# Remarks on quantum deformation of quasi-exactly solvable problems

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**Abstract**

We show a general method of $q$-deforming quasi-exactly solvable problems with underlying Su(2) symmetry. In some cases we find quasi-exactly solvable Schrödinger equations (with potentials depending on the deformation parameter) which are $Su_q(2)$ symmetric.

In recent years quantum groups or q-deformed algebras have been studied from many points of view [1]. In particular quantum deformation of quantum mechanics has been studied by many authors [2–10]. Generally there are two approaches to $q$-deformed quantum mechanical problems. In the first approach, deformation is carried out at the level of operators [2,3] or symmetry algebra [8,9] while in the second approach deformation is carried out at the level of Schrödinger-like equations [4–7,10]. In all these cases deformation is performed in quantum mechanical systems which are exactly solvable.

On the other hand, quasi-exactly solvable problems are a class of problems which are neither completely solvable nor unsolvable [11–15]. A part of the spectrum of quasi-exactly solvable problems can always be solved and in most cases these problems have a certain underlying symmetry. Here our objective is to study quantum deformation of quasi-exactly solvable problems following both the approaches mentioned above.

Let us now consider the Schrödinger equation

$$H\Psi = \left( -\frac{d^2}{dx^2} + V(x) \right) \Psi = E\Psi . \qquad (1)$$

Now we perform the following transformation,

$$\Psi(x) = \exp\left( -\int W(x)\, dx \right) \psi(x) \qquad (2)$$

and obtain from Eq. (1)

$$H_G \psi = \left( -\frac{d^2}{dx^2} + 2W\frac{d}{dx} - (W^2 - W' - V) \right) \psi = E\psi . \qquad (3)$$

Now putting $\xi = f(x)$ in the above equation we find

$$H_G\phi = \left( -f''(x)\frac{d}{d\xi} - [f'(x)]^2 \frac{d^2}{d\xi^2} + 2W(x)f'(x)\frac{d}{d\xi} - [W^2(x) - W'(x) - V(x)] \right)\phi = E\phi. \tag{4}$$

We now consider the Su(2) algebra. The generators of this algebra (acting in a space of polynomials of degree $\xi^{j}$) can be realised as

$$j_+ = \frac{1}{\sqrt{2}}\left(2j\xi - \xi^2\frac{d}{d\xi}\right), \quad j_0 = \left(-j + \xi\frac{d}{d\xi}\right), \quad j_- = \frac{1}{\sqrt{2}}\frac{d}{d\xi}, \tag{5}$$

where $j$ is the spin. It can be easily checked that $j_+$, $j_-$ and $j_0$ satisfy the relations

$$[j_+, j_-] = j_0, \quad [j_0, j_\pm] = \pm j_\pm. \tag{6}$$

The next task is to write $H_G$ in (4) in terms of a combination of the generators in (5). To be more specific, let us now choose the potential to be

$$V(x) = \lambda x^6 + \mu x^4 + \nu x^2. \tag{7}$$

The function $W(x)$ corresponding to (7) is taken to be

$$W(x) = \sqrt{\lambda}\, x^3 + \frac{\mu}{2\sqrt{\lambda}} x. \tag{8}$$

Then from (3) we find

$$\left[ -\frac{d^2}{dx^2} + 2\left(\lambda x^3 + \frac{\mu}{2\sqrt{\lambda}}x\right)\frac{d}{dx} - \left(\frac{\mu^2}{4\lambda} - 3\sqrt{\lambda} - \nu\right)x^2 + \frac{\mu}{2\sqrt{\lambda}} \right]\psi = E\psi. \tag{9}$$

Now if we take

$$\xi = f(x) = x^2, \tag{10}$$

the equation corresponding to (4) becomes

$$H_G\phi = \left[ -4\xi\frac{d^2}{d\xi^2} + \left(4\lambda\xi^2 + \frac{2\mu}{\sqrt{\lambda}} - 2\right)\frac{d}{d\xi} - \left(\frac{\mu^2}{4\lambda} - 3\sqrt{\lambda} - \nu\right)\xi + \frac{\mu}{2\sqrt{\lambda}} \right]\phi = E\phi. \tag{11}$$

Now we have to write $H_G$ in terms of a combination of the generators $j_-$ and $j_0$. To this end we write

$$H_G\phi = (A j_0 j_- + B j_- + C j_0 + D j_- + K)\phi = E\phi, \tag{12}$$

where $A$, $B$, $C$, $D$ and $K$ are constants to be determined later. Then using the expressions for $j_+$ and $j_0$ as given in (5) in the form of $H_G$ in (12) and comparing the resultant expression with (11) we find

$$A = -4\sqrt{2}, \quad B = -4\sqrt{2\lambda}, \quad C = \frac{2\mu}{\sqrt{\lambda}},$$
$$D = -\sqrt{2}(4j+2) \quad K = \frac{\mu}{2\sqrt{\lambda}}(4j+1),$$
$$\nu = \frac{\mu^2}{4\lambda} - 3\sqrt{\lambda} - 8j\sqrt{\lambda}. \tag{13}$$

Now to determine the spectrum we can proceed as in Refs. [13,14]. However, a slightly different way is to treat the eigenvalue problem corresponding to (12) as a matrix eigenvalue problem. For this we have to consider the $(2j+1)\times(2j+1)$ matrix representation of the operators $j_\pm$ and $j_0$. For instance, let us consider the case $j = 1$. In this case we have

$$j_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad j_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad j_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{14}$$

Then using (13) and (14) in Eq. (12) we find the following equation for the eigenvalues,

$$(C + K - E)\left[(K-E)^2 - C(K-E) + (BA - BD)\right] - BD(K - E - C) = 0. \tag{15}$$

Eq. (15) is a cubic equation in $E$ and $2j+1 = 3$ roots of this equation give us the eigenvalues. Solving Eq. (15) we find

$$E_0 = \frac{5\mu}{2\sqrt{\lambda}} - 2z\cos(\tfrac{1}{3}\theta),$$
$$E_2 = \frac{5\mu}{2\sqrt{\lambda}} - z\left[-\cos(\tfrac{1}{3}\theta) + \sqrt{3}\sin(\tfrac{1}{3}\theta)\right],$$
$$E_4 = \frac{5\mu}{2\sqrt{\lambda}} + z\left[\cos(\tfrac{1}{3}\theta) + \sqrt{3}\sin(\tfrac{1}{3}\theta)\right], \tag{16}$$

where $z$ and $\theta$ are given by

$$z=\frac{2}{\sqrt{3}}\left(\frac{\mu^2}{\lambda}+16\sqrt{\lambda}\right)^{1/2},$$

$$\tan\theta=\left(\frac{\mu^2/\lambda+16\sqrt{\lambda}}{432\mu}-1\right)^{1/2}. \quad (17)$$

We now consider $q$-deformation of quasi-exactly solvable problems and treat the potential in (7). The method is to replace the generators $j_\pm$ and $j_0$ by $J_\pm$ and $J_0$ which generate the $\mathrm{Su}_q(2)$ algebra. However, among the various realisations of the $\mathrm{Su}_q(2)$ algebra, the most useful for our purpose is the one considered by Curtright and Zachos [16,17]. In this case the $\mathrm{Su}_q(2)$ generators are given by [16,17]

$$J_0=j_0,$$

$$J_+=\sqrt{\frac{[j'+j_0][j'-j_0-1]}{(j'+j_0)(j'-j_0-1)}}\,j_+,$$

$$J_-=(J_+)^+, \quad (18)$$

where $[x]=(q^x-q^{-x})/(q-q^{-1})$ and the operator $j'$ is given by

$$j'(j'+1)=2j_+j_-+j_0(j_0-1). \quad (19)$$

Also it can be shown that $J_\pm$ and $J_0$ satisfy the relations

$$[J_0,J_\pm]=\pm J_\pm,\quad [J_+,J_-]=\tfrac{1}{2}[2J_0]. \quad (20)$$

It is not difficult to show that if $j_\pm$ and $j_0$ are given by $(2\times2)$ matrices (corresponding to the spin $j=\frac{1}{2}$) then $J_\pm$ and $J_0$ are given by the same matrices [16,17] and thus for $j=\frac{1}{2}$ the deformed and the undeformed problems are the same. However the case $j=1$ (in fact all $j>\frac{1}{2}$ cases) is nontrivial and we have

$$J_0=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\quad J_+=\sqrt{\frac{[2]}{2}}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},$$

$$J_-=\sqrt{\frac{[2]}{2}}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}. \quad (21)$$

Then from (15) we get

$$H_0\phi=(AJ_0J_-+BJ_++CJ_0+DJ_-+K)\phi=E_q\phi \quad (22)$$

and using (21) we obtain

$$(C+K-E_q)\left[(K-E_q)^2-C(K-E_q)+(BA-BD)f^2\right]-BDf^2(K-E_q-C)=0, \quad (23)$$

where $f$ is a constant and is given by

$$f=\sqrt{\frac{[2]}{2}}. \quad (24)$$

Now solving (23) for $E_q$ we find

$$E_{q,0}=\frac{5\mu}{2\sqrt{\lambda}}-2z(q)\cos\left[\tfrac{1}{3}\theta(q)\right],$$

$$E_{q,2}=\frac{5\mu}{2\sqrt{\lambda}}-z(q)\left\{-\cos\left[\tfrac{1}{3}\theta(q)\right]+\sqrt{3}\sin\left[\tfrac{1}{3}\theta(q)\right]\right\},$$

$$E_{q,4}=\frac{5\mu}{2\sqrt{\lambda}}+z(q)\left\{\cos\left[\tfrac{1}{3}\theta(q)\right]+\sqrt{3}\cos\left[\tfrac{1}{3}\theta(q)\right]\right\}, \quad (25)$$

where $z(q)$ and $\theta(q)$ are now given by

$$z(q)=\frac{2}{\sqrt{3}}\left(\frac{\mu^2}{\lambda}+16f^2\sqrt{\lambda}\right)^{1/2},$$

$$\tan\theta(q)=\left(\frac{\mu^2/\lambda+16f^2\sqrt{\lambda}}{432\mu f^2}-1\right)^{1/2}. \quad (26)$$

Thus the eigenvalues in (25) are those of the deformed sextic anharmonic oscillator. It is clear that as $q\to1$ we recover the undeformed eigenvalues. Evidently, if we proceed in a similar manner, it is possible to deform any quasi-exactly solvable problem with Su(2) symmetry and for all spins $j>\frac{1}{2}$. It may be noted that if we take higher values of $j$ then we shall obtain more eigenvalues and the equation corresponding to (23) will then have terms like $(E_q)^{2j+1}$. (In this context we refer the reader to Refs. [16,17] for a representation of $J_\pm$ and $J_0$ by higher order matrices and to Ref. [13] for a list of quasi-exactly solvable problems).

Till now we have confined ourselves to the formulation and solution of the problem at the level of operators. We do not know what potential or

Schrödinger equation corresponds to the spectrum (25). To obtain a Schrödinger equation with a $\mathrm{Su}_q(2)$ symmetry we consider the $j = 1$ case. Next we note that the operators

$$J_- = \frac{\sqrt{[2]}}{2}\left(2\xi - \xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\right),$$

$$J_- = \frac{\sqrt{[2]}}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right),$$

$$J_0 = \xi\frac{\mathrm{d}}{\mathrm{d}\xi} - 1 \tag{27}$$

span a representation of the $\mathrm{Su}_q(2)$ algebra in a space of polynomials of the form $\varphi(\xi) = a + b\xi + c\xi^2$. We now go back to (22) again. Using the expressions of $J_+$ and $J_0$ in (27) in Eq. (22) we find

$$\left[\frac{Af}{\sqrt{2}}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \left(\frac{Df}{\sqrt{2}} + C\xi - \frac{Bf\xi^2}{\sqrt{2}} - \frac{Af}{\sqrt{2}}\right)\frac{\mathrm{d}}{\mathrm{d}\xi} + \left(\frac{2Bf\xi}{\sqrt{2}} - C + K\right)\right]\phi = E_q\phi \tag{28}$$

and on using the values of $A$, $B$, $C$, $D$, $K$ (cf. (13)) Eq. (28) reduces to

$$\left[-4f\xi\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \left(-2f + \frac{2\mu\xi}{\sqrt{\lambda}} + 4\sqrt{\lambda}f\xi^2\right)\frac{\mathrm{d}}{\mathrm{d}\xi} - 8\sqrt{\lambda}f\xi - \frac{2\mu}{\sqrt{\lambda}} + \frac{5\mu}{2\sqrt{\lambda}}\right]\phi = E_q\phi. \tag{29}$$

We now make the following change of variables,

$$\xi = fx^2. \tag{30}$$

Then Eq. (29) becomes

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\mu}{\sqrt{\lambda}}x + 2\sqrt{\lambda}f^2x^3\right)\frac{\mathrm{d}}{\mathrm{d}x} - 8\sqrt{\lambda}f^2x^2 + \frac{\mu}{2\sqrt{\lambda}}\right]\psi = E_q\psi. \tag{31}$$

Eq. (31) is still not in the Schrödinger form. To bring it to the Schrödinger form we use the transformation

$$\psi(x) = \exp\left(\int W(x)\,\mathrm{d}x\right)\Psi(x) \tag{32}$$

and obtain from Eq. (31)

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\Psi(x) = E_q\Psi(x), \tag{33}$$

where the potential $V(x)$ is given by

$$V(x) = \lambda f^4x^6 + \mu f^2x^4 + \left(\frac{\mu^2}{4\lambda} - 11\sqrt{\lambda}f^2\right)x^2. \tag{34}$$

Thus the Schrödinger equation (33) with the potential (34) represents a $q$-deformed quasi-exactly solvable problem with $\mathrm{Su}_q(2)$ symmetry and its solutions are given by (25) and the corresponding eigenfunctions are given by

$$\Psi_{q,i} = \exp\left(-\frac{\sqrt{\lambda}}{4}f^2x^4 - \frac{\mu}{4\sqrt{\lambda}}x^2\right) \times \left(f^2x^4 - \frac{E_i - 9\mu/2\sqrt{\lambda}}{4\sqrt{2\lambda}}fx^2 + \frac{E_i - 9\mu/2\lambda}{2\sqrt{\lambda}\,(\mu/2\sqrt{\lambda} - E_i)}\right), \tag{35}$$

$i = 0, 2, 4.$

From the above considerations it is clear that we can obtain a Schrödinger equation corresponding to the number of $\mathrm{Su}_q(2)$ symmetric potentials [13] which are quasi-exactly solvable.

In this article we have considered a method of deforming quasi-exactly solvable problems with Su(2) symmetry, the underlying symmetry of the deformed problem being $\mathrm{Su}_q(2)$. Also the method is general enough to be applicable to a large number of potentials. Furthermore, we have obtained Schrödinger equations which are $\mathrm{Su}_q(2)$ symmetric and quasi-exactly solvable. Viewed as just Schrödinger equations, the remarkable feature of these problems is that they exhibit a nonlinear spectrum (for a semiclassical approach to a similar problem we refer the reader to Ref. [18]). Finally we note that it would be interesting to obtain Schrödinger (-like) equations corresponding to these potentials for $j > 1$.

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