Achievement and improvement in living standards

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Abstract

Kakwani (1993) suggested some postulates for an index of improvement in the level of living of a population. He also suggested a particular class of improvement indices. In this paper we characterize axiomatically the entire Kakwani class of indices. We also derive a sufficient condition to avoid a non-comparability problem pointed out by Kakwani.

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1. Introduction

Sen (1985, 1987) defined the concept of standard of living in terms of (i) functioning, which indicates 'achievement' of different attributes such as health, education, housing etc. and (ii) capability, which is the 'ability' to achieve. In a pioneering paper Kakwani (1993) has introduced a class of 'achievement indices' to represent the actual 'levels' of standard of living and a class of 'improvement indices' to represent the improvement in the standard of living of a country, using an axiomatic approach.

One of Kakwani's axioms is concerned with comparison of improvement indices of two populations. He shows that only one member of his class satisfies this comparability axiom. Extending Kakwani's set-up, in this paper we attempt to provide a sufficient condition that will overcome the non-comparability problem. A characterization of the entire Kakwani family of indices is also carried out in the paper.

2. The formal framework

Let x stand for the 'life expectancy at birth' of the country under consideration. This particular indicator of achievement is taken only for concreteness. The formal analysis applies equally well to any other measure of living standards. Denote the lower and upper bound of x by M_1 and M_2 respectively, that is $x \in [M_1, M_2]$. Suppose now that the value of x changes from x_1 to x_2. An improvement index Q is a real valued function that associates a value $Q(x_1, x_2, M_1, M_2)$ to each pair $(x_1, x_2) \in [M_1, M_2]^2$ indicating the country's improvement in life expectancy when longevity changes from x_1 to x_2, where $[M_1, M_2]^2$ is the two-fold Cartesian product of $[M_1, M_2]$.

Kakwani (1993) suggested some interesting axioms for an improvement index. The first axiom is

Range Subdivision (RS).

$$Q(x_1, x_2, M_1, M_2) \quad \begin{aligned} &< 0 \quad \text{if} \quad x_2 < x_1, \\ &= 0 \quad \text{if} \quad x_1 = x_2, \\ &> 0 \quad \text{if} \quad x_2 > x_1. \end{aligned}$$

This property subdivides the range of Q depending on the relationship between x_1 and x_2. More precisely, the index takes a negative, zero or positive value according as longevity decreases, remains constant or increases. The second postulate is a normalization condition. It states that Q should take on the highest value when the increase in longevity is maximum, that is, when $x_1 = M_1$ and $x_2 = M_2$.

Normalization (NM).

$$Q(x_1, x_2, M_1, M_2) = 1 \quad \text{if} \quad x_1 = M_1 \text{ and } x_2 = M_2. \tag{1}$$

It is reasonable to expect that Q increases when x_2 increases. Similarly Q should decrease when x_1 increases. We thus have

Monotonicity (MO). Given other things, (i) $Q(x_1, x_2, M_1, M_2)$ is increasing in x_2; (ii) $Q(x_1, x_2, M_1, M_2)$ is decreasing in x_1.

The next property is an additivity condition. It states that if there is a change in longevity from x_1 to x_2 and then further to x_3, the improvement index for the change from x_1 to x_3 should equal the sum of changes from x_1 to x_2 and from x_2 to x_3. Formally, we have

Additivity (AD). For all $x_1, x_2, x_3 \in [M_1, M_2]$,

$$Q(x_1, x_3, M_1, M_2) = Q(x_1, x_2, M_1, M_2) + Q(x_2, x_3, M_1, M_2).$$

To state the fifth property let us consider two situations. In the first situation the life expectancy increases from 70 to 80 while in the second it increases from 50 to 60. Clearly, it is harder to increase longevity from 70 to 80 than from 50 to 60. Consequently, the index should attach a higher numerical value for improvement in the former case than in the latter case (see also Sen, 1981; Dasgupta, 1993). The above argument is formalized as

Upward Sensitivity (US).

$$Q(y_1, y_1 + \delta, M_1, M_2) > Q(x_1, x_1 + \delta, M_1, M_2)$$
$$\text{for all } \delta > 0 \text{ and } y_1 > x_1.$$

The final property considered by Kakwani is concerning comparability of improvement indices of two populations.

Comparability (CO). Given x_1, x_2 and y_1 less than M_2 such that

$$Q(x_1, x_2, M_1, M_2) = Q(y_1, y_2, M_1, M_2),$$

1. y_2 must always exist (be a real number).
2. y_2 should be greater (less) than x_2 when y_1 is greater (less) than x_1.
3. y_2 must never exceed M_2.
4. Given $y_1 > x_1$, as x_2 approaches M_2, y_2 must also approach M_2.

In addition to the above axioms we consider two more properties. The first one is

Continuity (CN). Q is a continuous function.

Continuity ensures that small changes in one or more arguments of Q will generate small changes in its functional value. Thus, a continuous index will not be oversensitive to minor observational errors in longevity. The second axiom is dimensionality. This postulate demands insensitivity of an improvement index to changes in the unit of measurement of longevity levels. That is, if longevity values as well as the upper and lower bounds are measured in months instead of in years, then the index value should remain unchanged.

Dimensionality (DM). $Q(x_1, x_2, M_1, M_2) = Q(\alpha x_1, \alpha x_2, \alpha M_1, \alpha M_2)$ for all $\alpha > 0$.

It is easy to see that if Q satisfies AD, then it can be written as

$$Q(x_1, x_2, M_1, M_2) = f(x_2, M_1, M_2) - f(x_1, M_1, M_2) \tag{2}$$

for some real valued function f. Kakwani (1993) referred to f as an achievement index. In order to ensure that the value of f lies between 0 and 1, he assumed the following form of f:

$$f(x, M_1, M_2) = \frac{g(M_2 - M_1) - g(M_2 - x)}{g(M_2 - M_1)}, \tag{3}$$

where $g(0) = 0$, g is continuous and increasing. Since the argument of g is nonnegative, it will always assume a nonnegative value under the assumption of increasingness along with $g(0) = 0$. One can easily check that for Q to satisfy US we require strict concavity of g. We assume that g is twice differentiable.

For illustrative purposes Kakwani assumed the following forms of g:

$$g(x) = \frac{x^{1-r}}{1-r}, \quad 0 < r < 1, \tag{4}$$

$$g(x) = \log(x), \quad r = 1, \tag{5}$$

which in turn generate the forms of Q given by

$$Q(x_1, x_2, M_1, M_2) = \frac{(M_2 - x_1)^{1-r} - (M_2 - x_2)^{1-r}}{(M_2 - M_1)^{1-r}}, \quad 0 < r < 1, \tag{6}$$

$$Q(x_1, x_2, M_1, M_2) = \frac{\log(M_2 - x_1) - \log(M_2 - x_2)}{\log(M_2 - M_1)}, \quad r = 1. \tag{7}$$

As r rises, increase in Q resulting from an increase in longevity is larger for higher values of initial longevity. When $r = 0$, the function $g(.)$ in (4) is concave, but not strictly so. In this case Q in (6) is related to Sen's index (S) (Sen, 1981) by $Q = S(M_2 - M_1)/(M_2 - x_1)$.

In the following proposition we axiomatically derive the entire Kakwani class of improvement indices.

Proposition 1. Suppose that the achievement function f underlying the improvement index Q is of the type (3). Then Q satisfies DM if and only if Q is of the forms given by (6) and (7).

Proof. In view of the assumption that the achievement function f underlying Q is of the form (3), we have

$$Q(x_1, x_2, M_1, M_2) = \frac{g(M_2 - x_1) - g(M_2 - x_2)}{g(M_2 - M_1)}. \tag{8}$$

Since (8) holds for all $x_1, x_2 \in [M_1, M_2]$, without loss of generality we can put $x_2 = M_2$. Given that $g(0) = 0$, Q in (8) now becomes $g(M_2 - x_1)/g(M_2 - M_1)$. By DM, Q is homogeneous of degree zero in its arguments. The postulate DM in this particular case means that

$$\frac{g(M_2 - x_1)}{g(M_2 - M_1)} = \frac{g(\alpha(M_2 - x_1))}{g(\alpha(M_2 - M_1))} \quad \text{for all } \alpha > 0. \tag{9}$$

Differentiating both sides of (9) with respect to x_1 and making some rearrangements we have

$$\frac{\mathrm{d}g(M_2 - x_1)}{\mathrm{d}x_1} \Big/ \frac{\mathrm{d}g(\alpha M_2 - \alpha x_1)}{\mathrm{d}(\alpha x_1)} = \alpha \frac{g(M_2 - M_1)}{g(\alpha(M_2 - M_1))}. \tag{10}$$

Since the right-hand side of (10) is independent of x_1, the left-hand side (l.h.s.) should be as well. Now we can write the numerator of the l.h.s. in (10) as $-\mathrm{d}g(M_2 - x_1)/\mathrm{d}(M_2 - x_1)$ and the denominator as $-\mathrm{d}g(\alpha(M_2 - x_1))/\mathrm{d}(\alpha(M_2 - x_1))$. Thus, for the l.h.s. of (10) to be independent of x_1 we must have the requirement that $g'(p)/g'(q)$ will depend on p/q where $p = M_2 - x_1$ and $q = \alpha(M_2 - x_1)$. Since α and x_1 are arbitrary and g' is continuous, for all u, v, $g'(u)/g'(v)$ will be of the form $h(u/v)$ for some continuous function h. This implies that $g'(u) = Bu^{-r}$, where $B \neq 0$ is a constant (see Aczél, 1966, p. 144). Integrating g' we get $g(u) = A + Bu^{1-r}/(1-r)$, where A is the constant of integration. The condition $g(0) = 0$ along with the increasingness and strict concavity of g require that $A = 0$, $B > 0$ and $0 < r \leq 1$. For $r = 1$, $g(u) = B \log u$. Substituting these specifications of g in (8), we get the desired forms of Q. This establishes the necessity part of the proposition. The sufficiency part is easy to verify. □.

Let us now suppose that g is of the form

$$g(x) = 1 - e^{-x}. \tag{11}$$

g in (11) is continuous, increasing, strictly concave and $g(0) = 0$. However, the form of Q underlying this specification of g does not meet DM. It, therefore, shows the importance of DM in the characterization of the Kakwani class given by (6) and (7). Furthermore, the specification in (11) supports Kakwani's statement "There may exist other functions which could satisfy all these conditions ..." (Kakwani, 1993, p. 314).

Assuming that Q is given by (6) and (7), Kakwani equates $Q(x_1, x_2, M_1, M_2)$ with $Q(y_1, y_2, M_1, M_2)$ and shows that for given x_1, x_2, y_1 a (real) value of y_2 need not exist for $r < 1$. That is, Q in (6) need not satisfy axiom CO. However, the axiom is fully satisfied for $r = 1$, that is, when Q is given by (7). In the following proposition we give a sufficient condition, for given values of y_1, x_1 and x_2, that will enable us to avoid the non-comparability problem, like the one that arises in the context of (6).

Proposition 2. Suppose that Q satisfies axioms MO and CN. Then a sufficient condition for Q to fulfill axiom CO is

$$Q(x_1, x_2, M_1, M_2) < Q(y_1, M_2, M_1, M_2) \tag{12}$$

or

$$Q(x_1, x_2, M_1, M_2) > Q(y_1, M_2, M_1, M_2) \tag{13}$$

according as $y_1 > x_1$ *or* $y_1 < x_1$.

Proof. The proof of this proposition relies on the intermediate value theorem which states that for a real valued continuous function h defined on the interval $[a, b]$ if $h(a) < h(b)$ and c is a number such that $h(a) < c < h(b)$, then there exists a point $z \in (a, b)$ such that $h(z) = c$ (Rudin, 1976, p. 93).

Assume that $y_1 > x_1$. In view of MO, we have $Q(y_1, x_2, M_1, M_2) < Q(x_1, x_2, M_1, M_2)$. This inequality along with the condition (12) gives

$$Q(y_1, x_2, M_1, M_2) < Q(x_1, x_2, M_1, M_2) < Q(y_1, M_2, M_1, M_2). \tag{14}$$

Since y_1, M_1 and M_2 are given, we can treat $Q(y_1, y_2, M_1, M_2)$ as a function of y_2 and write it as $H(y_2)$. Then from inequality (14) we get $H(x_2) < H(M_2)$. Now applying the intermediate value theorem for the continuous function H over the interval $[x_2, M_2]$, we see that a value of $y_2 \in (x_2, M_2)$, say y_2^*, must exist such that $Q(x_1, x_2, M_1, M_2) = H(y_2^*) = Q(y_1, y_2^*, M_1, M_2)$. A similar demonstration can be given for the case $y_1 < x_1$. Thus, given x_1, x_2, y_1 satisfying conditions stated in CO, there will exist a value of y_2 under the sufficient condition (12) (or (13)). □

The sufficient condition (12) states that the negative impact due to increase in the first argument of Q (from x_1 to y_1) will get outweighed by the positive impact resulting from the increase in its second argument (from x_2 to M_2). Clearly, if Q satisfies axiom US, then such a condition will be fulfilled in the particular case when $y_1 - x_1 = M_2 - x_2$. We can interpret condition (13) in an analogous manner. It may be noted that condition (12) (or (13)) is quite general in the sense that it does not depend on any specific functional form of Q.

We conclude by observing that for $r = 1$ the Kakwani index satisfies all the axioms stated in the paper. For $0 < r < 1$ the index meets all the axioms except CO, which is satisfied under the sufficient condition stated in Proposition 2.

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