Uncertainty Relations and Time-Frequency Distributions for Unsharp Observables

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ABSTRACT

This paper deals with a new framework in analyzing the formal mathematical correspondence between quantum mechanics and time-frequency representations of a signal. It is also shown that joint time-frequency distributions have a close link with Heisenberg uncertainty relations if the observables are taken as fuzzy entities. This result contradicts the arguments of Cohen [*IEEE Proc.* 77(7):941 (1989)] regarding the time-frequency distributions and the uncertainty relation. It is postulated that these mechanisms will be of crucial importance in highly fragmented computation structures, such as neural networks, as they may exhibit a strong mutual interaction between data and operator.

1. INTRODUCTION

Gabor [1] published a pioneering paper in 1946 on time-frequency representations of signals in the context of communication theory. It was Wigner [2] who first investigated the possibility of constructing joint distribution functions in phase-space and their importance in the domain of quantum mechanics. Later on, Ville [3] studied the joint distribution functions in signal analysis in the spirit of Wigner. In 1965, Cohen [4] suggested a generalized approach for constructing the joint distributions in phase-space. Most of the joint distributions are shown to be special cases in Cohen's class of distributions. Since the publications by Gabor, Wigner, and Ville, a large number of interesting works [5] on joint distributions have been reported in the context of quantum mechanics, optics, and in signal processing. In the 1980s, the interpretation of the uncertainty relation in signal processing and its relation to the Heisenberg uncertainty relation renewed the interest of the signal processing community. It was started after the publication of papers by Wilson et al. [6, 7] and Daugman et al. [8] in the context of computer vision. Daugman tried to analyze the psychophysiological evidence of receptive fields in the mammalian visual cortex and discussed the existence of the uncertainty relation in case of a 2D-signal representation. Recently, Cohen made an excellent review [9] on time-frequency distributions and their relation to the Heisenberg uncertainty relation, especially with reference to signal processing. He tried to make a systematic study on time-frequency analysis of signals and investigated the correspondence between quantum mechanics and signal analysis. He claimed that the uncertainty principle is a relationship concerning the marginals only and has no bearing on the existence of joint distributions. He has also criticized the mathematical correspondence between quantum mechanics and signal analysis as often discussed by various authors [8]. The recent developments of the theory of unsharp measurements [10–12] in quantum mechanics opens up a new possibility of constructing joint distributions of unsharp observables and studying their relevance to the Heisenberg uncertainty relation. This theory is a natural extension of the usual Hilbert space formalism of quantum theory. This development is quite interesting not only in quantum mechanics but also in the context of signal processing, especially in image processing and pattern recognition. If one considers the case of object-background segmentation in a gray level image, a great deal of ambiguity arises in a decision process due to the continuous gray level distribution from object to background. This kind of situation could be handled more efficiently if fuzzy set theoretical formalism is used instead of ordinary set theory [13]. Similarly, in extending the usual Hilbert space formalism of quantum mechanics to describe unsharp observables, it is generally required to introduce the concept of fuzzy sets in place of ordinary sets [10]. Consequently, Cohen's analysis needs reinvestigation in the light of this extended framework of quantum mechanics. In this paper, we shall briefly discuss Cohen's main arguments regarding joint distributions and their relation to the Heisenberg uncertainty relation in Section 2. In Section 3, we shall try to reinvestigate the whole issue within the framework of unsharp measurements. It will be shown that Cohen's analysis is not true in the case of unsharp observables, which is much more relevant in signal analysis. In our opinion, a more detailed investigation considering all other aspects is needed to settle the issue on the applicability of quantum-mechanical concepts in signal processing and analysis, as is discussed in Section 4.

2. TIME-FREQUENCY DISTRIBUTIONS AND COHEN'S ANALYSIS

The joint distribution in the time-frequency plane was originally considered by Gabor and Ville. Gabor was motivated by the development of quantum mechanics and the formal resemblance between the time-frequency uncertainty relation and Heisenberg's time-energy uncertainty relation. He also introduced the concept of analytic signal. Ville derived a distribution that Wigner studied in quantum-statistical mechanics. The Wigner-Ville distribution is

$$W(t,w) = \frac{1}{2\pi}\int s^*\left(t - \frac{1}{2}\tau\right)e^{-j\tau w}s\left(t + \frac{1}{2}\tau\right)d\tau \tag{1}$$

for a signal $s(t)$ at time t and frequency ω.

The fundamental goal for the derivation of a joint distribution of time and frequency is to represent the energy in terms of intensity per unit time per unit frequency. In general, for a joint distribution $p(t,w)$, we have

$$p(t,w) = \text{intensity at time } t \text{ and frequency } \omega,$$

or

$$p(t,w)\,\Delta t\,\Delta w = \text{fractional energy}$$

in time-frequency cell $\Delta t\,\Delta w$ at t, ω.

Summing up of the energy distribution for all frequencies at a particular time would give the instantaneous energy, and the summing up over all times at a particular frequency would give the energy density spectra

$$\int p(t,w)\,dw = |s(t)|^2, \tag{2}$$

$$\int p(t,w)\,dt = |s(w)|^2. \tag{3}$$

The total energy E can be given in terms of the distribution as

$$E = \int p(t, w)\, dw\, dt. \tag{4}$$

This will be equal to the total energy of the signal if the marginals (2) and (3) are satisfied. At this juncture Cohen raised the following questions:

(a) Do there exist joint time-frequency distributions that would satisfy our intuitive ideas of a time varying spectrum?

(b) Can their interpretation be as true densities of distribution?

(c) How can such functions be constructed?

(d) Do they really represent correlations between time and frequency?

(e) What reasonable conditions can be imposed to obtain such functions?

(f) Is there any distribution that is the best, or do different distribution need to be used in different situations? Are there inherent limitations to a joint time-frequency distribution?

Cohen tried to answer these questions in his above-mentioned review and raised a number of other questions which should be investigated thoroughly in future research. What concern us in this paper are Cohen's analysis on the uncertainty principle and joint distributions, and the remarks on the formal mathematical correspondence between quantum mechanics and signal analysis.

With respect to the second issue, Cohen first pointed out that while quantum mechanics is inherently probabilistic, in signal theory the signal is inherently deterministic. He also pointed out that the probabilistic character of quantum mechanics is not due to ignorance of the initial conditions. At this point, the physicists working in quantum mechanics are not in accord. Bohm [14] proposed and elaborated a theory in 1952, called the hidden variable theory in quantum mechanics. According to this theory, the basic equations are fully deterministic and the concept of probability can be interpreted as due to ignorance. All the tenets of the hidden variable theory have not yet been discarded [15] and investigations are still going on. Moreover, in many world interpretations of quantum mechanics as expounded by Everett and elaborated by Wheeler et al. in [16], the equations are also deterministic in nature. Our main contention is that the issue of the interpretation of probability is not yet settled in quantum mechanics itself and it requires further investigations to settle the issue. Then Cohen pointed out another important difference. In quantum mechanics, observables are associated with operators. For example, the position and momentum can be associated to self-adjoint operators. In signal analysis, time and frequency are not considered as operators. Moreover, in quantum mechanics, q and p are continuous variables, while q^2+p^2 is never continuous. On the other hand, in signal analysis for time and frequency, t^2+w^2 is never continuous. This observation is very important for making a mathematical analogy with quantum mechanics. If we consider the energy-time uncertainty relation in quantum mechanics, the issue of the time operator is not yet fully resolved. Moreover, the interpretation of this uncertainty relation in comparison with the position and momentum uncertainty relation has raised a great deal of controversy since its very inception. In a series of recent papers, Busch et al. [17] tried to give a consistent interpretation of the time-energy uncertainty relation within the framework of the theory of unsharp measurements in quantum mechanics.

Cohen made a comprehensive analysis regarding the uncertainty principle and joint distributions. The uncertainty principle expresses a fundamental relationship between the standard deviation of a function and the standard deviation of its Fourier transform. In particular, the standard deviations are defined by

$$(\Delta t)^2=\int(t-\bar{t})^2|s(t)|^2\,dt,$$
$$(\Delta w)^2=\int(w-\bar{w})^2|s(w)|^2\,dw, \tag{5}$$

where $\bar{t}$ and $\bar{w}$ are the mean time and mean frequency. The uncertainty principle is

$$\Delta t\,\Delta w\geqslant\tfrac{1}{2} \tag{6}$$

for any signal. Δt and Δw are called the duration and bandwidth of a signal. Cohen has shown that the uncertainty principle is a relationship concerning the marginals only and has no bearing on the existence of joint distributions. He concluded that any joint distribution that yields the marginals will give the uncertainty principle and it has nothing to do with correlations between time and frequency or measurements for small times and frequencies. It says that marginals are functionally dependent but it does not imply correlation between the variables and has nothing to do with the existence or nonexistence of joint distributions. Recently, Busch and Lahti [18] discussed uncertainty relations and complementarity of canonical conjugate variables in quantum mechanics with respect to some general coupling properties of a function and its Fourier transform. They have clearly disproved Gibbins (and Popper's) conclusion. In 1981, Gibbins [19] pointed out that uncertainty relations have nothing to do with impossibility of joint localization of a particle on bounded positions and momentum value sets, and he concludes that uncertainty relations can be interpreted only as statistical scatter relations. So we shall discuss the theory of unsharp localizations.

3. UNSHARP MEASUREMENTS, UNCERTAINTY RELATIONS, AND JOINT DISTRIBUTIONS

Busch and Lahti [18] questioned the exclusive validity of statistical interpretation of uncertainty relations. They tried to justify an individualistic interpretation introducing the concept of unsharp observable where a joint measurement of position and momentum is a logically tenable hypothesis.

It is the transition from ordinary sets to fuzzy sets that enables one to describe this type of unsharpness. In ordinary set theory, the element relations $x \in E$ can be represented by a characteristic function X_E which is either 0 or 1, while in the case of a fuzzy set, E_X is replaced by a more general function μ_E with the following properties:

$$0 \leqslant \mu_E \leqslant 1 \tag{7}$$

for any x; $\mu_E(x)$ is a membership function which indicates the extent of membership [13].

The main difference from a characteristic function X_E is that μ_E may assume any value in the real interval [0, 1]. Now we proceed with the corresponding change with the description of observables. A spectral projection $Q(E)$ is defined in the configuration space representation by means of the equation

$$(Q(E)\phi)(q) = X_E(q)\phi(q). \tag{8}$$

Replacing X_E by μ_E results in a new "effect"

$$(a(E)\phi)(q) = \mu_E(q)\phi(q), \tag{9}$$

where a: $E \to a(E)$ is a positive operator valued (POV) measure called unsharp position observable or fuzzy observable. We shall assume that to

μ_{E} there corresponds a density function f_q such that

$$\mu_E(q) = \int_E d_q f_q(q'). \tag{10}$$

Furthermore, the functions f_q shall be supposed to be essentially the same for all q, i.e.,

$$f_a(q') = f(q' - q), \tag{11}$$

which means that the position measuring device which corresponds to the fuzzy observable a also works equally well in all regions of space. Let us define joint measurements of unsharp observables. A joint position momentum observable $a_{f,g}$ is defined as a POV measure on a phase space $\Gamma = \mathbb{R}^2$,

$$a_{f,g}(\Delta) = \int_\Delta dq\,dp\,\mathscr{F}(q,p), \tag{12}$$

with a continuous positive phase space distribution

$$(q,p) \to \mathscr{F}(q,p), \qquad 0 \leqslant \mathscr{F}(q,p), \tag{13}$$

bounded and with marginals

$$a_{f,g}(E \times \mathbb{R}) = Q_f(E), \qquad a_{f,g}(\mathbb{R} \times F) = P_g(F) \tag{14}$$

being unsharp position and momentum observable, respectively; the measures

$$\Delta \to t_r\left[w, a_{f,g}(\Delta)\right] \tag{15}$$

are positive definite phase-space measures and thus may be interpreted as joint probability distributions. Before discussing joint probability distributions and uncertainty relations, let us look into the theory of Fourier integrals. Two important theorems in Fourier integral theory are (1) the Paley-Wiener theorem and (2) the bandwidth theorem. One refers to the support property and the other to the dispersion property of a function and its Fourier transform. If ϕ is a smooth function on the real line $\Re$ with its Fourier transform $\bar{\phi}$

$$\bar{\phi}(y) = (F\phi)(y) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \bar{e}^{iyx}\phi(x)\,dx, \qquad \forall y \in \Re, \tag{16}$$

then the following properties hold.

Support property. If $\operatorname{supp}(\phi) = \operatorname{cl}\{x \in \mathbb{R} : \phi(x) \neq 0\}$ is compact (i.e., bounded), then $\operatorname{supp}(\overline{\phi})$ is the whole real line $\mathbb{R}$, and conversely. cl indicates closure.

Dispersion Property.

$$\Delta|\phi|^2 \Delta|\overline{\phi}|^2 \geq \frac{1}{2}. \tag{17}$$

Here, $\Delta|\phi|^2$ denotes the standard deviation of the absolute square $|\phi|^2$ of ϕ, i.e.,

$$\Delta|\phi|^2 = \left\{ \int_{-\infty}^{+\infty} x^2 |\phi(x)|^2 \, dx - \left(\int_{-\infty}^{+\infty} x |\phi(x)|^2 \, dx \right)^2 \right\}^{1/2},$$

assuming

$$\|\phi\|^2 = \int_{-\infty}^{+\infty} |\phi(x)|^2 \, dx = 1$$

and

$$\int_{\mathbb{R}} |\overline{\phi}(y)|^2 \, dy = 1.$$

The dispersion property is an immediate consequence of the well-known Cauchy-Schwartz-Buniakowski inequality. The Fourier transform of a compactly supported function is analytic, so the standard results in function theory give the support property.

There are pairs of physical quantities which share both of the properties in (16) and (17). Such quantities frequently appear in optics, electrodynamics, solid-state physics, and communication theory as well as in quantum mechanics. In quantum mechanics, canonically conjugate position and momentum Q and P of a physical system are represented as a Fourier couple. Thus they share both the support property as well as the dispersion property as defined above.

Let us define the quantity $\Pi^Q(x)$ for each Borel set X of the real line $\mathbb{R}$, with

$$\Pi^Q(x)\phi = \chi_X \phi, \qquad \forall \phi \text{ in } L_2(\mathbb{R}).$$

where χ_E is the characteristic function of the set $X \in B(\mathbb{R})$. Similarly, we define

$$\Pi^P(Y) = (h/2\pi) F^{-1} \Pi^Q(Y) F$$

for any $Y \in B(\mathbb{R})$. The projections $\Pi^P(Y)$ are then exactly the spectral projections of P.

Let X and Y be bounded Borel sets and assume

$$(\phi, \Pi^Q(x)\phi) = 1 = (\phi, \Pi^P(Y)\phi) \quad \text{for some } \phi \in L_2(\mathbb{R}).$$

Then for any two bounded X and Y in $B(\mathbb{R})$, the intersection of the (closed) subspaces X and Y,

$$\Pi^Q(x)(L_2(\mathbb{R})) = \{\phi \in L_2 \in \mathbb{R};\ \Pi^Q(x)\phi = \phi\}$$

and $\Pi^P(Y)(L_2(\mathbb{R}))$ is the null space zero, i.e., the greatest lower bound, $\Pi^Q(X) \wedge \Pi^P(Y)$ of the projections $\Pi^Q(x)$, and $\Pi^P(Y)$ is the zero operator 0. We thus arrive at the relation

$$\Pi^Q(x) \wedge \Pi^P(Y) = 0 \tag{18}$$

for all bounded X *and* $Y \in B(\mathbb{R})$. This is a special case of support property defined earlier.

In discussing the ideal measurements of properties (described as projections) of a physical system, Ludwig [21] mentions that

$$\Pi^Q(X) \wedge \Pi^P(Y)$$

as an example of a property is not possible and thus not measurable. Ludwig introduces the notion of complementarity as: Two properties (projections) P_1 and P_2 are complementary if $P_1 \wedge P_2 = 0$. He also claims that the properties $\Pi^Q(X)$ and $\Pi^P(Y)$, with bounded X and Y, are complementary properties. It may be mentioned that Weizsacker also gave a definition of complementarity similar to that of Ludwig. According to Weizsacker [22], two elementary propositions are complementary if and only if they cannot be detected simultaneously. However, Busch and Lahti [23] showed that this claim of Ludwig is wrong and that Ludwig's notion of complementarity is too strong.

The notion of complementary physical quantities was systematically analyzed by Lahti [23] in 1979. His starting point was with the result

$$\Pi^{Q}(X) \wedge \Pi^{P}(Y) = 0,$$

which expresses the complementarity of position and momentum observables in the sense of natural exclusiveness of the experimental arrangements that permit the unambiguous definitions of the quantities. The general concept of complementarity, expressed as a generalization of $\Pi^{Q}(X) \wedge \Pi^{P}(Y) = 0$, was then found to be logically independent of uncertainty, expressed in terms of the dispersion property, i.e.,

$$\Delta|\phi|\cdot\Delta|\bar{\phi} \geq \tfrac{1}{2}.$$

Also, the invalidity of the "localization interpretation" of the uncertainty relations was pointed out there. The claim that the dispersions $\Delta(A, \phi)$ can only be interpreted as statistical spreads has been shown to be erroneous. It would imply that the uncertainty relations only can be interpreted as a statistical relation. The statistical interpretation of quantum-mechanical probabilities is, however, not the only one. In fact, if we look at the interpretation originally proposed by Heisenberg [24], i.e.,

> The dispersion property expresses as an uncertainty relation the limits of accuracy within which joint measurements of position and momentum are possible.

it appears that Heisenberg was originally motivated by an interpretation which is different from probabilistic interpretation. Of course, critics are divided in two major groups on the Heisenberg issue. One group claims that there are no joint measurements of position and momentum at all. They are right if one thinks of "sharp" measurements as localizations in the sense of support property $\Pi^{Q}(X) \wedge \Pi^{P}(Y) = 0$.

The question of inaccurate or unsharp measurement has never been explicitly discussed in this viewpoint. Another claim is that joint measurements of position and momentum are possible only with arbitrary accuracy, irrespective of the uncertainty relations. Busch et al. [25] have strongly challenged this claim. They have systematically studied the situation, and an alternative interpretation, i.e., "individualistic interpretation," of quantum probabilities (that is, of the states as dispersion of an individual system) was proposed and elaborated in a series of recent papers [20]. Such an interpretation leads quite naturally to the Heisenberg interpretation of the dispersion property. Especially, the founders [18] of quantum mechanics have (probably) always thought of individual systems. According to this individualistic interpretation, the state vectors in Hilbert space seem to play a double role; the wave function $\phi(x)$, which describes position indeterminacy, may also be imagined to have been prepared by a kind of "unsharp" position measurement. This means that the probability densities $|\phi(x)|^2$ may not only be used as probability densities for the possible values of (sharp) position measurements (first role of ϕ), but also as descriptions of unsharply defined measuring values (second role of ϕ). It was an intuitive notion of unsharp measurements which led to the famous Gedanken experiment illustrating the Heisenberg interpretation of dispersion property.

If a system possesses position and momentum only with indetermination to the extent given by the probabilistic uncertainty relations, then joint measurements should be impossible with accuracies violating the uncertainty relation. With the introduction of the so-called unsharp (fuzzy) observables, a proper joint probability distribution for position and momentum has been developed by Busch et al. The existence of joint probability distributions is stated in the following theorem.

THEOREM. *For any vector space $\phi \in L_2(\mathbb{R})$, there exists a joint probability distribution for (unsharp) position and momentum, namely,*

$$B(\mathbb{R}) \to [0,1] \ni (\phi, A_\psi(Z)\phi) = \int\int |\phi_\psi(x, y)|^2 \, dx\, dy, \qquad (19)$$

with

$$\phi_\psi(x, y) = \int_{-\infty}^{\infty} \psi^*(x')\phi(x')\, dx',$$

where ψ is any unit vector in $L_2(\mathbb{R})$ with average or the expected values, i.e., $E(Q, \psi)$ and $E(P, \psi)$, equal to zero and finite standard deviation $\Delta(Q, Y)$ and $\Delta(P, \psi)$.

Consequently, vectors ψ represent unsharply defined points in phase space or $|\psi(x')|^2$ and $|\psi(y')|^2$ represent distribution functions contained at X and Y, respectively.

Marginal distributions are

$$X \to (\phi, A_\psi(X \times \mathbb{R})\phi) = \left((\phi, \Pi_f^Q(x)\phi), f(x)\right) = |\psi(x)|^2, \qquad (20)$$

$$Y \to (\phi, A_\psi(\mathbb{R} \times Y)\phi) = \frac{h}{2\pi}|\psi(y)|^2 = (\phi, \Pi^P_{g(x)}, g(Y)). \qquad (21)$$

Moreover,

$$\Delta f \cdot \Delta g = \Delta|\psi|^2 \frac{h}{2\pi} \Delta|\tilde{\psi}|^2 \geqslant \frac{h}{4\pi}, \tag{22}$$

so that the inaccuracies Δf and Δg of the Fourier couple (Q_f, P_g) obey the uncertainty relation.

It is evident that the probability measure [18] is built up from two vectors ϕ and ψ; ϕ is used as the state vector representing the preparation of the measured object, whereas ψ is used to define measuring inaccuracies for unsharp (joint) observables.

Busch and Lahti correctly formulated the double role of vectors in Hilbert space formation as the following.

> According to the use of Hilbert space vectors in quantum theory as generators of probability measures (description of states) or as generators of unsharp (joint) observables (descriptions of measuring inaccuracies), there are two possible interpretations of the dispersion property as a probabilistic relation characterizing the approximate range of possible position and momentum values or as an uncertainty relation limiting the accuracy of joint position and momentum measurements.

Cohen's claim regarding the relationship of the uncertainty principle and joint distributions may be valid only for sharp observables as well as if we take only the statistical uncertainty relations. Busch [17] studied the interpretation of the time-energy uncertainty relation in this framework of unsharp measurement theory. He has shown that it is the mutual influence between measuring instruments (observer and the observed) which is manifested through the uncertainty relation. Thus according to Busch's work, the uncertainty relation determines the lower limit of (individual) unsharpness of measuring results which one necessarily is subjected to with joint measurements. He has analyzed Heisenberg's slit experiment within a model of quantum measurement and concluded that within the framework of unsharp observables, it is the mutual disturbance of the measuring device which is responsible for the occurrence of the uncertainty relation for incompatible observables (like position and momentum or time and energy). It may be mentioned that Wotters and Zurek [26] studied the complementarity principle analyzing the double slit experiment with the help of information theory, and they clearly indicated the existence of unsharp observables. In fact, if we analyze the above-mentioned double slit experiment with photons (say), then we meet with a difficulty, i.e., on one hand, the photon always chooses one of the two paths; on the other hand, it behaves as if it had passed both ways. Niels Bohr, in a long epistemological discussion with Einstein, pointed out, "It is just arguments of this kind which recall the impossibility of subdividing quantum phenomena and reveal the ambiguity in ascribing physical attribu··s to atomic objects."

4. IMPLICATIONS IN IMAGE PROCESSING

In the context of spatial information processing, which often implies the processing of visual information, there are several issues which have important bearings on quantum-mechanical formalism. To start with, sampling, or any operation whatever upon spatial data, gives rise to an ambiguity which is reflected in the uncertainty of measured properties in two complementary domains [6]. This can be interpreted as a property of the filter used, or it can be interpreted as a more fundamental limitation. The trivial example for interpretation is that integration over a larger area will give a more reliable feature estimate. On the other hand, integration over a larger area will give an increased uncertainty about the position at which the feature estimate is valid.

This gives a limitation of, e.g., the ability to determine the position of an edge, given a certain frequency function of the filter used. The same type of argument can be used for other image analysis operations, such as for segmentation, recognition, etc. What varies among the different operations are the domains of uncertainty according to our interpretation. There is inevitably an uncertainty in the complementary features derived.

There are now ways to improve this lack of resolution in typical situations, e.g., through a combination of statements of several filters, using adaptive methods [27]. This should not be viewed as a way to fool the uncertainty principle; rather, it is an application of the uncertainty principle in that we can nonisotropically decrease the uncertainty in one direction of the uncertainty domain.

This ultimate limit of uncertainty for features may not have a large influence for the bulk of classical image operations used. In this context, an interpretation can be given which is more similar to classical mechanics than to quantum mechanics. This is due to the fact that we in this case do not generally assume a mutual interaction between data and operator [17]. We might say that we are dealing with systems having the equivalent of a larger inertia, which brings us into classical mechanics. Although data changes the output of an operator, it does not modify the operator itself.

Besides the uncertainty arising out of spatial sampling, another kind of uncertainty one may encounter in image processing is due to grayness ambiguity. This uncertainty arises out of the multivalued nature of the pixel intensity [13]. According to the fuzzy set theoretic notion, this indeterminacy is due to inherent vagueness rather than randomness. Incertitude in image pattern can be explained in terms of grayness ambiguity, spatial ambiguity, or both. Grayness ambiguity means indefiniteness in deciding whether a pixel is white or black. Spatial ambiguity refers to indefiniteness in shape and geometry of the region within the gray level image.

The conventional approach to image analysis and recognition consists of segmenting the meaningful regions, and extracting the various features and properties (for example, area, parameter, centroid, etc.) and primitives, and of relationship among the regions. As the regions in the image (multilevel) are not always crisply defined, uncertainty can arise in every phase of the aforesaid tasks. Any decision made at a particular level will have a definite impact on all subsequent levels of activities. A recognition system should have sufficient provisions for representing and manipulating the uncertainty involved in every processing stage, so that the system can retain as much of the original information as possible. If this is done, the ultimate output of the system will possess minimal uncertainty. Let us consider an example, the problem of object extraction from a scene.

Now the question is "How can one define exactly the target or object region in a scene when its boundary is ill defined?" (ill defined in the sense of multivalueoness or grayness). Any hard decision making (thresholding) for the extraction of the object will propagate unassociated uncertainty to subsequent stages. This might affect the subsequent analysis and recognitions. This is convenient, natural, and appropriate to avoid commission to a specific hard decision by allowing a segment or contours to be a fuzzy subset of the image, the subsets being characterized by their possibilities or degree to which each pixel belongs to them.

Now it is curious to note that in extracting the image, the background plays an important role in the sense that it introduces some sort of indeterminacy similar to that evoked during the preparation of a state vector of an object in the theory of unsharp measurement in quantum mechanics. It is also worth mentioning that Slepian and his collaborators in [6] explained the role of uncertainty in simultaneous windowing operations. This work has found very important and fruitful applications in areas such as sampling theory, spectrum estimation, image coding, and filter design. The works of Marr, Granlund, Jacobson, Wechsler, and Wilson either explicitly or implicitly acknowledged the role of uncertainty among various Fourier couple other than only time-frequency variables. So, in the field of image processing and computer visions at one level, the background introduces uncertainty like the unsharpness introduced in preparing the state of the object in quantum mechanics, and uncertainty is also involved at the level of measurement. This has a striking similarity with the two roles of the state vector as already discussed in the framework of unsharp measurement theory. It has been emphasized that the unsharpness due to preparation has a profound impact on the consistent interpretations of the Heisenberg uncertainty relation.

The situation may well be different for the types of distributed or heavily fragmented computation architectures, which are visualized for the future. This is, e.g., the case in what is termed neural computation. For this case, postulates have been made that for a flexible context control of processing, it is necessary that information can appear interchangeably as data or as operator in traditional terms [28]. This implies that there is a pronounced mutual influence between data and operator, or in quantum-mechanical terms, between the observer and the observed [17]. For this case, it is postulated that uncertainty effects will not only be significant, but also play an important role of the computation mechanism itself. It is outside the scope of this article to go into this issue in detail.

This raises the possibility that a framework of unsharp observables may be constructed to find relevant application within spatial data processing, such as signal processing and computer vision. As alluded to in the preceding paragraph, it is believed that the uncertainty relation in this context benefits from an individualistic interpretation, rather than a statistical one. Lastly, we think that the role of context (e.g., background of an object) in image analysis and computer vision should be thoroughly studied for the interpretation of the various uncertainty relations already found in these fields.

5. DISCUSSION

It is evident from the above analysis of the joint distribution and uncertainty relations, that Cohen's arguments are not tenable, at least in the context of unsharp observables and individualistic interpretation of uncertainty relations. It has a striking correspondence with the phenomena in the domain of image processing and computer vision. It not only sharpens the debate regarding the formal similarity of time-frequency uncertainty relations in signal processing and the Heisenberg time-energy uncertainty relation, but also raises a new possibility to describe the ambiguity in an image by using fuzzy observables. It is worth mentioning that the unsharp observables are also introduced in quantum mechanics so as to describe the semiclassical behavior of a quantum system, i.e., to describe the trajectory of a quantum particle (say, electron). Moreover, this trajectory is like a classical trajectory, i.e., it can be described by the

Ehrenfast theorem. As a result, the theory of unsharp observables as formulated by Busch, Lahti, and Mittlestaedt [20] might be also a suitable framework for mesoscopic systems (i.e., systems between classical and quantum systems).

It is postulated that these mechanisms will be of crucial importance in highly fragmented computation structures, such as neural networks, as they may exhibit a mutual interaction between data and operator.

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