ON MEASUREMENT OF INCOME INEQUALITY AND POVERTY

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P R E F A C E

This thesis is concerned with measurement of inequality, poverty and tax progressivity. We begin in Chapter 1 with a brief survey of the methodological work available in the literature on inequality and poverty. Chapters 2-4 make different approaches to the measurement of poverty and the next two chapters concentrate on the measurement of inequality. Finally, Chapter 7 is concerned with the measurement of tax progressivity. Throughout these chapters, the emphasis is on construction of families of indices possessing different sets of properties desirable from the point of view of welfare economics.

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CHAPTER 0

S U M M A R Y

The subject of this thesis is the measurement of income inequality, poverty and tax progressivity. Here we briefly summarise the main results presented in the thesis.

Let us consider a community S consisting of n earning units, where the ith unit has income $y_i \geq 0$ $(i = 1, 2, \ldots n)$, which indicates its economic position in the society. A state of the distribution of income in the society is given by a vector $\underline{y} = (y_1, y_2, \ldots, y_n)'$. Let $\lambda > 0$ be the mean value : $\lambda = \frac{1}{n} \sum_{i=1}^{n} y_i$.

The term 'inequality', as observed by Bauer and Prest (1973) is generally applied to cases where incomes (or wealth) are simply different, just as one might refer to two persons being of unequal height. Inequality signifies departure from the state of equality. The tools that an economist employs for measuring the extent of such departures are known as measures of inequality. The first chapter in Part I, makes a brief review of the methodological work available in the literature on economic inequality. First, we lay down a number of postulates for selecting a good inequality index. Then we discuss the measures of inequality that have been proposed in the literature. These measures fall into two categories, viz., (i) the positive measures which make no explicit use of any concept of social welfare, and (ii) normative measures which are based on some explicit formulation of the social welfare function. We also discuss the implications of the positive measures of inequality in terms of the underlying social welfare functions. Finally, we briefly mention the significance of Lorenz curve comparisons of income distributions.

Studies on the incidence of poverty in absolute or relative sense have been very frequently carried out during the last decade or so. In the second part of Chapter 1 we review the existing measures of poverty in the light of different axioms. These measures generally pre-suppose that an exogenously given level of income (called the poverty line) represents the minimum norm or standard of living and that all other incomes are to be compared with this norm. Suppose that in a society consisting of n earning units, q-units are poor (that is, below the poverty line z). In symbols, let $y_1 \leqq y_2 \leqq \ldots\ldots \leqq y_q < z \leqq y_{q+1} \leqq \ldots\ldots \leqq y_n$. A poverty measure is required to combine the deprivations of the q poor units into one overall indicator.

Chapter 2 introduces new relative measures of poverty based on the utility gaps of the poor. In terms of symbols already introduced, the utility gap of unit i is $U(z)-U(y_i)$, where $U(.)$ is the utility function common for all units. U is assumed to be increasing and strictly concave. For a given income configuration $\underline{y}$, the poverty of $\underline{y}$ is measured by using the normalised sum of the utility gaps of the q poor units. The unknown form of $U(.)$ is determined by imposing the restriction that the measure is invariant under affine transformations of $U(.)$ and the axiom of scale irrelevance. The resulting index satisfies Sen's monotonicity and transfer axioms [Sen (1976)]. The measure attaches greater weight to transf rs of income lower down the income scale. In this respect the new index seems to be superior to Sen's index using ordinal rank weights. Moreover, if the population is partitioned into a number of groups according to one or more attributes, the measure can be easily decomposed into components that reflect that partition.

We also propose in Chapter 2 a class of relative measures based on weighted sum of the utility gaps. Here the utility function has been assumed to be concave. This class of measures also satisfies the basic postulates of a poverty measure.

The third chapter explores the usefulness of 'representative income gaps' in the measurement of poverty. Let $g_i = z - y_i$, $i = 1, 2, \ldots, q$, denote the income gap of the ith poor unit. Suppose that $F(g_1, g_2, \ldots, g_q)$ is the group deprivation function of the poor, where - F is continuous, non-decreasing in y_i's and also S-concave. We define the representative income gap g_e of the poor corresponding to a given income profile as that level of gap which if shared by every poor unit would make the existing distribution of gaps socially indifferent (as measured by F):

$$F(g_e, g_e, \ldots, g_e) = F(g_1, g_2, \ldots g_q) \quad \ldots \quad (0.1)$$

Then a general relative poverty index corresponding to a given profile is defined as

$$P = \frac{q}{n} \cdot \frac{g_e}{z} \quad \ldots \quad (0.2)$$

We also define an absolute poverty index as

$$Q = \frac{q}{n} \cdot g_e \quad \ldots \quad (0.3)$$

For every homothetic group deprivation function of the poor we get a relative poverty index of the form (0.2), and for every group deprivation function possessing only the properties mentioned above there corresponds an absolute poverty index of the form (0.3).

We have proposed another class of poverty measures in Chapter 3 assuming that the group deprivation function is the sum of identical individual deprivation functions of the poor units. In fact, a particular form of these functions is also assumed. The class of measures can be written as

$$P(\alpha) = \left[\frac{1}{n} \sum_{i=1}^{q} \left(1 - \frac{y_i}{z}\right)^{\alpha} \right]^{1/\alpha} \qquad \ldots \qquad (0.4)$$

where $\alpha \geq 1$ is a parameter.

For a disequalising transfer enabling its recipient to cross the poverty line, $P(\alpha)$ increases unambiguously, but the general measures P and Q may decrease. This might be taken as a deficiency of the latter mentioned measures, but Sen [(1979) p. 302] has argued that such behaviour of P and Q actually be treated as reasonable.

The fourth chapter introduces measures of poverty using Atkinson — Kolm — Sen representative income [Atkinson (1970); Kolm (1969); Sen (1973)] of a community corresponding to the censored income profile.[1] The representative income y_f^* is that level of income which if enjoyed by every unit would make the existing censored income profile socially

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1/ The censored income profile corresponding $\underline{y}$ is $\underline{y}^* = (y_1, y_2, \ldots, y_q, z, z, \ldots, z)' = (y_1^*, y_2^*, \ldots, y_n^*)'$.

or ethically indifferent. That is,

$$W\,(y_f^*, y_f^*, \ldots, y_f^*) \;=\; W\,(y_1^*, y_2^*, \ldots, y_n^*) \qquad \ldots \qquad (0.5)$$

The social welfare functions W(.) that we employ here are S-concave. This means, if two censored income profiles have the same mean income and if one is unambiguously at least as unequal as the other (by the Lorenz criterion), then the former is ranked as no worse than the latter by the social welfare function. We introduce relative as well as absolute measures based on this approach. The general relative measure introduced in this chapter is

$$P \;=\; 1 - \frac{y_f^*}{z} \qquad \ldots \qquad (0.6)$$

The corresponding absolute measure is defined as

$$Q \;=\; z - y_f^* \qquad \ldots \qquad (0.7)$$

If P is to be scale irrelevant, the social welfare function W must be homothetic. For Q to be invariant with respect to translation of z and $\underline{y}^*$, W must be translatable. For every homothetic/translatable social welfare function there exists a relative/absolute poverty index. Further, for every relative/absolute poverty index, there exists a family of homothetic/translatable social welfare functions. The member of either family are ordinally equivalent.

The chapter also includes a discussion of consistent, neutral and ethical aggregation. If a population is partitioned into k groups, then using the notion of group representative income, we investigate the resolution of both relative and absolute indices into group indices. The indices that do this in an ethically consistent way are the measures underlying the symmetric mean of order β ($\beta \leq 1$) and the Kolm — Pollak social welfare function. These two classes of measures are axiomatised in Section 5 of the chapter with a view to bringing out the assumptions implicitly made by their choices. Section 6 is concerned with the so-called compromise indices.

Chapter 5 presents two axiomatisations of the Theil entropy measure of inequality. First, we define the normalised value of the amount by which aggregate welfare of an income profile $\underline{y}$ (as measured by $\sum_{i=1}^{n} U(y_i)$, where U is an increasing, concave utility function) falls short of its maximum as a measure of inequality. This measure in a limiting situation is shown to coincide with the entropy measure normalised in the interval $[0, 1]$. We next axiomatise a social welfare function, which for a given mean income and a given population size ranks all possible income profiles in exactly the opposite way as the Theil entropy measure.

Chapter 6 suggests two modifications of the Gini coefficient, each of which ranks income profiles in the same way as some strictly quasi-concave swf. Suppose $\lambda > 0$ is the mean of the income variable y(assumed non-negative) with distribution function F, then the first modification suggested is

$$I = \left[\int_0^1 [p - L(p)]^{\alpha} \, dp \right]^{1/\alpha} \quad \ldots \quad (0.8)$$

where L(P) is the ordinate of the Lorenz curve of the distribution against abscissa p. For $\alpha > 1$ the ranking of income profiles by the measure I is the same as that based on some strictly concave social welfare function. For $\alpha = 1$, the measure equals a multiple of the Gini coefficient, and fails to possess such property.

Another modification of the Gini measure proposed for the same purpose is the following index:

$$\bar{I} = 1 - \frac{1}{\lambda} \left[\frac{\sum_{i=1}^{n} a_i \, y_i^{\,r}}{\sum_{i=1}^{n} a_i} \right]^{1/r} \qquad r < 1 \qquad \ldots \quad (0.9)$$

This inequality measure corresponds to the homothetic social welfare function having the image

$$\bar{W}_r(\underline{y}) = \left[\frac{\sum_{i=1}^{n} a_i \, y_i^{\,r}}{\sum_{i=1}^{n} a_i} \right]^{1/r} , \quad r < 1 \quad \ldots \quad (0.10)$$

Here $y_1 \geqq y_2 \geqq \ldots \geqq y_n$ and the sequence $\{ a_1, a_2, \ldots \}$ is positive and non-decreasing. The Gini coefficient corresponds to the special case where $r = 1$ and $a_i = (2i - 1)$, $i = 1, \ldots, n$. On the other hand, the Atkinson measures correspond to the subclass where $a_i = \frac{1}{n}$.

In Chapter 7 the attention is focussed on the related problem of measurement of tax progressivity. First, we prove the intuitively obvious result that if the post-tax income profile is an increasing, concave function of the pre-tax income profile, then the former is Lorenz superior to the latter. Then we review the available indices of tax progressivity and propose a new index. The tax progressivity index is regarded as the normalised value of the difference between the welfare value of the post-tax income profile (as measured by $\sum_{i=1}^{n} U(y_i)$, where U is an increasing, concave utility function) and that of the profile that would result if the same amount of tax were realised through a proportional tax scheme.

A form of utility function that enables the measure to satisfy the axiom of scale irrelevance yields a class of computable measures. This in a limiting case is simply the proportionate gap between the Theil inequality measure for the pre-tax profile and that for the post-tax profile.

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CHAPTER 1

SURVEY OF LITERATURE - I: MEASUREMENT OF INEQUALITY

1.1.1 Introduction

Kuznets in his pioneering study stated that: 'When we say "income inequality", we mean simply differences in income, without regard to their desirability as a system of reward or undesirability as a scheme running counter to some ideal of equality of economic opportunity' [Kuznets (1953), p. XXVII]. Following Kuznets we can say that a measure of inequality could roughly be defined as a scalar representation of inter-personal income differences within a given population. Measures of inequality are employed to study and to compare the commonly recognised phenomenon of inequality in personal distribution of income or wealth which exist at different times and in different places. We shall be concerned with the inequality of income and not directly with wealth.[1/]

[1/] There are several studies based on attributes other than income or wealth, which provide interesting material for comparison. A few notable studies may be mentioned here. Jencks (1973) puts income inequality in the much wider context of social inequality; Addo (1973) considers international inequality in such things as school enrolment, calorie/energy consumption and numbers of physicians; Alker (1965) discusses a quantification of voting power; Russet (1964) relates inequality in land ownership to political instability.

This part of the chapter aims at discussing different measures of income inequality. The next section sets out the different postulates by which measures of inequality can be selected. In Section 3 we discuss the different measures of inequality that have been proposed in the literature.

1.1.2 Postulates for the Selection of Good Measures of Inequality

We assume that no ambiguity arises in connection with the definitions of income and of the income earning unit and the choice of the reference period over which income is observed. The conceptual problems connected with these definitions/choices have been discussed in Atkinson (1974, 1975) and in Tinbergen (1975). We consider a community S consisting of n units. An income profile of the community is given by the vector of incomes of units, denoted by $\underline{y}$,

$$\underline{y} = (y_1, y_2, \ldots, y_n)' \qquad \ldots \qquad (1.1)$$

where $y_i \geq 0$ stands for the income of the ith unit and the prime denotes transposition. Let $\lambda > 0$ be the mean income, i.e.,

$$\lambda = \frac{1}{n} \sum_{i=1}^{n} y_i \qquad \ldots \qquad (1.2)$$

Henceforth Y, the set of all possible income profiles will be referred to as the income-space. A measure of income inequality is a scalar function defined on Y. We shall denote this measure by $I(\underline{y})$.

We shall now set out the criteria that measures of inequality may be required to satisfy. Discussions along these lines have been made by Dalton (1920, 1925), Atkinson (1970), Sen (1973), Champernowne (1974), Cowel and Kuga (1976, 1977), Kolm (1976), Cowel (1977), Kurabayashi and Yatsuka (1977), Fields and Fei (1978). The suggested postulates are listed below:

(P1) Zero at Equality

This postulate requires that for all admissible λ's

$$I(\lambda \underline{1}) = 0 \qquad \ldots \qquad (1.3)$$

where $$\underline{1} = (1, 1, \ldots, 1)' \qquad \ldots \qquad (1.4)$$

(P2) Positivity out of Equality

We write this postulate as

$$I(\underline{y}) > 0 \qquad \ldots \qquad (1.5)$$

if $\underline{y} \neq \lambda \underline{1}$.

(P3) <u>Impartiality</u>

I remains unchanged by a permutation of y_i's, i.e.,

$$I(P\,\underline{y}) = I(\underline{y}) \qquad \ldots \qquad (1.6)$$

where P is any permutation matrix of order n [2,3]

(P4) <u>Principle of Population</u>

$$\text{Let } N = \{1, 2, 3, \ldots\} \qquad \ldots \qquad (1.7)$$

Let $\underline{z}$ stand for an nk dimensional income vector defined by

$$\underline{z} = (z_{ir})', \text{ where } z_{ir} = y_i,\ i = 1, \ldots, n;$$
$$r = 1, \ldots, k.$$

Measures of inequality corresponding to $\underline{y}$ and $\underline{z}$ are represented by $I_n(\underline{y})$ and $I_{nk}(\underline{z})$ respectively.

I satisfies Dalton's principle of population [Dalton (1920)] if and only if

$$I_n(\underline{y}) = I_{nk}(\underline{z}) \qquad \ldots \qquad (1.8)$$

for any $n, k \in N$ such that $n \geq 2$.

2/ A non-negative square matrix $B = (b_{ij})_{n \times n}$ is said to be bistochastic if all row and column sums are unity. A bistochastic matrix of order n is said to be a permulation matrix if it has one and only one positive element in each row and in each column.

3/ The criterion of anonymity proposed by May (1952) is analogous to this criterion.

(P5) Principle of Transfers

Suppose an income profile $\underline{y}$ is transformed into another profile by an operation of the form:

$$x_i = y_i + \delta < x_j$$

$$x_j = y_j - \delta \qquad \dots \qquad (1.9)$$

where $\delta > 0$, $x_k = y_k \;\forall\; k \neq i, j$. Then the principle of transfers - also known as Pigou - Dalton criterion [Pigou (1912), Dalton (1920)] and rectifiance [Kolm (1976, 1976a)] requires that

$$I(\underline{y}) > I(\underline{x}) \qquad \dots \qquad (1.10)$$

Conversely, if some amount of income is transferred from a poor unit to a richer unit then I must increase.

In fact, it has been argued that the impact of each transfer should be greater if it takes place at lower income levels [See for example, Dalton (1920), Atkinson (1970), Sen (1973), Kolm (1976a)]. The point can be made clearer as follows:

Consider a couple of pairs of units (i, j) and (i', j') such that

$$y_i > y_j, \quad y_{i'} > y_{j'}, \quad y_i > y_{i'} \quad \text{and} \quad y_i - y_j = y_{i'} - y_{j'} \qquad \dots \qquad (1.11)$$

The measure of inequality I is said to satisfy the principle of diminishing transfers if

$$I(\underline{y}) - I(\underline{y}^T(i, j)) > I(\underline{y}) - I(y^T(i', j')) \quad \ldots(1.12)$$

where $\underline{y}^T(i, j)$ and $\underline{y}^T(i', j')$ are respectively the income profiles resulting from $\underline{y}$ after a transfer of $\delta > 0$ amount of income from y_j to y_i and from y_j' to y_i' respectively, where δ is so small that such transfers donot alter the ranking of the affected units.

(P6) Scale Invariance

This property requires that

$$I(\underline{y}) = I(c\,\underline{y}) \quad \ldots \quad (1.13)$$

for all $c > 0$.4/ This takes care of changes in the unit of money, and in the general price level.

4/ Taussig (1939) and Blau (1977) feel that a variation of all incomes in the sume proportion should not change the measure of inequality. Kolm (1976), on the other hand, argues that proportionate increases in income represent increase in inequality. Taking the opposite point of view, Dalton (1920) and Sen (1973) suggest that inequality should decrease when all incomes are increased proportionately.

(P6') Translation Invariance

An inequality measure is said to be translation invariant if

$$I(\underline{y}) = I(\underline{y} + \alpha \underline{1}) \quad \ldots \quad (1.14)$$

for all real α [5] such that $\underline{y} + \alpha \underline{1}$ is in the domain of definition of I.

An inequality measure will be called a relative measure or an absolute measure according as it satisfies (P6) or (P6').

In addition to above a few authors like Champernowne (1974) have laid down some stipulations regarding the upper bound of an inequality measure. An inequality measure should take the maximum value when the richest unit monopolises the whole income and all others have zero income. Champernowne (1974) stipulates that in the limit as the number of incomes n increases while one unit always gets all the income, the measure should tend to the value one. But for some of the widely used measures of inequality the upper bound tend to infinity in the limit as $n \to \infty$. However, the upper bound need not be a criterion for preferring one inequality measure to another, because, simple (non-unique) transformations can produce any desired upper bound. If we want to make the upper bound of an inequality measure independent of n, we can

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[5] While Kolm (1976) argues that an inequality measure should be invariant under equal absolute changes in its arguments, Dalton (1920) and Cannan (1930), however, feel that an equal addition to all incomes decreases inequality.

divide the measure by its maximum attainable value. But it can be easily shown that such normalization of a rectifiant inequality measure is obtained at the cost of the principle of population.

1.1.3 Measures of Inequality and Their Properties

The inequality measures that have been proposed in the literature fall into two classes, viz., (i) positive measures which makes no explicit use of any concept of social welfare and (ii) normative measures which are based on explicit consideration of social welfare and loss of welfare incurred through unequal distribution. Some of these measures have been discussed by Dalton (1920), Theil (1967), Atkinson (1970) Kuga (1973), Sen (1973), Champernowne (1974), Kondor (1975), Szal and Robinson (1975), Maddala and Singh (1977) and Allison (1978). We shall organise the discussion in several subsections. The measures of positive type and normative type will be discussed in subsections 1 and 2 respectively. In subsection 3 we shall discuss normative aspects of different positive measures of inequality and significance of Lorenz curve comparisons.

1.1.3.1 Positive Measures of Inequality

(a) The Range

The range, R, the simplest positive measure of inequality is defined as

$$R = \frac{\max_i y_i - \min_i y_i}{\lambda} \qquad \ldots \quad (1.15)$$

If $\underline{y} = \lambda \underline{1}$, $R = 0$ and if one unit receives all the income, $R = n$. By concentrating on extreme values only R fails to possess many desirable properties of an inequality measure.

(b) Measures Based on Income Shares of Selected Ordinal Groups

The most widely used measures belonging to this category are the ratios of shares of total income held by two fractile groups, such as the upper and the lower 25 per cent (quartiles), 20 per cent (quintiles), 10 per cent (deciles) and 5 per cent [Wiles (1974), Tinbergen (1975, 1975a)].

Measures of this type and also of type (c) mentioned below are advocated by those who argue that measures based on the entire income profile may not reflect changes in the lower/upper brackets of the profile which are of primary interest for policy. Such measures donot, however, meet the principle of transfers.

(c) Measures Based on Fractiles

Slightly different from the approach mentioned in (b) above is the method of percentiles used, among others, by Lydall (1959) to look at changes in the pattern of distribution in its different ranges over years. A formalisation of this approach for comparative purposes was suggested by Esberger and Malmquist (1972). Bhattacharya and Iyengar (1961) used ratios of selected percentiles to the arithmetic mean to study intertemporal changes in the distribution of persons by monthly per capita consumer expenditure. Bowely (1937) used $\frac{Q_3-Q_1}{Q_3+Q_1}$ as a measure of inequality, where Q_i is the ith quartile.

We shall now concentrate on measures based on the entire income profile. Sometimes measures of skewness have been used as measures of inequality [Young (1917)], but this is essentially a confusion of inequality with symmetry. An unskewed symmetric profile need not be an equal one [Stark (1972), pp. 139-140].

(d) The Relative Mean Deviation

This measure discussed by Yntema (1933) and earlier by Ricci (1916) is defined as follows:

$$M = \sum_{i=1}^{n} \left| y_i - \lambda \right| / n\lambda \qquad \ldots \qquad (1.16)$$

where M is the relative mean deviation. Sometimes referred to as Kuznets' ratio [Kuznets (1963)] this has often been used extensively [vide Schutz (1951), Rosenbluth (1951), McCabe (1974), Kondor (1975a)]. The UN Economic Commission for Europe (1957) called it the 'maximum equalisation percentage', since, when expressed as a percentage, it gives the percentage of total income which has to be transferred from earners above the mean income level to those below it in order to achieve perfect equality of the incomes. It is a member of the family of inequality measures proposed by Mehran (1976).

Clearly $M = 0$ for perfect equality and $M = \frac{2(n-1)}{n}$ if one unit receives all the income. However, M violates the principle of transfers — it is not affected at all if income is transferred between two units on the same side of the mean.

(e) The Elteto & Frigyes Measures

The Elteto & Frigyes measures [Elteto & Frigyes (1968)] may be defined in terms of three ratios:

$$E_1 = \frac{\lambda}{\lambda_1}, \quad E_2 = \frac{\lambda_2}{\lambda_1}, \quad E_3 = \frac{\lambda_2}{\lambda} \qquad \ldots \qquad (1.17)$$

where λ_1 and λ_2 are respectively the means of observations below the mean income and incomes at or above the mean. E_2 is a measure of inequality for the entire profile. It is possible to compress the essential informations contained in E_1, E_2 and E_3 in the following measure [Kondor (1971)]:

$$T = \frac{(E_1 - 1)(E_3 - 1)}{(E_2 - 1)}$$

$$= \frac{1}{2} \text{[Relative Mean Deviation]} \quad \ldots(1.18)$$

None of Elteto & Frigyes' measures satisfies the principle of transfers.

(f) <u>The Variance and the Coefficient of Variation</u>

The variance V^2 of a set of incomes $y_1, y_2, \ldots, y_n$ is defined by

$$V^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \lambda)^2 \quad \ldots \quad (1.19)$$

V^2 is an absolute measure of inequality. But one profile may show greater relative variation than another and still end up having a lower variance if the mean income around which variations take place is sufficiently smaller in the former profile than in the latter [Sen (1973), p. 27]. A relative measure of inequality that does not suffer from this deficienc is the coefficient of variation, defined by

$$CV = \frac{V}{\lambda} \quad \ldots \quad (1.20)$$

CV takes the minimum value zero when all the incomes are equal and its upper bound is $\sqrt{n-1}$ which is attained when one unit receives all the income. The CV (and hence V^2) satisfies the principle of transfers but not the principle of diminishing transfers. It has the characteristic of attaching equal weightage to transfers of income at different income levels.[6]

(g) The Standard Deviation of Logarithms

The standard deviation of logarithms of a set of incomes $y_1, y_2, \ldots, y_n$ is defined as

$$H = \left[\frac{1}{n} \sum_{i=1}^{n} (\log (y_i/\lambda))^2 \right]^{1/2} \quad \ldots \quad (1.21)$$

Unlike variance or its positive square root, the standard deviation of logarithms is scale invariant. The measure is more sensitive to transfers at lower income levels than to

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6/ A one-parameter family of inequality measures that looks similar to the coefficient of variation is given by

$$I(\alpha) = \frac{1}{\lambda} \left(\sum_{i=1}^{n} y_i^{\alpha} \right)^{1/\alpha} - n^{1/\alpha}, \quad \alpha > 1.$$

This family of measures satisfies the principle of diminishing transfers if $1 < \alpha < 2$ [see Kurabayashi and Yatsuka (1977)].

transfers at higher income levels. But at very high levels of income, the measure actually decreases instead of increasing with a transfer from a relatively poor to a richer unit. Creedy (1977) argues that the extent to which the standard deviation of logarithms violate the principle of transfers is very minor for empirical profiles.

(h) The Relative Mean Difference and the Gini Coefficient

The most frequently used measure of inequality is perhaps the Gini Coefficient attributed to Gini (1912) and analysed among others by Dalton (1920), Atkinson (1970), Newbery (1970), Sheshinski (1972), Kats (1972), Sen (1973, 1973a, 1974, 1976), Pyatt (1976), Blackorby & Donaldson (1978), Michal (1978), Takayama (1979), Dorfman (1979), Weymark (1979), Yitzhaki (1979, 1980), Donaldson & Weymark (1980, 1980a), Hey & Lambert (1980), Thon (1980). The measure is also known as the Lorenz ratio [Lorenz (1905)].

The Gini coefficient G is defined as

$$G = \frac{\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| y_i - y_j \right|}{2\lambda} \qquad \ldots \qquad (1.22)$$

The numerator of the expression (1.22) is called the Gini mean difference (with repetition). If divided by λ, this gives the relative mean deviation. We can rewrite G as [Sen (1973)]:

$$G = 1 + \frac{1}{n} - \frac{2}{n^2 \lambda} \sum_{i=1}^{n} i\, y_i \qquad \ldots \qquad (1.23)$$

where $y_1 \geq y_2 \geq \ldots \geq y_n$.

An illuminating manner of viewing the Gini coefficient is in terms of the Lorenz curve due to Lorenz (1905). The construction of the Lorenz curve is explained in many places [vide Levine and Singer (1970), Sen (1973)]. The Gini coefficient turns out to be the ratio of the area enclosed between the egalitarian line of the Lorenz box and the Lorenz curve to the area of the triangular region underneath the egalitarian line. The Gini coefficient is peculiar in that its sensitivity to transfers depends on the difference in size ranks rather than on the absolute incomes of the units concerned. For a typical income profile the Gini coefficient tends to be most sensitive to transfers around the middle of the profile and less to transfers among the very rich or the very poor. This is clearly undesirable. However, this index is very firmly established through its long usage. Its relation to the Lorenz curve adds considerably to it's attraction.7/

7/ A general class of inequality measures that contains the Gini coefficient and the coefficient of variation as special

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(i) Gastwirth's Measure

Gastwirth (1974) suggested the use of

$$G_1 = \binom{n}{2}^{-1} \sum_{i<j} \frac{|y_i - y_j|}{(y_i + y_j)} \qquad \ldots \qquad (1.24)$$

as a measure of inequality. The direct computation of the measure is complicated since all the $\binom{n}{2}$ pairs must be used. Gastwirth, however, suggested an approximation of the measure when the data are arranged in (k+1) groups with n_i observations in group i and suppose that λ_i, the mean of the ith group is also given. Then G_1 is approximated by

$$\hat{G}_1 = 2 \sum_{i<j} \frac{|\lambda_i - \lambda_j|}{(\lambda_i \quad \lambda_j)} \cdot \frac{n_i}{n} \cdot \frac{n_j}{n} \qquad \ldots \qquad (1.25)$$

In the ungrouped case the measure meets the principle of transfers. But in the grouped case the measure ignores income variations within a group.

(j) The Theil Entropy Measure

The Theil measure of inequality [Theil (1967)] derived from the notion of entropy in information theory is given by

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cases is given by $\bar{I}(\alpha) = \frac{1}{\lambda} \left(\frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|^{\alpha} \right)^{1/\alpha}$.

The class of measures satisfies the principle of transfers for $\alpha \geq 1$ [see Allison (1978)].

$$T = \sum_{i=1}^{n} x_i \log(nx_i) \qquad \ldots \qquad (1.26)$$

where $x_i = \frac{y_i}{n\lambda}$, the income share of unit i. T takes its minimum value zero when $x_1 = x_2 = \ldots = x_n = \frac{1}{n}$; and its maximum value log n when $x_i = 1$ for some i and $x_j = 0$ for all $j \neq i$. While for V^2 the effect of a transfer of income between two units depends on the difference between the two incomes, for T it depends approximately on the ratio of these incomes. The measure is thus more sensitive to transfers at lower income levels than at higher income levels. However, the Theil formula has been regarded as somewhat arbitrary [<u>vide</u> Sen (1973), pp. 35-36]. Therefore it will be interesting to examine more carefully the logical foundations of this measure. An attempt to do this is made in Chapter 5 of the thesis.

Sometimes it is assumed that some continuous type distribution fits the personal income distribution. In such cases some parameter appearing in the density of the fitted distribution is considered as a measure of inequality. A notable example of this approach is Pareto's α [Pareto (1897)]. α is the slope of the line showing the comulative frequency of persons with incomes above each stated level plotted on a double

-log scale against the size of income. Samuelson (1965) show that within the Pareto —— Levy family [vide Mandelbrot(1960) the coefficient α is not a valid measure of inequality in the usual sense of the word.[8] Chipman (1974) confines himsel to the Pareto —— Levy family and shows that for distribution with the same subsistence level an increase in α actually decreases social welfare. The result is due to the lowering of the mean income with increase in α which outweighs the effect of the associated reduction of inequality. Chipman prov this result assuming that the social welfare function is in essence a sum of individual utilities and that common utility function for all individuals is twice continuously differentiable, nondecreasing and concave. Gibrat (1931) demonstrated that in many situations income follows three-parameter lognormal distribution [vide Aitchison and Brown (1957)] and pro-

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8/ Samuelson (1965) considered the class of stable (Pareto —— Levy) distributions, which is a four-parameter family with location parameter μ, scale parameter β, skewness paramete γ and kurtosis parameter α. The parameter α ranges from 2 for the normal distribution to 1 for the Cauchy distributio It is known [Feller (1971), p. 576] that the stable densitie have the property $f(x \mid \mu, \beta, \gamma, \alpha) = O(x^{-\alpha-1})$ as $x \to \infty$ hence if the skewness parameter γ is positive and sufficien large, the stable distributions are well approximated in the upper tail by the Pareto distributions with Pareto coefficient α, as long as α is not too close to 2 [cf. Mandelbrot (1960].

posed standard deviation of $\log(x - x_o)$ as a measure of inequality, where x_o is the threshold parameter. This is clearly a measure of inequality of the absolute variety.

With this we conclude the review of the wellknown positive measures of inequality.[9/] In the subsequent discussion, we will take into account only the coefficient of variation, the standard deviation of logarithms, the Gini coefficient and the Theil measure.[10/]

1.1.3.2 The Normative Measures

Dalton (1920) rightly pointed out that the choice of any inequality measure involves an implicit normative judgement as to whether one income profile is to be preferred in some sense to another.[11/] He then argued that the normative criteria concerning measures of inequality should be made explicit through the use of a social welfare function (swf)

9/ Some of these measures have interesting large sample properties [Iyengar (1960), Gastwirth (1974a), McDonald and Jensen (1979)].

10/ The estimation problems associated with these measures have been discussed by different authors [vide Gastwirth (1972, 1975), Kakwani and Poddar (1973, 1976), Mehran (1975), Gastwirth and Glauberman (1976), Kakwani (1976), Krieger (1979), Petersen (1979)].

11/ On the other hand, it has sometimes been argued that normative exercises involve some ambiguities and answer questions different from the purely descriptive one that was originally asked [vide Bentzel (1970), Esteban (1976), Hansson (1977) and Sen (1978)].

which simply ranks all possible states of the society (i.e., income profiles) in the order of the society's preference.[12,13] This approach makes the following prescription : From individual preferences we should derive a social ranking of different social states, and hence some implicit measure of inequality based on social values.

We now discuss the particular measure that Dalton suggested.

(a) Dalton's Measure

Let us assume that U_i denotes the utility function of the ith unit and that it depends only on its income y_i. Further assume that $U_i(.) = U(.)$ and also that U is increasing and concave. Dalton chose the utititarian form of swf in which the sum of individual utitities is taken as a measure

12/ In the inequality literature the swf is often called the social evaluation function [Kolm (1976a), Blackorby and Donaldson (1980, 1980a)].

13/ In contrast to Arrow's (1951) classical demonstration of the non-existence of a swf based on a set of mild-looking restrictions, Sen (1973) states that if the approach of swf is to give us any substantial help in measuring inequality, then the framework must be broadened to include interpersonal comparisons of welfare. Hammond (1976) shows that interpersonal comparisons allow a very much richer set of possible social orderings and the construction of normative measures of inequality. Alternative frameworks for interpersonal comparability were explored in Sen (1970). Also see Sen (1972).

of social welfare. Therefore welfare value of an income profile $\underline{y}$ is given by

$$W(\underline{y}) = \sum_{i=1}^{n} U(y_i) \qquad \ldots \quad (1.27)$$

By Jensen's inequality we have $nU(\lambda) \geq W(\underline{y})$ for all $y \in Y \ni \sum_{i=1}^{n} y_i = n$

Taking the utility levels to be all positive, Dalton's measure of inequality is given by

$$D = 1 - \frac{\sum_{i=1}^{n} U(y_i)}{n\,U(\lambda)} \qquad \ldots \quad (1.28)$$

The measure tells us by how much (in relative terms) we can increase social welfare by distributing incomes more equally among the units. This approach was largely ignored by later researchers until Aigner and Heins (1967) proposed some alternative functions for W and derived corresponding measures of inequality. However, many of Dalton's ideas were really revived with the publication of the seminal article by Atkinson [Atkinson (1970)].

Atkinson (1970) pointed out that Dalton's measure is not invariant under affine transformations of U(.). He modified Dalton's measure to remedy this defect.

(b) Atkinson's Measure

Atkinson defined the 'equally distributed equivalent (ede) income' of a given profile of a total income as that level of income which if enjoyed by every unit would make total welfare exactly equal to the total welfare generated by the actual profile.[14]

Following Dalton he restricted his attention to the utilitarian form of swf. Let us denote the ede income corresponding to a given $\underline{y}$ by y_e. Then

$$y_e = y \mid nU(y) = \sum_{i=1}^{n} U(y_i) \quad \ldots \quad (1.29)$$

By concavity of $W(\underline{y})$, $y_e \leq \lambda$.

Atkinson replaced the Dalton measure of inequality by

$$A = 1 - \frac{y_e}{\lambda} \quad \ldots \quad (1.30)$$

14/ An earlier use of the concept of 'ede income' can be found in Champernowne (1952), where one of the measures proposed was 'the proportion of total income that is absorbed in compensating for the loss of aggregate satisfaction due to inequality' (op.cit., p. 610). Kolm (1969) called 'ede income' the 'equal equivalent income', whereas in the subsequent literature [Blackorby and Donaldson (1980, 1980a)] the ede income of a given profile has been referred to as the "representative income". The idea of equating the total welfare of the profile where every unit enjoys a particular level of income to the total welfare of the given profile has been called ethical or social indifference. The vector of ede incomes $(y_e, y_e, \ldots, y_e)'$ is considered to be ethically or socially indifferent to the given income vector $(y_1, y_2, \ldots, y_n)'$.

The measure has the convenient property of lying between zero (complete equality) and unity (complete inequality).

Drawing upon results obtained by Pratt (1964) and Arrow (1965) in another context, Atkinson showed that A satisfies the postulate of scale invariance if and only if U(.) is of the form

$$U(y) = C + D\frac{y^{1-\epsilon}}{1-\epsilon} \quad \text{for} \quad \epsilon \neq 1$$

$$= \log y \quad \text{for} \quad \epsilon = 1 \qquad \ldots \quad (1.31)$$

where $\epsilon \geq 0$ for concavity. Using the form of U(.) given by (1.31), we get

$$A = 1 - \frac{\left(\frac{1}{n}\sum_{i=1}^{n} y_i^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}}}{\lambda}, \quad \epsilon \neq 1$$

$$= 1 - \frac{\prod_{i=1}^{n} y_i^{1/n}}{\lambda}, \quad \epsilon = 1 \qquad \ldots \quad (1.32)$$

Here ϵ is said to represent the degree of inequality aversion or the relative sensitivity of A to transfers of income at different income levels. As ϵ rises, greater weight is attached to transfers at the lower end of the profile and less weight to transfers at the top. As $\epsilon \to \infty$, $A \to 1 - \frac{\min_i \{y_i\}}{\lambda}$, which corresponds to the Rawlsian maximin rule [Rawls (1958, 1971)], that ranks social states in terms of the welfare of the worst-off unit in the state. The association of Rawls' maximin rule with a swf exhibiting extreme inequality aversion is discussed in Arrow (1973), Sen (1974) and Hammond (1975).[15]

Atkinson in the same paper made an extremely interesting application of his measure to inter-country comparisons of income inequality. Atkinson's approach and his measure have also been used by others to study various problems of inequality measurement [Allingham (1972), Bruno (1974), Muellbauer (1974), Singh (1974), Harrison (1975), Bartels and Nijkamp (1976), Bruno and Habib (1976), Williamson (1977), Shorrocks (1978), Ulph (1978), Von Weizsacker (1978), Birchenhall and Grout (1979) and Kanbur (1979)].

However, the form of swf assumed by Atkinson seems to be unnecessarily restrictive. It can be substantially generalised and this would lead to a more general normative measure of inequality based on the ede income approach. To see this,

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[15] Maximin is not, however, the only possible limit as $\epsilon \to \infty$. Sen's lexicographic extension of the maximin criterion [Sen (1970a, 1977, 1977a)] is also a possible limit, as $\epsilon \to \infty$, of a preference ordering represented by the swf underlying A. [See Hammond (1975)].

we come to the formulation adopted by Sen (1973).

(c) Sen's Approach

Sen (1973) adopted a more general form of swf W defined on y_i's. He assumed that W satisfies the following conditions :

(A1) W is increasing in individual incomes.[16]

(A2) W is symmetric in $y_1, y_2, \ldots, y_n$; i.e.,

$$W(P\underline{y}) = W(\underline{y}) \qquad \ldots \qquad (1.33)$$

for any permutation matrix P of order n.

(A3) W is quasi-concave ; i.e.,

$$W[\theta \underline{x} + (1-\theta)\underline{y}] \geq \text{Min}[W(\underline{x}), W(\underline{y})] \qquad (1.34)$$

for all x, y $\in$ Y and for all θ, $0 < \theta < 1$.

For strict quasi-concavity the sign $\geq$ should be replaced by $>$. Quasi-concavity is weaker than concavity. To incorporate the egalitarian bias into distributional judgements it is sufficient

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16/ A swf W is called benevolent or non-malevolent according as $\frac{\delta W}{\delta y_i} > 0$ or $\frac{\delta W}{\delta y_i} \geq 0$, for all i = 1, 2, ..., n [Kolm (1976a)].

to consider strict quasi-concavity.[17/]

Sen (1973) defines the generalised ede income y_f as that level of income which if enjoyed by every unit would produce the same level of W as that generated by the actual profile, that is,

$$y_f = y \mid [W(y, y, \ldots, y) = W(y_1, y_2, \ldots, y_n)] \quad \ldots \quad (1.35)$$

Sen then defines a more general measure of inequality as

$$S = 1 - \frac{y_f}{\lambda} \quad \ldots \quad (1.36)$$

From now on the ede income y_f will be called the Atkinson — Kolm — Sen (AKS) representative income, and the index S will be referred to as the AKS relative inequality index.

The measure S is scale invariant if and only if W is homothetic [Muellbauer (1974a), Blackorby and Donaldson (1978); see also Atkinson (1970)]. This means W should be

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17/ In fact further weakening of the assumption is possible; it is sufficient to assume S-concavity of the swf. A swf W is said to be S-concave if $W(Q\underline{y}) \geqq W(\underline{y})$ for all $\underline{y}$ and for all bistochastic matrices Q of order n. W is strictly S-concave if the inequality is strict whenever the vector $Q\underline{y}$ is not a permutation of $\underline{y}$. It can be shown that [Berge (1963)] S-concavity implies symmetry; and that quasi-concavity and symmetry imply S-concavity but the converse is not true.

of the form

$$W(\underline{y}) = \phi[\bar{W}(\underline{y})] \quad \ldots \quad (1.37)$$

where $\bar{W}(\underline{y})$ is positively linearly homogeneous and ϕ is increasing in its argument. Therefore, to every homothetic swf of the form (1.37) there corresponds a different relative inequality index.

(d) Kolm's Absolute Measure

Kolm (1976) suggested the following absolute measure of inequality :

$$K = \frac{1}{\alpha} \log\left[\frac{1}{n} \sum_{i=1}^{n} e^{\alpha(\lambda - y_i)}\right] \quad \ldots \quad (1.38)$$

where $\alpha > 0$. The index K is increasing in α. As α increases we attach more weight to income transfers at the lower end of the income profile and less to transfers at the top. As $\alpha \to \infty$, $K \to \lambda - \min_i \{y_i\}$, the absolute maximin index.

Blackorby and Donaldson (1980) investigate the swf underlying K by the procedure outlined below:

They define a continuous, nondecreasing, S-concave swf W on Y. Let y_e be the AKS representative income corresponding to a given $\underline{y}$ according to W. They define an absolute inequality measure analogous to the AKS relative measure by

$$A(\underline{y}) = \lambda - y_e \quad \ldots \quad (1.39)$$

A is translation invariant if and only if W is translatable, i.e.,

$$W = \emptyset [\bar{W} (\underline{y})] \quad \ldots \quad (1.40)$$

where $\emptyset$ is increasing in its argument and $\bar{W}(\underline{y})$ is unit-translatable, i.e.,

$$\bar{W} (\underline{y} + \theta \underline{1}) = \bar{W} (\underline{y}) + \theta \quad \ldots \quad (1.41)$$

where θ is any scalar such that $\underline{y} + \theta \underline{1}$ is in the domain of definition of A.[18/] It is clear that to every translatable swf of the form (1.40) there corresponds a different absolute inequality index.

Pollak (1971) established the form of utility functions that are additively separable and homothetic to minus infinity. The simplest symmetric representation is

$$W_P (\underline{y}) = - \sum_{i=1}^{n} e^{-\alpha y_i} \quad \ldots \quad (1.42)$$

where $\alpha > 0$.

We can write this function as an increasing transformation of $\bar{W}_K (\underline{y})$ where

$$\bar{W}_K (\underline{y}) = - \frac{1}{\alpha} \log[\frac{1}{n} \sum_{i=1}^{n} e^{-\alpha y_i}] \quad \ldots \quad (1.43)$$

18/ The property of translatability can be thought of as being 'homothetic to minus infinity' [Chipman (1965), Pollak (1971)].

the translation function in (1.40) associated with K. It can be seen that the Blackorby — Donaldson absolute index for the particular swf (1.42) is the same as Kolm's measure. From now on K will be called the Kolm — Pollak measure of inequality and the corresponding swf, the Kolm — Pollak swf.

(e) Van Praag's Log-Marginal-Welfare Variance

Let $U(y_i)$ stand for the ith units utility function and assume $U(y_i)$ depends only on y_i. Van Praag (1977, 1978) suggests

$$VP = \frac{1}{n} \sum_{i=1}^{n} [\log U'(y_i) - \overline{\log U'}]^2 \quad \ldots \quad (1.44)$$

as a measure of welfare inequality. Here $U'(y_i)$ is the marginal utility of unit i and $\overline{\log U'}$ is the mean of log-marginal-utility. Depending on the form of U(.) the measure is scale invariant or translation invariant.

Thus, if $$U(y_i) = 1 - \exp(-\alpha y_i) \quad \ldots \quad (1.45)$$

then $$VP = \frac{\alpha^2}{n} \sum_{i=1}^{n} (y_i - \lambda)^2 \quad \ldots \quad (1.46)$$

Again if $U(y_i)$ is given by (1.31), then

$$VP = \frac{\epsilon^2}{n} \sum_{i=1}^{n} (\log y_i - \mu)^2 \quad \ldots \quad (1.47)$$

where μ = mean log-income.

The merits and demerits of indices in (1.46) and (1.47) are wellknown and need not be recapitulated. According to Sen (1978a), the view of welfare based on comparison of marginal utilities is rather limited, since it is concerned with concepts of optimality rather than inequality. Anyway, Van Praag's formulation throws some light on a number of existing measures of relative and absolute inequality.

We may now consider the normative aspects of different positive measures of inequality, because 'even if we take inequality as an objective notion, our interest in its measurement must relate to our normative concern with it' [Sen (1973), p. 3].

1.1.3.3 Positive Measures and their Meaning in terms of Social Welfare.

Blackorby and Donaldson (1978) using the formulation adopted by Sen (1973) [*vide* 1.1.3.2c *supra*] established continuous, monotonic, S-concave homothetic swf's that are implied by a variety of inequality measures which have been used in the literature. The swf's that correspond to the coefficient of variation, the Theil index (both appropriately normalised) and the Gini index have the following images respectively :

$$\bar{W}_{CV}(\underline{y}) = \frac{1}{n} \sum_{i=1}^{n} y_i - \left[\frac{1}{n} \sum_{i=1}^{n} (y_i - \lambda)^2 \right]^{1/2} \quad \ldots \quad (1.48)$$

$$\bar{W}_T(\underline{y}) = \frac{1}{n \log n} \left[n\lambda \log(n\lambda) - \sum_{i=1}^{n} y_i \log y_i \right] \quad (1.49)$$

$$\bar{W}_G(\underline{y}) = \frac{1}{n^2} \sum_{i=1}^{n} (2i - 1) y_i \quad \ldots \quad (1.50)$$

where $y_1 \geqq y_2 \geqq \ldots \geqq y_n$.

However, it would perhaps be more interesting to axiomatise these indices as measures of inequality rather than look at the existence of swf's that are implied by them. This existence is in fact obvious from S-convexity of the formulae 1.19, 1.26 and 1.23. The axiomatisation for the Theil measure has been carried out in Chapter 5. The corresponding work for the Gini coefficient was done by Sen (1973a, 1974).

Weymark (1979) replaces the coefficients in (1.50) by general coefficients $a_1^n, a_2^n, \ldots, a_n^n$ where $a_i^n > 0 \ \forall_i$ and $a_1^n \leqq a_2^n, \ldots, \leqq a_n^n$. This procedure yields a class of generalised Gini indices with representative income given by

$$E_G^n(\underline{y}) = \frac{\sum_{i=1}^{n} a_i^n y_i}{\sum_{i=1}^{n} a_i^n} \quad \ldots \quad (1.51)$$

The family of measures based on $E_G^n(\underline{y})$ donot, in general, satisfy the principle of population. However, Donaldson and Weymark (1980) demonstrate that in a special case $E_G^n(\underline{y})$ satisfies the principle of population if and only if

$$a_m^n = \frac{g(\frac{m}{n}) - g(\frac{m-1}{n})}{g(\frac{1}{n})} \quad \ldots \quad (1.52)$$

where g is defined on non-negative rational numbers with g(0) = 0 and <u>either</u> (a) g is an arbitrary convex function with g(x) > 0 for x > 0 <u>or</u> (b) g(x) is an arbitrary concave function with g(x) < 0 for x > 0.

If in particular we write $g(x) = x^\delta$, $\delta \geq 1$ in (1.52), then we get the representative income

$$E_\delta(\underline{y}) = \frac{1}{n^\delta} \sum_{i=1}^{n} [i^\delta - (i-1)^\delta] y_i \quad \ldots \quad (1.53)$$

This defines a single-parameter family of measures analogous to the Gini coefficient.

But $E_\delta(\underline{y})$ is not strictly quasi-concave on individual incomes and the ranking of income profiles by the AKS relative inequality index corresponding to $E_\delta(\underline{y})$ cannot be reflected by any swf (additive or not) if the latter is strictly quasi-concave on individual incomes. This is a generalisation of what is known about ranking based on the Gini index[19] which is a special case of the AKS index corresponding to $E_{\delta}(\underline{y})$.

Finally, we may mention that the standard deviation of logarithms, a commonly used measure of inequality, is not S-convex.

1.1.3.4 Significance of Lorenz Curve Comparisons

To discuss the significance of comparisons of Lorenz curves of different income profiles, let xLy stand for the relation between the two profiles $\underline{x}$ and $\underline{y}$ when the Lorenz curve of $\underline{x}$ is nowhere outside that of $\underline{y}$ and at some places (at least) strictly inside the latter. Sometimes this is mentioned as $\underline{x}$ strictly Lorenz dominates $\underline{y}$, while $\underline{x}$ Lorenz dominates $\underline{y}$ means Lorenz curve of $\underline{x}$ is nowhere outside that of $\underline{y}$. A remarkable consequence of such a relation was proved by Atkinson (1970). The result says:

If we have two income profiles $\underline{x}$ and $\underline{y}$ over the same number of units with the same mean income, then xLy implies that the $\underline{x}$-profile gives a higher social welfare level (as measured by $W(\underline{x}) = \sum_{i=1}^{n} U(x_i)$ where U is any strictly

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19/ See Dasgupta, Sen and Starrett (1973) and Rothschild and Stiglitz (1973).

concave utility function) than the y- profile without knowing which precise U function is used. The converse is also true, i.e., if we have W(x) > W(y) irrespective of which U function is chosen (U is strictly concave), then xLy.[20] But if the two curves intersect we can find two strictly concave U functions that will rank x and y differently in regard to the level of social welfare.

Atkinson's choice of the form of W may seem to be quite restrictive. However, the above result has been considerably generalised by Dasgupta, Sen and Starrett (1973) and by Rothschild and Stiglitz (1973). The generalised versions donot require W to be a sum of individual U functions. W may be defined directly on the income profile. Dasgupta, Sen and Starrett (1973) take swf F to be any strictly S-concave function, and prove that, if for two different profiles x and y over the same number of units and with the same mean income we have xLy, then F(x) > F(y). The converse is also true. And if *not* xLy, then for some F, F(x) ≤ F(y).[21]

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[20] This condition of Lorenz domination is also equivalent to the condition that x is obtained from y by a finite sequence of transformations where each transformation transfers some income from a richer unit to a poorer unit.

[21] This result will be referred to as Lorenz quasi-ordering and the corresponding weak version as weak Lorenz quasi-ordering.

So far the results mentioned apply to comparisons of income profiles over the same number of units. To compare income profiles with different population sizes, Dasgupta, Sen and Starrett (1973) propose the following axiom:

If r communities with the same number of units and identical income profiles are considered together, then the mean welfare of the whole must be equal to the mean welfare of each part. With this axiom they prove that, given the level of mean income, Lorenz domination implies a higher mean welfare level even for variable population sizes.

Rothschild and Stiglitz (1973) demonstrate that for two income profiles $\underline{x}$ and $\underline{y}$ over the same number of units and with the same mean income $\underline{x}$ Lorenz dominates $\underline{y}$ is equivalent to saying that $\underline{x}$ is preferable to $\underline{y}$ under all real-valued, locally equality preferring monotonic functions W which satisfy $W(z) \geq W(\pi(z))$ for π any permutation of z.[22/]

22/ A function $W(\underline{x})$ is locally equality preferring if for every vector $x \in Y$

$$W(\underline{x}) \leq W(\alpha\, \underline{y} + (1 - \alpha)x) \quad \text{for} \quad 0 \leq \alpha \leq 1$$

where $y_i = x_i, \quad i \neq k, \ell.$

$$y_k = y_\ell = \frac{x_k + x_\ell}{2}.$$

It is clear that locally equality preferring is a weaker concept than quasi-concavity, for while every quasi-concave function is locally equality preferring the converse is not true.

Therefore from the above results we can say that given the mean income, if x Lorenz dominates y, we can be sure that x is preferable to y under a broad class of social welfare functions. But if the two Lorenz curves intersect, then the welfare-ranking of the two profiles is no longer unambiguous. In this sense, the Lorenz curve comparisons lead to a partial ordering of income profiles. These theorems provide the theoretical foundation for the Lorenz curve and the associated measures of inequality.[23/]

Before concluding this brief review, we may mention that some of the measures discussed above have been used by different authors interested in decomposition of the overall degree of inequality. In this context, two particular applications stand out. The first concerns a partition of the population into disjoint subsets, such as groups by age, sex, race, region, etc., and the researcher is interested in examining how the overall degree of inequality can be appropriately resolved into contributions due to (i) inequality within each of the groups and (ii) inequality between groups, that is, due

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23/ Actually, these measures computed from empirical data tend to overestimate the actual inequality of income profiles. This is because the observed incomes deviate from true incomes due to the presence of errors in observations. This is demonstrated in Appendix I, under certain realistic assumptions regarding the errors.

to variation in average levels of income across these groups [Soltow (1960), Bhattacharya and Mahalanobis (1967), Theil (1967, 1972), Love and Wolfson (1970), Mehran (1974, 1975a), Mangahas (1975), Paglin (1975), Pyatt (1976), Blackorby, Donaldson and Auersperg (1978), Handerson and Rowley (1978), Bourguignon (1979), Murray (1979), Cowel (1980), Shorrocks (1980)]. The other main application disaggregates total income of each unit into amounts earned from different sources (or factor components) and examines the impact of each of these sources on the overall degree of inequality [Mahalanobis (1960), Rao (1969), Kakwani (1977), Fei et al (1978, 1979), Fields (1979, 1979a), Layard and Zabalza (1980), Pyatt, Chen and Fei (1980), Shorrocks (1980a)].

CHAPTER 1

SURVEY OF LITERATURE - II : MEASUREMENT OF POVERTY

1.2.1 Introduction

In the decades since the end of World War II, lot of attention has been paid by economists to the problem of development of the third world countries and the associated problems of poverty. Even in the developed countries poverty remains one of the major issues of current economic and social policy. To understand the threat that the problem of poverty poses, it is necessary to know its dimensions and the process through which it seems to be perpetrated. In the present dissertation we are interested solely in the quantification of poverty, and this part of this chapter makes a review of existing methodological work in this area.

The quantification or the measurement of poverty requires the solution of two distinct problems, viz., (a) the problem of identification of the poor among the total population and (b) the problem of aggregation which requires some method of combining the degrees of deprivation of different poor units into an overall indicator. The problem of identification involves the selection of an appropriate poverty line. In the literature on poverty, broadly, there are two approaches regarding definition of a poverty line. The first approach makes an attempt to define a poverty line in terms of an absolute standard that represents a minimum standard of living Many experts, however, doubt the possibility of one's being able to define a poverty line (representing subsistence level) in this absolute sense. They hold the view that the only meaningful concept of a poverty line is a relative one, where the line is defined in relation to social conventions and contemporary living standards of a society.24/ In either approach a unit is said to be poor if its income is below the poverty line. The second problem, the problem of aggregation is the main issue of our discussion. There are various methods for combining the deprivations of the poor units into a single indicator. In the next section we shall briefly discuss these procedures.25/

24/ The problems regarding determination of an appropriate poverty line are discussed in Rowntree (1901), Orshansky (1965), Rein (1968), Atkinson (1969, 1975), Kilpatrick (197 Rainwater (1974), Rudra (1974), Goedhart, Halberstadt, Kapteyn and Van Praag (1977), Riffault and Rabier (1977), Srinivasan (1977), Sen (1979) and Van Praag, Goedhart and Kapteyn (1980).

25/ Discussions along these lines can also be found in Sen (197 Chakravarty (1980), Thon (1980a, 1980b).

1.2.2 A Review of Poverty Measures

To discuss various poverty measures proposed in the literature, let us assume that the poverty line $z > 0$ is given exogenously. We further assume that out of n units in the society q units are poor ($q \leq n$). For convenience, we assume that the incomes are arranged in non-decreasing order, i.e.,

$$y_1 \leq y_2 \leq \ldots \leq y_q < z \leq y_{q+1} \leq \ldots \leq y_n \quad \ldots \quad (1.54)$$

1.2.2.1 The Standard Measures

The simplest and most widely used measure of poverty is given by the proportion of the total population that happens to fall below the specified poverty line z. This index, known as the 'head-count ratio', is written

$$H = \frac{q}{n} \quad \ldots \quad (1.55)$$

Ever since the quantification of poverty began, the head-count ratio has been used explicitly or by implication [Booth (1889), Rowntree (1901)]. The measure seems to be still the mainstay of poverty statistics on which poverty programmes are based [see Orshansky (1965, 1966), Abel-Smith and Townsend (1965)]. The measure has also been used to study the trends in the incidence of poverty [See, for example Indian Studies on poverty. Ojha (1970), Dandekar and Rath (1971), Minhas (1970, 1971), Bardhan (1970, 1971, 1973), Mukherjee, Bhattacharya and Chatterjee (1972), Vaidyanathan (1974) and Lal (1976)]. For international comparisons of poverty this measure has been used by Chenery, Ahluwalia, Bell, Duloy and Jolly (1974).

In his pioneerting article Sen (1976) sought to treat the problem of measurement of poverty from the welfare theoret approach as has been adopted for the measurement of inequality He raised objections against the measure H on the ground that it violates both the following axioms:

Monotonicity Axiom (M)

Given other things, a reduction of income of a unit below the poverty line must increase the poverty measure.

Transfer Axiom (T)

Given other things, a pure transfer of income from a unit below the poverty line to anyone who is richer must increase the poverty measure.

Sen (1976) raised objections against another standard measure of poverty, the 'poverty gap', Pg, which is defined as

$$P_g = \sum_{i=1}^{q} (z-y_i) \quad \ldots \quad (1.56)$$

on the ground that although it satisfies the monotonicity axiom it violates the transfer axiom.26/

1.2.2.2 More Refined Measures of Poverty

We shall now concentrate on measures that were proposed keeping the axioms mentioned above in mind. Following Sen (1976, 1979) and Hamada & Takayama (1977) we may add another axiom which one can intuitively expect a poverty measure to satisfy:

Axiom M' :

Income variations of any unit above the poverty line donot change the poverty measure unless the unit falls below the poverty line.

(a) Sen's Measure

Sen (1976) can be regarded as the first rigorous work on the conceptual problems connected with the measurement of poverty. Sen takes the view that the two basic axioms noted above will be satisfied anyway by incorporating a more demanding axiomatic structure. Starting from a specified set of axioms he arrives at a new measure of poverty. His measure P_S

26/ This measure has been used by the Social Security Administration of the U.S.A. to study the incidence of poverty [Batchelder (1971)]. A discussion of the poverty gap approach can be found also in Beckerman (1977).

is the normalised weighted sum of the income gaps $(z-y_i)$ of all the units below the poverty line, the weight of $(z-y_i)$ being the rank order of i in the interpersonal welfare ordering of the poor.27, 28, 29/ In symbols, Sen's measure30/ is written

$$P_S = \frac{2}{(q+1)nz} \sum_{i=1}^{q} (z-y_i)(q+1-i) \quad \ldots \quad (1.57)$$

The measure has already been used extensively to study the incidence of poverty in different countries [See, for example, Bhatty (1974), Seastrand and Diwan (1975), Alamgir (1976), Anand (1977), Kakwani (1977a), Ahluwalia (1978), Dutta (1978), Osmany (1978), Sastry (1978) and Thon (1981)].

27/ This weighting rule is in the same spirit as Borda's (1781) famous rank order method of decisions, choosing equal distances in the absence of a convincing case for any alternative assumption. The rank order weighting has been used extensively in voting theory [See, for example, Black (195[illegible]) Fishburn (1973), Hansson (1973), Fine & Fine (1974), Gardenfors (1974)].

28/ The approach of attaching greater weight to the income shortfall of a poorer unit than to the income short-fall of a richer unit relates closely to the evaluation of real national income in terms of 'named good vectors' presented in Sen (1976a).

29/ This weighting scheme is taken in Sen (1976) as an axiom, though it is easy to derive it from more primitive axioms [See, Sen (1973a, 1974)].

30/ A game theoretic interpretation of the measure was provided by Pyatt (1980).

The measure satisfies the monotonicity axiom. But if in the process of the kind of transfers considered in axiom T, the recipient unit goes above the poverty line, then the measure may not change in the desired direction, i.e., the measure may decrease instead of increasing. This happens due to the weighting scheme that Sen proposes. If one changes the weight of the income gap $(z-y_i)$ from $(q+1-i)$ to $(n+1-i)$, the rank order of the unit i in the total population and also makes a slight modification of Sen's normalisation axiom, then one gets a measure that satisfies the transfer axiom in all cases.[31]

Sen (1979) however argues that the behaviour of P_S described above may not be regarded as a shortcoming. His argument is given in the next paragraph. He proposes the following modified version of the transfer axiom[(Sen, 1977b)]:

Given other things, a pure transfer of income from a unit below the poverty line to anyone who is richer must increase the poverty measure unless the number of units below the poverty line is strictly reduced by the transfer.

We denote the modified version of the transfer axiom by T'. P_S satisfies this axiom, but not the original

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31/ This is demonstrated in Chapter 2, where this modification of Sen's measure is obtained as a special case of a class of measures.

axiom T. Obviously if a transfer of the type mentioned above is considered there are two underlying effects. One is that the donor of the transfer is becoming poorer and the other is that the recipient is becoming richer. Sen (1979) in this context says : 'In so far as the index of poverty is interpreted to represent the condition of the poor in the nation – their prevalence and their penury – a good case can perhaps be made for permitting the possibility that a reduction of prevalence of poverty might under some circumstances compensate a rise in the extent of penuary of those who remain below the poverty line' (op. cit., p. 302). The measure P_S admits this possibility.

This view of Sen can naturally be controverted. Suppose we wish to rank the income profiles in terms of the welfare of the worst-off unit [Rawls' maximin criterion [Rawls (1958, 1971)]. Then evaluated by P_S a transfer from the worst-off unit to the richest unit below the poverty line (assume that it crosses the poverty line as a result of the transfer) can very well indicate an increase in welfare. This is clearly a violation of the maximin criterion.

(b) Anand's Measure

Anand (1977) proposes a measure to assess the society's potential ability to eliminate poverty. The difficulty of alleviation of poverty in a given society may depend on a large number of factors other than income and any attempt at measuring it by concentrating only on the existing income profile is then obviously defective. Let us denote Anand's measure by P_A. Then P_A is defined :

$$P_A = \frac{z}{\lambda} \cdot P_S \qquad \ldots \qquad (1.58)$$

As a measure of poverty P_A is not a suitable candidate at all, since it has got a number of defects. P_A violates axiom M'. It violates axiom M also. With respect to the transfer axiom the measure has got the same defect as Sen's measure.

(c) Kakwani's Measures

Kakwani (1980) derives several poverty indices, investigates the effect of negative income tax schemes with the help of these indices and gives a numerical illustration based on Malaysian data [Some of these measures were presented earlier in Kakwani (1977a)].

Assume that the income a unit is a random variable X with distribution function F. Let λ_p be the mean income of the poor. Kakwani proposes

$$P_K = F(z)\left[\frac{z - \lambda_p}{\lambda}\right] \qquad \ldots \qquad (1.59)$$

as a measure of poverty.

But P_K is completely insenstitive to the transfer of income so long as both the recipient and the donor of the transfer are below the poverty line. It also violates axioms M' and M

Kakwani (1980) presented another class of poverty measures

$$P_f = \frac{F(z)}{\lambda}[z - \lambda_P f(G_P)] \quad \ldots \quad (1.60)$$

where G_P is the Gini coefficient of the income profile of the poor and $f(G_P)$ is a monotonic function G_P such that $0 \leq f(G_P) \leq$ $f(G_P) = 1$ if $G_P = 0$, $f'(G_P) < 0$.

Two particular cases of P_f are given and used in the empirical illustration:

$$P_1 = \frac{F(z)}{\lambda}[z - \lambda_P(1 - G_P)] \quad \ldots \quad (1.61)$$

$$P_2 = \frac{F(z)}{\lambda}\left[z - \frac{\lambda_P}{1 + G_P}\right] \quad \ldots \quad (1.62)$$

Both P_1 and P_2 can be shown to violate the transfer axiom when the recipient crosses the poverty line. They also violate axioms M and M'.

In another paper, Kakwani (1980a) modifies Sen's axiom of ordinal rank weights to provide a more general structure than Sen's axiom would permit. Essentially, Kakwani's axiom makes the weight on income short-fall of a unit equal to the rth power $(r \geq 1)$ of the income rank of that unit among the poor. The class of measures proposed is:

$$P(r) = \frac{q}{n\, z\, \theta_q(r)} \sum_{i=1}^{q} (z - y_i)(q + 1 - i)^r \quad \ldots (1.63)$$

where $$\theta_q(r) = \sum_{i=1}^{q} (i)^r \quad \ldots (1.64)$$

Sen's measure corresponds to the special case where $r = 1$.

The motivation for introducing $P(r)$ is to enable it to satisfy alternative axioms about transfer sensitivity. The sensitivity axioms considered are:

Axiom of Monotonic Sensitivity (M.S.)

If $(\Delta P)_i$ represents the increase in the poverty measure due to a small reduction in the income of the ith poor unit (by a fixed amount $\delta > 0$), then $(\Delta P)_i > (\Delta P)_j$ if $j > i$.

Axiom of Transfer Sensitivity I (T. S. I)

For any positive integer $h > 0$ and any pair of units ranked i and j, if $j > i$, then $(\Delta P)_{i,\, i+h} > (\Delta P)_{j,\, j+h}$

where $(\Delta P)_{i,\ i+h}$ is the increase in poverty measure due to a transfer of a fixed amount of income from the ith poor unit to the (i+h)th poor unit.

The axiom implies that the sensitivity of the poverty measure depends on the position of the donor of the transfer in the ordering of the poor units when the rank difference between the donor and the recipient is fixed. The poorer the donor, the greater should be the increase in the poverty measure.

Axiom of Transfer Sensitivity II (T.S.II)

If a transfer of income takes place from the ith poor unit with income y_i to a unit with income $y_i + h$, then for a given $h > 0$, the magnitude of increase in poverty measure decreases as i increases.

This axiom gives more weight to transfers of income at the lower end of the profile than at the higher end. There is considerable discussion of a similar axiom in the literature of income inequality [Dalton (1920), Atkinson (1970), Sen (1973), Kolm (1976, 1976a)].

The index P(r) clearly satisfies the axioms M.S. and T.S.I. However, one might object to T.S.I as a desirable axiom and choose T.S.II in its place. In this case, it is quite unlikely that an arbitrary value of $r > 1$ in (1.63) will guarantee the required sensitivity for all income profiles. For a given profile a value of r can of course be found which ensures that $P(r)$ meets T.S.II, but as r is supposed to measure preferences regarding relative transfer sensitivity as between the top and bottom of the income profile amongst the poor, the need to search for a sufficiently large value of r just to yield greater sensitivity at the bottom than at the top is particularly inconvenient [Clark, Hemming and Ulph (forthcoming)]. Moreover, with respect to transfer, $P(r)$ suffers from the same sort of discontinuity as P_S. Anyway, for attaching more weight to income transfers lower down the profile it is not essential to raise the rank weights arbitrarily upto the rth power $(r > 1)$ as proposed by Kakwani. The problem can as well be tackled by introducing a strictly concave interpersonally comparable cardinal welfare function. This is what is done in Chakravarty (1980). We shall pursue this idea in Chapter 2.

(d) <u>The approach of Takayama and Hamada and Takayama</u>

With a view to accommodating deprivation of the poor units relative to units above the poverty line Takayama (1979) defines the censored income vector $\underline{y}^*$ truncated from above by the poverty line z corresponding to a given income profile

$\underline{y}$ as

$$\underline{y}^* = (y_1^*, y_2^*, \ldots, y_n^*)'$$

where

$$\begin{aligned} y_i^* &= y_i \quad \text{if} \quad y_i < z \\ &= z \quad \text{if} \quad y_i \geq z \end{aligned} \qquad \ldots \quad (1.65)$$

He then defines the Gini coefficient of the censored income profile $\underline{y}^*$ as the Gini coefficient of poverty of the profile $\underline{y}$, as

$$G_c = \frac{1}{2n^2 \lambda^*} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| y_i^* - y_j^* \right| \qquad \ldots \quad (1.66)$$

where

$$\lambda^* = \frac{1}{n} \sum_{i=1}^{n} y_i^* \qquad \ldots \quad (1.67)$$

Takayama also provided an axiomatisation of his measure G_c. Other measures of inequality had been applied to the censored income profile to derive corresponding measures of poverty in Hamada and Takayama (1977).

The Takayama and Hamada and Takayama formulae are neat applications of measures of inequality to the measurement of poverty. But the simplicity of these formulae is achieved at some real cost. The main drawback of these measures lies in their violation of the monotonicity axiom.

To see this, consider a situation where all units have the same positive income below the poverty line. In this case, the indices proposed by Takayama and by Hamada & Takayama take the value zero, irrespective of the common value of the incomes.

Again, suppose that all the incomes of the original income profile are less than z and these are multiplied by a a constant $c > 0$ such that all the units remain poor. Then a poverty measure should increase or decrease according as c is less than or greater than unity. But the Takayama and Hamada and Takayama measures remain invariant under such circumstances. This difficulty can be overcome if we consider the following index of poverty

$$P = \left(\frac{z}{\lambda^*} \right) I_c' \qquad \ldots \qquad (1.68)$$

where I_c is an inequality measure of the censored income profile. But even this measure is not free from defects. It takes the zero value if all units are poor and have the same positive income.

Finally, we have the following proposition:

<u>Proposition 1.1</u>

Let $\underline{y} = (y_1, y_2, \ldots, y_n)'$ be an income profile such that $y_1 \leq y_2 \leq \ldots\ldots \leq y_n < z$. Let I be a rectifiant, scale invariant inequality measure. Suppose that y_n increases by an amount $\delta > 0$ such that $y_n + \delta < z$. Then $I(\underline{y}) < I(\hat{\underline{y}})$ where $\hat{\underline{y}} = (y_1, y_2, \ldots, y_{n-1}, y_n + \delta)'$ and $I(.)$ is the inequality measure based on the corresponding profile.

Proof : $I(y_1, y_2, \ldots, y_{n-1}, y_n + \delta)$

$$> I\left(y_1 + \frac{\delta\, y_1}{\sum_{i=1}^{n} y_i},\ y_2 + \frac{\delta\, y_2}{\sum_{i=1}^{n} y_i},\ \ldots,\ y_n + \frac{\delta\, y_n}{\sum_{i=1}^{n} y_i}\right)$$

$$= I(cy_1, cy_2, \ldots, cy_n) \quad \text{where } c = 1 + \frac{\delta}{\sum_{i=1}^{n} y_i}$$

$$= I(y_1, \ldots, y_n).$$

□

But as suggested by the monotonicity axiom a poverty measure in these circumstances should decrease.

It thus appears that the measure of inequality based on the censored income profile proposed so far may not behave as a satisfactory index of poverty in many situations. The broad question that arises is whether we can propose measures based on censored income profiles (containing inequality indices as components) that would meet the monotonicity and the transfer axioms. An attempt to answer this question is made in Chapter 4 of the thesis.

(e) Blackorby and Donaldson's Approach

In a recent paper Blackorby and Donaldson (1980a) offered an alternative interpretation and a generalisation of Sen's measure as an ethical measure. They assume a continuous, S-concave and non-decreasing swf. They define the swf over the incomes of the poor. Then the representative income of the poor is defined as that level of income which, if given to each poor unit, would prove ethically equivalent to the given income profile.[32]

They call a poverty index relative or absolute according as it satisfies the axiom of scale irrelevance or translation invariance. The two axioms are stated as follows:

Axiom of Scale Irrelevance (SI)

The value of the poverty index remains unchanged when all the incomes and the poverty line itself are multiplied by a positive scalar.

Axiom of Translation Invariance (TI)

The value of the poverty index remains unchanged when the same amount of income is added to or subtracted from all the incomes and the poverty line itself.

32/ For the population as a whole, this representative income is the AKS representative income.

As a general relative poverty index they consider the following:

$$P_{BD} = \frac{q}{n}\left[1 - \frac{\xi^P}{z}\right] \quad \ldots \quad (1.69)$$

where ξ^P is the representative income of the poor.

But this poverty measure will satisfy the axiom of scale irrelevance if and only if the underlying swf is homothetic.

Given (1.69) it is clear that to every homothetic swf there corresponds a relative poverty index.33/

Assume that the homothetic swf W^P of the poor is given by

$$W^P(\underline{y}^P) = \emptyset\left[\frac{2}{q(q+1)} \sum_{i=1}^{q} y_i(q+1-i)\right] \quad \ldots \quad (1.70)$$

where $\underline{y}^P$ = income profile of the poor; and $\emptyset$ is increasing in its argument. Then the corresponding poverty measure is the Sen measure. Therefore if any general remark is made on the Blackorby and Donaldson measures given by (1.69), it will be valid for Sen's measure also.

33/ Blackorby and Donaldson point out that the swf must be completely strictly recursive in the sense that the ordering over any group of poor units must be separable from (the income of) anyone who is richer. [A detailed discussion on recursivity can be found in Blackorby, Primont and Russell (1978, Chapter 6)].

We now have the following proposition:

Proposition 1.2

Suppose some amount of income is transferred from a poor unit to the richest unit below the poverty line such that the recipient crosses the poverty line. Then there exist income profiles for which the aforesaid transfer would cause P_{BD} given by (1.69) to decrease, violating the (original version of the) transfer axiom formulated by Sen (1976).

Proof: Without loss of generality, assume that an amount $\delta > 0$ is to be transferred from the poorest unit to the richest poor unit.

Let $x_1 = (z - \delta) > 0$ and $x_q = (z - \frac{\delta}{q})$. Then we can get $0 < x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_{q-1} \leq x_q < z$. Let us suppose $x^P = (x_1, \ldots, x_q)'$ is the income vector of the poor in a community. Transfer $\delta > 0$ amount of income from x_1 to x_q. New income profile of the poor is given by $\underline{x}^P(\delta) = (x_1(\delta), \ldots, x_{q-1}(\delta))'$

$$\begin{aligned} \text{where} \quad x_i(\delta) &= x_i \quad \text{for} \quad 2 \leq i \leq q-1 \\ &= x_1 - \delta \text{ for } \quad i = 1 . \end{aligned} \qquad \ldots \qquad (1.71)$$

Denote the corresponding representative incomes (according to some homothetic swf) by ξ^P and $\xi^P(\delta)$ respectively. Since the swf is continuous and $\delta > 0$ is arbitrary, we can make

$\left| \xi^P - \xi^P(\delta) \right|$ sufficiently small so that the inequality

$$\frac{q}{n}\left[1 - \frac{\xi^P}{z}\right] < \frac{q-1}{n}\left[1 - \frac{\xi^P(\varepsilon)}{z}\right] \qquad \ldots \qquad (1.72)$$

does not hold.

□

Blackorby and Donaldson also proposed absolute measures of poverty. As a general absolute index they introduce the measure

$$Q(\underline{y}^P) = q\left[z - \xi^P\right] \qquad \ldots \qquad (1.73)$$

This measure will remain invariant with respect to translation of z and $\underline{y}^P$ if and only if the underlying swf is translatable.

Therefore for every translatable swf there is a single absolute measure of poverty of this type that is invariant with respect to translation of z and $\underline{y}^P$.

But the difficulty pointed out in Proposition 1.2 remains valid for absolute measures also. Also the measure is completely insensitive to the population size of the community. The fourth chapter of the thesis introduces general relative and absolute measures that meet the basic criteria of a poverty measure.

(f) Thon's Measures

Thon (1980c) proposed a one-parameter family of relative poverty measures and gave numerical illustration on measurement of poverty in rural India. He opted in a somewhat arbitrary way for an additive from of measure and for an equidistant Gini-like pattern of the system of weights for individual incomes.

The family of measures is given by

$$P_C = \sum_{i=1}^{q} (z - y_i) \frac{Cn + 1 - 2i}{(C - 1)n^2 z} \qquad \ldots(1.74)$$

where $C \geq 2$.

P_C satiesfies the monotonicity and the transfer axioms. But P_C is more sensitive to transfers around the middle of the income profile (among the poor) and less sensitive to transfers among the very poor. [The same is true for Sen's and Takayama's measures]. However, it will be shown that P_C coincides with the general relative measure proposed in Chapter 4 of the thesis if the homothetic swf takes the form $\emptyset\left[\frac{1}{(C-1)n^2} \sum_{i=1}^{n} y_i^*(Cn+1-2i)\right]$ where $\emptyset$ is increasing in its argument, and $y^* = (y_1^*,\ldots,y_n^*)'$ is the censored income profile.

(g) Measures of Clark, Hemming and Ulph

Clark, Hemming and Ulph (forthcoming) assume that the identical individual deprivation functions take the form

$$d(g_i) = \frac{1}{\alpha} g_i^{\alpha} \quad \ldots \quad (1.75)$$

where $g_i = z - y_i$, $i = 1, 2, \ldots, q$, and $\alpha \geq 1$ for concavity in income. The swf is assumed to be additively separable and can therefore be written

$$- W(\underline{g}, \alpha) = \sum_{i=1}^{q} d(g_i) \quad \ldots \quad (1.76)$$

where $\underline{g} = (g_1, \ldots, g_q)'$ ... (1.77)

The welfare of the poor is assumed to be separable from that of the non-poor.

To measure inequality in the distribution of poverty gaps they define the 'equally distributed equivalent poverty gap', which is that poverty gap, which if shared by all the poor would be regarded as yielding the same level of welfare of the poor as the existing level and distribution of gaps. This is given by

$$g^* = \left[\frac{1}{q} \sum_{i=1}^{q} g_i^{\alpha} \right]^{1/\alpha} \quad \ldots \quad (1.78)$$

Their measure P^* is given by

$$P^* = \frac{q}{n} \cdot \frac{g^*}{z} \quad .. \quad (1.79)$$

However, this approach to the measurement of poverty is essentially the same as that of Blackorby and Donaldson who average the incomes of the poor instead of the gaps in a similar manner [See, eqn. (1.69) supra].

To see how the measure P^* violates the transfer axiom when a disequalising transfer enables its recipient to cross the poverty line, let us consider the following example:

$\underline{y} = (1, 2, 2.6, 4, 5)'$

Let $z = 3$ and $\alpha = 2$.

Then $q = 3$, $g^* = 1.320$ and $P^* = 0.264$.

Now transfer 0.5 amount of income from the poorest unit to the richest unit below the poverty line. The new profile is $(0.5, 2, 3.1, 4, 5)'$.

Therefore, now $q = 2$, $g^* = 1.905$, $P^* = 0.254$.

In Chapter 3 of the thesis we will propose a new class of poverty measures based on this 'equally distributed equivalent poverty gap' approach. Unlike the measure P^*, the proposed class will satisfy the transfer axiom in all cases. We will then show how this approach can be generalised. Finally, Sen's measure will be interpreted in this framework.

Clark, Hemming and Ulph also consider the proportionate gap between the poverty line and the ede income of the censored income profile based on the additively separable swf $W = \frac{1}{\beta} \sum_{i=1}^{n} (y_i^*)^{\beta}$, $\beta < 1$, as a measure of poverty. The measure satisfies the two basic axioms. A transfer from unit i to unit j will increase the measure by a greater amount the richer is unit j. It will be shown in Chapter 4 that this measure coincides with the general relative measure introduced there if the homothetic swf has the image $[\frac{1}{n} \sum_{i=1}^{n} (y_i^*)^{\beta}]^{1/\beta}$.

CHAPTER - 2

MEASURES OF POVERTY BASED ON UTILITY GAPS

2.1 Introduction

It has been shown in Chapter 1, Part II, that most of the existing measures of poverty have one or more shortcomings. The present chapter aims at constructing new measures of poverty based on the utility gaps of the poor. The indices proposed seem to satisfy all the desirable axioms for poverty measures. The next section explains the set-up and chooses an additive form of the poverty index, Section 3 lays down a number of axioms or criteria to be satisfied by the poverty measure. Section 4 derives the class of measures of the chosen form starting from these axioms. Section 5 makes some remarks on the class of measures derived in Section 4. In Section 6 we propose alternative measures of poverty when the utility functions are assumed to be concave, instead of strictly concave as assumed in Sections 2 to 5.

2.2 Formulation and Definitions

Without loss of generality, we can assume that the incomes are arranged in non-decreasing order, i.e., $y_1 \leq y_2 \leq \ldots \leq y_n$ and that out of the n units, q are poor, i.e., below the poverty line z.

Let $U_i(.)$ stand for the ith unit's utility function. We shall assume that U_i depends only on y_i, the income that the ith unit earns. We further assume that U_i's are the same for all, i.e., $U_i(.) = U(.)$, and that U is increasing and strictly concave.

Consider $h_i = U(z) - U(y_i)$ ... (2.1)

= Utility gap of unit i.

Obviously, h_i is positive for the poor and non-positive for others. Also, the more the income of a unit falls below the poverty line, the more is its utility gap. Therefore, if $y_i < y_j < z$, then $h_i > h_j > 0$.

Let S(x) denote the set of individual units with income less than $x \geq 0$. Then we have :

Definition 2.1 : For a given income configuration $\underline{y}$, the 'aggregate gap' G(x) of the set S(x) of units is defined as the normalised sum of gaps h_i over units in S(x):

$$G(x) = A(z, n) \sum_{i \in S(x)} h_i \quad \ldots \quad (2.2)$$

where $A(z, n) > 0$ is the coefficient of normalisation. A(z, n) can be determined by a set of axioms to be proposed in the next section.

Definition 2.2 : For a given income configuration $\underline{y}$, the index of poverty P is defined to be the maximal value of the aggregate gap G(x) for all x.

$$P = \max_x G(x) \quad \ldots \quad (2.3)$$

From the two definitions given above it is obvious that

$$P = G(z) \quad \ldots \quad (2.4)$$

$$= A(z, n) \sum_{i \in S(z)} h_i \quad \ldots \quad (2.5)$$

$$= A(z, n) \sum_{i=1}^{q} h_i \quad \ldots \quad (2.6)$$

That is, the index of poverty is given by the normalised value of the aggregate utility gap of the poor units. Therefore given z, P can be regarded as a measure of distance separating the income profile of the poor under study from the social state $\underline{z}$ with implications for social welfare, where $\underline{z}$ is the q-coordinated vector (z, z , ..., z).

2.3 Desiderata of a Satisfactory Measure: The measure given by (2.6) should remain invariant under affine transformations of U(.). Below we propose an axiom of normalisation that is found to ensure its invariance under affine transformations of U(.). The axiom is:

Axiom of Normalisation (N 1)

If all the n units are poor and have zero income then the value of the poverty index is unity.

It turns out that P attains its highest value in the extreme case noted above. In general, it lies between zero and unity.

The poverty index given by (2.6) is intended to be a relative index. Therefore it should satisfy the following:

<u>Axiom of Scale Irrelevance (SI)</u>

The value of the poverty index should remain unchanged when all the incomes and the poverty line itself are multiplied by a positive scalar.

2.4 <u>The Class of Poverty Measures Derived</u>:

<u>Theorem 2.1</u>: The only class of poverty measures satisfying axioms (N1) and (SI) is given by

$$P = \frac{1}{n} \sum_{i=1}^{q} \left[1 - \left(\frac{y_i}{z}\right)^{\epsilon}\right] \quad \ldots \quad (2.7)$$

where $0 < \epsilon < 1$.

<u>Proof</u>: In the special case when all the n units have zero income, we have from (2.6),

$$P = A(z, n)\, n[U(z) - U(o)] \quad \ldots \quad (2.8)$$

But according to axiom (N1), for this special case, we must have

$$P = 1 \quad \ldots \quad (2.9)$$

Therefore from (2.8) and (2.9),

$$A(z, n) = \frac{1}{n[U(z) - U(o)]} \qquad \ldots \qquad (2.10)$$

From (2.6) and (2.10) it follows that

$$P = \frac{1}{n[U(z) - U(o)]} \sum_{i=1}^{q} [U(z) - U(y_i)] \quad \ldots \qquad (2.11)$$

Define $f(x) = U(x) - U(o)$ $\ldots$ (2.12)

$$\therefore \quad P = \frac{1}{nf(z)} \sum_{i=1}^{q} [f(z) - f(y_i)]$$

$$= \frac{1}{n} \sum_{i=1}^{q} \left[1 - \frac{f(y_i)}{f(z)}\right] \qquad \ldots \qquad (2.13)$$

For $n = 1$ and $y_1 < z$, by axiom (SI) the measure depends only on y_1/z. But for this special case the measure given by (2.13) is

$$P = 1 - \frac{f(y_1)}{f(z)} \qquad \ldots \qquad (2.14)$$

Therefore $\frac{f(y_1)}{f(z)}$ depends only on $\frac{y_1}{z}$.

Let $g(s, t) = \frac{f(s)}{f(t)}, \quad f(t) \neq 0.$

$\therefore$ $\frac{f(s)}{f(t)}$ is homogeneous of degree zero.

By Euler's theorem of homogeneous functions we have

$$\frac{s\,f'(s)}{f(t)} - \frac{t\,f(s)\quad f'(t)}{(f(t))^2} = 0$$

or, $$\frac{s\,f'(s)}{f(s)} = \frac{t\,f'(t)}{f(t)}$$

$\therefore$ $\dfrac{s\,f'(s)}{f(s)}$ is constant.

Let $$\frac{s\,f'(s)}{f(s)} = \epsilon .$$

$$\therefore \quad \log f(s) = \log s^{\epsilon} + K$$

where K is constant of integration.

$$\therefore \quad f(s) = B\,s^{\epsilon}, \quad B > 0.$$

This gives

$$\begin{aligned} U(x) &= U(o) + f(x) \\ &= U(o) + B\,x^{\epsilon} \\ &= A + B\,x^{\epsilon} \quad \text{(say)} \qquad \ldots \qquad (2.15) \end{aligned}$$

Since U(.) is increasing and strictly concave in its argument, we must have $0 < \epsilon < 1$.

Substituting U(.) given by (2.15) in (2.11) we have

$$P = \frac{1}{n z^{\epsilon}} \sum_{i=1}^{q} [z^{\epsilon} - y_i^{\epsilon}]$$

$$= \frac{1}{n} \sum_{i=1}^{q} [1 - (\frac{y_i}{z})^{\epsilon}].$$

This establishes the necessity part of the theorem. The sufficiency can be verified easily.

□

2.5 Some Remarks on P

(1) P lies in the closed interval [0, 1]. The lower limit is attained when incomes of all the units are at or above the poverty line, while the upper limit is reached in the situation mentioned in the axiom of normalisation (N1).

(2) For a given income configuration $\underline{y}$ and a poverty line z , P increases as ϵ increases. $(\epsilon - 1)$, it may be noted, is the constant elasticity of marginal utility with respect to income.

(3) The measure P can be written as the product of the head-count ratio H and the ratio

$$I_{\epsilon} = \frac{1}{q} \sum_{i=1}^{q} \frac{z^{\epsilon} - y_i^{\epsilon}}{z^{\epsilon}},$$

i.e., $P = HI_{\epsilon}$. P is increasing both in H and I_{ϵ}. If one has information on the two components viz., H and I_ϵ , then one can judge how far the two factors contribute to variations in P.

(4) P satisfies the monotonicity axiom.

(5) When $0 < \epsilon < 1$, -P embodies a swf which is strictly concave in the incomes of the poor and the transfer axiom is satisfied in all cases, even if the recipient goes above the poverty line. A transfer from unit i to unit j (both units are poor) will increase P by a larger amount, the richer the unit j (for fixed i). The sensitivity of P to fixed equidistant transfers depends upon the difference in marginal utilities of the units concerned and so the sensitivity is greater at lower income levels. As ϵ decreases, P becomes more sensitive to transfers lower down the income profile.

(6) When all the poor have the same positive income the measure is not independent of ϵ . Therefore the poverty index P (unlike the class of measures of Clark , Hemming and Ulph given by (1.79)) is affected by changes in the value of the parameter. The other class of measures proposed by the same authors (vide Subsection 1.2.2.2g of Chapter 1) become independent of the corresponding parameter when all the n units are poor and have the same positive income. The measure P does not have this defect.

(7) When all the q poor units have zero income the measure takes the value H, the head-count ratio.

(8) The value of the index remains unchanged if the income profile is censored, so that all incomes above z are replaced by z.

(9) In practice, most of the data on personal income/expenditure are available in grouped form i.e., distributed over income or expenditure classes. So we have to adopt the formula given by (2.7) to grouped data of the form indicated below :

Income classes	Frequency	Relative Frequency	Class means
$0 - Y_1$	f_1	$p_1 = \frac{f_1}{n}$	$\bar{y}_1$
$Y_1 - Y_2$	f_2	$p_2 = \frac{f_2}{n}$	$\bar{y}_2$
⋮	⋮	⋮	⋮
$Y_{m-1} - Y_m$	f_m	$p_m = \frac{f_m}{n}$	$\bar{y}_m$

Here m = Number of income classes below the poverty line. We assume that the poverty line coincides with Y_m, the end point of the interval (Y_{m-1}, Y_m).

A good approximation to formula (2.7) may be given by

$$P = \frac{1}{n} \sum_{i=1}^{m} f_i \left[1 - \left(\frac{\bar{y}_i}{z}\right)^{\epsilon}\right] \quad \ldots \quad (2.16)$$

where $0 < \epsilon < 1$.

Since $\bar{y}_i$ is the mean income of the ith class, the measures given by (2.16) will differ from (2.7) to the extent variations of income within income classes are ignored.

(10) Ranking income profiles by summary measures is just a first step in the analysis of poverty. For deeper analysis we should inquire into the factors contributing to poverty using time series or cross-section data. We have already mentioned in connection with decomposition of inequality measures that we can investigate to what extent the income differences between regions, age - sex groups, races, occupations, educational categories etc., contribute to the overall degree of inequality. We can apply the same kind of decomposition to poverty measures also. This becomes important when one implements policies for reduction of poverty.

Suppose the population is partitioned into k groups with respect to certain characteristics. Then the poverty index proposed in Section 2.4 for the whole population is equal to the weighted average of the poverty indices for different groups the weights being the population shares of the different groups. That is,

$$P = \sum_{j=1}^{k} \frac{n_j}{n} \cdot P_j \qquad \ldots \qquad (2.17)$$

where, n_j : Cardinality of group j.

P_j : Poverty index for group j.

Clearly, such decomposability is a desirable property if one wishes to analyse the influence of poverty within each group on the overall poverty in the population.

(11) Suppose income y has a continuous type distribution with distribution function F. Then the expression for the poverty measure is given by

$$P = \int_0^z [1 - (\frac{y}{z})^{\epsilon}] \, dF(y) \qquad \ldots \qquad (2.18)$$

To illustrate the formula, we consider as an example, the lognormal distribution with parameters μ and σ^2.

Since $t = \frac{\log y - \mu}{\sigma} \sim N(0, 1)$, we get

$$F(y) = \int_0^y dF(x) = \phi(t) \qquad \ldots \qquad (2.19)$$

where ϕ is the distribution function of $N(0, 1)$.

Now, $$P = \int_0^z \left[1 - \left(\frac{y}{z}\right)^\epsilon\right] dF(y) \quad \ldots \quad (2.20)$$

$$= \int_0^z dF(y) - \frac{1}{z^\epsilon} \int_0^z y^\epsilon \, dF(y)$$

$$= \int_{-\infty}^{z'} d\,\phi\,(t) - \frac{1}{z^\epsilon} \int_{-\infty}^{z'} e^{\epsilon\,(\mu + t\sigma)}\, d\,\phi\,(t) \quad \ldots \quad (2.21)$$

where $$z' = \frac{\log z - \mu}{\sigma}.$$

Expression (2.21) on simplification yields

$$P = \phi\,(z') - \frac{e^{\epsilon\mu + \epsilon^2 \frac{\sigma^2}{2}}}{z^\epsilon}\,\phi\,(z' - \epsilon\sigma) \quad (2.22)$$

Therefore, the poverty measure in this case depends on the poverty line z, the parameters of the lognormal distribution and also on ϵ .

2.6 Alternative Measures of Poverty

The measure in Section 2.4 is in essence a straight sum of the utility gaps of the poor units. It appears to be rewarding to construct poverty indices based on weighted sum of the utility gaps, with weights increasing as one goes down the scale of income. This is attempted in the present section

2.6.1 The Class of Measures

For a given income configuration $\underline{y}$, we define the poverty measure $\bar{P}$ as the normalised weighted sum of the utility gaps of the poor units using non-negative weights $V_i(z, \underline{y})$:

$$\bar{P} = \bar{A}(z, n) \sum_{i=1}^{q} h_i \; V_i(z, \underline{y}) \qquad \ldots \qquad (2.23)$$

where $\bar{A}(z, n) > 0$ is the coefficient of normalisation. We will determine $\bar{A}$ and the V_i's by a set of axioms to be proposed in this section.

In line with the motivation of the transfer axiom we can, following Sen (1976), have the following:

Axiom of Relative Equity (E): Greater weightage is to be given to the utility gap of a unit with less income than to the gap of a richer unit.

Since a unit with less income has more utility gap (that is, less welfare) than a unit with more income, axiom E reflects the view that if in the income configuration $\underline{y}$ unit j is worse off than unit i, then the weight $V_j(z, \underline{y})$ of h_j should be more than $V_i(z, \underline{y})$, the weight of h_i.[1]

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[1] On various aspects of equity considerations in welfare economics, see Runciman (1966), Graaff (1967), Sen (1970a) and Pattanaik (1971).

Axiom E gives expression to a mild requirement of equity. There can be a variety of axioms by which we can specify $V_i(z, \underline{y})$ and which incorporate axiom E. The following axiom is substantially demanding and incorporates axiom E.

Axiom of Ordinal Rank Weights (R1)

The weight $V_i(z, \underline{y})$ of the utility gap of unit i equals the number of units in S who are at least as well off as unit i.

This axiom gives a relativist view of poverty. As a unit's position becomes lower in the welfare scale its incidence to the poverty measure should be greater and its welfare rank among others may be considered to indicate the weight to be placed on its utility gap.

Then we have

Theorem 2.2 : The only class of poverty measures satisfying axioms (R1), (N1), (SI) is given by

$$\bar{P} = \frac{2}{n(n+1)} \sum_{i=1}^{q} [1 - (\frac{y_i}{z})^{\epsilon}](n+1-i) \quad \ldots \quad (2.24)$$

$$0 < \epsilon \leq 1$$

Proof: Axiom (R1) requires that the weight V_i on the utility gap of unit i should be (n+1-i). Therefore from (2.23) we have

$$\bar{P} = \bar{A}(z, n) \sum_{i=1}^{q} h_i (n+1-i) \qquad \ldots \qquad (2.25)$$

The remaining part of the proof is completely analogous to that of Theorem 2.1 and hence omitted.[2/]

□

2.6.2 Properties of $\bar{P}$

(1) $\bar{P}$ is invariant under affine transformations of U (.).

(2) For a given income profile $\underline{y}$ and a given poverty line z, $\bar{P}$ increases as ϵ increases.

(3) For large number of the poor, $\bar{P}$ as given by (2.24), has the following approximate form :

$$\bar{P} \simeq 2 H I_\epsilon (1-H) + H^2 [1 - (1-I_\epsilon)(1-G_\epsilon)] \qquad \ldots \qquad (2.26)$$

where I_ϵ = the ratio $\frac{1}{q} \sum_{i=1}^{q} \left(\frac{z^\epsilon - y_i^\epsilon}{z^\epsilon} \right)$,

$$G_\epsilon = \frac{1}{2q^2 m_\epsilon} \sum_{i=1}^{q} \sum_{j=1}^{q} |y_i^\epsilon - y_j^\epsilon| \qquad \ldots \qquad (2.27)$$

2/ If there are more than one unit having the same income, then the weighted sum of the form $\sum_i h_i (n+1-i)$ remains the same if one splits the ties at random or allots the average rank to all the tied units.

= The Gini coefficient of the profile $(y_1^{\epsilon}, y_2^{\epsilon}, \ldots y_q^{\epsilon})$.

$$m_{\epsilon} = \frac{1}{q} \sum_{i=1}^{q} y_i^{\epsilon} \qquad \ldots \qquad (2.28)$$

<u>Proof</u>:

$$\bar{P} = \frac{2}{n(n+1)} \sum_{i=1}^{q} [1 - (\frac{y_i}{z})^{\epsilon}](n+1-i)$$

$$= \frac{2}{n(n+1)} \sum_{i=1}^{q} [1 - (\frac{y_i}{z})^{\epsilon}](n-q) + \frac{2}{n(n+1)} \sum_{i=1}^{q} [1 - (\frac{y_i}{z})^{\epsilon}](q+1-i)$$

$$= \bar{P}_1 + \bar{P}_2 \text{ (say).}$$

$$\bar{P}_1 = \frac{2}{n(n+1)} \sum_{i=1}^{q} [1 - (\frac{y_i}{z})^{\epsilon}](n-q)$$

$$= \frac{2q(n-q)}{n(n+1)} I_{\epsilon}$$

$$\simeq 2H(1-H) I_{\epsilon} \qquad \ldots \qquad (2.29)$$

$$\bar{P}_2 = \frac{2}{n(n+1)} \sum_{i=1}^{q} [1 - (\frac{y_i}{z})^{\epsilon}](q+1-i)$$

$$= \frac{2}{n(n+1)} \cdot \frac{q(q+1)}{2} - \frac{2}{n(n+1)} \sum_{i=1}^{q} (\frac{y_i}{z})^{\epsilon} (q+1-i)$$

$$\simeq H^2 - \frac{2}{n(n+1)} \sum_{i=1}^{q} (\frac{y_i}{z})^{\epsilon} (q+1-i) .$$

Now, $$G_\epsilon = \frac{1}{2q^2 m_\epsilon} \sum_{i=1}^{q} \sum_{j=1}^{q} |y_i{}^\epsilon - y_j{}^\epsilon|$$

$$= 1 + \frac{1}{q} - \frac{2}{q^2 m_\epsilon} \sum_{i=1}^{q} y_i{}^\epsilon \ (q+1-i) .$$

$$\therefore \quad q^2 m_\epsilon (G_\epsilon - 1 - \frac{1}{q}) = -2 \sum_{i=1}^{q} y_i{}^\epsilon \ (q+1-i) .$$

$$\bar{P}_2 \simeq H^2 + \frac{1}{n(n+1) \ z^\epsilon} q^2 m_\epsilon \ (G_\epsilon - 1 - \frac{1}{q})$$

$$\simeq H^2 - H^2 \left[\frac{1}{q} \sum_{i=1}^{q} (1 - (\frac{y_i}{z})^\epsilon) - 1\right](G_\epsilon - 1 - \frac{1}{q}).$$

$$\simeq H^2 - H^2 (1 - I_\epsilon)(1 - G_\epsilon) \quad \ldots \quad (2.30)$$

From (2.29) and (2.30) we have

$$\bar{P} \simeq 2 H I_\epsilon \ (1-H) + H^2 [1 - (1-I_\epsilon)(1-G_\epsilon)] .$$

$\bar{P}$ given by (2.26) is found to be an increasing function of G_ϵ and also of I_ϵ and H, which seems to be quite reasonable.

(4) When $\epsilon = 1$, so that the individual utility function is not strictly concave

$$\bar{P} = \frac{2}{n(n+1)} \sum_{i=1}^{q} [1 - (\frac{y_i}{z})](n+1-i) \quad \ldots \quad (2.31)$$

$\bar{P}$ given by (2.31) is Thon's measure [Thon (1979)]. Therefore

Thon's measure is a member of the class of measures given by (2.24)[3/].

(5) $\bar{P}$ satisfies the monotonicity axiom. The sensitivity of $\bar{P}$ to fixed equi-distant transfers depends on the difference in the product of the marginal utilities and the corresponding rank weights. However, for $\epsilon = 1$ the sensitivity depends solely on the rank differences and thus in this case the measure is more sensitive to transfers around the middle of the income profile (among the poor) and less sensitive to transfers among the very poor. With $0 < \epsilon < 1$, as ϵ decreases $\bar{P}$ attaches more weight to transfers at the lower end of the profile.

3/ This particular case could be arrived at through a different formulation: Define the poverty index as the normalised value of $\sum_{i=1}^{q} g_i V_i$, where $V_i = n+1-i$ and where the normalisation axiom is: 'If all the n units are poor and have the same income then the poverty index $= \frac{1}{nz} \sum_{i=1}^{n} g_i$'. While V_i is a modification of ordinal rank weights used by Sen, the normalisation axiom is the same as his if all the n units are poor. The corresponding absolute index $\frac{2}{n(n+1)} \sum_{i=1}^{q} (z-y_i)(n+1-i)$ is reached if the normalisation axiom is: 'If all the n-units are poor and have the same income then the poverty index should equal $\frac{1}{n} \sum_{i=1}^{n} g_i$'. This absolute measure will be discussed in Chapter 4.

(6) $\bar{P}$ lies in the closed interval [0, 1]. It is affected by changes in the value of the parameter ϵ $(0 < \epsilon < 1)$ when all the poor units have the same positive income. The value of the index remains the same for both censored and uncensored profiles.

(7) When income data are available in grouped form (see Section 5 of this chapter), the following expression for $\bar{P}$ may be used:

$$\bar{P} = \frac{2}{(1 + \frac{1}{n})} \sum_{i=1}^{m} p_i [1 - (\frac{\bar{y}_i}{z})^{\epsilon}][1 + \frac{1}{2n} - \sum_{j=1}^{i} p_j + \frac{p_i}{2}] \quad \ldots \quad (2.32)$$

This expression is obtained by using the average of the ranks of incomes within a class as the weight for the average of incomes in the class. The formula ignores income variations within an income class. Moreover, since the rank weight is larger for lower incomes, the measure tends to underestimate the value of $\bar{P}$ that would be obtained for ungrouped data.

(8) Suppose income y has a continuous type distribution with distribution function F. To arrive at an expression for $\bar{P}$ in this continuous case, we have to establish a correspondence between ranks in the discrete case and the area under the probability density

curve. We know that the reverse cumulative distribution function $H(x) = 1-F(x)$ represents the proportion of units with incomes greater than x. Now in the interpersonal welfare ordering of units the relative position of a unit in the profile $(y_1^{\epsilon}, y_2^{\epsilon} \ldots y_n^{\epsilon})$ remains the same as that in the profile $(y_1, y_2, \ldots, y_n)$. In terms of welfare, individual units (among the total population) with incomes above x are better off than the unit with income x. Therefore H(x) will form the weight of the gap $[1 - (\frac{x}{z})^{\epsilon}]$. The total of weights is obtained by integrating $H(x) = 1 - F(x)$ over $(0, \infty)$. Therefore the measure given by (2.24) takes the form

$$\bar{P} = \frac{\int_0^z [1 - (\frac{y}{z})^{\epsilon}](1 - F(y))\,dy}{\int_0^{\infty} (1 - F(y))\,dy} \quad \ldots \quad (2.33)$$

But

$$\int_0^{\infty} (1 - F(y))\,dy = \lambda \quad \ldots \quad (2.34)$$

where λ is the mean of y [vide Rao (1974), pp. 94-95]. Here λ is assumed to be finite.

$$\therefore \quad \bar{P} = \frac{1}{\lambda} \int_0^z [1 - (\frac{y}{z})^{\epsilon}](1 - F(y))\,dy \quad \ldots \quad (2.35)$$

To illustrate, let $y \sim \Lambda(\mu, \sigma^2)$.

Then $\bar{P} = \frac{1}{\lambda} \int_0^z [1 - (\frac{y}{z})^\epsilon](1 - F(y))\, dy$

$$= \frac{1}{\lambda} \int_0^z [1 - (\frac{y}{z})^\epsilon](1 - \phi\,(t))\, dy$$

[where $t = \frac{\log y - \mu}{\sigma}$ and ϕ is the distribution function of N(O, 1)]

$$= \frac{1}{\lambda} [\int_0^z dy - \int_0^z (\frac{y}{z})^\epsilon dy - \int_0^z \phi\,(t)\, dy + \frac{1}{z^\epsilon} \int_0^z y^\epsilon\, \phi\,(t)\, dy].$$

$$= \frac{1}{\lambda} [\frac{\epsilon z}{(1+\epsilon)} - \frac{\sigma}{z^\epsilon} \int_{-\infty}^{z'} \phi\,(t)\, e^{\mu+t\sigma} dt + \frac{\sigma}{z^\epsilon} \int_{-\infty}^{z'} e^{\epsilon(\mu+t\sigma)} \cdot \phi\,(t)\, e^{\mu+t\sigma} dt]$$

which on simplification yields

$$\bar{P} = \frac{1}{e^{\mu + \sigma^2/2}} [\frac{\epsilon z}{1 + \epsilon} + (\frac{e^{(1+\epsilon)(\mu+z'\sigma)}}{(1 + \epsilon)\, z^\epsilon} - e^{(\mu+z'\sigma)}) \phi\,(z')$$

$$+ e^{\mu+\sigma^2/2} \phi\,(z' - \sigma) - \frac{e^{(1+\epsilon)[\mu+\frac{\sigma^2}{2}(1+\epsilon)]}}{(1+\epsilon)\, z^\epsilon} \phi\,(z' - \sigma(1+\epsilon))]$$

... (2.36)

2.7 Conclusion

Assigning higher weights to incomes further down the scale, as proposed by Sen (1976), has the consequence of making the poverty index sensitive to transfers among the poor. The same effect is obtained if instead of weighted income gaps we consider utility gaps of the poor and use unweighted sum of these gaps. Use of utility gaps yields the class of measures HI_{ϵ} (vide Section 4 of this chapter), which may be adequate for practical purposes. However, if we are interested in attaching more weight to transfers lower down the profile, the approach of utility gap seems to be more appropriate than the ordinal rank weight approach. In Section 6 we have used both the techniques in conjunction and have derived another class of poverty measures. This class contains Thon's measure as a special case. The class (like HI_{ϵ}) meets the basic criteria of a poverty measure.

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CHAPTER 3

REPRESENTATIVE INCOME GAP APPROACH TO THE MEASUREMENT OF POVERTY

3.1 Introduction

This chapter attempts to construct measures of poverty based on the notion of equally distributed equivalent (or representative) income gap of the poor. It was observed in Chapter 1 that a measure of this type has been proposed by Clark, Hemming and Ulph (forthcoming). It was pointed out there that this measure may decrease instead of increasing if a disequalising transfer of income among the poor units enables its recipient to cross the poverty line. In this chapter we attempt to modify the measure of Clark *et al* in such a way that the modified measure satisfies the transfer axiom in all cases. This is done in Section 2.

The third section generalises the approach of Clark *et al* and shows that their measure coincides with the general measure proposed here if the underlying group deprivation function of the poor is given by the symmetric mean of order α ($\alpha \geq 1$) of the income gaps of the poor.[1] Sen's measure is

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[1] It should be noted that the general measure proposed here suffers from the limitation mentioned above for the measure of Clark *et al*. See, however, Sen's argument [Sen (1979), p. 302] supporting measures with this defect.

also interpreted in this section. In Section 4, we propose absolute measures based on the same approach and observe that there is a major advantage of these measures over the general absolute measures introduced by Blackorby and Donaldson (1980a). The measures proposed here do not need the assumption of translatability of the group deprivation function of the poor.

3.2 A New Class of Poverty Measures

For the sake of simplicity let us assume that

$$y_1 \leq y_2 \leq \ldots \leq y_q < z \leq y_{q+1} \ldots \leq y_n \quad \ldots(3.1)$$

Let $g_i = (z - y_i)$ be the income gap of the ith poor unit $(i = 1, 2, \ldots, q)$.

Throughout the chapter, we will assume that the welfare of the poor is separable from that of the non-poor.

The measure that Clark et al proposed is given by

$$P^* = \left(\frac{q}{n}\right) \frac{\left(\frac{1}{q} \sum_{i=1}^{q} g_i^{\alpha}\right)^{1/\alpha}}{z} \quad \ldots \quad (3.2)$$

where $\alpha \geq 1$.

But as mentioned in Section 3.1 above, the measure P^* has got some defect with respect to the transfer axiom. Therefore we attempt to modify the measure.

Following Clark et al let us assume that the individual deprivation function takes the form

$$d(g_i) = \frac{1}{\alpha} g_i^{\alpha} \quad \ldots \quad (3.3)$$

where $\alpha \geq 1$.

Overall welfare function of the poor is assumed to be symmetric and additively separable and is therefore written as

$$-W(\underline{g}, \alpha) = \sum_{i=1}^{q} d(g_i) \quad \ldots \quad (3.4)$$

where $$\underline{g} = (g_1, g_2, \ldots, g_q)' \quad \ldots \quad (3.5)$$

The representative income gap of the poor is given by

$$g_e = \left[\frac{1}{q} \sum_{i=1}^{q} g_i^{\alpha} \right]^{1/\alpha} \quad \ldots \quad (3.6)$$

Poverty can then be measured using the normalised value of g_e as the poverty index.

The axiom of normalisation proposed here is: If all the poor units have zero income, the value of the poverty index is given by $H^{1/\alpha}$, where $H = q/n$.

This axiom does not make the poverty index insensitive to the inequality aversion parameter α in the situation to which it refers. In fact, the index in that particular situation increases as α increases.

It is easy to see that the poverty index is given by

$$P(\alpha) = [\frac{1}{n} \sum_{i=1}^{q} (1 - \frac{y_i}{z})^{\alpha}]^{1/\alpha} \quad \ldots \quad (3.7)$$

Properties of $P(\alpha)$

(1) The measure lies in the closed interval [0, 1], the lower and upper limits being attained, respectively (i) when there is no unit below the poverty line and (ii) when all the n-units have zero income.

(2) $P(\alpha)$ satisfies the monotonicity axicm.

(3) $P(\alpha)$ can be written as

$$P(\alpha) = H^{1/\alpha} . P_r.(\frac{g_e}{P_a}) \quad \ldots \quad (3.8)$$

where

$$P_r = \frac{1}{qz} \sum_{i=1}^{q} g_i \quad \ldots \quad (3.9)$$

$$= \text{Average relative poverty gap}$$

and

$$P_a = \frac{1}{q} \sum_{i=1}^{q} g_i \quad \ldots \quad (3.10)$$

= Average absolute poverty gap.

$P(\alpha)$ is increasing in each of the components shown in (3.8), viz., H, P_r and $\frac{g_e}{P_a}$.

(4) $P(\alpha) = HP_r$ when $\alpha = 1$. $P(\alpha)$ increases as α increases. As $\alpha \rightarrow \infty$, only the largest gap matters.

(5) Unlike the measure given by (3.2), $P(\alpha)$ satisfies the transfer axiom in all cases. For $\alpha > 1$, a transfer from unit i to a richer unit j will increase $P(\alpha)$ by a larger amount, the richer is unit j. The sensitivity of $P(\alpha)$ to fixed equidistant transfers depends on the difference in the marginal social valuations of the income gaps of the units concerned. The larger the value of α the greater is the sensitivity of $P(\alpha)$ to transfers.

(6) If all the n-units are poor and have the same positive income, then $P(\alpha)$ becomes independent of α. This appears to be a small defect of $P(\alpha)$.

(7) Suppose that the population is partitioned into k groups with respect to certain characteristics. Let n_i be the number of units in group i, out of which q_i units are poor

$$\left(\sum_{i=1}^{k} n_i = n, \quad \sum_{i=1}^{k} q_i = q \right).$$

Then the overall poverty index satisfies the relation

$$P(\alpha) = \left[\sum_{i=1}^{k} \frac{n_j}{n} \cdot (P_j(\alpha))^{\alpha} \right]^{1/\alpha} \quad \ldots \quad (3.11)$$

where $P_j(\alpha)$ is the poverty index for group j. This result is useful if one is interested in analysing the influence of poverty within individual groups on the overall incidence of poverty.

3.3 A More General Approach

In this section we generalise the approach adopted in the preceding section. For this, let the group deprivation function of the poor be given by

$$F = F(g_1, g_2, \ldots, g_q) \quad \ldots \quad (3.12)$$

where $-F$ is assumed to be continuous, non-decreasing in y_i's (i = 1, 2, ..., q) and also S-concave.

Let us define the representative income gap g_e of the poor as that gap which if shared by every poor unit would make the distribution of income gaps socially indifferent to the observed distribution.

In symbols g_e is defined by

$$F(g_e, g_e, \ldots, g_e) = F(g_1, g_2, \ldots, g_q) \quad \ldots(3.13)$$

For a given income configuration $\underline{y}$, the relative poverty index P_R is defined as the product of the head-count ratio and the ratio of the representative income gap of the poor to the poverty line, that is,

$$P_R(\underline{g}) = \frac{q}{n} \cdot \frac{g_e}{z} \quad \ldots \quad (3.14)$$

In words, P_R is the aggregate income gap of the poor in a situation where all the poor units have the same income and which yields the same level of deprivation as the actual distribution of the gaps generate, expressed as a proportion of the aggregate gap when each member of the community has a zero income.

The measure given by (3.14) satisfies the axiom of scale irrelevance if and only if F is homothetic, that is,

$$F = \emptyset[\bar{F}(g_1, g_2, \ldots, g_q)] \quad \ldots \quad (3.15)$$

where $\bar{F}$ is positively linearly homogeneous and $\emptyset$ is increasing in its argument. Therefore, g_e, the representative income gap of the poor will be determined as

$$g_e = \frac{\bar{F}(g_1, g_2, \ldots, g_q)}{\bar{F}(1, 1, \ldots, 1)} \quad \ldots \quad (3.16)$$

Comparing (3.15) with (3.13) we see that in (3.15) we have some additional restriction on F which need not be satisfied by F in (3.13).

Examples

(1) Suppose that the group deprivation function of the poor has the image

$$\frac{2}{q(q+1)} \sum_{i=1}^{q} g_i(q+1-i) \quad \ldots \quad (3.17)$$

Then $P_R = \frac{q}{n} \cdot \frac{2}{zq(q+1)} \sum_{i=1}^{q} g_i(q+1-i)$

$$= \frac{2}{n(q+1)z} \sum_{i=1}^{q} g_i(q+1-i) \quad \ldots \quad (3.18)$$

= Sen's measure.

(2) Let the deprivation function of the poor be given by the symmetric mean of order α ($\alpha \geq 1$). Then

$$\bar{F}(g_1, g_2, \ldots, g_q) = \left(\frac{1}{q} \sum_{i=1}^{q} g_i^{\alpha}\right)^{1/\alpha} \quad \ldots \quad (3.19)$$

where $\alpha \geq 1$.

$$\text{Then} \quad P_R = \frac{q}{n} \cdot \frac{\left(\frac{1}{q} \sum_{i=1}^{q} g_i^{\alpha}\right)^{1/\alpha}}{z} \quad \ldots \quad (3.20)$$

= The Clark, Hemming and Ulph measure.

Given (3.14) we note that for every homothetic group deprivation function of the poor these exists a corresponding relative poverty index. These indices differ only in the way in which the relative deprivation of the poor units are taken into account. The measure is also sensitive to the head-count ratio.

This index P_R is a generalisation of the Clark, Hemming and Ulph index of poverty given by (3.2).

3.4 Absolute Measures of Poverty

For many policy purposes it might be necessary to introduce absolute measures of poverty which are invariant with respect to translation of z and $\underline{y}^P$.

Let F be the group deprivation function of the poor where - F is assumed to be continuous, non-decreasing and S- concave in y_i's $(i = 1, 2, \ldots, q)$.

For a given income profile $\underline{y}$, we may define the absolute poverty index Q as the product of the head-count ratio and the representative income gap of the poor corresponding to $\underline{y}$, that is,

$$Q(\underline{g}) = \frac{q}{n} \cdot g_e \qquad \ldots \qquad (3.21)$$

Now, the value of the deprivation function remains unaltered when the same amount of income is added to or subtracted from the incomes of the poor and to or from the poverty line z. Hence the measure given by (3.21) remains invariant with respect to translation of z and $\underline{y}^P$. Therefore, for this general measure we do not in particular need the assumption of translatability of the deprivation function which Blackorby and Donaldson needed regarding the social welfare function underlying the general measure given by equation (1.73) of Chapter 1, Part II. Moreover, unlike their absolute measure the general absolute measure introduced in this section is related to the head-count ratio instead of the actual number of poor units.

Examples

(1) Consider the representative income gap of the poor corresponding to the symmetric mean of order α.

Then

$$Q = \frac{q}{n} \left[\frac{1}{q} \sum_{i=1}^{q} (z - y_i)^{\alpha} \right]^{1/\alpha} \quad \ldots \quad (3.22)$$

where $\alpha \geq 1$.

(2) Let g_e correspond to the Kolm-Pollak group deprivation function of the poor.

Then

$$g_e = -\frac{1}{\alpha} \log\left[\frac{1}{q} \sum_{i=1}^{q} e^{-\alpha(z-y_i)} \right] \quad \ldots \quad (3.23)$$

where $\alpha > 0$.

Therefore

$$Q = -\left(\frac{q}{n}\right) \frac{1}{\alpha} \log\left[\frac{1}{q} \sum_{i=1}^{q} e^{-\alpha(z-y_i)} \right] \quad \ldots \quad (3.24)$$

Indeed, any group deprivation function $F = F(g_1, g_2, \ldots, g_q)$ where $-F$ is continuous, non-decreasing and S-concave in y_i's $(i = 1, 2, \ldots, q)$, will serve for constructing an absolute poverty index of the form (3.21). These indices will differ only in the way in which the deprivations of the poor units are accounted for. The approach yields a rich class of measures to choose from.

3.5 Conclusion

We have generalised the Clark, Hemming and Ulph index [*vide* equation (3.2)] in this chapter in several ways and have shown that (i) for every homothetic group deprivation of the poor there is one relative poverty index of the type proposed; (ii) Clark, Hemming and Ulph's index is the relative poverty index when the group deprivation function of the poor is the symmetric mean income gap (of the poor) of order α $(\alpha \geq 1)$;

(iii) Sen's measure can be interpreted in this general framework;

(iv) for every group deprivation function F with minimal properties there is an absolute poverty index of the type proposed here.

We have also introduced in Section 2, a relative measure based on a similar approach that does not share the shortcoming of the measures of Sections 3 and 4 with respect to Sen's transfer axiom [See footnote 1 of this chapter, and also Sen (1979) for an argument in favour of measures with this type of defect].

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CHAPTER 4

ETHICALLY FLEXIBLE MEASURES OF POVERTY

4.1 Introduction

This chapter proposes measures of poverty based on the notion of censored income profiles[1] introduced by Takayama (1979). In Chapter 1 we have briefly touched upon measures of poverty proposed by Takayama and by Hamada and Takayama based on this notion. It was observed there that these measures violate the monotonicity axiom (vide subsection 1.2.2.2d of Chapter 1). Keeping this defect in mind we propose in this chapter new ethical measures of poverty. The essential idea underlying these measures is the notion of representative income of the community corresponding to the censored income profile. We propose measures of relative as well as absolute variety. These measures are found to satisfy both the transfer and the monotonicity axioms. They are closely related to the AKS relative inequality indices (or the Blackorby — Donaldson absolute inequality indices) if applied to the censored income profile.

Section 2 introduces the new type of relative measures and Section 3 the corresponding measures of the absolute variety. Section 4 considers the problem of their consistency in ethical aggregation. In Section 5 we provide axiomatisations of two different classes of poverty measures (one relative and

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1/ The censored income profile doctors the income profile by ignoring the information on the actual incomes of the units above the poverty line, but counts them in with the poverty line income.

the other, absolute) which have especially attractive properties in aggregation. This is done to bring out their economic implications. Finally, Section 6 gives two illustrations of so-called compromise absolute indices of poverty. These yield relative indices of poverty when divided by the poverty line (z).

4.2 Relative Measures of Poverty

We use the following symbols:

$y_i \geqq 0$: income of unit i $(i = 1, 2, \ldots, n)$;

$z > 0$: the poverty line (given exogenously).

Assume that the units are numbered in a non-decreasing order of income, i.e.,

$$y_1 \leqq y_2 \leqq \ldots \leqq y_n \qquad \ldots(4.1)$$

Let q : the number of poor units, and $\underline{y}^* = (y_1^*, y_2^*, \ldots, y_n^*)$ censored income profile corresponding to the income profile $\underline{y} = (y_1, y_2, \ldots, y_n)'$, where

$$\begin{aligned} y_i^* &= y_i \quad \text{if} \quad y_i < z \text{ (i.e., } i \leqq q) \\ &= z \quad \text{if} \quad y_i \geqq z \text{ (i.e., } i > q) \qquad \ldots(4.2) \end{aligned}$$

Let W be a continuous, non-decreasing and S-concave swf defined on the censored income profile. The representative income y_f^* of the community S corresponding to the censored income profile $\underline{y}^*$ is that level of income which if enjoyed by every unit would be ethically or socially indifferent to the income profile $\underline{y}^*$.

For a given income profile $\underline{y}$, the relative poverty index P is defined as the proportionate gap between the poverty line z and the representative income corresponding to $\underline{y}^*$, the censored income profile, that is,

$$P = 1 - \frac{y_f^*}{z} \quad \ldots \qquad (4.3)$$

The measure P has the convenient property of lying between zero (no poverty i.e., $y_i \geq z \ \forall \ i = 1, 2, \ldots, n$) and unity (extreme poverty where $y_i = 0 \ \forall \ i = 1, 2, \ldots, n$).

Let $I(\underline{y}^*)$ be the AKS relative inequality index based on the censored income profile $\underline{y}^*$, i.e.,

$$I(\underline{y}^*) = 1 - \frac{y_f^*}{\lambda^*} \quad \ldots \qquad (4.4)$$

where $\lambda^* = \frac{1}{n} \sum_{j=1}^{n} y_j^*$.

We can rewrite (4.3) as

$$P = 1 - \frac{\lambda^*(1-I)}{z} \quad \ldots \qquad (4.5)$$

Therefore we have the following immediate but interesting conclusion:

Let $\underline{x}^*$ and $\underline{y}^*$ be two censored income profiles with the same mean λ^*, then

$$I(\underline{x}^*) > I(\underline{y}^*) \iff P(\underline{x}^*) > P(\underline{y}^*) \quad \ldots \quad (4.6)$$

We can go further and state that for two censored income profiles $\underline{x}^*$ and $\underline{y}^*$ with the same mean, if x^* is at least as unequal as $\underline{y}^*$ in the Lorenz sense (i.e., the Lorenz curve of $\underline{y}^*$ is nowhere below that of $\underline{x}^*$) then $\underline{x}^*$ is unambiguously ranked as more poverty-stricken than (or as much poverty-stricken as) $\underline{y}^*$ as measured by the general poverty index P. Thus, one can have a quasi-ordering of income profiles based on the relative measures of poverty.

The poverty measure given by (4.3) satisfies the axiom of scale irrelevance if and only if the swf W is homothetic, i.e.,

$$W(\underline{y}^*) = \emptyset\,[\,\bar{W}\,(\underline{y}^{\vee})\,] \quad \ldots \quad (4.7)$$

where $\emptyset$ is increasing in its argument, and $\bar{W}$ is positively linearly homogeneous.

Therefore y_f^* is now defined by

$$\emptyset\,[\,\bar{W}\,(y_f^*,\ldots,y_f^*)\,] = \emptyset\,[\,\bar{W}\,(y_1^*,\ldots,y_n^*)\,] \quad \ldots \quad (4.8)$$

or, $\bar{W}(y_f^* \cdot \underline{1}) = \bar{W}(\underline{y}^*)$ ... (4.9)

since $\emptyset' > 0$.

$\therefore \quad y_f^* \; \bar{W}(\underline{1}) = \bar{W}(\underline{y}^*)$ ... (4.10)

since $\bar{W}$ is positively linearly homogeneous.

Given (4.3), we observe that to every homothetic swf there corresponds a relative poverty index. These indices will differ only in the manner in which the amount of relative inequality in the censored income profile is taken into account. Therefore the measure P is a fairly natural translation of a relative inequality index of a censored income profile into a relative poverty index.

Examples

(1) Let the swf be given by the symmetric mean of order β ($\beta \leq 1$) corresponding to $\underline{y}^*$.

Then

$$\bar{W}_\beta(\underline{y}^*) = \left[\frac{1}{n} \sum_{i=1}^{n} (y_i^*)^\beta \right]^{1/\beta}, \quad \beta \neq 0$$

... (4.11)

$$= \prod_{i=1}^{n} (y_i^*)^{1/n}, \quad \beta = 0$$

Therefore

$$P_\beta(\underline{y}^*) = 1 - \frac{\left[\frac{1}{n}\sum_{i=1}^{n}(y_i^*)^\beta\right]^{1/\beta}}{z}, \quad \beta \neq 0$$

... (4.12)

$$= 1 - \frac{\prod_{i=1}^{n}(y_i^*)^{1/n}}{z}, \quad \beta = 0$$

which is the measure of Clark, Hemming and Ulph (forthcoming).

For $\beta = 1$, we have $\bar{W}_1(\underline{y}^*) = \lambda^*$ and $P_1(\underline{y}^*) = HP_r$

where $P_r = \frac{1}{qz}\sum_{i=1}^{q}(z - y_i)$.

As $\beta \longrightarrow -\infty$, the limiting form of $\bar{W}_\beta(\underline{y}^*)$ is $W_{-\infty}(\underline{y}^*) = \min_i \{y_i^*\}$. The corresponding index is

$$P_{-\infty}(\underline{y}^*) = 1 - \frac{\min_i \{y_i^*\}}{z},$$ the relative maximin index.

The measure P_β can be written as

$$P_\beta = 1 - [H((1 - A_p)(1 - P_r))^\beta + (1 - H)]^{1/\beta}$$

... (4.13)

where

$$A_p = 1 - [\frac{1}{q} \sum_{i=1}^{q} y_i^{\beta}]^{1/\beta} / \lambda_p \quad \ldots \quad (4.14)$$

the Atkinson measure of inequality for the poor, λ_p being the mean income of the poor.

P_β therefore measures the incidence of poverty in terms of the proportion of population which is below the poverty line (H), the average relative shortfall of incomes of the poor compared to the poverty line (P_r) and the inequality in their incomes (A_p).

P_β is increasing in each of the three components — H, P_r and A_p. The transfer sensitivity properties of P (α) with $\alpha > 1$ (vide Chapter 3, Section 2) all go through for P_β when $\beta < 1$.

(2) Let the homothetic swf be given by

$$\phi [\bar{W} (\underline{y}^*)] = \phi [\frac{1}{(C-1)n^2} \sum_{i=1}^{n} (Cn + 1 - 2i) y_i^*] \quad (4.15)$$

where $C \geq 2$.

Then the relative poverty index of equation (4.3) becomes

$$P_C = 1 - \frac{1}{(C-1)n^2 z} \sum_{i=1}^{n} (Cn + 1 - 2i) y_i^*$$

$$= \frac{1}{(C-1)n^2 z} \sum_{i=1}^{q} (z-y_i)(Cn + 1 - 2i) \quad \ldots \quad (4.16)$$

which is the same as Thon's measure [Thon, 1980c]. This measure was discussed thoroughly in Chapter 1.

(3) Suppose that the swf has the image

$$\bar{W}(y^*) = \frac{1}{\left(\sum_{i=1}^{n} a_i\right)} \sum_{i=1}^{n} a_i y_i^* \qquad \ldots \qquad (4.17)$$

where $\{a_1, a_2, \ldots, \}$ is an arbitrary non-increasing sequence, normalised so that $a_1 = 1$, $a_i > 0 \ \forall\, i \in N$, where N is the set of natural numbers.2/

Let a_i's correspond to the single - parameter Ginis [Donaldson and Weymark (1980)]. Then

$$a_i = i^{\delta} - (i-1)^{\delta} \qquad \ldots \qquad (4.18)$$

where $0 < \delta \leq 1$.

$$\therefore \quad \bar{W}_{\delta}(y^*) = \frac{1}{n^{\delta}} \sum_{i=1}^{n} [i^{\delta} - (i-1)^{\delta}]\, y_i^* \qquad \ldots \qquad (4.19)$$

We then get the poverty index

$$P_{\delta} = \frac{1}{n^{\delta} z} \sum_{i=1}^{q} (z-y_i)[i^{\delta} - (i-1)^{\delta}] \qquad \ldots \qquad (4.20)$$

2/ The requirement that $\{a_1, a_2, \ldots\}$ is non-increasing is necessary and sufficient for $\bar{W}$ to be quasi-concave or S-concave for each n. S-concavity is the requirement that $\bar{W}$ agree with (weak) Lorenz quasi-ordering. The restriction $0 < \delta \leq 1$ in (4.19) serves the same purpose.

where $0 < \delta \leq 1$.[3]

When $\delta = 1$, $\bar{W} = \lambda^*$, the associated poverty index can be looked upon as the product of the average relative poverty gap and the head-count ratio. When $\delta \to 0$, $\bar{W} \to \min \{ y_i^* \}$, the maximin swf and the corresponding poverty index is the relative maximin index.

We can construct another single-parameter class by employing a welfare ranked permutation $\hat{\underline{y}}$ of $\underline{y}$ i.e., $\hat{y}_1 \geq \hat{y}_2 \geq \cdots \geq \hat{y}_n$ ($\hat{y}_i \geq z$ for $i = 1, \ldots, n-q$; $\hat{y}_i < z$ for $i = n-q+1, \ldots, n$).

The censored income profile in this case is given by $\hat{\underline{y}}^* = (\hat{y}_1^*, \ldots, \hat{y}_n^*)'$, where

$$\begin{aligned} \hat{y}_i^* &= z \quad \text{if} \quad i \leq (n-q) \\ &= \hat{y}_i \quad \text{if} \quad i > (n-q) \end{aligned}$$

The sequence $\{ a_1, a_2, \ldots, \}$ (normalised so that $a_1 = 1$) has to be non-decreasing in order that

$$\bar{W}(\hat{\underline{y}}^*) = \frac{1}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \hat{y}_i^* \qquad \ldots \quad (4.21)$$

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[3] However, $\bar{W}_\delta(\underline{y}^*)$ is not strictly quasi-concave on individual incomes.

is quasi-concave or S-concave. Now let a_i's correspond to the single-parameter Ginis, then

$$\bar{W}_{\delta}(\hat{\underline{y}}^*) = \frac{1}{n^{\delta}} \sum_{i=1}^{n} [i^{\delta} - (i-1)^{\delta}] \hat{y}_i^* \quad \ldots \quad (4.22)$$

where $1 \leq \delta < \infty$.

When $\delta = 1$, the associated poverty index $P_1(\hat{\underline{y}}^*) = HP_r$.

When $\delta = 2$, $\bar{W}_{\delta}(\hat{\underline{y}}^*)$ is the Gini swf for the censored income profile and the corresponding poverty index is

$$P_2(\hat{\underline{y}}^*) = \frac{1}{n^2 z} \sum_{i=n-q+1}^{n} (z-\hat{y}_i)(2i-1) \quad \ldots \quad (4.23)$$

Finally, when $\delta \rightarrow \infty$, we get the relative maximin index.

4.3 Absolute Measures of Poverty[4]

We define an absolute poverty index Q as the difference between the poverty line z and the representative income of the community corresponding to $\underline{y}^*$, that is,

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4/ As a first step, following Takayama (1979), one might consider the absolute measures of inequality, when applied to censored income profiles, as absolute measures of poverty. But the difficulty pointed out in Chapter 1, Part II, for relative measures of inequality based on censored income profiles arises also for absolute measures. Hence the aim is not a fruitful one.

$$Q(\underline{y}^*) = z - y_f^* \quad \ldots \quad (4.24)$$

Q lies between zero (no poverty) and z (extreme poverty).

It is desirable to make the measure invariant with respect to translation of z and $\underline{y}^*$. This is possible if and only if the underlying swf is translatable, i.e.,

$$W(\underline{y}^*) = \emptyset [\bar{W}(\underline{y}^*)] \quad \ldots \quad (4.25)$$

where $\emptyset' > 0$ and $\bar{W}(\underline{y}^*)$ is unit translatable, i.e.,

$$\bar{W}(\underline{y}^* + \alpha.\underline{1}) = \bar{W}(\underline{y}^*) + \alpha \quad \ldots \quad (4.26)$$

where α is a scalar such that $\underline{y}^* + \alpha.\underline{1}$ is in the domain of definition of Q. We can rewrite Q as

$$Q(\underline{y}^*) = z - \lambda^* + A(\underline{y}^*) \quad \ldots \quad (4.27)$$

where $A(\underline{y}^*)$ is the Blackorby — Donaldson absolute inequality index based on $\underline{y}^*$, i.e.,

$$A(\underline{y}^*) = \lambda^* - y_f^* \quad \ldots \quad (4.28)$$

Therefore we have the following: For two censored income profiles $\underline{x}^*$ and $\underline{y}^*$ with the same mean λ^*,

$$Q(\underline{y}^*) > Q(\underline{x}^*) \Longleftrightarrow A(\underline{y}^*) > A(\underline{x}^*) \quad \ldots \quad (4.29)$$

Thus, we can have a quasi-ordering of income profiles based on the absolute measures of poverty.

The measure Q $(\underline{y}^*)$ can be interpreted as the difference between the representative income of a community where every unit enjoys the subsistence income z and the representative income of the community S corresponding to $\underline{y}^*$. An absolute measure that depends on absolute differentials only exists for every translatable swf defined on censored income profiles.

Examples

(1) Suppose that the swf has the translation function

$$\frac{1}{n(n+1)} \sum_{i=1}^{n} y_i^* (n+1-i).$$

This swf is translatable as well as homothetic. The corresponding poverty index is

$$Q(\underline{y}^*) = \frac{2}{n(n+1)} \sum_{i=1}^{q} (z-y_i)(n+1-i) \quad \ldots (4.30)$$

Like $\bar{P}$ (for $\epsilon = 1$) (vide Chapter 2), Q is less sensitive to transfers among the very poor units than to transfers at some higher levels among the poor for typical income profiles.

(2) Another welfare function which is translatable as well as homothetic is that underlying the S - Ginis:

$$\bar{W}_{\delta}(\underline{y}^*) = \frac{1}{n^{\delta}} \sum_{i=1}^{n} [i^{\delta} - (i-1)^{\delta}]\, y_i^* , \quad 0 < \delta \leq 1$$

$$\therefore \quad Q_{\delta}(\underline{y}^*) = \frac{1}{n^{\delta}} \sum_{i=1}^{q} (z-y_i)[i^{\delta} - (i-1)^{\delta}] \quad \ldots (4.31)$$

where $0 < \delta \leq 1$.

For $\delta \rightarrow 0$, the measure $\rightarrow z - \min_i \{y_i\}$, the absolute maximin index of poverty; and for $\delta = 1$, we get the product of the head-count ratio and the average absolute poverty gap.

On the other hand, if $\hat{\underline{y}}$ is the welfare - ranked permutation of $\underline{y}$ (i.e., $\hat{\underline{y}} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)'$), and $\hat{\underline{y}}^*$ is the censored income profile corresponding to $\hat{\underline{y}}$, then we may consider the following:

$$\bar{W}_{\beta}(\hat{\underline{y}}^*) = \frac{1}{n^{\beta}} \sum_{i=1}^{n} [i^{\beta} - (i-1)^{\beta}]\, \hat{y}_i^* , \quad \beta \geq 1.$$

In this case

$$Q_{\beta}(\hat{\underline{y}}^*) = \frac{1}{n^{\beta}} \sum_{i=n-q+1}^{n} (z-\hat{y}_i)[i^{\beta} - (i-1)^{\beta}] \quad \ldots (4.32)$$

where $\beta \geq 1$.

For $\beta = 1$, $Q_{\beta}(\hat{\underline{y}}^*)$ is the product of the head-count ratio and the average absolute poverty gap. For $\beta = 2$, the index becomes the absolute Gini index of poverty and as $\beta \to \infty$, we get the absolute maximin index.

(3) Another alternative of interest arises from the Kolm - Pollak swf. Its implicit absolute poverty index is

$$Q_{KP}(\underline{y}^*) = z - [-\frac{1}{\alpha} \log (\frac{1}{n} \sum_{i=1}^{n} e^{-\alpha y_i^*})]$$

$$= \frac{1}{\alpha} \log [\frac{1}{n} \sum_{i=1}^{n} e^{\alpha(z-y_i^*)}] \qquad \ldots (4.33)$$

where $\alpha > 0$.

Here α is a free parameter which determines the curvature of the social indifference surfaces. As α increases, the measure attaches more weight to transfers lower down the income scale. As $\alpha \to \infty$, $Q_{KP} \to z - \min_i \{ y_i^* \}$, the absolute maximin index which corresponds to Rawls' maximin rule.

The general measure $Q(\underline{y}^*)$ introduced in this section incorporates an absolute measure of inequality of a censored income profile for purposes of measurement of poverty. To every translatable swf there corresponds a particular absolute index. These indices will differ in the way they take account of the absolute deprivation of the different poor units in the censored income profile.

4.4 Consistent, Neutral and Ethical Aggregation

Suppose we wish to construct poverty indices for some mutually exclusive groups of a population and use summary statistics from these groups to construct an overall poverty index. There are three different aspects of this aggregation problem. Let $\bar{M} = \{M_1, M_2, \ldots, M_k\}$ be a partition of the index set $M = \{1, 2, \ldots, n\}$. The income space Y is expressed as the cartesian product of $Y^1, Y^2, \ldots, Y^k$ where dimension of Y^i given by cardinality of M_i. The income vector $\underline{y} \in Y$ can be written as $\underline{y} = (\underline{y}(1), \underline{y}(2), \ldots, \underline{y}(k))$ where $\underline{y}(r)$ is the vector of incomes in group $r (r = 1,2,\ldots,k)$.

Following Blackorby and Donaldson (1980) let us define the following :

Definition 1

An aggregation procedure which uses k $(k \geq 2)$ summary statistics, each of which is computed from observations relating to a particular group, is said to be consistent if it generates the same summary statistic which would be arrived at by using the observations for the entire population.

Definition 2

The aggregation procedure is said to be neutral if it is consistent for every union of the elements of $\bar{M}$.

Definition 3

An aggregation procedure is said to be ethical if the same functional form is used for each aggregator and also for the overall function.

Let the cardinality of M_i be n_i. Out of n_i units in M_i, let q_i units be poor ($\sum_{i=1}^{k} n_i = n$, $\sum_{i=1}^{k} q_i = q$).

Let us consider a typical income vector $\underline{y} \in Y$.

$$\underline{y} = (y_{11}, y_{12}, \ldots, y_{1n_1}, y_{21}, \ldots, y_{2n_2}, \ldots\ldots, y_{kn_k})' \quad \ldots \quad (4.34)$$

where y_{ij} : income of unit j belonging to group i. Now the censored income profile corresponding to $\underline{y}$ is given by

$$\underline{y}^* = (y_{11}^*, y_{12}^*, \ldots, y_{1n_1}^*, y_{21}^*, \ldots, y_{kn_k}^*)' \ldots \quad (4.35)$$

where

$$\begin{aligned} y_{ij}^* &= y_{ij} \quad \text{if} \quad y_{ij} < z \\ &= z \quad \text{if} \quad y_{ij} \geq z \end{aligned} \quad \ldots \quad (4.36)$$

Let y_f^* be the representative income according to some homothetic/translatable swf corresponding $\underline{y}^*$. Now suppose $y_f^*(r)$

denotes the corresponding representative income obtained from the censored income profile $\underline{y}^*(r)$. Then $y_f^*(r)$ is defined as

$$W\,(y_f^*(r)\cdot \underline{1}) = W\,(y_{r1}^*, y_{r2}^*, \ldots, y_{rn_r}^*) \quad \ldots (4.37)$$

where $\underline{1}$ is the n_r- coordinated vector of ones. Therefore the condition for consistency in aggregation can be formally stated as :

$$W\,(y_{11}^*, \ldots, y_{kn_k}^*) = W\,(\underbrace{y_f^*(1), \ldots, y_f^*(1)}_{n_1 \text{ times}}, \ldots$$

$$\ldots, \underbrace{y_f^*(r), \ldots, y_f^*(r)}_{n_r \text{ times}}, \ldots, \underbrace{y_f^*(k), \ldots, y_f^*(k)}_{n_k \text{ times}})$$

$$\ldots (4.38)$$

It is evidently desirable to base the poverty indices on representative incomes of censored income profiles where the representative incomes are consistent in ethical aggregation for arbitrary partitions of the population. The poverty indices having such properties are those corresponding to (1) the symmetric mean of order β ($\beta \leq 1$) and (2) the Kolm - Pollak social welfare function for the censored income profile.[5]

5/ The proof of this assertion follows immediately from Eichhorn (1974) and Blackorby and Donaldson (1978, 1980).

It may be interesting to look at the relationship between the within group indices and the over all poverty index. Let us first consider the Kolm — Pollak index of poverty. The representative income corresponding to $\underline{y}^*$ is given by

$$y_f^* = -\frac{1}{\alpha} \log \left[\frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} e^{-\alpha y_{ij}^*} \right]$$

where the curvature parameter α is positive. The representative income for group r corresponding to $\underline{y}^*(r)$ is

$$y_f^*(r) = -\frac{1}{\alpha} \log \left[\frac{1}{n_r} \sum_{j=1}^{n_r} e^{-\alpha y_{rj}^*} \right].$$

Now the poverty index for group r is defined as

$$Q(\underline{y}^*(r)) = Q_r \text{ (say)} = z - y_f^*(r)$$

$$= z + \frac{1}{\alpha} \log \left[\frac{1}{n_r} \sum_{j=1}^{n_r} e^{-\alpha y_{rj}^*} \right]$$

$$= \frac{1}{\alpha} \log \left[\frac{1}{n_r} \sum_{j=1}^{n_r} e^{\alpha(z-y_{rj}^*)} \right]$$

or, $$e^{\alpha Q_r} = \frac{1}{n_r} \sum_{j=1}^{n_r} e^{\alpha(z-y_{rj}^*)} \qquad \ldots (4.39)$$

Now the overall poverty index is

$$Q(\underline{y}^*) = Q \text{ (say)} = z - y_f^*$$

$$= \frac{1}{\alpha} \log \left[\frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} e^{\alpha(z-y_{ij}^*)} \right]$$

$$= \frac{1}{\alpha} \log \left[\sum_{i=1}^{k} \frac{n_i}{n} \cdot \frac{1}{n_i} \sum_{j=1}^{n_i} e^{\alpha(z - y_{ij}^*)} \right]$$

$$= \frac{1}{\alpha} \log \left[\sum_{i=1}^{k} \frac{n_i}{n} \cdot e^{\alpha Q_i} \right] \qquad \text{(From 4.39)}$$

$$\therefore \quad e^{\alpha Q} = \sum_{i=1}^{k} \frac{n_i}{n} \cdot e^{\alpha Q_i} \quad \ldots \quad (4.40)$$

In order to decompose the relative index of poverty it is somewhat simpler to use the relative welfare index. The overall relative welfare index $R(\underline{y}^*)$ is defined as

$$R(\underline{y}^*) = R \text{ (say)} = \frac{y_f^*}{z} = \frac{\left(\frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij}^*)^\beta \right)^{1/\beta}}{z}$$

$$(\beta \leq 1)$$

$$\therefore \quad [R(\underline{y}^*)z]^\beta = \left[\frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij}^*)^\beta \right] \quad \ldots \quad (4.41)$$

Now the corresponding index for group i is defined as

$$R(\underline{y}^*(i)) = R_i \text{ (say)} = \left(\frac{1}{n_i} \sum_{j=1}^{n_i} (y_{ij}^*)^\beta \right)^{1/\beta} / z$$

$$\text{or} \quad [R_i \cdot z]^\beta = \frac{1}{n_i} \sum_{j=1}^{n_i} (y_{ij}^*)^\beta \text{ (say)} \quad \ldots \quad (4.42)$$

Substituting (4.42) in (4.41) we get

$$R^{\beta} = \sum_{i=1}^{k} \frac{n_i}{n} \cdot [R_i]^{\beta} \qquad \ldots \qquad (4.43)$$

If a population is divided into different groups according to certain socio-economic characteristics of units, results (4.40) and (4.43) would be useful if one is interested in assessing the influence of each group on the overall index.

4.5 Axiomatic Derivation of Two Classes of Poverty Measures

In this section we shall axiomatise the following classes of poverty measures:

$$P(\underline{y}^*) = 1 - \frac{\left[\frac{1}{n} \sum_{i=1}^{n} (y_i^*)^{\beta} \right]^{1/\beta}}{z}, \qquad \beta \leq 1$$

$$Q(\underline{y}^*) = \frac{1}{\alpha} \log \left[\frac{1}{n} \sum_{i=1}^{n} e^{\alpha(z-y_i^*)} \right], \qquad \alpha > 0$$

that is, we shall find for each of the above-noted measures a set of axioms which lead to it.

We define a relative (or an absolute) poverty index P (or Q) as a function of the censored income profile and of the poverty line z .

We consider the following axioms:

(1) Zero at equality of the incomes of the poor with the poverty line z.

This axiom says that if all the poor units enjoy subsistence level income z then the value of the poverty index should be zero.[6]

(2) Impartiality: A permutation of the incomes should leave the value of the index unchanged.[7]

To derive axiomatically the measures noted-above we introduce two more axioms.

(3) Welfare independence I: If P be the poverty index, then

6/ While this axiom is concerned with the minimum value of a poverty index, the upper bound of a poverty index (given z) is attained when all the n units have zero income. The measures discussed in Chapters 2, 3 and 4 attain both these bounds. On the other hand, the Takayama (1979) and the Hamada and Takayama (1977) formulae are not even defined when all the y_i's are equal to zero.

7/ In addition to these two axioms a poverty index may be required to satisfy Sen's monotonicity and transfer axioms, axiom M', axiom of diminishing transfers, the axiom of scale irrelevance (or translation invariance) and the population symmetry axiom. All these axioms are set out in Chapter 1 (Part II) with the exception of the population symmetry axiom which is stated as follows: If two or more indentical populations were pooled together, the poverty index for the whole and for each part should be the same.

$$\frac{\delta\,[(1-P)z]/\delta\, y_i^*}{\delta\,[(1-P)z]/\delta\, y_j^*} \quad \text{is independent of} \quad y_k^*$$

for all $y_i^* \neq y_j^* \neq y_k^*$.

(4) Welfare independence II : If Q be the poverty index, then

$$\frac{\delta\,[(z-Q)]/\delta\, y_i^*}{\delta\,[(z-Q)]/\delta\, y_j^*} \quad \text{is independent of} \quad y_k^*$$

for all $y_i^* \neq y_j^* \neq y_k^*$.

These two axioms require, in some sense, that the marginal social rate of substitution between the censored incomes accruing to unit i and unit j is independent of the censored income level of any other unit.

Theorem 1

(a) The set of axioms (1), (2), (3) and the axiom of scale irrelevance are satisfied together if and only if the (relative) poverty index has the form

$$P(\underline{y}^*) = 1 - \frac{\left[\frac{1}{n} \sum_{i=1}^{n} (y_i^*)^{\beta}\right]^{1/\beta}}{z} .$$

(b) These axioms plus Sen's transfer axiom hold if and only if, furthermore $\beta < 1$.

Theorem 2

(a) The set of axioms (1), (2), (4) and the axiom of translation invariance hold together if and only if the (absolute) poverty index is of the form

$$Q\,(\underline{y}^*) = \frac{1}{\alpha} \log \Big[\frac{1}{n} \sum_{i=1}^{n} e^{\alpha(z-y_i^*)} \Big].$$

(b) These axioms plus Sen's transfer axiom hold if and only if, furthermore $\alpha > 0$.

The proofs of these results are similar respectively to those of results 2^o(a), 2^o(b), 1^o(a) and 1^o(b) of Kolm (1976) and hence omitted.

4.6 Compromise Indices of Poverty

Kolm (1976a) suggested that it would be useful to have indices of inequality "which measure relative as well as absolute changes". One may consider a similar problem for poverty measurement. An absolute index $Q\,(\underline{y}^*)$ is a compromise absolute index of poverty if and only if

$$P\,(\underline{y}^*) = \frac{Q\,(\underline{y}^*)}{z} \qquad \ldots \qquad (4.44)$$

is a relative poverty index, i.e., it is homogeneous of degree zero in $\underline{y}^*$ and z.

Examples

(1) Suppose that the translation function associated with the swf is given by

$$\bar{W}(\underline{y}^*) = \frac{2}{n(n+1)} \sum_{i=1}^{n} y_i^* (n+1-i).$$

Then

$$Q(\underline{y}^*) = \frac{2}{n(n+1)} \sum_{i=1}^{q} (z-y_i)(n+1-i) .$$

This $Q(\underline{y}^*)$ is a compromise absolute index because

$$\frac{Q(\underline{y}^*)}{z} = P(\underline{y}^*) = \frac{2}{n(n+1)z} \sum_{i=1}^{q} (z-y_i)(n+1-i)$$

is a relative index of poverty.

(2) Suppose that the swf has the translation function corresponding to the single series absolute Ginis, that is,

$$\bar{W}_\delta(\underline{y}^*) = \frac{1}{n^\delta} \sum_{i=1}^{n} [i^\delta - (i-1)^\delta] y_i^*, \ 0 < \delta \leq 1.$$

Then the corresponding absolute index is

$$Q_{\delta}(\underline{y}^*) = \frac{1}{n^{\delta}} \sum_{i=1}^{q} (z-y_i)[i^{\delta} - (i-1)^{\delta}]$$

and the relative index associated with this compromise index is

$$P_{\delta}(\underline{y}^*) = \frac{1}{n^{\delta} z} \sum_{i=1}^{q} (z-y_i)[i^{\delta} - (i-1)^{\delta}].$$

4.7 Conclusion

While Takayama's index and Blackorby and Donaldson's ethical indices donot satisfy the monotonicity and the transfer axioms respectively, the general ethical indices introduced in this chapter satisfy both the axioms. These indices make use of the notion of the representative income of the censored income profile. To every homothetic swf there corresponds a relative poverty measure and for every translatable swf there exists a corresponding absolute poverty index.

Considerations of consistency in aggregation lead us to two indices whose underlying swf's are the symmetric mean of order $\beta(\beta \leq 1)$ and the Kolm - Pollak swf. These are known to be the only indices which are consistent, neutral and ethical in representative income aggregation. Therefore they are the most appropriate indices for studying the influence of group poverty on overall degree of poverty. In Section 5, the same indices are derived starting from two different sets of axioms.

Finally in Section 6 we look for absolute poverty measures that become relative measures when divided by the poverty line.

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CHAPTER 5

NORMATIVE APPROACHES LEADING TO THEIL'S ENTROPY MEASURE OF INEQUALITY

5.1 Introduction

This chapter reports two results on inequality measurement through the normative approach both leading to Theil's index. While the normative aspects of the Gini coefficient have been discussed extensively [see, for example, Newbery (1970), Sheshinski (1972), Kats (1972), Dasgupta, Sen and Starrett (1973), Rothschild and Stiglitz (1973)], Theil's ingenious measure [Theil (1967)] derived from the notion of entropy in information theory is yet to be discussed thoroughly. The next section proposes a set of axioms and derives from them a normative measure of inequality. It is then shown that this measure, in a limiting case, coincides with the normalised Theil's entropy measure. In Section 3 we propose a complete axiomatisation of a social welfare function such that the welfare ranking of all possible income profiles coincides with their ranking by the Theil index.

5.2 A General Normative Index of Inequality

5.2.1 Formulation and Definition

Let U_i stand for the utility function of the ith unit and assume that U_i depends only on y_i, the income that the ith unit receives. Further assume that the individual utility functions are the same for all, i.e., $U_i(.) = U(.)$, and that U is increasing and concave.

Following Dalton (1920) and Atkinson (1970) let us assume that the welfare value of an income vector $\underline{y} = (y_1, y_2, \ldots, y_n)'$ is given by

$$W(\underline{y}) = \sum_{i=1}^{n} U(y_i) \quad \ldots \quad (5.1)$$

In the context of measurement of inequality of income profiles the above from of swf has been used extensively.[1,2]

Since the individual utility functions are the same for all, the aggregate welfare will be at its maximum level when all the units have equal income.

1/ However, additive separability is a strong condition to impose on a general welfare function. Hamada (1973) investigates an axiomatic structure of additive separability.

2/ This form of swf $(n > 1)$ is found to be quasi-concave if and only if it is concave [Cox (1973)].

Definition

For a given income configuration $\underline{y}$, the index of inequality I is defined to be the normalised value of the amount by which the aggregate welfare of the actual profile falls short of its maximum.

In symbols, the index is

$$I = A(\lambda, n)[W(\underline{\lambda}) - W(\underline{y})] \quad \ldots \quad (5.2)$$

where $A(\lambda, n) > 0$ is the coefficient of normalisation and

$$\underline{\lambda} = (\lambda, \lambda, \ldots, \lambda)' \quad \ldots \quad (5.3)$$

In essence, the index I treats perfect equality of incomes as the benchmark and measures the distance separating the income profiles under study from the state of perfect equality.

5.2.2 Desiderata of a Satisfactory Measure

It is obvious from the definition of the inequality measure that it takes the value zero if all the incomes are equal.

In contrast to perfect equality we can conceptualise complete inequality in a situation described by the following axiom:

Axiom of Normalisation (A1)

If the richest unit of the community gets the total income and all the other units receive zero income, then the value of the inequality index is unity.

However, as mentioned in Chapter 1, an inequality measure satisfying this axiom does not meet Dalton's principle of population. But since our main aim is to axiomatise a standard measure of inequality rather than to introduce a new inequality index we can use the above normalisation axiom unambiguously.[3,4] The measure I attains its highest value in the extreme case noted above. In general it lies between zero and one.

The inequality index sought here is a relative index, that is, it is required to satisfy:

3/ A measure that satisfies axiom (A1) is the Gini coefficient without repetition, a variant of the more commonly mentioned Gini coefficient (with repetition). The former may be written as [Kendall and Stuart (1977)]:

$$G' = -\sum_{i=1}^{n} y_i(n+1-2i)/n(n-1)\lambda$$

where $y_1 \leq y_2 \leq \cdots \leq y_n$.

The same remark applies to the Atkinson measure in the special case where $\epsilon = 1$, so that $A = 1 - \frac{\prod_{i=1}^{n} (y_i)^{1/n}}{\lambda}$.

4/ See also Blackorby and Donaldson (1978) who normalised the Theil index in the interval [0, 1] to look at the homotheti[c] swf that corresponds to it.

Axiom of Scale Invariance (A2)

An equi-proportional variation in all incomes doesnot change the inequality.

5.2.3 The Result

Theorem 5.1: The inequality measure I satisfies axioms (A1) and (A2) if and only if it has the following form:

$$I = \frac{1}{1-n^{\epsilon-1}} \left[1 - \frac{1}{n} \sum_{i=1}^{n} (y_i/\lambda)^{\epsilon} \right] \quad \text{if} \quad 0 < \epsilon < 1 \qquad \ldots \qquad (5.4)$$

$$= \frac{1}{(\log n)} \cdot \frac{1}{n} \sum_{i=1}^{n} \left(\frac{y_i}{\lambda}\right) \log\left(\frac{y_i}{\lambda}\right) \quad \text{if} \quad \epsilon = 1.$$

Proof: Assume without loss of generality that

$$y_1 \leq y_2 \leq y_3 \leq \ldots\ldots \leq y_n \qquad \ldots \qquad (5.5)$$

From (5.2) we have

$$I = A(\lambda, n)[nU(\lambda) - \sum_{i=1}^{n} U(y_i)] \qquad \ldots \qquad (5.6)$$

In the special case when the richest unit receives all the income and all the other units receive zero income, we have

$$I = A(\lambda, n)[nU(\lambda) - (n-1)\,U(0) - U(n\lambda)] \quad ..(5.7)$$

But according to axiom (A1) we have for this extreme case

$$I = 1 \quad \ldots \quad (5.8)$$

From (5.7) and (5.8) it follows that

$$A(\lambda, n) = \frac{1}{nU(\lambda) - (n-1)\,U(0) - U(n\lambda)} \quad \ldots (5.9)$$

From (5.6) and (5.9) we get

$$I = \frac{1}{nU(\lambda) - (n-1)\,U(0) - U(n\lambda)} \left[nU(\lambda) - \sum_{i=1}^{n} U(y_i)\right].^{5/}$$

$$\ldots \quad (5.10)$$

Define $$f(x) = U(x) - U(0) \quad \ldots \quad (5.11)$$

I can be rewritten as

$$I = \frac{1}{nf(\lambda) - f(n\lambda)} \left[\sum_{i=1}^{n} (f(\lambda) - f(y_i))\right] \ldots \quad (5.12)$$

$$I = \frac{1 - \frac{1}{n}\sum_{i=1}^{n}\frac{f(y_i)}{f(\lambda)}}{1 - \frac{f(n\lambda)}{nf(\lambda)}} \quad \ldots \quad (5.13)$$

Invoke axiom (A2). It can be seen that for I to be homogeneous of degree zero in $(y_1, \ldots, y_n)$ we need $\frac{f(u)}{f(v)}$ to be a function of u/v .

5/ Axiom (A1) keeps I invariant under affine transformations of U(.).

This shows

$$f(x) = Bx^{\epsilon}, \quad B > 0$$ [*vide* Proof of Theorem 2.1 in Chapter 2].

$$U(x) = U(0) + f(x)$$

$$= U(0) + Bx^{\epsilon}$$

$$= A + Bx^{\epsilon} \quad \text{(say)} \qquad \ldots \qquad (5.14)$$

where $B > 0$.

Since $U(.)$ is increasing and concave in its agrument, we need $0 < \epsilon \leq 1$ and $\epsilon < 1$ for strict concavity.[6] Substituting $U(.)$ given by (5.14) in (5.10) we get

$$I = \frac{1}{n\lambda^{\epsilon} - (n\lambda)^{\epsilon}} \sum_{i=1}^{n} [\lambda^{\epsilon} - y_i^{\epsilon}] . \quad 0 < \epsilon < 1.$$

$$= \frac{1}{1 - n^{\epsilon-1}} [1 - \frac{1}{n} \sum_{i=1}^{n} (y_i/\lambda)^{\epsilon}] . \quad 0 < \epsilon < 1.$$

$$\ldots \qquad (5.15)$$

As $\epsilon \to 1$, by L'Hospital's rule we get

$$I = \frac{1}{(\log n)} \cdot \frac{1}{n} \sum_{i=1}^{n} (\frac{y_i}{\lambda}) \log(\frac{y_i}{\lambda}) \qquad \ldots \qquad (5.16)$$

which is the normalised entropy measure.

6/ $(\epsilon - 1)$ is the constant elasticity of marginal utility of income.

The sufficiency part can be verified easily.[7/]

□

5.3 An Alternative Axiomatisation of the Entropy Measure

In Section 2 of this chapter we have obtained the normalised Theil measure as a limiting case of a general normative measure of inequality. Here we make a complete axiomatisation of a social welfare function that will rank all possible income profiles, with the same number of earners and the same mean of income, in the same way as the Theil measure.

For this purpose let us assume that s_i represents the income share of the ith unit of the community S ; $s_i = \frac{y_i}{n\lambda} \geq 0$, $\sum_{i=1}^{n} s_i = 1$. We assume that $\lambda > 0$ is finite and is the same for all income profiles under comparison.

$$\text{Let } \underline{s} = (s_1, s_2, \ldots, s_n)' \qquad \ldots \qquad (5.17)$$

Assume that $(\underline{s}, i)$ is the state of unit i in the social state $\underline{s}$. We choose the following weighted sum form of the social welfare function :

$$W(\underline{s}) = \sum_{i=1}^{n} s_i V(s_i) \qquad \ldots \qquad (5.18)$$

7/ A transfer of income from unit i to unit j will increase I by a larger amount, the richer the unit j (for fixed i). As ϵ decreases, I becomes more sensitive to transfers lower down the income scale.

The weight on s_i is given by $V(s_i) > 0$, a function of the income share of unit i.

We will now propose a set of axioms which seem to be appealing. These axioms correspond to exactly the entropy measure helping us to understand it.

Axiom M (Independent Monotonicity):

For all $\underline{s}$, all units regard $(\underline{s}, i)$ to be at least as good as $(\underline{s}, j)$ if and only if $s_i > s_j$.

Axiom M states that each unit decides its preference on the basis of its income alone and it is assumed that it prefers more to less.

Axiom E (Monotonic Equity):

If everyone prefers $(\underline{s}, i)$ to $(\underline{s}, j)$ then $V(s_i) < V(s_j)$.

This axiom gives a relativist view. It says that if in the social state $\underline{s}$, the position of a unit j is regarded to be worse off than that of anyother unit i then unit j deserves greater weightage in the evaluation scheme (5.18).

Axiom D (Weight Differential Information):

If everyone prefers $(\underline{s}, i)$ to $(\underline{s}, j)$ then the weight differential $V(s_j) - V(s_i)$ will depend only on s_i/s_j, the income share of the ith unit relative to that of unit j, that is, only on their relative income.

This axiom implies that so long as relative income of two units remain constant, that is, if their incomes are allowed to vary through a ray of constant proportions, the weight differential will remain constant.

Theorem 5.2 : A social welfare function of the form (5.18) satisfying axioms M , E and D must rank all possible income profiles, with a given total of income, over a given population in exactly the same way as the negative of Theil's entropy measure of inequality.

Proof : Consider any two units i and j. By axiom D,

$$V(s_j) - V(s_i) \text{ depends only on } \frac{s_i}{s_j}.$$

Therefore we can write

$$V(s_j) - V(s_i) = f\left(\frac{s_i}{s_j}\right) \qquad \ldots \qquad (5.19)$$

where f is some real valued function whose domain is the interval $(0, \infty)$.

Choose $c \in (0, 1)$.

$$f\left(s_i \cdot \frac{1}{s_j}\right) = f\left(\frac{s_i}{s_j}\right)$$

$$= f\left(\frac{Cs_i}{Cs_j}\right)$$

$$= V(Cs_j) - V(Cs_i)$$

$$= V(Cs_j) - V(C) + V(C) - V(Cs_i)$$

$$= [V(C) - V(Cs_i)] + [V(Cs_j) - V(C)]$$

$$= f\left(\frac{Cs_i}{C}\right) + f\left(\frac{C}{Cs_j}\right)$$

$$= f(s_i) + f\left(\frac{1}{s_j}\right) \qquad \ldots \qquad (5.20)$$

Equation (5.20) is the well-known Cauchy equation [see Eichhorn (1978), pp: 12-13].

Equation (5.20) alongwith axioms M and E implies that

$$V(s_j) = A \log\left(\frac{1}{s_j}\right) \qquad \ldots \qquad (5.21)$$

where $A > 0$ is independent of s_j.

This yields,

$$W(\underline{s}) = \sum_{i=1}^{n} s_i \left(A \log \frac{1}{s_i}\right)$$

$$= A \sum_{i=1}^{n} s_i \log\left(\frac{1}{s_i}\right) \qquad \ldots \qquad (5.22)$$

The Theil entropy measure of inequality based on $\underline{s}$ can be written as

$$T(\underline{s}) = \log n - \sum_{i=1}^{n} s_i \log\left(\frac{1}{s_i}\right) \quad \ldots \quad (5.23)$$

It is clear from (5.22) and (5.23) that given n and λ, the ordering of S yielded by $W(\underline{s})$ is exactly the opposite that given by $T(\underline{s})$.

□

The theorem provides a complete axiomatisation of the Theil measure of inequality. It is interesting to note that given the weight differential information axiom, the other axioms are necessary and sufficient for the group welfare function to be a negative transformation of T. In this sense the axiom system specifies a set of necessary and sufficient conditions for the welfare interpretation of the entropy measure.

5.4 Conclusion

This chapter proposes two axiomatisations of Theil's entropy measure of inequality which appears to have been somewhat neglected in comparison with the other measures of inequality, partly because its nature and properties are not so obvious.

In the first, we have chosen the utilitarian form of swf. A normalised value of the amount by which the social welfare level of the egalitarian income profile dominates that of the actual profile has been regarded as a measure of inequality. The measure is to remain invariant under affine transformations of the common utility function. This approach leads to a measure which in a limiting case coincides with the Theil entropy measure.

In the second part, we consider a swf which is the weighted sum of individual incomes and require the weights to satisfy some axioms incorporating equity considerations. For a given population size and a given mean income, this swf is found to rank all income profiles in the same way as the negative of the entropy measure.

These results may lend some theoretical support to the Theil measure of inequality.

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CHAPTER 6

SOME MODIFICATIONS OF THE GINI MEASURE OF INEQUALTY

6.1 Introduction

The Gini coefficient, the most frequently used positive measure of income inequality, is known to be incapable of ranking income profiles according to some strictly quasi-concave swf [Dasgupta, Sen and Starrett (1973), Rothschild and Stiglitz (1973)]. This chapter proposes a parametric generalisation of the measure which throws up some indices not suffering from this deficiency. The next section suggests the generalisation and the third section makes some remarks on the measure.

Section 4 examines the consequences of an alternative formulation leading to a generalisation of the Weymark (1979) measure of inequality. For an interesting subclass of this generalised measure we get a one-parameter class of indices which contains the Gini coefficient as a special case. Many members of this subclass possess the property of ranking income profiles according to some strictly quasi-concave swf. Another subclass becomes the well-known Atkinson indices of inequality.

6.2 A Parametric Generalization of the Gini Coefficient

We know that given an ordered set of incomes $y_1 \leq y_2 \leq \ldots$ $\ldots \leq y_n$, the empirical Lorenz curve generated by them is based on the points (i/n, L(i/n)), where i = 0, 1 ...n and L(0) = 0, $L(i/n) = \frac{z_i}{z_n}$ where $z_i = \sum_{j=1}^{i} y_j$. If n is finite, the empirical Lorenz curve, L(p), is then defined for all p in the interval (0 , 1) by joining the points defined above successively by straight lines. Broadly speaking, L(p) represents the fraction of total income that the earners forming the bottom 100 p% of the population possess. We will, however, concentrate on the case where n is quite large and the income distribution will be treated as continuous. For this purpose we consider the numbers y_i as a sample drawn from a distribution y. Let F be the cumulative distribution function of y (F(u) is the proportion of population receiving income less than or equal to u) on the internal $(0, \infty)$. We assume F(0) = 0 $F(\infty) = 1$ and F is piecewise continuous. Since F is monotonically increasing, it has, at most, a countable number of points of discontinuity. This assumption is compatible with Stieltjes or Lebesgue integration.

We define the inverse $F^{-1}(t)$ of F as

$$F^{-1}(t) = \inf_u \{ u : F(u) \geq t \} \quad \ldots \quad (6.1)$$

This gives the income of a unit at the t-th quantile.

We assume that the mean of the distribution, λ, is finite :

$$\lambda = \int_0^\infty y \, dF(y) \qquad \ldots \qquad (6.2)$$

Now the Lorenz curve corresponding to any continuous type random variable $y \geq 0$ with finite mean having distribution function F is defined as

$$L(p) = \lambda^{-1} \int_0^p F^{-1}(t) \, dt \qquad \ldots \qquad (6.3)$$

[Gastwirth (1971)].

The curve $L(p)$ is a convex function of p and $L(p) \leq p \;\forall\; 0 \leq p \leq 1$. The straight line $L(p) = p$ represents the position of the curve in the case of perfect equality. On the other hand, $L(p) = 0$, $0 \leq p < 1$ and $L(1) = 1$ represents the position of the curve in a situation of extreme inequality, where only one earner has a positive income.

Thus the divergence between the Lorenz curve for an income distribution and the line of perfect equality is given by the function

$$H(p) = p - L(p) \qquad \ldots \qquad (6.4)$$

The well-known Gini coefficient for the random variable y is given by

$$2 \int_0^1 [p - L(p)]\,dp \quad \text{1/} \qquad \ldots \qquad (6.5)$$

As noted above, this measure has been rightly criticised for not ranking the income distributions according to any strictly quasi-concave swf.

In this chapter we seek to generalize or modify the Gini coefficient so as to take care of this deficiency. Let the divergence function for the random variable y be given by

$$d(p, \alpha) = \frac{1}{\alpha} [p - L(p)]^{\alpha} \qquad \ldots \qquad (6.6)$$

Later we will show that $\alpha > 1$ is needed to ensure the strict convexity of the measure proposed here.

Now define the equal equivalent divergence d_e of the distribution of y according to the divergence function (6.6) as that level of divergence which is independent of p and for which the following equality holds:

1/ The Gini coefficient G of a random variable $y \geq 0$ with finite mean λ having distribution function F is also shown to be

$$G = 1 - \frac{1}{\lambda} \int_0^{\infty} (1-F(t))^2\,dt.$$

This formula was derived by Dorfman (1979). For a simpler proof of this result see Appendix II.

$$\int_0^1 \frac{1}{\alpha} [d_e]^\alpha \, dp = \int_0^1 d(p, \alpha) \, dp \quad \ldots \quad (6.7)$$

This gives

$$d_e = [\int_0^1 [p - L(p)]^\alpha \, dp]^{1/\alpha} \quad \ldots \quad (6.8)$$

We finally define the new inequality index I of the distribution as

$$I = d_e = [\int_0^1 [p - L(p)]^\alpha \, dp]^{1/\alpha} \quad \ldots \quad (6.9)$$

In essence, we are measuring the extent of inequality in a distribution by averaging the divergence function by using a general averaging method. If $\alpha = 1$, the measure coinsides with half of the Gini coefficient.

6.3 Some Remarks on the Generalized Coefficient

(1) To show that I ranks income distributions according to a strictly concave swf it is enough to show that I is strictly convex in incomes. For this let us consider two income distributions with distribution functions F_1 and F_2 respectively such that

$$\int_0^\infty x_1 \, dF_1(x_1) = \int_0^\infty x_2 \, dF_2(x_2) = \lambda \quad \ldots \quad (6.10)$$

where $0 < \lambda < \infty$.

We now generate an income distribution with distribution function F_3 by considering a convex combination of the two distributions mentioned above:

$$F_3^{-1}(t) = \theta F_1^{-1}(t) + (1-\theta) F_2^{-1}(t) \quad \ldots \quad (6.11)$$

where $0 < \theta < 1$, and $0 \leq t \leq 1$.[2]

$$\therefore \quad \frac{1}{\lambda} F_3^{-1}(t) = \frac{\theta}{\lambda} F_1^{-1}(t) + \frac{(1-\theta)}{\lambda} F_2^{-1}(t) .$$

$$\therefore \quad \frac{1}{\lambda} \int_0^p F_3^{-1}(t)dt = \frac{\theta}{\lambda} \int_0^p F_1^{-1}(t)dt + \frac{(1-\theta)}{\lambda} \int_0^p F_2^{-1}(t)dt$$

or, $$L_3(p) = \theta L_1(p) + (1-\theta) L_2(p) \quad \text{(say)}.$$

$$\therefore \quad p-L_3(p) = \theta(p-L_1(p)) + (1-\theta)(p-L_2(p)) \quad \ldots \quad (6.12)$$

---

[2] In the discrete case consider two income vectors $\underline{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n})$, $\underline{x}_2 = (x_{21}, x_{22}, \ldots, x_{2n})$ where $x_{11} \leq x_{12} \leq \ldots \leq x_{1n}$, $x_{21} \leq x_{22} \leq \ldots \leq x_{2n}$ and $\sum_{i=1}^{n} x_{1i} = \sum_{i=1}^{n} x_{2i}$. Consider a third income vector $\underline{x}_3 = (x_{31}, x_{32}, \ldots, x_{3n})$ where $x_{3i} = \theta x_{1i} + (1-\theta) x_{2i}$. If F_1 and F_2 are the distribution functions for $\underline{x}_1$ and $\underline{x}_2$ respectively, then F_3 is the distribution function for $\underline{x}_3$.

We shall show that

$$I_3 < \theta I_1 + (1-\theta) I_2$$

where

$$I_i = [\int_0^1 [p - L_i(p)]^\alpha dp]^{1/\alpha}, \quad i = 1, 2, 3.$$

$$I_3 = [\int_0^1 [p - L_3(p)]^\alpha dp]^{1/\alpha}$$

$$= [\int_0^1 [\theta(p-L_1(p)) + (1-\theta)(p - L_2(p))]^\alpha dp]^{1/\alpha}$$

$$< [\int_0^1 [\theta(p-L_1(p))]^\alpha dp]^{1/\alpha} + [\int_0^1 [(1-\theta)(p-L_2(p))]^\alpha dp]^{1/\alpha}$$

[by Minkowski's inequality if $\alpha > 1$ [3]]

$$= \theta[\int_0^1 [p-L_1(p)]^\alpha dp]^{1/\alpha} + (1-\theta)[\int_0^1 [p-L_2(p)]^\alpha dp]^{1/\alpha}$$

$$= \theta I_1 + (1-\theta) I_2 \qquad \ldots \qquad (6.13)$$

(2) I is increasing in α. As $\alpha \rightarrow \infty$,

---

[3] See Hardy, Littlewood and Polya (1964), p. 146.

$$I \longrightarrow \sup_{p \in [0,1]} [p - L(p)]^{4/} \quad \ldots \quad (6.14)$$

(3) To illustrate the formula, we consider two simple examples, the pareto distribution and the exponential distribution.

In the case of the pareto distribution,

$$F(y) = 0 \quad \text{if} \quad y < A \quad , \quad A > 0$$
$$= 1 - \left(\frac{A}{y}\right)^{\Theta} \quad \text{if} \quad A \leq y < \infty \quad \ldots \quad (6.15)$$

where $\Theta > 1$.

$$\lambda = A\,\Theta/(\Theta - 1) \quad \ldots \quad (6.16)$$

$$F^{-1}(t) = \frac{A}{(1-t)^{1/\Theta}} \quad (0 < t < 1) \quad \ldots \quad (6.17)$$

$$L(p) = \frac{1}{\lambda} \int_0^p \frac{A}{(1-t)^{1/\Theta}}\,dt$$
$$= 1 - (1-p)^{\frac{\Theta - 1}{\Theta}} \quad \ldots \quad (6.18)$$

4/ See Rudin (1966), p. 73.

$$\therefore \quad p - L(p) = (1-p)^{\frac{\theta-1}{\theta}} - (1-p) \quad \ldots \quad (6.19)$$

$$\therefore \quad I = [\int_0^1 [p - L(p)]^{\alpha} dp]^{1/\alpha}$$

$$= [\int_0^1 [(1-p)^{\frac{\theta-1}{\theta}} - (1-p)]^{\alpha} dp]^{1/\alpha}$$

$$= [\theta B(\alpha + 1, \alpha\theta + \theta - \alpha)]^{1/\alpha} \quad \ldots \quad (6.20)$$

where $B(m, n)$ is the Beta function defined as

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \ldots \quad (6.21)$$

Levine and Singer (1970) considered the Lorenz curve of the exponential distribution. Here

$$F(y) = 0 \quad \text{if} \quad y < 0$$

$$= 1-e^{-\beta y} \quad \text{if} \quad y \geq 0 \quad \ldots \quad (6.22)$$

where $\beta > 0$. We get

$$\lambda = \beta^{-1} \quad \ldots \quad (6.23)$$

$$F^{-1}(t) = -\beta^{-1} \log(1-t) \quad \ldots \quad (6.24)$$

$$L(p) = \int_o^p [-\log(1-t)]dt$$

$$= p - (1-p) \log[\frac{1}{(1-p)}] \quad \ldots \quad (6.25)$$

$$\therefore \quad p-L(p) = (1-p) \log[\frac{1}{(1-p)}] \quad \ldots \quad (6.26)$$

In this case we get

$$I = [\int_o^1 [(1-p) \log \frac{1}{(1-p)}]^{\alpha} dp]^{1/\alpha}$$

$$= [\frac{\sqrt{\alpha + 1}}{(\alpha + 1)^{\alpha+1}}]^{1/\alpha} \quad \ldots \quad (6.27)$$

(4) For discrete case we may use the expression

$$I = [\frac{1}{n} \sum_{i=1}^{n} (i/n - L(i/n))^{\alpha}]^{1/\alpha} \quad \ldots \quad (6.28)$$

as an analogue of I given by (6.9).

For straight line joining of successive points (i/n, L(i/n)) trapezoidal rule gives [see Bhattacharya and Mahalanobis (1967) for a discussion on this point]:

$$\text{Gini coefficient} = 1 - 2 \cdot \frac{1}{n} \sum_{i=1}^{n} \frac{L(i/n) + L(\frac{i-1}{n})}{2}$$

$$= 1 - \frac{1}{n} \sum_{i=1}^{n} [L(i/n) + L(\frac{i-1}{n})]$$

$$= 1 - \frac{2}{n} \sum_{i=1}^{n} L(i/n) + \frac{1}{n} \cdot 1$$

$$= 1 + \frac{1}{n} - \frac{2}{n} \sum_{i=1}^{n} L(i/n) \quad \ldots \quad (6.29)$$

On the other hand, I given by (6.28) for $\alpha = 1$ gives

$$I = \frac{1}{n} \sum_{i=1}^{n} (i/n) - \frac{1}{n} \sum_{i=1}^{n} L(i/n)$$

$$= \frac{1}{2} (1 + \frac{1}{n}) - \frac{1}{n} \sum_{i=1}^{n} L(i/n)$$

which is half of the expression in (6.29).

So we can use $I = [\frac{1}{n} \sum_{i=1}^{n} (i/n - L(i/n))^{\alpha}]^{1/\alpha}$ with $\alpha > 1$

as a modification of the Gini coefficient for the discrete case.

6.4 Consequences of an Alternative Formulation

In this section the income distribution will be treated as discrete and let us assume that n incomes are arranged in non-increasing order, that is,

$$y_1 \geq y_2 \geq \dots \geq y_n \qquad \dots \qquad (6.30)$$

The Gini coefficient [Sen (1973)] is

$$G = 1 + \frac{1}{n} - \frac{2}{n^2 \lambda} \sum_{i=1}^{n} i\, y_i \qquad \dots \qquad (6.31)$$

$$= 1 - \frac{1}{n^2 \lambda} \sum_{i=1}^{n} (2i - 1) y_i \qquad \dots \qquad (6.32)$$

The Gini social welfare function has the image [Blackorby and Donaldson (1978)]:

$$\bar{W}_G(\underline{y}) = \frac{1}{n^2} \sum_{i=1}^{n} (2i - 1) y_i \qquad \dots \qquad (6.33)$$

It may be noted that if $\underline{y} = \underline{1}$, then $\bar{W}_G(\underline{1}) = 1$. The coefficients in the Gini index are the first n odd numbers and their sum is n^2. This observation will be used in defining a generalised family of indices.

A priori there seems to be no special status accorded to the weights $\{1, 2, \dots, (2i - 1) \dots\}$ found in (6.33). We can replace the coefficients in (6.33) by general coefficients $\{ a_1, a_2, \ldots, a_i, \ldots \}$ where $0 < a_1 \leqq a_2 \leqq \ldots \leqq a_n$ for all $n \in N$.

This procedure yields a class of generalised Gini indices, with representative income given by [Weymark (1979)]:

$$E_G(\underline{y}) = \frac{1}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i y_i \quad \ldots \quad (6.34)$$

However, $E_G(\underline{y})$ is not strictly quasi-concave on individual incomes. To remove this deficiency, let the (homothetic) swf be given by the weighted mean of order r $(r < 1)$ with a_i's as weights.

That is,

$$\bar{W}_r(\underline{y}) = \left[\frac{1}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i y_i^r\right]^{1/r} \quad \ldots \quad (6.35)$$ 5/

The corresponding AKS relative inequality index is

$$\bar{I} = 1 - \frac{1}{\lambda}\left[\frac{1}{\sum_{i=1}^{n} a_i} \cdot \sum_{i=1}^{n} a_i y_i^r\right]^{1/r}, \quad r < 1, \ r \neq 0 \quad \ldots (6.36)$$

$$= 1 - \frac{1}{\lambda} \prod_{i=1}^{n} (y_i^{a_i})^{\frac{1}{\sum_{i=1}^{n} a_i}}, \qquad r = 0 .$$

5/ However, $\bar{W}_r(\underline{y})$ is not unit-translatable.

As $r \rightarrow -\infty$, $\bar{I} \rightarrow 1 - \frac{\min_i \{y_i\}}{\lambda}$, the relative maximin index. On the other hand, for $r = 1$ we get

$$\bar{I} = 1 - \frac{1}{\lambda \left(\sum_{i=1}^{n} a_i \right)} \cdot \sum_{i=1}^{n} a_i \, y_i \quad \ldots \quad (6.37)$$

which is the measure suggested by Weymark.

The measure in (6.36) is a generalisation of the Weymark measure. The measure is strictly quasi-convex in individual incomes if $r < 1$. Further, it is simple weighted average of incomes of order r with non-decreasing weights placed on lower incomes. The lower the value of the parameter r, the closer are the implicit ethics to the maximin rule.

Two Illustrations

(1) Let $a_i = (2i - 1)$.

Then $$\bar{I} = 1 - \frac{1}{\lambda} \left[\frac{\sum_{i=1}^{n} (2i - 1) y_i^{\,r}}{n^2} \right]^{1/r}, \quad r < 1 \quad \ldots \quad (6.38)$$

With this weighting scheme $\bar{I}$ is a parametric generalisation of the Gini coefficient. For $r = 1$, $\bar{I}$ coincides with the Gini coefficient which is quasi-convex but not strictly so.

$$\int_0^1 \frac{1}{\alpha} [d_e]^\alpha \, dp = \int_0^1 d(p, \alpha) \, dp \quad \ldots \quad (6.7)$$

This gives

$$d_e = [\int_0^1 [p - L(p)]^\alpha \, dp]^{1/\alpha} \quad \ldots \quad (6.8)$$

We finally define the new inequality index I of the distribution as

$$I = d_e = [\int_0^1 [p - L(p)]^\alpha \, dp]^{1/\alpha} \quad \ldots \quad (6.9)$$

In essence, we are measuring the extent of inequality in a distribution by averaging the divergence function by using a general averaging method. If $\alpha = 1$, the measure coinsides with half of the Gini coefficient.

6.3 Some Remarks on the Generalized Coefficient

(1) To show that I ranks income distributions according to a strictly concave swf it is enough to show that I is strictly convex in incomes. For this let us consider two income distributions with distribution functions F_1 and F_2 respectively such that

$$\int_0^\infty x_1 \, dF_1(x_1) = \int_0^\infty x_2 \, dF_2(x_2) = \lambda \quad \ldots \quad (6.10)$$

where $0 < \lambda < \infty$.

distributional judgements. It is hoped that this new family will be an important tool in the investigation of inequality.

CHAPTER 7

THE EFFECT OF PROGRESSIVE TAXATION AND THE MEASUREMENT OF TAX PROGRESSIVITY

7.1 Introduction

This chapter reports two results, one rather obvious on the effect of progressive taxation on the degree of inequality and the second, on the measurement of degree of tax progression. The first result shows that if the post-tax income (y) is an increasing, concave function of the pre-tax income (x), then the y-profile is Lorenz superior to the x-profile. This is done in Section 2.

In Section 3 we briefly review the existing indices of tax progressivity which intend to measure the deviation of a tax system from proportionality. We then propose a new measure of tax progressivity starting from a set of axioms. This measure is the normalised value of the degree of superiority of the welfare value (as measured by an additively separable swf) of the actual post-tax income profile over that of the hypothetical post-tax income profile that would result if the same aggregate amount of tax were realised through a proportional tax system[1/]

1/ A somewhat similar approach was employed by Blackorby and Donaldson (1979) to generalize the Musgrave and Thin (1948) index of tax progressivity.

The measure remains invariant under affine transformations of the common utility function. The measure in its general form contains the unknown utility function $U(.)$. To remove this deficiency we impose the axiom of scale invariance and find a form of $U(.)$ for which the measure satisfies this axiom; but it not quite clear whether this form is unique. For this particular form of $U(.)$, we get a computable measure involving only one unknown parameter r ($r \leq 1$) associated with concavity of $U(.)$. The measure, interestingly, in a limiting case equals the proportionate gap between the Theil entropy measure of inequality of the pre-tax profile and of the post-tax profile.

7.2 Lorenz Comparisons of Pre- and Post-tax Incomes

In this section we shall compare the inequality of pre-tax incomes with that of post-tax incomes for a specific tax scheme. Let us denote pre-tax income by x and post-tax income by y. We define a tax scheme as follows :

If $x \in [0, C_1)$, then $y = x$

If $x \in [C_1, C_2)$, then $y = C_1 + (1-t_1)(x-C_1)$, $0 < t_1 < 1$

If $x \in [C_2, C_3)$, then $y = C_1 + (1-t_1)(C_2-C_1) + (1-t_2)(x-C_2)$, $0 < t_1 < t_2 < 1$

and so on.

Finally, we have

If $x \in [C_n, \infty)$, then $y = C_1 + (1-t_1)(C_2-C_1) + (1-t_2)(C_3-C_2) + \ldots + (1-t_{n-1})(C_n-C_{n-1}) + (1-t_n)(x-C_n), \; 0 < t_1 < t_2 < \ldots < t_n < 1$

So y is a continuous function of x with non-differentiable points $C_1, C_2, \ldots, C_n$. We assume that the probability of income taking on the value C_i $(i = 1, 2, \ldots, n)$ is zero. We can treat y as an increasing, concave function of x. Moreover, $y \leq x$.

<u>Theorem</u>: In the set-up formulated above $y L x$.

<u>Proof</u>: Write $y = f(x)$, where f is increasing and concave in x. To show

$$F_x(z_1) = F_y(z_2) = r \Rightarrow F_1^{y}(z_2) > F_1^{x}(z_1) \quad \ldots \quad (7.1)$$

where

$$F_x(z_1) = \int_0^{z_1} p_1(x)dx \quad \ldots \quad (7.2)$$

$$F_y(z_2) = \int_0^{z_2} p_2(y)dy \quad \ldots \quad (7.3)$$

$$F_1^{x}(z_1) = \frac{\int_0^{z_1} x\, p_1(x)dx}{\int_0^{\infty} x\, p_1(x)dx} \quad \ldots \quad (7.4)$$

$$F_1^{y}(z_2) = \frac{\int_0^{z_2} y\, p_2(y)dy}{\int_0^{\infty} y\, p_2(y)dy} \quad \ldots \quad (7.5)$$

Here $p_1(x)$ and $p_2(y)$ are the density functions of x and y respectively.

Now, obviously,

$$F_y(z_2) = F_x(z_1) \iff z_2 = f(z_1) \ldots \quad (7.6)$$

To show

$$\frac{\int_0^{z_2} y\, p_2(y)dy}{\int_0^{\infty} y\, p_2(y)dy} > \frac{\int_0^{z_1} x\, p_1(x)dx}{\int_0^{\infty} x\, p_1(x)dx} \quad \ldots \quad (7.7)$$

or, $$\frac{\int_0^{f^{-1}(z_2)} f(x)\, p_2(f(x))\, f'(x)dx}{\int_0^{\infty} y\, p_2(y)dy} > \frac{\int_0^{z_1} x\, p_1(x)dx}{\int_0^{\infty} x\, p_1(x)dx}$$

or,
$$\frac{\int_0^{z_1} f(x)\, p_1(x)dx}{\int_0^{z_1} x\, p_1(x)dx} > \frac{\int_0^{\infty} y\, p_2(y)dy}{\int_0^{\infty} x\, p_1(x)dx} .$$

Therefore it suffices to show that

$$g(t) = \frac{\int_0^{t} f(x)\, p_1(x)dx}{\int_0^{t} x\, p_1(x)dx}$$ is decreasing in t over $(0, \infty)$.

Now,
$$g'(t) = \frac{[\int_0^t x\, p_1(x)dx] f(t)\, p_1(t) - [\int_0^t f(x)\, p_1(x)dx] t p_1(t)}{[\int_0^t x\, p_1(x)dx]^2}$$

Clearly $g'(t) \leq 0$ if and only if $g(t) \geq \frac{f(t)}{t}$.

Since f is increasing, concave and $f(0) = 0$, $\frac{f(t)}{t}$ is decreasing in t over $(0, \infty)$[vide Lemma in Appendix III].

Take $x \in (0, t)$.

$$\therefore \quad \frac{f(x)}{x} \geq \frac{f(t)}{t} .$$

$$\therefore \quad \int_0^t f(x)\, p_1(x)dx = \int_0^t \frac{f(x)}{x} \cdot x\, p_1(x)dx$$

$$\geq \int_0^t \frac{f(t)}{t} \cdot x\, p_1(x)dx .$$

$$\therefore \quad g(t) \geq \frac{f(t)}{t} .$$

Hence $g(t)$ is decreasing in t over $(0, \infty)$. Therefore $y L x$.

Broadly speaking a tax scheme is said to be progressive if it brings down the degree of inequality. Therefore the tax scheme we have taken into account is progressive. In the next section we make an attempt to suggest an index of tax progressivity starting from a set of axioms incorporating welfare theoretic considerations.

7.3 A New Normative Measure of Tax Progressivity

Before introducing a new index of tax progressivity let us discuss the existing measures proposed in this context.

For this, let

x_i : income of unit i $(i = 1, 2, \ldots, n)$,

t_i : amount of tax that unit i has to pay,

$y_i = x_i - t_i$: post-tax income of unit t.

The pre-tax income vector is $\underline{x} = (x_1, \ldots, x_n)'$, the tax vector is $\underline{t} = (t_1, \ldots, t_n)'$ and the post-tax income vector is $\underline{y} = (y_1, \ldots, y_n)'$.

Among the contributions to the theory of measurement of tax progressivity are those of Dalton (1936), Musgrave and Thin (1948) , Jakobsson (1976), Khetan and Poddar (1976), Kakwani (1977b) and Blackorby and Donaldson (1979).

The index that Musgrave and Thin (1948) suggested is

$$T_{MT} = \frac{1 - G(\underline{y})}{1 - G(\underline{x})} \quad \ldots \quad (7.8)$$

where G(.) is the Gini coefficient of the corresponding income profile. The measure takes the value 1 for proportional taxation and a value greater (less) than unity if the tax scheme is progressive (regressive). Blackorby and Donaldson (1979) proposed a generalization of the Musgrave and Thin index and this generalised index involves an arbitrary AKS inequality measure which is implied by a homothetic S-concave social welfare function. Their index is

$$T_{BD} = \frac{I(\underline{x}) - I(\underline{y})}{1 - I(\underline{x})} \quad \ldots \quad (7.9)$$

where I(.) is the AKS inequality measure.

Kakwani's index [Kakwani (1977b)] is

$$T_K = G(\underline{t}) - G(\underline{x}) \quad \ldots \quad (7.10)$$

It is very difficult to give any welfare interpretation of $G(\underline{t})$ in this context. However, $G(\underline{t})$ (hence T_K) remains invariant for any permutation of $(t_1, t_2, \ldots, t_n)$. This means T_K would be the same for very different schemes of taxation, progressive or regressive. Moreover, T_K remains unchanged when the tax vector $\underline{t}$ is multiplied by a positive scalar. This also is clearly undesirable.

Khetan and Poddar (1976) suggested the following index:

$$T_{KP} = \frac{1 - G(\underline{x})}{1 - G(\underline{t})} \quad \ldots \quad (7.11)$$

This index shares all the shortcomings of Kakwani's measure mentioned above.

To propose a new index of tax progressivity we view the proportional tax scheme as our benchmark. A measure of tax progressivity is required to show the deviation of a given tax system from proportionality. We shall consider our new tax progressivity index as the normalised value of the degree of dominance of the welfare level associated with the post-tax income profile over that of the profile that would have been obtained from a proportional tax scheme.

For this purpose, let U_i denote the utility function of unit i. Assume $U_i(.)$ depends only on the income of unit i and $U_i(.) = U(.)$. Further assume that U is increasing and concave. We assume that the swf is of the form

$$W(\underline{x}) = \sum_{i=1}^{n} U(x_i) \quad \ldots \quad (7.12)$$

Therefore our tax progressivity measure is

$$TP = A(\underline{x}, \underline{y})\left[\sum_{i=1}^{n} \left(U(y_i) - U((1 - \bar{t})\, x_i) \right) \right] \quad .. \quad (7.13)$$

where $A > 0$ is the coefficient of normalization and

$$\bar{t} = \frac{\sum_{i=1}^{n} t_i}{\sum_{i=1}^{n} x_i} .$$

$$\therefore \quad 1 - \bar{t} = 1 - \frac{\sum_{i=1}^{n} t_i}{\sum_{i=1}^{n} x_i} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i} = \frac{\lambda_y}{\lambda_x}$$

where λ_x and λ_y are the means of the income profiles $\underline{x}$ and $\underline{y}$ respectively.

$$\therefore \quad TP = A(\underline{x}, \underline{y})\left[\sum_{i=1}^{n} \left(U(y_i) - U\left(\frac{\lambda_y}{\lambda_x} x_i\right) \right) \right] \quad \ldots \quad (7.14)$$

The measure should remain invariant under affine transformations of U(.).

For this we require a normalisation axiom that will serve this purpose.

We shall call a tax scheme most progressive if the post-tax income profile is perfectly equal. Since U_i's are independent of i, the swf $\sum_{i=1}^{n} U(y_i)$ will generate maximum welfare (given the mean of y_i's, that is, given total tax proceeds) when y_i's are all equal i.e., when the tax scheme is most progressive.

Axiom of Normalisation

For a most progressive tax scheme the value of the progressivity index is unity.

The tax progressivity index is then given by

$$TP = \frac{\sum_{i=1}^{n} [U(y_i) - U(\frac{\lambda_y}{\lambda_x} \cdot x_i)]}{\sum_{i=1}^{n} [U(\lambda_y) - U(\frac{\lambda_y}{\lambda_x} \cdot x_i)]} \qquad \ldots (7.15)$$

Obviously, the measure in (7.15) remains invariant under affine transformations of U(.). The measure is positive for progressivity, zero for proportionality and negative for regressivity.

We may now propose another axiom.

Axiom of Scale Invariance

If all the pre-tax and all the post-tax incomes are multiplied by the same positive schlar C, then the value of the index remains unchanged.

A form of utility function that keeps TP in (7.15) invariant to scale changes described by the above axiom is

$$U(m) = A + \frac{1}{r} m^r \qquad \ldots \qquad (7.16)$$

where A is independent of m and for concavity $r \leq 1$.

Substituting U(.) given by (7.16) in (7.15) we get

$$TP = \frac{\frac{1}{n} \sum_{i=1}^{n} [(\frac{y_i}{\lambda_y})^r - (\frac{x_i}{\lambda_x})^r]}{1 - \frac{1}{n} \sum_{i=1}^{n} (\frac{x_i}{\lambda_x})^r} \qquad \ldots \qquad (7.17)$$

where $r < 1$.

In the limiting case as $r \to 0$,

$$TP \to 1 - \frac{\sum_{i=1}^{n} \log(\frac{y_i}{\lambda_y})}{\sum_{i=1}^{n} \log(\frac{x_i}{\lambda_x})} \qquad \ldots \qquad (7.18)$$

When $r \to 1$,

$$TP \rightarrow 1 - \frac{\frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{\lambda_y} \log(\frac{y_i}{\lambda_y})}{\frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{\lambda_x} \log(\frac{x_i}{\lambda_x})} \quad \ldots \quad (7.19)$$

$$= 1 - \frac{\text{Theil inequality index for } \underline{y}}{\text{Theil inequality index for } \underline{x}} .$$

Therefore the measure in a limiting case is simply the proportionate gap between the Theil inequality index of the pre-tax income profile and of the post-tax income profile.

7.4 Conclusion

In this chapter we have presented two results. The first result shows that if post-tax income is an increasing, concave function of the pre-tax income, then the profile of post-tax incomes Lorenz dominates the profile of pre-tax incomes. We then propose a normative index of tax progressivity. According to this measure, a tax scheme is regarded as progressive if the index takes some value in the interval (0, 1]. However, the measure in its general form contains the unknown common utility function U(.). A form of U has been found for which the measure has the desirable property of being unchanged when all the pre-tax and the post-tax incomes are multiplied by the same positive scalar C.

Assuming that U(.) is of this form we get a computable index which may be convenient for practical purposes. The measure in a limiting case is simply the amount by which the ratio of Theil inequality index of the post-tax profile to that of the pre-tax profile falls short of unity. This again makes Theil's index of inequality attractive from the welfare point of view.

APPENDIX - I

Errors in Variables and Measurement of Inequality

In this appendix we recognize the presence of errors in observation in data on personal income and study their impact on the measures of income inequality.

Assumption and notation: Let x's denote true values of income and y's represent observed values. We assume that the conditional expectation of y given $x = x_o$ i.e., $E(y \mid x = x_o)$ is equal to x_o, where $E(.)$ stands for expectation.

If we write $y = x + u$, $y \geq 0$, then u can be interpreted as the error in the income variable. Let λ_x and λ_y denote the means of x and y respectively.

Theorem I

Under the assumptions stated above we have xLy.

To prove this result we first need to prove a lemma stated as follows:

Lemma I

Let z(.) be a strictly concave function and let

$$E(z(x)) = \int_0^\infty z(x)\, dF(x),$$ where F(.) is the distribution

function of x. Then $E(z(y)) \leq E(z(x))$.

Proof: By Jensen's inequality, we have

$$E(z(y) \mid x) \leq z(E(y \mid x)).$$

But $$z(E(y \mid x)) = z(x).$$

Taking expectations on both sides with respect to x, we have

$$E(z(y)) \leq E(z(x)).$$ 1/

□

Proof of Theorem I

We have $\lambda_x = \lambda_y$. Therefore by Atkinson's Theorem (*vide* Chapter 1) and lemma I, we have xLy.

□

From Theorem I it is evident that the inequality of the true income distribution will be less than that of the observed income distribution for any S-convex inequality measure.

1/ Lemma I, together with Theorem 4' of Hadar and Russell (1969), shows that the true distribution second order stochastic dominates the observed one.

APPENDIX - II

Alternative Proof of an Expression for the Gini Coefficient due to Dorfman

Let $y \geq 0$ be a random variable with finite mean λ having distribution function F. Dorfman (1979) showed that the Gini coefficient G of y can be written as

$$G = 1 - \frac{1}{\lambda} \int_0^\infty (1 - F(t))^2 \, dt .$$

Below we give a much shorter derivation of this result. The Gini coefficient of y can be written as

$$G = 1 - 2 \int_0^\infty F_1(y) dF(y)$$

where $$F_1(y) = \frac{1}{\lambda} \int_0^y t \, dF(t).$$

Now $$\int_0^\infty F_1(y) \, dF(y)$$

$$= \frac{1}{\lambda} \int_0^\infty \left[\int_0^y t \, dF(t) \right] dF(y)$$

$$= \frac{1}{\lambda} \int_0^\infty \left[\int_t^\infty dF(y) \right] t \, dF(t)$$

$$= \frac{1}{\lambda} \int_0^\infty [1 - F(t)]t \, d\, F(t)$$

$$= - \frac{1}{\lambda} \int_0^\infty t\, [1 - F(t)] \, d(1 - F(t))$$

$$= - \left[\frac{t(1 - F(t))^2}{2\lambda} \right]_0^\infty + \frac{1}{2\lambda} \int_0^\infty (1 - F(t))^2 \, dt$$

$$= \frac{1}{2\lambda} \int_0^\infty (1 - F(t))^2 \, dt .$$

$$\therefore \; G = 1 - \frac{1}{\lambda} \int_0^\infty (1 - F(t))^2 \, dt .$$

APPENDIX - III

A Lemma

Lemma

If f is an increasing, concave function over $(0, \infty)$ and $f(0) = 0$, then $\frac{f(x)}{x}$ is decreasing over $(0, \infty)$.

Proof: Fix $x \in (0, \infty)$.

Take $0 < z < x$.

Let $s = \frac{z}{x}$, $0 < s < 1$.

Due to concavity of f

$$f((1 - s).0 + sx) \geq (1 - s) f(0) + sf(x)$$

or, $f(sx) \geq sf(x)$.

$$\therefore \frac{f(z)}{z} \geq \frac{f(x)}{x}.$$

$$\therefore \frac{f(x)}{x} \text{ is decreasing over } (0, \infty).$$

□

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