White Noise Theory of Robust Nonlinear Filtering with Correlated State and Observation Noises

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Abstract. In the existing "direct" white noise theory of nonlinear filtering, the state process is still modelled as a Markov process satisfying an Itô stochastic differential equation, while a "finitely additive" white noise is used to model the observation noise. We remove this asymmetry by modelling the state process as the solution of a (stochastic) differential equation with a "finitely additive" white noise as the input. This enables us to introduce correlation between the state and observation noises, and to obtain robust nonlinear filtering equations in the correlated noise case.

1 Introduction

Nonlinear filtering equations are usually derived using the theory of stochastic differential equations as developed by Itô [1]. These equations hold, as is common in probability theory, almost surely in the space of continuous functions. The difficulty in applying this theory in practice is that the set of all possible real data has measure zero! In technical terms, nonlinear filtering equations obtained directly from Itô's theory are not robust. There are essentially two approaches to circumvent this difficulty. In one approach due to Davis [2] the Itô equations of nonlinear filtering are rewritten in an equivalent pathwise form which does not involve differential of the observation process that appear in Itô's framework. This observation process is actually the integrated version of real data. The nonlinear filtering equation is then shown to be continuous (in some topology) with respect to the observation and, therefore, it can be extended by continuity to the actual sample paths. In the other approach, due originally to Balakrishnan [3], one tries to model the observation process directly with a white noise error term. Although modeling white noise directly is intuitively appealing, it brings a host of mathematical complications. Kallianpur and Karandikar [4] developed the theory of nonlinear filtering in this framework. The advantage of this approach is that, once the mathematical difficulties are resolved, one always obtains results already in the robust form.

One drawback of the existing "direct" white noise theory is that the state process in this theory is still modelled as a Markov process satisfying an Itô stochastic differential equation. The state and observation noises are then jointly modelled in an appropriate product space which makes them necessarily uncorrelated. One purpose of this paper is to remove this asymmetry in the theory by modelling the state process as the solution of a stochastic differential equation with a "directly modelled" white noise as the input. We show that the solution of such a stochastic differential equation is, indeed, a Markov process. With this formulation, it is straightforward to introduce correlation between the state and observation noises. The other purpose of the paper is to obtain robust nonlinear filtering equation for stochastic dynamical systems with correlated state and observation noises.

For lack of space, we omit the mathematical preliminaries. Interested reader may find the necessary material in [4].

2 Markov Property

We begin with some notations. Let H, H_1, H_2 denote the Hilbert spaces

$$H = L_2([0,T], \mathbf{R}^d), H_1 = L_2([0,s], \mathbf{R}^d), H_2 = L_2([s,T], \mathbf{R}^d)$$

and B, B_1, B_2 denote the Banach spaces

$$B = C_0([0,T]; \mathbf{R}^d), B_1 = C_0([0,s], \mathbf{R}^d), B_2 = C_0([s,T], \mathbf{R}^d).$$

We define $\gamma : H \to B$ by

$$\gamma(\eta)(t) = \int_0^t \eta(\sigma) d\sigma \tag{1}$$

and define $\gamma_1 : H_1 \to B_1, \gamma_2 : H_2 \to B_2$ by

$$\gamma_1(\eta_1)(t) = \int_0^t \eta_1(\sigma) d\sigma, \gamma_2(\eta_2)(t) = \int_s^t \eta_2(\sigma) d\sigma.$$

Let us define $Q_1 : H \to H_1, Q_2 : H \to H_2, Q_1^\sim : B \to B_1, Q_2^\sim : B \to B_2$ by

$$Q_1(\eta)(t) = \eta(t), 0 \le t \le s; Q_2(\eta)(t) = \eta(t), s \le t \le T$$

$$Q_1^\sim(\zeta)(t) = \zeta(t), 0 \le t \le s; Q_2^\sim(\zeta)(t) = \zeta(t) - \zeta(s), s \le t \le T.$$

Finally, define $Q : H_1 \times H_2 \to H, Q^\sim : B_1 \times B_2 \to B$ by

$$Q(\eta_1, \eta_2)(t) = \begin{cases} \eta_1(t), t \le s \\ \eta_2(t), t > s \end{cases} ; Q^\sim(\zeta_1, \zeta_2)(t) = \begin{cases} \zeta_1(t), t \le s \\ \zeta_1(s) + \zeta_2(t), t > s. \end{cases}$$

Let m be the canonical Gauss measure on H and $m_1 = m \circ Q_1^{-1}, m_2 = m \circ Q_2^{-1}$ are Gauss measures on H_1 and H_2, respectively (see [4]).

Let $u : \mathbf{R}^d \to \mathbf{R}^d$ be a Lipschitz function. For $x_0 \epsilon \mathbf{R}^d$ and $\eta \epsilon H$, consider the following differential equation

$$\frac{dx_t(\eta)}{dt} = u(x_t(\eta)) + e_t(\eta) \tag{2}$$

where e defined by $e_t(\eta) = \eta(t)$ is a white noise on H. We shall see presently that equation (2) admits a unique solution. First note that (2) is equivalent to

$$x_t(\eta) = x_0 + \int_0^t u(x_r(\eta)) dr + \int_0^t e_r(\eta) dr \tag{3}$$

More generally, consider for $x_0 \epsilon \mathbf{R}^d$ and $\zeta \epsilon B = C([0,T], \mathbf{R}^d)$,

$$X_t = x_0 + \int_0^t u(X_r) dr + \zeta(t) \tag{4}$$

Existence and uniqueness of solution to this equation follows from the Lipschitz condition on u. Denoting the solution of (4) by $X_t(\zeta)$, it can be proved that $\zeta \to X_t(\zeta)$ is continuous from B into $\mathbf{R}$ and $\zeta \to X.(\zeta)$ is continuous from B into B. Moreover,

$$x_t(\eta) = X_t(\gamma(\eta)) \tag{5}$$

with γ defined by (1). Furthermore, $x_t \epsilon \mathcal{L}(H, \mathcal{C}, m)$ and $x. \epsilon \mathcal{L}(H, \mathcal{C}, m; B)$. See [4] for the definition of $\mathcal{L}$, and also that of conditional expectation appearing in the following theorem. We are now in a position to establish the Markov property.

Theorem 1 Let g be a bounded continuous function on $\mathbf{R}^d$. Then, for $t \geq s$,

$$E_m[g(x_t) \mid Q_1] = g_1(x_s) \tag{6}$$

where

$$g_1(x) = \int g(\Gamma_{st}(x, \eta_2))dm_2(\eta_2) \tag{7}$$

with $x_t' \triangleq \Gamma_{st}(x, \eta_2)$ being the unique solution of

$$x_t' = x + \int_s^t u(x_\tau')d\tau + \int_s^t e_\tau(\eta)d\tau, \quad t \geq s. \tag{8}$$

3 Robust Filtering with Correlated State and Observation Noises

We now consider the filtering problem where the state is the solution of the stochastic differential equation studied in the preceding section and the observation is corrupted by another white noise , possibly correlated with the state noise. We study only the scalar case of $d = 1$.

We take $H = L_2([0,T];\mathbf{R}^2)$ and for $\eta \epsilon H$ expressed as $\eta(t) = (\eta_1(t), \eta_2(t))$, we define

$$\begin{aligned} e_{1t}(\eta) &\triangleq \eta_1(t) \\ e_{2t}(\eta) &\triangleq \eta_2(t) \end{aligned}$$

We consider the following filtering model. The signal process $\{x_t(\eta)\}$ is given by

$$\begin{aligned} \dot{x}_t(\eta) &= u(x_t(\eta)) + \alpha e_{1t}(\eta) + \delta e_{2t}(\eta) \\ x_0(\eta) &= x_0 \end{aligned} \tag{9}$$

where $u : \mathbf{R} \to R$ is a bounded Lipschitz function and $\alpha > 0, \delta > 0$ are real numbers. The observation process $\{y_t(\eta)\}$ is given by

$$y_t(\eta) = h(x_t(\eta)) + e_{1t}(\eta) \tag{10}$$

where h is assumed to be a twice continuously differentiable function on $\mathbf{R}$ such that h and h' are bounded. As noted in the previous section, the solution process $\{x_t\}$ of (9) is a Markov process on $(H, \mathcal{C}, m)$, m being the canonical Gauss measure on H. Nonlinear filtering problem consists of obtaining a formula for

$$\pi_t(f, y) \triangleq E_m[f(x_t) \mid Q_t y] \tag{11}$$

where $Q_t : H \to H$ is defined by $(Q_t\eta)(s) = \eta(s)1_{[0,t]}(s)$. This is intuitively clear as the right hand side of (11) is the "white noise" analog of the expression $E[f(x_t) \mid y_s; 0 \leq s \leq t]$ calculated in the usual set-up. Taking a cue from the case of independent signal and noise in the white noise approach adopted here, as well as standard results on nonlinear filtering in the usual "countably additive" approach, one expects the following result. Let

$$\begin{aligned} \rho_t(\eta) \triangleq \exp\Big\{&\textstyle\int_0^t h(x_s(\eta))y_s(\eta)ds \\ &-\tfrac{1}{2}\alpha \textstyle\int_0^t h'(x_s(\eta))ds - \tfrac{1}{2}\int_0^t h^2(x_s(\eta))ds\Big\} \end{aligned} \tag{12}$$

Then

$$\pi_t(f, y) = \sigma_t(f, y)/\sigma_t(1, y) \tag{13}$$

where

$$\sigma_t(f, y) \triangleq E_0[f(x_t)\rho_t \mid Q_t y] \tag{14}$$

where $E_0(\cdot \mid Q_t y)$ is the conditional expectation under the measure m_0, defined by

$$m_0(C) = \int_C \rho_T^{-1} dm, \quad C \epsilon \mathcal{C} \tag{15}$$

To evaluate the desired conditional expectation, we follow the approach of Davis to robust filtering in the "countably additive" setting. We introduce a new process $\{z_t(\eta)\}$ by

$$\begin{aligned} \dot{z}_t(\eta) &= \dot{x}_t(\eta) - \alpha y_t(\eta) \\ z_0(\eta) &= x_0 \end{aligned} \tag{16}$$

Then

$$z_t(\eta) = v_t(y(\eta), e_2(\eta)) \tag{17}$$

where, for $\phi_1, \phi_2 \epsilon \overline{H} = L_2([0,T] : \mathbf{R})$, $v_t \equiv v_t(\phi_1, \phi_2)$ is the unique solution of

$$\begin{aligned} v_t = x_0 &+ \textstyle\int_0^t [u(v_s + \alpha\hat{\phi}_1(s)) - \alpha h(v_s + \alpha\hat{\phi}_1(s))]ds \\ &+ \delta \textstyle\int_0^t \phi_2(s)ds \end{aligned} \tag{18}$$

with $\hat{\phi}_1(s) \triangleq \int_0^s \phi_1(\tau)d\tau$. Let $\overline{m}$ be the canonical Gauss measure on $\overline{H}$. Below follows our main result on robust nonlinear filtering for the case of correlated state and observation noises.

Theorem 2 For $\overline{\phi} \epsilon \overline{H}$, define

$$\begin{aligned} \sigma_t(f, \overline{\phi}) = &\textstyle\int_{\overline{H}} f(v_t(\phi_1, \phi_2) + \alpha\hat{\phi}_1(t)) \\ &\exp\Big\{\textstyle\int_0^t h(v_s(\phi_1, \phi_2) + \alpha\hat{\phi}_1(s))\phi_1(s)ds \\ &-\tfrac{1}{2}\textstyle\int_0^t (\alpha h' + h^2)v_s(\phi_1, \phi_2) + \alpha\hat{\phi}_1(s)ds\Big\} d\overline{m}(\overline{\phi}) \end{aligned} \tag{19}$$

Then we have

$$E_m[f(x_t) \mid Q_t y] = \frac{\sigma_t(f, y)}{\sigma_t(1, y)} \tag{20}$$

4 Conclusion

We stated, without proof, two important results on the direct modelling of white noise in stochastic systems. We showed that if we model the state process directly with the "finitely additive" white noise as the forcing term, the solution of the resulting stochastic differential equation is a Markov process. This led to the modelling of a stochastic dynamical system with correlated state and observation noises. We then obtained the robust nonlinear filtering formula for this correlated case. It may be interesting to study Radon-Nikodym derivatives for the case of correlated state and observation noises.

References

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