{\underline{H}})\}$ iff it has a Neyman factorization

$$\frac{dP_\theta}{dm} = g_\theta(x)\ h(x) \quad [m]$$

where $g_\theta(x)$ is $\underline{\underline{B}}$ measurable for each θ and h(x) is a non-negative $\underline{\underline{A}}$ measurable function.

J.K. Ghosh had asked us whether Theorem 4.2.1 is true without the assumption of localizability. Below we give an example to show that this is not so. In our example each P_θ admits a density with respect to a measure m, but there exists a pairwise sufficient σ-algebra containing supports of $\{P_\theta : \theta \varepsilon (\underline{\underline{H}})\}$ for which factorisation of the form described in Theorem 4.2.1 does not hold.

Our example is based on a theorem of Fremlin (page 166 [11]). Note that "Maharam" spaces in Fremlin's terminology is same as "localizable" in ours. Further a measure space $(X, \underline{\underline{A}}, \mu)$ is decomposable if there are pairwise disjoint sets $\{F_\gamma : \gamma \in \Gamma\}$ of positive finite μ - measure, such that for all $A \in \underline{\underline{A}}$, $\mu(A) = \sum_{\gamma \in \Gamma} \mu(F_\gamma \cap A)$. For a formulation of product measures in the context of localizable measures we refer to Fremlin [11].

<u>Proposition</u> [Fremlin]. A complete locally determined measure space $(X, \underline{\underline{A}}, \mu)$ is decomposable iff its complete locally determined product with every probability space is Maharam.

In [11] Fremlin gives an example of a complete locally determined measure space which is Maharam but not decomposable. Now by the preceding proposition, there is a probability space such that its product with Fremlin's example is not Maharam or equivalently localizable. This product space is an instance of a space $(X, \underline{\underline{A}}, m)$ satisfying the properties listed in Example 4.2.2 below.

<u>Example 4.2.2</u>.

Assume (i) $(X, \underline{\underline{A}}, m)$ is a non-localizable, locally determined (hence non locally localizable) measure space with the finite subset property.

(ii) $\underline{\underline{B}} \subseteq \underline{\underline{A}}$ such that $(X, \underline{\underline{B}}, m)$ is localizable and has the finite subset property.

Let $\{ E_\gamma : \gamma \in \Gamma_1 \}$ be a collection of sets of positive finite measure in $\underline{\underline{A}}$, such that $\{ E_\gamma : \gamma \in \Gamma_1 \}$ does not have an essential supremum in $\underline{\underline{A}}$. Without loss of generality $\{ E_\gamma : \gamma \in \Gamma_1 \}$ can be taken to be almost disjoint i.e. $m(E_{\gamma_1} \cap E_{\gamma_2}) = 0$ whenever $\gamma_1 \neq \gamma_2$.

Define μ' as $\mu'(A) = \sum_{\gamma \in \Gamma_1} m(A \cap E_\gamma)$

μ' is easily seen to be a measure.

Define μ as $\mu = m + \mu'$.

Then m is equivalent to μ and $\frac{dm}{d\mu}$ does not exist.

Now let $\underline{\underline{F}} = \{ F \in \underline{\underline{B}} : 0 < m(F) < \infty \}$.

Define $\mathbb{P} = \{ P_F : F \in \underline{\underline{F}} \}$ where $P_F = \frac{m}{m(F)}$.

Then

(i) $\mathbb{P} \equiv m \equiv \mu$ on $\underline{\underline{A}}$

(ii) $\underline{\underline{B}}$ is pairwise sufficient for $(X, \underline{\underline{A}}\ \mathbb{P})$ and contains support of $\{ P_F : F \in \underline{\underline{F}} \}$.

We will show now that $\frac{dP_F}{d\mu}$ cannot be factorised.

Suppose $\frac{dP_F}{d\mu} = g_F(x) . h(x)$ where $g_F(x)$ is $\underline{\underline{B}}$ measurable and $h(x) > 0$.

$$\frac{dP_F}{d\mu} = \frac{1}{m(F)} \frac{dm}{d\mu}\Big|_F = g_F(x).h(x)$$

or $\frac{dm}{d\mu}\Big|_F = m(F)\ g_F(x)\ h(x)$.

Now $\{ m(F)\ g_F(x) : F \varepsilon \underline{\underline{F}} \}$ is a $\underline{\underline{B}}$ measurable cross section. Therefore there is a g such that

$$I_F(x)\ g(x) = g_F(x)\ m(F)$$

$$\therefore \quad g(x)\ h(x) = \frac{dm}{d\mu}. \quad \text{Contradiction !}$$

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