Ranking Opportunity Sets and Arrow Impossibility Theorems: Correspondence Results*

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The paper demonstrates the formal similarity between the problem of ranking opportunity sets and an Arrovian aggregation problem. This parallel is exploited to axiomatize several rules for ranking opportunity sets. The standard framework is also extended to allow for considerations of the diversity of available choices. *Journal of Economic Literature* Classification Number: D71.

1. INTRODUCTION

Several papers have recently pursued the axiomatic approach to the ranking of *opportunity sets*. (See, for instance, [2, 4, 5, 7, 8].) An opportunity set is the set of outcomes from which an agent can choose an element. The ranking of such sets may be based on several alternative criteria. A somewhat simplistic and naive approach would be to base the ranking solely in terms of a comparison of most preferred elements in each set. A richer approach is to take into account the *degree of freedom* associated with different opportunity sets since freedom is an important component of a person's well-being.

In an interesting paper [2], Bossert, Pattanaik, and Xu (henceforth BPX) discuss the axiomatic characterization of several alternative rankings of opportunity sets. The two criteria used for ranking these opportunity sets are the agent's *preference* on the universal set of alternatives and the freedom of choice associated with opportunity sets. Different sets of axioms are used to characterize alternative rankings, each set representing a particular trade-off between the alternative criteria.

The main purpose of this paper is to emphasize the close formal similarity between the ranking of opportunity sets and the more familiar social choice theory problem of *aggregating* individual preferences to

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obtain social preferences. In particular, individual components of well-being, such as utility or freedom of choice, induce natural (individual) rankings of alternative opportunity sets. For example, the *freedom ranking* may compare alternative opportunity sets in terms of the *size* of each set, whereas the utility ranking may compare sets according to the *rank* of the maximal element of each set. The freedom and utility rankings have to be *aggregated* into an overall ranking of opportunity sets. So, rankings induced by individual components of well-being correspond to individual preference orderings, whereas the overall ranking of opportunity sets is the analogue of the social preference ordering.

It is worth pointing out that in the present framework, for each component of well-being such as freedom of choice or utility, there can be only *one* ranking. Hence, the exercise of aggregating the component rankings into an overall ranking of opportunity sets is analogous to the *single profile* aggregation problem in the Arrovian problem. A general discussion of the distinction between single profile and multi-profile aggregation problems can be found in [6]. We adopt an approach which is considerably more general than simply assuming that the overall ranking of opportunity sets is aggregated from the component rankings. This turns out be a *consequence* of our set of axioms.[1] Recognition of the formal similarity between the two frameworks enables us to exploit results which are well known in the Arrovian context. For instance, if individual utilities are *ordinal* and *non-comparable*, then the only Paretian social welfare ordering must be dictatorial. This result translates into a dictatorship of the freedom or utility ranking and enables us to obtain a more general, joint characterization of the two rankings characterized by BPX.

The "dictatorship" result rules out any non-trivial trade-off between freedom of choice and utility criteria in the ranking of opportunity sets. A second main result is to characterize a *utilitarian* type ranking. In the Arrovian context, this requires that individual utilities be *cardinal* and *unit comparable*. An extensive discussion of these terms can be found, once again, in [6].

Our proofs rely heavily on the proof techniques used in the Arrovian framework, in particular, the approach developed in [1]. However, as we point out later, the structure of opportunity sets necessitates a somewhat more complicated proof. We also emphasize here that we have sacrificed generality in order to make the proofs simpler.

It should be clear to the reader that our approach can easily accommodate criteria other than freedom of choice and utility or preference. Indeed, we indicate how our framework can be extended to allow for issues

[1] The crucial axioms and a lemma which is needed for this result are actually borrowed from BPX. However, they do not seem to have noticed the analogy.

such as *diversity* of choice in the ranking of opportunity sets in the last section.

2. NOTATION AND DEFINITIONS

Let X be a finite set of alternatives, with $|X| = n > 3$. The set $\pi(X)$ is the set of all *non-empty* subsets of X. Let R be a *linear* ordering on X (that is, a reflexive, complete, antisymmetric and transitive binary relation on X). The assumption of a linear ordering is made only for convenience. The analysis is not changed in any substantive way if indifference is permitted. For all $A \in \pi(X)$, $\max(A)$ denotes the (unique) R-best element in A.

The set X is the universal set of opportunities available to an agent, while elements of $\pi(X)$ are possible opportunity sets. The ordering R is to be interpreted as the "preference" ranking over elements of X. Our purpose is to use this ranking as well as other information to generate an ordering $\succcurlyeq$ over $\pi(X)$. The asymmetric and symmetric factors of $\succcurlyeq$ are denoted by $\succ$ and $\sim$, respectively.

For all $A \in \pi(X)$, let $r(A) = |\{b \in X \mid \max(A)\, Rb\}|$. Thus, $r(A)$ is the number of elements in X that are "defeated" by the maximal element of A. Since R is reflexive, note that $x \in \{b \in X \mid xRb\}$ for all $x \in X$.

Let $Q = \{(a, b) \in \{1, 2, ..., n\}^2 \mid a \geqslant b\}$. Define the map $\lambda: \pi(X) \to Q$ as follows:

$$\forall A \in \pi(X), \qquad \lambda(A) = (r(A), |A|). \tag{2.1}$$

Now, take any $(a, b) \in Q$. Choose $x \in X$ such that $|\{y \in X \mid xRy\}| = a$. Since $a \geqslant b$, one can always choose $B \subseteq X - \{x\}$ such that $|B| = b - 1$ and xRy for all $y \in B$. Letting $A = B \cup \{x\}$, we have $\lambda(A) = (a, b)$. Hence, λ is *onto* Q.

We denote by Φ the set of all monotonic functions from $\mathscr{R}$ to itself. We will sometimes let ϕ denote $(\phi_1, \phi_2) \in \Phi^2$, and $\phi(a, b) = (\phi_1(a), \phi_2(b))$. We also use $\hat{\Phi}^2 \subset \Phi^2$ to represent the set of $\phi = (\phi_1, \phi_2)$ such that $\phi_i(a) = t_i + \varepsilon a$ for some $\varepsilon > 0$.

3. SOME AXIOMS ON $\succcurlyeq$

In this section, we specify some axioms which will be used in our characterization results later on.

3.1. *Preliminary Axioms*

We will first present a set of axioms used by BPX to identify the cardinality and the rank of the maximal element in an opportunity set as the initial characteristics which determine the ordering $\succcurlyeq$. They represent, respectively, the degree of freedom and the utility associated with an opportunity set.

Simple Indirect Indifference Principle (SIIP). $\forall$ distinct $x, y, z \in X$, $xRyRz \Rightarrow \{x, y\} \sim \{x, z\}$.

Weak Independence (WIND). $\forall A, B \in \pi(X), \forall x \in X-(A \cup B)$, $[\max(A)$ Rx and $\max(B)\, Rx] \Rightarrow [A \cup \{x\} \succcurlyeq B \cup \{x\} \Leftrightarrow A \succcurlyeq B]$.

SIIP implies that the best element plays a dominent role in ranking two element sets. WIND states that if x is ranked lower than the maximal elements of two sets A and B, then the addition of x to the two sets does not change the ranking of the sets.

BPX prove the following results.

Lemma 3.1. *Let $\succcurlyeq$ be an ordering on $\pi(X)$ satisfying* SIIP *and* WIND. *Then, for all $A, B \in \pi(X)$, $\lambda(A) = \lambda(B) \Rightarrow A \sim B$.*

Proof. See Lemma 5.2, BPX.

In the traditional Arrovian framework, *Welfarism* refers to the property by virtue of which alternative social states can be ranked by the social preference relation solely on the basis of the vector of utilities associated with each state. Thus, non-utility information can be disregarded not only within a single utility profile but also across profiles. This permits the social ordering to be represented by a single ordering of $\mathscr{R}^n$, the space of utility N-types, where N is the number of individuals in the society.

We now show that Lemma 3.1 can be used to arrive at a somewhat similar conclusion.

Lemma 3.2. *Suppose $\succcurlyeq$ is an ordering over $\pi(X)$ satisfying* SIIP *and* WIND. *Then there is an ordering R^* over Q such that $A \succcurlyeq B$ only if $\lambda(A)\, R^* \lambda(B)$.*

Proof. Construct R^* as follows:

For all $(a, b), (c, d) \in Q$, $(a, b)\, R^*(c, d) \leftrightarrow \exists A, B \in \pi(X)$ such that $\lambda(A) = (a, b)$, $\lambda(B) = (c, d)$, and $A \succcurlyeq B$.

We show that R^* is well defined and is an ordering. Take any pair $(a, b), (c, d) \in Q$ and suppose $(a, b)\, R^*(c, d)$. Then, there exists $A, B \in \pi(X)$ such that $\lambda(A) = (a, b)$, $\lambda(B) = (c, d)$ and $A \succcurlyeq B$. Let $C, D \in \pi(X)$ be such that $\lambda(C) = (a, b)$ and $\lambda(D) = (c, d)$. Then, from Lemma 3.1, $C \sim A$ and $D \sim B$. Transitivity of $\succcurlyeq$ ensures $C \succcurlyeq D$. Hence, R^* is welldefined. Since λ is onto, R^* is complete. Reflexivity and transitivity of R^* follow from corresponding properties of $\succcurlyeq$. ∎

Remark 3.1. An important difference between the Arrovian framework and the present one is worth emphasizing. As Lemma 3.2 makes clear, the domain of R^* is Q, a *subset* of $\mathscr{R}^2$. However, in the Arrovian framework, the corresponding domain would be the entire $\mathscr{R}^2$ since the condition of *unrestricted domain* ensures that all possible points in $\mathscr{R}^2$ can be "realised" as permissible utility 2-types. The smaller domain makes our proof more difficult.

3.2. *Some More Axioms*

In this section, we list the other axioms which will be used in our characterization results. We also mention some axioms used by BPX.

Simple Dominance (SD). $\forall$ distinct $x, y \in X$, $xRy \Rightarrow \{x\} \succ \{y\}$.

Simple Monotonicity (SM). $\forall$ distinct $x, y \in X$, $\{x, y\} \succ \{x\}$.

Simple Priority of Freedom (SPF). $\forall$ distinct $x, y, z \in X$, $xRyRz \Rightarrow \{y, z\} \succ \{x\}$.

Indirect Preference Principle (IPP). $\forall A \in \pi(X)$ with $|A| > 1$, $\{\max(A)\} \succ A - \{\max(A)\}$.

Weak Priority of Freedom (WPF). $\exists$ distinct $x, y, z \in X$ such that $xRyRz \Rightarrow \{y, z\} \succ \{x\}$.

Non-negative Preference Principle (NPP). $\exists A \in \pi(X)$ such that $\{\max(A)\} \succ A - \{\max(A)\}$.

Non-trivial Indifference (NTI). $\exists A, B \in \pi(X)$ such that $\lambda(A) \neq \lambda(B)$ and $A \sim B$.

Ordinal Non-comparibility (ONC). $\forall A, B, C, D \in \pi(X)$, $\forall \phi = (\phi_1, \phi_2) \in \Phi^2$, $[A \succcurlyeq B, \lambda(C) = \phi(\lambda(A)), \lambda(D) = \phi(\lambda(B))$, and $\lambda(C), \lambda(D) \in Q] \rightarrow C \succcurlyeq D$.

Cardinal Unit Comparibility (CUC). $\forall A, B, C, D \in \pi(X)$, $\forall \phi \in \hat{\Phi}^2$, $[A \succcurlyeq B, \lambda(C) = \phi(\lambda(A), \lambda(D) = \phi(\lambda(B))$, and $\lambda(C), \lambda(D) \in Q] \rightarrow C \succcurlyeq D$.

The definitions SD, SM, IPP, and SPF are discussed in BPX. SD and SM are relatively non-controversial, although it is argued in [5] that if opportunity sets are to be ranked *solely* on the basis of the freedom of choice, then the agent should be indifferent between such sets since none of them offer any freedom of choice. For a contrary view, see [7]. IPP and SPF are extreme assumptions. The former hardly allows any role to freedom of choice, while the latter is heavily biased *in favour* of freedom of choice. For instance, IPP would dictate that the *singleton set* $\{\max(X)\}$ is preferred to $X - \{\max(X)\}$. On the other hand, even if x and y are the *worst* and next-to-worst alternatives (according to R) in X, SPF requires that $\{x, y\}$ be preferred to $\{\max(X)\}$.

WPF and NPP are much weaker and hence more acceptable versions of SPF and IPP, respectively. WPF requires that there be *some* occasion on which freedom of choice prevails over preference considerations, while NPP requires that freedom of choice should not *always* dictate.

ONC and CUC are "borrowed" from the social choice literature on interpersonal comparibility. Both conditions imply that the ranking of opportunity sets be *invariant* over certain classes of opportunity sets. In particular, ONC requires that if A is ranked higher than B, and if C and D can be derived from A and B, respectively, through independent monotonic transformations of rank and cardinality, then C must be ranked higher than D. Intuitively, this means that what are important in the ranking of two sets are the *ordinal* comparisons of ranks and sizes. CUC imposes a *weaker* invariance restriction. The ranking of two sets is permitted to be sensitive to *differences* in ranks and cardinalities (and not just to ordinal comparisons).

There are two features of our definition of ONC and CUC which deserve special comment. The first point is best illustrated by means of an example. Assume that $\succcurlyeq$ satisfies ONC and consider the points $(3, 3), (4, 1), (5, 2) \in Q$. Observe that the ordinal rankings of rank and size between $(4, 1)$ and $(3, 3)$ is the same as that between $(5, 2)$ and $(3, 3)$. The spirit of ONC therefore requires $(4, 1)$ and $(5, 2)$ to be ranked in the same way relative to $(3, 3)$.

One way to establish this formally is to find a pair of monotonic transformations $\phi = (\phi_1, \phi_2) \in \Phi^2$ such that $\phi(4, 1) = (5, 2)$ and $\phi(3, 3) = (3, 3)$. (We could, of course, have $\phi(5, 2) = (4, 1)$ and $\phi(3, 3) = (3, 3)$ instead.) Then, if, for example, $(4, 1) \succcurlyeq (3, 3)$, we have $(5, 2) = \phi(4, 1) \succcurlyeq \phi(3, 3) = (3, 3)$. However, if $\phi_2(1) = 2$ and $\phi_2(3) = 3$, it must be the case that $2 < \phi_2(2) < 3$. Thus, $\phi_2(2)$ is not an integer and $\phi(5, 2) \notin Q$. Indeed, since Q is finite, it is not possible to find *any* non-trivial transformation ϕ: $Q \to Q$. In our definition of ONC and CUC, we allow *all* monotonic transformations $\phi_i: \mathscr{R} \to \mathscr{R}$, $i = 1, 2$, but make inferences only when the transformed values characteristics lie in the set Q.

In the example cited above, since we can find monotonic transformations $\phi_i: \mathscr{R} \to \mathscr{R}$, $i = 1, 2$ such that $\phi(4, 1), \phi(3, 3) \in Q$, $(4, 1) \succcurlyeq (3, 3)$ implies $(5, 2) = \phi(4, 1) \succcurlyeq \phi(3, 3) = (3, 3)$, etc. We simply ignore the fact that $\phi(5, 2) \notin Q$ because $\phi(5, 2)$ does not figure in any of the comparisons we wish to make. Thus, a subtle modification of the conventional definitions of ONC and CUC is necessary in order to deal with the special structure of Q.

The second point is that our formulation of ONC and CUC seems to assume directly that freedom is measured by cardinality and utility by the rank of the maximal element. However, this is done solely to facilitate exposition. It is possible to define ONC and CUC with respect to any set of characteristics which measure freedom and utility. ONC and CUC only bear on the possibility of trade-offs between these characteristics. We could then assume WIND separately, apply Lemma 3.1, and then go on to Theorem 4.1. However, this approach would require considerably more complicated notation for generality, which we are not going to use. Our approach also has the additional advantage of allowing us to deduce WIND, rather than assume it separately.

Indeed, we show below that CUC (and hence ONC) implies WIND.[2]

LEMMA 3.3. *$\succcurlyeq$ satisfies* WIND *if it satisfies* CUC.

Proof. Choose any $A, B \in \pi(x)$. Then, $\forall x \in X - (A \cup B)$, if $\max(A)\, Rx$ and $\max(B)\, Rx$, then $\lambda(A) = (r(A), |A|)$, $\lambda(B) = (r(B), |B|)$, $\lambda(A \cup \{x\}) = (r(A), |A|+1)$, $\lambda(B \cup \{x\}) = (r(B), |B|+1)$. Let $\phi_1(r(A)) = r(A)$ and $\phi_1(r(B) = r(B)$. Also, let $\phi_2(k) = k+1\ \forall k$. Then, $[A \succcurlyeq B$ implies $A \cup \{x\} \succcurlyeq B \cup \{x\}]$ follows from CUC immediately. The implication in the other direction follows easily. ∎

3.3. *Characterization Results*

In this section, we will characterize three alternative rankings of opportunity sets. The first two rankings have been characterized by BPX. These are the following:

The *cardinality first lexicographic* relation $\succcurlyeq_C$ is defined by:

$$\forall A, B \in \pi(X), \qquad A \succcurlyeq_C B \leftrightarrow [|A| > |B|]$$

or

$$[|A| = |B| \text{ and } \max(A)\, R \max(B)]. \tag{3.1}$$

The *preference first lexicographic* relation $\succcurlyeq_P$ is defined by:

$$\forall A, B \in \pi(X), \qquad A \succcurlyeq_P B \leftrightarrow [\max(A)\, R \max(B)]$$

or

$$\max(A) = \max(B) \qquad \text{and} \qquad |A| \geqslant |B|. \tag{3.2}$$

The orderings $\succcurlyeq_C$ and $\succcurlyeq_P$ do not allow for any trade-off between freedom of choice and preference considerations. The following rule does permit non-trivial trade-offs between these two components of well-being.

[2] We are indebted to a referee for pointing this out to us.

The *generalized utilitarian* relation $\succcurlyeq_U$ is defined by:

$$\begin{gathered}\exists \rho_1, \rho_2 > 0 \qquad \text{such that} \quad \forall A, B \in \pi(X), \\ A \succcurlyeq_U B \leftrightarrow \rho_1 r(A) + \rho_2 |A| \geqslant \rho_1 r(B) + \rho_2 |B|.\end{gathered} \tag{3.3}$$

We first provide a joint characterization of $\succcurlyeq_C$ and $\succcurlyeq_P$.

THEOREM 4.1. *$\succcurlyeq$ satisfies* SM, SD, SIIP, *and* ONC *iff $\succcurlyeq$ is either $\succcurlyeq_C$ or $\succcurlyeq_P$.*

Proof. It is clear that both $\succcurlyeq_P$ and $\succcurlyeq_C$ satisfy the stated conditions. So, we only need to prove the "only if" part.

Let $\succcurlyeq$ satisfy SM, SD, SIIP, and ONC. Our proof consists of several steps.

STEP 1. $\forall A, B \in \pi(X)$, $\lambda(A) > \lambda(B)$[3] $\Rightarrow A \succ B$.

Proof. See Lemma 3.3 above and Lemmas 5.3 and 5.4 of BPX.

Note that Step 1 implies that $(a, b) > (c, d) \Rightarrow (a, b)\, P^*(c, d)$. For any $(a, b) \in Q$, let $SE(a, b) = \{(c, d) \in Q \mid c > a, d < b\}$ and $NW(a, b) = \{(c, d) \in Q \mid c < a, d > b\}$.

Let (a^*, b^*) be a generic element of Q.

STEP 2. (i) *$\forall (a, b), (c, d) \in NW(a^*, b^*)$, either $[(a, b)\, P^*(a^*, b^*)$ and $(c, d)\, P^*(a^*, b^*)]$ or $[(a^*, b^*)\, P^*(a, b)$ and $(a^*, b^*)\, P^*(c, d)]$.*

(ii) *$\forall (a, b), (c, d) \in SE(a, b)$, either $[(a, b)\, P^*(a^*, b^*)$ and $(c, d)\, P^*(a^*, b^*)]$ or $[(a^*, b^*)\, P^*(a, b)$ and $(a^*, b^*)\, P^*(c, d)]$.*

Proof. The proof consists of an extension of arguments in [1].

First, suppose $NW(a^*, b^*)$ contains at least two distinct pairs. Then, it can be checked that $NW(a^*, b^*)$ contain *distinct* pairs $(a, b), (c, d)$ such that either $(a, b) > (c, d)$ or $(c, d) > (a, b)$. W.l.o.g. assume $(a, b) > (c, d)$. From Step 1, $(a, b)\, P^*(c, d)$.

Now, we put ONC to use. Define monotonic transformations (ϕ_1, ϕ_2) such that $\phi_1(a^*) = a^*$, $\phi_1(c) = a$, $\phi_2(b^*) = b^*$, $\phi_2(d) = b$. Such transformations clearly exist. From ONC, $(a, b)\, R^*(a^*, b^*)$ iff $(c, d)\, R^*(a^*, b^*)$. Transitivity of R^* and $(a, b)\, P^*(c, d)$ rules out the possibility of $(a^*, b^*)\, I^*(c, d)$.

Clearly, exactly the same argument goes through if $SE(a^*, b^*)$ contains two distinct pairs.

[3] For $x, y \in \mathbb{R}^2$, $x > y$ if $x_i \geqslant y_i$ for $i = 1, 2$ and $x \neq y$.

The only remaining possibility is that $NW(a^*, b^*)$ [respectively, $SE(a^*, b^*)$] contains a single pair (a, b). Then, since $n > 3$ by assumption, $SE(a, b)$ [respectively, $NW(a, b)$] must contain distinct pairs (a^*, b^*) and (c, d). From previous arguments, either $(a, b)\, P^*(a^*, b^*)$ or $(a^*, b^*)\, P^*(a, b)$.

This completes the proof of Step 2.

STEP 3. *Consider the points* $(n-1, n-1)$ *and* $(n, n-2)$. *From Step 2, either* $(n, n-2)\, P^*(n-1, n-1)$ *or* $(n-1, n-1)\, P^*(n, n-2)$.

Case 1. Suppose $(n, n-2)\, P^*(n-1, n-1)$. We shall show that in this case $\succcurlyeq = \succcurlyeq_P$.

Proof. We prove the general induction step: if $(n-k+1, n-k-1)\, P^*(n-k, n-k)$, then $(n-k, n-k-2)\, P^*(n-k-1, n-k-1)$ for $k = 1, 2, ..., n-3$.

Suppose $(n-k+1, n-k-1)\, P^*(n-k, n-k)$, but $(n-k-1, n-k-1)\, P^*(n-k, n-k-2)$. Since $(n, 1), (n-k, n-k-2) \in SE(n-k-1, n-k-1)$, we must have $(n-k-1, n-k-1)\, P^*(n, 1)$. Since $(n-k, n-k) > (n-k-1, n-k-1)$, $(n-k, n-k)\, P^*(n-k-1, n-k-1)$ from Step 1. Transitivity of P^* implies $(n-k, n-k)\, P^*(n, 1)$.

On the other hand, $(n, 1)$ and $(n-k+1, n-k-1)$ are in $SE(n-k, n-k)$. So, Step 2 and the hypothesis that $(n-k+1, n-k-1)\, P^*(n-k, n-k)$ imply $(n, 1)\, P^*(n-k, n-k)$. This contradiction establishes the induction argument.

Observe that $(n-k+1, n-k+j)\, P^*(n-k, n-k)$, $j = 0, 1, ...,$ from Step 1. Also, $(n-k+1, n-k-j)\, P^*(n-k, n-k)$, $j = 2, ..., n-k-1$ since $(n-k+1, n-k-1)$, $(n-k+1, n-k-1) \in SE(n-k, n-k)$. Therefore, all points in column $(k+1)$ are strictly preferred to all points in column k. Transitivity and Step 1 establish that $\succcurlyeq = \succcurlyeq_P$.

Case 2. Suppose $(n-1, n-1)\, P^*(n, n-2)$. We shall show that $\succcurlyeq = \succcurlyeq_C$.

Proof. Once the induction step is established, the rest of the argument follows exactly along the lines of the one used in Step 1.

The induction step is the following: if $(n-k, n-k)\, P^*(n-k+1, n-k-1)$, then $(n-k-1, n-k-1)\, P^*(n-k, n-k-2)$ for all $k = 1, 2, ..., n-3$.

Suppose not, so that for some k, $(n-k, n-k)\, P^*(n-k+1, n-k-1)$ and $(n-k, n-k-2)\, P^*(n-k-1, n-k-1)$. Since $(n, 1)$, $(n-k, n-k-2) \in SE(n-k-1, n-k-1)$, $(n, 1)\, P^*(n-k-1, n-k-1)$ from Step 2. Since $(n-k, n-k)$, $(n-k-1, n-k-1) \in NW(n, 1)$, $(n, 1)\, P^*(n-k, n-k)$.

On the other hand, $(n, 1), (n-k+1, n-k-1) \in SE(n-k, n-k)$. So, Step 2 and the assumption that $(n-k, n-k)\ P^*(n-k+1, n-k-1)$ imply $(n-k, n-k)\ P^*(n, 1)$. This contradiction establishes the induction argument.

This concludes the proof of Theorem 4.1. ∎

The following follow easily from Theorem 4.1.

THEOREM 4.2. $\succcurlyeq$ *satisfies* SM, SD, SIIP, ONC, *and* WPF *iff* $\succcurlyeq = \succcurlyeq_C$.

THEOREM 4.3. $\succcurlyeq$ *satisfies* SM, SD, SIIP, ONC, *and* NNP *iff* $\succcurlyeq = \succcurlyeq_P$.

The proofs of these results follow from Theorem 4.1 and the fact that $\succcurlyeq_P$ does not satisfy WPF and $\succcurlyeq_C$ does not satisfy NNP.

Theorems 4.1–4.3 should be compared to the characterization result of BPX.

THEOREM 4.2′ (BPX). $\succcurlyeq$ *satisfies* SM, SD, WIND, SIIP, *and* SPF *iff* $\succcurlyeq = \succcurlyeq_C$.

THEOREM 4.3′ (BPX). $\succcurlyeq$ *satisfies* SM, SD, WIND, SIIP, *and* IPP *iff* $\succcurlyeq = \succcurlyeq_P$.

We have noted earlier that SPF and IPP are very strong conditions. SPF declares *all* two-element sets to be strictly preferred to one-element sets, irrespective of the ranks of the alternatives. In other words, freedom of choice dominates in this case. Given the help of WIND, it is not surprising that freedom of choice prevails in all comparisons (Theorem 4.2′). Moreover, since IPP rules out any consideration of freedom of choice, Theorem 4.3′ is also a natural consequence. In contrast, WPF (respectively, NNP) only requires that freedom of choice (respectively, preference considerations) matter in *some* comparison. Despite these relatively weak requirements we are still able to isolate the rankings $\succcurlyeq_P$ and $\succcurlyeq_C$. Theorem 4.1 makes it clear that the "driving force" underlying the characterisations is ONC. It has been pointed out in the traditional social choice literature that the informational parsimony of the Arrovian framework (individual utilities being ordinal and noncomparable) precipitates the impossibility results. Clearly, the same forces are at work in the present context.

We now turn to an axiomatization of the utilitarian rule.

THEOREM 4.4. $\succcurlyeq$ *satisfies* SD, SM, SIIP, NTI, *and* CUC *iff* $\succcurlyeq = \succcurlyeq_U$.

Proof. Again, it is obvious that $\succcurlyeq_U$ satisfies the stated conditions.

From NTI, there exist $(a, b), (c, d) \in Q$ such that $(a, b)\, I^*(c, d)$. It follows from arguments in [1] that all points in Q lying on the line joining (a, b) and (c, d) are indifferent to them. Clearly, this line may be regarded as an *indifference curve*.

Now, we need to show that all indifference curves are parallel to each other. Observe that by CUC, $(a+k_1, a+k_2)\, I^*(c+k_1, d+k_2)$ for all $k_1, k_2 \in \mathbb{R}$ such that $(a+k_1, b+k_2)$, $(c+k_1, d+k_2)$ belong to Q. Indifference curves so constructed must be parallel to the line joining (a, b) and (c, d).

As Fig. 1 makes clear, this may leave isolated points such as (α, β). Note that $(\gamma, \delta) > (\alpha, \beta) > (c, d)$. Hence, from Step 1 of Theorem 4.1, (γ, δ) $P^*(\alpha, \beta)\, P^*(c, d)$. Therefore, one can draw a "pseudo-indifference curve" through (α, β) which is parallel to the other indifference curves. ∎

Obviously, the arguments in this section can be extended to other invariance restrictions. For example, it is easy to provide an axiomatization of the Rawlsian ranking rule $\succcurlyeq_R$, where $\succcurlyeq_R$ is defined as follows:

$$\forall A, B \in \pi(X), \qquad A \succcurlyeq_R B \qquad \text{iff} \quad \min \lambda(A) \geqslant \min \lambda(B). \tag{3.4}$$

We omit these details.

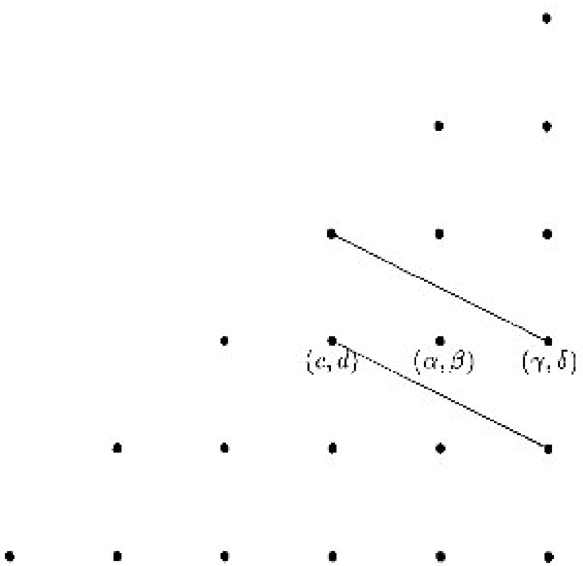

FIGURE 1

4. DIVERSITY OF CHOICE

In this section, we briefly describe an extension of the framework to allow for *diversity* of choice. Pattanaik and Xu in [5] argued that an axiom like WIND may not be reasonable since it ignores possible "interdependence" between various opportunities. For example, consider a situation where $X = \{\text{blue car, red car, train}\}$. An agent may prefer [blue car] to [train]. WIND implies that the agent prefers [blue car, red car] to [train, red car]. However Pattanaik and Xu argue that the set [train, red car] offers greater diversity of choice than the set [blue car, red car], and hence the former set may well be ranked higher than the latter.

Let $\{X_1, ..., X_K\}$ be a partition of X. Elements in X_k, $k = 1, ..., K$, represent opportunities which are "similar" to each other. For instances, cars of different colours are *different* opportunities but are similar. Hence, they may be put in the same X_l. Now, interpreting each X_l as consisting of 'informationally equivalent alternatives', the analysis developed in the previous sections can be readily applied to accomodate diversity of choice. The reader is referred to [3] for details.

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