NORTH-HOLLAND

Permanents, Max Algebra and Optimal Assignment

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ABSTRACT

The max algebra consists of the set of real numbers together with $-\infty$, equipped with two binary operations, maximization and addition. For a square matrix, its permanent over the max algebra is simply the maximum diagonal sum of the matrix. Several results are proved for the permanent over the max algebra which are analogs of the corresponding results for the permanent of a nonnegative matrix. These include Alexandroff inequality, Bregman's inequality, Cauchy-Binet formula and a Bebiano-type expansion.

1. PRELIMINARIES

The *max algebra* consists of the set $\mathscr{M} = R \cup \{-\infty\}$, where R is the set of real numbers, equipped with two binary operations, denoted by $\oplus$ and $\otimes$ (and to be referred to as addition and multiplication over the max algebra), respectively. The operations are defined as follows:

$$a \oplus b = \max(a, b) \quad \text{and} \quad a \otimes b = a + b.$$

The notation $\otimes$ should not be confused with the Kronecker product. We denote $x_1 \oplus \cdots \oplus x_n$ by $\sum_{\oplus\, i=1}^{n} x_i$. In general, $\sum_{\oplus}$ will denote summation over max algebra as opposed to $\sum$ which denotes the usual summation.

The max algebra is useful in describing in a linear fashion phenomena which are nonlinear in the usual algebra. For an introduction to matrices over the max algebra and for some basic properties, see [7, 2, 13] and the references contained therein.

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If $A = [a_{ij}]$, $B = [b_{ij}]$ are $m \times n$ matrices over $\mathscr{M}$, then $A + B$ is the $m \times n$ matrix with (i, j)-entry $a_{ij} \oplus b_{ij}$. If $c \in \mathscr{M}$, then $c \otimes A$ is the matrix $[c \otimes a_{ij}] = [c + a_{ij}]$. If A is $m \times n$ and B is $n \times p$, then $A \otimes B$ is the $m \times p$ matrix with (i, j)-entry

$$\sum_{k=1}^{n}{}_{\oplus}\, a_{ik} \otimes b_{kj} = \max_{k} \left(a_{ik} + b_{kj} \right).$$

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the permanent of A, denoted by per A, is defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the group of permutations of $1, 2, \ldots, n$.

Let A be an $n \times n$ matrix over $\mathscr{M}$. The permanent of A over the max algebra, denoted by $\pi(A)$, is obtained by replacing the sum and the product in the definition of permanent by $\oplus$ and $\otimes$, respectively. Clearly,

$$\pi(A) = \max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i\sigma(i)}.$$

For $\sigma \in S_n$, we refer to the set $\{a_{1\sigma(1)}, \ldots, a_{n\sigma(n)}\}$ as the diagonal corresponding to σ and $\sum_{i=1}^{n} a_{i\sigma(i)}$ is called a diagonal sum. Thus $\pi(A)$ is the maximum diagonal sum in A.

Let A be an $n \times n$ matrix over $\mathscr{M}$. Imagine that there are n individuals and n jobs or tasks. The worth (possibly negative) of the i-th individual, when engaged in the j-th job, is given to a_{ij}. It is desired to assign the jobs, exactly one per individual, such that the total worth is maximum. Then it is clear that $\pi(A)$ is the maximum worth possible. Any permutation $\sigma \in S_n$ can be thought of as an assignment. An assignment that corresponds to $\pi(A)$ is called an optimal assignment, see [8].

The purpose of this paper is to obtain some results for the permanent over the max algebra. These are motivated by analogous results for the permanent and can often be derived from those results. However, in some cases we give a proof of the max algebra analog based on the duality theorem of linear programming and it has a completely different flavor.

The idea of translating formulas from conventional algebra to max algebra has been used earlier. As an example, we refer to Gaubert [9], who terms it the "Transfer Principle" and, in particular, proves an analog of the Cauchy-

Binet formula for the determinant over commutative semirings. A similar technique had also been used in [14]. We also remark that some results for the permanent over the max algebra have been obtained in [13, 18].

If $A = [a_{ij}]$ is an $n \times n$ matrix over $\mathscr{M}$ and if $\alpha > 0$, then we define α^A to be the matrix $[\alpha^{a_{ij}}]$. Here $\alpha^{-\infty}$ is defined to be zero. We make the convention that $\log 0 = -\infty$. By a nonnegative (or positive) matrix we mean a matrix with nonnegative (or positive) entries.

The following simple fact is the key observation which lets one pass from a result for the permanent of a nonnegative matrix to a result for the permanent over max algebra.

LEMMA 1.1. *Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix over* $\mathscr{M}$. *Then*

$$\lim_{\alpha \to \infty} \frac{\log \operatorname{per} \alpha^A}{\log \alpha} = \pi(A). \tag{1}$$

Proof. If $\pi(A) = -\infty$, then the left-hand side of (1) is also seen to be $-\infty$. We therefore assume that $\pi(A) > -\infty$. The proof now follows by an application of L'Hospital's Rule. ∎

We will need the following special case of the duality theorem of linear programming (see [8]).

LEMMA 1.2. *Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix over* $\mathscr{M}$ *and suppose* $\pi(A) > -\infty$. *Then there exist real numbers* u_i, v_i, $i = 1, \dots, n$, *such that*

$$a_{ij} \leqslant u_i + v_j, \qquad i, j = 1, \dots, n \tag{2}$$

and

$$\pi(A) = \sum_{i=1}^{n} (u_i + v_i). \tag{3}$$

We remark that if the numbers u_i, v_i, $i = 1, \dots, n$, satisfy (2), then $\pi(A) \leqslant \sum_{i=1}^{n} (u_i + v_i)$. The following well-known results (see, for example, [6]) will also be used. The first result is essentially the Forbenius-Konig Theorem stated in terms of Max algebra.

THEOREM 1.3. *Let A be a nonnegative $n \times n$ matrix over $\mathcal{M}$. Then $\pi(A) = -\infty$ if and only if A has a zero submatrix of order $r \times s$ with each entry $-\infty$ and with $r + s = n + 1$.*

Recall that an $n \times n$ nonnegative matrix is *doubly stochastic* if each row sum and each column sum of the matrix is 1.

THEOREM 1.4 (Birkhoff-Von Neumann). *Let A be a nonnegative $n \times n$ matrix. Then A is doubly stochastic if and only if it is a convex combination of permutation matrices of order n.*

It follows from Theorem 1.4 that the permanent of a doubly stochastic matrix is positive.

2. ALEXANDROFF INEQUALITY

Suppose $a_1, \dots, a_{n-2}$ are positive vectors in R^n. Then it is well known (see [17]) that R^n, equipped with the inner product.

$$\langle x, y \rangle = \operatorname{per}(a_1, \dots, a_{n-2}, x, y),$$

is a Lorentz space, and this fact leads to the Alexandroff inequality for the permanent of a nonnegative matrix. We now state an alternative formulation of the same result; see [17] for a proof.

LEMMA 2.1. *Let U be a positive $n \times (n-2)$ matrix and let $x_1, \dots, x_m$ be vectors in R^n. Then the symmetric $m \times m$ matrix $Q = [q_{ij}]$ with*

$$q_{ij} = \operatorname{per}(U, x_i, x_j), \qquad i, j = 1, \dots, m$$

has at most one, simple, positive eigenvalue.

We now introduce some definitions. A real, symmetric $n \times n$ matrix A is said to be conditionally negative definite (c.n.d.) if for any vector $x \in R^n$ with $\sum_{i=1}^n x_i = 0$, it is true that

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j \leq 0.$$

If A is a symmetric matrix over $\mathscr{M}$, and if $a_{ii} = -\infty$ for some i, then we make the convention that A is c.n.d.

The class of c.n.d. matrices is closely related to that of distance matrices and is important in many applications such as limit theorems in probability theory and numerical interpolation. We refer to [16] for some results bringing out the connection between distance matrices and Lorentz space. We will need the following result from [1].

LEMMA 2.2. *Let U be a symmetric, positive $n \times n$ matrix with exactly one, simple, positive eigenvalue. Then the matrix* $[\log u_{ij}]$ *is c.n.d.*

The following is one of the main results of this section.

THEOREM 2.3. *Let A be an $n \times (n-2)$ matrix over $\mathscr{M}$ and let $y_1, \dots, y_m$ be vectors of order n over $\mathscr{M}$. Let $S = [s_{ij}]$ be the $n \times n$ matrix given by*

$$s_{ij} = \pi(A, y_i, y_j), \qquad i, j = 1, \dots, m.$$

Then S is c.n.d.

Proof. First suppose that $s_{ij} = -\infty$ for some i, j. If $i = j$, then S is c.n.d., so suppose $i \neq j$. By Theorem 1.3, the matrix (A, y_i, y_j) must contain a $p \times q$ submatrix with each entry $-\infty$ and with $p + q = n + 1$. Then it is clear that at least one of the two matrices (A, y_i, y_i) and (A, y_j, y_j) must contain a $p \times q$ submatrix with each entry $-\infty$ and with $p + q = n + 1$, and therefore, either s_{ii} or s_{jj} is $-\infty$. Again, S is c.n.d. We therefore assume that each s_{ij} is finite. If A has some entries equal to $-\infty$, then we may replace them by a sufficiently large negative number so that the matrix S remains unchanged. We thus assume, without loss of generality, that each a_{ij} is finite. For $\alpha > 0$, let $T^{(\alpha)} = [t_{ij}^{(\alpha)}]$ be the $m \times m$ matrix with

$$t_{ij}^{(\alpha)} = \operatorname{per}(\alpha^{A}, \alpha^{y_i}, \alpha^{y_j}), \qquad i, j = 1, \dots, m.$$

Then by Lemma 2.1, $T^{(\alpha)}$ has exactly one simple, positive eigenvalue and by Lemma 2.2, the matrix $[\log t_{ij}^{(\alpha)}]$ is c.n.d. By Lemma 1.1, the matrix $(1/\log \alpha)[\log t_{ij}^{(\alpha)}]$ converges to S as $\alpha \to \infty$ and therefore S is c.n.d. That completes the proof. ∎

It follows from elementary properties of c.n.d. matrices that for the matrix S in Theorem 2.3,

$$2s_{ij} \geq s_{ii} + s_{jj}, \qquad i, j = 1, \dots, m,$$

and this can be interpreted as Alexandroff inequality over the max algebra. We now prove a more general result. First, we introduce some notation.

If n, r are positive integers, then we set

$$\mathscr{K}_{n,r} = \left\{ k = (k_1, \dots, k_n) : k_i \text{ non-negative integers, } \sum_{i=1}^{n} k_i = r \right\}.$$

If A is an $n \times n$ matrix and if $k, l \in \mathscr{K}_{n,r}$, then we define

$$A(k, l) = A(k_1, \dots, k_n; l_1, \dots, l_n)$$

to be the $r \times r$ matrix obtained by repeating k_i times the i-th row, l_j times the j-th column, $i, j = 1, \dots, n$. To be more precise, $A(k, l)$ is the block matrix

$$A(k, l) = \begin{bmatrix} a_{11}J_{k_1 l_1} & a_{12}J_{k_1 l_2} & \cdots & a_{1n}J_{k_1 l_n} \\ a_{21}J_{k_2 l_1} & a_{22}J_{k_2 l_2} & \cdots & a_{2n}J_{k_2 l_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}J_{k_n l_1} & a_{n2}J_{k_n l_2} & \cdots & a_{nn}J_{k_n l_n} \end{bmatrix},$$

where $J_{k_i l_j}$ is the $k_i \times l_j$ matrix with each entry equal to 1. (If k_i or l_j is zero, then $J_{k_i l_j}$ is vacuous.)

THEOREM 2.4. *Let A be an $n \times n$ matrix over $\mathscr{M}$. Let $K = [k_{ij}]$, $L = [l_{ij}]$ be $n \times m$ matrices of nonnegative integers such that each row sum of K, L is m and each column sum of K, L is n. Then*

$$\pi(A) \geq \frac{1}{m} \sum_{j=1}^{m} \pi\big(A(k_{1j}, \dots, k_{nj}; l_{1j}, \dots, l_{nj})\big). \tag{4}$$

Proof. First suppose that $\pi(A) = -\infty$. Then by Theorem 1.3 we may assume, without loss of generality, that

$$A = \begin{bmatrix} -\infty & C \\ D & E \end{bmatrix},$$

where $-\infty$ denotes a matrix with each entry $-\infty$ and is of order $p \times q$ with $p + q = n + 1$. Suppose, for $j = 1, \ldots, m$,

$$\sum_{i=1}^{p} k_{ij} + \sum_{i=1}^{q} l_{ij} \leq n.$$

Then it follows that

$$\sum_{i=1}^{p} \sum_{j=1}^{m} k_{ij} + \sum_{i=1}^{q} \sum_{j=1}^{m} l_{ij} = pm + qm \leq nm,$$

which is a contradiction. Therefore there exists $j \in \{1, \ldots, m\}$ such that

$$\sum_{i=1}^{p} k_{ij} + \sum_{i=1}^{q} l_{ij} > n.$$

Then, by Theorem 1.3,

$$\pi\big(A(k_{1j}, \ldots, k_{nj}; l_{1j}, \ldots, l_{nj})\big) = -\infty,$$

and (4) is proved. We now assume that $\pi(A) > -\infty$. By Lemma 1.2, there exist real numbers u_i, v_i, $i = 1, \ldots, n$, such that (2), (3) hold. It follows from (2) that

$$\pi\big(A(k_{1j}, \ldots, k_{nj}; l_{1j}, \ldots, l_{nj})\big) \leq \sum_{i=1}^{n} (k_{ij} u_i + l_{ij} v_i).$$

Therefore

$$\begin{aligned} \sum_{j=1}^{m} \pi\big(A(k_{1j}, \ldots, k_{nj}; l_{1j}, \ldots, l_{nj})\big) &\leq \sum_{i=1}^{n} \sum_{j=1}^{m} (k_{ij} u_i + l_{ij} v_i) \\ &= m \sum_{i=1}^{n} (u_i + v_i) \\ &= m\pi(A), \end{aligned}$$

and (4) is proved. ■

The Alexandroff inequality for the permanent over max algebra is derived next.

COROLLARY 2.5. *Let $A = (a_1, \dots, a_n)$ be an $n \times n$ matrix over $\mathscr{M}$. Then*

$$2\pi(A) \geq \pi(a_1, \dots, a_{n-2}, a_{n-1}, a_{n-1}) + \pi(a_1, \dots, a_{n-2}, a_n, a_n).$$

Proof. The result follows as a special case of Theorem 2.4 when $m = 2$, each $k_{ij} = 1$, and

$$L = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}. \qquad \blacksquare$$

3. BREGMAN'S INEQUALITY

If U is an $n \times n$ matrix, then $U(i, j)$ will denote the submatrix obtained by deleting row i and column j of U. Let U be a nonnegative $n \times n$ matrix with per $U > 0$. We define $f(U)$ to be the $n \times n$ matrix with (i, j)-entry

$$f_{ij}(U) = \frac{u_{ij} \operatorname{per} U(i, j)}{\operatorname{per} U}, \qquad i, j = 1, \dots, n.$$

Then $f(U)$ is doubly stochastic. Several interesting properties of the map f have been obtained by Brégman [5] (also see [15]). In particular, the following inequality is contained in [5].

THEOREM 3.1. *Let U be a nonnegative $n \times n$ matrix with* per $U > 0$ *and let $V = f(U)$. Then for any nonnegative $n \times n$ matrix W,*

$$\operatorname{per} W \geq \operatorname{per} U \prod_{i,j=1}^{n} \left(\frac{w_{ij}}{u_{ij}} \right)^{v_{ij}}.$$

Let A, B be $n \times n$ matrices over $\mathscr{M}$ with $\pi(A) > -\infty$. If we apply Theorem 3.1 to the matrices $U = [\alpha^{a_{ij}}]$, $W = [\alpha^{b_{ij}}]$, take log, divide by $\log \alpha$ and let $\alpha \to \infty$, then we obtain an inequality for the permanent over the max algebra. We state the inequality as the next result and give a different proof based on Lemma 1.2. We say that a diagonal of A is *optimal* if the corresponding diagonal sum is maximum, i.e, equals $\pi(A)$.

Let A be an $n \times n$ matrix over $\mathscr{M}$. Define the $n \times n$ $0-1$ matrix $Q = [q_{ij}]$ as follows: $q_{ij} = 1$ if and only if a_{ij} is contained in an optimal diagonal of A. Let $C = f(Q)$ so that

$$c_{ij} = \frac{q_{ij} \operatorname{per} Q(i,j)}{\operatorname{per} Q}, \qquad i,j = 1,\dots,n.$$

Then c_{ij} is the *proportion* of optimal diagonals of A which pass through (i,j). Observe that C is doubly stochastic.

THEOREM 3.2. *Let A, B be $n \times n$ matrices over $\mathscr{M}$ with $\pi(A) > -\infty$. Then*

$$\pi(B) \geq \pi(A) + \sum_{i,j=1}^{n} c_{ij}(b_{ij} - a_{ij}), \tag{5}$$

where c_{ij} is the proportion of optimal diagonals of A which pass through (i,j).

Proof. First consider the case where $\pi(B) = -\infty$. Since C is doubly stochastic, there must exist (i,j) such that $c_{ij} > 0$ and $b_{ij} = -\infty$, for otherwise, per C would be zero, contradicting the observation made following Theorem 1.4. Thus in this case equality holds in (5). Therefore we assume that $\pi(B) > -\infty$. By Lemma 1.2, there exist real numbers u_i, v_i, $i = 1,\dots,n$, such that (2), (3) hold. Observe that when $c_{ij} > 0$, we have $a_{ij} > -\infty$. When $c_{ij} = 0$ we define $c_{ij}(b_{ij} - a_{ij}) = 0$. Now

$$\begin{aligned} \sum_{i,j=1}^{n} c_{ij}(b_{ij} - a_{ij}) &= \sum_{i,j=1}^{n} c_{ij}(b_{ij} - u_i - v_j) \\ &= \sum_{i,j=1}^{n} c_{ij} b_{ij} - \sum_{i,j=1}^{n} c_{ij}(u_i + v_j). \end{aligned} \tag{6}$$

Since C is doubly stochastic, an application of Theorem 1.4 shows that

$$\sum_{i,j=1}^{n} c_{ij} b_{ij} \leq \pi(B). \tag{7}$$

Again, the fact that C is doubly stochastic and (3) lead to

$$\begin{aligned}\sum_{i,j=1}^{n} c_{ij}(u_i + v_j) &= \sum_{i=1}^{n} u_i \sum_{j=1}^{n} c_{ij} + \sum_{j=1}^{n} v_j \sum_{i=1}^{n} c_{ij} \\ &= \sum_{i=1}^{n} (u_i + v_i) \\ &= \pi(A), \end{aligned} \tag{8}$$

in view of (3). Substituting (7), (8) in (6), we get

$$\sum_{i,j=1}^{n} c_{ij}(b_{ij} - a_{ij}) \leqslant \pi(B) - \pi(A),$$

and the proof is complete. ∎

We remark that Theorem 3.2 can be viewed as a perturbation result and may be of interest in sensitivity analysis for the optimal assignment problem. Thus suppose that we have solved the optimal assignment problem for the matrix A and have computed the dual variables. Then we can find the matrix C in Theorem 3.2. Let B be a perturbation of A. Then (5) provides a lower bound for $\pi(B)$ which is easily computable.

4. SOME MISCELLANEOUS RESULTS

We denote by $\mathbf{1}$ the column vector of the appropriate size with each entry equal to 1. The next result is the Cauchy-Binet formula for the permanent (see [11]).

THEOREM 4.1. *Let U, V be $r \times n$ real matrices. Then*

$$\operatorname{per} UV^T = \sum_{k \in \mathscr{K}_{n,r}} \frac{1}{k_1! \cdots k_n!} \operatorname{per} U(\mathbf{1}, k) \operatorname{per} V(\mathbf{1}, k).$$

The corresponding formula over the max algebra is stated next.

THEOREM 4.2. *Let A, B be $r \times n$ matrices over $\mathscr{M}$. Then*

$$\pi(A \otimes B^T) = \sum_{\substack{\otimes \\ k \in \mathscr{K}_{n,r}}} \pi(A(\mathbf{1}, k)) \otimes \pi(B(\mathbf{1}, k)).$$

To prove Theorem 4.2 first apply Theorem 4.1 to per $\alpha^U(\alpha^V)^T$, then take log and divide by log α, and finally let $\alpha \to \infty$. The result follows by Lemma 1.1. The following consequence of Theorem 4.2 has been noted in [13].

COROLLARY 4.3. *Let A, B be $n \times n$ matrices over $\mathscr{M}$. Then*

$$\pi(A \otimes B) \geq \pi(A) \otimes \pi(B).$$

The following expansion has been obtained by Bebiano [3], see also Blokhuis and Seidel [4]. We have changed the notation to suit the present paper.

THEOREM 4.4. *Let U be a real $n \times n$ matrix and let $v, w \in R^n$. Then for any positive integer r,*

$$\frac{\left(\sum_{i,j=1}^n u_{ij} v_i w_j\right)^r}{r!} = \sum_{k, l \in \mathscr{K}_{n,r}} \frac{v_1^{k_1} \cdots v_n^{k_n}}{k_1! \cdots k_n!} \frac{w_1^{l_1} \cdots w_n^{l_n}}{l_1! \cdots l_n!} \operatorname{per} U(k, l).$$

Using the same proof technique as before, the max algebra analog of Theorem 4.4 can be obtained as follows.

THEOREM 4.5. *Let A be an $n \times n$ matrix and let x, y be vectors of order n over $\mathscr{M}$. Then*

$$x^T \otimes A \otimes y = \frac{1}{r} \sum\nolimits_{\oplus}{}_{k,l \in \mathscr{K}_{n,r}} \left(\sum_{i=1}^{n} (x_i k_i + y_i l_i) \right) \otimes \pi(A(k,l)).$$

We illustrate Theorem 4.5 by a numerical example.

EXAMPLE. Let $n = r = 2$, and let

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}, \qquad x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad y = \begin{bmatrix} 9 \\ 1 \end{bmatrix}.$$

The following table gives the values of $(\sum_{i=1}^{n}(x_i k_i + y_i l_i)) \otimes \pi(A(k,l))$ (given in the last row) corresponding to $k = (k_1, k_2)$, $l = (l_1, l_2)$ for all $k, l \in \mathscr{K}_{2,2}$:

k_1	1	2	0	1	1	2	2	0	0
k_2	1	0	2	1	1	0	0	2	2
l_1	1	1	1	2	0	2	0	2	0
l_2	1	1	1	0	2	0	2	0	2
	27	19	32	33	18	28	10	38	26

It can be seen that the maximum element in the last row of the table is 38. Since

$$x^T \otimes A \otimes y = \max_{i,j} (a_{ij} + x_i + y_j) = 19 = \frac{38}{2},$$

Theorem 4.5 is demonstrated.

We conclude with the max algebra analog of a result related to rook polynomials. Suppose U is a nonnegative $n \times n$ matrix. For $m = 1, \dots, n$, let β_m denote the sum of all $m \times m$ subpermanents of U. We set $\beta_0 = 1$. The polynomial $\sum_{m=0}^{n} \beta_m \lambda^m$ is called the *rook polynomial* associated with the matrix A and it has only real roots (see [12, 10]). As a consequence, the sequence $\beta_0, \beta_1, \dots, \beta_n$ is *log-concave*, i.e., it satisfies $\beta_m^2 \geq \beta_{m-1} \beta_{m+1}$,

$m = 1, \dots, n-1$. This latter result leads to a corresponding result over the max algebra which is stated next. We give a different, easier proof, again based on Lemma 1.2.

THEOREM 4.6. *Let A be an $n \times n$ matrix over $\mathscr{M}$ and for $m = 1, \dots, n$, let*

$$\gamma_m = \sum\nolimits_{\oplus} \pi(B),$$

where the sum is over all $m \times m$ submatrices B of A. Set $\gamma_0 = 0$. Then

$$\gamma_m \geqslant \frac{\gamma_{m-1} + \gamma_{m+1}}{2}, \qquad m = 1, \dots, n-1. \tag{9}$$

Proof. We first prove (9) when $m = n-1$. If $\pi(A) = -\infty$, then, for $m = n-1$, (9) clearly holds. We therefore assume that $\pi(A) > -\infty$. Consider the bordered $(n+2) \times (n+2)$ matrix C defined as

$$C = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & 0 \\ 0 & \cdots & 0 & -\infty & -\infty \\ 0 & \cdots & 0 & -\infty & -\infty \end{bmatrix}.$$

Let D be the matrix obtained by deleting the last row and column of C. Observe that $\pi(C) = \gamma_{n-2}$, $\pi(D) = \gamma_{n-1}$. Apply Lemma 1.2 to D to get optimal dual variables $u_1, \dots, u_n, w$ corresponding to the rows and $v_1, \dots, v_n, z$ corresponding to the columns. Then

$$\begin{aligned} \gamma_{n-1} &= \pi(D) \\ &= \sum_{i=1}^{n} u_i + w + \sum_{i=1}^{n} v_i + z \\ &= \frac{1}{2} \sum_{i=1}^{n} (u_i + v_i) + \frac{1}{2} \left\{ \sum_{i=1}^{n} u_i + 2w + \sum_{i=1}^{n} v_i + 2z \right\} \\ &\geq \frac{\gamma_n + \gamma_{n-2}}{2}, \end{aligned}$$

and (9) is proved when $m = n - 1$. For $1 \leq m < n - 1$, the proof is similar except that we must border A with $n - m + 1$ rows and columns to construct the matrix C. ■

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